Linear Algebra Game

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Chapter 1

VectorSpaces

1.1 ZeroScalarMultiplication

Definition 1. A *vector space* is a space over a field K with an abelian group V. It has four main properties:

- **smul_add**: $\forall a \in K, x, y \in V, a \bullet (x + y) = a \bullet x + a \bullet y$
- **add_smul**: $\forall a, b \in K, x \in V, (a + b) \bullet x = a \bullet x + b \bullet x$
- **mul_smul**: $\forall a, b \in K, x \in V, (a * b) \bullet x = a \bullet (b \bullet x)$
- **one_smul**: $\forall x \in V, 1 \in K \bullet x = x$

Theorem 2. *In any vector space V over K, the scalar 0 multiplied by any vector gives the zero vector: (0 : K) • w = (0 : V)*

1.2 MultiplyingByTheZeroVector

Theorem 3. *In any vector space V over K, any scalar a multiplied by the zero vector gives the zero vector:*$a \in K, 0 \in V : a \bullet 0 = 0$

1.3 ScalingByNegativeOne

Theorem 4. *In any vector space V over K, multiplying a vector by -1 gives its additive inverse:* $v \in V, -1 \in K : -1 \bullet v = -v$

1.4 ZeroMustBelong

Definition 5. A *subspace* is a subset of a vector space that acts similarly to a vector space itself. It has three main properties:

- **Nonempty**: W.Nonempty
- **Closure_Under_Addition**: $\forall (xy : V), x \in W \to y \in W \to x + y \in W$
- **Closure_Under_Scalar_Multiplication**: $\forall (a : K)(x : V), x \in W \to a \bullet x \in W$

Theorem 6. *This is a proof that any subspace contains the zero vector:*$(0 : V) \in W$

1.5 NegativesInSubspace

Theorem 7. *This is a proof that if a subspace contains a vector 'x', it also contains '-x':*$\forall x \in V, x \in W \to (-x) \in W$

Chapter 2

LinearIndependenceSpan

2.1 LinearCombinations

Definition 8. $\exists (s : FinsetV)(f : V \to K), (\uparrow s \subseteq S) \wedge (x = Finset.sums(funv \Rightarrow fv \bullet v))$

Theorem 9. *If $v \in S$, then v is a linear combination of S: $S : SetVv : V(hv : v \in S) : is_linear_combinationKVSv$*

2.2 IntroducingSpan

Definition 10. $x \in V | linear_combinationKVSx$

Theorem 11. *If $v \in S$, then $v \in spanKVS$: $S : SetVv : V(hv : v \in S) : v \in spanKVS$*

2.3 MonotonicityOfSpan

Theorem 12. *The span of sets is monotonic. Simply, this means that if you have $h : A \subseteq B$, then $span_mono$ K V h is a proof that $spanKVA \subseteq spanKVB$.: $AB : SetV(hAB : A \subseteq B) : spanKVA \subseteq spanKVB$*

2.4 LinearIndependence

Definition 13. A set of vectors S is **linearly independent** if no vector in S can be written as a linear combination of the others. Equivalently, the only solution to a linear combination of elements of S equaling zero is the trivial solution (all coefficients zero). Here we formalize this condition: $\forall (s : FinsetV)(f : V \to K), (\uparrow s \subseteq S) \to (Finset.sums(funv \mapsto fv \bullet v) = 0) \to (\forall v \in s, fv = 0)$

Theorem 14. *The empty set is linearly independent: $linear_independent_vKV(\emptyset : SetV)$*

2.5 LinearIndependenceOfSubsets

Theorem 15. *If A is a linearly independent set, and we have $B \subseteq A$, then B is also linearly independent: $AB : SetV(hBsubA : B \subseteq A)(hA : linear_independent_vKVA)$*

2.6 SupersetsSpanTheWholeSpace

Theorem 16. *If a set A spans the whole space V, then any superset of A also spans V:* $ABT : SetV(hT : \forall(x : V), x \in T)(hA : T = spanKVA)(hAsubB : A \subseteq B) : T = spanKVB$

2.7 UniquenessOfLinearCombinations

Theorem 17. $S : SetV(hS : linear_independent_vKVS)(st : FinsetV)(hs :\uparrow s \subseteq S)(ht :\uparrow t \subseteq S)(fg : V \to K)(hf\emptyset : \forall v \notin s, fv = 0)(hg0 : \forall v \notin t, gv = 0)(heq : Finset.sums(funv \Rightarrow fv \bullet v) = Finset.sumt(funv \Rightarrow gv \bullet v)) : f = g$

2.8 LinearIndependenceOfSetWithInsertion

Theorem 18. $S : SetVv : V(hS : linear_independent_vKVS)(hv_not_span : v \notin spanKVS) : linear_independent_vKV(S \cup v)$

2.9 SpanAfterRemovingElements

Definition 19. $x \in V | linear_combinationKVSx$

Theorem 20. $S : SetVv : V(hS : linear_independent_vKVS)(hv_not_span : v \notin spanKVS) : linear_independent_vKV(S \cup v)$