Surface area of revolution of a polar curve

We can find the surface area of the object created when we rotate a polar curve around either the x-axis or the y-axis using the formulas

$$S_x = \int_{\alpha}^{\beta} 2\pi y \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\ d\theta \qquad \text{around the } x\text{-axis}$$

$$S_y = \int_{\alpha}^{\beta} 2\pi x \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\ d\theta \qquad \text{around the } y\text{-axis}$$

where r is the equation of the polar curve

$\frac{dr}{d\theta}$ is the derivative of the polar curve

$[\alpha, \beta]$ is the interval

We can solve for x and y as needed using the conversion formulas

$$x = r\cos\theta$$

$$y = r\sin\theta$$

Example

Find the surface area of revolution of the polar curve over the given interval.

$$r = 5\cos\theta$$

$$0 \le \theta \le \pi$$

around the y-axis

Before we can plug into the formula, we need to find x and $dr/d\theta$.

Since $x = r\cos\theta$, we get

$$x = 5\cos\theta\cos\theta$$

$$x = 5\cos^2\theta$$

To find $dr/d\theta$, we'll take the derivative of the given polar equation.

$$r = 5\cos\theta$$

$$\frac{dr}{d\theta} = -5\sin\theta$$

We'll plug everything into the formula for the surface area of revolution about the y-axis.

$$S_y = \int_\alpha^\beta 2\pi x\sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\ d\theta$$

$$S_y = \int_0^\pi 2\pi\left(5\cos^2\theta\right)\sqrt{\left(5\cos\theta\right)^2 + \left(-5\sin\theta\right)^2}\ d\theta$$

$$S_y = 10\pi\int_0^\pi \cos^2\theta\sqrt{25\cos^2\theta + 25\sin^2\theta}\ d\theta$$

$$S_y = 10\pi \int_0^{\pi} \cos^2\theta \sqrt{25\left(\cos^2\theta + \sin^2\theta\right)}\ d\theta$$

Since $\cos^2\theta + \sin^2\theta = 1$,

$$S_y = 50\pi \int_0^{\pi} \cos^2\theta \sqrt{1}\ d\theta$$

$$S_y = 50\pi \int_0^{\pi} \cos^2\theta\ d\theta$$

Since $\cos^2\theta = \frac{1}{2}\left[1 + \cos(2\theta)\right]$,

$$S_y = 50\pi \int_0^{\pi} \frac{1}{2}\left[1 + \cos(2\theta)\right]\ d\theta$$

$$S_y = 50\pi \int_0^{\pi} \frac{1}{2} + \frac{1}{2}\cos(2\theta)\ d\theta$$

$$S_y = 25\pi \int_0^{\pi} 1 + \cos(2\theta)\ d\theta$$

$$S_y = 25\pi \left[\theta + \frac{1}{2}\sin(2\theta)\right]\Bigg|_0^{\pi}$$

$$S_y = 25\pi\theta + \frac{25\pi}{2}\sin(2\theta)\Bigg|_0^{\pi}$$

$$S_y = 25\pi(\pi) + \frac{25\pi}{2}\sin(2\pi) - \left[25\pi(0) + \frac{25\pi}{2}\sin(2(0))\right]$$

$$S_y = 25\pi^2 + \frac{25\pi}{2}(0) - 25\pi(0) - \frac{25\pi}{2}(0)$$

$$S_y = 25\pi^2$$