Topic: Surface area of revolution of a parametric curve, horizontal axis

Question: Find the surface area of revolution of the parametric curve rotated about the given axis.

$$x = \frac{7}{4}t$$

$$y = t + 3$$

$$0 \le t \le 4$$

about the x-axis

Answer choices:

A 22π

B 56π

C $32\pi\sqrt{59}$

D $10\pi\sqrt{65}$

Solution: D

The formula for surface area of a parametric curve revolved about the x-axis on the given interval is

$$S=\int_0^4 2\pi y\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\ dt$$

We'll calculate the derivatives of x and y so that we can plug them into the formula.

$$x=\frac{7}{4}t$$

$$\frac{dx}{dt}=\frac{7}{4}$$

and

$$y=t+3$$

$$\frac{dy}{dt}=1$$

Plugging these into our formula, we get

$$S=\int_0^4 2\pi(t+3)\sqrt{\left(\frac{7}{4}\right)^2+(1)^2}\ dt$$

$$S=\int_0^4 2\pi(t+3)\sqrt{\frac{49}{16}+\frac{16}{16}}\ dt$$

$$S=2\pi\sqrt{\frac{65}{16}}\int_0^4 t+3\ dt$$

$$S = \frac{\pi\sqrt{65}}{2}\left(\frac{1}{2}t^2 + 3t\right)\Bigg|_0^4$$

$$S = \frac{\pi\sqrt{65}}{2}\left[\left(\frac{1}{2}(4)^2 + 3(4)\right) - \left(\frac{1}{2}(0)^2 + 3(0)\right)\right]$$

$$S = \frac{\pi\sqrt{65}}{2}(8 + 12)$$

$$S = 10\pi\sqrt{65}$$

Topic: Surface area of revolution of a parametric curve, horizontal axis

Question: Find the surface area of revolution of the parametric curve rotated about the given axis.

$$x = 3e^{3t} - 9t$$

$$y = 12e^{\frac{3t}{2}}$$

$$1 \le t \le 2$$

about the x-axis

Answer choices:

A $\quad 48\pi\left(e^9 + 3e^3 - e^{\frac{9}{2}} - 3e^{\frac{3}{2}}\right)$

B $\quad 48\pi\left(e^9 + 3e^3\right)$

C $\quad 48\pi\left(e^9 - 3e^3\right)$

D $\quad 32\pi\left(e^9 + 3e^3 - e^{\frac{9}{2}} - 3e^{\frac{3}{2}}\right)$

Solution: A

The formula for surface area of a parametric curve revolved about the x-axis on the given interval is

$$S=\int_1^2 2\pi y\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\ dt$$

We'll calculate the derivatives of x and y so that we can plug them into the formula.

$$x=3e^{3t}-9t$$

$$\frac{dx}{dt}=9e^{3t}-9$$

and

$$y=12e^{\frac{3t}{2}}$$

$$\frac{dy}{dt}=18e^{\frac{3t}{2}}$$

Plugging these into our formula, we get

$$S=\int_1^2 2\pi\left(12e^{\frac{3t}{2}}\right)\sqrt{(9e^{3t}-9)^2+\left(18e^{\frac{3t}{2}}\right)^2}\ dt$$

$$S=24\pi\int_1^2 e^{\frac{3t}{2}}\sqrt{81e^{6t}-162e^{3t}+81+324e^{3t}}\ dt$$

$$S=24\pi\int_1^2 e^{\frac{3t}{2}}\sqrt{81e^{6t}+162e^{3t}+81}\ dt$$

$$S = 24\pi \int_1^2 e^{\frac{3t}{2}} \sqrt{81\left(e^{6t} + 2e^{3t} + 1\right)} \ dt$$

$$S = 216\pi \int_1^2 e^{\frac{3t}{2}} \sqrt{e^{6t} + 2e^{3t} + 1} \ dt$$

$$S = 216\pi \int_1^2 e^{\frac{3t}{2}} \sqrt{\left(e^{3t} + 1\right)^2} \ dt$$

$$S = 216\pi \int_1^2 e^{\frac{3t}{2}} \left(e^{3t} + 1\right) \ dt$$

$$S = 216\pi \int_1^2 e^{\frac{9t}{2}} + e^{\frac{3t}{2}} \ dt$$

$$S = 216\pi \left(\frac{2}{9}e^{\frac{9t}{2}} + \frac{2}{3}e^{\frac{3t}{2}}\right)\Bigg|_1^2$$

$$S = 144\pi \left(\frac{1}{3}e^{\frac{9t}{2}} + e^{\frac{3t}{2}}\right)\Bigg|_1^2$$

$$S = 48\pi \left(e^{\frac{9t}{2}} + 3e^{\frac{3t}{2}}\right)\Bigg|_1^2$$

$$S = 48\pi \left[\left(e^{\frac{9(2)}{2}} + 3e^{\frac{3(2)}{2}}\right) - \left(e^{\frac{9(1)}{2}} + 3e^{\frac{3(1)}{2}}\right)\right]$$

$$S = 48\pi \left(e^9 + 3e^3 - e^{\frac{9}{2}} - 3e^{\frac{3}{2}}\right)$$

Topic: Surface area of revolution of a parametric curve, horizontal axis

Question: A circle is defined by the parametric functions $x = 3\cos t$ and $y = 3 + 3\sin t$. The curve is revolved around the x-axis. Which of the following pieces of surface area is the smallest?

Area A_1 is between $t = 0$ and $t = \pi$

Area A_2 is between $t = 0$ and $t = 2\pi$

Area A_3 is between $t = \pi/6$ and $t = 7\pi/6$

Area A_4 is between $t = \pi/2$ and $t = 3\pi/2$

Answer choices:

A A_1

B A_2

C A_3

D A_4

Solution: D

Differentiate both functions, and square the results.

$$x = 3\cos t$$

$$\frac{dx}{dt} = -3\sin t$$

$$\left(\frac{dx}{dt}\right)^2 = \left(-3\sin t\right)^2$$

$$\left(\frac{dx}{dt}\right)^2 = 9\sin^2 t$$

and

$$y = 3 + 3\sin t$$

$$\frac{dy}{dt} = 3\cos t$$

$$\left(\frac{dy}{dt}\right)^2 = (3\cos t)^2$$

$$\left(\frac{dy}{dt}\right)^2 = 9\cos^2 t$$

Plug the values we've found into the surface area of revolution formula:

$$S = \int_1^2 2\pi y\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\ dt$$

For A_1 between $t=0$ and $t=\pi$:

$$A_1=\int_0^{\pi} 2\pi\left(3+3\sin t\right)\sqrt{9\left(\sin^2+\cos^2 t\right)}\ dt$$

$$A_1=\int_0^{\pi} 6\pi\left(3+3\sin t\right)\ dt$$

$$A_1=18\pi\int_0^{\pi}\left(1+\sin t\right)\ dt$$

$$A_1=18\pi\left(t-\cos t\right)\Big|_0^{\pi}$$

$$A_1=18\pi\left[(\pi-\cos\pi)-(0-\cos 0)\right]$$

$$A_1=18\pi(\pi+1+1)$$

$$A_1=18\pi(\pi+2)$$

For A_2 between $t=0$ and $t=2\pi$:

$$A_2=\int_0^{2\pi} 2\pi\left(3+3\sin t\right)\sqrt{9\left(\sin^2+\cos^2 t\right)}\ dt$$

$$A_2=18\pi\left(t-\cos t\right)\Big|_0^{2\pi}$$

$$A_2=18\pi\left[(2\pi-1)-(0-1)\right]$$

$$A_2=36\pi^2$$

For A_3 between $t=\pi/6$ and $t=7\pi/6$:

$$A_3 = \int_{\frac{\pi}{6}}^{\frac{7\pi}{6}} 2\pi\left(3 + 3\sin t\right)\sqrt{9\left(\sin^2 + \cos^2 t\right)}\ dt$$

$$A_3 = 18\pi\left(t - \cos t\right)\Big|_{\frac{\pi}{6}}^{\frac{7\pi}{6}}$$

$$A_3 = 18\pi\left[\left(\frac{7\pi}{6} - \cos\frac{7\pi}{6}\right) - \left(\frac{\pi}{6} - \cos\frac{\pi}{6}\right)\right]$$

$$A_3 = 18\pi\left(\frac{7\pi}{6} + \frac{\sqrt{3}}{2} - \frac{\pi}{6} + \frac{\sqrt{3}}{2}\right)$$

$$A_3 = 18\pi\left(\pi + \sqrt{3}\right)$$

For A_4 between $t = \pi/2$ and $t = 3\pi/2$:

$$A_4 = \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} 2\pi\left(3 + 3\sin t\right)\sqrt{9\left(\sin^2 + \cos^2 t\right)}\ dt$$

$$A_4 = 18\pi\left(t - \cos t\right)\Big|_{\frac{\pi}{2}}^{\frac{3\pi}{2}}$$

$$A_4 = 18\pi\left[\left(\frac{3\pi}{2} - \cos\frac{3\pi}{2}\right) - \left(\frac{\pi}{2} - \cos\frac{\pi}{2}\right)\right]$$

$$A_4 = 18\pi\left(\frac{3\pi}{2} - 0 - \frac{\pi}{2} + 0\right)$$

$$A_4 = 18\pi^2$$

If we compare the amount of area in each region,

$$A_1 = 18\pi(\pi + 2) \approx 290.75$$

$$A_2 = 36\pi^2 \approx 355.31$$

$$A_3 = 18\pi\left(\pi + \sqrt{3}\right) \approx 275.60$$

$$A_4 = 18\pi^2 \approx 177.65$$

we can see that A_4 is the smallest region of area.