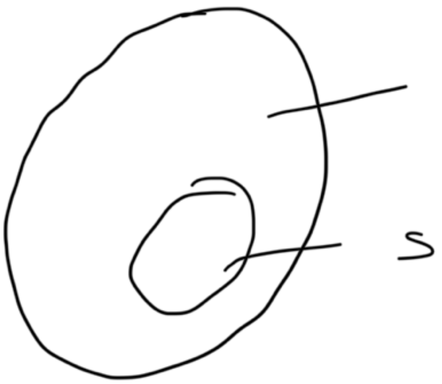
Linear Algebra Exam 2 Review



If $a \in S$ then $c \cdot a \in S$

If $a \in S$ & $b \in S$ then $a + b \in C$

The set of matrices of the form $\begin{bmatrix} a & b \\ -b & c \end{bmatrix}$ is a subspace of $\mathbb{R}^{2\times 2}$ - True

If $v_1 = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$ & $v_2 = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$ find their span

$$\begin{bmatrix} a & b \\ -b & c \end{bmatrix} \begin{bmatrix} 5 \\ 1 \end{bmatrix} = \begin{bmatrix} 5a + b \\ -5b + c \end{bmatrix} \quad \begin{bmatrix} -b \\ -c \end{bmatrix}$$

$5a$

$$\begin{bmatrix} a & b \\ -b & c \end{bmatrix} \begin{bmatrix} 0 \\ -1 \end{bmatrix} = \begin{bmatrix} -b \\ -c \end{bmatrix}$$

Subset U of V is called a subspace of V if U is also a vector space with the same additive idubity addition and scalar multiplication.

example:

$$S = \{ (x,y) \in \mathbb{R}^2 \mid 3x - y = 0$$ is a subspace of $\mathbb{R}^2$

1.) Additive identity

Check for $(0,0)$

$$3(0) + 0 = 0$$

2) Addition

$u = (u_1, u_2) \quad v = (v_1, v_2)$

$3u_1 - u_2 = 0 \qquad 3v_1 - v_2 = 0$

Check $u+v$

$$(u_1, u_2) + (v_1, v_2) = (u_1 + v_1, u_2 + v_2)$$

$$= 3(u_1 + v_1) - (u_2 + v_2) = (3u_1 - u_2) + (3v_1 - v_2) = 0 + 0 = 0$$

Closure under scalar Multiplication

$u = (u_1, u_2)$ from S where $3u_1 - u_2 = 0$

$$c \cdot u = (cu_1, cu_2)$$

$$3(C(u_1) - C(u_2)) = C(3u_1 - u_2) = C \cdot 0 = 0$$

Sum Of Subspaces

Given two subspaces U and W of a vector space V, their sum $U+W$ is defined as:

$$U + W = \{ u + w \mid u \in U, w \in W \}$$

This means $U+W$ consists of all possible vectors you can get by adding a vector from U to a vector from W

Proving $U + W$ is a subspace

1) Additive identity → zero vector in both U & W Thus, $0 = 0 + 0 \in U + W$

2.) Closure under addition: Take two vectors $x = u_1 + w_1$ and $y = u_2 + w_2$ from $U+W$, where $u_1, u_2 \in U$ & $w_1, w_2 \in W$. Their sum is

$$x + y = (u_1 + w_1) + (u_2 + w_2) = (u_1 + u_2) + (w_1 + w_2)$$

Since U is closed under addition $u_1 + u_2 \in U$ & $w_1 + w_2 \in W$ ...

$\in W$, Thus $x+y \in U+W$

3) Closure under scaler Multiplication: Take a vector $x = u+w \in U+W$ and a scalar c

Then:

$$cx = c(u+w) = cu + cw$$

$cu \in U$ & $cw \in W$ hence $cx \in U+W$

Dimension of the Sum

$$\dim(U+W) = \dim(U) + \dim(W) - \dim(U \cap W)$$

Thm 1.40

Suppose $V_1, \dots, V_m$ are subspaces of V. Then $V_1 + \dots + V_m$ is the smallest subspace of V containing $V_1, \dots, V_m$

Eg. let's Compute the sum of two subspaces in $\mathbb{R}^3$

$\to U = \{(x, y, 0) \mid x \in \mathbb{R}\}$ (the x-axis dimension 1)

$\to W = \{(0, y, 0) \mid y \in \mathbb{R}\}$ (the y-axis dim 1)

Aditive identity

$$(0,0,0) = (0,0,0) + (0,0,0) \in U+W$$

Addition

$$(x_1, y_1, 0) + (x_2, y_2, 0) = (x_1 + x_2, y_1 + y_2, 0)$$

Scalar Multiplicat

$$c \cdot (x, y, 0) = (cx, cy, 0) \in U + W$$

Direct Sums

definition:

Suppose $V_1, \ldots, V_m$ are subspaces of V

→ The sum $V_1 + \cdots + V_m$ is called a direct sum if each element of $V_1 + \cdots + V_m$ can be written in only one way as a sum $v_1 + \cdots + v_m$ where each $v_k \in V_k$

→ If $V_1 + \cdots + V_m$ is a direct sum then $V_1 \oplus \cdots \oplus V_m$ denotes $V_1 + \cdots + V_m$ with the $\oplus$ notation showing it is a direct sum

Thm 1.45

Suppose $V_1, \ldots, V_m$ are subspaces

$V_1 + \cdots + V_m$ is a direct sum if and only if the only way to write 0 as a sum $v_1 + \cdots + v_m$, where each $v_k \in V_k$ is by taking each $v_k = 0$

Thm 1.46

Suppose U and W are subspaces of V. Then

$U + W$ is a direct sum $\Longleftrightarrow U \cap W = \{0\}$

2A Span and Linear Independence

Linear Combination & Span

defn 2.2 linear combination

A linear combination of a list $v_1, \ldots, v_m$ of vectors in V is a vector of the form

$$a_1 v_1 + \cdots a_m v_m$$

e.g

$(17, -4, 2)$ is a linear combination of $(2, 1, -3)$ & $(1, -2, 4)$

$$(17, -4, 2) = 6(2, 1, -3) + 5(1, -2, 4)$$

degn Span

Set of all linear combination of a list of vectors $v_1, \ldots, v_m$ in V is called the span of $v_1, \ldots, v_m$ denoted by $\operatorname{span}(v_1, \ldots, v_m)$

$$\operatorname{span}(v_1, \ldots, v_m) = \{a_1 v_1 + \cdots + a_m v_m : a_1, \ldots, a_m \in F$$

Thm 2.6 Span is the smallest containing subspace

The span of a list of vectors in V is the smallest subspace of V containing all vectors in the list.

Proof

Suppose

$v_1, \ldots, v_m$ is a list of vectors in V

W.T.S

$\rightarrow$ $\operatorname{span}(v_1, \ldots, v_m)$ is a subspace of V.

Additive idebity

$$0 = 0v_1 + \cdots + 0v_m$$

Closure under a delition

$$(a_1 v_1 + \cdots + a_m v_m) + (c_1 v_1 + \cdots + c_m v_m)$$
$$= (a_1 + c_1) v_1 + \cdots + (a_m + c_m) v_m$$

scalar multiplication

$$\lambda(a_1 v_1 + \cdots + a_m v_m) = \lambda a_1 v_1 + \cdots + \lambda a_m v_m$$

$\ldots a_1 v_1 + \cdots + a_m v_m.$

defn 2.7 spans

If span $(v_1, \ldots, v_m)$ equals V we say that the list $v_1, \ldots v_m$ spans V.

Suppose n is a positive integer. We want to show that

$$(1, 0, \ldots 0), (0, 1, 0 \ldots 0), \ldots, (0, \ldots 0, 1)$$

Span F^n. Here the k^{th} vector in the List above has 1 in k^{th} scot \$ 0 on other

Suppose $(x_1, \ldots x_n) \in F^n$

$$(x_1, \ldots, x_n) = x_1 (1, 0, \ldots 0) + x_2 (0, 1, 0 \ldots 0) + \cdots + x_n (0, \ldots 0, 1)$$

thus

defn 2.9 finite dimensional V.S

A vector space is called finite-dimensional if some List of vectors in it span the space.

defn 2-10 polynomials P

A function $p : F \to F$ is called a polynomial with coefficients F if there exists $a_0, \ldots, a_m \in F$ such that

$$p(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_m z^m$$

for all $z \in F$

2.13 definition:

A vector space is called infinite dimensional if it is not finite dimensional

Linear Independence

Suppose $v_1, \ldots, v_m \in V$ and $v \in \operatorname{span}(v_1, \ldots, v_m)$

By definition of span, there exist $a_1, \ldots, a_m \in F$ such that

$$v = a_1 v_1 + \cdots + a_m v_m$$

Another set

$$v = c_1 v_1 + \cdots + c_m v_m$$

Subtracting

$$0 = (a_1 - c_1) v_1 + \cdots + (a_m - c_m) v_m.$$

2.15 defn

→ A list $v_1, \ldots, v_m$ of vectors in V is called linearly independent if the only choice of $a_1, \ldots, a_m$

$\in F$ that makes

$$a_1 v_1 + \cdots + a_m v_m = 0$$

is $a_1 = \cdots a_m = 0$

- Empty list is declared as Linearly independent

Eg.

$(1,0,0,0), (0,1,0,0), (0,0,1,0)$

$$a_1(1,0,0,0) + a_2(0,1,0,0) + a_3(0,0,1,0)$$

$$= (a_1, a_2, a_3, 0) = (0,0,0,0)$$

$a_1 = 0 \quad a_2 = 0 \quad a_3 = 0$ Linearly independent

Definition : Linearly dependent

→ A list of vectors in V is called Linearly dependent if it is not Linearly independent

→ In other words, a list $v_1 \cdots v_m$ of Vectors in V is Linear dependent if there exists $a_1, \cdots a_m \in F$ not all 0 such that $a_1 v_1 + \cdots a_m v_m = 0$

$$2(2,3,1) + 3(1,-1,2) + -1(7,3,8) = (0,0,0)$$

$(4, 6, 2) + (3, -3, 6) + (-7, -3, -8) = (0, 0, 0)$

Lemma 2.19 Linear dependance lemma

Suppose $v_1, \ldots, v_m$ is a linearly dependent list in V. then there exists $k \in \{1, 2, \ldots, m\}$ such that

$$v_k \in \operatorname{span}(v_1, \ldots, v_{k-1}).$$

Furthermore if k satisfies the condition above & kth term is removed from $v_1, \ldots, v_m$ then the span of the remaining list equals $\operatorname{span}(v_1, \ldots, v_m)$.

Proof:

Because the list $v_1, \ldots, v_m$ is linearly dependent there exists numbers $a_1, \ldots, a_m \in \mathbb{F}$ not all 0 such that

$$a_1 v_1 + \cdots a_m v_m = 0$$

let k be the largest element of $(1, \ldots m)$ such that $a_k \neq 0$. Then

$$v_k = -\frac{a_1}{a_k} v_1 - \cdots - \frac{a_{k-1}}{a_k} v_{k-1},$$

which proves that $v_k \in \operatorname{Span}(v_1, \ldots, v_{k-1})$ as desired.

Thm 2.22 length of linearly independent $\leq$ length of spanning list

In a finite-dimensional vector space, the length of every linearly independent list of vectors is less than or equal to the length of every spanning list of vectors

Thm 2.25 finite-dimensional subspaces

Every subspace of a finite-dimensional vector space is finite dimensional.

2B Bases

defn 2.26 basis

A basis of V is a list of vectors in V that is linearly independent and spans V.

Thm 2.28 Criterion for basis

A list $v_1, \dots, v_n$ of vectors in V is a basis of V if and only every $v \in V$ can be written uniquely in the form.

$$v = a_1 v_1 + \dots + a_n v_n$$

where $a_1, \dots a_n \in F$.

Thm 2.30 every spanning list con...

contains a basis

Every spanning list in vector space can be reduced to a basis of the vector space.

Thm 2.31

Every finite-dimensional vector space has a basis

Thm 2.32

Every Linearly independent list of vectors in a finite dimensional vector space can be extended to a basis of the vector space.

Thm 2.33 every subspace of V is part of a direct sum equal to V

2C Dimensions

Thm 2.34 basis length does not depend on basis

Any two bases of a finite dimensional vector space have the same length.

defn. dimension $\dim V$

→ The dimension of a finite dimensional vector space is the length of any basis of the vector space

→ denoted by $\dim V$

Th

Thm 2.37 dimension of subspace
If V is finite-dimensional & U is a subspace of V, then
$\dim U \leq \dim V$

Thm 2.38 - Linearly independent list of the right length is a basis

Suppose V is finite-dimensional. Then every linearly independent list of vectors in V of length $\dim V$ is a basis of V

Thm 2.39 - Subspace of full dimension equals the whole space

Suppose that V is finite-dimensional and U is a subspace of V such that $\dim U = \dim V$. then $U = V$

Thm 2.42 dimension of a sum

Suppose V is finite dimensional. Then every spanning list of vectors in V of length $\dim V$ is a basis of V

Thm 2.43

If V_1 & V_2 are subspaces of a finite dimensional vector space, then

$$\dim(V_1 + V_2) = \dim V_1 + \dim V_2 - \dim(V_1 \cap V_2)$$

3A Vector Space of linear maps

of Linear Maps

A Linear map from V to W is a function $T: V \to W$ with the following properties:

additivity

$$T(u+v) = Tu + Tv \text{ for all } u, v \in V$$

Homogenity

$$T(\lambda v) = \lambda(Tv) \text{ for al } \lambda \in F \text{ and for all } v \in V$$

→ set of linear maps from V to W → $\mathcal{L}(V,W)$

" " " " V to V → $\mathcal{L}(V)$

$$\mathcal{L}(V) = \begin{pmatrix} v_2 \\ v_1 + v_2 \\ v_1 - v_2 \end{pmatrix}$$

3.4 Linear Map Lemma

Suppose $v_1, \ldots, v_n$ is a basis of V and $w_1, \ldots, w_n \in W$. Then there exists a unique linear map $T: V \to W$ such that

$$Tv_k = w_k$$

for each $k = 1, \ldots, n$.

3.5 definition: Addition and Scalar multiplication on $\mathcal{L}($

suppose $S, T \in \mathcal{L}(V, W)$ and $\lambda \in F$. The sum $S+T$ and the product λT are linear maps from V to W defined by

$$(S+T)(v) = Sv + Tv \quad \& \quad (\lambda T)v = \lambda(Tv)$$

Thm 3.6

With the operations of addition and scalar multiplication, $\mathcal{L}(V, W)$ is a vector space.

defn 3.7 Product of linear maps

If $T \in (U, V)$ and $S \in \mathcal{L}(V, W)$, then the product $ST \in \mathcal{L}(U, W)$ is defined by

$$(ST)(u) = S(Tu)$$

for all $u \in U$.

Thm 3.8

associativity

$$(T_1 T_2)T_3 = T_1(T_2 T_3)$$

Idabity

$TI = IT = T$

distributive property

$(S_1 + S_2)T = S_1T + S_2T$

Thm 3.10

Suppose T is a linear map from V to W. Then $T(0) = 0$

3B

3.11 defin

Null Spaces and Ranges

for $T \in L(V, W)$ the null space of T, denoted by null T, is the subset of V consisting of those vectors that T maps to 0:

$$\text{null } T = \{v \in V : Tv = 0\}$$

Theorem 3.13 Null space is a subspace

Suppose $T \in L(V, W)$. Then null T is a subspace of V.

defn 3.14 injective

3.14 Injective

A function $T: V \to W$ is called injective if $Tu = Tv$ implies $u = v$.

Thm 3.15 Injectivity Null space equals $\{0\}$

Let $T \in \mathcal{L}(V, W)$. Then T is injective if and only if null $T = \{0\}$

Range and Surjectivity

defn 3.16 range:

for $T \in \mathcal{L}(V, W)$ the range of T is the subset of W consisting of those vectors that are equal to Tv for some $v \in V$

$$\text{range } T = \{Tv : v \in V\}$$

Thm 3.18 Range is a subspace

If $T \in \mathcal{L}(V, W)$, then range T is a subspace of W

defn 3.19 Surjective

A function $T: V \to W$ is called surjective iff it's range equals W.

Fundamatal Theorem of Linear Maps

Suppose V is finite dimensional and $T \in \mathcal{L}(V, W)$.
Then range T is finite-dimensional

$$\dim V = \text{null } T + \text{range } T$$

Theorem 3.22 Linear map to a lower-dimensional space is not injective

Suppose V and W are finite-dimensional vector spaces such ~~that~~ $\dim V > \dim W$. Then no linear map from V to W is injective

Theorem 3.24 Linear map to a higher dimension space is not surjective

Suppose V and W are finite dimensional vector space such that $\dim V > \dim W$. Then no linear map from V to W is surjective.

Thm 3.26 homogenous system of linear equations

A homogenous system of linear equations with more variables than equations has nonzero solutions

Thm 3.28 inhomogenous system of linear equations

An inhomogenous system of linear equations with more equations than variables has no solution for some choice of constant terms.