Topic: Improper integrals, case 2

Question: Evaluate the improper integral.

$$\int_{-\infty}^{0} \frac{dx}{(2x-1)^3}$$

Answer choices:

A $\frac{1}{4}$

B $-\frac{1}{4}$

C $-\frac{1}{2}$

D $\frac{1}{2}$

Solution: B

Using an arbitrary variable b, first take the limit of the integral as $b \to -\infty$.

$$\int_{-\infty}^{0} \frac{dx}{(2x-1)^3} = \lim_{b \to -\infty} \int_{b}^{0} \frac{dx}{(2x-1)^3}$$

$$\lim_{b \to -\infty} \int_{b}^{0} (2x-1)^{-3}\ dx$$

$$\lim_{b \to -\infty} \left[-\frac{1}{2}(2x-1)^{-2} \cdot \frac{1}{2} \right] \Bigg|_{b}^{0}$$

$$\lim_{b \to -\infty} \left[-\frac{1}{4(2x-1)^2} \right] \Bigg|_{b}^{0}$$

$$\lim_{b \to -\infty} \left[-\frac{1}{4(2(0)-1)^2} + \frac{1}{4(2(b)-1)^2} \right]$$

$$\lim_{b \to -\infty} \left[-\frac{1}{4} + \frac{1}{4(2b-1)^2} \right]$$

$$-\frac{1}{4} + \frac{1}{4(2(-\infty)-1)^2}$$

$$-\frac{1}{4} + \frac{1}{\infty}$$

$$-\frac{1}{4} + 0$$

$$-\frac{1}{4}$$

Topic: Improper integrals, case 2

Question: Evaluate the improper integral.

$$\int_{-\infty}^{2} \frac{7}{7x-16}\,dx$$

Answer choices:

A 0

B $-\infty$

C ∞

D $\ln 26$

Solution: B

The integral in this problem is considered to be an improper integral, case 2, because the lower limit of integration is $-\infty$ and the upper limit is a constant. Evaluating this type of improper integral follows this general rule:

$$\int_{-\infty}^{b} f(x)\ dx = \lim_{a \to -\infty} \int_{a}^{b} f(x)\ dx$$

We basically ignore the lower limit by replacing it with a and using a limit process. Then, once we integrate, finding the anti-derivative, we use the limit to finish the evaluation. Let's begin by rewriting the integral as a limit.

$$\int_{-\infty}^{2} \frac{7}{7x-16}\ dx = \lim_{a \to -\infty} \int_{a}^{2} \frac{7}{7x-16}\ dx$$

Use a u-substitution on the integrand.

$$u = 7x - 16$$

$$du = 7\ dx$$

$$dx = \frac{du}{7}$$

Substitute into the integral.

$$\lim_{a \to -\infty} \int_{x=a}^{x=2} \frac{7}{u} \left(\frac{du}{7} \right)$$

$$\lim_{a \to -\infty} \int_{x=a}^{x=2} \frac{1}{u}\ du$$

Integrate and then back-substitute. Then evaluate over the interval.

$$\lim_{a\to-\infty} \ln|u| \Big|_{x=a}^{x=2}$$

$$\lim_{a\to-\infty} \ln|7x-16| \Big|_{a}^{2}$$

$$\lim_{a\to-\infty} \left[\ln|7(2)-16| - \ln|7(a)-16|\right]$$

$$\lim_{a\to-\infty} \left[\ln 2 - \ln|7a-16|\right]$$

When we take the limit, $\ln|7a-16|$ becomes ∞. Therefore, we essentially have

$$\ln 2 - \infty$$

$$-\infty$$

Topic: Improper integrals, case 2

Question: Evaluate the improper integral.

$$\int_{-\infty}^{5} \frac{1}{e^x + e^{-x}} \, dx$$

Answer choices:

A $-\infty$

B 0

C ∞

D $\frac{\pi}{2}$

Solution: D

The integral in this problem is considered to be an improper integral, case 2, because the lower limit of integration is $-\infty$ and the upper limit is a constant. Evaluating this type of improper integral follows this general rule:

$$\int_{-\infty}^{b} f(x)\ dx = \lim_{a \to -\infty} \int_{a}^{b} f(x)\ dx$$

We basically ignore the lower limit by replacing it with a and using a limit process. Then, once we integrate, finding the anti-derivative, we use the limit to finish the evaluation. Let's begin by rewriting the integral as a limit.

$$\int_{-\infty}^{5} \frac{1}{e^x + e^{-x}}\ dx = \lim_{a \to -\infty} \int_{a}^{5} \frac{1}{e^x + e^{-x}}\ dx$$

Rewrite the integrand.

$$\lim_{a \to -\infty} \int_{a}^{5} \frac{1}{e^{-x}\left(e^{2x} + 1\right)}\ dx$$

$$\lim_{a \to -\infty} \int_{a}^{5} \frac{e^x}{(e^x)^2 + 1}\ dx$$

Use a u-substitution on the integral.

$$u = e^x$$

$$du = e^x\ dx$$

$$dx = \frac{du}{e^x}$$

Substitute into the integral.

$$\lim_{a\to-\infty}\int_{x=a}^{x=5}\frac{u}{u^2+1}\left(\frac{du}{e^x}\right)$$

$$\lim_{a\to-\infty}\int_{x=a}^{x=5}\frac{u}{u^2+1}\left(\frac{du}{u}\right)$$

$$\lim_{a\to-\infty}\int_{x=a}^{x=5}\frac{1}{u^2+1}\ du$$

Integrate and then back-substitute. Then evaluate over the interval.

$$\lim_{a\to-\infty}\tan^{-1}u\Big|_{x=a}^{x=5}$$

$$\lim_{a\to-\infty}\tan^{-1}e^x\Big|_{a}^{5}$$

$$\lim_{a\to-\infty}\left(\tan^{-1}e^5-\tan^{-1}e^a\right)$$

Taking the limit essentially gives us

$$\tan^{-1}e^5-\tan^{-1}e^{-\infty}$$

$$\tan^{-1}e^5-\tan^{-1}\frac{1}{e^\infty}$$

$$\tan^{-1}e^5-\tan^{-1}\frac{1}{\infty}$$

$$\tan^{-1}e^5-\tan^{-1}0$$

$$\frac{\pi}{2} - 0$$

$$\frac{\pi}{2}$$