Linear Algebra Game

Adam Kern Justin Morrill Letian Yang Huiyu Chen

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Chapter 1

VectorSpaces

1.1 ZeroScalarMultiplication

Definition 1. A *vector space* is a space over a field K with an abelian group V. It has four main properties:

- **smul_add**: $\forall a \in K, x, y \in V, a \bullet (x + y) = a \bullet x + a \bullet y$
- **add_smul**: $\forall a, b \in K, x \in V, (a + b) \bullet x = a \bullet x + b \bullet x$
- **mul_smul**: $\forall a, b \in K, x \in V, (a * b) \bullet x = a \bullet (b \bullet x)$
- **one_smul**: $\forall x \in V, 1 \in K \bullet x = x$

Theorem 2. *In any vector space V over K, the scalar 0 multiplied by any vector gives the zero vector: (0 : K) • w = (0 : V)*

1.2 MultiplyingByTheZeroVector

Theorem 3. *In any vector space V over K, any scalar a multiplied by the zero vector gives the zero vector:*$(a : K) : a \bullet (0 : V) = (0 : V)$

1.3 ScalingByNegativeOne

Theorem 4. *In any vector space V over K, multiplying a vector by -1 gives its additive inverse:* $(v : V) : (-1 : K) \bullet v = -v$

1.4 ZeroMustBelong

Definition 5. A *subspace* is a subset of a vector space that acts similarly to a vector space itself. It has three main properties:

- **Nonempty**: W.Nonempty
- **Closure_Under_Addition**: $\forall (xy : V), x \in W \to y \in W \to x + y \in W$
- **Closure_Under_Scalar_Multiplication**: $\forall (a : K)(x : V), x \in W \to a \bullet x \in W$

Theorem 6. *This is a proof that any subspace contains the zero vector:*$(0 : V) \in W$

1.5 NegativesInSubspace

Theorem 7. *This is a proof that if a subspace contains a vector 'x', it also contains '-x':* $\forall (x : V), x \in W \to (-x) \in W$

Chapter 2

LinearIndependenceSpan

2.1 LinearCombinations

Definition 8. $\exists(s : FinsetV)(f : V \to K), (\uparrow s \subseteq S) \wedge (x = Finset.sums(funv \Rightarrow fv \bullet v))$

Theorem 9. *If $v \in S$, then v is a linear combination of S:* $S : SetVv : V(hv : v \in S) : is_linear_combinationKVSv$

2.2 IntroducingSpan

Definition 10. $x \in V | linear_combinationKVSx$

Theorem 11. *If $v \in S$, then $v \in spanKVS$:* $S : SetVv : V(hv : v \in S) : v \in spanKVS$

2.3 MonotonicityOfSpan

2.4 LinearIndependence

2.5 LinearIndependenceOfSubsets

2.6 SupersetsSpanTheWholeSpace

2.7 UniquenessOfLinearCombinations

2.8 LinearIndependenceOfSetWithInsertion

2.9 SpanAfterRemovingElements