Estimating indefinite integrals

Sometimes we'll be given an indefinite integral to evaluate, and we can't easily evaluate it with our normal integration techniques, like u-substitution, integration by parts, and partial fractions. In this case, we might be able to replace the function in the integral with its power series representation, whose expanded form is just a polynomial that's much easier to integrate.

As usual, we'll try to find a well-known power series that's similar to the given function, and then manipulate the power series until it matches the function. Most often, we'll use the power series

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

Once we've modified this equation so that the left-hand side matches our function, we'll expand the power series through the first few terms. Then we'll integrate the individual terms from the expanded form to find a power series representation for the indefinite integral.

If we need to, we can go back to the power series representation that we found originally and use it to find the radius and interval of convergence.

Example

Evaluate the indefinite integral as a power series, then find the radius of convergence.

$$\int \frac{r}{1 - r^6}\ dr$$

We need to manipulate the function we've been given in this integral until it matches the power series

$$\frac{1}{1 - x} = \sum_{n=0}^{\infty} x^n$$

We'll start by factoring r out of the numerator.

$$\frac{r}{1 - r^6} = (r)\frac{1}{1 - r^6}$$

Comparing the remaining fraction to the standard form of the power series, we can see that $x = r^6$. Therefore, we'll have to substitute r^6 for x, and multiply by the r that we factored out.

$$r\sum_{n=0}^{\infty} \left(r^6\right)^n$$

$$r^1\sum_{n=0}^{\infty} r^{6n}$$

$$\sum_{n=0}^{\infty} r^{6n+1}$$

This is the power series representation of the given function. If we expand the series, we get

$$\sum_{n=0}^{\infty} r^{6n+1} = r^1 + r^7 + r^{13} + r^{19} + \ldots + r^{6n+1}$$

Now we can replace the original function with the expanded power series, and the integral becomes

$$\int \frac{r}{1-r^6}\ dr = \int r^1 + r^7 + r^{13} + r^{19} + \ldots + r^{6n+1}\ dr$$

$$\int \frac{r}{1-r^6}\ dr = \frac{r^2}{2} + \frac{r^8}{8} + \frac{r^{14}}{14} + \frac{r^{20}}{20} + \ldots + \frac{r^{6n+2}}{6n+2} + C$$

That means the indefinite integral

$$\int \frac{r}{1-r^6}\ dr$$

is equal to the sum of the series

$$\sum_{n=0}^{\infty} \frac{r^{6n+2}}{6n+2} + C$$

To find the radius of convergence, we'll use the power series representation we found earlier,

$$\sum_{n=0}^{\infty} r^{6n+1}$$

We'll identify a_n and a_{n+1} as

$$a_n = r^{6n+1}$$

$$a_{n+1} = r^{6n+7}$$

We'll plug both of these into the limit formula from the ratio test, and get

$$L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$$

$$L = \lim_{n\to\infty} \left| \frac{r^{6n+7}}{r^{6n+1}} \right|$$

$$L = \lim_{n\to\infty} \left| r^{(6n+7)-(6n+1)} \right|$$

$$L = \lim_{n\to\infty} \left| r^{6n+7-6n-1} \right|$$

$$L = \lim_{n\to\infty} \left| r^6 \right|$$

The limit only effects n, not r, and we only have r left over in the limit. Therefore, we can remove the limit.

$$L = \left| r^6 \right|$$

The ratio test tells us that that series converges when $L < 1$. Since we know $L = \left| r^6 \right|$, we'll say that the series converges when

$$\left| r^6 \right| < 1$$

$$-1 < r^6 < 1$$

$$0 < r < 1$$

This tells us that the radius of convergence is $R = 1$.