Integration by parts

Unlike differentiation, integration is not always straightforward and we can't always express the integral of every function in terms of neat and clean elementary functions.

When your integral is too complicated to solve without a fancy technique and you've ruled out u-substitution, integration by parts should be your next approach for evaluating your integral. If you remember that the product rule was your method for finding derivatives of functions that were multiplied together, you can think about integration by parts as the method often used for *integrating* functions that are multiplied together.

Suppose you want to integrate the following

$$\int xe^{-x}\ dx$$

How can you integrate the above expression quickly and easily? You can't, unless you're a super human genius. But hopefully you can recognize that you have two functions multiplied together inside of this integral, one being x and the other being e^{-x}. If you try u-substitution, you won't find anything to cancel in your integral, and you'll be no better off, which means that your next step should be an attempt at integrating with our new method, integration by parts.

The formula we'll use is derived by integrating the product rule formula, and looks like this:

$$\int u\ dv = uv - \int v\ du$$

In the formula above, everything to the left of the equals sign represents your original function, which means your original function must be composed of u and dv. Your job is to identify which part of your original function will be u, and which will be dv.

My favorite technique for picking u and dv is to assign u to the function in your integral whose derivative is simpler than the original u. Consider again the example from earlier:

$$\int xe^{-x}\ dx$$

I would assign u to x, because the derivative of x is 1, which is much simpler than x. If you have $\ln x$ in your integral, that's usually a good bet for u because the derivative of $\ln x$ is $1/x$; much simpler than $\ln x$. Once you pick which of your functions will be represented by u, the rest is easy because you know that the other function will be represented by dv.

Using this formula can be challenging for a lot of students, but the hardest part is identifying which of your two functions will be u and which will be dv. That's the very first thing you have to tackle with integration by parts, so once you get that over with, you'll be home free.

After completing this first crucial step, you take the derivative of u, called du, and the integral of dv, which will be v. Now that you have u, du, v and dv, you can plug all of your components into the right side of the integration by parts formula. Everything to the right of the equals sign will be part of your answer. If you've correctly assigned u and dv, the integral on the right should now be much easier to integrate.

Example

Evaluate the integral.

$$\int xe^{-x}\ dx$$

Our integral is comprised of two functions, x and e^{-x}. One of them must be u and the other dv. Since the derivative of x is 1, which is much simpler than the derivative of e^{-x}, we'll assign u to x.

$u = x$ → differentiate → $du = 1\ dx$

$dv = e^{-x}\ dx$ → integrate → $v = -e^{-x}$

Plugging all four components into the right side of our formula gives the following transformation of our original function:

$$(x)(-e^{-x}) - \int (-e^{-x})(1\ dx)$$

$$-xe^{-x} + \int e^{-x}\ dx$$

Now that we have something we can work with, we integrate.

$$-xe^{-x} + (-e^{-x}) + C$$

The answer is therefore

$$-xe^{-x} - e^{-x} + C$$

Or factored, we have

$$-e^{-x}(x+1) + C$$