**Math 231 — Hw 1**
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1. **Floating-point** numbers are a way to represent real numbers in computers using a *finite* number of bits, expressed in scientific notation as a combination of a sign, a significand, and an exponent. This format allows representation of a wide range of values but introduces rounding errors and precision limits, leading to potential inaccuracies in arithmetic operations. As was stated in class, this space of numbers are not associative.

   Within your python terminal, demonstrate that floating point numbers are not associative using the following values: $a = 1.0, b = 10^8, c = -10^8$.

   **Solution:** For this problem to work, you may have had to increase the exponent. Typically, 8 bits are used for the exponent of the number and one bit is used for the sign. The remaining is used for the significand in scientific notation. Due to limited precision, when we add $a + b = 1.0 + 10^8 = 10^8$. Why? Over the real numbers, the sum should be 10000001. In binary, this number requires 24 bits. If your system does not have enough memory allocated for this, then it would simply store it as $1.0 \times 10^8$. Once $c$ is added, the answer becomes 0.0.

   If we switch the order, $a + (b + c) = 1.0 + 0.0 = 1.0$. Since $1.0 \neq 0.0$, floating point numbers are not associative.

2. The example given in class used $\{1, 2, 3\}$ with $a \circ b = \max\{a, b\}$ and $a * b = \min\{a, b\}$. Demonstrate why 2 has no inverse under both operations (you can do this exhaustively).

   **Solution:** For $\circ$ the identity is 1. Observe that

   (a) $2 \circ 1 = 2$
   (b) $2 \circ 2 = 2$
   (c) $2 \circ 3 = 3$

   None of these values is the identity.

   For $*$ the identity is 3. Observe that

   (a) $2 \circ 1 = 1$
   (b) $2 \circ 2 = 2$
   (c) $2 \circ 3 = 2$

   None of these values is the identity.

   Hence, 2 has no inverse under either operation.

3. Consider the space with $\{0,1,2\}$ with operation $a \circ b = (a+b) \mod 3$. What axioms are satisfied? (You can skip the distributive property since there is only one operation.)

First, let's verify that the set is closed under the operation. We do this usually with an operation table. This shows us all possible combinations and the results should all be the set. If there exists a result that is not in the set, then the set is not closed under the operation.

| $\circ$ | 0 | 1 | 2 |
|---|---|---|---|
| 0 | 0 | 1 | 2 |
| 1 | 1 | 2 | 0 |
| 2 | 2 | 0 | 1 |

So the set is closed under the operation.

The axioms satsified by this operation are associativity, commutativity, identity, and inverses. Using the table above, the easiest to demonstrate is the identity. We clearly see 0 is the identity of this operation. Using the table again, we can identiy the inverses of each element:

(a) the inverse of 0 is 0,

(b) the inverse of 1 is 2, and

(c) the inverse of 2 is 1.

Although we didn't need to prove these, let's prove the last two (just to show what it would look like).

**Commutativity:** First, let's observe that for any choice of $a$ and $b$ in our set, we know that $a$ and $b$ are also in $\mathbb{Z}$. If $a, b \in \mathbb{Z}$, $a+b = 3k+r$ where $r \in \{0,1,2\}$, so $a+b \equiv r \mod 3$. Because addition is commutative over $\mathbb{Z}$, the same is true for $b+a$. Hence $a \circ b = b \circ a$.

**Associativity:** Now let's show associativity. Notice that as elements of $\mathbb{Z}$,

$$a+b+c = 3x+r$$

for some $x \in \mathbb{Z}$ and $r \in \{0,1,2\}$. Associativity holds for $\mathbb{Z}$, meaning

$$(a+b)+c = a+(b+c).$$

This also means that $(a+b)+c \mod 3 = a+(b+c) \mod 3 = r$. To show that our operation is associative, we need to show that

$$(a+b)+c \mod 3 = (a+b \mod 3)+c \mod 3$$

because the portion to the right of the equals sign corresponds to $(a \circ b) \circ c)$–our target operation. Since $a, b, c \in \{0,1,2\}$, $a+b = 3y+s$ where $y \leq x$. And it must be true that $3y+s+c = 3x+r$. Modulo 3, this equation shows that $s+c \equiv r$ and since $s = a+b \mod 3$, we have that

$$(a+b)+c \mod 3 = (a+b \mod 3)+c \mod 3.$$

The argument to show

$$a + (b + c) \mod 3 = a + (b + c \mod 3) \mod 3$$

is exactly the same and therefore omitted. Thus, we have that

$$(a \circ b) \circ c = (a + b) + c \mod 3 = a + (b + c) \mod 3 = a \circ (b \circ c).$$