Invisibility and Maxwell's Equations

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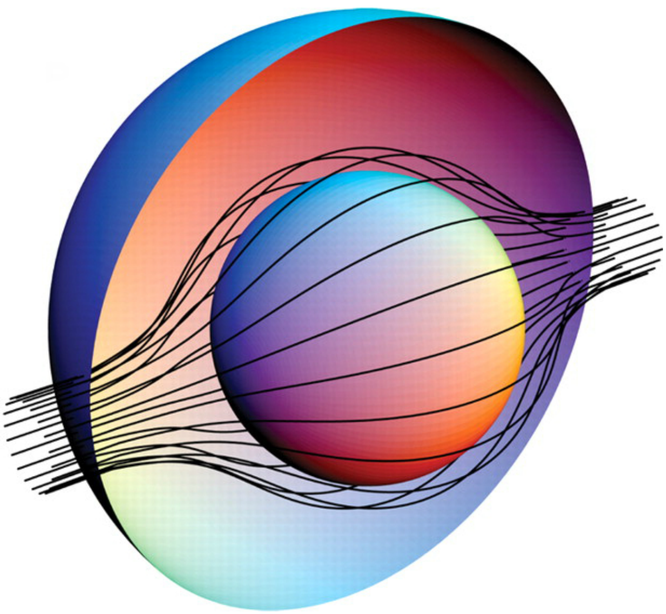
Abstract

We investigate the origin of the form invariance of Maxwell's Equations and see how this relates to the theory of transformation optics, in which a particular geometry is related to the equivalent electromagnetic properties required in Euclidian space to create the geometry. This idea is used to reformulate transformation optics in the more natural language of differential geometry and to derive the electromagnetic properties associated with arbitrary transformations using these techniques. The theory of transformation optics is then generalised to non-linear electromagnetic media and to a spacetime formulation.



From Controlling Electromagnetic Fields, J.B. Pendry et al. [1]

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Contents

1 Introduction

It is well known that Maxwell's Equations are coordinate independent, as any physical equations should be, but what is often less appreciated is that they are in fact form invariant. This means that not only do the equations hold in any coordinate system, but they also take on the same form, so long as the variables are appropriately transformed.

Only a decade ago, in 2006, was it recognised that this form invariance implies that active coordinate transformations can instead be interpreted as a transformation of the electromagnetic properties of the space [1]. This resulted in the foundation of the field of transformation optics [2].

The best known application of this is to create the ultimate illusion: invisibility. If a transformation is chosen such that all electromagnetic fields are distorted around an object with the boundary fields of the region left unaffected, then any external observer only able to measure the boundary fields will not be able to detect the presence of the object, rendering it invisible. The electromagnetic properties corresponding to this transformation can then be calculated, and a material fashioned with these properties would effectively form an invisibility cloak.

This may seem farfetched, but with the impressive rate of developments in the field of metamaterials since the turn of the century, this concept is of more than just academic interest.

Metamaterials make possible novel electromagnetic properties, such as a negative index of refraction, through intricate sub-wavelength structures which appear as an effectively homogeneous medium to incident radiation [3]. The development of these materials has been such that the construction of a material in which the permittivity and permeability are varied independently and arbitrarily throughout the medium is now a feasible reality, allowing the electromagnetic fields to be controlled through the medium like a fluid.

In fact, in November 2006, less than six months after the first paper was published recognising the theoretical possibility for invisibility using metamaterials, a metamaterial cloak was constructed which could conceal a cylindrical object from microwaves in a particular plane [4]. More recently, in September 2015, an ultra thin invisibility skin cloak, just 80nm thick, was fabricated which could be wrapped around an arbitrarily shaped 3D object to cloak it from both wide-field imaging and phase sensitive detection methods, by reflecting any incident light as if the object was a flat mirror [5].

Therefore the theoretical development of the theory of transformation optics is very relevant, with any theoretical possibilities likely to soon be realised by developments in metamaterials.

2 Project Overview

Current literature on transformation optics mostly uses tensor analysis to calculate the electromagnetic properties required to achieve a given coordinate transformation, often requiring unwieldy calculations and giving little geometrical insight [6, 7, 8]. Therefore, one of the main goals of this project will be to reformulate the theory of transformation optics in terms of modern differential geometry, in order to gain a better geometrical understanding of the transformations, as well as to simplify and generalise the current theory.

One motive for this comes from the naturalness with which modern differential geometry, in particular differential forms, can be applied to electromagnetism [9, 10]. Another comes from the well recognised equivalence between transformation optics and general relativity [2, 11]. In general relativity, spacetime geometries are induced by mass distributions, causing light to follow a curved geodesic path in regions where the metric depends on position. Similarly in transformation optics, spatial geometries are induced by electromagnetic properties, causing light to follow a curved geodesic path in regions where the refractive index depends on position. Since differential geometry is the natural language of general relativity and lends itself very readily to electromagnetism and coordinate transformations, there is a strong motivation to apply these techniques to transformation optics.

In addition, when using tensor analysis, the representation of the constitutive relations in terms of permittivity and permeability tensors implicitly requires linearity, which may not hold in all materials, or for stronger fields. Recent work on metamaterials has begun to consider the inclusion of non-linear components to enhance their non-linear response and give, for example, an intensity dependent electromagnetic response [12, 13]. Therefore, another aim of this work is to generalise the current theory of transformation optics to allow for non-linear constitutive relations. This will allow the theory of transformation optics to be applied to more realistic non-linear materials including memory dependent electromagnetic continua.

We begin, in section 3, by considering transformations of the integral forms of Maxwell's Equations. In this formulation the physical interpretation of Maxwell's Equations is explicit, so we aim to get a better understanding of the origin of the form invariance of the equations.

In section 4, we then reformulate Maxwell's Equations in terms of differential forms and consider how these equations transform. We will be able to use the basic results from the integral transformations to verify our more general differential geometry equations.

Finally in section 5, we will generalise to a differential geometry spacetime formulation of Maxwell's Equations and transformation optics, and discover the most general non-linear constitutive relation that is consistent with Lorentz invariance.

3 Transformation of the Integral Formulation of Maxwell's Equations

3.1 Transformations of Integral Equations

In the literature, Maxwell's Equations are usually transformed in differential form, using either vector or tensor calculus [6, 7, 8]. Here we transform the integral equations, in the hope of gaining a better insight into the origin of the form invariance.

Consider some smooth bounded domain $M \subseteq \mathbb{R}^n$ with $n > 2$ and boundary ∂M. Let $\xi : \widetilde{M} \to M$ be a diffeomorphism (a smooth mapping with a smooth inverse) which defines the related domain $\widetilde{M}$ [14].

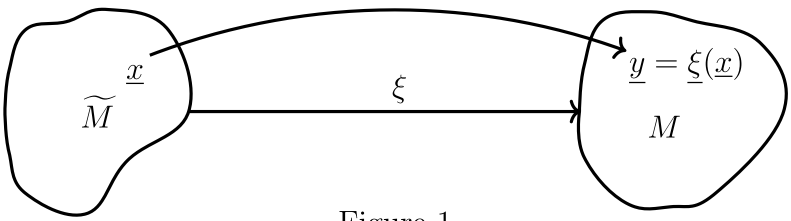

Figure 1.

First consider the transformation of the basis vectors.

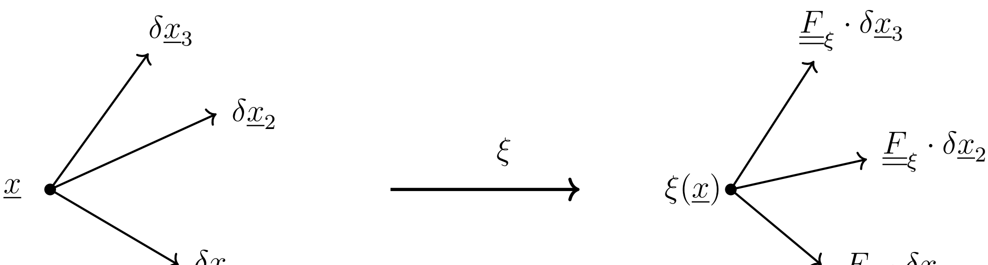

Figure 2.

$$\underline{x} \mapsto \underline{\xi}(\underline{x}) := \underline{y} \tag{1}$$

$$\underline{x} + \delta\underline{x} \mapsto \underline{\xi}(\underline{x} + \delta\underline{x})$$
$$\Rightarrow [\underline{x} + \delta\underline{x}]_i \mapsto \underline{\xi}_i(\underline{x}) + \frac{\partial \xi_i}{\partial x_j}(\delta x)_j$$
$$\therefore \delta\underline{x} \mapsto \underline{\underline{F}}_\xi \cdot \delta\underline{x} \qquad \text{where} \qquad \underline{\underline{F}}_\xi := (\nabla\underline{\xi})^T \tag{2}$$

The transformation of a line integral under the diffeomorphism ξ is now trivial.

$$\int_L \underline{f}_l(\underline{y}) \cdot \mathrm{d}\underline{l} = \int_{\tilde{L}} \underline{f}_l[\underline{\xi}(\underline{x})] \cdot \underline{\underline{F}}_\xi \cdot \mathrm{d}\underline{\tilde{l}} = \int_{\tilde{L}} \underline{\tilde{f}}_l(\underline{x}) \cdot \mathrm{d}\underline{\tilde{l}} \tag{3}$$

where

$$\underline{\tilde{f}}_l(\underline{x}) := \underline{\underline{F}}_\xi^T \cdot \underline{f}_l[\underline{\xi}(\underline{x})].$$

Surface and volume integrals transform similarly, as shown in appendix A.

$$\int_{\partial M} \underline{f}_S(\underline{y}) \cdot \mathrm{d}\underline{S} = \int_{\partial \widetilde{M}} \underline{\widetilde{f}}_S(\underline{x}) \cdot \mathrm{d}\underline{\widetilde{S}} \tag{4}$$

$$\int_{M} f_V(\underline{y}) \cdot \mathrm{d}V = \int_{\widetilde{M}} \widetilde{f}_V(\underline{x}) \cdot \mathrm{d}\widetilde{V} \tag{5}$$

where

$$\underline{\widetilde{f}}_S(\underline{x}) = J_\xi(\underline{x})\ \underline{\underline{F}}_\xi^{-1} \cdot \underline{f}_S[\underline{\xi}(\underline{x})]$$
$$\widetilde{f}_V(\underline{x}) = J_\xi(\underline{x})\ f_V[\underline{\xi}(\underline{x})]$$

$$J_\xi(\underline{x}) := \det \underline{\underline{F}}_\xi.$$

3.2 Transformation of Maxwell's Equations

Now let us consider the integral form of Maxwell's Equations, defined on some domain M.

$$\oint_S \underline{D} \cdot \mathrm{d}\underline{S} = \int_V \rho \, \mathrm{d}V \tag{6}$$

$$\oint_S \underline{B} \cdot \mathrm{d}\underline{S} = 0 \tag{7}$$

$$\oint_L \underline{E} \cdot \mathrm{d}\underline{l} = -\frac{\partial}{\partial t} \int_S \underline{B} \cdot \mathrm{d}\underline{S} \tag{8}$$

$$\oint_L \underline{H} \cdot \mathrm{d}\underline{l} = \int_S \underline{J} \cdot \mathrm{d}\underline{S} + \frac{\partial}{\partial t} \int_S \underline{D} \cdot \mathrm{d}\underline{S} \tag{9}$$

Applying the transformation rules (3)-(5), we obtain Maxwell's Equations defined on the related domain $\widetilde{M}$.

$$\oint_{\widetilde{S}} \underline{\widetilde{D}} \cdot \mathrm{d}\underline{\widetilde{S}} = \int_{\widetilde{V}} \widetilde{\rho} \, \mathrm{d}\widetilde{V} \tag{10}$$

$$\oint_{\widetilde{S}} \underline{\widetilde{B}} \cdot \mathrm{d}\underline{\widetilde{S}} = 0 \tag{11}$$

$$\oint_{\widetilde{L}} \underline{\widetilde{E}} \cdot \mathrm{d}\underline{\widetilde{l}} = -\frac{\partial}{\partial t} \int_{\widetilde{S}} \underline{\widetilde{B}} \cdot \mathrm{d}\underline{\widetilde{S}} \tag{12}$$

$$\oint_{\widetilde{L}} \underline{\widetilde{H}} \cdot \mathrm{d}\underline{\widetilde{l}} = \int_{\widetilde{S}} \underline{\widetilde{J}} \cdot \mathrm{d}\underline{\widetilde{S}} + \frac{\partial}{\partial t} \int_{\widetilde{S}} \underline{\widetilde{D}} \cdot \mathrm{d}\underline{\widetilde{S}} \tag{13}$$

The new fields are given by

$$\widetilde{\underline{D}}(\underline{x}) = J_\xi \ \underline{\underline{F}}_\xi^{-1} \cdot \underline{D}[\underline{\xi}(\underline{x})] \tag{14}$$
$$\widetilde{\underline{E}}(\underline{x}) = \underline{\underline{F}}_\xi^{T} \cdot \underline{E}[\underline{\xi}(\underline{x})] \tag{15}$$
$$\widetilde{\underline{B}}(\underline{x}) = J_\xi \ \underline{\underline{F}}_\xi^{-1} \cdot \underline{B}[\underline{\xi}(\underline{x})] \tag{16}$$
$$\widetilde{\underline{H}}(\underline{x}) = \underline{\underline{F}}_\xi^{T} \cdot \underline{H}[\underline{\xi}(\underline{x})] \tag{17}$$
$$\widetilde{\rho}(\underline{x}) = J_\xi \ \rho[\underline{\xi}(\underline{x})] \tag{18}$$
$$\widetilde{\underline{J}}(\underline{x}) = J_\xi \ \underline{E}^{-1} \cdot \underline{J}[\underline{\xi}(\underline{x})]. \tag{19}$$

We see that the form of Maxwell's Equations is unchanged. This arises because of the integral interpretation of the equations. The electric field E and the magnetic field H are interpreted as quantities per unit length which must be transformed as line integral variables, while the displacement field D and the magnetic flux B are interpreted as quantities per unit area (fluxes) which must be transformed as surface integral variables. These interpretations lead to consistent transformations of the fields. Also, we have required that the E and D fields, and the H and B fields, are independently variable, requiring arbitrary electromagnetic properties in the considered domain. It is these features that have allowed Maxwell's equations to be written in a form invariant way.

By choosing a linear constitutive relation on our original domain M of the form

$$\underline{D}(\underline{x}) = \underline{\underline{\epsilon}} \cdot \underline{E}(\underline{x}), \tag{20}$$

we obtain

$$\widetilde{\underline{D}}(\underline{x}) = J_\xi \ \underline{\underline{F}}_\xi^{-1} \cdot \underline{\underline{\epsilon}} \cdot \underline{\underline{F}}_\xi^{-T} \cdot \widetilde{\underline{E}}(\underline{x}). \tag{21}$$

So we have

$$\widetilde{\underline{\underline{\epsilon}}} = J_\xi \ \underline{\underline{F}}_\xi^{-1} \cdot \underline{\underline{\epsilon}} \cdot \underline{\underline{F}}_\xi^{-T}. \tag{22}$$

And equivalently

$$\widetilde{\underline{\underline{\mu}}} = J_\xi \ \underline{\underline{F}}_\xi^{-1} \cdot \underline{\underline{\mu}} \cdot \underline{\underline{F}}_\xi^{-T}. \tag{23}$$

Note that $\widetilde{\underline{\underline{\epsilon}}}$ and $\widetilde{\underline{\underline{\mu}}}$ are symmetric if $\underline{\underline{\epsilon}}$ and $\underline{\underline{\mu}}$ are symmetric. This is expected since energy conservation requires the permittivity and permeability to be Hermitian, and therefore symmetric if the material is lossless. Also the transformed medium is impedance matched, $\widetilde{\underline{\underline{\epsilon}}} = \widetilde{\underline{\underline{\mu}}}$, if the reference medium is impedance matched, $\underline{\underline{\epsilon}} = \underline{\underline{\mu}}$, thus preventing reflections from the new medium [15]. These are in fact general properties of transformation optics.

The form invariance of Maxwell's Equations has interesting consequences for the inverse problem of determining the fields inside a domain given complete measurements of the surface fields [16]. Defining an arbitrary diffeomorphism ξ from M onto itself subject to the condition that $\underline{\xi}(\underline{x}) = \underline{x}$ for $\underline{x} \in \partial M$, new fields $\widetilde{\underline{E}}, \widetilde{\underline{H}}, \widetilde{\underline{D}}, \widetilde{\underline{B}}$ can be obtained which will satisfy Maxwell's Equations in the new domain $\widetilde{M}$ and which will satisfy the same boundary measurements on ∂M. As there are an infinite number of diffeomorphisms of this form, there will be an infinite degeneracy of possible fields inside the domain which satisfy the same boundary measurements [14]. Since boundary measurements correspond to what is observed, for example via electromagnetic radiation, this degeneracy leads to the possibility of perfect electromagnetic cloaking through the application of a carefully chosen diffeomorphism.

It is also worth noting here that if we were restricted to a homogeneous isotropic medium, where the permittivity and permeability are scalars, Maxwell's Equations would not be form invariant as it would be impossible to consistently redefine the fields to recover the original form of the equations. Therefore there would only be a single set of field solutions in the domain. This is equivalent to the well known uniqueness theorems of electrostatics and magnetostatics, which show that an isotropic material with given boundary conditions has a unique interior field solution.

3.3 Non-Linear Electromagnetic Media

In the above we chose a linear constitutive equation in the original domain and so found a linear constitutive relation in the transformed domain. However, there are many materials for which the constitute equations are non-linear, and even linear materials often become non-linear for strong fields. Now let us consider how transformation optics applies in these materials.

In general the electromagnetic response of a material is defined by

$$\underline{D} = \frac{\partial U}{\partial \underline{E}} := \underline{\epsilon}(\underline{E}), \tag{24}$$

$$\underline{B} = \frac{\partial U}{\partial \underline{H}} := \underline{\mu}(\underline{H}), \tag{25}$$

which in turn defines the generalised permittivity and permeability operators, $\underline{\epsilon}$ and $\underline{\mu}$.

Similarly in another domain $\widetilde{M}$ related by a diffeomorphism ξ we have

$$\underline{\widetilde{D}} = \underline{\widetilde{\epsilon}}(\underline{\widetilde{E}}), \tag{26}$$

$$\underline{\widetilde{B}} = \underline{\widetilde{\mu}}(\underline{\widetilde{H}}). \tag{27}$$

From the field transformations under the diffeomorphism ξ derived earlier we have

$$\underline{\widetilde{D}}(\underline{x}) = J_\xi \; \underline{\underline{F}}_\xi^{-1} \cdot \underline{D}(\underline{\xi}(\underline{x})) \qquad := \xi_D \underline{D}, \tag{28}$$

$$\underline{\widetilde{E}}(\underline{x}) = \underline{\underline{F}}_\xi^{T} \cdot \underline{E}(\underline{\xi}(\underline{x})) \qquad := \xi_E \underline{E}, \tag{29}$$

$$\underline{\widetilde{B}}(\underline{x}) = J_\xi \; \underline{\underline{F}}_\xi^{-1} \cdot \underline{B}(\underline{\xi}(\underline{x})) \qquad := \xi_B \underline{B}, \tag{30}$$

$$\underline{\widetilde{H}}(\underline{x}) = \underline{\underline{F}}_\xi^{T} \cdot \underline{H}(\underline{\xi}(\underline{x})) \qquad := \xi_H \underline{H}. \tag{31}$$

Therefore the fields in the two domains must be related by the following commutative diagrams.

$$
\begin{tikzcd}
\underline{E} \arrow[r, "\epsilon"] \arrow[d, "\xi_E"] & \underline{D} \arrow[d, "\xi_D"] \\
\underline{\widetilde{E}} \arrow[r, "\widetilde{\epsilon}"] & \underline{\widetilde{D}}
\end{tikzcd}
\qquad
\begin{tikzcd}
\underline{H} \arrow[r, "\mu"] \arrow[d, "\xi_H"] & \underline{B} \arrow[d, "\xi_B"] \\
\underline{\widetilde{H}} \arrow[r, "\widetilde{\mu}"] & \underline{\widetilde{B}}
\end{tikzcd}
$$

Figure 3.

We can now derive the relations between the generalised electromagnetic response functions simply by requiring commutativity of the diagrams. For the permittivity response operator we have

$$\widetilde{\epsilon} \circ \xi_E = \xi_D \circ \epsilon$$

$$\implies \underline{\widetilde{\epsilon}}\ [\underline{\widetilde{E}}(\underline{x})] = J_\xi\ \underline{\underline{F}}_\xi^{-1} \cdot \underline{\epsilon}\ [\underline{\underline{F}}_\xi^{-T} \cdot \underline{\widetilde{E}}(\underline{x})]. \tag{32}$$

Equivalently for the permeability response operator,

$$\widetilde{\mu} \circ \xi_H = \xi_B \circ \mu$$

$$\implies \underline{\widetilde{\mu}}\ [\underline{\widetilde{E}}(\underline{x})] = J_\xi\ \underline{\underline{F}}_\xi^{-1} \cdot \underline{\mu}\ [\underline{\underline{F}}_\xi^{-T} \cdot \underline{\widetilde{H}}(\underline{x})]. \tag{33}$$

These equations allow the generalised permittivity and permeability operators required to achieve a given diffeomorphism of the fields to be calculated in terms of the original operators in the domain in which they are known. Importantly, this demonstrates that Maxwell's Equations can still be satisfied under any diffeomorphism in a non-linear material, and that the diffeomorphism preserves the nature of the non-linearity (the non-linear functional dependence on the E and H fields at every point is unchanged by the transformation). This opens up the possibility of electromagnetic cloaking using non-linear materials, the implications of which are discussed in section 7.

Note that in the case where the operators are linear these reduce to the above equations found by directly manipulating the fields in the linear case.

3.4 Hysteretic Electromagnetic Media

Now that we have derived general non linear expressions for the transformations of the permittivity and permeability operators, it is only a small step to generalise to memory dependent (hysteretic) electromagnetic materials. These materials do not just depend on the instantaneous applied fields but on the entire time dependent functions $E(t)$ and $H(t)$.

$$\underline{D}(\underline{x}, t) = \underline{\epsilon}\ [\underline{x},\ \underline{E}(\underline{x}, \tau)] \tag{34}$$
$$\underline{B}(\underline{x}, t) = \underline{\mu}\ [\underline{x},\ \underline{H}(\underline{x}, \tau)] \tag{35}$$

We require only that this time dependence satisfies causality and time translational symmetry [17]. Causality requires that the induced fields can only depend on the applied fields at earlier times, $\tau < t$. Time translational symmetry requires that

$$\underline{D}(\underline{x},\ t+t_0) = \underline{\epsilon}\ [\underline{x}, \underline{E}(\underline{x},\ \tau + t_0)]. \tag{36}$$

The most general time dependence that satisfies these properties is $\tau = t - t'$ for $t' \in [0, \infty]$. Now let us introduce the notation [17]

$$\underline{E}^t(t') := \underline{E}(t - t'), \tag{37}$$
$$\underline{H}^t(t') := \underline{H}(t - t'). \tag{38}$$

where $t' \in [0, \infty]$. Therefore we can write

$$\underline{D}(\underline{x}, t) = \underline{\epsilon}\ [\underline{x},\ \underline{E}^t(t')], \tag{39}$$
$$\underline{B}(\underline{x}, t) = \underline{\mu}\ [\underline{x},\ \underline{H}^t(t')]. \tag{40}$$

Now since the diffeomorphisms so far considered only affect the spatial variables, this additional temporal dependence of the permittivity and permeability operators does not affect the transformation rules derived for them previously. Therefore we can straight away write down the transformed permittivity and permeability operators for a non-linear hysteretic material in a transformed geometric domain.

$$\widetilde{\underline{\epsilon}}\ [\widetilde{\underline{E}}^t(\underline{x},\ t')] = J_\xi\ \underline{\underline{F}}_\xi^{-1} \cdot \underline{\epsilon}\ [\underline{\underline{F}}_\xi^{-T} \cdot \widetilde{\underline{E}}^t(\underline{x},\ t')] \tag{41}$$
$$\widetilde{\underline{\mu}}\ [\widetilde{\underline{H}}^t(\underline{x},\ t')] = J_\xi\ \underline{\underline{F}}_\xi^{-1} \cdot \underline{\mu}\ [\underline{\underline{F}}_\xi^{-T} \cdot \widetilde{\underline{H}}^t(\underline{x},\ t')] \tag{42}$$

As before, these equations show the existence of an infinite degeneracy of electromagnetic fields which solve Maxwell's equations inside a general non-linear hysteretic material and which satisfy the same boundary conditions. This leads to the theoretic possibility of perfect invisibility even inside a non-linear hysteretic material.

If the hysteretic response of the material is linear, we can simply sum the contributions of the applied fields at time t' using the Boltzmann superposition principle.

$$\underline{D}(t) = \int \underline{\underline{\epsilon}}(t, t') \cdot \underline{E}(t')\ \mathrm{d}t' \tag{43}$$
$$\underline{B}(t) = \int \underline{\underline{\mu}}(t, t') \cdot \underline{H}(t')\ \mathrm{d}t' \tag{44}$$

Causality requires $t' < t$, while time-translational symmetry requires $\underline{\underline{\epsilon}}(t, t') = \underline{\underline{\epsilon}}(t - t')$, $\underline{\underline{\mu}}(t, t') = \underline{\underline{\mu}}(t - t')$, so we have

$$\underline{D}(t) = \int_{-\infty}^{t} \underline{\underline{\epsilon}}(t - t') \cdot \underline{E}(t')\ \mathrm{d}t' \quad := \underline{\epsilon}[\underline{E}^t(t')], \tag{45}$$
$$\underline{B}(t) = \int_{-\infty}^{t} \underline{\underline{\mu}}(t - t') \cdot \underline{H}(t')\ \mathrm{d}t' \quad := \underline{\mu}[\underline{H}^t(t')]. \tag{46}$$

Therefore in the case of a linear hysteretic material we can interpret the permittivity and permeability operators as the integral operators defined above, and the corresponding transformed operators under a diffeomorphism of the fields are given by

$$\underline{\widetilde{D}}(\underline{x},t) = \widetilde{\epsilon}[\underline{\widetilde{E}}^t(\underline{x},t')] \quad = J_\xi \, \underline{\underline{F}}_\xi^{-1} \cdot \int_{-\infty}^{t} \underline{\underline{\epsilon}} \, (t-t') \cdot \underline{\underline{F}}_\xi^{-T} \cdot \underline{\widetilde{E}}(\underline{x},t') \, \mathrm{d}t', \tag{47}$$

$$\underline{\widetilde{B}}(\underline{x},t) = \underline{\widetilde{\mu}} \, [\underline{\widetilde{H}}^t(\underline{x},t')] \quad = J_\xi \, \underline{\underline{F}}_\xi^{-1} \cdot \int_{-\infty}^{t} \underline{\underline{\mu}} \, (t-t') \cdot \underline{\underline{F}}_\xi^{-T} \cdot \underline{\widetilde{H}}(\underline{x},t') \, \mathrm{d}t'. \tag{48}$$

In the non-hysteretic case we simply have

$$\underline{\underline{\epsilon}}(t-t') = \delta(t-t')\underline{\underline{\epsilon}},$$
$$\underline{\underline{\mu}}(t-t') = \delta(t-t')\underline{\underline{\mu}}$$

and the equations reduce to the instantaneous linear response equations derived previously.

4 Differential Geometry Formulation and Transformation of Maxwell's Equations

4.1 Formulation of Maxwell's Equations in terms of Differential Forms

In considering the transformation of the integral formulation of Maxwell's Equations, it became clear that the form invariance of the equations followed from the interpretation of the E and H fields as physical quantities per unit length which must be integrated over a one-dimensional path, and the D and B fields as physical quantities per unit area which must be integrated over a two-dimensional surface. This idea leads to a more fundamental representation of Maxwell's Equations which is independent of coordinate system.

This representation involves the use of differential forms. A brief overview of differential forms is given in appendix B, but there are many good texts containing a more extensive coverage [10, 18, 19, 20]. Here we begin by defining the forms

$$E = E_1 \, \mathrm{d}x^1 + E_2 \, \mathrm{d}x^2 + E_3 \, \mathrm{d}x^3, \tag{49}$$
$$H = H_1 \, \mathrm{d}x^1 + H_2 \, \mathrm{d}x^2 + H_3 \, \mathrm{d}x^3, \tag{50}$$
$$D = D_1 \, \mathrm{d}x^2 \wedge \mathrm{d}x^3 + D_2 \, \mathrm{d}x^3 \wedge \mathrm{d}x^1 + D_3 \, \mathrm{d}x^1 \wedge \mathrm{d}x^2, \tag{51}$$
$$B = B_1 \, \mathrm{d}x^2 \wedge \mathrm{d}x^3 + B_2 \, \mathrm{d}x^3 \wedge \mathrm{d}x^1 + B_3 \, \mathrm{d}x^1 \wedge \mathrm{d}x^2, \tag{52}$$
$$J = J_1 \, \mathrm{d}x^2 \wedge \mathrm{d}x^3 + J_2 \, \mathrm{d}x^3 \wedge \mathrm{d}x^1 + J_3 \, \mathrm{d}x^1 \wedge \mathrm{d}x^2, \tag{53}$$
$$\rho = \varrho \, \mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3, \tag{54}$$

where x^1, x^2, x^3 represent the coordinates of an $n = 3$ dimensional space, but can be any kind of generalised local curvilinear coordinates defined on a manifold M. In this formulation the electric field E is represented as a 1-form since it can be integrated over a one-dimensional path to give a physically meaningful quantity, specifically the work done per unit charge. Similarly the current density J is a 2-form because integrating it over a surface gives the current, while the charge density ρ is a 3-form because integrating it over a volume gives the charge enclosed.

These definitions allow Maxwell's Equations to be written

$$\mathrm{d}D = \rho, \tag{55}$$

$$\mathrm{d}B = 0, \tag{56}$$

$$\mathrm{d}E = -\dot{B}, \tag{57}$$

$$\mathrm{d}H = J + \dot{D}. \tag{58}$$

Here d is the exterior derivative and subsumes the gradient, curl and divergence operations into one operator which is independent of the coordinate system in which it is evaluated (see section 4.2). A brief overview of the exterior derivative and the equivalence of these equations with the traditional vector calculus formulation of Maxwell's Equations is given in appendix B.

The linear constitutive equations in this formulation are given by [9]

$$D = \star(\epsilon E), \tag{59}$$

$$B = \star(\mu H), \tag{60}$$

where $\star$ is the Hodge star operator and ϵ and μ are linear operators. The Hodge star is effectively a unique mapping from a k-form to an $n-k$ form, or in this case a 1-form to a 2-form, and is defined in appendix B. It is worth noting that in this formulation, while Maxwell's Equations themselves do not depend upon the nature of the manifold or the coordinate charts used, it is through the Hodge star that the geometry of the space enters the equations.

4.2 Pullback of Maxwell's Equations

We now consider how this form of Maxwell's Equations transforms under a diffeomorphism to a different geometrical domain.

To transform a differential form we must use the pullback operation, which maps k-forms on a manifold M to k-forms on a related manifold $\widetilde{M}$.

Crucially this operation commutes with the exterior derivative, such that if ω is a k-form on M, we have

$$\xi^* \mathrm{d}\omega = \mathrm{d}(\xi^* \omega). \tag{61}$$

This vital relation is proved in appendix C and demonstrates the earlier statement that the exterior derivative of a differential form is independent of the coordinate system in which it is computed [10].

Therefore, applying the pullback to Maxwell's Equations defined on M given above, and using the commutativity of the pullback with the exterior derivative we obtain

$$\mathrm{d}\widetilde{D} = \widetilde{\rho}, \tag{62}$$

$$\mathrm{d}\widetilde{B} = 0, \tag{63}$$

$$\mathrm{d}\widetilde{E} = -\dot{\widetilde{B}}, \tag{64}$$

$$\mathrm{d}\widetilde{H} = \widetilde{J} + \dot{\widetilde{D}}, \tag{65}$$

where we have defined

$$\widetilde{E}(\underline{x}) = \xi^* E[\underline{\xi}(\underline{x})], \tag{66}$$
$$\widetilde{H}(\underline{x}) = \xi^* H[\underline{\xi}(\underline{x})], \tag{67}$$
$$\widetilde{D}(\underline{x}) = \xi^* D[\underline{\xi}(\underline{x})], \tag{68}$$
$$\widetilde{B}(\underline{x}) = \xi^* B[\underline{\xi}(\underline{x})], \tag{69}$$
$$\widetilde{J}(\underline{x}) = \xi^* J[\underline{\xi}(\underline{x})], \tag{70}$$
$$\widetilde{\rho}(\underline{x}) = \xi^* \rho[\underline{\xi}(\underline{x})]. \tag{71}$$

These equations represent the differential geometry equivalents of equations (14)-(19), which can be demonstrated by applying the definition of the pullback (given in appendix C). The form invariance of Maxwell's equations is now explicit, and follows simply from the commutativity of the exterior derivative with the pullback operation, giving a much better geometrical insight into its origin.

4.3 Pullback of the Linear Constitutive Equations

Consider again the linear constitutive equations:

$$D = \star(\epsilon E), \tag{72}$$
$$B = \star(\mu H) \tag{73}$$

where ϵ and μ are linear operators.

To pullback these equations we need to find the appropriate transformation of the Hodge star operator. We find that the pullback of the Hodge star of a k-form β is given by

$$\xi^*(\star\beta) = \star[J_\xi(\bigwedge^k T)\xi^*\beta], \tag{74}$$

where we have defined

$$\begin{aligned} T &= \widetilde{g}(\xi^* g)^{-1}, \\ J_\xi &= |T|^{-\frac{1}{2}}. \end{aligned}$$

The derivation of this result is given in appendix D.

Now let us apply this to the constitutive relation (72).

$$\begin{aligned} \widetilde{D} &= \xi^* D \\ &= \xi^*[\star(\epsilon E)] \\ &= \star[J_\xi \ (\bigwedge^1 T)\xi^*(\epsilon E)] \\ &= \star[J_\xi \ T(\xi^*\epsilon)(\xi^* E)] \\ &= \star[J_\xi \ T(\xi^*\epsilon)\widetilde{E}] \\ &\equiv \star(\widetilde{\epsilon}\widetilde{E}) \end{aligned}$$

$$\implies \widetilde{\epsilon} = J_\xi \ \widetilde{g}(\xi^* g)^{-1}(\xi^*\epsilon) \tag{75}$$

Now we need to find the pullback of the metric tensor g and the linear operator ϵ. We find

$$(\xi^* g)_x = {F_x}^* g_{\xi(x)} F_x \tag{76}$$

and

$$(\xi^* \epsilon)_x = {F_x}^* \epsilon_{\xi(x)} {F_x}^{*-1} \tag{77}$$

where we have defined the deformation gradient

$$F_x := T\xi|_x : T_x \widetilde{M} \to T_{\xi(x)} M \tag{78}$$

and its dual

$$F_x{}^* : \; T_{\xi(x)}{}^* M \to T_x{}^* \widetilde{M}. \tag{79}$$

These results are derived in appendix E.

Using these results we obtain

$$\begin{aligned}
\widetilde{\epsilon} &= J_\xi \; \widetilde{g}_x \; (\xi^* g)_x^{-1} (\xi^* \epsilon) \\
&= J_\xi \; \widetilde{g}_x \; (F_x^* \; g_{\xi(x)} \; F_x)^{-1} \; (F_x^* \; \epsilon_{\xi(x)} \; F_x^{*-1}) \\
&= J_\xi \; \widetilde{g}_x \; F_x^{-1} \; g_{\xi(x)}^{-1} \; F_x^{*-1} \; F_x^* \; \epsilon_{\xi(x)} \; F_x^{*-1} \\
&= J_\xi \; \widetilde{g}_x \; F_x^{-1} (g_{\xi(x)}^{-1} \; \epsilon_{\xi(x)} \; F_x^{*-1})
\end{aligned}$$

And equivalently for $\widetilde{B} = \star(\widetilde{\mu}\widetilde{H})$. So we have

$$\widetilde{\epsilon} = J_\xi \; \widetilde{g}_x \; F_x^{-1} (g_{\xi(x)}^{-1} \; \epsilon_{\xi(x)} \; F_x^{*-1}), \tag{80}$$

$$\widetilde{\mu} = J_\xi \; \widetilde{g}_x \; F_x^{-1} (g_{\xi(x)}^{-1} \; \mu_{\xi(x)} \; F_x^{*-1}). \tag{81}$$

These results are the differential geometry equivalent to the earlier equations

$$\widetilde{\epsilon} = J_\xi \; \widetilde{g}_x \; F_x^{-1} (g_{\xi(x)}^{-1} \; \epsilon_{\xi(x)} \; F_x^{*-1}) \Leftrightarrow J_\xi \; \underline{\underline{F}}_\xi^{-1} \cdot \underline{\underline{\epsilon}}_{\xi(x)} \cdot \underline{\underline{F}}_\xi^{-T} = \underline{\underline{\widetilde{\epsilon}}},$$

$$\widetilde{\mu} = J_\xi \; \widetilde{g}_x \; F_x^{-1} (g_{\xi(x)}^{-1} \; \mu_{\xi(x)} \; F_x^{*-1}) \Leftrightarrow J_\xi \; \underline{\underline{F}}_\xi^{-1} \cdot \underline{\underline{\mu}}_{\xi(x)} \cdot \underline{\underline{F}}_\xi^{-T} = \underline{\underline{\widetilde{\mu}}},$$

which clearly agree in the case of a Euclidian metric in Cartesian coordinates.

However, unlike equations (22)-(23), these equations can now be directly applied in any coordinate system or in any spatial geometry. The metric factors now automatically take care of any coordinate transformations from Cartestian coordinates in Euclidian space.

4.4 Differential Geometry Formulation of Non-Linear Constitutive Equations

In vector calculus the defining electric constitutive relation is given by

$$\underline{D} = \frac{\partial U}{\partial \underline{E}}. \tag{82}$$

Now consider how to formulate this in terms of differential forms defined on a manifold. Here we will simply justify the form of the general constitutive relation by appealing to reason; we will prove this relation more formally in section 5.5 once a relativistic formulation of the electromagnetic field has been established.

Since differential forms are defined on the cotangent bundle, let us define U as a scalar function of the 1-forms E and H defined on the cotangent bundle of some manifold M,

$$U \in C^{\infty}(T^*M, T^*M). \tag{83}$$

Since the constitutive equation must be local, we want to vary U with respect to E with the position on the manifold fixed. Therefore we need to vary U along the vertical fibre bundle defined to be in some sense 'perpendicular' to the manifold at each point. To do this we need to use the partial fibre derivative $\mathbb{F}_E U(E, H)$ [18], defined by

$$\langle \mathbb{F}_E U(E, H),\ \delta E \rangle = \frac{\mathrm{d}}{\mathrm{d}s} U(E + s\ \delta E, H) \bigg|_{s=0} \tag{84}$$

$$\text{where} \quad \begin{aligned} & U \in C^{\infty}(T^*M, T^*M) \\ & E, \delta E, H \in (T^*_x M). \end{aligned}$$

$\mathbb{F}_E U(E, H)$ is therefore a vector on the tangent bundle

$$\mathbb{F}_E U \in T_x M. \tag{85}$$

This can be mapped to a differential form (a 1-form) simply by 'flattening' the vector to the cotangent bundle (applying the lowering metric),

$$\mathbb{F}_E U^{\flat} \equiv [g_{\mu\nu}\ \mathbb{F}_E U^{\nu}] \in T^*_x M. \tag{86}$$

This is now a 1-form, but D is a 2-form. Therefore to complete the constitutive relation, we must apply the Hodge star operator to give

$$D = \star[\mathbb{F}_E U(E, H)^{\flat}]. \tag{87}$$

This is the general non-linear constitutive relation in the differential geometry formulation.

And of course equivalently, the magnetic constitutive relation

$$\underline{B} = \frac{\partial U}{\partial \underline{H}} \tag{88}$$

becomes

$$B = \star[\mathbb{F}_H U(E, H)^{\flat}]. \tag{89}$$

4.5 Pullback of the Non-Linear Constitutive Equations

To relate these constitutive equations to the electromagnetic properties of a medium by applying the pullback, we must first find the pullback of the fibre derivative. This is derived in appendix F, and gives

$$\xi^* \mathbb{F}U(E[\xi(x)]) = \mathbb{F}\big[(\xi^* U)(\widetilde{E}(x))\big]. \tag{90}$$

Together with the previous results for the pullback of the Hodge star and the metric, we are now able to pullback the general non-linear constitutive relation. For simplicity we write $U(E, H) = U(E)$ here, but the results are identical for a general electromagnetic internal energy function $U(E, H)$ so long as we take the appropriate partial fibre derivatives of section 4.4.

$$\begin{aligned}
\widetilde{D}(x) &= \xi^* D[\xi(x)] \\
&= \xi^*[\star\, \mathbb{F}U(E[\xi(x)])^\flat] \\
&= \star[J_\xi\ \widetilde{g}(\xi^* g)^{-1}\ \xi^*\{\mathbb{F}U(E[\xi(x)])^\flat\}] \\
&= \star[J_\xi\ \widetilde{g}(\xi^* g)^{-1}(\xi^* g)\ \xi^*\mathbb{F}U(E[\xi(x)])] \\
&= \star[J_\xi\ \widetilde{g}\ \xi^*\mathbb{F}U(E[\xi(x)])] \\
&= \star[J_\xi\ \widetilde{g}\ \mathbb{F}\{(\xi^* U)(\widetilde{E}(x))\}] \\
&= \star[\widetilde{g}\ \mathbb{F}\widetilde{U}(\widetilde{E})] \\
&= \star[\mathbb{F}\widetilde{U}(\widetilde{E})^\flat]
\end{aligned}$$

where we have defined

$$\widetilde{U} = (\tau^* J_\xi)(\xi^* U) = (J_\xi \circ \tau)(U \circ F^{*-1}) \tag{91}$$

where τ is just the projection $\tau : T^*\widetilde{M} \to \widetilde{M}$.

Therefore given the internal energy functional for a non-linear material $U(E)$, we are able to calculate the related internal energy functional $\widetilde{U}(E)$ which will cause the fields to transform under the diffeomorphism ξ. As before we see that the nature of the non-linearity of the energy functional of the material is unchanged by the transformation.

Now using the above, we write the expression for $\widetilde{D}$ as

$$\widetilde{D} = \star[J_\xi\ \widetilde{g}_x\ F_x^{-1}\mathbb{F}U(F_{\xi(x)}^{*-1}\widetilde{E})]. \tag{92}$$

And defining

$$D = \star[\mathbb{F}U(E)^\flat] \quad := \star[\epsilon(E)] \tag{93}$$

$$\widetilde{D} = \star[\mathbb{F}\widetilde{U}(\widetilde{E})^\flat] \quad := \star[\tilde{\epsilon}(\widetilde{E})] \tag{94}$$

where ϵ and $\tilde{\epsilon}$ are non general non-linear operators, we obtain

$$\tilde{\epsilon}[\widetilde{E}] = J_\xi\ \widetilde{g}_x\ F_x^{-1}\ g_{\xi(x)}^{-1}\ \epsilon[F_{\xi(x)}^{*-1}\widetilde{E}], \tag{95}$$

which is clearly equivalent to the earlier tensorial result (32) for a generalised non-linear electromagnetic response in the case of a Euclidian metric in Cartesian coordinates. Again the results are all equivalent for the permeability operator.

$$\tilde{\mu}[\widetilde{H}] = J_\xi\ \widetilde{g}_x\ F_x^{-1}\ g_{\xi(x)}^{-1}\ \mu[F_{\xi(x)}^{*-1}\widetilde{H}], \tag{96}$$

4.6 Example of a Spherical Cloak using the Differential Geometry Formulation

As a pedagogical example often considered in the transformation optics literature [1, 6], consider the case of a spherical invisibility cloak. We begin by defining the diffeomorphism

$$\xi : \quad \widetilde{M} \to M, \tag{97}$$

where M is free space labelled by spherical polar coordinates r, θ, ϕ and $\widetilde{M}$ is the distorted space labelled by $\widetilde{r}, \widetilde{\theta}, \widetilde{\phi}$, which are related via [1]

$$\left.\begin{aligned} \widetilde{r} &= \tfrac{r_2 - r_1}{r_2} r + r_1 \\ \\ \widetilde{\theta} &= \theta \\ \widetilde{\phi} &= \phi \end{aligned}\right\} \quad \begin{aligned} r &= \tfrac{r_2}{r_2 - r_1}(\widetilde{r} + r_1) \\ \\ \theta &= \widetilde{\theta} \\ \phi &= \widetilde{\phi}. \end{aligned}$$

This diffeomorphism has been chosen such that all fields in the region $r < r_1$ in free space are compressed into the region $r_1 < r < r_2$ in the distorted space. The spatial distortion can then be represented as a transformation of the electromagnetic properties of Euclidian space in the region $r_1 < r < r_2$ using the theory of transformation optics. This means that for an object placed in the region $r < r_1$, the above spatial distortion effectively created by an electromagnetic cloak existing in the region $r_1 < r < r_2$ will render the object invisible to electromagnetic radiation. In addition, since the diffeomorphism has been chosen such that $r = \widetilde{r}$ when $r = r_2$, the transformation is continuous across the edge of the electromagnetic cloak and therefore will also render the cloak invisible.

From the differential geometry formulation of Maxwell's Equations, it was found that the transformation of the permittivity tensor (assuming linearity) is given by

$$\widetilde{\epsilon} = J_\xi \; \widetilde{g} \; F_x^{-1} g_{\xi(x)}^{-1} \epsilon_{\xi(x)} F_x^{*-1}. \tag{98}$$

Taking M to be free space, we have $\epsilon = \epsilon_0$ everywhere on M.

In spherical polar coordinates the line element on M is given by

$$\mathrm{d}s^2 = \mathrm{d}r^2 + r^2 \mathrm{d}\theta^2 + r^2 \sin^2\theta \; \mathrm{d}\phi^2 \tag{99}$$

$$\implies g = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2 \sin^2\theta \end{pmatrix}$$

$$\implies g^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{r^2} & 0 \\ 0 & 0 & \frac{1}{r^2 \sin^2\theta} \end{pmatrix}$$

$$\therefore \quad g_{\xi(x)}^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \left[\frac{r_2 - r_1}{r_2(\widetilde{r} - r_1)}\right]^2 & 0 \\ 0 & 0 & \left[\frac{r_2 - r_1}{r_2(\widetilde{r} - r_1)\sin\theta}\right]^2 \end{pmatrix}.$$

And similarly the line element on $\widetilde{M}$ in spherical polar coordinates is given by

$$\mathrm{d}s^2 = \mathrm{d}\widetilde{r}^2 + \widetilde{r}^2\mathrm{d}\widetilde{\theta}^2 + \widetilde{r}^2\sin^2\widetilde{\theta}\,\mathrm{d}\widetilde{\phi}^2 \tag{100}$$

$$\implies \widetilde{g} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \widetilde{r}^2 & 0 \\ 0 & 0 & \widetilde{r}^2\sin^2\widetilde{\theta} \end{pmatrix}.$$

Since the deformation gradient is linear and expressed in a matrix representation, from the definition of the dual operator, its dual is equal to its transpose. This gives

$$F_x = F_x^* = \begin{pmatrix} \frac{r_2}{r_2-r_1} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\implies F_x^{-1} = F_x^{*-1} = \begin{pmatrix} \frac{r_2-r_1}{r_2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

We also have

$$\begin{aligned} J_\xi &= (\det T)^{-\frac{1}{2}} \\ &= \det(\widetilde{g}(\xi^* g)^{-1})^{-\frac{1}{2}} \\ &= \det(\widetilde{g}(F_x^* g_{\xi(x)} F_x)^{-1})^{-\frac{1}{2}} \\ &= \det(\widetilde{g} F_x^{-1} g_{\xi(x)}^{-1} F_x^{*-1})^{-\frac{1}{2}} \end{aligned}$$

$$= \det\begin{bmatrix} \left(\frac{r_2-r_1}{r_2}\right)^2 & 0 & 0 \\ 0 & \left(\frac{(r_2-r_1)\widetilde{r}}{r_2(\widetilde{r}-r_1)}\right)^2 & 0 \\ 0 & 0 & \left(\frac{(r_2-r_1)\widetilde{r}}{r_2(\widetilde{r}-r_1)}\right)^2 \end{bmatrix}^{-\frac{1}{2}}$$

$$= \left(\frac{r_2}{r_2-r_1}\right)^3 \left(\frac{\widetilde{r}-r_1}{\widetilde{r}}\right)^2.$$

Combining these results together using $\widetilde{\epsilon} = J_\xi\ \widetilde{g}\ F_x^{-1} g_{\xi(x)}^{-1} \epsilon_{\xi(x)} F_x^{*-1}$ gives

$$\widetilde{\epsilon} = \begin{pmatrix} \frac{r_2}{r_2-r_1}\left(\frac{\widetilde{r}-r_1}{\widetilde{r}}\right)^2 & 0 & 0 \\ 0 & \frac{r_2}{r_2-r_1} & 0 \\ 0 & 0 & \frac{r_2}{r_2-r_1} \end{pmatrix} \epsilon_0.$$

And equivalently

$$\widetilde{\mu} = \begin{pmatrix} \frac{r_2}{r_2-r_1}\left(\frac{\widetilde{r}-r_1}{\widetilde{r}}\right)^2 & 0 & 0 \\ 0 & \frac{r_2}{r_2-r_1} & 0 \\ 0 & 0 & \frac{r_2}{r_2-r_1} \end{pmatrix} \mu_0.$$

Therefore the electromagnetic properties of the spherical cloak existing in the annulus $r_1 < r < r_2$ must be

$$\widetilde{\epsilon}_{\widetilde{r}} = \widetilde{\mu}_{\widetilde{r}} = \frac{r_2}{r_2 - r_1}\left(\frac{\widetilde{r} - r_1}{\widetilde{r}}\right)^2 \tag{101}$$

$$\widetilde{\epsilon}_{\widetilde{\theta}} = \widetilde{\mu}_{\widetilde{\theta}} = \frac{r_2}{r_2 - r_1} \tag{102}$$

$$\widetilde{\epsilon}_{\widetilde{\phi}} = \widetilde{\mu}_{\widetilde{\phi}} = \frac{r_2}{r_2 - r_1}. \tag{103}$$

These are in agreement with the properties found in the founding papers of transformation optics [1, 6], but we have arrived at the results in an arguably much more direct way using our differential geometry formulation when compared with undertaking coordinate transformations in terms of Cartesian tensors as is done in the literature [6, 7]. This method can easily be applied to any curvilinear coordinate system and any diffeomorphism in an identical manner.

A computer simulation using a ray tracing program to calculate ray trajectories through a cloak with these properties is shown below in figure 4 (reproduced from *Controlling Electromagnetic Fields*, J.B. Pendry et al. [1]). The cloak behaves as expected.

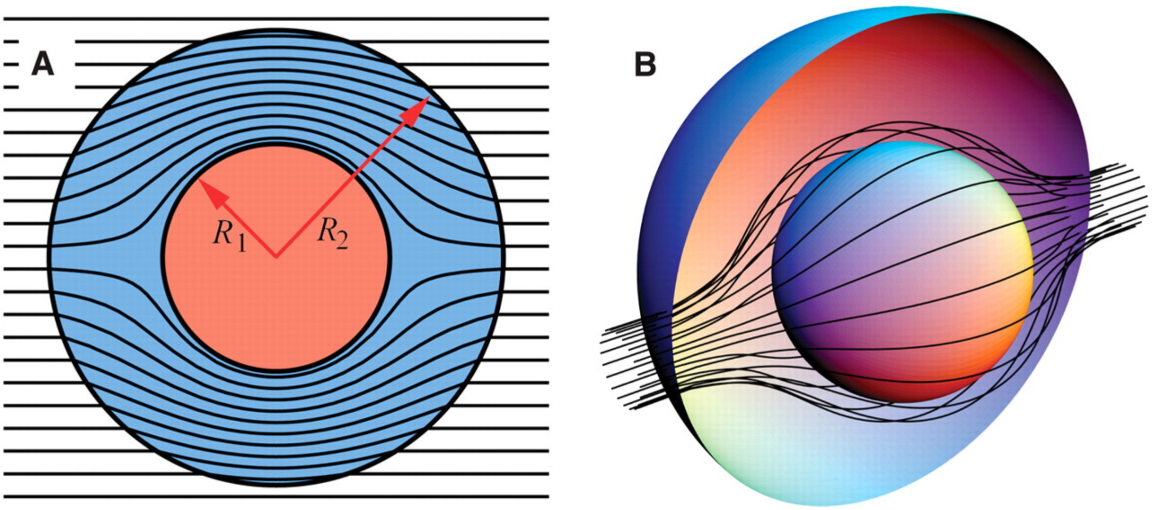

Figure 4: The cloak exists between R1 and R2 and refracts any incident light around the central sphere of radius R1. This means any objects placed within R1 cannot be observed by an external observer at a radius greater than R2 [1].

5 Spacetime Differential Geometry Formulation and Transformation of Maxwell's Equations

In section 4 we demonstrated the effectiveness with which differential geometry can be applied to electromagnetism, and in particular transformation optics. This already generalises to curved spaces in 3-dimensions, but we now consider how to generalise to a fully covariant 4-dimensional spacetime representation of Maxwell's Equations using differential geometry. We begin by considering free Minkowski spacetime.

5.1 Formulation in Minkowski Spacetime

In special relativity, electromagnetism is a single unified phenomenon with the distinction between the electric and magnetic fields only arising in a particular inertial frame of reference. So in Minkowski spacetime, we begin by combining E and B into a single object known as the Faraday 2-form [9, 21]

$$F = E \wedge \mathrm{d}t + B. \tag{104}$$

Consider taking the exterior derivative of this object in 4D Cartesian coordinates. Setting $\mathrm{d}F = 0$ and equating the coefficients of the 3-form components gives 4 equations

$$\begin{aligned}
\Big(\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z}\Big) + \frac{\partial B_x}{\partial t} &= 0, \\
\Big(\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x}\Big) + \frac{\partial B_y}{\partial t} &= 0, \\
\Big(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}\Big) + \frac{\partial B_z}{\partial t} &= 0, \\
\frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z} &= 0.
\end{aligned} \tag{105}$$

These clearly correspond to the components of the two homogeneous Maxwell Equations.

To write the inhomogeneous Maxwell Equations in a covariant and coordinate independent form, we similarly combine the H and D fields into a single object known as the Maxwell 2-form [9, 21]

$$G = H \wedge \mathrm{d}t - D. \tag{106}$$

We may also combine the charge density ρ and the current density J into a single relativistically covariant spacetime object known as the source 3-form

$$j = J \wedge \mathrm{d}t - \rho. \tag{107}$$

By analogy with (105), we find that we can recover the inhomogeneous Maxwell Equations simply by setting $\mathrm{d}G = j$.

Note that charge conservation simply requires that j is closed, meaning $\mathrm{d}j = 0$, which follows trivially from $\mathrm{d}G = j$ using $\mathrm{d}^2 = 0$ (appendix B, equation (149)).

So in a spacetime differential geometry formulation, Maxwell's Equations can be written

$$\mathrm{d}F = 0 \tag{108}$$
$$\mathrm{d}G = j \tag{109}$$

where

$$F = E \wedge \mathrm{d}t + B, \quad G = H \wedge \mathrm{d}t - D \quad \text{and} \quad j = J \wedge \mathrm{d}t - \rho.$$

These are clearly equivalent to the Minkowski form of Maxwell's Equations

$$\begin{aligned} \nabla_{[\mu} F_{\nu\sigma]} &= 0 \\ \nabla_\mu F^{\mu\nu} &= \mu_o\, j^\nu. \end{aligned}$$

But now the expressions are coordinate independent, and the antisymmetrisation of both the electromagnetic field tensor $F^{\mu\nu}$ and the Bianchi identity is inherent to the exterior algebra, again demonstrating the naturalness with which differential forms can be applied to electromagnetism.

Finally, to complete the formulation, we just need a constitutive equation to relate the Faraday and Maxwell 2-forms. In the Euclidian formulation the applied and induced fields were related by the Hodge star operator, so let us consider the operation of the Hodge star on F. We note here that as we are now in Minkowski spacetime, taking the metric convention as $(-,+,+,+)$ we must introduce additional negative signs in some Hodge duals, as discussed in appendix B.

We find

$$\star(\mathrm{d}x^i \wedge \mathrm{d}t) = -\mathrm{d}x^j \wedge \mathrm{d}x^k, \tag{110}$$
$$\star(\mathrm{d}x^j \wedge \mathrm{d}x^k) = \mathrm{d}x^i \wedge \mathrm{d}t. \tag{111}$$

So we have (leaving the wedge products implicit),

$$\begin{aligned} \star F &= \star(E_x \mathrm{d}x\mathrm{d}t + E_y \mathrm{d}y\mathrm{d}t + E_z \mathrm{d}z\mathrm{d}t + B_x \mathrm{d}y\mathrm{d}z + B_y \mathrm{d}z\mathrm{d}x B_z \mathrm{d}x\mathrm{d}y) \\ &= B_x \mathrm{d}x\mathrm{d}t + B_y \mathrm{d}y\mathrm{d}t + B_z \mathrm{d}z\mathrm{d}t - E_x \mathrm{d}y\mathrm{d}z - E_y \mathrm{d}z\mathrm{d}x - E_z \mathrm{d}x\mathrm{d}y. \end{aligned}$$

And therefore in free space, using natural geometrised units such that $\epsilon_o = \mu_o = c = 1$ and therefore $\underline{B} = \underline{H}, \underline{D} = \underline{E}$, we have simply [21]

$$G = \star F. \tag{112}$$

But in general, the linear constitutive relation can be written

$$G = \star[f(F)] \tag{113}$$

where the linear operator f encompasses both the permittivity and permeability as a single object which fully defines the electromagnetic response of the material.

5.2 Pullback of the Spacetime Formulation

These equations can now be pulled back along a diffeomorphism onto any related geometry in an identical manner to before. Applying the pullback and using the commutativity of the pullback with the exterior derivative as before, we obtain

$$\mathrm{d}\widetilde{F} = 0 \tag{114}$$
$$\mathrm{d}\widetilde{G} = \widetilde{j} \tag{115}$$

where

$$\widetilde{F} = \xi^* F,$$
$$\widetilde{G} = \xi^* G,$$
$$\widetilde{j} = \xi^* j.$$

These equations appear completely equivalent to those derived in section 4.3 in the Euclidian spatial formulation. However, the important difference is that now these equations have been formulated on a 4-dimension spacetime manifold, the pullback may be along a 4-dimension diffeomorphism. Therefore these equations allow for the possibility of general spacetime transformations of Maxwell's Equations, including temporal distortions in addition to spatial distortions.

5.3 Pullback of the Linear Constitutive Relation

To actually obtain the correspondence between the spacetime transformations and the equivalent electromagnetic properties, we now need to pullback the linear constitutive equation

$$G = \star[f(F)]. \tag{116}$$

This is almost completely equivalent to the pullback of the spatial constitutive equations as derived in section 4.3, the only difference being that f is now a linear operator acting on a 2-form rather than a 1-form as for ϵ and μ. This simply results in the need to take the second exterior product of the transformation operators. Therefore we have

$$\widetilde{f} = J_\xi \bigwedge^2 (\widetilde{g}\; F_x^{-1} g_{\xi(x)}^{-1}) f_{\xi(x)} \bigwedge^2 (F_x^{*-1}). \tag{117}$$

5.4 Variational Principle Derivation of Free Space Maxwell Equations

Now that we have formulated a relativistic theory of the electromagnetic field using differential forms, we will use a variational principle to re-derive Maxwell's Equations in free space. It is hoped that this will give some insight into how to generalise the constitutive equation to non-linear electromagnetic responses.

The Lagrangian for classical electrodynamics, using units where $\epsilon_o = \mu_o = c = 1$, is given by [15]

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - A_\mu J^\mu. \tag{118}$$

The first term represents the energy stored in the electromagnetic field and follows simply from the requirement that it is the lowest order term that obeys both gauge invariance and Lorentz invariance. The gauge invariance requires that the term only contains the electromagnetic field tensor $F_{\mu\nu}$, and Lorentz invariance requires that all of the indices are contracted, while the negative sign is required so the kinetic terms give a positive contribution to the Lagrangian. This is just the lowest order possibility required in vacuum, but we shall be investigating the possibilities for higher order terms in section 5.5. The second term follows simply from including a linear coupling of the field with the source in order to obtain the correct form of Maxwell's Equations, since the source is not a fundamental field.

By analogy, consider the differential form Lagrangian [21]

$$\mathcal{L} = -\frac{1}{2} F \wedge \star F + A \wedge j, \tag{119}$$

or equivalently

$$\mathcal{L} = \frac{1}{2}(F, F)\sigma + A \wedge j. \tag{120}$$

We see that this similarly satisfies all of the requirements discussed above.

Now since $\mathrm{d}F = 0$, using Poincaré's lemma (appendix B), there must exist some 1-form A such that $F = \mathrm{d}A$. Therefore we have the 4-form Lagrangian

$$\mathcal{L}(A) = -\frac{1}{2}\mathrm{d}A \wedge \star \mathrm{d}A + A \wedge j. \tag{121}$$

The associated action function is given by the 4-dimensional integral over the spacetime manifold X.

$$S[A] = \int_X \mathcal{L}(A) \tag{122}$$

Taking a variation α of A, where α vanishes on the boundary ∂X, we have

$$\delta S = \mathrm{d}S[A] \cdot \alpha = \left.\frac{\mathrm{d}}{\mathrm{d}\epsilon}\right|_{\epsilon=0} \Big(S[A + \epsilon\alpha] - S[A] \Big) \tag{123}$$

$$= \int_X \alpha \wedge \big[-\mathrm{d} \star \mathrm{d}A + j \big]. \tag{124}$$

And from Hamilton's principle we require $\delta S = 0 \quad \forall \alpha$,

$$\begin{aligned} \implies \mathrm{d} \star \mathrm{d}A &= j \\ \mathrm{d}G &= j \qquad \text{as expected.} \end{aligned}$$

Therefore we have recovered Maxwell's Equations in free space from a variational principle.

5.5 Variational Principle Derivation of Maxwell's Equations in a General Electromagnetic Medium

As explained in section 5.4, in free space the electromagnetic Lagrangian can be written

$$\mathcal{L} = -\frac{1}{2} F \wedge \star F + A \wedge j = \frac{1}{2}(F, F)\sigma + A \wedge j. \tag{125}$$

The inner product of F with itself represents the electromagnetic field energy, while the interaction term $A \wedge j$ represents the linear coupling between the field and its source. Since (F, F) is clearly quadratic in the electromagnetic fields, the resulting equations of motion are linear, and so this Lagrangian clearly only applies for linear media (although as we shall see, there is even a more general possibility in linear media).

To generalise to an arbitrary non-linear medium, consider replacing the quadratic inner-product (F, F) with a general energy functional of the Faraday 2-form, or equivalently of the differential of the electromagnetic potential 1-form A.

$$\mathcal{L} = U(\mathrm{d}A)\sigma + A \wedge j \tag{126}$$

Here $U(F)$ is a general non-linear functional representing the energy stored in the material at each point.

As before, $S[A] = \int_X \mathcal{L}(A)$ and take a variation α of A which vanishes on the boundary ∂X.

$$\begin{aligned}
\delta S = \mathrm{d}S[A] \cdot \alpha &= \left.\frac{\mathrm{d}}{\mathrm{d}\epsilon}\right|_{\epsilon=0} \Big(S[A + \epsilon\alpha] - S[A] \Big) \\
&= \left.\frac{\mathrm{d}}{\mathrm{d}\epsilon}\right|_{\epsilon=0} \int_X U[\mathrm{d}A + \epsilon \mathrm{d}\alpha]\sigma + \epsilon\alpha \wedge j \\
&= \int_X \sigma \left.\frac{\mathrm{d}}{\mathrm{d}\epsilon}\right|_{\epsilon=0} U[\mathrm{d}A + \epsilon\mathrm{d}\alpha] + \int_X \alpha \wedge j
\end{aligned}$$

For variations which keep the position on the manifold fixed, or equivalently variations along the associated vertical fibre bundle, this is just the definition of the fibre derivative.

$$\langle \mathbb{F}U(\mathrm{d}A), \mathrm{d}\alpha \rangle = \left.\frac{\mathrm{d}}{\mathrm{d}\epsilon}\right|_{\epsilon=0} U[\mathrm{d}A + \epsilon\mathrm{d}\alpha] \tag{127}$$

Therefore we have

$$\begin{aligned}
\delta S &= \int_X \sigma \, \langle \mathbb{F}U(\mathrm{d}A), \mathrm{d}\alpha \rangle + \alpha \wedge j \\
&= \int_X \sigma \, \big(\mathbb{F}U(\mathrm{d}A)^\flat, \mathrm{d}\alpha\big)_{\Omega^1 M} + \alpha \wedge j,
\end{aligned}$$

where $\mathbb{F}U(\mathrm{d}A)^\flat \equiv g\mathbb{F}U(\mathrm{d}A) \in T^*M$. By definition of the Hodge star,

$$= \int_X -\mathrm{d}\alpha \wedge \star\mathbb{F}U(\mathrm{d}A)^\flat + \alpha \wedge j.$$

Using the generalised product rule (148) and the generalised Stokes' theorem (151), this can be written

$$= \int_X \Big(-\alpha \wedge \mathrm{d} \star \mathbb{F}U(\mathrm{d}A)^\flat + \alpha \wedge j \Big) - \Big[\alpha \wedge \star\mathbb{F}U(\mathrm{d}A)^\flat \Big]_{\partial X}.$$

Since α vanishes on the boundary ∂X, we have effectively integrated by parts to give

$$= \int_X \alpha \wedge \Big[-\mathrm{d}\big[\star \mathbb{F}U(\mathrm{d}A)^\flat \big] + j \Big].$$

From Hamilton's principle, we require $\delta S = 0 \quad \forall \alpha$,

$$\implies \mathrm{d}\big[\star \mathbb{F}U(\mathrm{d}A)^\flat \big] = j,$$

and from Maxwell's Equations we require $\mathrm{d}G = j$,

$$\implies G = \star[\mathbb{F}U(F)^\flat]. \tag{128}$$

This gives the general non-linear constitutive relation in the differential geometry spacetime formulation, and confirms the form of the constitutive relation of section 4.4.

$$G = \star[\mathbb{F}U(F)^\flat] \quad := \star[f(F)] \tag{129}$$

5.6 Requirement for Lorentz Invariance of General Energy Functional

We have now found the general non-linear constitutive relation in terms of the electromagnetic energy functional $U(F)$. However, we also require that the Lagrangian, and therefore the functional $U(F)$, is Lorentz invariant. Equivalently, while the electric and magnetic fields may change between inertial frames, the electomagnetic energy stored in the medium must not. This is highly restrictive on the possible form of $U(F)$.

To find possible forms of $U(F)$, we first need to find invariants of the Faraday 2-form.

We already know that $F \wedge \star F = (F, F)\sigma$ must be Lorentz invariant, as the inner product must be a Lorentz scalar and Lorentz transforms must preserve the 4-volume σ. Consider the interpretation of this term in terms of the physical fields E and B.

$$\begin{aligned} F \wedge \star F &= (E \wedge \mathrm{d}t - B) \wedge \star(E \wedge \mathrm{d}t - B) \\ &= E \wedge \mathrm{d}t \wedge \star(E \wedge \mathrm{d}t) - B \wedge \star(E \wedge \mathrm{d}t) - E \wedge \mathrm{d}t \wedge \star B + B \wedge \star B \\ &= (B^2 - E^2)\sigma \\ \text{where} \quad E^2 = E^i E_i, &\quad B^2 = B^i B_i. \end{aligned}$$

Also, let us consider the term $F \wedge F = -(F, \star F)\sigma$, which must similarly also be Lorentz invariant. This can be written

$$\begin{aligned} F \wedge F &= (E \wedge \mathrm{d}t - B) \wedge (E \wedge \mathrm{d}t - B) \\ &= -B \wedge E \wedge \mathrm{d}t - E \wedge \mathrm{d}t \wedge B \\ &= -2(E \cdot B)\sigma \\ \text{where} \quad E \cdot B &= E^i B_i. \end{aligned}$$

Since these are the only two independent 4-forms which can be constructed from F, we expect these two invariants to be fundamental, in the sense that any other invariants of the electromagnetic field must be a function of these two. Indeed, $(B^2 - E^2)$ and $E \cdot B$ are fundamental invariants, and a more elementary proof of this result is given in appendix G. This leads us to the following theorem.

Theorem: *Any function $U(F) \in \mathbb{R}$ which satisfies $U((\bigwedge^2 L)F) = U(F)$, where L is a Lorentz transformation operator, can be written $U(F) = W(F \wedge \star F, F \wedge F)$.*

We will now use this result to find the most general possible non-linear constitutive relation.

5.7 General Expression for the Non-Linear Constitutive Relation Given a Lorentz Invariant Energy Functional

It was shown in section 5.6 that any Lorentz invariant functional of the Faraday 2-form F can be written

$$U(F) = W'(F \wedge \star F, F \wedge F). \tag{130}$$

This can equivalently be written

$$U(F) = W(\alpha, \beta), \tag{131}$$

where

$$\alpha = \star(F \wedge \star F) \qquad = (F, F), \tag{132}$$
$$\beta = \star(F \wedge F) \qquad = -(F, \star F). \tag{133}$$

To apply the general non-linear constitutive relation

$$G = \star[\mathbb{F}U(F)^\flat], \tag{134}$$

we first need to find the fibre derivative of the internal energy functional $U(F)$.

$$\begin{aligned} \mathbb{F}U(F) &= \mathbb{F}W(\alpha, \beta) && (135)\\ &= \frac{\partial W}{\partial \alpha}\mathbb{F}\alpha + \frac{\partial W}{\partial \beta}\mathbb{F}\beta && (136) \end{aligned}$$

Now evaluating the fibre derivatives of α and β,

$$\begin{aligned} \left.\frac{\mathrm{d}}{\mathrm{d}\epsilon}\right|_{\epsilon=0} \alpha[F + \epsilon\delta F] &= 2(F, \delta F) \\ &= 2\left\langle (\textstyle\bigwedge^2 g^{-1})F, \delta F \right\rangle \\ &= \left\langle 2F^\sharp, \delta F \right\rangle \\ &\equiv \left\langle \mathbb{F}\alpha, \delta F \right\rangle \end{aligned}$$

$$\implies \mathbb{F}\alpha = 2F^\sharp, \tag{137}$$

$$
\begin{aligned}
\left.\frac{\mathrm{d}}{\mathrm{d}\epsilon}\right|_{\epsilon=0} \beta[F+\epsilon\delta F] &= -2(\star F, \delta F) \\
&= -2\Big\langle \big(\bigwedge^2 g^{-1}\big)(\star F), \delta F\Big\rangle \\
&= \Big\langle -2(\star F)^\sharp, \delta F\Big\rangle \\
&\equiv \Big\langle \mathbb{F}\beta, \delta F\Big\rangle \\
\implies \mathbb{F}\beta &= -2(\star F)^\sharp. \qquad (138)
\end{aligned}
$$

So the fibre derivative of the internal energy functional is given by

$$
\mathbb{F}U = \frac{\partial W}{\partial \alpha} 2F^\sharp - \frac{\partial W}{\partial \beta} 2(\star F)^\sharp. \qquad (139)
$$

Therefore the most general form of the Maxwell 2-form in any kind of generalised electromagnetic medium is given by

$$
G = \star[\mathbb{F}U^\flat] = 2\frac{\partial W}{\partial \alpha}(\star F) + 2\frac{\partial W}{\partial \beta} F. \qquad (140)
$$

As a quick check of this result, consider again free space for which

$$
\mathcal{L} = -\frac{1}{2} F \wedge \star F + A \wedge j = \frac{1}{2}(F, F)\sigma + A \wedge j.
$$

Therefore,

$$
\begin{aligned}
U(F) &= \frac{1}{2}(F, F) \qquad (141) \\
\implies W(\alpha, \beta) &= \frac{1}{2}\alpha. \qquad (142)
\end{aligned}
$$

And so we have,

$$
G = \star F \quad \text{as expected.}
$$

6 Summary

In section 3, we have gained a deeper insight into the origin of the form invariance of Maxwell's Equations and confirmed the one to one equivalence between geometries and electromagnetic media, the founding principle of transformation optics.

In section 4, we have reformulated transformation optics in its natural language of differential geometry, and through equations (80)-(81), have provided a way of evaluating the electromagnetic properties corresponding to a given transformation directly in any coordinate system or geometry. Equations (95)-(96) then provide the natural generalisation of transformation optics to non-linear media, and demonstrate that perfect invisibility is still possible in these more general media.

Finally in section 5, we have formulated a spacetime theory of transformation optics which allows for general spacetime transformations through equation (117). We have also found the most general constitutive relation possible for electromagnetic media that is consistent with Lorentz invariance, through equation (140).

7 Applications and Further Work

The central results of section 3, equations (80)-(81), allow the electromagnetic properties required to achieve any geometry of light propagation or electromagnetic field distortion to be calculated directly in any coordinates in a relatively simple way. This is most obviously applied to invisibility, by allowing the properties of any geometry of invisibility cloak to be calculated in an identical manner to the case of the spherical cloak as shown in section 4.6. However, this could also be applied to many other metamaterial applications such as absorbers and waveguides. For example, the equations (80)-(81) could also be used to calculate the properties required to focus incident electromagnetic fields into a point for more efficient energy absorption, for example in a new generation of solar panel. The concept of a metamaterial black hole with a 99% absorption rate in microwave frequencies has already been described by Cheng et al. [22].

The non-linear transformation equations (95)-(96) of section 3, together with the most general Lorentz invariant constitutive relation (140) derived in section 5, may also lead to new applications as non-linear metamaterials are developed. As an example, the original electromagnetic cloaks described by transformation optics, such as the spherical cloak described in section 4.6, would actually be detectable by phase sensitive detection due to the additional phase accumulated through the cloak in circumnavigating the cloaked object relative to free space. By using a non-linear material as the cloaking medium, it may be possible for the non-linear components to modify the phase of radiation passing through the cloak in order to make the cloak undetectable even by phase sensitive detection.

The central result of the spacetime formulation of transformation optics, equation (117), may lead to new possibilities for transformation optics since electromagnetic properties can now be derived which will not only distort ray paths but will also be able to slow down light rays at different positions in the material. This could be used to create new kinds of light manipulations and illusions.

8 Conclusions

We have successfully formulated transformation optics in terms of its natural language of differential geometry, and generalised the theory to non-linear electromagnetic media and curved relativistic spacetimes. It is clear that as the possibilities for metamaterials continue to develop, the full potential of transformation optics will eventually be realised.

Appendices

Appendix A - Transformation of Surface and Volume Integrals under a Diffeomorphism

First consider transformation of a surface area element given by

$$\mathrm{d}\underline{\widetilde{S}} = \delta \underline{x}_1 \times \delta \underline{x}_2.$$

This is deformed into

$$\begin{aligned}\mathrm{d}\underline{S} &= (\underline{\underline{F}}_\xi \cdot \delta \underline{x}_1) \times (\underline{\underline{F}}_\xi \cdot \delta \underline{x}_2) \\ &= J_\xi(\underline{x}) \underline{\underline{F}}_\xi^{-T} \cdot \mathrm{d}\underline{\widetilde{S}},\end{aligned}$$

where

$$J_\xi(\underline{x}) := \det \underline{\underline{F}}_\xi.$$

So we have

$$\int_{\partial M} \underline{f}(\underline{y}) \cdot \mathrm{d}\underline{S} = \int_{\partial \widetilde{M}} J_\xi(\underline{x}) \underline{\underline{F}}_\xi^{-1} \cdot \underline{f}[\underline{\xi}(\underline{x})] \cdot \mathrm{d}\underline{\widetilde{S}} = \int_{\partial \widetilde{M}} \underline{\widetilde{f}}(\underline{x}) \cdot \mathrm{d}\underline{\widetilde{S}},$$

where

$$\underline{\widetilde{f}}_S(\underline{x}) = J_\xi(\underline{x}) \underline{\underline{F}}_\xi^{-1} \cdot \underline{f}_S[\underline{\xi}(\underline{x})].$$

Similarly consider the transformation of a volume element,

$$\mathrm{d}\underline{\widetilde{V}} = \delta \underline{x}_1 \cdot [\delta \underline{x}_2 \times \delta \underline{x}_3].$$

This is deformed into

$$\mathrm{d}V = J_\xi(\underline{x}) \mathrm{d}\underline{\widetilde{V}}.$$

So we have

$$\int_M f_V(\underline{y}) \cdot \mathrm{d}V = \int_{\widetilde{M}} \widetilde{f}_V(\underline{x}) \cdot \mathrm{d}\widetilde{V},$$

where

$$\widetilde{f}_V(\underline{x}) = J_\xi(\underline{x}) \cdot f_V[\underline{\xi}(\underline{x})].$$

Appendix B - Overview of Differential Forms

Definition

For a given manifold M, vectors can be defined on the tangent bundle $TM := \bigcup_x T_xM,\ \forall\ x \in M$ (the disjoint union of the tangent space of every point on M). Differential 1-forms are then defined on the cotangent bundle T^*M (the dual bundle to TM) to be local linear functionals from the tangent bundle to the reals, $\alpha_x : T_xM \to \mathbb{R}$. A differential k-form is defined by generalisation as an alternating (entirely antisymmetric) multilinear map from the k'th outer product of tangent vectors to the reals, $\omega_x : \bigotimes_1^k T_xM \to \mathbb{R}$, where $\omega \in \Omega^k(M)$. In terms of the more familiar notion of tensors, this means a k-form is just an entirely antisymmetric covariant tensor of rank k, defined in a coordinate independent way [18].

Exterior Algebra

We find that these objects naturally satisfy an exterior (or Grassmann) algebra. The product of this algebra is the exterior product, or wedge product, $\wedge : (\bigwedge^p L) \bigotimes (\bigwedge^q L) \to \bigwedge^{p+q} L$, defined for a p-form α and q-forms β, γ by [10]

$$\begin{aligned} &1. \quad \alpha \wedge (\beta + \gamma) = \alpha \wedge \beta + \alpha \wedge \gamma && \text{Distributivity} && (143)\\ &2. \quad \alpha \wedge (\beta \wedge \gamma) = (\alpha \wedge \beta) \wedge \gamma && \text{Associativity} && (144)\\ &3. \quad \alpha \wedge \beta = (-1)^{p+q} \beta \wedge \alpha && \text{Anticommutativity.} && (145) \end{aligned}$$

Notably for 1-forms we see the product is antisymmetric. This allows us to write a general monomial (single term) k-form ω defined at a point P on a manifold to be written in terms of the local coordinates as

$$\omega = A \ \mathrm{d}x^1 \wedge ... \wedge \mathrm{d}x^k \tag{146}$$

for constant A, where the $\mathrm{d}x^i$ are 1-forms. Clearly there are $\binom{n}{k}$ independent k-forms on a n-dimensional manifold, and so $dim\{\Omega^k(M)\} = \binom{n}{k}$.

Differentiation

In exterior calculus there is a single derivative operator d which subsumes the gradient, curl and divergence operators. It is uniquely defined by [10]

$$1.\ \mathrm{d}(\alpha + \beta) = \mathrm{d}\alpha + \mathrm{d}\beta \tag{147}$$

$$2.\ \mathrm{d}(\alpha \wedge \beta) = \mathrm{d}\alpha \wedge \beta + (-1)^p \alpha \wedge \mathrm{d}\beta \tag{148}$$

$$3.\ \mathrm{d}(\mathrm{d}\omega) = 0 \tag{149}$$

$$4.\ \mathrm{d}f = \sum \frac{\partial f}{\partial x^i} \mathrm{d}x^i \quad \text{for any function } f. \tag{150}$$

Property 3 is often written $\mathrm{d}^2 = 0$ and follows simply from the symmetry of mixed partial derivatives. This generalises the well-known results $\vec{\nabla} \cdot (\vec{\nabla} \times \vec{v}) = 0, \vec{\nabla} \times (\vec{\nabla} f) = 0$. The converse of this result is also true: for any k-form ω such that $\mathrm{d}\omega = 0$, we can always write $\omega = \mathrm{d}\alpha$ for some $k-1$ form α. Equivalently it can be stated that any closed form ($\mathrm{d}\omega = 0$) is exact ($\omega = \mathrm{d}\alpha$). This is known as Poincaré's Lemma [18].

Integration

The above notation for ω is appropriately suggestive that a k-form can be integrated over a k-dimensional manifold. So 1-forms appear in line integrals, 2-forms in surface integrals and so on. For example, in 2D Cartesian coordinates (x, y), the surface area element is $\mathrm{d}x\mathrm{d}y$. The lack of $\mathrm{d}x\mathrm{d}x, \mathrm{d}y\mathrm{d}y$ terms is indicative of the antisymmetry of an implicit exterior product between 1-forms $\mathrm{d}x$ and $\mathrm{d}y$ (and the neglected sign represents the orientation of the integral).

This leads to the generalised Stokes' formula

$$\int_{\partial\Omega} \omega = \int_{\Omega} \mathrm{d}\omega \tag{151}$$

where ω is a k-form, Ω is a $(k+1)$-dimensional manifold and $\partial\Omega$ is its k-dimensional boundary. This subsumes the classical Stokes' theorem, the divergence theorem, Green's theorem and the fundamental theorem of calculus into a single result [18].

Hodge Star

Since $\binom{n}{k} = \binom{n}{n-k}$, we are able to define a unique mapping from a k-form α to an $(n-k)$-form β via the relation [10]

$$\alpha \wedge \star\beta = (-1)^{\frac{1}{2}(n-t)}\ (\alpha, \beta)\sigma, \tag{152}$$

where n is the dimension of the manifold M and t is its signature, which is defined simply as the trace of the metric for our purposes. The volume element σ is the only independent n-form on M. In 3D Euclidian space we have simply $n = t$ so $(-1)^{\frac{1}{2}(n-t)} = 1$, while in 4D Minkowski spacetime we have $n = 4, t = 2$ so $(-1)^{\frac{1}{2}(n-t)} = -1$.

Maxwell's Equations using Differential Forms

Taking the definitions of the differential forms given by equations (49)-(54), applying the definition of the exterior derivative d in Cartesian coordinates, we find equations (55)-(58) can be written

$$\left[\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z}\right]\mathrm{d}x\mathrm{d}y\mathrm{d}z = \rho\ \mathrm{d}x\mathrm{d}y\mathrm{d}z$$

$$\left[\frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z}\right]\mathrm{d}x\mathrm{d}y\mathrm{d}z = 0$$

$$\left[\left(\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z}\right)\right]\mathrm{d}y\mathrm{d}z + \text{cyclic perms.} = -\left[\frac{\partial B_x}{\partial t}\right]\mathrm{d}y\mathrm{d}z + \text{cyclic perms.}$$

$$\left[\left(\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z}\right)\right]\mathrm{d}y\mathrm{d}z + \text{cyclic perms.} = \left[J + \frac{\partial D_x}{\partial t}\right]\mathrm{d}y\mathrm{d}z + \text{cyclic perms.}$$

These are clearly equivalent to the vector calculus expressions for Maxwell's Equations.

Appendix C - Derivation of Coordinate Independence of Exterior Derivative

Let ξ be a diffeomorphism from a manifold $\widetilde{M}$ to an assoicated manifold M.

$$\xi : \widetilde{M} \to M$$

Let $x^1...x^m$ be a local coordinate chart on $\widetilde{M}$, and $y^1...y^n$ be a local coordinate chart on M.

Define the pullback ξ^* as a map taking k-forms on M to k-forms on $\widetilde{M}$.

$$\xi^* : F^k(\widetilde{M}) \to F^k(M)$$

Therefore for a function g defined on M, $g : M \to \mathbb{R}$, we have the following.

$$\begin{tikzcd} \widetilde{M} \arrow[r, "\xi"] \arrow[dr, "g \cdot \xi = \xi^* g"'] & M \arrow[d, "g"] \\ & \mathbb{R} \end{tikzcd}$$

This is achieved using subsititution of coordinate functions,

$$\mathrm{d}y^i = \sum \frac{\partial y^i}{\partial x^j} \mathrm{d}x^j.$$

So for a 1-form ω on M,

$$\begin{aligned} \omega &= \sum a_i(\underline{y}) \mathrm{d}y^i \quad \in \; F^1(V) \\ \implies \quad \xi^*\omega &= \sum a_i(\underline{y}(\underline{x})) \frac{\partial y^i}{\partial x^j} \mathrm{d}x^j \quad \in \; F^1(U). \end{aligned}$$

This trivially generalises to k-forms on M.

Theorem: The pullback commutes with the exterior derivative such that for any k-form ω on M,

$$\mathrm{d}(\xi^*\omega) = \xi^* \mathrm{d}\omega.$$

This can be proved inductively [10].

Proof: First take a 0-form g on M,

$$\mathrm{d}g = \sum \frac{\partial g}{\partial y^j} \mathrm{d}y^j$$

$$\begin{aligned} \implies \xi^* \mathrm{d}g &= \sum \frac{\partial g(\underline{y}(\underline{x}))}{\partial y^j} \frac{\partial y^j}{\partial x^i} \mathrm{d}x^i \\ &= \sum \frac{\partial g(\underline{y}(\underline{x}))}{\partial x^i} \mathrm{d}x^i \\ &= \sum \frac{\partial (\xi^* g)}{\partial x^i} \mathrm{d}x^i \\ &= \mathrm{d}(\xi^* g). \end{aligned}$$

Now assume true for a $k-1$-form. Consider a monomial k-form:

$$\omega = g\mathrm{d}\eta$$

where η is a $k-1$-form.

$$\begin{aligned}
\xi^*\omega &= (\xi^* g)(\xi^*\mathrm{d}\eta) \\
&= (\xi^* g)\mathrm{d}(\xi^*\eta)
\end{aligned}$$

$$\begin{aligned}
\implies \mathrm{d}(\xi^*\omega) &= \mathrm{d}(\xi^* g) \wedge \mathrm{d}(\xi^*\eta) + (\xi^* g) \wedge \mathrm{d}^2(\xi^*\eta) \\
&= \mathrm{d}(\xi^* g) \wedge \mathrm{d}(\xi^*\eta)
\end{aligned}$$

Also,

$$\begin{aligned}
\mathrm{d}\omega &= \mathrm{d}g \wedge \mathrm{d}\eta + g\mathrm{d}^2\eta \\
&= \mathrm{d}g \wedge \mathrm{d}\eta
\end{aligned}$$

$$\begin{aligned}
\implies \xi^*(\mathrm{d}\omega) &= \xi^*\mathrm{d}g \wedge \xi^*\mathrm{d}\eta \\
&= \mathrm{d}(\xi^* g) \wedge \mathrm{d}(\xi^*\eta).
\end{aligned}$$

Therefore $\mathrm{d}(\xi^*\omega) = \xi^*\mathrm{d}\omega$ for a k-form ω if true for a $k-1$ form η.

Since this is true for 0-forms, by induction this must be true for all monomial k-forms. And since a general k-form is just a sum over monomial k-forms, this holds for all k-forms.

Now consider the case where M and $\widetilde{M}$ are equivalent manifolds. Interpreting the coordinates $\underline{x}$ of $\widetilde{M}$ as new coordinates on M, this result means that the exterior derivative of a differential form is independent of the coordinate system in which it is evaluated.

Appendix D - Pullback of Hodge Star

General method obtained from Dr Al-Attar by personal communication.

The Hodge star is defined by

$$\alpha \wedge \star\beta = (-1)^{\frac{1}{2}(n-t)}\ (\alpha, \beta)\sigma$$

$$\begin{aligned} \text{where} \quad \alpha, \beta \in\ & \Omega^k(M) \quad \text{and } (\alpha,\beta) \text{ is their inner product,} \\ \sigma \in\ & \Omega^n(M) \quad \text{is the volume element.} \end{aligned}$$

Here assume a Euclidian spacetime for simplicity such that $(-1)^{\frac{1}{2}(n-t)} = 1$. However note that for a Lorentzian spacetime, the extra sign does not affect the final result derived here.

Apply the pullback to this definition

$$\begin{aligned} \xi^*(\alpha \wedge \star\beta) &= \xi^*[(\alpha, \beta)_{\bigwedge^k M}\ \sigma] \\ \xi^*\alpha \wedge \xi^*(\star\beta) &= (\alpha, \beta)_{\bigwedge^k M}\ \xi^*\sigma. \end{aligned}$$

The transformation of the volume element is defined by the Jacobian

$$\xi^*\sigma = J_\xi\ \widetilde{\sigma}$$

where $\widetilde{\sigma}$ is the volume element on $\widetilde{M}$. This is evident since there is only one linearly independent n-form on an n-dimensional manifold, so the pullback of σ must be related to $\widetilde{\sigma}$ by a constant.

Now the inner product is a scalar so its value is unaffected by pullback.

$$\begin{aligned} (\alpha, \beta)_{\bigwedge^k M} &= \xi^*(\alpha, \beta)_{\bigwedge^k M} \\ &= \xi^*\langle \alpha, (\textstyle\bigwedge^k g^{-1})\beta\ \rangle \end{aligned}$$

where g is the metric defined on M and $\langle\cdot,\cdot\rangle$ represents the duality product

$$\begin{aligned} &= \Big\langle \xi^*\alpha, \xi^*\big[(\textstyle\bigwedge^k g^{-1})\beta\big]\ \Big\rangle \\ &= \Big\langle \xi^*\alpha, \textstyle\bigwedge^k (\xi^* g^{-1})\ \xi^*\beta\ \Big\rangle \\ &= \Big\langle \xi^*\alpha, \textstyle\bigwedge^k (\xi^* g)^{-1}\ \xi^*\beta\ \Big\rangle \\ &= \Big\langle \xi^*\alpha, (\textstyle\bigwedge^k\ \widetilde{g}^{-1})(\textstyle\bigwedge^k\ \widetilde{g})\textstyle\bigwedge^k (\xi^* g^{-1})\ \xi^*\beta\ \Big\rangle \end{aligned}$$

where $\widetilde{g}$ is the metric on $\widetilde{M}$

$$\begin{aligned} &= \Big\langle \xi^*\alpha, \textstyle\bigwedge^k\ \widetilde{g}^{-1}\textstyle\bigwedge^k \big[\widetilde{g}(\xi^* g)^{-1}\big]\ \xi^*\beta\ \Big\rangle \\ &= \Big\langle \xi^*\alpha, (\textstyle\bigwedge^k\ \widetilde{g}^{-1})(\textstyle\bigwedge^k\ T)\ \xi^*\beta\ \Big\rangle \\ &= \Big(\xi^*\alpha, (\textstyle\bigwedge^k\ T)\ \xi^*\beta\ \Big)_{\bigwedge^k\ \widetilde{M}} \end{aligned}$$

where

$$T := \widetilde{g}(\xi^* g)^{-1}.$$

So we have written the inner product on M as an inner product on $\widetilde{M}$. Therefore we have

$$\xi^*\alpha \wedge \xi^*(\star\beta) = J_\xi\Big(\xi^*\alpha, (\bigwedge^k T)\ \xi^*\beta\ \Big)_{\bigwedge^k\ \widetilde{M}}\ \widetilde{\sigma}.$$

By defintion of the Hodge star, the inner product can be written

$$\Big(\xi^*\alpha, (\bigwedge^k T)\xi^*\beta\Big)_{\bigwedge^k M}\ \widetilde{\sigma} = \xi^*\alpha \wedge \star\big[(\bigwedge^k T)\xi^*\beta\big].$$

So we have

$$\xi^*\alpha \wedge \xi^*(\star\beta) = \xi^*\alpha \wedge \star\big[J_\xi(\bigwedge^k T)\xi^*\beta\big].$$

And since this is true for all k-forms α,

$$\xi^*(\star\beta) = \star\big[J_\xi(\bigwedge^k T)\xi^*\beta\big]\Big].$$

Now we just need to determine the determinant factor J_ξ. Consider the case where $\beta = \sigma$,

$$\xi^*(\star\sigma) = \star\big[J_\xi(\bigwedge^n T)\xi^*\sigma\big].$$

Since $\star\sigma = 1$ and $\xi^*\sigma \equiv J_\xi\widetilde{\sigma}$,

$$\begin{aligned} \implies\ \xi^*(1) &= \star\big[J_\xi(\bigwedge^n T)J_\xi\widetilde{\sigma}\big] \\ 1 &= J_\xi^2 \star \big[(\bigwedge^n T)\widetilde{\sigma}\big]. \end{aligned}$$

Now,

$$\begin{aligned} (\bigwedge^n T)\widetilde{\sigma} &= T\mathrm{d}\widetilde{x}_1\ \wedge\ T\mathrm{d}\widetilde{x}_2\ ...\ \wedge\ T\mathrm{d}\widetilde{x}_n \\ &\equiv |T|\mathrm{d}\widetilde{x}_1\ \wedge\ \mathrm{d}\widetilde{x}_2\ ...\ \wedge\ \mathrm{d}\widetilde{x}_n \\ &= |T|\widetilde{\sigma} \end{aligned}$$

where $|T|$ is the determinant of T. So we have

$$\begin{aligned} 1 &= J_\xi^2|T| \star \widetilde{\sigma} \\ \implies\ J_\xi &= |T|^{-\frac{1}{2}} \end{aligned}$$

Therefore we have

$$\xi^*(\star\beta) = \star[J_\xi(\bigwedge^k T)\xi^*\beta],$$

where we have defined

$$\begin{aligned} T &= \widetilde{g}(\xi^* g)^{-1}, \\ J_\xi &= |T|^{-\frac{1}{2}}. \end{aligned}$$

Appendix E - Pullback of Metric and Linear Operators

Consider manifolds M and $\widetilde{M}$ with associated metrics g and $\widetilde{g}$ respectively. Define diffeomorphism

$$\begin{aligned} \xi: &\quad \widetilde{M} \to M \\ \implies T\xi: &\quad T\widetilde{M} \to TM. \end{aligned}$$

And define deformation gradient

$$F(x) := T\xi|_x : T_x\widetilde{M} \to T_{\xi(x)}M.$$

For $v \in T_x\widetilde{M}$ and $\alpha \in T^*_{\xi(x)}M$, define dual operator F^* by

$$\langle F_x v, \alpha \rangle_M = \langle v, F^*_x \alpha \rangle_{\widetilde{M}}$$

$$\implies F^*_x : T^*_{\xi(x)}M \to T^*_x\widetilde{M}.$$

So we now have the mappings

$$\begin{aligned} F_x :&\ T_x\widetilde{M} \to T_{\xi(x)}M, \\ F_x^{-1} :&\ T_{\xi(x)}M \to T_x\widetilde{M}, \\ F^*_x :&\ T^*_{\xi(x)}M \to T^*_x\widetilde{M}, \\ F_x^{*-1} :&\ T^*_x\widetilde{M} \to T^*_{\xi(x)}M. \end{aligned}$$

Pullback of Metric

Require the pullback of the metric to satisfy

$$(\xi^* g)_x\ (v, w) = g_{\xi(x)}(T\xi|_x v, T\xi|_x w)$$

for $v, w \in T_x\widetilde{M}, T\xi|_x v \in T_{\xi(x)}M$, such that

$$\begin{aligned} \big\langle (\xi^* g)_x v, w \big\rangle_{\widetilde{M}} &= \big\langle g_{\xi(x)} F_x v, F_x w \big\rangle_M \\ &= \big\langle F^*_x g_{\xi(x)} F_x v, w \big\rangle \\ \implies (\xi^* g)_x &= F^*_x g_{\xi(x)} F_x. \end{aligned}$$

This is the pullback of the metric. This has the effect

$$T_x\widetilde{M} \to T_{\xi(x)}M \to T^*_{\xi(x)}M \to T^*_x\widetilde{M}$$

as expected.

Pullback of Linear Operators

For a differential form linear operator ϵ,

$$\begin{aligned} \epsilon : T^*M \to T^*M \quad &\text{(homomorphism)} \\ \implies \quad \xi^*\epsilon : T^*\widetilde{M} \to T^*\widetilde{M}. & \end{aligned}$$

Now define the pullback of ϵ by

$$\begin{aligned}
\left\langle (\xi^*\epsilon)_x \alpha, v \right\rangle_{\widetilde{M}} &= \left\langle \epsilon_{\xi(x)} F^{x-1}\alpha, F_x v \right\rangle_M \\
&= \left\langle F_x^* \epsilon_{\xi(x)} F^{x-1}\alpha, v \right\rangle_{\widetilde{M}} \\
\implies (\xi^*\epsilon)_x &= F_x^* \epsilon_{\xi(x)} F_x^{*-1}.
\end{aligned}$$

This is the pullback of a linear operator ϵ. This has the effect

$$T_x^*\widetilde{M} \to T_{\xi(x)}^* M \to T_{\xi(x)}^* M \to T_x^*\widetilde{M}$$

as expected.

Appendix F - Pullback of Fibre Derivative

General method obtained from Dr Al-Attar by personal communication.

Let $\tau : T^*\widetilde{M} \to \widetilde{M}$ and $\pi : T^*M \to M$ be projections from vector bundles $T^*\widetilde{M}$ and T^*M onto manifolds $\widetilde{M}$ and M. Let $F^{*-1} : T^*\widetilde{M} \to T^*M$ be a smooth vector bundle isomorphism along a diffeomorphism $\xi : \widetilde{M} \to M$, and $F^* : T^*M \to T^*\widetilde{M}$ be the associated dual vector bundle isomorphism defined along $\xi^{-1} : M \to \widetilde{M}$ such that we have the following commutative diagrams.

$$\begin{tikzcd} T^*\widetilde{M} \arrow[r, "F^{*-1}"] \arrow[d, "\tau"] & T^*M \arrow[d, "\pi"] \\ \widetilde{M} \arrow[r, "\xi"] & M \end{tikzcd} \qquad \begin{tikzcd} T^*M \arrow[r, "F^{*}"] \arrow[d, "\pi"] & T^*\widetilde{M} \arrow[d, "\tau"] \\ M \arrow[r, "\xi^{-1}"] & \widetilde{M} \end{tikzcd}$$

Consider function $U \in C^\infty(T^*M)$. Pullback of this function is defined by

$$\xi^*U := U \circ F^{*-1} \;\in\; C^\infty(T^*\widetilde{M}).$$

By definition of the fibre derivative

$$\left\langle \mathbb{F}\big[(\xi^*U)(\widetilde{E})\big], \widetilde{E}' \right\rangle = \frac{\mathrm{d}}{\mathrm{ds}}\big[(\xi^*U)(\widetilde{E} + s\widetilde{E}')\big]\Big|_{s=0}$$

for $\widetilde{E}, \widetilde{E}' \in T^*_x\widetilde{M}$ where $x \in \widetilde{M}$ (so $\widetilde{E}, \widetilde{E}'$ exist in the same fibre on M).

$$\begin{aligned} \text{Now,}\quad \xi^*U(\widetilde{E} + s\widetilde{E}') &= U \circ F^{*-1}(\widetilde{E} + s\widetilde{E}') \\ &= U(F^{*-1}\widetilde{E} + sF^{*-1}\widetilde{E}'). \end{aligned}$$

$$\begin{aligned} \text{Therefore,}\quad \left\langle \mathbb{F}\big[(\xi^*U)(\widetilde{E})\big], \widetilde{E}' \right\rangle &= \frac{\mathrm{d}}{\mathrm{ds}} U(F^{*-1}\widetilde{E} + sF^{*-1}\widetilde{E}')\Big|_{s=0} \\ &= \left\langle \mathbb{F}U(F^{*-1}\widetilde{E}), F^{*-1}\widetilde{E}' \right\rangle \\ &= \left\langle F^{-1}\mathbb{F}U(F^{*-1}\widetilde{E}), \widetilde{E}' \right\rangle \end{aligned}$$

$$\begin{aligned} \implies \mathbb{F}\big[(\xi^*U)(\widetilde{E})\big] &= F^{-1}\mathbb{F}U(F^{*-1}\widetilde{E}) \\ &= F^{-1}\mathbb{F}U(E_{\xi(x)}). \end{aligned}$$

And since the pullback of the fibre derivative $\mathbb{F}U(E_{\xi(x)}) \in T_{\xi(x)}M$ is defined by

$$\xi^*\mathbb{F}U(E_{\xi(x)}) := F_x^{-1}\mathbb{F}U(E_{\xi(x)}),$$

we therefore have

$$\xi^*\mathbb{F}U(E_{\xi(x)}) = \mathbb{F}\big[(\xi^*U)(\widetilde{E}_x)\big]$$

Appendix G - Fundamental Invariants of Electromagnetic Field

Proof taken from The Classical Theory of Fields, Volume 2, Landau and Lifshitz. [23]

Consider the 3-vector

$$\underline{F} = \underline{E} + i\underline{B}.$$

Now consider the behaviour of this vector under Lorentz transformations. The electromagnetic fields can be shown to transform under a Lorentz boost as

$$\begin{aligned} \underline{E}'_{\parallel} &= \underline{E}_{\parallel} \qquad \underline{E}'_{\perp} = \gamma[\underline{E} + \underline{v} \times \underline{B}]_{\perp}, \\ \underline{B}'_{\parallel} &= \underline{B}_{\parallel} \qquad \underline{B}'_{\perp} = \gamma\Big[\underline{B} - \frac{1}{c^2}(\underline{v} \times \underline{E})\Big]_{\perp}. \end{aligned}$$

Using the rapidity parameter ψ, defined by $\tanh\psi = \beta$, the Lorentz transformation of $\underline{F}$ due to a boost along the x-axis can therefore be written (in natural units)

$$\underline{F}' = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cosh\psi & -i\sinh\psi \\ 0 & i\sinh\psi & \cosh\psi \end{pmatrix} \underline{F}.$$

So we see that Lorentz transformations of the 3-vector $\underline{F}$ can be represented by rotations through complex angles. Real angles correspond to Euclidian rotations, while imaginary angles correspond to boosts via the rapidity parameter ψ.

Now, the only invariant of a vector with respect to rotations is its length, given by its square.

$$\underline{F}^2 = (\underline{E}^2 - \underline{B}^2) + 2i(\underline{E} \cdot \underline{B})$$

Since the 3-vector $\underline{F}$ is clearly uniquely defined for given $\underline{E}$ and $\underline{B}$ fields, the only invariants of the 3 vector $\underline{F}$ will also be the only invariants of the electromagnetic field. Therefore the only invariants of the electromagnetic field are $\underline{E}^2 - \underline{B}^2$ and $\underline{E} \cdot \underline{B}$.

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