Topic: Simpson's rule error bound

Question: Calculate area under the curve on the interval $[0,4]$. Then use Simpson's Rule with $n = 4$ to approximate area on the same interval and determine the error of the Simpson's Rule approximation.

$$f(x) = \frac{1}{2}(e^x + e^{-x})$$

Answer choices:

A	$A \approx 27.2899$	$\lvert E_S \rvert \approx 0.1354$
B	$A \approx 27.2899$	$\lvert E_S \rvert \approx 0.0192$
C	$A \approx 27.3091$	$\lvert E_S \rvert \approx 0.1354$
D	$A \approx 27.3091$	$\lvert E_S \rvert \approx 0.0192$

Solution: A

Find actual area under the curve on the interval $[0,4]$.

$$A=\int_0^4 \frac{1}{2}(e^x+e^{-x})\ dx$$

$$A=\frac{1}{2}\int_0^4 e^x+e^{-x}\ dx$$

$$A=\frac{1}{2}(e^x-e^{-x})\Big|_0^4$$

$$A=\frac{1}{2}\left[(e^4-e^{-4})-(e^0-e^{-0})\right]$$

$$A=\frac{1}{2}(e^4-e^{-4}-(1-1))$$

$$A=\frac{1}{2}\left(e^4-\frac{1}{e^4}\right)$$

$$A\approx 27.2899$$

With $n=4$,

$$\Delta x=\frac{b-a}{n}=\frac{4-0}{4}=\frac{4}{4}=1$$

The subinterval widths are all 1, so the boundaries of the subintervals are

$$x_0=0 \qquad x_1=1 \qquad x_2=2 \qquad x_3=3 \qquad x_4=4$$

Evaluate the function at each of these boundaries.

$$f(0) = \frac{1}{2}\left(e^0 + \frac{1}{e^0}\right) = 1$$

$$f(1) = \frac{1}{2}\left(e + \frac{1}{e}\right) \approx 1.5431$$

$$f(2) = \frac{1}{2}\left(e^2 + \frac{1}{e^2}\right) \approx 3.7622$$

$$f(3) = \frac{1}{2}\left(e^3 + \frac{1}{e^3}\right) \approx 10.0677$$

$$f(4) = \frac{1}{2}\left(e^4 + \frac{1}{e^4}\right) \approx 27.3082$$

Then Simpson's Rule approximates the area as

$$S_n = \frac{\Delta x}{3}\left[f(x_0) + 4f(x_1) + 2f(x_2) + \ \cdots \ + 2f(x_{x-2}) + 4f(x_{x-1}) + f(x_n)\right]$$

$$S_4 = \frac{\Delta x}{3}[f(0) + 4f(1) + 2f(2) + 4f(3) + f(4)]$$

$$S_n \approx \frac{1}{3}[1 + 4(1.5431) + 2(3.7622) + 4(10.0677) + 27.3082]$$

$$S_n \approx \frac{1}{3}(1 + 6.1724 + 7.5244 + 40.2708 + 27.3082)$$

$$S_n \approx \frac{1}{3}(82.2758)$$

$$S_n \approx 27.4253$$

The error is the actual area minus the estimated area, as an absolute value.

$$|E_S| \approx |27.2899 - 27.4253|$$

$$|E_S| \approx |-0.1354|$$

$$|E_S| \approx 0.1354$$

Topic: Simpson's rule error bound

Question: Calculate the error bound of Simpson's Rule for the function $f(x)$ with $n = 4$ on the interval $[0,4]$.

$$f(x) = \frac{1}{2}(e^x + e^{-x})$$

Answer choices:

A $|E_S| \le 0.3793$

B $|E_S| \le 0.0038$

C $|E_S| \le 1.0379$

D $|E_S| \le 0.6068$

Solution: D

To find the error bound of Simpson's Rule on the interval $[a, b]$, we use

$$|E_S| \le k\frac{(b-a)^5}{180n^4}$$

The value of k is the maximum absolute value of the function's fourth derivative in the interval, so to find k, we'll start by finding the function's fourth derivative.

$$f(x) = \frac{1}{2}(e^x + e^{-x})$$

$$f'(x) = \frac{1}{2}(e^x - e^{-x})$$

$$f''(x) = \frac{1}{2}(e^x + e^{-x})$$

$$f'''(x) = \frac{1}{2}(e^x - e^{-x})$$

$$f^{(4)}(x) = \frac{1}{2}(e^x + e^{-x})$$

To determine where the fourth derivative reaches its maximum absolute value, we'll find the derivative of the fourth derivative function,

$$f^{(5)}(x) = \frac{1}{2}(e^x - e^{-x})$$

and set it equal to 0 to find critical numbers.

$$\frac{1}{2}(e^x - e^{-x}) = 0$$

$$e^x - e^{-x} = 0$$

$$e^x = e^{-x}$$

$$x = 0$$

The only critical number is at the left edge of the interval $[0,4]$, so we only need to determine whether the fourth derivative function is increasing or decreasing in the interval, which we can do by evaluating the fifth derivative at a point in the interval. We'll choose $x = 1$.

$$f^{(5)}(1) = \frac{1}{2}(e^1 - e^{-1})$$

$$f^{(5)}(1) = \frac{1}{2}(e - e^{-1})$$

$$f^{(5)}(1) \approx 1.1752$$

Because we get a positive value, the fourth derivative function is increasing everywhere on $[0,4]$, which means the fourth derivative function will reach its maximum absolute value at the far right edge of the interval, at $x = 4$.

$$k = f^{(4)}(4) = \frac{1}{2}(e^4 + e^{-4})$$

$$k = f^{(4)}(4) \approx 27.3082$$

Then the error bound formula gives

$$|E_S| \le k\frac{(b-a)^5}{180n^4}$$

$$|E_S| \le 27.3082\left(\frac{(4-0)^5}{180(4)^4}\right)$$

$$|E_S| \le 27.3082\left(\frac{4^5}{180(4)^4}\right)$$

$$|E_S| \le 27.3082\left(\frac{4}{180}\right)$$

$$|E_S| \le 27.3082\left(\frac{1}{45}\right)$$

$$|E_S| \le 0.6068$$

Topic: Simpson's rule error bound

Question: Find the smallest value of n that keeps the Simpson's rule error bound within 0.00001 for the function $f(x)$ on the interval $[0,4]$.

$$f(x) = \frac{1}{2}(e^x + e^{-x})$$

Answer choices:

A $\quad n = 62$

B $\quad n = 63$

C $\quad n = 64$

D $\quad n = 62.7813$

Solution: C

To find the error bound of Simpson's Rule on the interval $[a, b]$, we use

$$|E_S| \leq k\frac{(b-a)^5}{180n^4}$$

The value of k is the maximum absolute value of the function's fourth derivative in the interval, so to find k, we'll start by finding the function's fourth derivative.

$$f(x) = \frac{1}{2}(e^x + e^{-x})$$

$$f'(x) = \frac{1}{2}(e^x - e^{-x})$$

$$f''(x) = \frac{1}{2}(e^x + e^{-x})$$

$$f'''(x) = \frac{1}{2}(e^x - e^{-x})$$

$$f^{(4)}(x) = \frac{1}{2}(e^x + e^{-x})$$

To determine where the fourth derivative reaches its maximum absolute value, we'll find the derivative of the fourth derivative function,

$$f^{(5)}(x) = \frac{1}{2}(e^x - e^{-x})$$

and set it equal to 0 to find critical numbers.

$$\frac{1}{2}(e^x - e^{-x}) = 0$$

$$e^x - e^{-x} = 0$$

$$e^x = e^{-x}$$

$$x = 0$$

The only critical number is at the left edge of the interval $[0,4]$, so we only need to determine whether the fourth derivative function is increasing or decreasing in the interval, which we can do by evaluating the fifth derivative at a point in the interval. We'll choose $x = 1$.

$$f^{(5)}(1) = \frac{1}{2}(e^1 - e^{-1})$$

$$f^{(5)}(1) = \frac{1}{2}(e - e^{-1})$$

$$f^{(5)}(1) \approx 1.1752$$

Because we get a positive value, the fourth derivative function is increasing everywhere on $[0,4]$, which means the fourth derivative function will reach its maximum absolute value at the far right edge of the interval, at $x = 4$.

$$k = f^{(4)}(4) = \frac{1}{2}(e^4 + e^{-4})$$

$$k = f^{(4)}(4) \approx 27.3082$$

Then the error bound formula gives

$$|E_S| \leq k\frac{(b-a)^5}{180n^4}$$

$$|E_S| \leq 27.3082\left(\frac{(4-0)^5}{180n^4}\right)$$

$$|E_S| \leq 27.3082\left(\frac{4^5}{180n^4}\right)$$

$$|E_S| \leq 27.3082\left(\frac{4^4}{45n^4}\right)$$

Since we want to bound the error at 0.00001,

$$27.3082\left(\frac{4^4}{45n^4}\right) \leq 0.00001$$

$$27.3082\left(\frac{4^4}{45}\right) \leq 0.00001n^4$$

$$\frac{27.3082}{0.00001}\left(\frac{4^4}{45}\right) \leq n^4$$

$$\sqrt[4]{\frac{27.3082}{0.00001}\left(\frac{4^4}{45}\right)} \leq n$$

$$62.7813 \leq n$$

Because n is the number of subintervals, n has to be a whole number, and because we're using Simpson's rule n has to be an *even* whole number. Therefore, $n = 64$.