Math 231 — Hw 20
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1. Suppose S is the half circle of radius 1 under a basis $\{v_1, v_2\}$ of $\mathbb{R}^2$. Let A is a matrix that changes this basis to the standard normal basis. If the determinant of A is 4, what is the area of AS?

2. What is the determinant of the following matrix:

$$\begin{pmatrix} 1 & 1 & -2 \\ 2 & 1 & -1 \\ 1 & 2 & -1 \end{pmatrix}.$$

   Based on the determinant, compute the null space of this map.

3. Suppose you wish to find the area of the ellipse

$$\left(\frac{x+2y}{3}\right)^2 + \left(\frac{y-x}{2}\right)^2 = 1.$$

   After thinking about this for a moment, you realize that you can characterize this as a linear change of coordinates (a change of basis!!) (see the figures below). You first define a linear transformation T that would characterize transforming the ellipse into the unit circle: from $\left(\frac{x+2y}{3}, \frac{y-x}{2}\right)$ to (x, y). This is equivalent to "translating" the ellipse's basis, which we can denote as $\{v_1, v_2\}$, to the standard normal basis $\{e_1, e_2\}$:

$$Tv_1 = \frac{1}{3}e_1 - \frac{1}{2}e_2$$

   and

$$Tv_2 = \frac{2}{3}e_1 + \frac{1}{2}e_2$$

   Here, the standard normal basis helps us represent the cartesian coordinate system with points being (x, y).

   Find a matrix representation of T and compute its determinant. Use its determinant to find the area of the ellipse. (Hint: What is the area of the image? How should you use the determinant to figure out the ellipse, which is your domain?)

4. The inverse of a 2×2 matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

   has the following formula:

$$A^{-1} = \frac{1}{\det A}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

   Based on the matrix you found in the previous problem, find the matrix representation of T^{-1}.

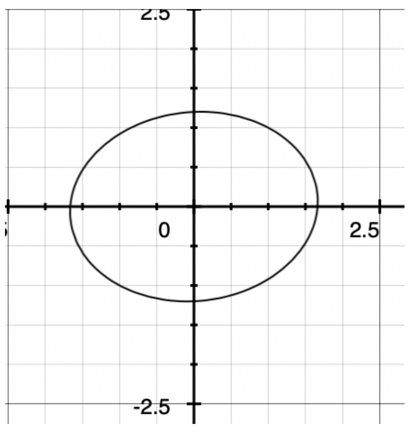

Figure 1: Original Ellipse

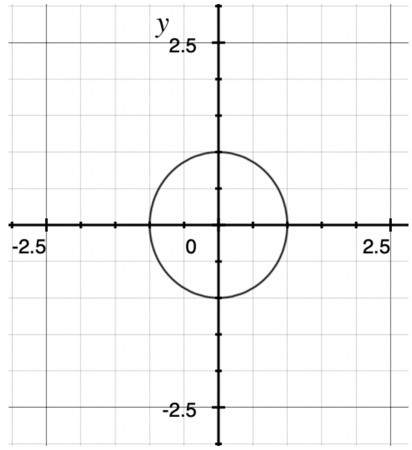

Figure 2: Unit Circle

5. Verify the formula of the inverse matrix by multiplying the matrices from the two previous problems and show that you get the identity matrix:

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$