Topic: Surface area of revolution of a parametric curve, vertical axis

Question: Find the surface area of revolution of the parametric curve rotated about the given axis.

$$x = 2t$$

$$y = 5t^2$$

$$0 \le t \le 1$$

about the y-axis

Answer choices:

A $\quad \frac{8\pi}{75}\left(26^{\frac{3}{2}} + 1\right)$

B $\quad \frac{8\pi}{75}\left(26^{\frac{3}{2}} - 1\right)$

C $\quad \frac{4\pi}{75}\left(26^{\frac{3}{2}} + 1\right)$

D $\quad \frac{4\pi}{75}\left(26^{\frac{3}{2}} - 1\right)$

Solution: B

The formula for surface area of a parametric curve revolved about the y-axis on the given interval is

$$S = \int_0^1 2\pi x\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\ dt$$

We'll calculate the derivatives of x and y so that we can plug them into the formula.

$$x = 2t$$

$$\frac{dx}{dt} = 2$$

and

$$y = 5t^2$$

$$\frac{dy}{dt} = 10t$$

Plugging these into our formula, we get

$$S = \int_0^1 2\pi(2t)\sqrt{(2)^2 + (10t)^2}\ dt$$

$$S = 4\pi\int_0^1 t\sqrt{4 + 100t^2}\ dt$$

$$S = 4\pi\int_0^1 t\sqrt{4\left(1 + 25t^2\right)}\ dt$$

$$S=8\pi\int_0^1 t\sqrt{1+25t^2}\ dt$$

Using u-substitution with

$$u=1+25t^2$$

$$du=50t\ dt$$

$$dt=\frac{du}{50t}$$

we'll substitute and get

$$S=8\pi\int_{t=0}^{t=1} t\sqrt{u}\ \frac{du}{50t}$$

$$S=\frac{8\pi}{50}\int_{t=0}^{t=1}\sqrt{u}\ du$$

$$S=\frac{8\pi}{50}\int_{t=0}^{t=1}u^{\frac{1}{2}}\ du$$

$$S=\frac{8\pi}{50}\left(\frac{2}{3}u^{\frac{3}{2}}\right)\Big|_{t=0}^{t=1}$$

$$S=\frac{8\pi}{25}\left(\frac{1}{3}u^{\frac{3}{2}}\right)\Big|_{t=0}^{t=1}$$

$$S=\frac{8\pi}{75}u^{\frac{3}{2}}\Big|_{t=0}^{t=1}$$

Back-substituting so that we can evaluate over the interval, we get

$$S=\frac{8\pi}{75}\left(1+25t^2\right)^{\frac{3}{2}}\Big|_{t=0}^{t=1}$$

$$S=\frac{8\pi}{75}\sqrt{\left(1+25t^2\right)^3}\Big|_{t=0}^{t=1}$$

$$S=\frac{8\pi}{75}\sqrt{\left[1+25(1)^2\right]^3}-\frac{8\pi}{75}\sqrt{\left[1+25(0)^2\right]^3}$$

$$S=\frac{8\pi}{75}\sqrt{(26)^3}-\frac{8\pi}{75}\sqrt{(1)^3}$$

$$S=\frac{8\pi}{75}\left(26^{\frac{3}{2}}-1\right)$$

Topic: Surface area of revolution of a parametric curve, vertical axis

Question: Find the surface area of revolution of the parametric curve rotated about the given axis.

$$x = t$$

$$y = 2t^2 + 6$$

$$0 \le t \le 3$$

about the y-axis

Answer choices:

A $\frac{\pi}{6}\left(145^{\frac{3}{2}} - 1\right)$

B $\frac{\pi}{6}\left(145^{\frac{3}{2}} + 1\right)$

C $\frac{\pi}{24}\left(145^{\frac{3}{2}} - 1\right)$

D $\frac{\pi}{24}\left(145^{\frac{3}{2}} + 1\right)$

Solution: C

The formula for surface area of a parametric curve revolved about the y-axis on the given interval is

$$S=\int_0^3 2\pi x\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\ dt$$

We'll calculate the derivatives of x and y so that we can plug them into the formula.

$$x=t$$

$$\frac{dx}{dt}=1$$

and

$$y=2t^2+6$$

$$\frac{dy}{dt}=4t$$

Plugging these into our formula, we get

$$S=\int_0^3 2\pi t\sqrt{(1)^2+(4t)^2}\ dt$$

$$S=2\pi\int_0^3 t\sqrt{1+16t^2}\ dt$$

Using u-substitution with

$$u=1+16t^2$$

$$du=32t\ dt$$

$$dt=\frac{du}{32t}$$

we'll substitute and get

$$S=2\pi\int_{t=0}^{t=3}t\sqrt{u}\ \frac{du}{32t}$$

$$S=\frac{2\pi}{32}\int_{t=0}^{t=3}\sqrt{u}\ du$$

$$S=\frac{\pi}{16}\int_{t=0}^{t=3}u^{\frac{1}{2}}\ du$$

$$S=\frac{\pi}{16}\left(\frac{2}{3}u^{\frac{3}{2}}\right)\Bigg|_{t=0}^{t=3}$$

$$S=\frac{\pi}{8}\left(\frac{1}{3}u^{\frac{3}{2}}\right)\Bigg|_{t=0}^{t=3}$$

$$S=\frac{\pi}{24}\left(u^{\frac{3}{2}}\right)\Bigg|_{t=0}^{t=3}$$

Back-substituting so that we can evaluate over the interval, we get

$$S=\frac{\pi}{24}\left(1+16t^2\right)^{\frac{3}{2}}\Bigg|_0^3$$

$$S=\frac{\pi}{24}\left[1+16(3)^2\right]^{\frac{3}{2}}-\frac{\pi}{24}\left[1+16(0)^2\right]^{\frac{3}{2}}$$

$$S = \frac{\pi}{24}\left[1 + 16(9)\right]^{\frac{3}{2}} - \frac{\pi}{24}\left[1 + 16(0)\right]^{\frac{3}{2}}$$

$$S = \frac{\pi}{24}145^{\frac{3}{2}} - \frac{\pi}{24}$$

$$S = \frac{\pi}{24}\left(145^{\frac{3}{2}} - 1\right)$$

Topic: Surface area of revolution of a parametric curve, vertical axis

Question: Find the surface area of revolution of the parametric curve rotated about the given axis.

$$x = 5t^2$$

$$y = 3t^3$$

on the interval $0 \leq t \leq 4$

about the y-axis

Answer choices:

A $51{,}338.46$

B $61{,}338.46$

C $61{,}469.39$

D $81{,}338.46$

Solution: C

The formula for surface area of a parametric curve revolved about the y -axis on the given interval is

$$S=\int_{0}^{4} 2\pi x\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\ dt$$

We'll calculate the derivatives of x and y so that we can plug them into the formula.

$$x=5t^2$$

$$\frac{dx}{dt}=10t$$

and

$$y=3t^3$$

$$\frac{dy}{dt}=9t^2$$

Plugging these into our formula, we get

$$S=\int_{0}^{4} 2\pi\left(5t^2\right)\sqrt{(10t)^2+\left(9t^2\right)^2}\ dt$$

$$S=10\pi\int_{0}^{4} t^2\sqrt{t^2\left(100+81t^2\right)}\ dt$$

$$S=10\pi\int_{0}^{4} t^3\sqrt{100+81t^2}\ dt$$

Using u-substitution with

$$u=100+81t^2$$

$$du=162t\ dt$$

$$dt=\frac{du}{162t}$$

we'll substitute and get

$$S=10\pi\int_{t=0}^{t=4}t^3\sqrt{u}\ \frac{du}{162t}$$

$$S=\frac{10\pi}{162}\int_{t=0}^{t=4}t^2\sqrt{u}\ du$$

$$S=\frac{5\pi}{81}\int_{t=0}^{t=4}t^2\sqrt{u}\ du$$

Since we said earlier that $u=100+81t^2$, we can say

$$u=100+81t^2$$

$$u-100=81t^2$$

$$\frac{u-100}{81}=t^2$$

Make this substitution.

$$S=\frac{5\pi}{81}\int_{t=0}^{t=4}\frac{u-100}{81}\sqrt{u}\ du$$

$$S=\frac{5\pi}{81}\int_{t=0}^{t=4}\frac{u\sqrt{u}-100\sqrt{u}}{81}\ du$$

$$S=\frac{5\pi}{81}\int_{t=0}^{t=4}\frac{u^{\frac{3}{2}}-100u^{\frac{1}{2}}}{81}\ du$$

$$S=\frac{5\pi}{81^2}\int_{t=0}^{t=4}u^{\frac{3}{2}}-100u^{\frac{1}{2}}\ du$$

$$S=\frac{5\pi}{81^2}\left(\frac{2}{5}u^{\frac{5}{2}}-\frac{200}{3}u^{\frac{3}{2}}\right)\Bigg|_{t=0}^{t=4}$$

Back-substituting so that we can evaluate over the interval, we get

$$S=\frac{5\pi}{81^2}\left[\frac{2}{5}\left(100+81t^2\right)^{\frac{5}{2}}-\frac{200}{3}\left(100+81t^2\right)^{\frac{3}{2}}\right]\Bigg|_0^4$$

$$S=\frac{5\pi}{81^2}\left[\frac{2}{5}\left(100+81(4)^2\right)^{\frac{5}{2}}-\frac{200}{3}\left(100+81(4)^2\right)^{\frac{3}{2}}\right]-\frac{5\pi}{81^2}\left[\frac{2}{5}\left(100+81(0)^2\right)^{\frac{5}{2}}-\frac{200}{3}\left(100+81(0)^2\right)^{\frac{3}{2}}\right]$$

$$S=\frac{5\pi}{81^2}\left[\frac{2}{5}\left(100+81(16)\right)^{\frac{5}{2}}-\frac{200}{3}\left(100+81(16)\right)^{\frac{3}{2}}\right]-\frac{5\pi}{81^2}\left[\frac{2}{5}(100)^{\frac{5}{2}}-\frac{200}{3}(100)^{\frac{3}{2}}\right]$$

$$S=\frac{5\pi}{81^2}\left[\frac{2}{5}\left(100+1{,}296\right)^{\frac{5}{2}}-\frac{200}{3}\left(100+1{,}296\right)^{\frac{3}{2}}\right]-\frac{5\pi}{81^2}\left[\frac{2}{5}(100{,}000)-\frac{200}{3}(1{,}000)\right]$$

$$S=\frac{5\pi}{81^2}\left[\frac{2}{5}(1{,}396)^{\frac{5}{2}}-\frac{200}{3}(1{,}396)^{\frac{3}{2}}\right]-\frac{5\pi}{81^2}\left(\frac{200{,}000}{5}-\frac{200{,}000}{3}\right)$$

$$S=\frac{10\pi}{81^2}\left[\frac{1}{5}(1{,}396)^{\frac{5}{2}}-\frac{100}{3}(1{,}396)^{\frac{3}{2}}\right]-\frac{1{,}000{,}000\pi}{81^2}\left(\frac{1}{5}-\frac{1}{3}\right)$$

$$S=\frac{10\pi}{81^2}\left[\frac{1}{5}(1{,}396)^{\frac{5}{2}}-\frac{100}{3}(1{,}396)^{\frac{3}{2}}-100{,}000\left(\frac{1}{5}-\frac{1}{3}\right)\right]$$

$$S=\frac{10\pi}{81^2}\left[\frac{1}{5}(1{,}396)^{\frac{5}{2}}-\frac{100}{3}(1{,}396)^{\frac{3}{2}}-100{,}000\left(-\frac{2}{15}\right)\right]$$

$$S=\frac{10\pi}{81^2}\left[\frac{1}{5}(1{,}396)^{\frac{5}{2}}-\frac{100}{3}(1{,}396)^{\frac{3}{2}}+\frac{40{,}000}{3}\right]$$

$$S=\frac{10\pi}{81^2}(14{,}562{,}754.9418-1{,}738{,}628.8135+13{,}333.3333)$$

$$S=\frac{10\pi}{81^2}(12{,}837{,}459.4616)$$

$$S\approx 61{,}469.39$$