Linear Algebra Game

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Chapter 1

Vector Spaces

1.1 Zero Scalar Multiplication

Definition 1. We begin by defining a *vector space* V over a field K as an abelian group with four key axioms:

- **smul_add**: $\forall a \in K,\ x, y \in V, a \bullet (x + y) = a \bullet x + a \bullet y$
- **add_smul**: $\forall a, b \in K,\ x \in V, (a + b) \bullet x = a \bullet x + b \bullet x$
- **mul_smul**: $\forall a, b \in K,\ x \in V, (a * b) \bullet x = a \bullet (b \bullet x)$
- **one_smul**: $\forall x \in V, (1 : K) \bullet x = x$

Theorem 2. *In any vector space V over K, the scalar 0 multiplied by any vector gives the zero vector:*$\forall w \in V, (0 : K) \bullet w = (0 : V)$

1.2 Multiplying By The Zero Vector

Theorem 3. *In any vector space V over K, any scalar a multiplied by the zero vector gives the zero vector:*$\forall a \in K, a \bullet (0 : V) = (0 : V)$

1.3 Scaling By NegativeOne

Theorem 4. *In any vector space V over K, multiplying a vector by -1 gives its additive inverse:* $v \in V : (-1 : K) \bullet v = -v$

1.4 Zero Must Belong

Definition 5. A *subspace* is a subset of a vector space that acts similarly to a vector space itself. It has three main properties:

- **Nonempty**: W.Nonempty
- **Closure_Under_Addition**: $\forall (x, y \in V), x \in W \to y \in W \to x + y \in W$
- **Closure_Under_Scalar_Multiplication**: $\forall (a \in K)(x \in V), x \in W \to a \bullet x \in W$

Theorem 6. *This is a proof that any subspace contains the zero vector:*$(0 : V) \in W$

1.5 Negatives In Subspace

Theorem 7. *This is a proof that if a subspace contains a vector 'x', it also contains '-x':*$\forall x \in V, x \in W \rightarrow (-x) \in W$

Chapter 2

LinearIndependenceSpan

2.1 Linear Combinations

Definition 8. $\exists(s : (Finset\ V))(f : V \to K), (\uparrow s \subseteq S) \wedge (x = Finset.sum\ (\text{fun}\ v \Rightarrow f(v) \bullet v))$

Theorem 9. *If $v \in S$, then v is a linear combination of S:* $(S : (Set\ V))(v : V)(hv : v \in S) : is\_linear\_combination\ K\ V\ S\ v$

2.2 Introducing Span

Definition 10. $\{x \in V | linear\_combination\ K\ V\ S\ x\}$

Theorem 11. *If $v \in S$, then $v \in span\ K\ V\ S$: $S : SetVv : V(hv : v \in S) : v \in span\ K\ V\ S$*

2.3 Monotonicity Of Span

Theorem 12. *The span of sets is monotonic. Simply, this means that if you have $h : A \subseteq B$, then span_mono K V h is a proof that span K V $A \subseteq$ span K V B: A B : Set V (hAB : $A \subseteq B$) : span K V $A \subseteq$ span K V B*

2.4 Linear Independence

Definition 13. A set of vectors S is *inearly independent* if no vector in S can be written as a linear combination of the others. Equivalently, the only solution to a linear combination of elements of S equaling zero is the trivial solution (all coefficients zero). Here we formalize this condition: $\forall(s : FinsetV)(f : V \to K), (\uparrow s \subseteq S) \to (Finset.sums(funv \mapsto fv \bullet v) = 0) \to (\forall v \in s, fv = 0)$

Theorem 14. *The empty set is linearly independent: $linear\_independentKV(\emptyset : SetV)$*

2.5 Linear Independence Of Subsets

Theorem 15. *If A is a linearly independent set, and we have $B \subseteq A$, then B is also linearly independent: $AB : SetV(hBsubA : B \subseteq A)(hA : linear\_independent\_vKVA)$*

2.6 Supersets Span The Whole Space

Theorem 16. *If a set A spans the whole space V, then any superset of A also spans V:* $ABT : SetV(hT : \forall(x : V), x \in T)(hA : T = spanKVA)(hAsubB : A \subseteq B) : T = spanKVB$

2.7 Uniqueness Of Linear Combinations

Theorem 17. $S : SetV(hS : linear\_independentKVS)(st : FinsetV)(hs :\uparrow s \subseteq S)(ht :\uparrow t \subseteq S)(fg : V \to K)(hf\emptyset : \forall v \notin s, fv = 0)(hg0 : \forall v \notin t, gv = 0)(heq : Finset.sums(funv \Rightarrow fv \bullet v) = Finset.sumt(funv \Rightarrow gv \bullet v)) : f = g$

2.8 Linear Independence Of Set With Insertion

Theorem 18. $S : SetVv : V(hS : linear\_independentKVS)(hv\_not\_span : v \notin spanKVS) : linear\_independentKV(S \cup v)$

2.9 Span After Removing Elements

Definition 19. $x \in V|linear\_combinationKVSx$

Theorem 20. $S : SetVv : V(hS : linear\_independentKVS)(hv\_not\_span : v \notin spanKVS) : linear\_independentKV(S \cup v)$