Value that makes two tangent lines parallel

Now that we know how to find the equation of a tangent line, we can think about the relationship between multiple tangent lines of the same function.

For instance, we might be interested in the points at which two tangent lines of the function are parallel to one another. Remember that parallel lines have the same slope, so two parallel tangent lines will have the same slope.

To find the slope of a tangent line of a function, we differentiate the function, and then evaluate it at the point of tangency.

Example

Find the value of a such that the tangent lines of $f(x)$ at $x = a$ and $x = a + 1$ are parallel.

$$f(x) = x^3 + 2x^2 - 3x + 1$$

If the tangent lines are parallel, they must have the same slope. So, if two tangent lines at $x = a$ and $x = a + 1$ are parallel, it means their slopes are equal, which means the value of the function's derivative will be equal at $x = a$ and $x = a + 1$.

So we'll start by finding the derivative of $f(x)$.

$$f'(x) = 3x^2 + 4x - 3$$

Now we'll plug both $x = a$ and $x = a + 1$ into the derivative.

$$f'(a) = 3a^2 + 4a - 3$$

$$f'(a + 1) = 3(a + 1)^2 + 4(a + 1) - 3$$

These represent the slope of each tangent line, so we'll set them equal to one another.

$$3a^2 + 4a - 3 = 3(a + 1)^2 + 4(a + 1) - 3$$

$$3a^2 + 4a - 3 = 3(a^2 + 2a + 1) + 4a + 4 - 3$$

$$3a^2 + 4a - 3 = 3a^2 + 6a + 3 + 4a + 4 - 3$$

Collect like terms and solve for a.

$$4a - 3 = 6a + 3 + 4a + 4 - 3$$

$$4a - 3 = 6a + 4a + 4$$

$$4a - 3 = 10a + 4$$

$$-3 = 6a + 4$$

$$-7 = 6a$$

$$a = -\frac{7}{6}$$

If this is the value of a, then $a + 1$ is

$$a + 1 = -\frac{7}{6} + 1$$

$$a + 1 = -\frac{7}{6} + \frac{6}{6}$$

$$a + 1 = -\frac{1}{6}$$

Therefore, the function has parallel tangent lines one unit apart at $x = -7/6$ and $x = -1/6$.

Let's do another example where we find both tangent line equations, instead of just the points at which the tangent lines are parallel.

Example

Find the equations of the tangent lines to $f(x)$ at $x = a$ and $x = a + 2$, if those tangent lines are parallel.

$$f(x) = 2x^3 + 8x - 2$$

We'll start by finding the derivative of $f(x)$.

$$f'(x) = 6x^2 + 8$$

Now we'll plug both $x = a$ and $x = a + 2$ into the derivative.

$$f'(a) = 6a^2 + 8$$

$$f'(a + 2) = 6(a + 2)^2 + 8$$

These represent the slope of each tangent line, so we'll set them equal to one another.

$$6a^2 + 8 = 6(a+2)^2 + 8$$

$$6a^2 + 8 = 6(a^2 + 4a + 4) + 8$$

$$6a^2 + 8 = 6a^2 + 24a + 24 + 8$$

Collect like terms and solve for a.

$$8 = 24a + 24 + 8$$

$$0 = 24a + 24$$

$$24a = -24$$

$$a = -1$$

If this is the value of a, then $a + 2$ is

$$a + 2 = -1 + 2$$

$$a + 2 = 1$$

Therefore, the function has parallel tangent lines two units apart at $x = -1$ and $x = 1$.

Now that we know where the tangent lines are located, we can find their equations. For the tangent line at $a = -1$, we'll need $f(a)$ and $f'(a)$.

$$f(x) = 2x^3 + 8x - 2$$

$$f(-1) = 2(-1)^3 + 8(-1) - 2$$

$$f(-1) = 2(-1) - 8 - 2$$

$$f(-1) = -2 - 8 - 2$$

$$f(-1) = -12$$

and

$$f'(a) = 6a^2 + 8$$

$$f'(-1) = 6(-1)^2 + 8$$

$$f'(-1) = 6 + 8$$

$$f'(-1) = 14$$

Then the equation of the tangent line at $a = -1$ is

$$y = f(a) + f'(a)(x - a)$$

$$y = -12 + 14(x - (-1))$$

$$y = -12 + 14(x + 1)$$

$$y = -12 + 14x + 14$$

$$y = 14x + 2$$

For the tangent line at $a = 1$, we'll need $f(a)$ and $f'(a)$.

$$f(x) = 2x^3 + 8x - 2$$

$$f(1) = 2(1)^3 + 8(1) - 2$$

$$f(1) = 2 + 8 - 2$$

$$f(1) = 8$$

and

$$f'(a) = 6a^2 + 8$$

$$f'(1) = 6(1)^2 + 8$$

$$f'(1) = 6 + 8$$

$$f'(1) = 14$$

Then the equation of the tangent line at $a = 1$ is

$$y = f(a) + f'(a)(x - a)$$

$$y = 8 + 14(x - 1)$$

$$y = 8 + 14x - 14$$

$$y = 14x - 6$$

So the equations of the parallel tangent lines at $x = -1$ and $x = 1$ are

$$y = 14x + 2$$

$$y = 14x - 6$$

Remember that you can always double-check your answers to problems like these by graphing the given function and the two tangent line equations, to verify visually that the tangent lines look parallel.