Area between polar curves

In order to calculate the area between two polar curves, we'll

1. Find the points of intersection if the interval isn't given

2. Graph the curves to confirm the points of intersection

3. For each enclosed region, use the points of intersection to find upper and lower limits of integration $[\alpha, \beta]$

4. For each enclosed region, determine which curve is the outer curve and which is the inner

5. Plug this into the formula for area between curves,

$$A = \int_{\alpha}^{\beta} \frac{1}{2}(r_O^2 - r_I^2)\ d\theta$$

where $[\alpha, \beta]$ is the interval that defines the area, r_O is the outer curve, and r_I is the inner curve

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Example

Find the area between the polar curves $r = 2$ and $r = 3 + 2\sin\theta$.

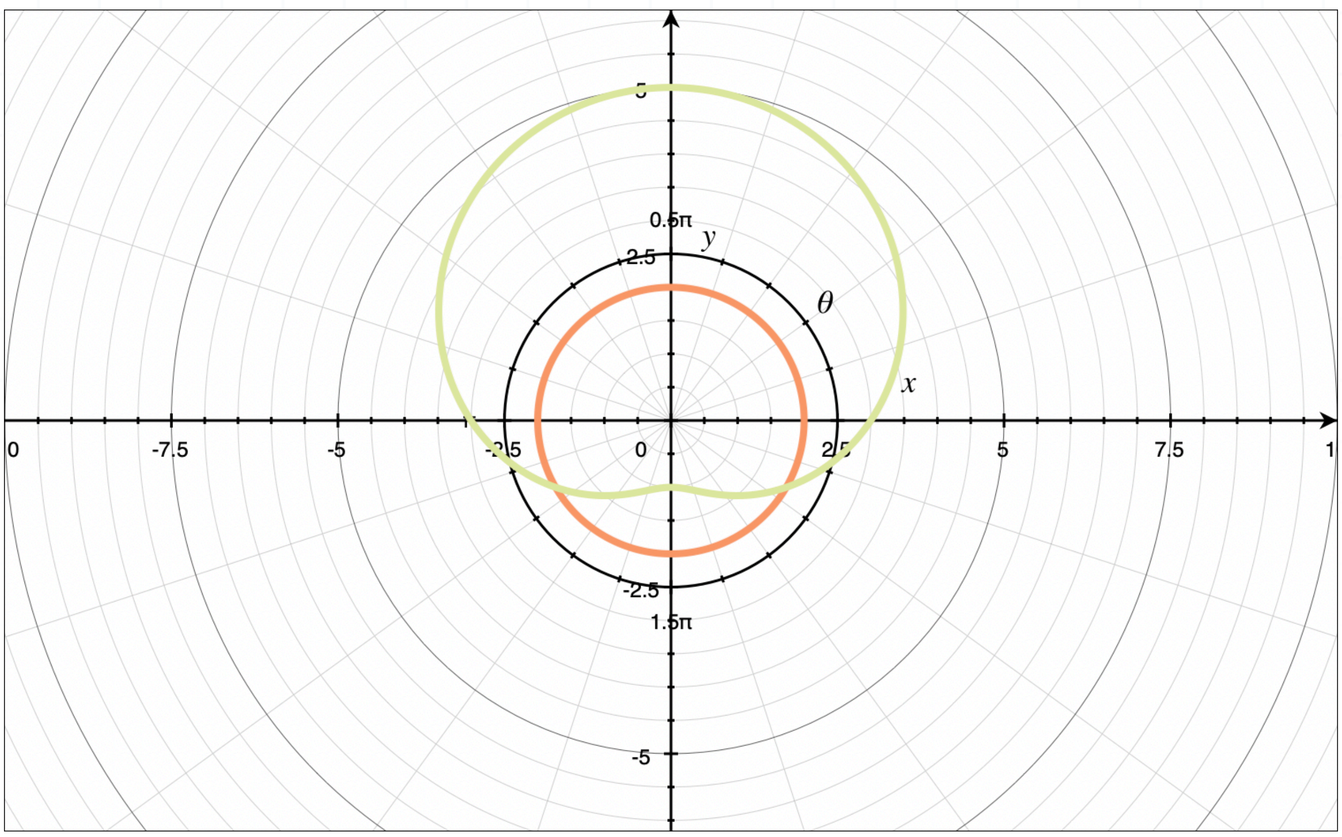
Since the problem doesn't give us an interval over which to evaluate the area, we'll need to find the points of intersection of the curves. We'll set the polar curves equal to each other and solve for θ.

$$3 + 2\sin\theta = 2$$

$$\sin\theta = -\frac{1}{2}$$

$$\theta = \frac{7\pi}{6}, \frac{11\pi}{6}$$

We'll graph the curves to confirm the points of intersection.



Based on the graph above, the area between curves is given by

$$A_T = A_1 + A_2$$

where A_T is total area, A_1 is the larger section, and A_2 is the smaller section. We always want to work in a counterclockwise direction, which means that, in order to find A_1 and A_2, we'll use the intervals

$$A_1 \qquad \left[\frac{11\pi}{6}, \frac{7\pi}{6}\right]$$

$A_2 \qquad \left[\frac{7\pi}{6}, \frac{11\pi}{6}\right]$

However, we always need $\alpha < \beta$ in our interval, so we'll change the interval for A_1 into its equivalent $-\theta$, and we'll get

$A_1 \qquad \left[-\frac{\pi}{6}, \frac{7\pi}{6}\right]$

$A_2 \qquad \left[\frac{7\pi}{6}, \frac{11\pi}{6}\right]$

We'll also need to use the graph to indicate which curve is the outer curve and which is the inner curve. We'll say

	Interval	**Outer**	**Inner**
A_1	$\left[-\frac{\pi}{6}, \frac{7\pi}{6}\right]$	$r_O = 3 + 2\sin\theta$	$r_I = 2$
A_2	$\left[\frac{7\pi}{6}, \frac{11\pi}{6}\right]$	$r_O = 2$	$r_I = 3 + 2\sin\theta$

Now we can plug everything we've found into the area formula.

$$A_T = A_1 + A_2$$

$$A_T = \int_{-\frac{\pi}{6}}^{\frac{7\pi}{6}} \frac{1}{2}((3 + 2\sin\theta)^2 - (2)^2)\ d\theta + \int_{\frac{7\pi}{6}}^{\frac{11\pi}{6}} \frac{1}{2}((2)^2 - (3 + 2\sin\theta)^2)\ d\theta$$

$$A_T = \frac{1}{2}\int_{-\frac{\pi}{6}}^{\frac{7\pi}{6}} (3 + 2\sin\theta)(3 + 2\sin\theta) - 4\ d\theta + \frac{1}{2}\int_{\frac{7\pi}{6}}^{\frac{11\pi}{6}} 4 - (3 + 2\sin\theta)(3 + 2\sin\theta)\ d\theta$$

$$A_T = \frac{1}{2}\int_{-\frac{\pi}{6}}^{\frac{7\pi}{6}} 9 + 12\sin\theta + 4\sin^2\theta - 4\ d\theta + \frac{1}{2}\int_{\frac{7\pi}{6}}^{\frac{11\pi}{6}} 4 - (9 + 12\sin\theta + 4\sin^2\theta)\ d\theta$$

$$A_T = \frac{1}{2}\int_{-\frac{\pi}{6}}^{\frac{7\pi}{6}} 4\sin^2\theta + 12\sin\theta + 5\ d\theta + \frac{1}{2}\int_{\frac{7\pi}{6}}^{\frac{11\pi}{6}} 4 - 9 - 12\sin\theta - 4\sin^2\theta\ d\theta$$

$$A_T = \frac{1}{2}\int_{-\frac{\pi}{6}}^{\frac{7\pi}{6}} 4\sin^2\theta + 12\sin\theta + 5\ d\theta - \frac{1}{2}\int_{\frac{7\pi}{6}}^{\frac{11\pi}{6}} 4\sin^2\theta + 12\sin\theta + 5\ d\theta$$

Since $2\sin^2\theta = 1 - \cos(2\theta)$,

$$A_T = \frac{1}{2}\int_{-\frac{\pi}{6}}^{\frac{7\pi}{6}} 2(1 - \cos(2\theta)) + 12\sin\theta + 5\ d\theta - \frac{1}{2}\int_{\frac{7\pi}{6}}^{\frac{11\pi}{6}} 2(1 - \cos(2\theta)) + 12\sin\theta + 5\ d\theta$$

$$A_T = \frac{1}{2}\int_{-\frac{\pi}{6}}^{\frac{7\pi}{6}} 2 - 2\cos(2\theta) + 12\sin\theta + 5\ d\theta - \frac{1}{2}\int_{\frac{7\pi}{6}}^{\frac{11\pi}{6}} 2 - 2\cos(2\theta) + 12\sin\theta + 5\ d\theta$$

$$A_T = \frac{1}{2}\int_{-\frac{\pi}{6}}^{\frac{7\pi}{6}} 12\sin\theta - 2\cos(2\theta) + 7\ d\theta - \frac{1}{2}\int_{\frac{7\pi}{6}}^{\frac{11\pi}{6}} 12\sin\theta - 2\cos(2\theta) + 7\ d\theta$$

$$A_T = \frac{1}{2}(-12\cos\theta - \sin(2\theta) + 7\theta)\Big|_{-\frac{\pi}{6}}^{\frac{7\pi}{6}} - \frac{1}{2}(-12\cos\theta - \sin(2\theta) + 7\theta)\Big|_{\frac{7\pi}{6}}^{\frac{11\pi}{6}}$$

Evaluate over the interval.

$$A_T = \frac{1}{2}\left[-12\cos\frac{7\pi}{6} - \sin\frac{14\pi}{6} + 7\left(\frac{7\pi}{6}\right) - \left(-12\cos\left(-\frac{\pi}{6}\right) - \sin\left(-\frac{2\pi}{6}\right) + 7\left(-\frac{\pi}{6}\right)\right)\right]$$

$$-\frac{1}{2}\left[-12\cos\frac{11\pi}{6}-\sin\frac{22\pi}{6}+7\left(\frac{11\pi}{6}\right)-\left(-12\cos\frac{7\pi}{6}-\sin\frac{14\pi}{6}+7\left(\frac{7\pi}{6}\right)\right)\right]$$

$$A_T=\frac{1}{2}\left[-12\cos\frac{7\pi}{6}-\sin\frac{7\pi}{3}+\frac{49\pi}{6}-\left(-12\cos\frac{11\pi}{6}-\sin\frac{5\pi}{3}-\frac{7\pi}{6}\right)\right]$$

$$-\frac{1}{2}\left[-12\cos\frac{11\pi}{6}-\sin\frac{11\pi}{3}+\frac{77\pi}{6}-\left(-12\cos\frac{7\pi}{6}-\sin\frac{7\pi}{3}+\frac{49\pi}{6}\right)\right]$$

$$A_T=\frac{1}{2}\left(-12\cos\frac{7\pi}{6}-\sin\frac{7\pi}{3}+\frac{49\pi}{6}+12\cos\frac{11\pi}{6}+\sin\frac{5\pi}{3}+\frac{7\pi}{6}\right)$$

$$-\frac{1}{2}\left(-12\cos\frac{11\pi}{6}-\sin\frac{11\pi}{3}+\frac{77\pi}{6}+12\cos\frac{7\pi}{6}+\sin\frac{7\pi}{3}-\frac{49\pi}{6}\right)$$

$$A_T=\frac{1}{2}\left(-12\cos\frac{7\pi}{6}-\sin\frac{7\pi}{3}+12\cos\frac{11\pi}{6}+\sin\frac{5\pi}{3}+\frac{28\pi}{3}\right)$$

$$-\frac{1}{2}\left(-12\cos\frac{11\pi}{6}-\sin\frac{11\pi}{3}+12\cos\frac{7\pi}{6}+\sin\frac{7\pi}{3}+\frac{14\pi}{3}\right)$$

Simplify the trigonometric functions.

$$A_T=\frac{1}{2}\left[-12\left(-\frac{\sqrt{3}}{2}\right)-\frac{\sqrt{3}}{2}+12\left(\frac{\sqrt{3}}{2}\right)-\frac{\sqrt{3}}{2}+\frac{28\pi}{3}\right]$$

$$-\frac{1}{2}\left[-12\left(\frac{\sqrt{3}}{2}\right)-\left(-\frac{\sqrt{3}}{2}\right)+12\left(-\frac{\sqrt{3}}{2}\right)+\frac{\sqrt{3}}{2}+\frac{14\pi}{3}\right]$$

$$A_T=\frac{1}{2}\left(\frac{12\sqrt{3}}{2}-\frac{\sqrt{3}}{2}+\frac{12\sqrt{3}}{2}-\frac{\sqrt{3}}{2}+\frac{28\pi}{3}\right)$$

$$-\frac{1}{2}\left(-\frac{12\sqrt{3}}{2}+\frac{\sqrt{3}}{2}-\frac{12\sqrt{3}}{2}+\frac{\sqrt{3}}{2}+\frac{14\pi}{3}\right)$$

$$A_T=\frac{12\sqrt{3}}{4}-\frac{\sqrt{3}}{4}+\frac{12\sqrt{3}}{4}-\frac{\sqrt{3}}{4}+\frac{28\pi}{6}+\frac{12\sqrt{3}}{4}-\frac{\sqrt{3}}{4}+\frac{12\sqrt{3}}{4}-\frac{\sqrt{3}}{4}-\frac{14\pi}{6}$$

$$A_T=\frac{44\sqrt{3}}{4}+\frac{14\pi}{6}$$

$$A_T=11\sqrt{3}+\frac{7\pi}{3}$$