Topic: Estimating indefinite integrals

Question: Evaluate the indefinite integral as a power series.

$$\int \frac{r}{1-r^2}\,dr$$

Answer choices:

A $\displaystyle\sum_{n=0}^{\infty} \frac{r^{2n+1}}{2n+1} + C$

B $\displaystyle\sum_{n=0}^{\infty} \frac{r^{2n+2}}{2n+2} + C$

C $\displaystyle\sum_{n=0}^{\infty} \frac{r^{n+2}}{n+2} + C$

D $\displaystyle\sum_{n=0}^{\infty} \frac{r^{n+1}}{n+1} + C$

Solution: B

When we use power series to integrate a function like the given function

$$f(r) = \frac{r}{1 - r^2}$$

we use the standard form of a power series

$$\frac{1}{1 - x} = 1 + x + x^2 + x^3 + \dots = \sum_{n=0}^{\infty} x^n$$

Our goal will be to start with the given function, and then manipulate it until it matches the far left side of the standard form. To get the given function to match the left side of the standard form, we'll first factor out an r.

$$\frac{r}{1 - r^2} = r\left(\frac{1}{1 - r^2}\right)$$

Matching this to the standard form, we can say that $x = r^2$. Making this substitution throughout the standard form, we get

$$\frac{1}{1 - r^2} = 1 + r^2 + \left(r^2\right)^2 + \left(r^2\right)^3 + \dots = \sum_{n=0}^{\infty} \left(r^2\right)^n$$

$$\frac{1}{1 - r^2} = 1 + r^2 + r^4 + r^6 + \dots = \sum_{n=0}^{\infty} r^{2n}$$

We can't forget about the r that we factored out of the given function. We have to add that back in.

$$r\left(\frac{1}{1 - r^2}\right) = r\left(1 + r^2 + r^4 + r^6 + \dots\right) = r\sum_{n=0}^{\infty} r^{2n}$$

$$\frac{r}{1-r^2} = r + r^3 + r^5 + r^7 + \dots = \sum_{n=0}^{\infty} r^{2n+1}$$

Now that the far left of the manipulated standard form matches the given function, we can say that the far right of the manipulated standard form,

$$\sum_{n=0}^{\infty} r^{2n+1}$$

is the power series representation of the given function.

And instead of evaluating the integral of the given function directly, we can use the expanded sum in its place. So the integral becomes

$$\int \frac{r}{1-r^2}\ dr = \int r + r^3 + r^5 + r^7 + \dots + r^{2n+1}\ dr$$

$$\int \frac{r}{1-r^2}\ dr = \frac{r^2}{2} + \frac{r^4}{4} + \frac{r^6}{6} + \frac{r^8}{8} + \dots + \frac{r^{2n+2}}{2n+2} + C$$

Then we take the last term and say that

$$\int \frac{r}{1-r^2}\ dr = \sum_{n=0}^{\infty} \frac{r^{2n+2}}{2n+2} + C$$

Topic: Estimating indefinite integrals

Question: Evaluate the indefinite integral as a power series.

$$\int \frac{r}{1+r}\,dr$$

Answer choices:

A $\displaystyle\sum_{n=0}^{\infty}(-1)^n\frac{r^{2n+1}}{2n+1}+C$

B $\displaystyle\sum_{n=0}^{\infty}(-1)^n\frac{r^{2n+2}}{2n+2}+C$

C $\displaystyle\sum_{n=0}^{\infty}(-1)^n\frac{r^{n+2}}{n+2}+C$

D $\displaystyle\sum_{n=0}^{\infty}(-1)^n\frac{r^{n+1}}{n+1}+C$

Solution: C

When we use power series to integrate a function like the given function

$$f(r)=\frac{r}{1+r}$$

we use the standard form of a power series

$$\frac{1}{1-x}=1+x+x^2+x^3+\ldots=\sum_{n=0}^{\infty}x^n$$

Our goal will be to start with the given function, and then manipulate it until it matches the far left side of the standard form. To get the given function to match the left side of the standard form, we'll first factor out an r.

$$\frac{r}{1+r}=r\left(\frac{1}{1+r}\right)$$

$$\frac{r}{1+r}=r\left[\frac{1}{1-(-r)}\right]$$

Matching this to the standard form, we can say that $x=-r$. Making this substitution throughout the standard form, we get

$$\frac{1}{1-(-r)}=1+(-r)+(-r)^2+(-r)^3+\ldots=\sum_{n=0}^{\infty}(-r)^n$$

$$\frac{1}{1+r}=1-r+r^2-r^3+\ldots=\sum_{n=0}^{\infty}(-1)^n r^n$$

We can't forget about the r that we factored out of the given function. We have to add that back in.

$$r\left(\frac{1}{1+r}\right)=r\left(1-r+r^2-r^3+\dots\right)=r\sum_{n=0}^{\infty}(-1)^n r^n$$

$$\frac{r}{1+r}=r-r^2+r^3-r^4+\dots=\sum_{n=0}^{\infty}(-1)^n r^{n+1}$$

Now that the far left of the manipulated standard form matches the given function, we can say that the far right of the manipulated standard form,

$$\sum_{n=0}^{\infty}(-1)^n r^{n+1}$$

is the power series representation of the given function.

And instead of evaluating the integral of the given function directly, we can use the expanded sum in its place. So the integral becomes

$$\int\frac{r}{1+r}\ dr=\int r-r^2+r^3-r^4+\dots+(-1)^n r^{n+1}\ dr$$

$$\int\frac{r}{1+r}\ dr=\frac{r^2}{2}-\frac{r^3}{3}+\frac{r^4}{4}-\frac{r^5}{5}+\dots+(-1)^n\frac{r^{n+2}}{n+2}+C$$

Then we take the last term and say that

$$\int\frac{r}{1-r^2}\ dr=\sum_{n=0}^{\infty}(-1)^n\frac{r^{n+2}}{n+2}+C$$

Topic: Estimating indefinite integrals

Question: Evaluate the indefinite integral as a power series.

$$\int \frac{r}{1-r^3}\ dr$$

Answer choices:

A $\quad \displaystyle\sum_{n=0}^{\infty} \frac{r^{3n+2}}{3n+2} + C$

B $\quad \displaystyle\sum_{n=0}^{\infty} \frac{r^{2n+3}}{2n+3} + C$

C $\quad \displaystyle\sum_{n=0}^{\infty} \frac{r^{n+3}}{n+3} + C$

D $\quad \displaystyle\sum_{n=0}^{\infty} \frac{r^{3n+1}}{3n+1} + C$

Solution: A

When we use power series to integrate a function like the given function

$$f(r) = \frac{r}{1 - r^3}$$

we use the standard form of a power series

$$\frac{1}{1 - x} = 1 + x + x^2 + x^3 + \dots = \sum_{n=0}^{\infty} x^n$$

Our goal will be to start with the given function, and then manipulate it until it matches the far left side of the standard form. To get the given function to match the left side of the standard form, we'll first factor out an r.

$$\frac{r}{1 - r^3} = r\left(\frac{1}{1 - r^3}\right)$$

Matching this to the standard form, we can say that $x = r^3$. Making this substitution throughout the standard form, we get

$$\frac{1}{1 - r^3} = 1 + r^3 + \left(r^3\right)^2 + \left(r^3\right)^3 + \dots = \sum_{n=0}^{\infty} \left(r^3\right)^n$$

$$\frac{1}{1 - r^3} = 1 + r^3 + r^6 + r^9 + \dots = \sum_{n=0}^{\infty} r^{3n}$$

We can't forget about the r that we factored out of the given function. We have to add that back in.

$$r\left(\frac{1}{1 - r^3}\right) = r\left(1 + r^3 + r^6 + r^9 + \dots\right) = r\sum_{n=0}^{\infty} r^{3n}$$

$$\frac{r}{1-r^3} = r + r^4 + r^7 + r^{10} + \dots = \sum_{n=0}^{\infty} r^{3n+1}$$

Now that the far left of the manipulated standard form matches the given function, we can say that the far right of the manipulated standard form,

$$\sum_{n=0}^{\infty} r^{3n+1}$$

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And instead of evaluating the integral of the given function directly, we can use the expanded sum in its place. So the integral becomes

$$\int \frac{r}{1-r^3}\ dr = \int r + r^4 + r^7 + r^{10} + \dots + r^{3n+1}\ dr$$

$$\int \frac{r}{1-r^3}\ dr = \frac{r^2}{2} + \frac{r^5}{5} + \frac{r^8}{8} + \frac{r^{11}}{11} + \dots + \frac{r^{3n+2}}{3n+2} + C$$

Then we take the last term and say that

$$\int \frac{r}{1-r^3}\ dr = \sum_{n=0}^{\infty} \frac{r^{3n+2}}{3n+2} + C$$