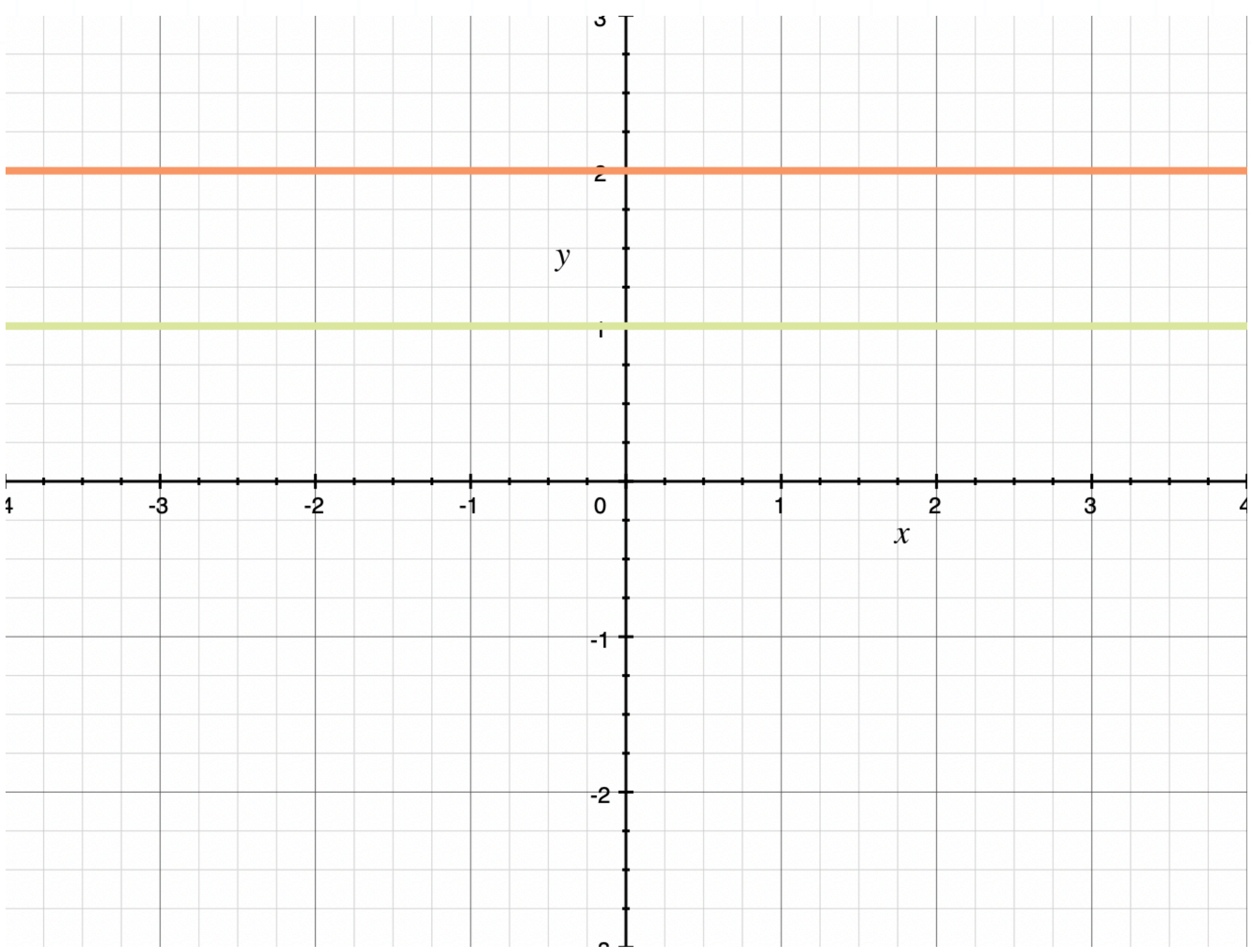
Area between curves

Based on the name of this application, finding area between curves is exactly what you'd expect it to be.

In these kinds of problems, you'll be given the equations of two curves, and asked to find the area between them. Finding the area is always going to require three pieces of information:

1. The orientation of your curves.

    a. Thinking broadly here, do they look more like this, where one curve is higher, and the other is lower?

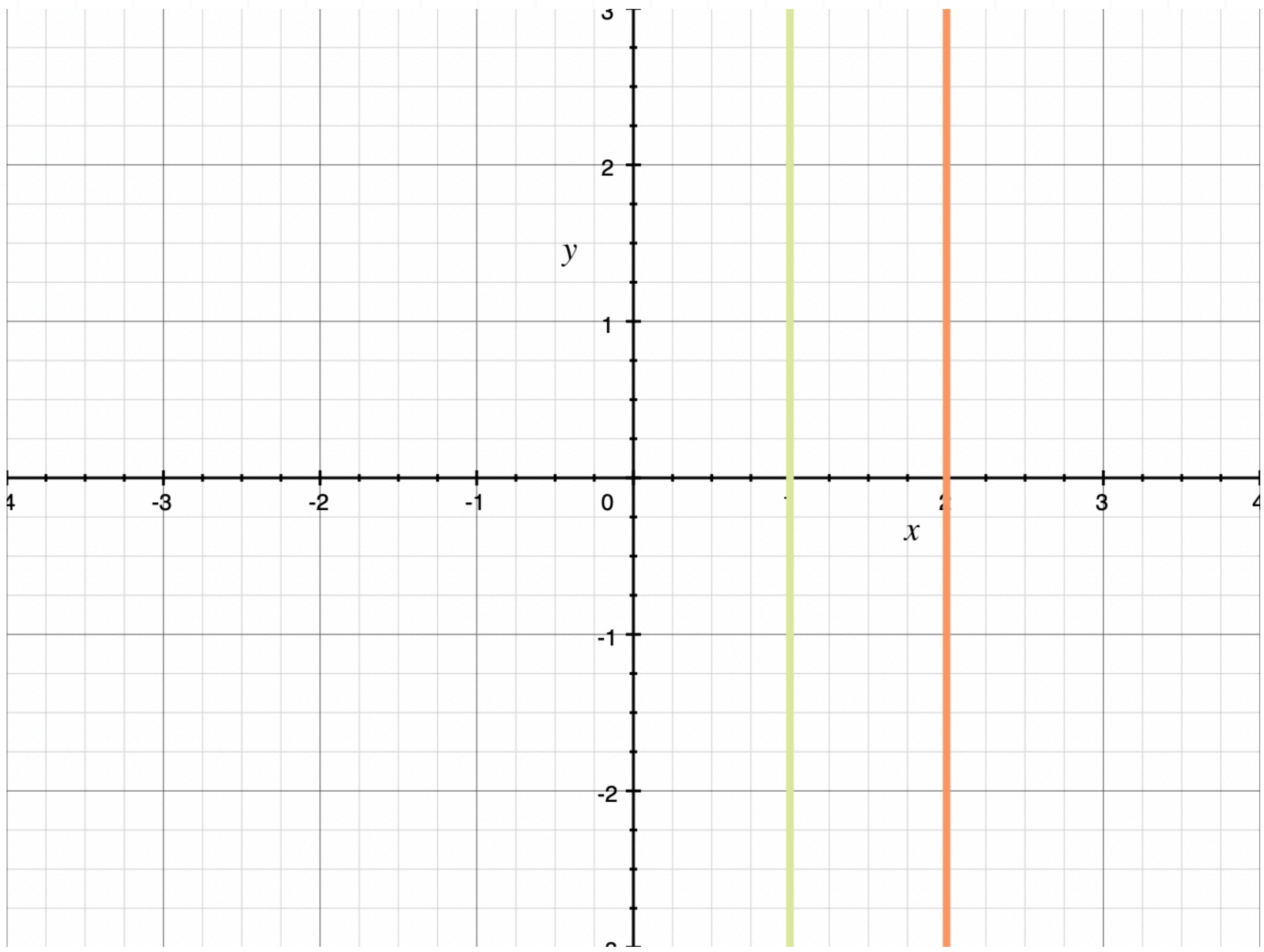


If so, we'll use the the area formula,

$$\textbf{[A]} \quad A = \int_{x=a}^{x=b} f(x) - g(x)\ dx$$

where $[a, b]$ is the interval on which we'll find area, $f(x)$ is the "higher" function, and $g(x)$ is the "lower" function.

b. Or do they look more like this, where one curve is on the left and the other is on the right?



If so, we'll use the area formula,

$$\textbf{[B]} \quad A = \int_{y=a}^{y=b} f(y) - g(y)\ dy$$

where $[a, b]$ is the interval on which we'll find area, $f(y)$ is the "right" function, and $g(y)$ is the "left" function.

2. The points where the curves intersect each other, or a given interval on which to evaluate area.

   a. If your problem just says "Find the area between the curves", but doesn't specify an interval, then you need to find the points where the curves intersect each other. Those points of intersection become your interval, $[a, b]$.

   b. If your problem says "Find the area between the curves on the interval $[a, b]$", then you need to check to see whether the curves intersect each other inside the given interval (more on this later). Points of intersection outside the given interval can be ignored.

3. Which curve is higher and which is lower, or which is on the left and which is on the right.

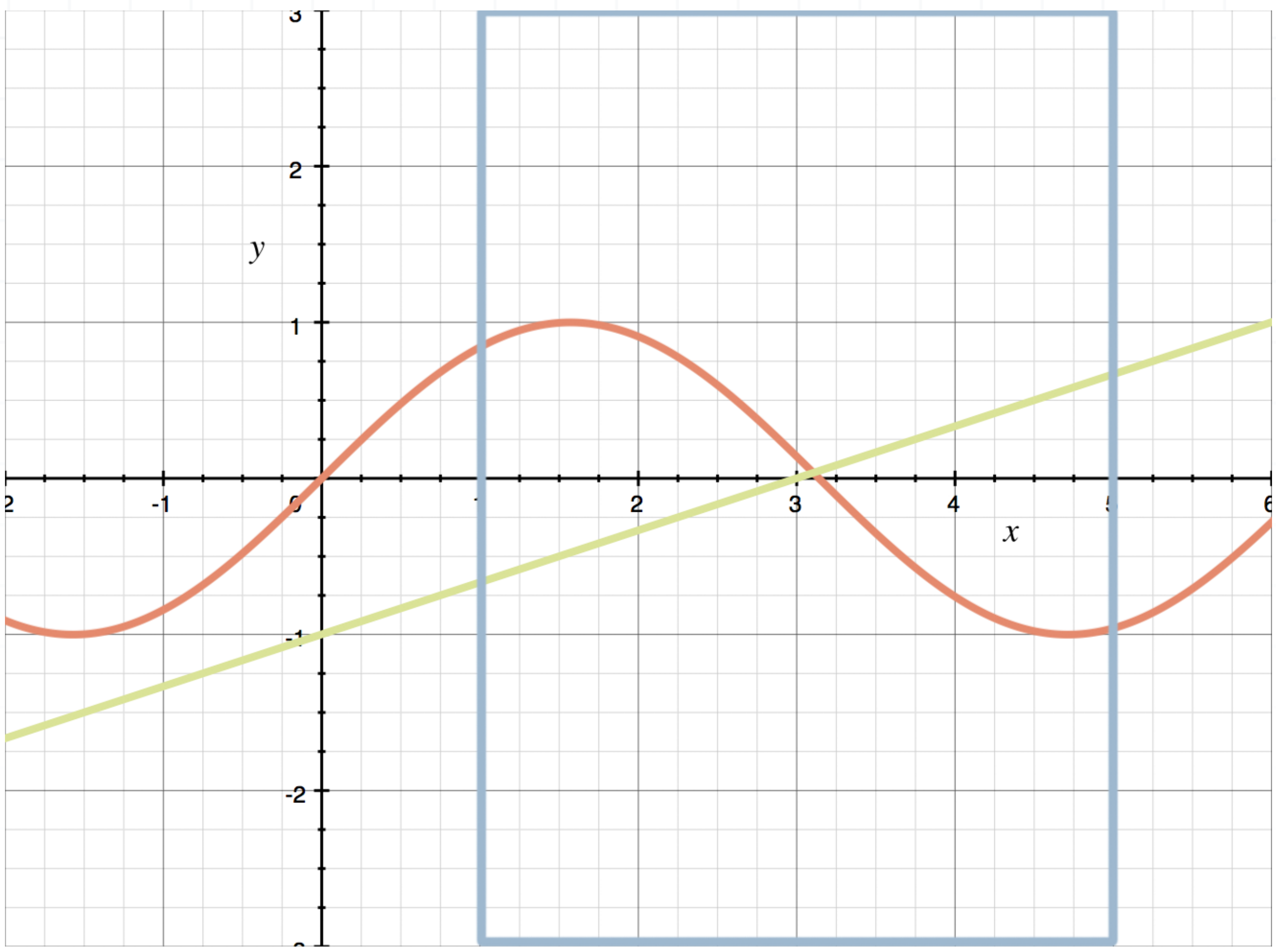
   a. If the orientation of your curves is "higher-lower", then you need to figure out which curve is higher and which curve is lower in the given interval.

   b. If the orientation of your curves is "left-right", then you need to figure out which curve is on the left and which is on the right in the given interval.

   c. Note: As we mentioned in (2b) above, if you have a point of intersection inside the given interval, the curves cross each other. Therefore, which curve is higher/lower or on-the-left/on-the-right will switch at the point of intersection.

1) If you have a higher/lower switch, you'll use the following area formula instead of [A]

[C] $$A=\int_{x=a}^{x=c} f(x)-g(x)\ dx+\int_{x=c}^{x=b} g(x)-f(x)\ dx$$



On the interval $[1,5]$, the sine function is higher in the first half of the interval than the linear function. Around $x=3$, the curves cross each other; the line becomes the higher function and the sine curve becomes lower function.

2) If you have a left/right switch, you'll use the following area formula instead of [B]

[D] $$A=\int_{y=a}^{y=c} f(y)-g(y)\ dy+\int_{y=c}^{y=b} g(y)-f(y)\ dy$$

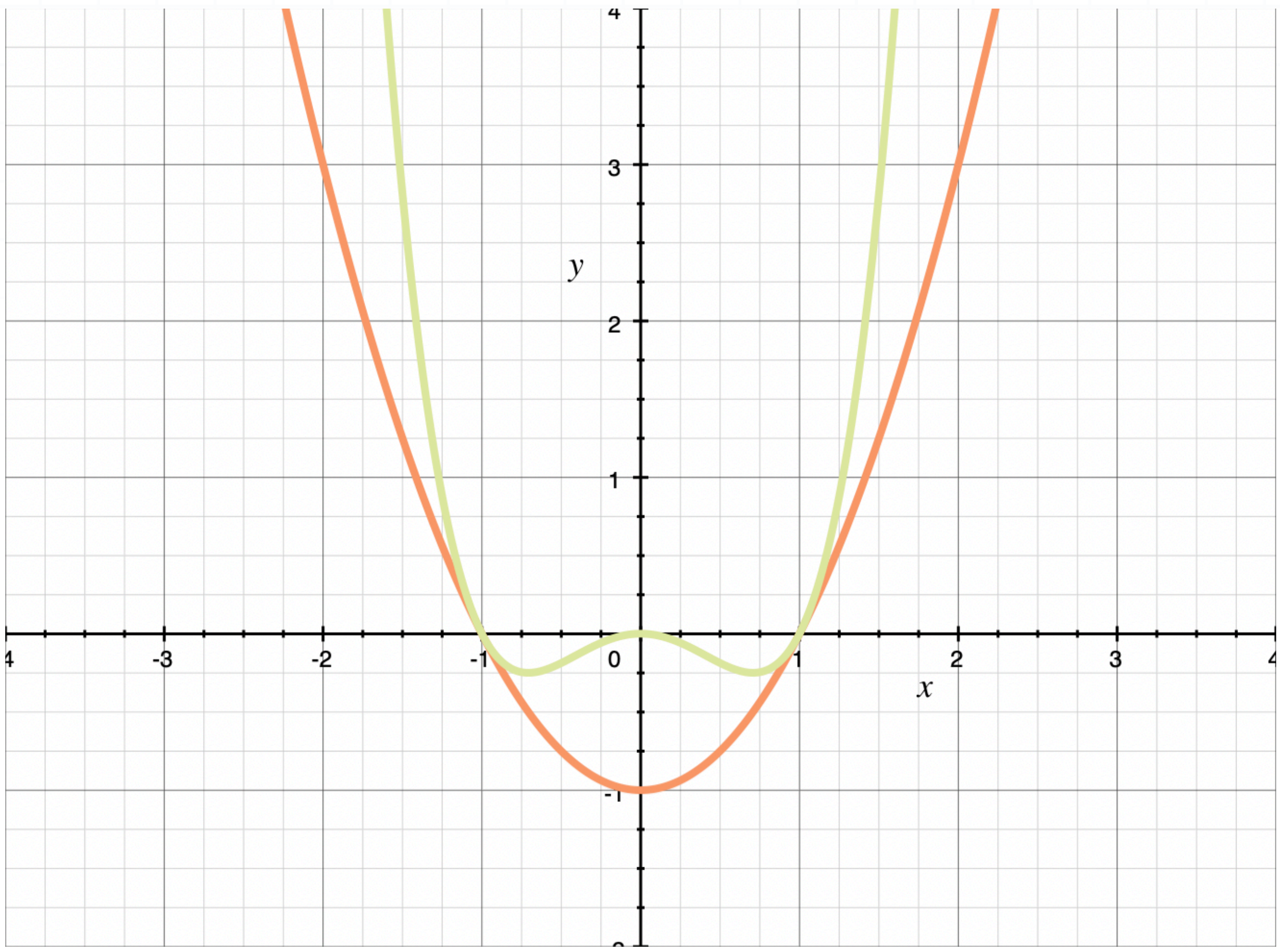
Example

Find the area between the curves.

$$y = x^2 - 1$$

$$y = x^4 - x^2$$

First, let's graph the functions to see their general orientation.



These look like higher-lower curves, with $y = x^4 - x^2$ being the "higher" function and $y = x^2 - 1$ being the "lower" function.

This means we'll be using formula **[A]** to find area.

$$A = \int_{x=a}^{x=b} f(x) - g(x)\ dx$$

Because the question didn't identify an interval, the second step is to find points of intersection. Since both curves are defined for y, we can set them equal to one another and then solve for x.

$$x^2 - 1 = x^4 - x^2$$

$$x^4 - 2x^2 + 1 = 0$$

$$\left(x^2 - 1\right)^2 = 0$$

$$x^2 - 1 = 0$$

$$x^2 = 1$$

$$x = \pm 1$$

Because the points of intersection are $x = -1$ and $x = 1$, we know we're trying to find the area enclosed by the curves between those points.

The third step is to figure out which curve is higher and which is lower between the points of intersection. Since there are only two points of intersection, $x = \pm 1$, and these are the endpoints of the interval, we know that there is no third point of intersection inside the interval where the curves cross each other. Which means that, across the full interval, one function will always be higher than the other one.

In order to figure out which one is higher, we'll pick an x-value between the points of intersection and plug it into the equations of the original functions. We'll pick $x=0$ since it lies between the points of intersection, $x=-1$ and $x=1$.

Plugging $x=0$ into $y=x^2-1$, we get

$$y=(0)^2-1$$

$$y=-1$$

Plugging $x=0$ into $y=x^4-x^2$, we get

$$y=(0)^4-(0)^2$$

$$y=0$$

Because $0>-1$, we know that $x^4-x^2>x^2-1$ between the points of intersection. Plugging this information into **[A]**, we get

$$A=\int_{-1}^{1}\left(x^4-x^2\right)-\left(x^2-1\right)\ dx$$

$$A=\int_{-1}^{1}x^4-2x^2+1\ dx$$

$$A=\frac{1}{5}x^5-\frac{2}{3}x^3+x\ \Big|_{-1}^{1}$$

$$A=\left[\frac{1}{5}(1)^5-\frac{2}{3}(1)^3+(1)\right]-\left[\frac{1}{5}(-1)^5-\frac{2}{3}(-1)^3+(-1)\right]$$

$$A=\left(\frac{1}{5}-\frac{2}{3}+1\right)-\left(-\frac{1}{5}+\frac{2}{3}-1\right)$$

$$A=\frac{1}{5}-\frac{2}{3}+1+\frac{1}{5}-\frac{2}{3}+1$$

$$A=\frac{2}{5}-\frac{4}{3}+2$$

$$A=\frac{6}{15}-\frac{20}{15}+\frac{30}{15}$$

$$A=\frac{16}{15}\text{ square units}$$

Let's look at another example where the orientation of the curves is left-right instead of higher-lower.

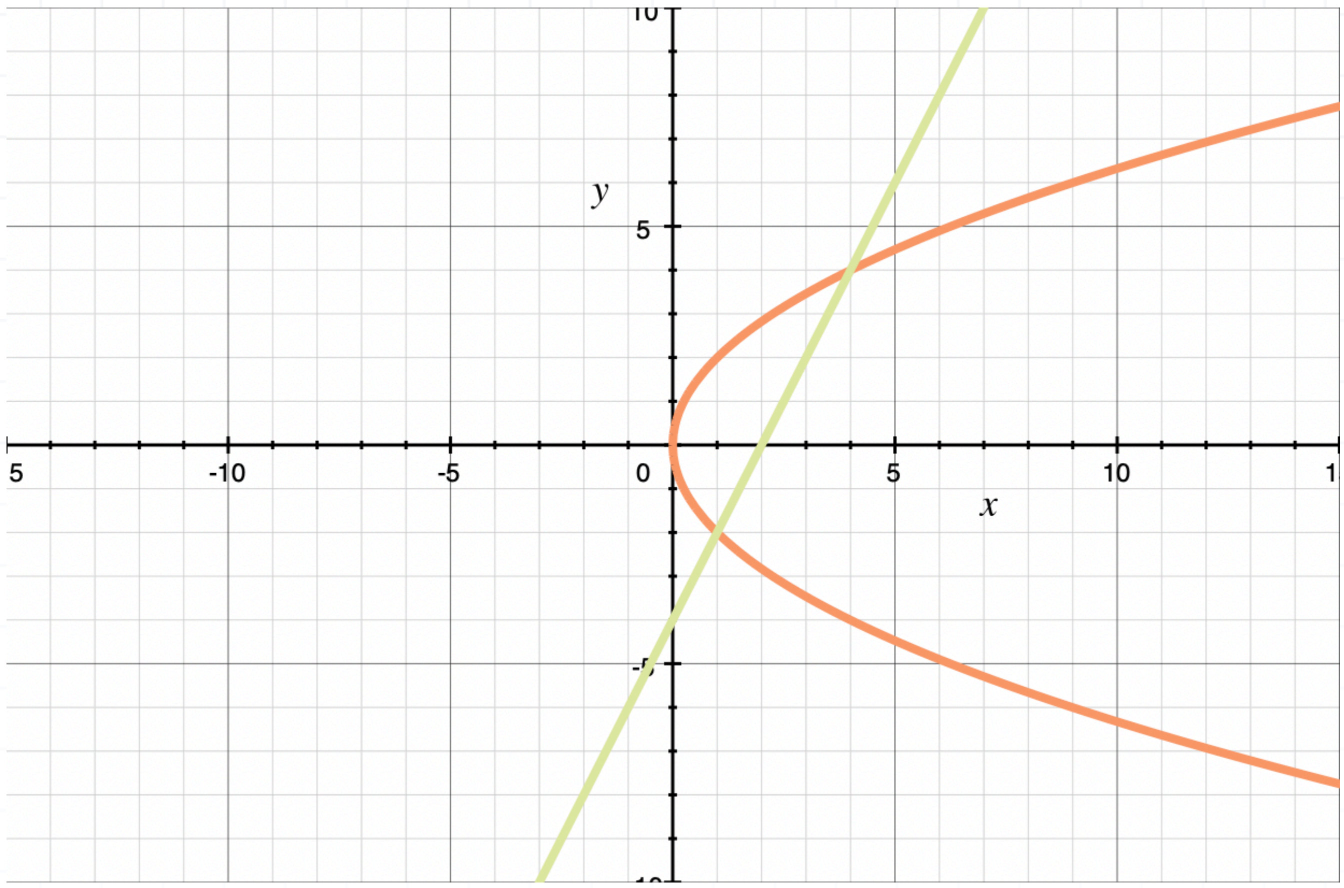
Example

Find the area between the curves.

$$y^2=4x$$

$$y=2x-4$$

First, let's graph the functions to see their general orientation.



These look like left-right curves, with $y = 2x - 4$ being the "right" curve and $y^2 = 4x$ being the "left" curve.

This means we'll be using formula **[B]** to find area.

$$A = \int_{y=a}^{y=b} f(y) - g(y)\ dy$$

Because the question didn't identify an interval, the second step is to find points of intersection. We'll solve both equations for x so that we can set them equal to one another and then solve for y to get points of intersection.

The first equation becomes

$$y^2 = 4x$$

[1] $x=\frac{y^2}{4}$

The second equation becomes

$$y=2x-4$$

[2] $x=\frac{y+4}{2}$

Setting them equal to each other gives

$$\frac{y^2}{4}=\frac{y+4}{2}$$

$$2y^2=4(y+4)$$

$$2y^2=4y+16$$

$$y^2=2y+8$$

$$y^2-2y-8=0$$

$$(y+2)(y-4)=0$$

$y=-2$ and $y=4$

Because the points of intersection are $y=-2$ and $y=4$, we know we're trying to find the area enclosed by the curves between those points.

The third step is to figure out which curve is on the right and which is on the left between the points of intersection. Since there are only two points of intersection, $y=-2$ and $y=4$, and these are the endpoints of the interval, we know that there is no third point of intersection inside the interval where the curves cross each other. Which means that, across the full interval, one function will always be on the right and the other will always be on the left.

In order to figure out which one is on the right, we'll pick a y-value between the points of intersection and plug it into [1] and [2]. We'll pick $y = 0$ since it lies between the points of intersection, $y = -2$ and $y = 4$.

Plugging $y = 0$ into [1], we get

$$x = \frac{(0)^2}{4}$$

$$x = 0$$

Plugging $y = 0$ into [2], we get

$$x = \frac{(0) + 4}{2}$$

$$x = 2$$

Because $2 > 0$, we know that [2]>[1] between the points of intersection. Plugging this information into [B], we get

$$A = \int_{-2}^{4} \frac{y + 4}{2} - \frac{y^2}{4} \ dy$$

$$A = \int_{-2}^{4} \frac{2(y + 4)}{4} - \frac{y^2}{4} \ dy$$

$$A = \frac{1}{4} \int_{-2}^{4} 2(y + 4) - y^2 \ dy$$

$$A=\frac{1}{4}\int_{-2}^{4}2y+8-y^2\ dy$$

$$A=\frac{1}{4}\left(y^2+8y-\frac{1}{3}y^3\right)\Big|_{-2}^{4}$$

$$A=\frac{1}{4}\left[(4)^2+8(4)-\frac{1}{3}(4)^3\right]-\frac{1}{4}\left[(-2)^2+8(-2)-\frac{1}{3}(-2)^3\right]$$

$$A=\frac{1}{4}\left[\left(16+32-\frac{64}{3}\right)-\left(4-16+\frac{8}{3}\right)\right]$$

$$A=\frac{1}{4}\left(16+32-\frac{64}{3}-4+16-\frac{8}{3}\right)$$

$$A=\frac{1}{4}\left(\frac{48}{3}+\frac{96}{3}-\frac{64}{3}-\frac{12}{3}+\frac{48}{3}-\frac{8}{3}\right)$$

$$A=\frac{1}{4}\left(\frac{108}{3}\right)$$

$$A=\frac{108}{12}$$

$A=9$ square units