Math 231 — Hw 8
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1. Let V be a vector space and $0 \in V$ the additive identity. Prove that $0 + 0 = 0$. Then prove that $0 + \ldots + 0 = 0$ for any finite number of sums.

Proof. We will prove this using induction.
Base case: Since 0 is the additive identity in V, we know that for any $v \in V$, we have:

$$v + 0 = v.$$

Since 0 is a vector as well, it follows that:

$$0 + 0 = 0.$$

Inductive step: Suppose $0 + 0 + \ldots + 0$ (k times) equals 0. Now consider $k + 1$ terms:

$$0 + 0 + \ldots + 0 + 0 = (0 + 0 + \ldots + 0) + 0.$$

By the inductive hypothesis, the first k terms sum to 0, so this reduces to

$$= 0 + 0.$$

This was shown in the base case to be 0. $\square$

2. Let $V = \mathbb{R}^3$ and consider the subspaces:

$$W_1 = \{(x, y, 0) \mid x, y \in \mathbb{R}\}, \quad W_2 = \{(0, 0, z) \mid z \in \mathbb{R}\}.$$

Prove that $V = W_1 \oplus W_2$ using the last theorem from class.

To prove that $V = W_1 \oplus W_2$, we must show two things:

(a) $V = W_1 + W_2$, meaning every vector in V can be written as the sum of a vector from W_1 and a vector from W_2.

(b) $W_1 \cap W_2 = \{0\}$.

Proof. Because W_1, W_2 are subspaces, it follows that $W_1 + W_2 \subseteq V$. Let $(a, b, c) \in V = \mathbb{R}^3$. We can write:

$$(a, b, c) = (a, b, 0) + (0, 0, c),$$

where $(a, b, 0) \in W_1$ and $(0, 0, c) \in W_2$. Hence, we have the opposite containment. Therefore, $V = W_1 + W_2$.

The intersection is the additive identity.
Suppose $(x, y, 0) \in W_1$ is also in W_2. Then it must also be of the form $(0, 0, z)$ for some z. The only vector that satisfies both forms is $(0, 0, 0)$. Therefore:

$$W_1 \cap W_2 = \{0\}.$$

By theorem 1.46,

$$V = W_1 \oplus W_2.$$

$\square$

3. Let $V = \mathbb{R}^3$. Consider the subspace $U = \{(x, y, 0) \mid x + y = 0\}$. Find a space W such that $V = U \oplus W$.

   For this problem, we need only define W to be any space that covers a different area in the xy-plane as well as the z direction. We will define it to be

   $$W = \{(0, y, z) \mid y, z \in \mathbb{R}\}.$$

   Claim: $V = U \oplus W$.

   Proof. Both W and U are subspaces of V, so the same holds for $W + U$, meaning $W + U \subseteq V$. Let $v \in V$. Then $v = (a, b, c)$ for some $a, b, c, \in \mathbb{R}$. Define vectors $u = (a, -a, 0)$ and $w = (0, b + a, c)$, which are vectors in U and W respectively. Observe that $v = u + w$. Hence $v \in U + W$. Lastly, observe that any vector in both U and W must be equal to $(0, 0, 0)$ since the first and last entry must be 0 to be in space W and U respectively. Consequently, the middle value must then be 0 for the vector to be in U. Thus, $U + W = W \oplus W$. $\square$