Calculus 2
Final Exam Solutions

krista king
MATH

Calculus 2 Final Exam Answer Key

1. (5 pts)	A	B	■	D	E
2. (5 pts)	A	B	C	D	■
3. (5 pts)	■	B	C	D	E
4. (5 pts)	A	B	C	■	E
5. (5 pts)	A	B	C	■	E
6. (5 pts)	A	■	C	D	E
7. (5 pts)	A	B	■	D	E
8. (5 pts)	A	■	C	D	E

9. (15 pts) $\ln\left|\sqrt{x^2-4x+5}\right|+C$

10. (15 pts) $V=\dfrac{168\pi}{5}$

11. (15 pts) $A=\pi+8$

12. (15 pts) $R=\dfrac{1}{2}$

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1. C. Before we can integrate, we need rewrite the integral by dividing each term in the numerator by the denominator.

$$\int \frac{4x^4 - 2x^2 - 5x}{x^2}\ dx$$

$$\int \frac{4x^4}{x^2} - \frac{2x^2}{x^2} - \frac{5x}{x^2}\ dx$$

$$\int 4x^2 - 2 - \frac{5}{x}\ dx$$

Integrate one term at a time.

$$\frac{4}{3}x^3 - 2x - 5\ln|x| + C$$

2. E. Use u-substitution, letting

$$u = x^3 + 3x$$

$$du = 3x^2 + 3\ dx$$

$$dx = \frac{du}{3(x^2 + 1)}$$

Substituting into the integral gives

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$$\int_0^2 (x^2+1)\sin(x^3+3x)\ dx$$

$$\int_{x=0}^{x=2} (x^2+1)\sin u \left(\frac{du}{3(x^2+1)}\right)$$

$$\frac{1}{3}\int_{x=0}^{x=2} \sin u\ du$$

Integrate, then back-substitute for u.

$$\frac{1}{3}(-\cos u)\Big|_{x=0}^{x=2}$$

$$-\frac{1}{3}\cos(x^3+3x)\Big|_0^2$$

Evaluate over the interval.

$$-\frac{1}{3}\cos(2^3+3(2)) - \left(-\frac{1}{3}\cos(0^3+3(0))\right)$$

$$-\frac{1}{3}\cos(14) + \frac{1}{3}\cos(0)$$

$$-\frac{1}{3}\cos(14) + \frac{1}{3}(1)$$

$$-\frac{1}{3}\cos(14) + \frac{1}{3}$$

$$\frac{1}{3} - \frac{1}{3}\cos(14)$$

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$$\frac{1}{3}(1 - \cos(14))$$

3. A. To find the average value of a function over a given interval, we use the integration formula

$$f_{avg} = \frac{1}{b-a}\int_a^b f(x)\ dx$$

where $f(x)$ is the function for which we want the average, and $[a, b]$ is the interval we're interested in. Plugging the given function and the interval into the formula, we get

$$f_{avg} = \frac{1}{3-1}\int_1^3 8x^3 + 4x\ dx$$

$$f_{avg} = \frac{1}{2}\int_1^3 8x^3 + 4x\ dx$$

Integrate, then evaluate over the interval.

$$f_{avg} = \frac{1}{2}\left(\frac{8x^4}{4} + \frac{4x^2}{2}\right)\Big|_1^3$$

$$f_{avg} = \frac{1}{2}(2x^4 + 2x^2)\Big|_1^3$$

$$f_{avg} = x^4 + x^2\Big|_1^3$$

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$$f_{avg} = (3^4 + 3^2) - (1^4 + 1^2)$$

$$f_{avg} = (81 + 9) - (1 + 1)$$

$$f_{avg} = 90 - 2$$

$$f_{avg} = 88$$

4. D. We'll separate a single sine factor and then replace the remaining sine factors using the Pythagorean identity $\sin^2 x = 1 - \cos^2 x$.

$$\int \sin^5(5x)\cos^2(5x)\ dx$$

$$\int \sin(5x)\sin^4(5x)\cos^2(5x)\ dx$$

$$\int \sin(5x)(\sin^2(5x))^2\cos^2(5x)\ dx$$

$$\int \sin(5x)(1 - \cos^2(5x))^2\cos^2(5x)\ dx$$

Using u-substitution with $u = \cos(5x)$, we get

$$u = \cos(5x)$$

$$du = -5\sin(5x)\ dx$$

$$\sin(5x)\ dx = \frac{du}{-5}$$

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Substitute into the integral.

$$\int \sin(5x)(1-u^2)^2u^2\ dx$$

$$\int (1-u^2)^2u^2\left(\frac{du}{-5}\right)$$

$$-\frac{1}{5}\int (1-2u^2+u^4)u^2\ du$$

$$-\frac{1}{5}\int u^2-2u^4+u^6\ du$$

$$-\frac{1}{5}\left(\frac{1}{3}u^3-\frac{2}{5}u^5+\frac{1}{7}u^7\right)+C$$

Back-substituting for u, we get

$$-\frac{1}{5}\left(\frac{1}{3}\cos^3(5x)-\frac{2}{5}\cos^5(5x)+\frac{1}{7}\cos^7(5x)\right)+C$$

$$-\frac{1}{15}\cos^3(5x)+\frac{2}{25}\cos^5(5x)-\frac{1}{35}\cos^7(5x)+C$$

5. D. If we rewrite answer choice D as

$$\sum_{n=1}^{\infty}\frac{1}{\sqrt[3]{n^5}}=\sum_{n=1}^{\infty}\frac{1}{(n^5)^{\frac{1}{3}}}=\sum_{n=1}^{\infty}\frac{1}{n^{\frac{5}{3}}}$$

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then we see that it's a p-series, so we can use a p-series test with $p = 5/3$. The p-series test tells us that a_n converges when $p > 1$, so we can say that this series converges.

6. B. With particle motion, acceleration, velocity, and position have the following relationships.

$$a(t) = v'(t) = x''(t)$$

$$v(t) = x'(t)$$

Which means that the given acceleration function $a(t) = 8t - 3$ is the second derivative of the position function $x(t)$ that we need to find. Integrate $a(t)$ to find $v(t)$.

$$v(t) = \int a(t)\ dt = \int 8t - 3\ dt$$

$$v(t) = 4t^2 - 3t + C$$

Now we'll use the initial condition for the velocity function $v(1) = 5$ to find the value of C.

$$5 = 4(1)^2 - 3(1) + C$$

$$5 = 4 - 3 + C$$

$$5 = 1 + C$$

$$4 = C$$

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So the velocity function is

$$v(t) = 4t^2 - 3t + 4$$

To find $x(t)$, we'll integrate the velocity function we just found.

$$x(t) = \int v(t)\ dt = \int 4t^2 - 3t + 4\ dt$$

$$x(t) = \frac{4}{3}t^3 - \frac{3}{2}t^2 + 4t + D$$

Now we'll use the initial condition for the position function $x(0) = 7$ to find the value of D.

$$7 = \frac{4}{3}(0)^3 - \frac{3}{2}(0)^2 + 4(0) + D$$

$$7 = D$$

So the position function is

$$x(t) = \frac{4}{3}t^3 - \frac{3}{2}t^2 + 4t + 7$$

7. C. In order to find the slope of the tangent line, we need to first find $dr/d\theta$.

$$r = 5\sin 2\theta$$

$$\frac{dr}{d\theta} = 10\cos 2\theta$$

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Plugging $dr/d\theta$ and the given polar equation $r = 5\sin 2\theta$ into the formula for the slope, then evaluating at $\theta = \pi/4$, we get

$$m = \frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta}$$

$$m = \frac{10\cos 2\theta\sin\theta + 5\sin 2\theta\cos\theta}{10\cos 2\theta\cos\theta - 5\sin 2\theta\sin\theta}$$

$$m = \frac{10\cos 2(\frac{\pi}{4})\sin(\frac{\pi}{4}) + 5\sin 2(\frac{\pi}{4})\cos(\frac{\pi}{4})}{10\cos 2(\frac{\pi}{4})\cos(\frac{\pi}{4}) - 5\sin 2(\frac{\pi}{4})\sin(\frac{\pi}{4})}$$

$$m = \frac{10\cos(\frac{\pi}{2})\sin(\frac{\pi}{4}) + 5\sin(\frac{\pi}{2})\cos(\frac{\pi}{4})}{10\cos(\frac{\pi}{2})\cos(\frac{\pi}{4}) - 5\sin(\frac{\pi}{2})\sin(\frac{\pi}{4})}$$

$$m = \frac{10(0)\left(\frac{\sqrt{2}}{2}\right) + 5(1)\left(\frac{\sqrt{2}}{2}\right)}{10(0)\left(\frac{\sqrt{2}}{2}\right) - 5(1)\left(\frac{\sqrt{2}}{2}\right)}$$

$$m = \frac{\frac{5\sqrt{2}}{2}}{-\frac{5\sqrt{2}}{2}}$$

$$m = \frac{5\sqrt{2}}{2}\left(-\frac{2}{5\sqrt{2}}\right)$$

$$m = -1$$

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To find (x_1, y_1), we'll plug $\theta = \pi/4$ and $r = 5\sin 2\theta$ into the conversion formulas.

$$x = r\cos\theta$$

$$x = 5\sin 2\theta\cos\theta$$

$$x_1 = 5\sin\left(\frac{\pi}{2}\right)\cos\left(\frac{\pi}{4}\right)$$

$$x_1 = 5(1)\left(\frac{\sqrt{2}}{2}\right)$$

$$x_1 = \frac{5\sqrt{2}}{2}$$

and

$$y = r\sin\theta$$

$$y = 5\sin 2\theta\sin\theta$$

$$y_1 = 5\sin\left(\frac{\pi}{2}\right)\sin\left(\frac{\pi}{4}\right)$$

$$y_1 = 5(1)\left(\frac{\sqrt{2}}{2}\right)$$

$$y_1 = \frac{5\sqrt{2}}{2}$$

Plugging m and (x_1, y_1) into the point-slope formula for the equation of a line, we get

$$y - y_1 = m(x - x_1)$$

$$y - \frac{5\sqrt{2}}{2} = -1\left(x - \frac{5\sqrt{2}}{2}\right)$$

$$y = -x + \frac{5\sqrt{2}}{2} + \frac{5\sqrt{2}}{2}$$

$$y = -x + 5\sqrt{2}$$

8. B. To find the area under one arc or loop of a parametric curve, we'll need to use

$$A = \int_a^b y(t)x'(t)\ dt$$

where $[a, b]$ is the interval that contains the loop and $x'(t)$ is the derivative of $x(t)$.

First we'll find the bounds we need to use by setting up a table for θ, x, and y.

θ	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π
x	6	7	6	5	6
y	2	0	-2	0	2

Because $(x, y) = (6,2)$ at $\theta = 0$, and we don't get back to $(6,2)$ until $\theta = \pi$, we know the first loop of the parametric curve is enclosed by $\theta = [0,\pi]$.

Before we can plug everything into our area formula, we'll need to find the derivative of $x(\theta)$.

$$x'(\theta) = 2\cos 2\theta$$

Plugging everything into the area formula, we get

$$A = \int_0^\pi (2\cos 2\theta)(2\cos 2\theta)\ d\theta$$

$$A = 4\int_0^\pi \cos^2 2\theta\ d\theta$$

Before we can integrate, we need to do a substitution for $\cos^2 2\theta$ using the formula

$$\cos^2\theta = \frac{1}{2}(1 + \cos 2\theta)$$

$$\cos^2 2\theta = \frac{1}{2}(1 + \cos 4\theta)$$

We'll make the substitution.

$$A = 4\int_0^\pi \frac{1}{2}(1 + \cos 4\theta)\ d\theta$$

$$A = 2\int_0^\pi 1 + \cos 4\theta\ d\theta$$

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$$A = 2\left(\theta + \frac{1}{4}\sin 4\theta\right)\Big|_0^{\pi}$$

$$A = 2\left(\pi + \frac{1}{4}\sin 4\pi\right) - 2\left(0 + \frac{1}{4}\sin(0)\right)$$

$$A = 2\left(\pi + \frac{1}{4}(0)\right) - 2\left(0 + \frac{1}{4}(0)\right)$$

$$A = 2\pi$$

9. For this particular integral, we need to start by completing the square in the denominator.

$$\int \frac{x-2}{-x^2+4x-3}\ dx$$

$$\int \frac{x-2}{-(x^2-4x)-3}\ dx$$

$$\int \frac{x-2}{-(x^2-4x+4-4)-3}\ dx$$

$$\int \frac{x-2}{-(x^2-4x+4)+4-3}\ dx$$

$$\int \frac{x-2}{1-(x-2)^2}\ dx$$

Because the denominator is the difference of two squares, we can try trigonometric substitution to evaluate the integral. Setting up trigonometric substitution by comparing $a^2 - u^2$ with $1 - (x - 2)^2$, we get $a = 1$ and $u = x - 2$. Then the sine substitution $u = a\sin\theta$ is

$$x - 2 = \sin\theta$$

$$x = \sin\theta + 2$$

$$dx = \cos\theta \ d\theta$$

In a right triangle, the side opposite of the angle θ will have length $x - 2$, the side adjacent to the angle θ will have length 1, and the hypotenuse of the triangle will have length $\sqrt{(x-2)^2 + 1}$. Substituting the values we've found into the integral, we get

$$\int \frac{\sin\theta}{1 - (\sin\theta + 2 - 2)^2}(\cos\theta \ d\theta)$$

$$\int \frac{\sin\theta\cos\theta}{1 - \sin^2\theta} \ d\theta$$

Since we know from the Pythagorean identity that $1 - \sin^2\theta = \cos^2\theta$, we can make a substitution into the denominator.

$$\int \frac{\sin\theta\cos\theta}{\cos^2\theta} \ d\theta$$

$$\int \frac{\sin\theta}{\cos\theta} \ d\theta$$

Use u-substitution with $u = \cos\theta$ and $du = -\sin\theta \ d\theta$.

$$\int \frac{\sin\theta}{u}\left(\frac{du}{-\sin\theta}\right)$$

$$-\int \frac{1}{u}\ du$$

$$-\ln|u|+C$$

Back-substitute for u.

$$-\ln|\cos\theta|+C$$

$$\ln|(\cos\theta)^{-1}|+C$$

$$\ln\left|\frac{1}{\cos\theta}\right|+C$$

Since the cosine of an angle is equal to the quotient of the adjacent side and the hypotenuse, we get

$$\ln\left|\frac{1}{\frac{1}{\sqrt{(x-2)^2+1}}}\right|+C$$

$$\ln\left|\sqrt{(x-2)^2+1}\right|+C$$

$$\ln\left|\sqrt{x^2-4x+4+1}\right|+C$$

$$\ln\left|\sqrt{x^2-4x+5}\right|+C$$

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10. Because we're rotating about the x-axis, and because our slices of volume must always be perpendicular to the axis of rotation, we'll be taking vertical slices of volume.

Therefore, the width of each infinitely thin slice of volume can be given by dx, which means we'll be integrating with respect to x, and the limits of integration will be $x = [1,4]$. The outer radius will be defined by $y = x^2 - 2x$. So the volume will be

$$V = \int_a^b \pi[f(x)]^2\ dx$$

$$V = \int_1^4 \pi(x^2 - 2x)^2\ dx$$

$$V = \int_1^4 \pi(x^4 - 4x^3 + 4x^2)\ dx$$

Integrate, then evaluate over the interval.

$$V = \pi\left(\frac{1}{5}x^5 - x^4 + \frac{4}{3}x^3\right)\Bigg|_1^4$$

$$V = \pi\left(\frac{1}{5}(4)^5 - 4^4 + \frac{4}{3}(4)^3\right) - \pi\left(\frac{1}{5}(1)^5 - 1^4 + \frac{4}{3}(1)^3\right)$$

$$V = \pi\left(\frac{1{,}024}{5} - 256 + \frac{256}{3}\right) - \pi\left(\frac{1}{5} - 1 + \frac{4}{3}\right)$$

$$V = \pi\left(\frac{3{,}072}{15} - \frac{3{,}840}{15} + \frac{1{,}280}{15}\right) - \pi\left(\frac{3}{15} - \frac{15}{15} + \frac{20}{15}\right)$$

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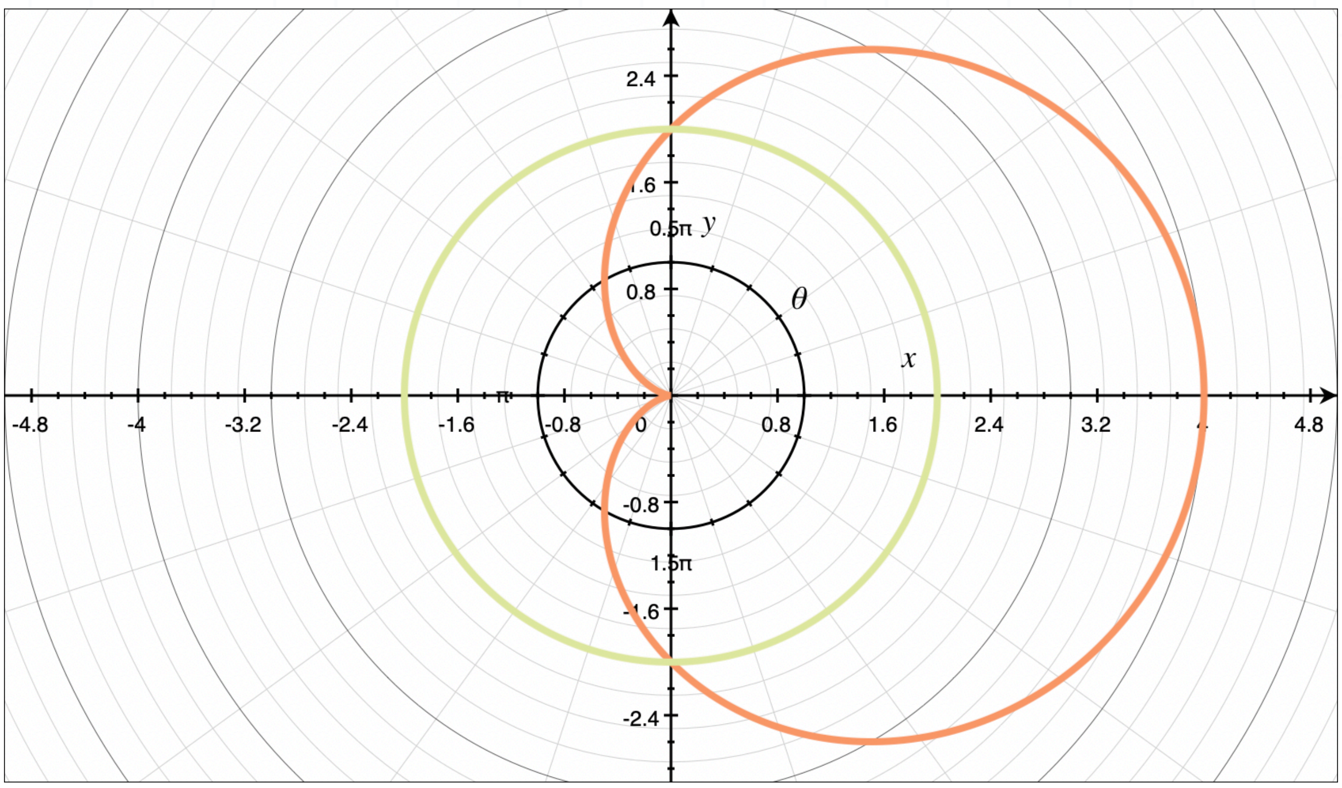
$$V = \pi\left(\frac{3{,}072}{15} - \frac{3{,}840}{15} + \frac{1{,}280}{15} - \frac{3}{15} + \frac{15}{15} - \frac{20}{15}\right)$$

$$V = \pi\left(\frac{504}{15}\right)$$

$$V = \frac{168\pi}{5}$$

11. It's helpful to sketch the graph of the polar curve.



The curves intersect at $(2,\pi/2)$ and $(2,3\pi/2)$. We can also write $(2,3\pi/2)$ as $(2,-\pi/2)$. So our area formula will be

$$A = \frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} r^2_{\text{cardioid}} - r^2_{\text{circle}}\ d\theta$$

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$$A=\frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(2+2\cos\theta)^2-2^2\ d\theta$$

$$A=\frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(4+8\cos\theta+4\cos^2\theta)-4\ d\theta$$

$$A=\frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}4+8\cos\theta+4\cos^2\theta-4\ d\theta$$

$$A=2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos^2\theta+2\cos\theta\ d\theta$$

Using the power reduction formula

$$\cos^2\theta=\frac{1}{2}+\frac{1}{2}\cos 2\theta$$

we get

$$A=2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{1}{2}+\frac{1}{2}\cos 2\theta+2\cos\theta\ d\theta$$

Integrate term by term.

$$A=2\left(\frac{1}{2}\theta+\frac{1}{4}\sin 2\theta+2\sin\theta\right)\Bigg|_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$$

$$A=\theta+\frac{1}{2}\sin 2\theta+4\sin\theta\Bigg|_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$$

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$$A = \frac{\pi}{2} + \frac{1}{2}\sin 2\frac{\pi}{2} + 4\sin\frac{\pi}{2} - \left(-\frac{\pi}{2} + \frac{1}{2}\sin 2\left(-\frac{\pi}{2}\right) + 4\sin\left(-\frac{\pi}{2}\right)\right)$$

$$A = \frac{\pi}{2} + \frac{1}{2}\sin\pi + 4\sin\frac{\pi}{2} + \frac{\pi}{2} - \frac{1}{2}\sin(-\pi) - 4\sin\left(-\frac{\pi}{2}\right)$$

$$A = \frac{\pi}{2} + \frac{1}{2}(0) + 4(1) + \frac{\pi}{2} - \frac{1}{2}(0) - 4(-1)$$

$$A = \pi + 8$$

12. In this case, the given series is similar to the known series

$$\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} x^n$$

We'll manipulate this series until it matches the given series. We'll just replace each x with $4x^2$, and then simplify.

$$\ln(1+4x^2) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} (4x^2)^n$$

$$\ln(1+4x^2) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} (2x)^{2n}$$

Since the left side of this manipulation now matches the given series, we can say that the power series representation of the function is

$$a_n = \frac{(-1)^{n+1}}{n}(2x)^{2n}$$

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In order to find L, we'll need to identify a_n and a_{n+1}.

$$a_n = \frac{(-1)^{n+1}}{n}(2x)^{2n}$$

$$a_{n+1} = \frac{(-1)^{n+1+1}}{n+1}(2x)^{2(n+1)} = \frac{(-1)^{n+2}}{n+1}(2x)^{2n+2}$$

Now we'll plug these values into the formula for L.

$$L = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|$$

$$L = \lim_{n\to\infty}\left|\frac{\frac{(-1)^{n+2}(2x)^{2n+2}}{n+1}}{\frac{(-1)^{n+1}(2x)^{2n}}{n}}\right|$$

$$L = \lim_{n\to\infty}\left|\frac{(-1)^{n+2}(2x)^{2n+2}}{n+1}\cdot\frac{n}{(-1)^{n+1}(2x)^{2n}}\right|$$

$$L = \lim_{n\to\infty}\left|\frac{(-1)^{n+2}}{(-1)^{n+1}}\cdot\frac{(2x)^{2n+2}}{(2x)^{2n}}\cdot\frac{n}{n+1}\right|$$

$$L = \lim_{n\to\infty}\left|(-1)^{n+2-(n+1)}(2x)^{2n+2-2n}\cdot\frac{n}{n+1}\right|$$

$$L = \lim_{n\to\infty}\left|(-1)(2x)^2\cdot\frac{n}{n+1}\right|$$

$$L = \lim_{n\to\infty}\left|(-4x^2)\cdot\frac{n}{n+1}\right|$$

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Since the limit only affects n, we can pull out $-4x^2$, as long as we keep it inside absolute value brackets.

$$L = |-4x^2| \lim_{n \to \infty} \left| \frac{n}{n+1} \right|$$

$$L = 4x^2 \lim_{n \to \infty} \left| \frac{1}{1 + \frac{1}{n}} \right|$$

$$L = 4x^2 \left| \frac{1}{1+0} \right|$$

$$L = 4x^2$$

The ratio test tells us that the series converges when $L < 1$.

$$4x^2 < 1$$

$$x^2 < \frac{1}{4}$$

$$|x| < \frac{1}{2}$$

So the radius of convergence of the Maclaurin series is $R = 1/2$.

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