16.4 Homework 21

1.) eigen values

$$A - \lambda I = \begin{pmatrix} 4-\lambda & 1 \\ 2 & 3-\lambda \end{pmatrix}$$

$$= (4-\lambda)(3-\lambda) - (1 \times 2)$$

$$= \lambda^2 - 7\lambda + 12 - 2$$

$$\lambda - 7\lambda + 10$$

$$\lambda_1 = 5 \quad \lambda_2 = 2$$

$\lambda_1 = 5$

$$A - 5I = \begin{pmatrix} 4-5 & 1 \\ 2 & 3-5 \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ 2 & -2 \end{pmatrix}$$

$$\begin{pmatrix} -1 & 1 \\ 2 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$v_1 = (1, 1)$$

$\lambda_2 = 2$

$$A - 2I = \begin{pmatrix} 4-2 & 1 \\ 2 & 3-2 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 2 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$v_2 = (1, -2)$$

eigen vectors = $(1, 1)$ & $(1, -2)$

2.) $B = \begin{pmatrix} 3 & 1 \\ 0 & 2 \end{pmatrix}$

$B - \lambda I = \begin{pmatrix} 3-\lambda & 1 \\ 0 & 2-\lambda \end{pmatrix}$

$= (3-\lambda)(2-\lambda) - (1 \cdot 0)$

$(3-\lambda)(2-\lambda) = 0$

$\lambda_1 = 3 \quad \lambda_2 = 2$

$B - 3I = \begin{pmatrix} 3-3 & 1 \\ 0 & 2-3 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix}$

$\begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$v_1 = (1, 0)$

$B - 2I = \begin{pmatrix} 3-2 & 1 \\ 0 & 2-2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$

$\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$v_2 = (1, -1)$

b) $D = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}$

c.) $P = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}$

d.) $P = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix} = P^{-1} = 1/\det(P) = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}$

$$DP^{-1} = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 3 & 3 \\ 0 & -2 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 3 & 3 \\ 0 & -2 \end{pmatrix} = \begin{pmatrix} 3 & 1 \\ 0 & 2 \end{pmatrix}$$

3.) $B = \begin{pmatrix} 3 & 1 \\ 0 & 2 \end{pmatrix}$

$$B^2 = \begin{pmatrix} 9 & 5 \\ 0 & 4 \end{pmatrix}$$

$$D = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}$$

$$D^2 = \begin{pmatrix} 9 & 0 \\ 0 & 4 \end{pmatrix}$$

$$D^2P^{-1} = \begin{pmatrix} 9 & 0 \\ 0 & 4 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 9 & 9 \\ 0 & -4 \end{pmatrix}$$

$$P(D^2P^{-1}) = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 9 & 9 \\ 0 & -4 \end{pmatrix} = \begin{pmatrix} 9 & 5 \\ 0 & 4 \end{pmatrix}$$

4.) By induction

$B = PDP^{-1}$

Base case $n = 1$

Assume holds for $n = k$

$$B^k = PD^kP^{-1}$$

for $n = k+1$

$$B^{k+1} = B^k \cdot B$$

$$B^{k+1} = (PD^kP^{-1})(PDP^{-1})$$

$$PD^k(P^{-1}P)DP^{-1} = PD^kIDP^{-1} = PD^kDP^{-1} = PD^{k+1}P^{-1}$$

hence holds for $k+1$

5.)

$$P = \begin{pmatrix} 0.99 & 0.05 \\ 0.01 & 0.95 \end{pmatrix}$$

$$x_0 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$P - \lambda I = \begin{pmatrix} 0.99-\lambda & 0.05 \\ 0.01 & 0.95-\lambda \end{pmatrix}$$

$$\det(P-\lambda I) = (0.99-\lambda)(0.95-\lambda) - (0.05)(0.01)$$

$$\lambda^2 - (0.99+0.95)\lambda + (0.99-0.95-0.0005)$$

$$\lambda^2 - 1.94\lambda + 0.94$$

$$\lambda_1 = 1 \quad \lambda_2 = 0.94$$

Eigen vectors

$\lambda = 1$

$$P - I = \begin{pmatrix} 0.99-1 & 0.05 \\ 0.01 & 0.95-1 \end{pmatrix} = \begin{pmatrix} -0.01 & 0.05 \\ 0.01 & -0.05 \end{pmatrix}$$

$$\begin{pmatrix} -0.01 & 0.05 \\ 0.01 & -0.05 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$v_1 = (5, 1)$$

$\lambda_2 = 0.94$

$$P - 0.94I = \begin{pmatrix} 0.99-0.94 & 0.05 \\ 0.01 & 0.95-0.94 \end{pmatrix} = \begin{pmatrix} 0.05 & 0.05 \\ 0.01 & 0.01 \end{pmatrix}$$

$$\begin{pmatrix} 0.05 & 0.05 \\ 0.01 & 0.01 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$v_2 = (1, -1)$$

$$D = \begin{pmatrix} 1 & 0 \\ 0 & 0.94 \end{pmatrix} \quad P = \begin{pmatrix} 5 & 1 \\ 1 & -1 \end{pmatrix}$$

$$\det P = (5)(-1) - (1 \times 1) = -6$$

$$P^{-1} = \frac{1}{-6}\begin{pmatrix} -1 & -1 \\ -1 & 5 \end{pmatrix} = \begin{pmatrix} 1/6 & 1/6 \\ 1/6 & -5/6 \end{pmatrix}$$

$$P^{30} = PD^{30}P^{-1}$$

$$D^{30} = \begin{pmatrix} 1^{30} & 0 \\ 0 & 0.94^{30} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0.94^{30} \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 0.1599 \end{pmatrix}\begin{pmatrix} 1/6 & 1/6 \\ 1/6 & -5/6 \end{pmatrix} = \begin{pmatrix} 1/6 & 1/6 \\ \frac{0.1599}{6} & -\frac{5 \times 0.1599}{6} \end{pmatrix}$$

$$P(D^{30}P^{-1}) = \begin{pmatrix} 5 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 0.1667 & 0.1667 \\ 0.02665 & -0.13325 \end{pmatrix}$$

$$= \begin{pmatrix} 0.86015 & 0.70025 \\ 0.14005 & 0.2995 \end{pmatrix}$$