



Calculus 1 Workbook Solutions

Continuity

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MATH

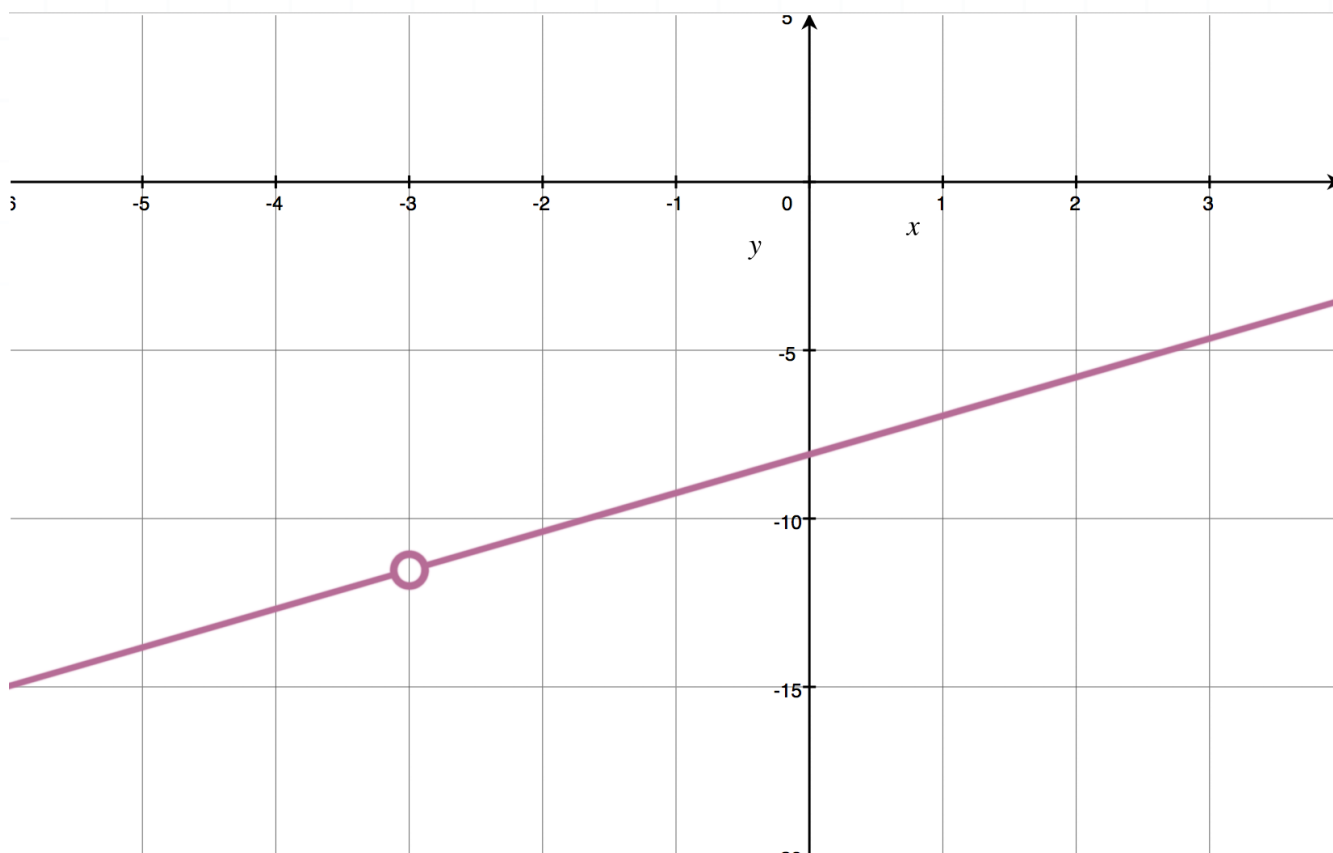
POINT DISCONTINUITIES

- 1. Redefine the function as a continuous piecewise function.

$$f(x) = \frac{x^2 - 6x - 27}{x + 3}$$

Solution:

The function is discontinuous at $x = -3$.



Factor and reduce to remove the discontinuity.

$$f(x) = \frac{x^2 - 6x - 27}{x + 3}$$



$$f(x) = \frac{(x+3)(x-9)}{x+3}$$

$$f(x) = x - 9$$

Evaluate $f(x)$ at $x = -3$.

$$f(-3) = -3 - 9 = -12$$

Therefore, to make the function continuous, we have to redefine it as

$$f(x) = \begin{cases} \frac{x^2 - 6x - 27}{x + 3} & x \neq -3 \\ -12 & x = -3 \end{cases}$$

We can see whether or not this function is continuous at $x = -3$ by looking at the limit as x approaches -3 .

$$\lim_{x \rightarrow -3} \frac{x^2 - 6x - 27}{x + 3} = \lim_{x \rightarrow -3} \frac{(x+3)(x-9)}{x+3} = -12$$

Since -12 is also the value of the function at $x = -3$, we see that this function is continuous.

■ 2. Identify the non-removable discontinuities of the function.

$$k(x) = \frac{x^3 + 3x^2 - 25x - 75}{x^2 + x - 12}$$

Solution:



Factor the function.

$$k(x) = \frac{x^3 + 3x^2 - 25x - 75}{x^2 + x - 12}$$

$$k(x) = \frac{(x + 5)(x - 5)(x + 3)}{(x + 4)(x - 3)}$$

No factors can be canceled. Which means the function has discontinuities at $x = -4$ and $x = 3$, both of which are non-removable.

■ 3. What is the set of removable discontinuities of the function?

$$j(\theta) = \frac{\cos^2\theta \cdot \sin^2\theta}{\tan^2\theta}$$

Solution:

We can rewrite the function as

$$j(\theta) = \frac{\cos^2\theta \cdot \sin^2\theta}{\tan^2\theta} = \frac{\cos^2\theta \cdot \sin^2\theta}{\frac{\sin^2\theta}{\cos^2\theta}} = \frac{\cos^2\theta \cdot \sin^2\theta \cdot \cos^2\theta}{\sin^2\theta} = \cos^4\theta$$

The removable discontinuities are the values of θ that make the sine function equal to 0, which are all the multiples of π .

$$\theta = \pm 0, \pm \pi, \pm 2\pi, \pm 3\pi, \pm 4\pi, \dots$$

$$\theta = n\pi, \text{ where } n \text{ is the set of all integers}$$



- 4. Examine whether or not the function is continuous at $x = 0$.

$$g(x) = \begin{cases} 2 - x^2 & x \leq 0 \\ x - 2 & x > 0 \end{cases}$$

Solution:

We can say that $f(x)$ is continuous at $x = a$ if $f(a)$ is defined and $\lim_{x \rightarrow a} f(x) = f(a)$.

Evaluate $f(x)$ at $x = 0$.

$$f(0) = 2 - 0^2 = 2$$

The right-hand limit at $x = 0$ is

$$\lim_{x \rightarrow 0^+} (x - 2) = 0 - 2 = -2$$

The right-hand limit at $x = 0$ isn't equivalent to the function's value at $x = 0$, so the function is not continuous there.

- 5. Where is the removable discontinuity in the graph of the function?

$$f(x) = \frac{x^3 + 27}{x + 3}$$

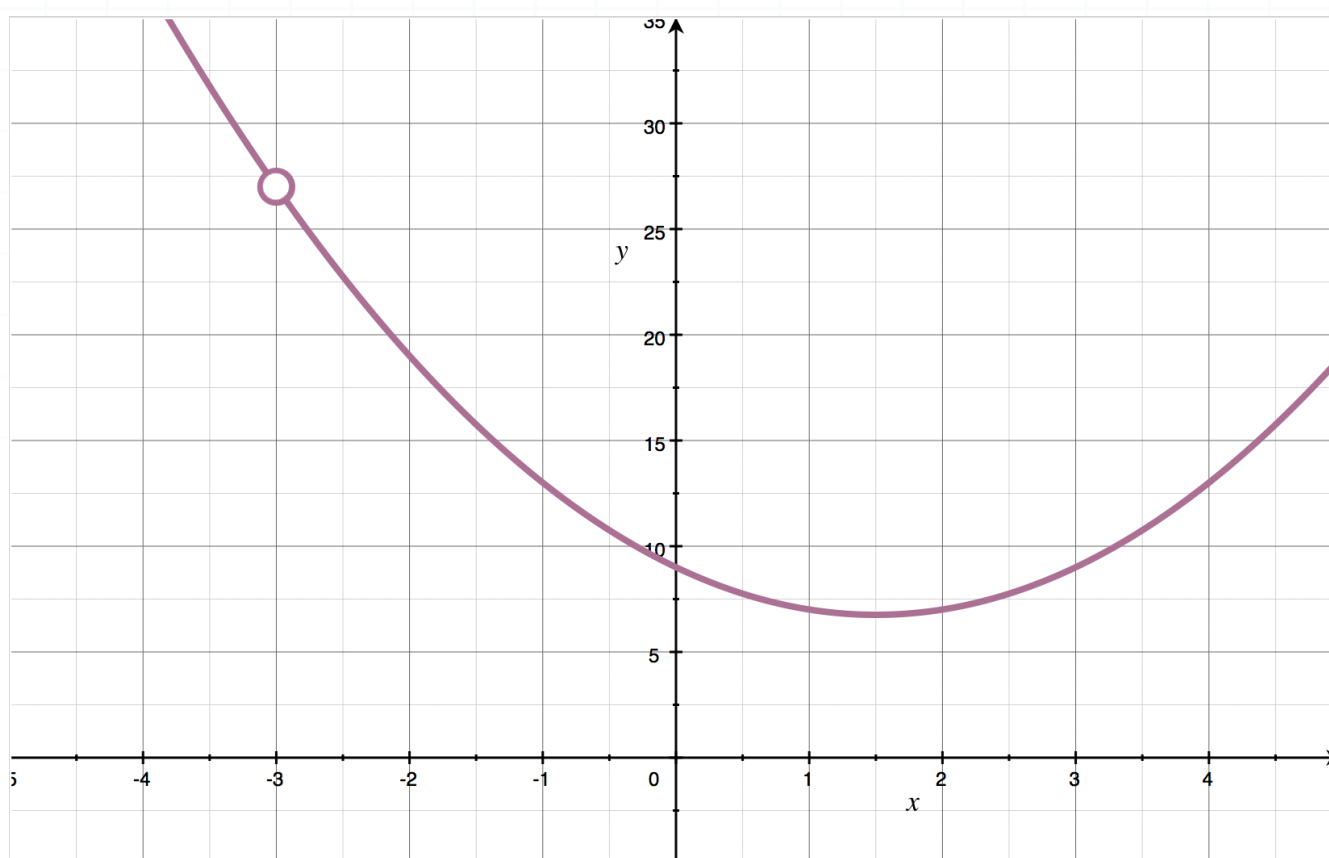
Solution:



If we factor the function, we can cancel a factor of $x + 3$. We could have also simplified the function by using polynomial long division to find the quotient.

$$f(x) = \frac{x^3 + 27}{x + 3} = \frac{(x + 3)(x^2 - 3x + 9)}{x + 3} = x^2 - 3x + 9$$

Because the factor of $x + 3$ cancels, the removable discontinuity is at $x + 3 = 0$, or $x = -3$.



■ 6. Identify the removable discontinuities in the function.

$$k(x) = \frac{x^4 - 2x^3 - 16x^2 + 2x + 15}{x^2 - 2x - 15}$$



Solution:

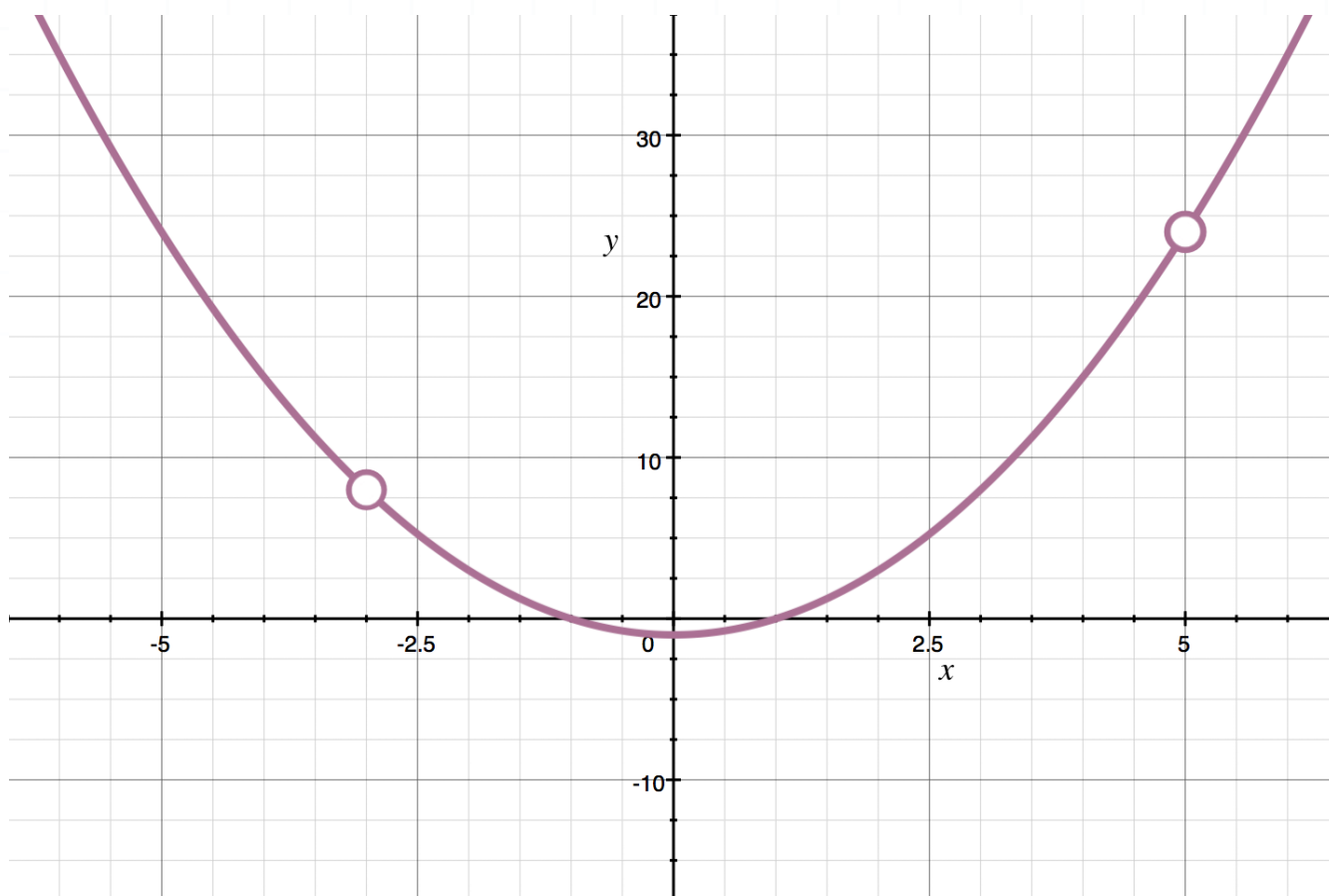
The function $k(x)$ has removable discontinuities at $x = -3$ and $x = 5$ because the function factors as

$$k(x) = \frac{(x+3)(x-5)(x+1)(x-1)}{(x+3)(x-5)}$$

and both factors from the denominator can be cancelled.

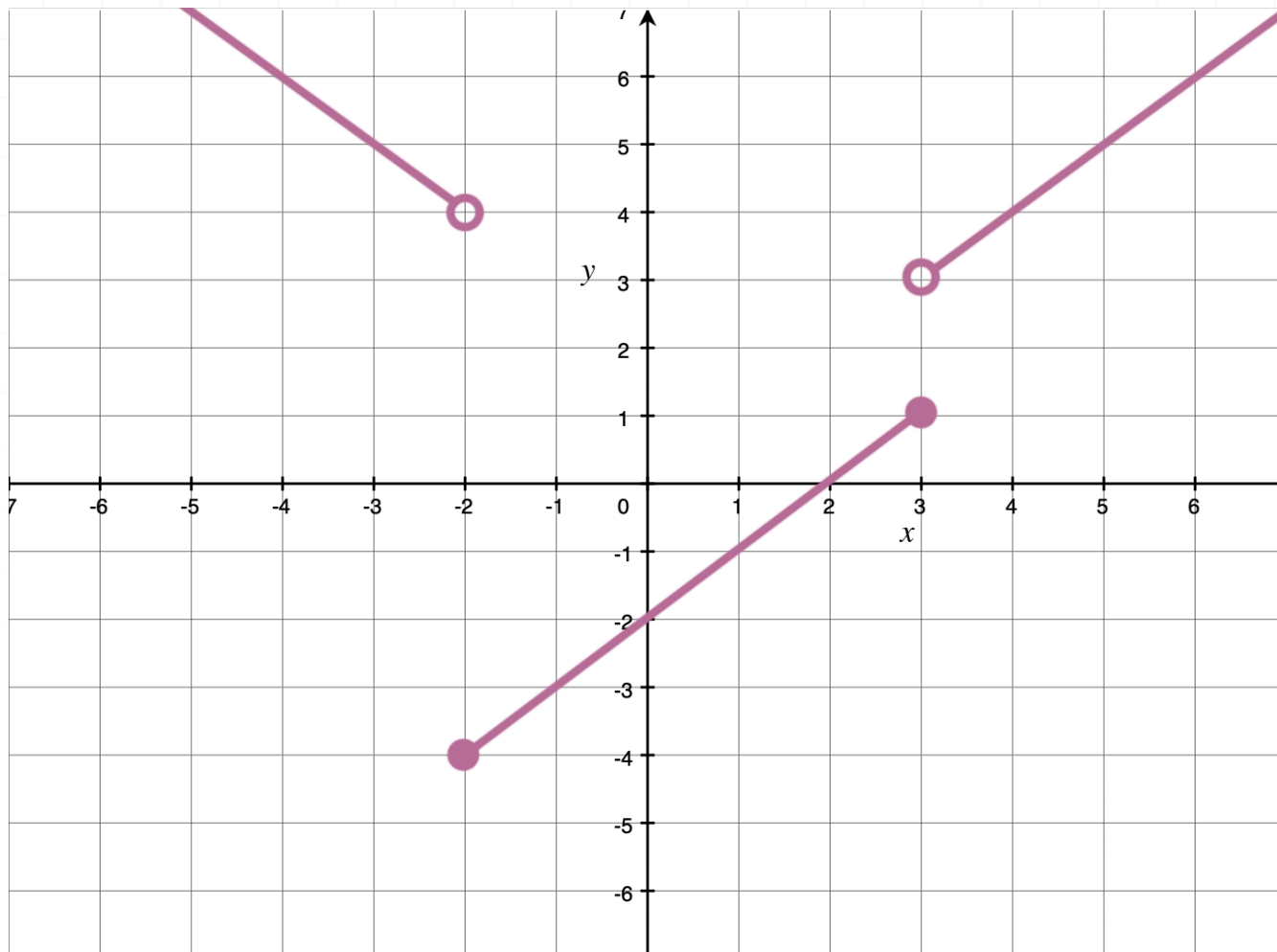
$$k(x) = (x+1)(x-1)$$

The graph is shown below.



JUMP DISCONTINUITIES

■ 1. What are the x -values where the graph of $f(x)$, shown below, has jump discontinuities?



Solution:

The function $f(x)$ has jump discontinuities at $x = -2$ and $x = 3$ because the left- and right-hand limits aren't equal at $x = -2$,

$$\lim_{x \rightarrow -2^-} f(x) = 4 \neq \lim_{x \rightarrow -2^+} f(x) = -4$$

and they aren't equal at $x = 3$.



$$\lim_{x \rightarrow 3^-} f(x) = 1 \quad \neq \quad \lim_{x \rightarrow 3^+} f(x) = 3$$

■ 2. Where are the jump discontinuities in the graph of the function?

$$h(x) = \begin{cases} -\frac{1}{3}x^2 + 2 & x < 0 \\ 3 & 0 \leq x \leq 1 \\ \frac{1}{3}x^2 + 4 & x > 1 \end{cases}$$

Solution:

The function $h(x)$ has jump discontinuities at $x = 0$ and $x = 1$ because the left- and right-hand limits aren't equal at $x = 0$,

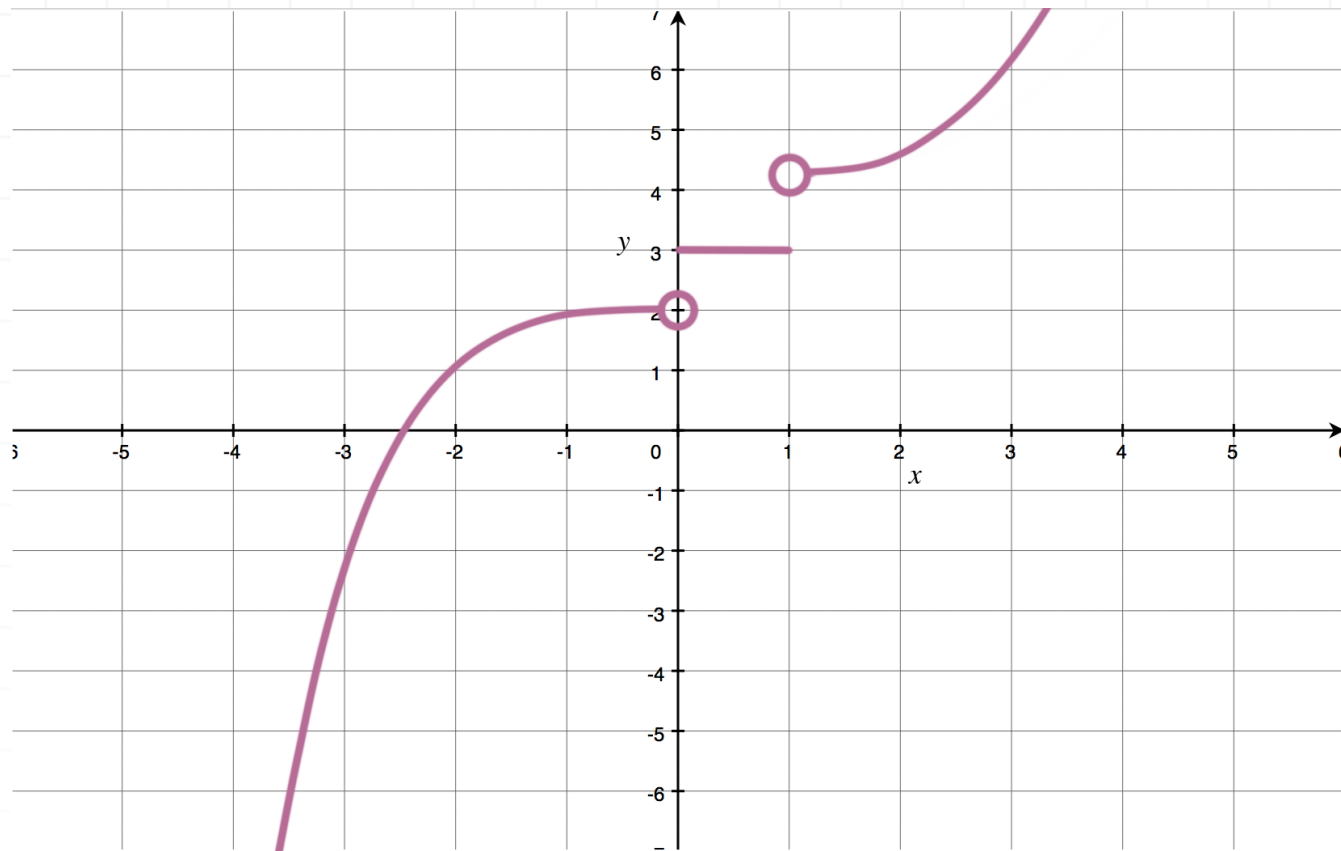
$$\lim_{x \rightarrow 0^-} f(x) = 2 \quad \neq \quad \lim_{x \rightarrow 0^+} f(x) = 3$$

or at $x = 1$.

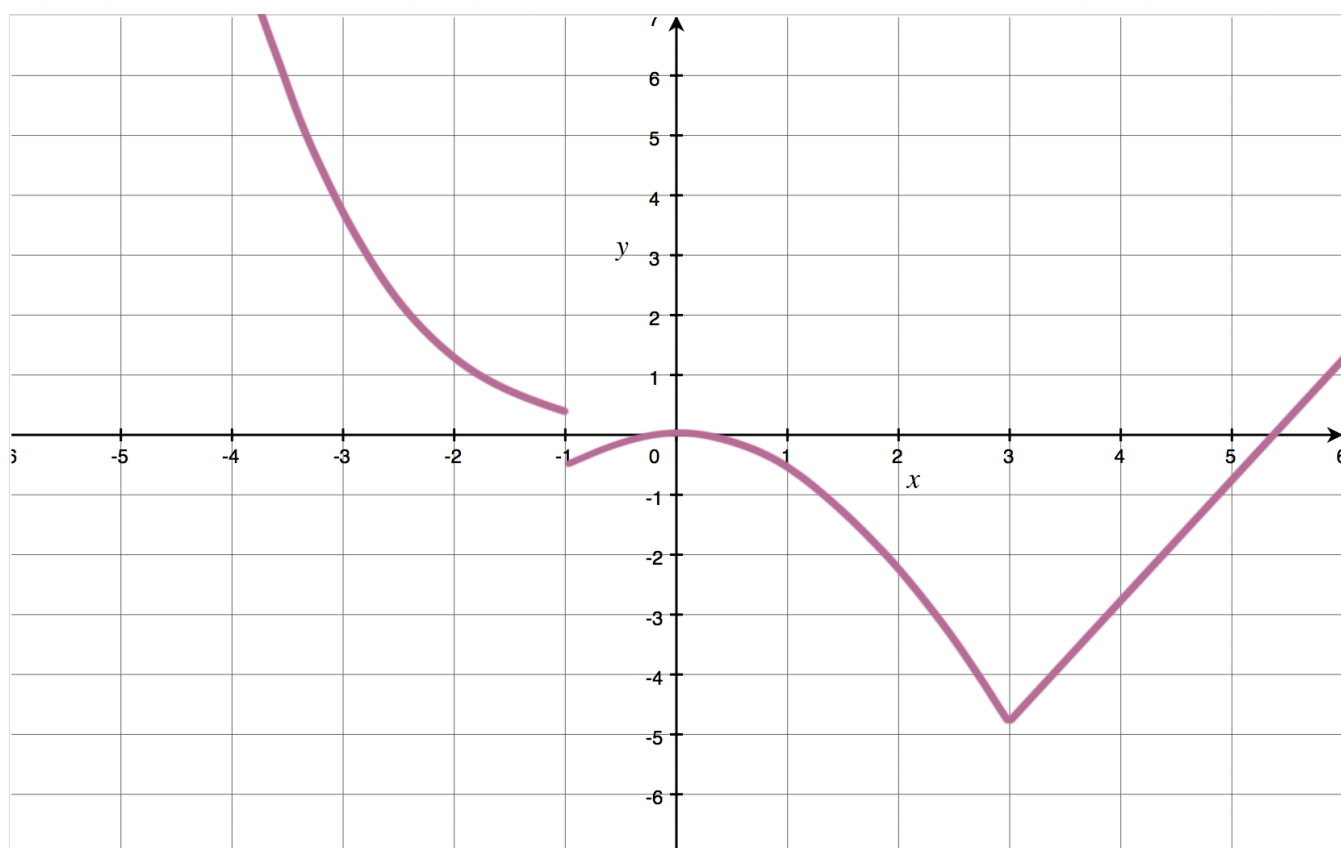
$$\lim_{x \rightarrow 1^-} f(x) = 3 \quad \neq \quad \lim_{x \rightarrow 1^+} f(x) = \frac{13}{3}$$

We can see the discontinuities in the function's graph, as well.





■ 3. What are the x -values where the graph of $g(x)$ has jump discontinuities?

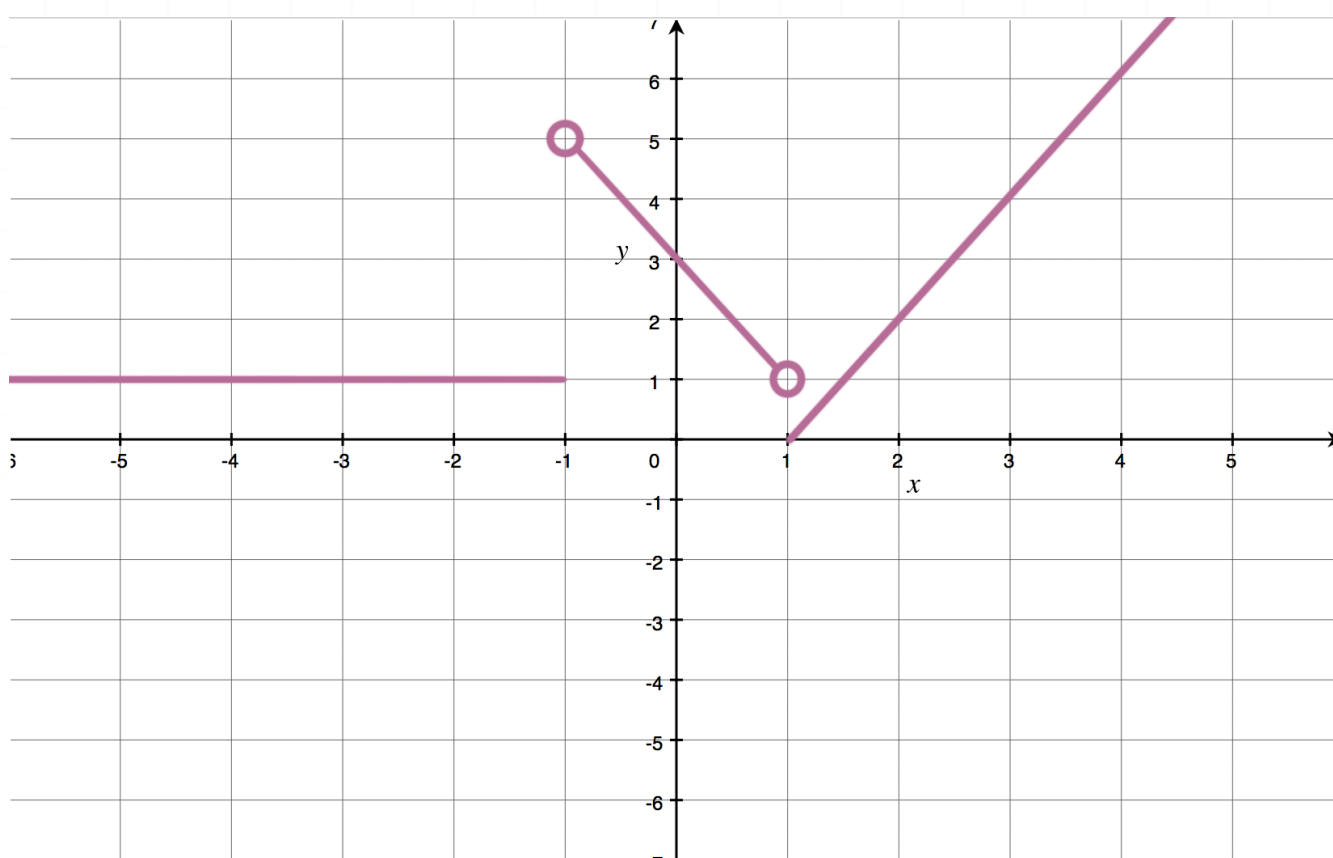


Solution:

The function $g(x)$ has a jump discontinuity at $x = -1$ because the left- and right-hand limits aren't equal there.

$$\lim_{x \rightarrow -1^-} f(x) = \frac{1}{3} \neq \lim_{x \rightarrow -1^+} f(x) = -\frac{1}{3}$$

■ 4. Show that $f(x)$ has jump discontinuity at $x = -1$ and $x = 1$.



Solution:

The function $f(x)$ has jump discontinuities at $x = -1$ and $x = 1$ because the left- and right-hand limits aren't equal at $x = -1$

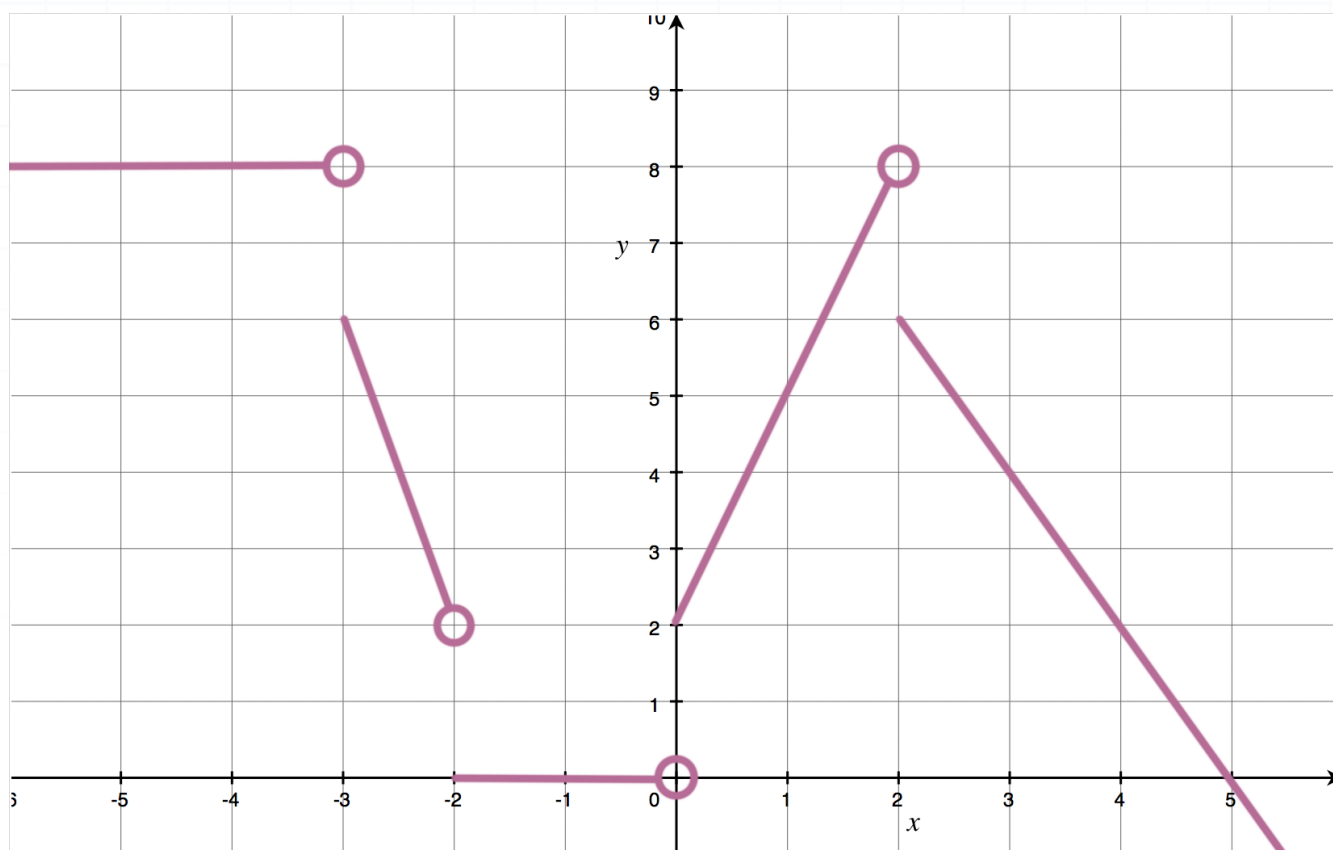
$$\lim_{x \rightarrow -1^-} f(x) = 1 \neq \lim_{x \rightarrow -1^+} f(x) = 5$$



or at $x = 1$.

$$\lim_{x \rightarrow 1^-} f(x) = 1 \neq \lim_{x \rightarrow 1^+} f(x) = 0$$

■ 5. Where are the jump discontinuities in the graph of the function shown below?

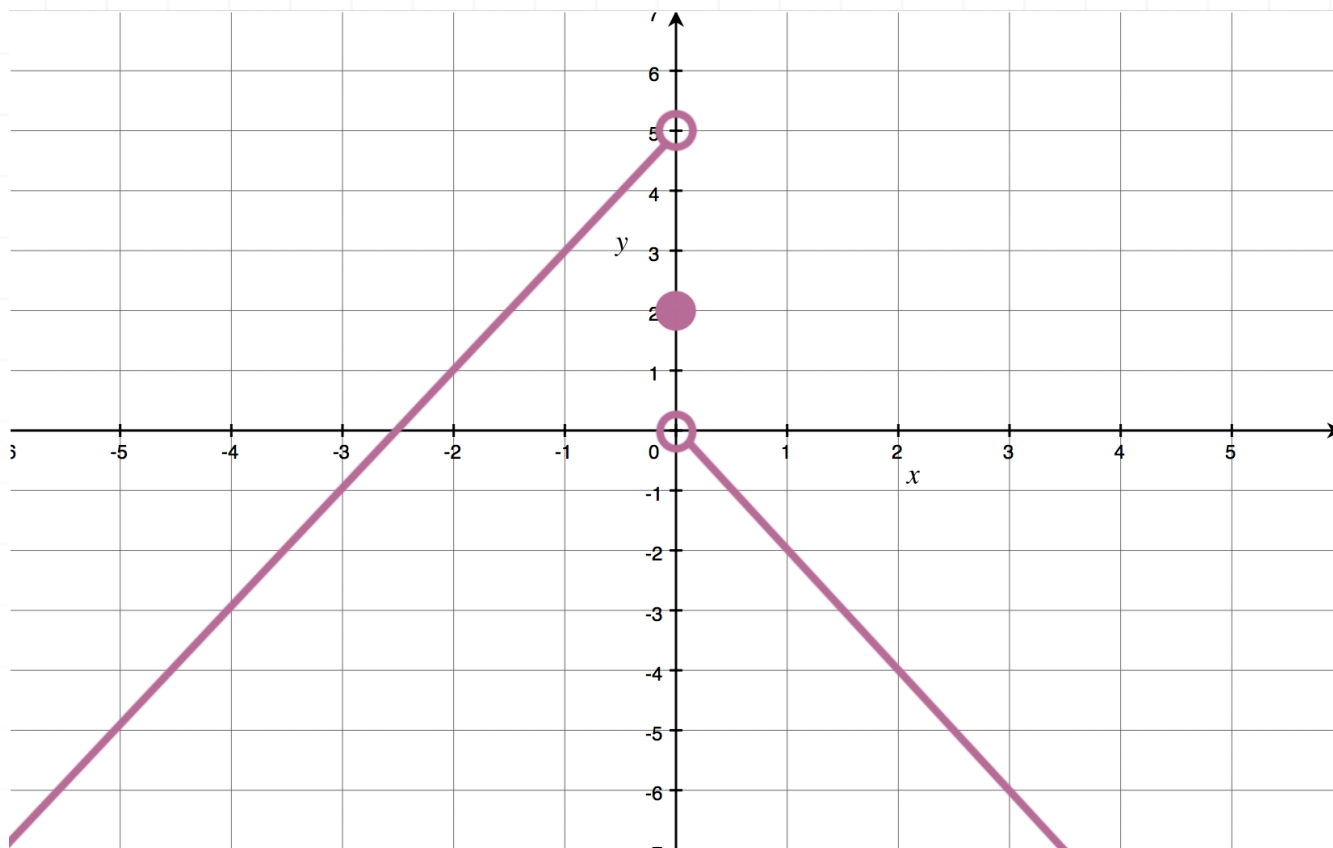


Solution:

The function has jump discontinuities at $x = -3$, $x = -2$, $x = 0$, and $x = 2$, because at each x -value, the left- and right-hand limits aren't equal.



■ 6. What are the x -values where the graph of $h(x)$, shown below, has jump discontinuities?



Solution:

The function $h(x)$ has a jump discontinuity at $x = 0$ because the left- and right-hand limits aren't equal there.

$$\lim_{x \rightarrow 0^-} f(x) = 5 \quad \neq \quad \lim_{x \rightarrow 0^+} f(x) = 0$$



INFINITE DISCONTINUITIES

- 1. At what x -values does the function have infinite discontinuities?

$$f(x) = \frac{x^2 + x - 12}{x^2 + x - 2}$$

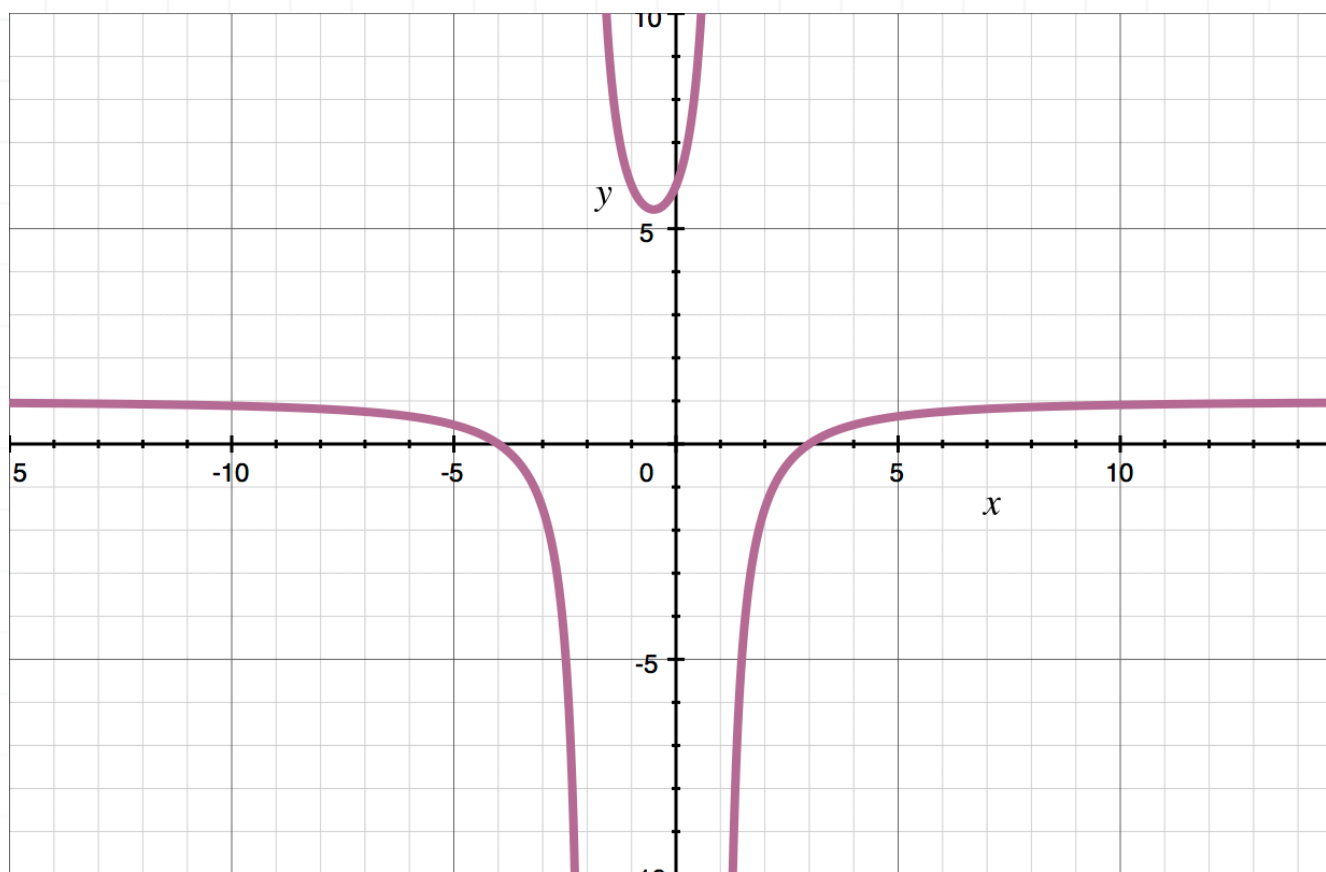
Solution:

Factor the function.

$$f(x) = \frac{x^2 + x - 12}{x^2 + x - 2} = \frac{(x + 4)(x - 3)}{(x + 2)(x - 1)}$$

None of these factors cancel, which means that $x + 2 = 0$ and $x - 1 = 0$ will both make the denominator equal to 0. Which means there are infinite discontinuities at $x = -2$ and $x = 1$.





■ 2. Where are the infinite discontinuities of the function?

$$h(x) = \frac{x^4 + 3x^3 - 8x - 24}{x^2 + 3x - 4}$$

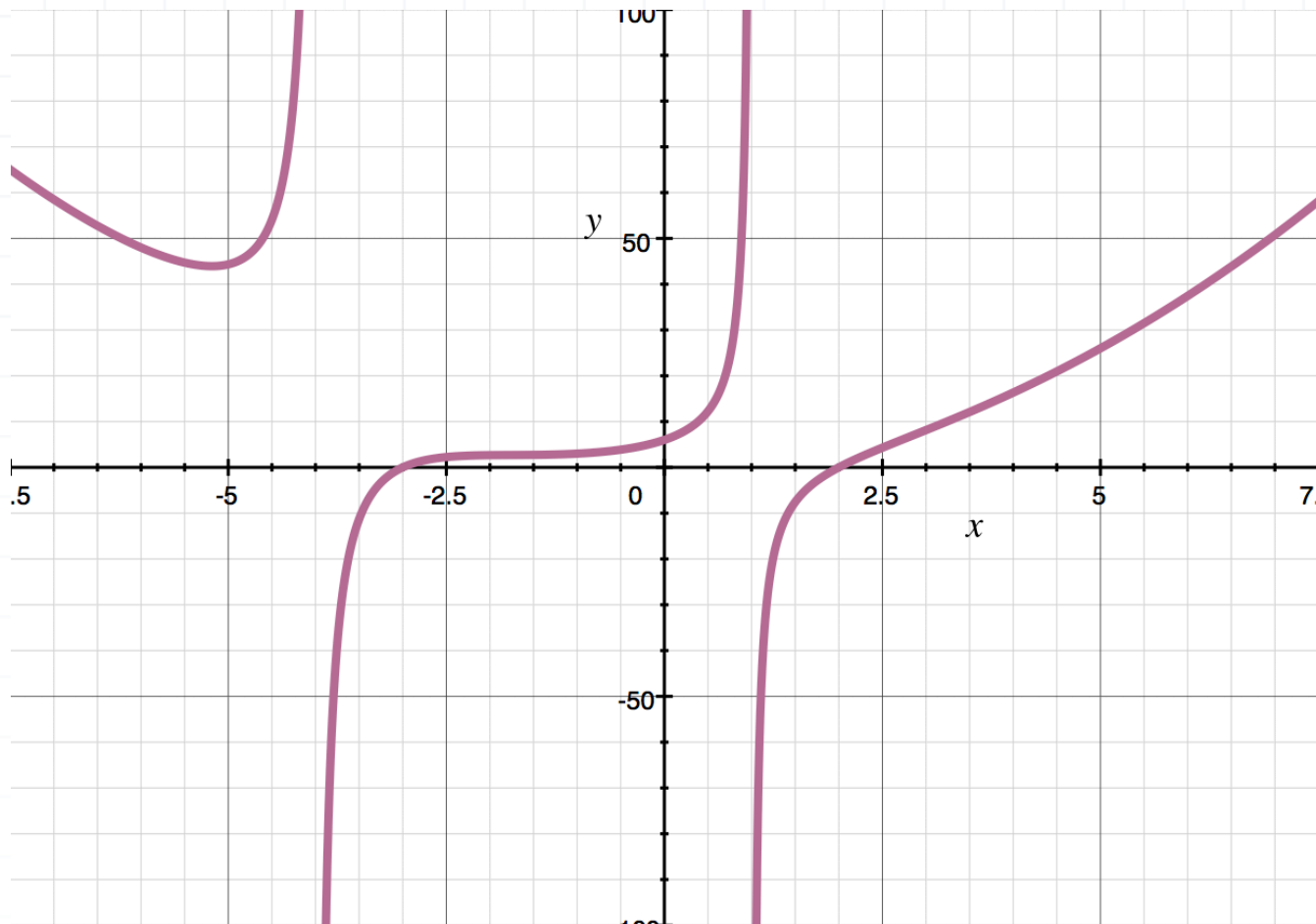
Solution:

Factor the function.

$$h(x) = \frac{x^4 + 3x^3 - 8x - 24}{x^2 + 3x - 4} = \frac{(x - 2)(x^2 + 2x + 4)(x + 3)}{(x + 4)(x - 1)}$$

None of these factors cancel, which means that $x + 4 = 0$ and $x - 1 = 0$ will both make the denominator equal to 0. Which means there are infinite discontinuities at $x = -4$ and $x = 1$.





■ 3. At what x -values does the function have infinite discontinuities?

$$g(x) = \frac{x^2 - 5x + 6}{x^2 - 1}$$

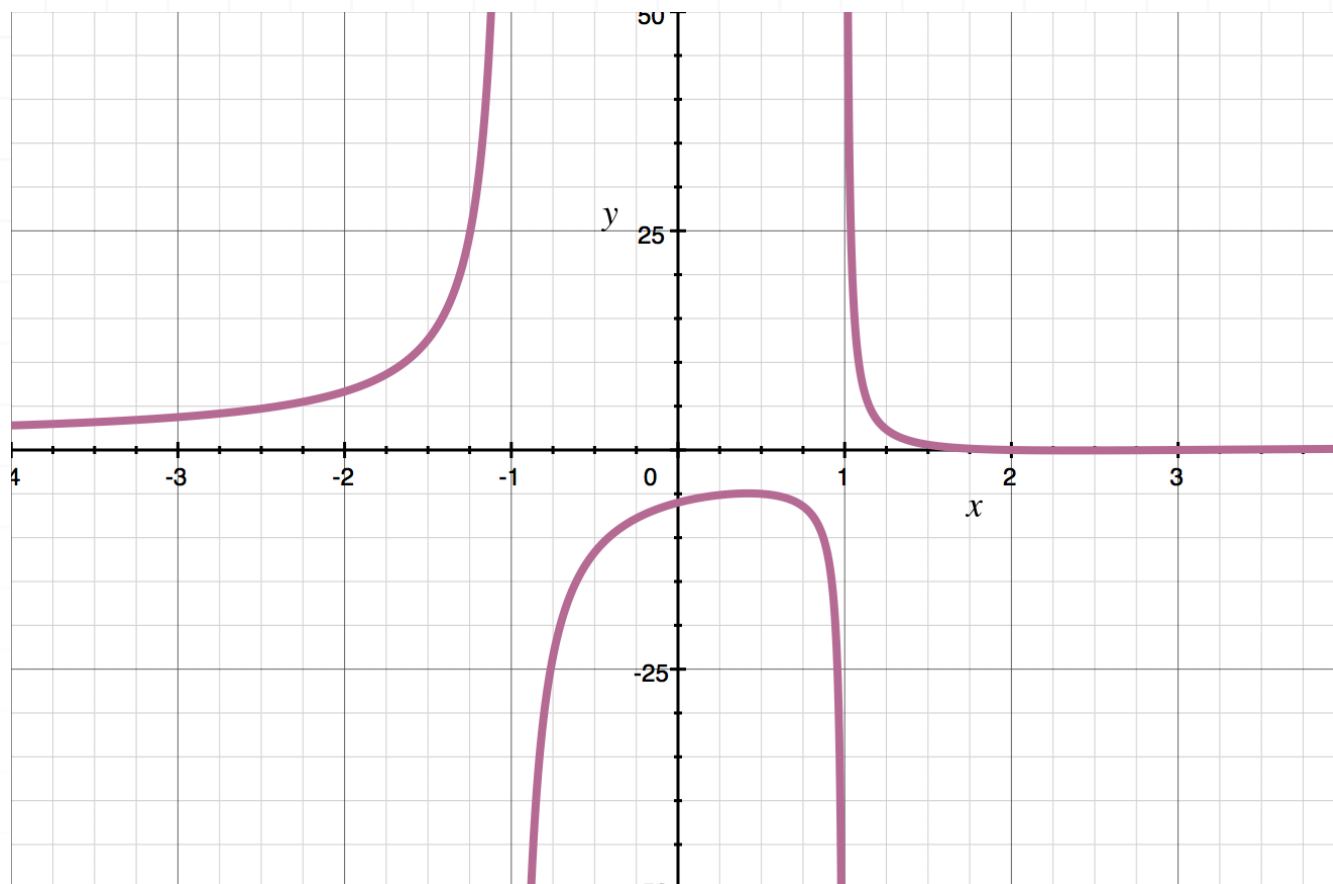
Solution:

Factor the function.

$$g(x) = \frac{x^2 - 5x + 6}{x^2 - 1} = \frac{(x - 3)(x - 2)}{(x + 1)(x - 1)}$$



None of these factors cancel, which means that $x + 1 = 0$ and $x - 1 = 0$ will both make the denominator equal to 0. Which means there are infinite discontinuities at $x = -1$ and $x = 1$.



■ 4. Where are the infinite discontinuities of the function?

$$h(x) = \frac{x^2 - 6x + 9}{x^2 - 4}$$

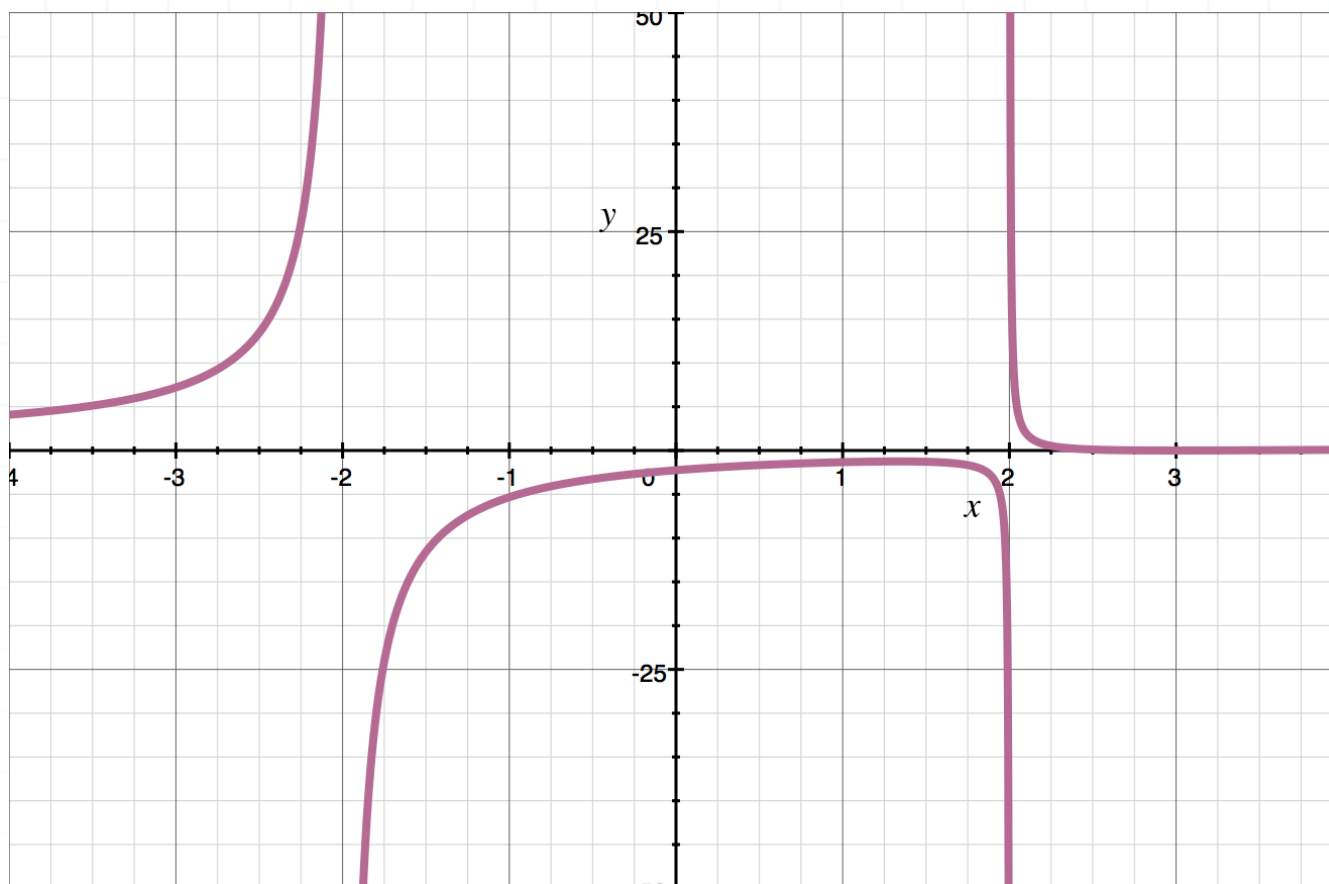
Solution:

Factor the function.

$$h(x) = \frac{x^2 - 6x + 9}{x^2 - 4} = \frac{(x - 3)^2}{(x + 2)(x - 2)}$$



None of these factors cancel, which means that $x + 2 = 0$ and $x - 2 = 0$ will both make the denominator equal to 0. Which means there are infinite discontinuities at $x = -2$ and $x = 2$.



■ 5. At what x -values does the function have infinite discontinuities?

$$h(x) = \frac{x^2 - 6x + 9}{x^2 + x - 12}$$

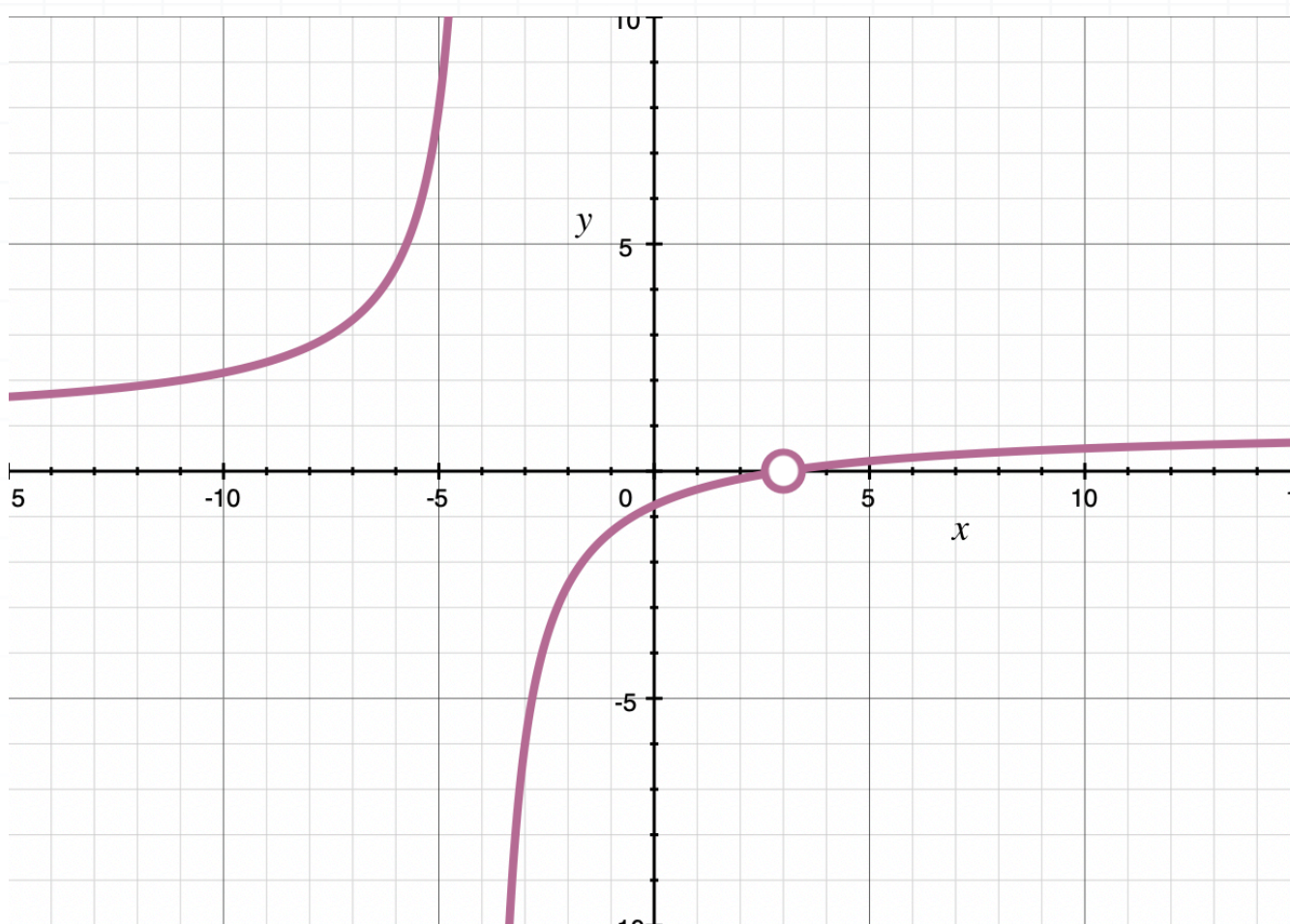
Solution:

Factor the function.

$$h(x) = \frac{x^2 - 6x + 9}{x^2 + x - 12} = \frac{(x - 3)^2}{(x + 4)(x - 3)} = \frac{x - 3}{x + 4}$$



In this form, we can see that the denominator is 0 at both $x = 3$ and $x = -4$. Because the $x - 3$ can be canceled, there's a point discontinuity at $x = 3$. Since the $x + 4$ can't be canceled, and, no matter how much we simplify the fraction, $x = -4$ will always make the denominator 0, that tells us there's a vertical asymptote at $x = -4$, and therefore an infinite discontinuity there.



■ 6. Classify the discontinuities of $f(x) = \cot x$ on the interval $[0, 2\pi]$.

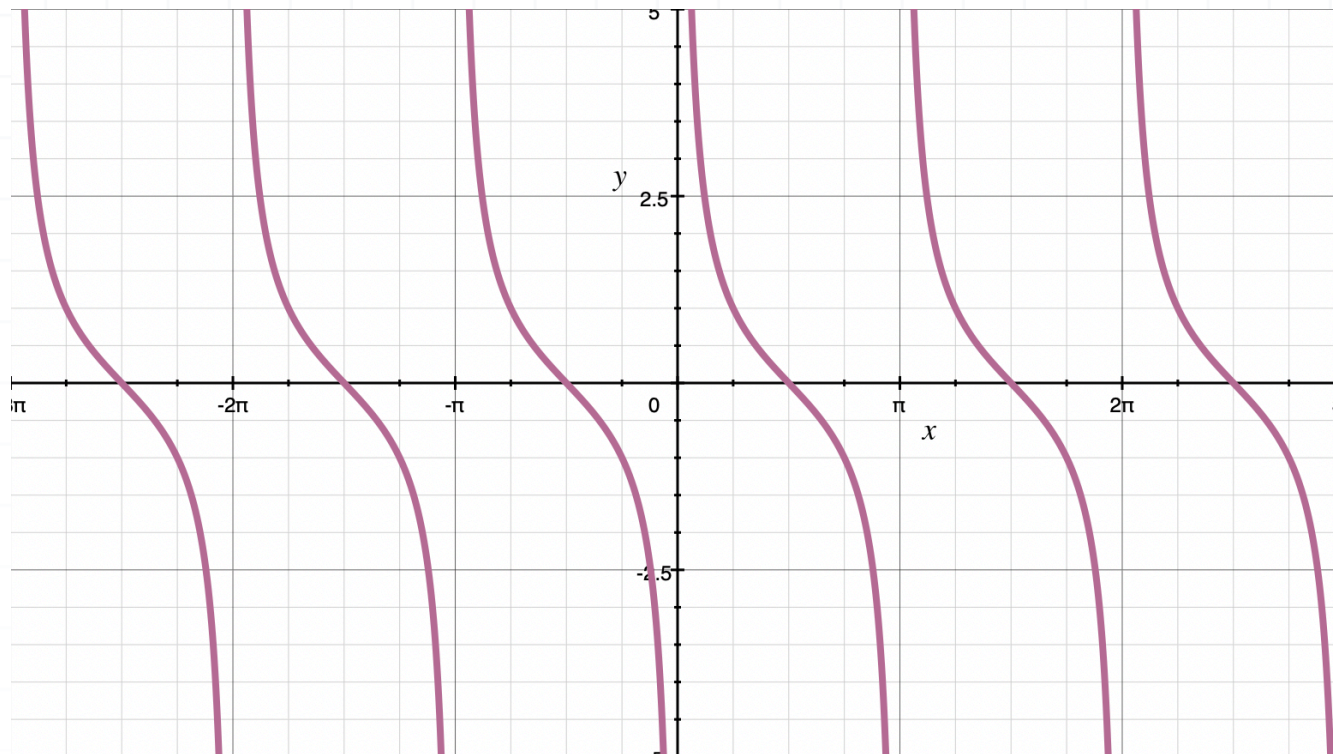
Solution:

Factor the function.

$$f(x) = \cot x = \frac{\cos x}{\sin x}$$



None of these factors cancel, which means that the denominator will be 0 whenever $\sin x = 0$. The value of $\sin x$ is 0 at all integer multiples of π , so $f(x)$ has an infinite discontinuity at three points in the interval $[0, 2\pi]$, which are at $x = 0$, $x = \pi$, and $x = 2\pi$.



ENDPOINT DISCONTINUITIES

- 1. What is the value of the limit on the interval $[0,3]$?

$$\lim_{x \rightarrow 3} -\sqrt{x+5}$$

Solution:

The limit does not exist because only the left-hand limit exists at $x = 3$. The right-hand limit does not exist, which means the one-sided limits are not equal.

$$\lim_{x \rightarrow 3^-} -\sqrt{x+5} = -2\sqrt{2} \quad \neq \quad \lim_{x \rightarrow 3^+} -\sqrt{x+5} = \text{DNE}$$

- 2. What is the value of the limit on the interval $[\pi, 2\pi]$?

$$\lim_{x \rightarrow \pi} \sin x$$

Solution:

The limit does not exist because only the right-hand limit exists at $x = \pi$. The left-hand limit does not exist, which means the one-sided limits are not equal.



$$\lim_{x \rightarrow \pi^+} \sin x = 0 \quad \neq \quad \lim_{x \rightarrow \pi^-} \sin x = \text{DNE}$$

- 3. What is the value of the limit on the interval $[4, \infty)$?

$$\lim_{x \rightarrow 4} - \frac{x + 7}{x^2 - 6x + 15}$$

Solution:

The limit does not exist because only the right-hand limit exists at $x = 4$. The left-hand limit does not exist, which means the one-sided limits are not equal.

$$\lim_{x \rightarrow 4^+} - \frac{x + 7}{x^2 - 6x + 15} = -\frac{11}{7} \quad \neq \quad \lim_{x \rightarrow 4^-} - \frac{x + 7}{x^2 - 6x + 15} = \text{DNE}$$

- 4. What is the value of the limit on the interval $[-9/2, 5/2]$?

$$\lim_{x \rightarrow \frac{5}{2}} \frac{x + 3}{x^2 + x + 1}$$

Solution:



The limit does not exist because only the left-hand limit exists at $x = 5/2$. The right-hand limit does not exist, which means the one-sided limits are not equal.

$$\lim_{x \rightarrow \frac{5}{2}^-} \frac{x+3}{x^2+x+1} = \frac{22}{39} \neq \lim_{x \rightarrow \frac{5}{2}^+} \frac{x+3}{x^2+x+1} = \text{DNE}$$

■ 5. What is the value of the limit on the interval $(-2, 2]$?

$$\lim_{x \rightarrow -2} \sqrt{2x+4}$$

Solution:

The limit does not exist because only the right-hand limit exists at $x = -2$. The left-hand limit does not exist, which means that the one-sided limits are not equal.

$$\lim_{x \rightarrow -2^+} \sqrt{2x+4} = 0 \neq \lim_{x \rightarrow -2^-} \sqrt{2x+4} = \text{DNE}$$

■ 6. What is the value of the limit on the interval $[-\pi, \pi]$?

$$\lim_{x \rightarrow \pi} -\frac{5 \cos x}{2}$$

Solution:



The limit does not exist because only the left-hand limit exists at $x = \pi$. The right-hand limit does not exist, which means the one-sided limits are not equal.

$$\lim_{x \rightarrow \pi^-} -\frac{5 \cos x}{2} = \frac{5}{2} \neq \lim_{x \rightarrow \pi^+} -\frac{5 \cos x}{2} = \text{DNE}$$



