Math 231 — Hw 17
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The null space is the space of all vectors that are sent to 0 by a matrix. For example, the null space of

$$\begin{pmatrix} 2 & 4 \\ -1 & -2 \end{pmatrix}$$

is the set of vectors of the form

$$\begin{pmatrix} 2x \\ -x \end{pmatrix}.$$

To demonstrate this, we see that

$$\begin{pmatrix} 2 & 4 \\ -1 & -2 \end{pmatrix} \begin{pmatrix} 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

1. Consider the following matrices. What is their null space? Based on their null space, do their column vectors form a basis?

   (a)

   $$\begin{pmatrix} 3 & 3 \\ -1 & -1 \end{pmatrix}$$

   The null space of this matrix is the set of vectors $\mathbf{v}$ such that:

   $$\begin{pmatrix} 3 & 3 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

   This simplifies to $3v_1 + 3v_2 = 0$, which implies $v_1 = -v_2$. Therefore, the null space is:

   $$\text{null space} = \text{span}\left\{ \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\}.$$

   Since the null space is nontrivial, the column vectors do not form a basis.

   (b)

   $$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 2 \end{pmatrix}$$

   The null space of this matrix is the set of vectors $\mathbf{v}$ such that:

   $$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

   This simplifies to the system:

   $$\begin{cases} v_1 + v_3 = 0 \\ v_2 + v_3 = 0 \\ v_1 + v_2 + 2v_3 = 0 \end{cases}$$

Solving this system, we find $v_1 = v_2 = -v_3$. Therefore, the null space is:

$$\text{null space} = \text{span}\left\{\begin{pmatrix}1\\1\\-1\end{pmatrix}\right\}.$$

Since the null space is nontrivial, the column vectors do not form a basis.

2. If a null space has more than just the 0 vector, we call it "nontrivial." Give the basis of the nontrivial null space of the following matrix

$$\begin{pmatrix}1 & -1 & 2\\3 & -3 & 6\end{pmatrix}$$

The null space of this matrix is the set of vectors $\mathbf{v}$ such that:

$$\begin{pmatrix}1 & -1 & 2\\3 & -3 & 6\end{pmatrix}\begin{pmatrix}v_1\\v_2\\v_3\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}.$$

This simplifies to the system:

$$\begin{cases}v_1 - v_2 + 2v_3 = 0\\3v_1 - 3v_2 + 6v_3 = 0\end{cases}$$

The second equation is a multiple of the first, so we only need to solve $v_1 - v_2 + 2v_3 = 0$. Let $v_2 = t$ and $v_3 = s$, then $v_1 = t - 2s$. Therefore, the null space is:

$$\text{null space} = \text{span}\left\{\begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}-2\\0\\1\end{pmatrix}\right\}.$$

3. Suppose M is a 3×3 matrix. We said in class that M can be thought of as changing the basis of the matrix. For this reason, the columns of M represent a basis. If the null space is nontrivial, then the vectors don't form a basis of the 3 dimensional vector space. What does that mean about dimensionality of the range of M?

   If the null space of M is nontrivial, it means that there exists at least one non-zero vector $\mathbf{v}$ such that $M\mathbf{v} = \mathbf{0}$. This implies that the columns of M are linearly dependent and do not span the entire 3-dimensional space. Therefore, the range (or column space) of M has a dimension less than 3.

4. Based on your answer to the previous question, what does it mean geometrically if a matrix has a nontrivial null space?

   Geometrically, if a matrix has a nontrivial null space, it means that the transformation represented by the matrix "collapses" or "squashes" the space into a lower-dimensional subspace. For example, in 3-dimensional space, the range of the matrix would be a plane or a line, rather than filling the entire space.

5. To capture rotations in two dimensions, we can use the following matrix:

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

Suppose a camera is pointed downward looking at a specimen located at $(2, 3)$. If the camera is rotated by 240 degrees in the positive direction, what is the vector that represents its location with this new orientation?

A rotation by 240 degrees corresponds to $\theta = 240°$. The rotation matrix is:

$$\begin{pmatrix} \cos 240° & -\sin 240° \\ \sin 240° & \cos 240° \end{pmatrix} = \begin{pmatrix} -1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{pmatrix}.$$

Applying this rotation to the vector $(2, 3)$, we get:

$$\begin{pmatrix} -1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{pmatrix} \begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} -1/2 \cdot 2 - \sqrt{3}/2 \cdot 3 \\ \sqrt{3}/2 \cdot 2 - 1/2 \cdot 3 \end{pmatrix} = \begin{pmatrix} -1 - 3\sqrt{3}/2 \\ \sqrt{3} - 3/2 \end{pmatrix}.$$

Thus, the new location vector is:

$$\begin{pmatrix} -1 - 3\sqrt{3}/2 \\ \sqrt{3} - 3/2 \end{pmatrix}.$$