Math 231 — Hw 4
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1. Show that $V = \mathbb{Z}_5^2$ is a vector space (with addition being modulo 5 and scalar multiplication also being modulo 5).

2. Let $P^3 = \{ax^3 + bx^2 + cx + d \mid a, b, c, d \in \mathbb{R}\}$ be the space of polynomials up to degree 3 over the field $\mathbb{R}$. Prove that P^3 is a vector space. (In other words, show that vector addition and scalar multiplication is closed. Then, do your best to show the other properties hold: associativity, additive identity, additive inverse, multiplicative identity, and distributivity.)

   To get you started, here is the proof of additive identity and the *start* of the proof for additive inverses:

   P^3 has an additive identity.

   Proof. Consider $0x^3 + 0x^2 + 0x + 0$, which we will write as 0_P and call the "zero polynomial." Observe that $0_P \in P^3$ since $0 \in \mathbb{R}$. In addition, for any choice of $ax^3 + bx^2 + cx + d \in P^3$,

$$\begin{aligned}\left(ax^3 + bx^2 + cx + d\right) + 0_P &= (a+0)x^3 + (b+0)x^2 + (c+0)x + (d+0) \quad \text{(by vector addition)}\\ &= ax^3 + bx^2 + cx + d \quad \text{(by additive identity of the field)}\end{aligned}$$

   Hence 0_P is the additive identity of P^3. □

   P^3 is closed under additive inverses.

   Proof. Let $ax^3 + bx^2 + cx + d \in P^3$ be an arbitrary element in the space. Because $\mathbb{R}$ is a field, there exists $-a, -b, -c \in \mathbb{R}$ such that $a + (-a) = 0$, $b + (-b) = 0$, and $c + (-c) = 0$. $\cdots$ finish the argument □