MacQuarries

Ⓐ Chapter II

complex #

$Z = x + iy$

real part imaginary part.

$Z^* = x - iy$

$ZZ^* = (x+iy)(x-iy)$

$x^2 - xiy + xiy - (-y^2)$

$\underbrace{x^2+y^2}$ } Real Number.

$|z|^2$

Euler's formula - helps relate exponential complex number to trigonometric complex numbers.

$$e^{ix} = \cos x + i\sin x$$

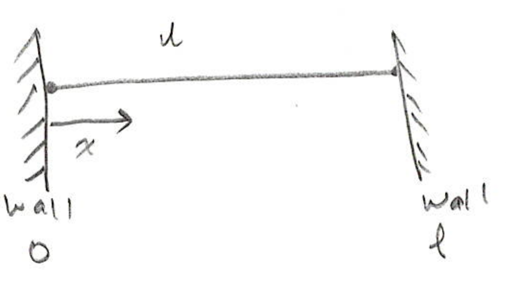
Ⓑ Classical wave Equation

- solving wave equations in classical sense.

* Vibrating string

u = how far string moves up and down

u(x,t) - string vibrating as a function of time



One possible solution

To describe this the string has to satisfy partial differential of this equation

$$\frac{\partial^2 u(x,t)}{\partial x} = \frac{1}{v^2}\frac{\partial^2 u(x,t)}{\partial t^2} \Big\}\text{ classical wave Equation.}$$

* ~~Boundary Conditions~~ Limits the number of solutions to our system.

↳ string is fixed

$u(0,t)=0$ ⟶ no displacement
when $x=0$

$u(l,t)=0$ ⟶ no displacement

} due to attachment to the wall.

Ⓒ $x=0$ $x=l$

Separation of Variables

assume we can "separate" into two part.

$u(x,t)=X(x)T(t)$ depends on each variable independently.

Assume that function depends on each variable — separate.

$$\frac{\partial^2 u(x,t)}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 u(x,t)}{\partial t^2}$$

$$\frac{\partial^2 X(x)T(t)}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 X(x)T(t)}{\partial t^2}$$

$$\underbrace{T(t)}_{\text{constant}}\frac{\partial^2 X(x)}{\partial x^2} = \frac{1}{v^2}\underbrace{X(x)}_{\text{constant}}\frac{\partial^2 T(t)}{\partial t^2}$$

Rearrange to have X's on left

$$\frac{1}{X(x)}\frac{d^2X(x)}{dx^2} = \frac{1}{V^2}\frac{1}{T(t)}\frac{d^2T(t)}{dt^2}$$

Remember x and t are independ so:

$$\frac{1}{X(x)}\frac{d^2X(x)}{dx^2} = \underline{\underline{k}} \quad \text{constant.} \qquad \frac{1}{V^2}\frac{1}{T(t)}\frac{d^2T(t)}{dt^2} = k$$

$$\frac{d^2X(x)}{dx^2} - k\,X(x) = 0$$

$$\frac{d^2T(t)}{dt^2} - kV^2T(t) = 0$$

Assuming $k = 0$.

$$\frac{d^2X(x)}{dx^2} = 0$$

$$\frac{d^2T(t)}{dt^2} = 0$$

A function that gives us a zero

$X(x) = a_1 x + b_1$ a and b are constants that disappear when we take the 2nd derivative.

$T(t) = a_2 t + b_2$

$$\left.\begin{aligned} \frac{d^2X(x)}{dx^2} &= \frac{d}{dx}(a_1x + b_1) = a_1 \\ \frac{d}{dx}(a_1) &= 0 \end{aligned}\right\} \text{proof}$$

To find the value of Constants lets use our boundary Conditions:

$u(0,t)=0 \quad X(0)T(0)=0 \quad u(\ell,t)=0 \quad X(\ell)T(t)=0.$

$T(t)\neq 0$ because t Increases with time.

$X(0)=0$, $X(\ell)=0$ $\Big\}$ for this to be true, the $X(x)$ should be equal to zero. This only useful when the string is not moving. This is called trivial solution.

(B) Non-Trivial Solutions; lets pick different value for K.

$K=-\beta^2 \rightarrow$ Assumption

$$\frac{d^2X(x)}{dx^2}+\beta^2X(x)=0 \longrightarrow \text{known solution is } X(x) = \text{Exponential } \underset{\text{alpha } x}{} = e^{\alpha x}$$

$$\frac{d^2 e^{\alpha x}}{dx^2}=\frac{d}{dx}\alpha e^{\alpha x}=\alpha^2 e^{\alpha x}$$

(So) substituting $X(x)$

$$\alpha^2 e^{\alpha x}+\beta^2 e^{\alpha x}=0 \rightarrow (\alpha^2+\beta^2)e^{\alpha x}=0 \text{ — Trivial}$$

- Not useful.

$$(\alpha^2+\beta^2)=0 \rightarrow \alpha^2=-\beta^2 \text{ or } \alpha=\sqrt{-\beta^2}$$

$$\underbrace{X(x)=C_1e^{i\beta x}+C_2e^{-i\beta x}} \qquad \underline{\alpha=\pm i\beta}$$

- This is what is known as linear Combination
- But we have Complex exponentials; lets take care of them:

Euler's formula.

$$e^{i\beta x} = \cos\beta x + i\sin\beta x$$

$$X(x) = C_1(\cos\beta x + i\sin\beta x) + C_2(\cos\beta x - i\sin\beta x)$$

combining terms

$$= \underbrace{(C_1 + C_2)}_{A}\cos\beta x + \underbrace{(iC_1 - iC_2)}_{B}\sin\beta x$$

$$= \underbrace{A\cos(\beta x) + B\sin(\beta x)}_{\text{Generic solution for X part.}} \longrightarrow \text{solution for the classical wave equation.}$$

So what is the solution for Our Specific Part with Boundary Condition

$X(0) = 0$ $\quad \cos(0) = 1 \longleftarrow$ Can never be zero.

$x(l) = 0$ $\quad \sin(0) = 0$

$$= \underbrace{A\cos(\beta x)} + B(\sin Bx)$$

$\longrightarrow$ since this part cannot be zero; we set A to zero to drive it to zero.

$\sin\frac{\pi}{2} = 1$

$A = 0$ and $B \neq 0$

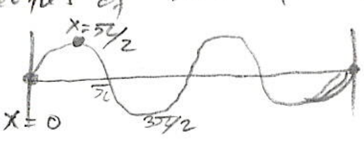
$\frac{\pi}{2} = 90^\circ$

$\sin \pi = 0$

$X(x) = B\sin(\beta x)$ $\quad \therefore \sin(\beta x) = 0$ when $(x = l)$

$\sin(\beta L) = 0$

Properties of sine function



The only time ~~B~~ $\sin x = 0$ is the Integer value of π

$$\sin(n\pi) = 0$$
$n = 0, 1, 2$
(integers)

$$\beta l = n\pi$$

Now lets solve for β

$$\beta = \frac{n\pi}{L}$$

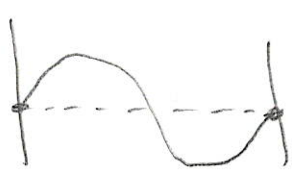
So our final solution:

$$X(x) = B \sin \frac{n\pi}{l} x$$

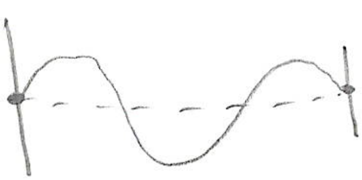
solution.

$n=1$

$n=2$



$n=3$



Similar to what we saw with Quantised n-values

Solving for $T(t)$

$$\frac{d^2 T(t)}{dt^2} + \beta^2 v^2 T(t) = 0$$

$$T(t) = D\cos \beta v t + E \sin \beta v t.$$

$$\beta = \frac{n\pi}{l}$$

let call $\beta_v = \frac{n\pi v}{l} = \omega_n$

$$\therefore \quad T(t) = D\cos \omega_n t + E \sin \omega t$$

The t part has no boundary condition.

$$u(x,t) = X(x)\,T(t) = \left(B \sin \frac{n\pi x}{l}\right)\left(D\cos \omega_n t + E \sin \omega_n t\right)$$

Combining Constants.

$$= (\underbrace{F}_{BD}\cos \omega_n t + \underbrace{G}_{BE}\sin \omega_n t)\sin \frac{n\pi x}{l}$$

$n = 1, 2, 3, 4$

Most General soln

$$u(x,t) = \sum_{n=1}^{\infty} \left(F\cos \omega_n t + G_n \sin \omega_n t\right) \sin \frac{n\pi x}{l}$$

Each value of n is what is known as normal mode of the string; the normal frequency.

Most conviniènt way is:

$$u(x,t) = \underbrace{\sum_{n=1}^{\infty} A_n \cos(\omega_n t + \phi_n) \sin \frac{n\pi x}{l}}_{\text{convenient way.}}$$

phase shift.

A_n = amplitude of mode n.

ϕ_n = phase.

$$\omega_n = \frac{n\pi v}{l}$$