Calculus 2 Workbook Solutions

Partial fractions

krista king
MATH

DISTINCT LINEAR FACTORS

■ 1. Use partial fractions to evaluate the integral.

$$\int \frac{4x+5}{x^2+5x+6}\ dx$$

Solution:

Factor the denominator, then do the partial fractions decomposition.

$$\frac{4x+5}{(x+2)(x+3)} = \frac{A}{x+2} + \frac{B}{x+3}$$

$$4x+5 = A(x+3) + B(x+2)$$

$$4x+5 = Ax + 3A + Bx + 2B$$

$$4x+5 = (A+B)x + (3A+2B)$$

Then the system of equations is

$$A + B = 4$$

$$3A + 2B = 5$$

Solve $A + B = 4$ for A.

$$A = 4 - B$$

Substitute $A = 4 - B$ into $3A + 2B = 5$.

$$3(4-B)+2B=5$$

$$12-3B+2B=5$$

$$12-B=5$$

$$12=5+B$$

$$7=B$$

Then plugging this back into $A=4-B$ gives

$$A=4-7$$

$$A=-3$$

Then the integral becomes

$$\int \frac{4x+5}{x^2+5x+6}\ dx$$

$$\int -\frac{3}{x+2}+\frac{7}{x+3}\ dx$$

$$-3\ln|x+2|+7\ln|x+3|+C$$

DISTINCT QUADRATIC FACTORS

1. Use partial fractions to evaluate the integral.

$$\int \frac{3x+6}{(x^2+2)(x^2+1)}\ dx$$

Solution:

Factor the denominator, then do the partial fractions decomposition.

$$\frac{3x+6}{(x^2+2)(x^2+1)} = \frac{Ax+B}{x^2+2} + \frac{Cx+D}{x^2+1}$$

$$3x+6 = (Ax+B)(x^2+1) + (Cx+D)(x^2+2)$$

$$3x+6 = Ax^3 + Ax + Bx^2 + B + Cx^3 + 2Cx + Dx^2 + 2D$$

$$3x+6 = (A+C)x^3 + (B+D)x^2 + (A+2C)x + (B+2D)$$

Then the system of equations is

$$A + C = 0$$

$$B + D = 0$$

$$A + 2C = 3$$

$$B + 2D = 6$$

Solve the system

$$A + C = 0$$

$$A + 2C = 3$$

Solve $A + C = 0$ for A to get $A = -C$. Plug this into $A + 2C = 3$ to get

$$-C + 2C = 3$$

$$C = 3$$

Then $A = -3$. Now solve the system

$$B + D = 0$$

$$B + 2D = 6$$

Solve $B + D = 0$ for B to get $B = -D$. Plug this into $B + 2D = 6$ to get

$$-D + 2D = 6$$

$$D = 6$$

Then $B = -6$. Then the integral becomes

$$\int \frac{3x + 6}{(x^2 + 2)(x^2 + 1)}\ dx$$

$$\int \frac{-3x - 6}{x^2 + 2} + \frac{3x + 6}{x^2 + 1}\ dx$$

$$\int -\frac{3x}{x^2 + 2} - \frac{6}{x^2 + 2} + \frac{3x}{x^2 + 1} + \frac{6}{x^2 + 1}\ dx$$

$$-\int \frac{3x}{x^2 + 2}\ dx - \int \frac{6}{x^2 + 2}\ dx + \int \frac{3x}{x^2 + 1}\ dx + \int \frac{6}{x^2 + 1}\ dx$$

Use u-substitution.

$$u = x^2 + 2$$

$$\frac{du}{dx} = 2x, \text{ so } du = 2x\ dx, \text{ so } dx = \frac{du}{2x}$$

and

$$u = x^2 + 1$$

$$\frac{du}{dx} = 2x, \text{ so } du = 2x\ dx, \text{ so } dx = \frac{du}{2x}$$

Substituting into the integral gives

$$-\int \frac{3x}{u}\left(\frac{du}{2x}\right) - \int \frac{6}{x^2+2}\ dx + \int \frac{3x}{u}\left(\frac{du}{2x}\right) + \int \frac{6}{x^2+1}\ dx$$

$$-\frac{3}{2}\int \frac{1}{u}\ du - \int \frac{6}{x^2+2}\ dx + \frac{3}{2}\int \frac{1}{u}\ du + \int \frac{6}{x^2+1}\ dx$$

$$-\frac{3}{2}\ln u - \int \frac{6}{x^2+2}\ dx + \frac{3}{2}\ln u + \int \frac{6}{x^2+1}\ dx$$

$$-\frac{3}{2}\ln(x^2+2) - \int \frac{6}{x^2+2}\ dx + \frac{3}{2}\ln(x^2+1) + \int \frac{6}{x^2+1}\ dx$$

Rewrite the integral.

$$-\frac{3}{2}\ln(x^2+2) - \int \frac{3}{\frac{x^2}{2}+1}\ dx + \frac{3}{2}\ln(x^2+1) + \int \frac{6}{x^2+1}\ dx$$

$$-\frac{3}{2}\ln(x^2+2)-3\int\frac{1}{\frac{x^2}{2}+1}\ dx+\frac{3}{2}\ln(x^2+1)+\int\frac{6}{x^2+1}\ dx$$

$$-\frac{3}{2}\ln(x^2+2)-3\int\frac{1}{\left(\frac{x}{\sqrt{2}}\right)^2+1}\ dx+\frac{3}{2}\ln(x^2+1)+6\int\frac{1}{x^2+1}\ dx$$

Use inverse tangent rules to integrate.

$$-\frac{3}{2}\ln(x^2+2)-3\sqrt{2}\arctan\frac{x}{\sqrt{2}}+\frac{3}{2}\ln(x^2+1)+6\arctan x+C$$

REPEATED LINEAR FACTORS

1. Use partial fractions to evaluate the integral.

$$\int \frac{5x-3}{(x+2)^2}\ dx$$

Solution:

Factor the denominator, then do the partial fractions decomposition.

$$\frac{5x-3}{(x+2)(x+2)} = \frac{A}{(x+2)^2} + \frac{B}{x+2}$$

$$5x-3 = A + B(x+2)$$

$$5x-3 = A + Bx + 2B$$

$$5x-3 = (B)x + (A+2B)$$

Then the system of equations is

$$B = 5$$

$$-3 = A + 2B$$

Substitute $B = 5$ into $-3 = A + 2B$.

$$-3 = A + 2(5)$$

$$-3 = A + 10$$

$$-13 = A$$

Then the integral becomes

$$\int \frac{5x-3}{(x+2)(x+2)}\ dx$$

$$\int \frac{-13}{(x+2)^2} + \frac{5}{x+2}\ dx$$

$$-13\int \frac{1}{(x+2)^2}\ dx + 5\int \frac{1}{x+2}\ dx$$

$$-13\int (x+2)^{-2}\ dx + 5\int \frac{1}{x+2}\ dx$$

Integrate.

$$13(x+2)^{-1} + 5\ln|x+2| + C$$

$$\frac{13}{x+2} + 5\ln|x+2| + C$$

2. Use partial fractions to evaluate the integral.

$$\int \frac{x+12}{(3x-2)^2}\ dx$$

Solution:

Factor the denominator, then do the partial fractions decomposition.

$$\frac{x+12}{(3x-2)(3x-2)} = \frac{A}{(3x-2)^2} + \frac{B}{3x-2}$$

$$x + 12 = A + B(3x - 2)$$

$$x + 12 = A + 3Bx - 2B$$

$$x + 12 = (3B)x + (A - 2B)$$

Then the system of equations is

$$3B = 1$$

$$A - 2B = 12$$

Then $B = 1/3$, and

$$A - 2 \cdot \frac{1}{3} = 12$$

$$A - \frac{2}{3} = 12$$

$$A = \frac{36}{3} + \frac{2}{3}$$

$$A = \frac{38}{3}$$

Then the integral becomes

$$\int \frac{x+12}{(3x-2)(3x-2)}\ dx$$

$$\int \frac{\frac{38}{3}}{(3x-2)^2} + \frac{\frac{1}{3}}{3x-2}\ dx$$

$$\frac{38}{3}\int \frac{1}{(3x-2)^2}\ dx + \frac{1}{3}\int \frac{1}{3x-2}\ dx$$

$$\frac{38}{3}\int (3x-2)^{-2}\ dx + \frac{1}{3}\int \frac{1}{3x-2}\ dx$$

Integrate.

$$-\frac{38}{9}(3x-2)^{-1} + \frac{1}{9}\ln|3x-2| + C$$

$$-\frac{38}{9(3x-2)} + \frac{1}{9}\ln|3x-2| + C$$

■ 3. Use partial fractions to evaluate the integral.

$$\int \frac{7x-4}{(5x+1)^2}\ dx$$

Solution:

Factor the denominator, then do the partial fractions decomposition.

$$\frac{7x-4}{(5x+1)(5x+1)} = \frac{A}{(5x+1)^2} + \frac{B}{5x+1}$$

$$7x-4 = A + B(5x+1)$$

$$7x - 4 = A + 5Bx + B$$

$$7x - 4 = (5B)x + (A + B)$$

Then the system of equations is

$$5B = 7$$

$$A + B = -4$$

Then $B = 7/5$, and we can substitute $B = 7/5$ into $A + B = -4$

$$A + \frac{7}{5} = -4$$

$$A = -\frac{20}{5} - \frac{7}{5}$$

$$A = -\frac{27}{5}$$

Then the integral becomes

$$\int \frac{7x - 4}{(5x + 1)(5x + 1)}\ dx$$

$$\int \frac{-\frac{27}{5}}{(5x + 1)^2} + \frac{\frac{7}{5}}{5x + 1}\ dx$$

$$-\frac{27}{5}\int \frac{1}{(5x + 1)^2}\ dx + \frac{7}{5}\int \frac{1}{5x + 1}\ dx$$

$$-\frac{27}{5}\int (5x + 1)^{-2}\ dx + \frac{7}{5}\int \frac{1}{5x + 1}\ dx$$

Integrate.

$$\frac{27}{25}(5x+1)^{-1}+\frac{7}{25}\ln|5x+1|+C$$

$$\frac{27}{25(5x+1)}+\frac{7}{25}\ln|5x+1|+C$$

■ 4. Use partial fractions to evaluate the integral.

$$\int \frac{12x+9}{(2x+7)^2}\ dx$$

Solution:

Factor the denominator, then do the partial fractions decomposition.

$$\frac{12x+9}{(2x+7)(2x+7)}=\frac{A}{(2x+7)^2}+\frac{B}{2x+7}$$

$$12x+9=A+B(2x+7)$$

$$12x+9=A+2Bx+7B$$

$$12x+9=(2B)x+(A+7B)$$

Then the system of equations is

$$2B=12$$

$$A+7B=9$$

Then $B=6$, and we can substitute $B=6$ into $A+7B=9$.

$$A+7(6)=9$$

$$A+42=9$$

$$A=-33$$

Then the integral becomes

$$\int \frac{12x+9}{(2x+7)(2x+7)}\ dx$$

$$\int \frac{-33}{(2x+7)^2}+\frac{6}{2x+7}\ dx$$

$$-33\int \frac{1}{(2x+7)^2}\ dx+6\int \frac{1}{2x+7}\ dx$$

$$-33\int (2x+7)^{-2}\ dx+6\int \frac{1}{2x+7}\ dx$$

Integrate.

$$\frac{33}{2}(2x+7)^{-1}+3\ln|2x+7|+C$$

$$\frac{33}{2(2x+7)}+3\ln|2x+7|+C$$

5. Use partial fractions to evaluate the integral.

$$\int \frac{24x+41}{(3x+4)^2}\ dx$$

Solution:

Factor the denominator, then do the partial fractions decomposition.

$$\frac{24x+41}{(3x+4)(3x+4)} = \frac{A}{(3x+4)^2} + \frac{B}{3x+4}$$

$$24x+41 = A + B(3x+4)$$

$$24x+41 = A + 3Bx + 4B$$

$$24x+41 = (3B)x + (A+4B)$$

Then the system of equations is

$$3B = 24$$

$$A + 4B = 41$$

Then $B = 8$ and we can substitute $B = 8$ into $A + 4B = 41$.

$$A + 4(8) = 41$$

$$A + 32 = 41$$

$$A = 9$$

Then the integral becomes

$$\int \frac{24x+41}{(3x+4)(3x+4)}\ dx$$

$$\int \frac{9}{(3x+4)^2} + \frac{8}{3x+4}\ dx$$

$$9\int (3x+4)^{-2}\ dx + 8\int \frac{1}{3x+4}\ dx$$

Integrate.

$$-3(3x+4)^{-1} + \frac{8}{3}\ln|3x+4| + C$$

$$-\frac{3}{3x+4} + \frac{8}{3}\ln|3x+4| + C$$

REPEATED QUADRATIC FACTORS

1. Rewrite the integral using partial fractions, but do not evaluate it.

$$\int \frac{x^2 - 3x + 2}{(x^2 + 2)^2} \ dx$$

Solution:

Factor the denominator, then do the partial fractions decomposition.

$$\frac{x^2 - 3x + 2}{(x^2 + 2)(x^2 + 2)} = \frac{Ax + B}{(x^2 + 2)^2} + \frac{Cx + D}{x^2 + 2}$$

$$x^2 - 3x + 2 = Ax + B + (Cx + D)(x^2 + 2)$$

$$x^2 - 3x + 2 = Ax + B + Cx^3 + 2Cx + Dx^2 + 2D$$

$$x^2 - 3x + 2 = (C)x^3 + (D)x^2 + (A + 2C)x + (B + 2D)$$

Then the system of equations is

$$C = 0$$

$$D = 1$$

$$A + 2C = -3$$

$$B + 2D = 2$$

Substituting $C=0$ into $A+2C=-3$ gives

$$A+2(0)=-3$$

$$A=-3$$

Substituting $D=1$ into $B+2D=2$ gives

$$B+2(1)=2$$

$$B+2=2$$

$$B=0$$

Then the integral becomes

$$\int \frac{x^2-3x+2}{(x^2+2)(x^2+2)}\ dx$$

$$\int \frac{Ax+B}{(x^2+2)^2}+\frac{Cx+D}{x^2+2}\ dx$$

$$\int \frac{-3x+0}{(x^2+2)^2}+\frac{0x+1}{x^2+2}\ dx$$

$$-3\int \frac{x}{(x^2+2)^2}\ dx+\int \frac{1}{x^2+2}\ dx$$

■ 2. Rewrite the integral using partial fractions, but do not evaluate it.

$$\int \frac{x^2-4x+6}{(x^2+3)^2}\ dx$$

Solution:

Factor the denominator, then do the partial fractions decomposition.

$$\frac{x^2-4x+6}{(x^2+3)(x^2+3)}=\frac{Ax+B}{(x^2+3)^2}+\frac{Cx+D}{x^2+3}$$

$$x^2-4x+6=Ax+B+(Cx+D)(x^2+3)$$

$$x^2-4x+6=Ax+B+Cx^3+3Cx+Dx^2+3D$$

$$x^2-4x+6=(C)x^3+(D)x^2+(A+3C)x+(B+3D)$$

Then the system of equations is

$$C=0$$

$$D=1$$

$$A+3C=-4$$

$$B+3D=6$$

Substituting $C=0$ into $A+3C=-4$ gives

$$A+3(0)=-4$$

$$A=-4$$

Substituting $D=1$ into $B+3D=6$ gives

$$B+3(1)=6$$

$$B + 3 = 6$$

$$B = 3$$

Then the integral becomes

$$\int \frac{x^2 - 4x + 6}{(x^2+3)(x^2+3)}\ dx$$

$$\int \frac{-4x+3}{(x^2+3)^2} + \frac{0x+1}{x^2+3}\ dx$$

$$-4\int \frac{x}{(x^2+3)^2}\ dx + 3\int \frac{1}{(x^2+3)^2}\ dx + \int \frac{1}{x^2+3}\ dx$$

3. Rewrite the integral using partial fractions, but do not evaluate it.

$$\int \frac{4x^3 - 2x^2 + x + 1}{(2x^2+1)^2}\ dx$$

Solution:

Factor the denominator, then do the partial fractions decomposition.

$$\frac{4x^3 - 2x^2 + x + 1}{(2x^2+1)(2x^2+1)} = \frac{Ax+B}{(2x^2+1)^2} + \frac{Cx+D}{2x^2+1}$$

$$4x^3 - 2x^2 + x + 1 = Ax + B + (Cx+D)(2x^2+1)$$

$$4x^3 - 2x^2 + x + 1 = Ax + B + 2Cx^3 + Cx + 2Dx^2 + D$$

$$4x^3 - 2x^2 + x + 1 = (2C)x^3 + (2D)x^2 + (A + C)x + (B + D)$$

Then the system of equations is

$$2C = 4$$

$$2D = -2$$

$$A + C = 1$$

$$B + D = 1$$

Then $C = 2$ and $D = -1$. Substitute $C = 2$ into $A + C = 1$.

$$A + 2 = 1$$

$$A = -1$$

Substitute $D = -1$ into $B + D = 1$.

$$B - 1 = 1$$

$$B = 2$$

Then the integral becomes

$$\int \frac{4x^3 - 2x^2 + x + 1}{(2x^2 + 1)(2x^2 + 1)} \ dx$$

$$\int \frac{-1x + 2}{(2x^2 + 1)^2} + \frac{2x - 1}{2x^2 + 1} \ dx$$

$$-\int \frac{x}{(2x^2 + 1)^2} \ dx + 2\int \frac{1}{(2x^2 + 1)^2} \ dx + 2\int \frac{x}{2x^2 + 1} \ dx - \int \frac{1}{2x^2 + 1} \ dx$$

4. Rewrite the integral using partial fractions, but do not evaluate it.

$$\int \frac{x^3 - 2x^2 + 3x + 5}{(x^2 + 1)^3}\ dx$$

Solution:

Factor the denominator, then do the partial fractions decomposition.

$$\frac{x^3 - 2x^2 + 3x + 5}{(x^2 + 1)(x^2 + 1)(x^2 + 1)} = \frac{Ax + B}{(x^2 + 1)^3} + \frac{Cx + D}{(x^2 + 1)^2} + \frac{Ex + F}{x^2 + 1}$$

$$x^3 - 2x^2 + 3x + 5 = Ax + B + (Cx + D)(x^2 + 1) + (Ex + F)(x^2 + 1)^2$$

$$x^3 - 2x^2 + 3x + 5 = Ax + B + Cx^3 + Cx + Dx^2 + D + (Ex + F)(x^4 + 2x^2 + 1)$$

$$x^3 - 2x^2 + 3x + 5 = Ax + B + Cx^3 + Cx + Dx^2 + D$$

$$+Ex^5 + 2Ex^3 + Ex + Fx^4 + 2Fx^2 + F$$

$$x^3 - 2x^2 + 3x + 5 = (E)x^5 + (F)x^4 + (C + 2E)x^3 + (D + 2F)x^2$$

$$+(A + C + E)x + (B + D + F)$$

Then the system of equations is

$$E = 0$$

$$F = 0$$

$$C + 2E = 1$$

$$D + 2F = -2$$

$$A + C + E = 3$$

$$B + D + F = 5$$

Substitute $E = 0$ into $C + 2E = 1$.

$$C + 2(0) = 1$$

$$C = 1$$

Substitute $F = 0$ into $D + 2F = -2$.

$$D + 2(0) = -2$$

$$D = -2$$

Substitute $C = 1$ and $E = 0$ into $A + C + E = 3$.

$$A + 1 + 0 = 3$$

$$A = 2$$

Substitute $D = -2$ and $F = 0$ into $B + D + F = 5$.

$$B - 2 + 0 = 5$$

$$B = 7$$

Then the integral becomes

$$\int \frac{x^3 - 2x^2 + 3x + 5}{(x^2+1)(x^2+1)(x^2+1)}\ dx$$

$$\int \frac{2x+7}{(x^2+1)^3} + \frac{1x-2}{(x^2+1)^2} + \frac{0x+0}{x^2+1}\ dx$$

$$\int \frac{2x+7}{(x^2+1)^3}\ dx + \int \frac{x-2}{(x^2+1)^2}\ dx$$

$$2\int \frac{x}{(x^2+1)^3}\ dx + 7\int \frac{1}{(x^2+1)^3}\ dx + \int \frac{x}{(x^2+1)^2}\ dx - 2\int \frac{1}{(x^2+1)^2}\ dx$$

RATIONALIZING SUBSTITUTIONS

1. Use a rationalizing substitution to rewrite the integral in terms of u, but don't integrate it.

$$\int \frac{\sqrt{x+16}}{x}\ dx$$

Solution:

Set up the rationalizing substitution.

$$u = \sqrt{x+16}, \text{ so } u^2 = x+16, \text{ so } x = u^2 - 16$$

$$du = \frac{1}{2\sqrt{x+16}}\ dx, \text{ so } dx = 2\sqrt{x+16}\ du$$

Substitute into the integral.

$$\int \frac{u}{x} \cdot 2\sqrt{x+16}\ du$$

$$2\int \frac{u}{u^2-16} \cdot u\ du$$

$$2\int \frac{u^2}{u^2-16}\ du$$

2. Use a rationalizing substitution to rewrite the integral in terms of u, but don't integrate it.

$$\int \frac{\sqrt{3x+5}}{x}\ dx$$

Solution:

Set up the rationalizing substitution.

$u = \sqrt{3x+5}$, so $u^2 = 3x+5$, so $3x = u^2 - 5$ and $x = (u^2-5)/3$

$du = \dfrac{3}{2\sqrt{3x+5}}\ dx$, so $dx = \dfrac{2}{3}\sqrt{3x+5}\ du$

Substitute into the integral.

$$\int \frac{u}{x} \cdot \frac{2}{3}\sqrt{3x+5}\ du$$

$$\frac{2}{3}\int \frac{u}{\frac{u^2-5}{3}} \cdot u\ du$$

$$\frac{2}{3}\int \frac{3u^2}{u^2-5}\ du$$

$$2\int \frac{u^2}{u^2-5}\ du$$

3. Use a rationalizing substitution to rewrite the integral in terms of u, but don't integrate it.

$$\int \frac{\sqrt{7x-2}}{x}\ dx$$

Solution:

Set up the rationalizing substitution.

$u = \sqrt{7x-2}$, so $u^2 = 7x - 2$, so $7x = u^2 + 2$ and $x = (u^2+2)/7$

$$du = \frac{7}{2\sqrt{7x-2}}\ dx, \text{ so } dx = \frac{2}{7}\sqrt{7x-2}\ du$$

Substitute into the integral.

$$\int \frac{u}{\frac{u^2+2}{7}} \cdot \frac{2}{7}\sqrt{7x-2}\ du$$

$$\frac{2}{7}\int u \cdot \frac{7}{u^2+2} \cdot u\ du$$

$$2\int \frac{u^2}{u^2+2}\ du$$

W W W . K R I S T A K I N G M A T H . C O M