# Calculus 2 Workbook Solutions

Arc length

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MATH

## ARC LENTH OF Y=F(X)

1. Find the arc length of the curve over $[0,2]$.

$$y = \frac{4\sqrt{2}}{3}x^{\frac{3}{2}} + 6$$

*Solution:*

The derivative of the function is

$$f'(x) = \frac{3}{2} \cdot \frac{4\sqrt{2}}{3}x^{\frac{3}{2}-1}$$

$$f'(x) = 2\sqrt{2}x^{\frac{1}{2}}$$

$$f'(x) = 2\sqrt{2x}$$

Then the arc length over the interval is

$$L = \int_a^b \sqrt{1 + \left[f'(x)\right]^2}\ dx$$

$$L = \int_0^2 \sqrt{1 + \left[2\sqrt{2x}\right]^2}\ dx$$

$$L = \int_0^2 \sqrt{1 + 4(2x)}\ dx$$

$$L=\int_0^2\sqrt{1+8x}\ dx$$

Use substitution.

$$u=1+8x$$

$$\frac{du}{dx}=8,\text{ so } du=8\ dx,\text{ so }\ dx=\frac{du}{8}$$

Substitute, integrate, then back-substitute and evaluate over the interval.

$$L=\int_{x=0}^{x=2}\sqrt{u}\left(\frac{du}{8}\right)$$

$$L=\frac{1}{8}\int_{x=0}^{x=2}u^{\frac{1}{2}}\ du$$

$$L=\frac{1}{8}\left(\frac{2}{3}u^{\frac{3}{2}}\right)\Bigg|_{x=0}^{x=2}$$

$$L=\frac{1}{12}u^{\frac{3}{2}}\Bigg|_{x=0}^{x=2}$$

$$L=\frac{1}{12}(1+8x)^{\frac{3}{2}}\Bigg|_0^2$$

$$L=\frac{1}{12}(1+8(2))^{\frac{3}{2}}-\frac{1}{12}(1+8(0))^{\frac{3}{2}}$$

$$L=\frac{1}{12}(17)^{\frac{3}{2}}-\frac{1}{12}(1)^{\frac{3}{2}}$$

$$L=\frac{1}{12}(17)^{\frac{3}{2}}-\frac{1}{12}$$

$$L=\frac{17\sqrt{17}-1}{12}$$

■ 2. Find the arc length of the curve over $[-3,3]$. Round your answer to the nearest three decimal places.

$$y=x^2-3$$

*Solution:*

The derivative of the function is

$$f'(x)=2x$$

Then the arc length over the interval is

$$L=\int_a^b\sqrt{1+\left[f'(x)\right]^2}\ dx$$

$$L=\int_{-3}^3\sqrt{1+\left[2x\right]^2}\ dx$$

$$L=\int_{-3}^3\sqrt{1+4x^2}\ dx$$

Use trigonometric substitution.

$$a = 1$$

$$u = 2x$$

$2x = \tan\theta$, so $\theta = \arctan(2x)$

$$x = \frac{1}{2}\tan\theta$$

$$dx = \frac{1}{2}\sec^2\theta\ d\theta$$

Substitute.

$$L = \int_{x=-3}^{x=3} \sqrt{1 + \tan^2\theta} \left(\frac{1}{2}\sec^2\theta\ d\theta\right)$$

$$L = \frac{1}{2}\int_{x=-3}^{x=3} \sec^2\theta\sqrt{1 + \tan^2\theta}\ d\theta$$

$$L = \frac{1}{2}\int_{x=-3}^{x=3} \sec^2\theta\sqrt{\sec^2\theta}\ d\theta$$

$$L = \frac{1}{2}\int_{x=-3}^{x=3} \sec^3\theta\ d\theta$$

Integrate.

$$L = \frac{1}{2}\left(\frac{1}{2}\sec\theta\tan\theta + \frac{1}{2}\ln|\sec\theta + \tan\theta|\right)\Bigg|_{x=-3}^{x=3}$$

$$L = \frac{1}{4}\left(\sec\theta\tan\theta + \ln|\sec\theta + \tan\theta|\right)\Bigg|_{x=-3}^{x=3}$$

Back-substitute, then evaluate over the interval.

$$L=\frac{1}{4}\Big(\sec(\arctan(2x))\tan(\arctan(2x))$$

$$+\ln|\sec(\arctan(2x))+\tan(\arctan(2x))|\Big)\Big|_{-3}^{3}$$

$$L=\frac{1}{4}\left(\sqrt{(2x)^2+1}\cdot(2x)+\ln\left|\sqrt{(2x)^2+1}+(2x)\right|\right)\Big|_{-3}^{3}$$

$$L=\frac{1}{4}\left(2x\sqrt{4x^2+1}+\ln\left|\sqrt{4x^2+1}+2x\right|\right)\Big|_{-3}^{3}$$

$$L=\frac{1}{4}\left(2(3)\sqrt{4(3)^2+1}+\ln\left|\sqrt{4(3)^2+1}+2(3)\right|\right)$$

$$-\frac{1}{4}\left(2(-3)\sqrt{4(-3)^2+1}+\ln\left|\sqrt{4(-3)^2+1}+2(-3)\right|\right)$$

$$L=\frac{1}{4}\left(6\sqrt{37}+\ln\left|\sqrt{37}+6\right|\right)-\frac{1}{4}\left(-6\sqrt{37}+\ln\left|\sqrt{37}-6\right|\right)$$

$$L=\frac{3}{2}\sqrt{37}+\frac{1}{4}\ln(\sqrt{37}+6)+\frac{3}{2}\sqrt{37}-\frac{1}{4}\ln(\sqrt{37}-6)$$

$$L=3\sqrt{37}+\frac{1}{4}\ln(\sqrt{37}+6)-\frac{1}{4}\ln(\sqrt{37}-6)$$

$$L=3\sqrt{37}+\frac{1}{4}\left[\ln(\sqrt{37}+6)-\ln(\sqrt{37}-6)\right]$$

$$L = 3\sqrt{37} + \frac{1}{4} \ln \frac{\sqrt{37} + 6}{\sqrt{37} - 6}$$

$$L \approx 19.494$$

■ 3. Set up the arc length integral of the curve over $[-1,2]$. Do not evaluate the integral.

$$y = \frac{x^3}{3} + x^2 + 5$$

*Solution:*

The derivative of the function is

$$f'(x) = x^2 + 2x$$

Then the arc length over the interval is

$$L = \int_a^b \sqrt{1 + \left[f'(x)\right]^2}\ dx$$

$$L = \int_{-1}^2 \sqrt{1 + \left[x^2 + 2x\right]^2}\ dx$$

$$L = \int_{-1}^2 \sqrt{1 + x^4 + 4x^3 + 4x^2}\ dx$$

4. Set up the arc length integral of the curve over $[-\pi, \pi]$. Do not evaluate the integral.

$$y = \sin x - 5$$

*Solution:*

The derivative of the function is

$$f'(x) = \cos x$$

Then the arc length over the interval is

$$L = \int_{a}^{b} \sqrt{1 + \left[f'(x)\right]^2}\ dx$$

$$L = \int_{-\pi}^{\pi} \sqrt{1 + [\cos x]^2}\ dx$$

$$L = \int_{-\pi}^{\pi} \sqrt{1 + \cos^2 x}\ dx$$

5. Set up the arc length integral of the curve over $[-\pi/4, \pi/4]$. Do not evaluate the integral.

$$y = \tan x \sec x + 2$$

Solution:

The derivative of the function is

$$f'(x) = \sec^2 x \sec x + \tan x \sec x \tan x$$

$$f'(x) = \sec^3 x + \tan^2 x \sec x$$

Then the arc length over the interval is

$$L = \int_a^b \sqrt{1 + \left[f'(x)\right]^2}\ dx$$

$$L = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \sqrt{1 + (\sec^3 x + \tan^2 x \sec x)^2}\ dx$$

$$L = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \sqrt{1 + \tan^4 x \sec^2 x + 2\tan^2 x \sec^4 x + \sec^6 x}\ dx$$

## ARC LENTH OF X=G(Y)

1. Find the arc length of the curve on the interval $1 \le y \le 6$.

$$x = \frac{y^2}{2} - \frac{\ln y}{4} - 8$$

*Solution:*

The derivative of the function is

$$g'(y) = y - \frac{1}{4y}$$

Then the arc length over the interval is

$$L = \int_c^d \sqrt{1 + \left[g'(y)\right]^2}\ dy$$

$$L = \int_1^6 \sqrt{1 + \left[y - \frac{1}{4y}\right]^2}\ dy$$

$$L = \int_1^6 \sqrt{1 + y^2 - \frac{1}{2} + \frac{1}{16y^2}}\ dy$$

$$L = \int_1^6 \sqrt{y^2 + \frac{1}{16y^2} + \frac{1}{2}}\ dy$$

$$L = \int_1^6 \sqrt{\left(y + \frac{1}{4y}\right)^2}\ dy$$

$$L = \int_1^6 y + \frac{1}{4y}\ dy$$

Integrate, then evaluate over the interval.

$$L = \frac{1}{2}y^2 + \frac{1}{4}\ln|y|\Big|_1^6$$

$$L = \frac{1}{2}(6)^2 + \frac{1}{4}\ln|6| - \left(\frac{1}{2}(1)^2 + \frac{1}{4}\ln|1|\right)$$

$$L = 18 + \frac{1}{4}\ln 6 - \frac{1}{2} - \frac{1}{4}(0)$$

$$L = \frac{35}{2} + \frac{1}{4}\ln 6$$

■ 2. Find the arc length of the curve on the interval $0 \le y \le 4$.

$$x = \frac{1}{3}(y^2 + 2)^{\frac{3}{2}} + 5$$

*Solution:*

The derivative of the function is

$$g'(y) = \frac{1}{2}(y^2+2)^{\frac{1}{2}}(2y)$$

$$g'(y) = y(y^2+2)^{\frac{1}{2}}$$

$$g'(y) = y\sqrt{y^2+2}$$

Then the arc length over the interval is

$$L = \int_c^d \sqrt{1+\left[g'(y)\right]^2}\ dy$$

$$L = \int_0^4 \sqrt{1+\left[y\sqrt{y^2+2}\right]^2}\ dy$$

$$L = \int_0^4 \sqrt{1+y^2(y^2+2)}\ dy$$

$$L = \int_0^4 \sqrt{y^4+2y^2+1}\ dy$$

$$L = \int_0^4 \sqrt{(y^2+1)^2}\ dy$$

$$L = \int_0^4 y^2+1\ dy$$

Integrate, then evaluate over the interval.

$$L = \frac{1}{3}y^3+y\Big|_0^4$$

$$L=\frac{1}{3}(4)^3+4-\left(\frac{1}{3}(0)^3+0\right)$$

$$L=\frac{64}{3}+4$$

$$L=\frac{64}{3}+\frac{12}{3}$$

$$L=\frac{76}{3}$$

■ 3. Find the arc length of the curve on the interval $4 \le y \le 16$.

$$x=y^{\frac{3}{2}}+15$$

*Solution:*

The derivative of the function is

$$g'(y)=\frac{3}{2}y^{\frac{1}{2}}$$

$$g'(y)=\frac{3}{2}\sqrt{y}$$

Then the arc length over the interval is

$$L=\int_c^d \sqrt{1+\left[g'(y)\right]^2}\ dy$$

$$L=\int_4^{16}\sqrt{1+\left[\frac{3}{2}\sqrt{y}\right]^2}\ dy$$

$$L=\int_4^{16}\sqrt{1+\frac{9}{4}y}\ dy$$

Integrate, then evaluate over the interval.

$$L=\frac{2}{3}\cdot\frac{4}{9}\left(1+\frac{9}{4}y\right)^{\frac{3}{2}}\Bigg|_4^{16}$$

$$L=\frac{8}{27}\left(1+\frac{9}{4}y\right)^{\frac{3}{2}}\Bigg|_4^{16}$$

$$L=\frac{8}{27}\left(1+\frac{9}{4}(16)\right)^{\frac{3}{2}}-\frac{8}{27}\left(1+\frac{9}{4}(4)\right)^{\frac{3}{2}}$$

$$L=\frac{8}{27}(1+36)^{\frac{3}{2}}-\frac{8}{27}(1+9)^{\frac{3}{2}}$$

$$L=\frac{8}{27}(37)^{\frac{3}{2}}-\frac{8}{27}(10)^{\frac{3}{2}}$$

$$L=\frac{296\sqrt{37}}{27}-\frac{80\sqrt{10}}{27}$$

$$L=\frac{296\sqrt{37}-80\sqrt{10}}{27}$$

■ 4. Find the arc length of the curve on the interval $1\le y\le 8$.

$$x = \left(1 - y^{\frac{2}{3}}\right)^{\frac{3}{2}}$$

Solution:

The derivative of the function is

$$g'(y) = \frac{3}{2}\left(1 - y^{\frac{2}{3}}\right)^{\frac{1}{2}}\left(-\frac{2}{3}y^{-\frac{1}{3}}\right)$$

$$g'(y) = -y^{-\frac{1}{3}}\left(1 - y^{\frac{2}{3}}\right)^{\frac{1}{2}}$$

Then the arc length over the interval is

$$L = \int_c^d \sqrt{1 + \left[g'(y)\right]^2}\ dy$$

$$L = \int_1^8 \sqrt{1 + \left[-y^{-\frac{1}{3}}\left(1 - y^{\frac{2}{3}}\right)^{\frac{1}{2}}\right]^2}\ dy$$

$$L = \int_1^8 \sqrt{1 + y^{-\frac{2}{3}}\left(1 - y^{\frac{2}{3}}\right)}\ dy$$

$$L = \int_1^8 \sqrt{1 + \left(y^{-\frac{2}{3}} - y^0\right)}\ dy$$

$$L = \int_1^8 \sqrt{1 + y^{-\frac{2}{3}} - 1}\ dy$$

$$L = \int_1^8 \sqrt{y^{-\frac{2}{3}}} \ dy$$

$$L = \int_1^8 \left(y^{-\frac{2}{3}}\right)^{\frac{1}{2}} \ dy$$

$$L = \int_1^8 y^{-\frac{1}{3}} \ dy$$

Integrate, then evaluate over the interval.

$$L = \frac{3}{2}y^{\frac{2}{3}}\Big|_1^8$$

$$L = \frac{3}{2}(8)^{\frac{2}{3}} - \frac{3}{2}(1)^{\frac{2}{3}}$$

$$L = \frac{3}{2}(4) - \frac{3}{2}(1)$$

$$L = \frac{3}{2}(4-1)$$

$$L = \frac{3}{2}(3)$$

$$L = \frac{9}{2}$$

■ 5. Find the arc length of the curve on the interval $1 \le y \le 5$.

$$x = \frac{y^2}{8} - \ln y$$

*Solution:*

The derivative of the function is

$$g'(y) = \frac{y}{4} - \frac{1}{y}$$

Then the arc length over the interval is

$$L = \int_c^d \sqrt{1 + [g'(y)]^2}\ dy$$

$$L = \int_1^5 \sqrt{1 + \left[\frac{y}{4} - \frac{1}{y}\right]^2}\ dy$$

$$L = \int_1^5 \sqrt{1 + \frac{y^2}{16} - \frac{1}{2} + \frac{1}{y^2}}\ dy$$

$$L = \int_1^5 \sqrt{\frac{y^2}{16} + \frac{1}{2} + \frac{1}{y^2}}\ dy$$

$$L = \int_1^5 \sqrt{\left(\frac{y}{4} + \frac{1}{y}\right)^2}\ dy$$

$$L = \int_1^5 \frac{y}{4} + \frac{1}{y}\ dy$$

Integrate, then evaluate over the interval.

$$L=\frac{y^2}{8}+\ln|y|\Big|_1^5$$

$$L=\frac{5^2}{8}+\ln|5|-\frac{1^2}{8}-\ln|1|$$

$$L=\frac{25}{8}+\ln 5-\frac{1}{8}-0$$

$$L=\frac{24}{8}+\ln 5$$

$$L=3+\ln 5$$

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