Linear Algebra Game

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Chapter 1

Vector Spaces

1.1 Zero Scalar Multiplication

Definition 1. We begin by defining a *vector space* V over a field K as an abelian group with four key axioms:

- **smul_add**: $\forall a \in K,\ x, y \in V, a \bullet (x + y) = a \bullet x + a \bullet y$
- **add_smul**: $\forall a, b \in K,\ x \in V, (a + b) \bullet x = a \bullet x + b \bullet x$
- **mul_smul**: $\forall a, b \in K,\ x \in V, (a * b) \bullet x = a \bullet (b \bullet x)$
- **one_smul**: $\forall x \in V, (1 : K) \bullet x = x$

Theorem 2. *In any vector space V over K, the scalar 0 multiplied by any vector gives the zero vector:*$\forall w \in V, (0 : K) \bullet w = (0 : V)$

1.2 Multiplying By The Zero Vector

Theorem 3. *In any vector space V over K, any scalar a multiplied by the zero vector gives the zero vector:*$\forall a \in K, a \bullet (0 : V) = (0 : V)$

1.3 Scaling By Negative One

Theorem 4. *In any vector space V over K, multiplying a vector by -1 gives its additive inverse:* $v \in V : (-1 : K) \bullet v = -v$

1.4 Zero Must Belong

Definition 5. A *subspace* is a subset of a vector space that acts similarly to a vector space itself. It has three main properties:

- **Nonempty**: W.Nonempty
- **Closure_Under_Addition**: $\forall (x, y \in V), x \in W \to y \in W \to x + y \in W$
- **Closure_Under_Scalar_Multiplication**: $\forall (a \in K)(x \in V), x \in W \to a \bullet x \in W$

Theorem 6. *This is a proof that any subspace contains the zero vector:*$(0 : V) \in W$

1.5 Negatives In Subspace

Theorem 7. *This is a proof that if a subspace contains a vector 'x', it also contains '-x':*$\forall x \in V, x \in W \to (-x) \in W$

Chapter 2

Linear Independence Span

2.1 Linear Combinations

Definition 8. $\exists (s : (Finset\ V))(f : V \to K), (\uparrow s \subseteq S) \wedge (x = Finset.sum\ (\text{fun}\ v \Rightarrow f(v) \bullet v))$

Theorem 9. *If $v \in S$, then v is a linear combination of S:* $(S : (Set\ V))(v : V)(hv : v \in S) : is\_linear\_combination\ K\ V\ S\ v$

2.2 Introducing Span

Definition 10. $\{x \in V | linear\_combination\ K\ V\ S\ x\}$

Theorem 11. *If $v \in S$, then $v \in span\ K\ V\ S$: $S : SetV v : V(hv : v \in S) : v \in span\ K\ V\ S$*

2.3 Monotonicity Of Span

Theorem 12. *The span of sets is monotonic. Simply, this means that if you have $h : A \subseteq B$, then $span\_mono\ K\ V\ h$ is a proof that $span\ K\ V\ A \subseteq span\ K\ V\ B$: $A\ B : Set\ V\ (hAB : A \subseteq B) : span\ K\ V\ A \subseteq span\ K\ V\ B$*

2.4 Linear Independence

Definition 13. A set of vectors S is *linearly independent* if no vector in S can be written as a linear combination of the others. Equivalently, the only solution to a linear combination of elements of S equaling zero is the trivial solution (all coefficients zero). Here we formalize this condition: $\forall (s : FinsetV)(f : V \to K), (\uparrow s \subseteq S) \to (Finset.sums(funv \mapsto fv \bullet v) = 0) \to (\forall v \in s, fv = 0)$

Theorem 14. *The empty set is linearly independent: $linear\_independentKV(\emptyset : SetV)$*

2.5 Linear Independence Of Subsets

Theorem 15. *If A is a linearly independent set, and we have $B \subseteq A$, then B is also linearly independent: $AB : SetV(hBsubA : B \subseteq A)(hA : linear\_independent\_vKVA)$*

2.6 Supersets Span The Whole Space

Theorem 16. *If a set A spans the whole space V, then any superset of A also spans V:* $ABT : SetV(hT : \forall(x : V), x \in T)(hA : T = spanKVA)(hAsubB : A \subseteq B) : T = spanKVB$

2.7 Uniqueness Of Linear Combinations

Theorem 17. $S : SetV(hS : linear\_independentKVS)(st : FinsetV)(hs :\uparrow s \subseteq S)(ht :\uparrow t \subseteq S)(fg : V \to K)(hf\emptyset : \forall v \notin s, fv = 0)(hg0 : \forall v \notin t, gv = 0)(heq : Finset.sums(funv \Rightarrow fv \bullet v) = Finset.sumt(funv \Rightarrow gv \bullet v)) : f = g$

2.8 Linear Independence Of Set With Insertion

Theorem 18. $S : SetVv : V(hS : linear\_independentKVS)(hv\_not\_span : v \notin spanKVS) : linear\_independentKV(S \cup v)$

2.9 Span After Removing Elements

Definition 19. $x \in V|linear\_combinationKVSx$

Theorem 20. $S : SetVv : V(hS : linear\_independentKVS)(hv\_not\_span : v \notin spanKVS) : linear\_independentKV(S \cup v)$

Chapter 3

Inner Product World

3.1 Linear Combinations

Definition 21. – Properties are simpler for real case

Definition 22.

Lemma 23.

Lemma 24.

Lemma 25.

Lemma 26.

Lemma 27.

Lemma 28.

Lemma 29.

Lemma 30.

Lemma 31.

Lemma 32.

Lemma 33.

Definition 34.

Definition 35.

Lemma 36.

Lemma 37.

Theorem 38.

Theorem 39.

Theorem 40.

Theorem 41.

Theorem 42.

Theorem 43.

Theorem 44.

Theorem 45.

Theorem 46.

Theorem 47.

Theorem 48.