Math 231 — Hw 12
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1. Consider the vector space $\mathbb{F}_2^2$. What are all the possible bases for this space? (You do not need to prove this).

   - $\{(1,0),(0,1)\}$
   - $\{(1,1),(0,1)\}$
   - $\{(1,0),(1,1)\}$

2. Consider a linearly independent set in $\mathbb{R}^3$ and construct a basis by adding one element to it. Then prove that it is a basis. This question was clearly a typo. My bad. Here's just a random set of vectors in the space and a proof that they are a basis.

   Proof. Assume $a(1,0,1)+b(-1,0,1)+c(0,1,0)=(0,0,0)$. This gives us the system of equations:
   $$\begin{cases} a-b=0 \\ c=0 \\ a+b=0 \end{cases}$$
   Solving this system, we get $a=0$, $b=0$, and $c=0$. Thus, the vectors are linearly independent. Any vector $(x,y,z)\in\mathbb{R}^3$ can be written as a linear combination of these vectors:
   $$(x,y,z)=\frac{x+z}{2}(1,0,1)+\frac{(z-x)}{2}(-1,0,1)+y(0,1,0)$$
   $\square$

3. Consider the linearly independent set in $\mathbb{R}^3$ and construct a basis by adding one element to it. Then prove that it is a basis.
   $$\{(1,-1,1),(0,1,1)\}$$

   Proof. Assume $a(1,-1,1)+b(0,1,1)+c(1,0,0)=(0,0,0)$. This gives us the system of equations:
   $$\begin{cases} a+c=0 \\ -a+b=0 \\ a+b=0 \end{cases}$$

Solving this system, we get $a = 0$, $b = 0$, and $c = 0$. Thus, the vectors are linearly independent. Any vector $(x, y, z) \in \mathbb{R}^3$ can be written as a linear combination of these vectors:

$$(x, y, z) = \left(\frac{z-y}{2}\right)(1, -1, 1) + \left(\frac{y+z}{2}\right)(0, 1, 1) + \left(x + \frac{y-z}{2}\right)(1, 0, 0)$$

$\square$

4. In the previous two examples, you wrote two distinct bases of $\mathbb{R}^3$. Given a vector $(x, y, z) \in \mathbb{R}^3$, write a set of functions $f_i(a_1, a_2, a_3) = b_i$ where each function takes in the coefficients of the first basis and produces the i^{th} coefficient of the second basis. What kind of functions are these?

   In the first basis, (x, y, z) is computed using the coefficients

   $$a_1 = \frac{x+z}{2}, a_2 = \frac{(z-x)}{2}, a_3 = y.$$

   The second basis has coefficients

   $$b_1 = \left(\frac{z-y}{2}\right), b_2 = \left(\frac{y+z}{2}\right), b_3 = \left(x + \frac{y-z}{2}\right)$$

   This question asks us to write each of the coefficients of the second basis in terms of the first set of coefficients. Here are the solutions:

   - $b_1 = \frac{1}{2}(a_1 + a_2 - a_3)$
   - $b_2 = \frac{1}{2}(a_1 + a_2 + a_3)$
   - $b_3 = \frac{1}{2}(a_1 - 3a_2 + a_3)$

5. Prove or give a counterexample: If v_1, v_2, v_3, v_4 is a basis of V and U is a subspace of V such that $v_1, v_2 \in U$ and $v_3 \notin U$ and $v_4 \notin U$, then v_1, v_2 is a basis of U.

   Consider the vector space $V = \mathbb{R}^4$ with the basis:

   $$v_1 = (0, 0, 1, -1), \quad v_2 = (0, 0, 1, 1), \quad v_3 = (0, 1, 0, 0), \quad v_4 = (1, 0, 0, 0)$$

   Let U be the subspace of V defined as:

   $$U = \{(0, x, y, z) \mid x + y + z = 0\}$$

   Clearly, $v_1, v_2 \in U$ and $v_3, v_4 \notin U$. However, v_1 and v_2 do not form a basis of U. Consider the element $(0, 1, -1, 0)$. This cannot be constructed from v_1, v_2.