Math 231 — Hw 21
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1. Find the eigenvalues and corresponding eigenvectors of the matrix:

$$A = \begin{pmatrix} 4 & 1 \\ 2 & 3 \end{pmatrix}$$

2. Diagonal matrices are matrices where the only nonzero values are on the diagonal. We an "diagonalize" a matrix using its eigenvalues. Consider the following matrix and follow these steps.

$$B = \begin{pmatrix} 3 & 1 \\ 0 & 2 \end{pmatrix}$$

   (a) Find the eigenvalues and corresponding eigenvectors of the matrix.

   (b) Define a matrix D whose diagonal entries are λ_1 and λ_2

   (c) Define a matrix P whose columns are the eigenvectors you found, the first column corresponding λ_1 and the second corresponding to λ_2.

   (d) Compute P^{-1}.

   (e) Finally, show that $PDP^{-1} = B$.

3. In the previous example, suppose you needed to compute BB, which is also denoted as B^2. You realize the diagonalization of B actually makes this process easier because

$$B^2 = \left(PDP^{-1}\right)\left(PDP^{-1}\right) = PDIDP^{-1} = PD^2P^{-1}$$

   The first equality holds by substitution, the second holds by the definition of inverses $P^{-1}P = I$, and the third holds by definition of the identity map I. This is desirable because powers of diagonal matrices are easier to compute.

   Compute B^2 explicitly and then compute D^2. Show that $B^2 = PD^2P^{-1}$.

4. Prove that for linear maps $B, D, P \in \mathcal{L}(V)$, if $B = PDP^{-1}$, then $B^n = PD^nP^{-1}$ for any finite n using induction.

5. A central application of linear maps is modeling probabilistic systems that change over time. Suppose you work for a real estate firm and are tasked with modeling the percentage of houses for sale within a specific region. Based on historical data, you have determined the following probabilities:

   - If a house is currently listed for sale, there is a 0.05 probability that it will come off the market on any given day (due to a purchase or other circumstances).

- Conversely, if a house is listed for sale, there is a 0.95 probability that it will remain for sale the next day.
- If a house is not currently listed for sale, there is a 0.01 probability that it will be listed for sale the next day.
- If a house is not currently listed for sale, there is a 0.99 probability that it will remain off the market the next day.

Using this information, you can construct a transition matrix to model the probabilistic state of the housing market over time. Let's define the states as follows:

- State 1: The house is listed for sale.
- State 2: The house is not listed for sale.

The transition matrix P can be represented as:

$$P = \begin{pmatrix} .99 & 0.05 \\ 0.01 & .95 \end{pmatrix}$$

Where each entry p_{ij} represents the probability of transitioning from state j to state i. The first column represents houses NOT on the market and the second columns lists houses on the market. Suppose a house today IS on the market. What is the probability it is not on the market in 30 days?

(Hint: This is an application of what is discussed in the previous problems.)