Math 231 — Hw 15
Sara Jamshidi, Mar 19, 2025

In class, we discussed how we to use matrix representations of linear maps between vector spaces. We did two examples in class. Below is the complete version of the second one with a small change to make the math easier.

Suppose we have two vector spaces, $V = \mathbb{Z}_7^3$ and $W = \mathbb{R}^2$, with the bases

$$B_V = \{(1,1,1),(1,6,1),(0,1,1)\}$$

and $B_W = \{(1,0),(0,1)\}$. I will use v_i and w_i to denote these elements. And we define the following linear map T between them:

- $Tv_1 = 3w_1$
- $Tv_2 = w_1 + 2w_2$
- $Tv_3 = -w_2$

The matrix representation of T between these two spaces with those bases is

$$M(T) = \begin{pmatrix} 3 & 1 & 0 \\ 0 & 2 & \text{-1} \end{pmatrix}$$

The basis used for $\mathbb{Z}_7^3$ in defining T is not ideal, so we'd like to construct a map $U : \mathbb{Z}_7^3 \to \mathbb{Z}_7^3$ where we change the basis from the standard basis, $\{(1,0,0),(0,1,0),(0,0,1)\}$ to B_V. Then we can construct a map by composing $T \circ U$ to construct a map from $\mathbb{Z}_7^3$ to $\mathbb{R}^3$.
So we need to construct $(1,0,0)$ as a linear combination of the elements in B_V. In other words, we need to find elements x, y, z such that $x + y = 1$, $x + 6y + z = 0$, $x + y + z = 0$. From the first and third equation, we can deduce that $z = 6$. And then see that $x = 1$ and $y = 0$ works for our problem. Hence

$$U((1,0,0)) = v_1 + 6v_3.$$

By similar logic, we can get the remaining two:

- $U((1,0,0)) = v_1 + 6v_3.$
- $U((0,1,0)) = 4v_1 + 3v_2.$
- $U((0,0,1)) = v_3.$

The matrix representation of U is

$$M(U) = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Suppose we have the element $v = (2, 3, 1)$ in $\mathbb{Z}_7^3$. Where does T map this element to? We can use the matrix representations and compute $T \circ U$:

$$\begin{aligned} TUv &= \begin{pmatrix} 3 & 1 & 0 \\ 0 & 2 & \text{-1} \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} 3 & 1 & 0 \\ 0 & 2 & \text{-1} \end{pmatrix} \begin{pmatrix} 5 \\ 2 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} 17 \\ 3 \end{pmatrix} \end{aligned}$$

A tricky element here is that U is happening modulo 7 because of the spaces it is going between. Now it is your turn to try.

1. Consider the vector spaces $V = \mathbb{Z}_5^2$ and $W = \mathbb{R}^3$ with bases

$$B_V = \{(1, 2), (0, 1)\}$$

and $B_W = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$.Define the linear map $T : V \to W$ such that

- $Tv_1 = 2w_1 + w_3$
- $Tv_2 = w_2 - w_3$

Let $v = (3, 4)$ be a vector in V. Compute $Tv \in W$ using the matrix representation method.

1. The matrix representation of T with respect to the bases B_V and B_W is:

$$M(T) = \begin{pmatrix} 2 & 0 \\ 0 & 1 \\ 1 & -1 \end{pmatrix}$$

2. Express $v = (3, 4)$ as a linear combination of the basis vectors in B_V. Let $v = a(1, 2) + b(0, 1)$. This gives us the system of equations:

$$\begin{cases} a = 3 \\ 2a + b = 4 \end{cases}$$

Solving this system, we get $a = 3$ and $b = -2$. Thus, $v = 3v_1 - 2v_2$.

3. Compute Tv using the matrix representation of T:

$$Tv = M(T) \begin{pmatrix} 3 \\ -2 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 3 \\ -2 \end{pmatrix} = \begin{pmatrix} 6 \\ -2 \\ 5 \end{pmatrix}$$

So our answer is $(6, -2, 5)$.