Math 231 — Hw 13

Sara Jamshidi, Feb 26, 2025

1. Suppose p_0, p_1, p_2, p_3 is a basis of the space of polynomials of degree 3. Construct a basis where none of the polynomials are degree 2.

   Recall: the degree of a polynomial is the term with the maximum degree, for example this polynomial is degree 3: $x^3 + x^2 + x + 1$.

   We can choose the following polynomials as a basis where none of the polynomials are of degree 2:

   - $p_0 = 1$ (degree 0)
   - $p_1 = x$ (degree 1)
   - $p_2 = x^3$ (degree 3)
   - $p_3 = x^3 + x^2$ (degree 3, not 2)

   We know this will be a basis because the function $x^2 = p_3 - p_2$.

2. Suppose $\{v_1.v_2, v_3, v_4\}$ is basis of V. Prove that

   $$\{v_1 + v_2,\ v_2 + v_3,\ v_3 + v_4,\ v_4\}$$

   is also a basis of V.

   Proof. To prove that $\{v_1 + v_2,\ v_2 + v_3,\ v_3 + v_4,\ v_4\}$ is a basis, we need to show that these vectors are linearly independent and span V. Assume $a(v_1 + v_2) + b(v_2 + v_3) + c(v_3 + v_4) + d(v_4) = 0$. This gives us:

   $$av_1 + (a + b)v_2 + (b + c)v_3 + (c + d)v_4 = 0$$

   Since $\{v_1, v_2, v_3, v_4\}$ is a basis, the coefficients must all be zero:

   $$a = 0, \quad a + b = 0, \quad b + c = 0, \quad c + d = 0$$

   Solving this system, we get $a = b = c = d = 0$. Thus, the vectors are linearly independent. Since there are four linearly independent vectors in a four-dimensional space, they span V. □

3. Suppose $v_1.v_2, v_3, v_4$ is basis of V. Prove that

   $$\{v_1,\ v_1 + v_2,\ v_1 + v_2 + v_3,\ v_1 + v_2 + v_3 + v_4\}$$

   is also a basis of V.

Proof. To prove that $\{v_1,\ v_1+v_2,\ v_1+v_2+v_3,\ v_1+v_2+v_3+v_4\}$ is a basis, we need to show that these vectors are linearly independent and span V. Assume $a(v_1)+b(v_1+v_2)+c(v_1+v_2+v_3)+d(v_1+v_2+v_3+v_4)=0$. This gives us:

$$(a+b+c+d)v_1+(b+c+d)v_2+(c+d)v_3+dv_4=0$$

Since $\{v_1,v_2,v_3,v_4\}$ is a basis, the coefficients must all be zero:

$$a+b+c+d=0,\quad b+c+d=0,\quad c+d=0,\quad d=0$$

Solving this system, we get $a=b=c=d=0$. Thus, the vectors are linearly independent. Since there are four linearly independent vectors in a four-dimensional space, they span V. $\square$