Topic: Inverse hyperbolic integrals

Question: Use inverse hyperbolic functions to evaluate the integral.

$$\int_5^8 \frac{dx}{9 - x^2}$$

Answer choices:

A $\frac{1}{6} \ln \frac{20}{11}$

B $\frac{1}{6} \ln \frac{11}{20}$

C $4 \ln 5$

D $-4 \ln 2$

Solution: B

Rewriting the integral gives

$$\int_5^8 \frac{dx}{9 - x^2}$$

$$\int_5^8 \frac{1}{3^2 - x^2}\ dx$$

The integral is of the form

$$\int \frac{1}{a^2 - u^2}\ du = \frac{1}{a}\coth^{-1}\frac{u}{a} + C$$

Note that when the integrated function matches the form

$$\int \frac{1}{a^2 - u^2}\ du$$

the integration formula you'll use depends on the relationship between u and a. More specifically,

$$\int \frac{1}{a^2 - u^2}\ du = \frac{1}{a}\coth^{-1}\frac{u}{a} + C \qquad \text{when } u^2 > a^2$$

$$\int \frac{1}{a^2 - u^2}\ du = \frac{1}{a}\tanh^{-1}\frac{u}{a} + C \qquad \text{when } u^2 < a^2$$

Since the limits of integration are given as $[5,8]$, x will always be between 5 and 8, which means x^2, or u^2, will always be between $5^2 = 25$ and $8^2 = 64$. Since $a = 3$ and $3^2 = 9$, we can say that $u^2 > a^2$, we'll use the hyperbolic cotangent formula, and therefore the integral becomes

$$\frac{1}{3}\coth^{-1}\frac{x}{3}\Big|_5^8$$

$$\frac{1}{3}\coth^{-1}\frac{8}{3}-\frac{1}{3}\coth^{-1}\frac{5}{3}$$

Knowing that

$$\coth^{-1}x=\frac{1}{2}\ln\frac{x+1}{x-1}$$

we can apply the formula and get

$$\frac{1}{3}\left(\frac{1}{2}\ln\frac{\frac{8}{3}+1}{\frac{8}{3}-1}\right)-\frac{1}{3}\left(\frac{1}{2}\ln\frac{\frac{5}{3}+1}{\frac{5}{3}-1}\right)$$

$$\frac{1}{6}\ln\frac{\frac{8}{3}+\frac{3}{3}}{\frac{8}{3}-\frac{3}{3}}-\frac{1}{6}\ln\frac{\frac{5}{3}+\frac{3}{3}}{\frac{5}{3}-\frac{3}{3}}$$

$$\frac{1}{6}\ln\frac{\frac{11}{3}}{\frac{5}{3}}-\frac{1}{6}\ln\frac{\frac{8}{3}}{\frac{2}{3}}$$

$$\frac{1}{6}\ln\left(\frac{11}{3}\cdot\frac{3}{5}\right)-\frac{1}{6}\ln\left(\frac{8}{3}\cdot\frac{3}{2}\right)$$

$$\frac{1}{6}\ln\frac{11}{5}-\frac{1}{6}\ln\frac{8}{2}$$

$$\frac{1}{6}\ln\frac{\frac{11}{5}}{\frac{8}{2}}$$

$$\frac{1}{6}\ln\left(\frac{11}{5}\cdot\frac{2}{8}\right)$$

$$\frac{1}{6}\ln\frac{11}{20}$$

Topic: Inverse hyperbolic integrals

Question: Evaluate the integral using integration of inverse hyperbolic functions.

$$\int \frac{1}{\sqrt{x^2+4x+8}}\, dx$$

Answer choices:

A $\cosh^{-1}\left(\frac{x+2}{2}\right)+C$

B $\frac{1}{2}\tanh^{-1}\left(\frac{x+2}{2}\right)+C$

C $\sinh^{-1}\left(\frac{x+2}{2}\right)+C$

D $\frac{1}{2}\sinh^{-1}\left(\frac{x+2}{2}\right)$

Solution: C

An integral of inverse hyperbolic functions takes on one of these common patterns.

$$\int \frac{1}{\sqrt{1+u^2}}\ du = \sinh^{-1}u + C$$

$$\int \frac{1}{\sqrt{u^2-1}}\ du = \cosh^{-1}u + C$$

$$\int \frac{1}{1-u^2}\ du = \tanh^{-1}u + C$$

We'll start by manipulating the integrand. The first step is the change the denominator so it contains a squared binomial. If we split the 8 into two 4's, we can accomplish this. The new integral is

$$\int \frac{1}{\sqrt{4+\left(x^2+4x+4\right)}}\ dx$$

$$\int \frac{1}{\sqrt{4+(x+2)^2}}\ dx$$

Now the denominator needs to be changed to the form $1+u^2$ so if we divide the terms in the denominator by $\sqrt{4}$, we can convert it to what we want. However, if we divide the denominator by $\sqrt{4}$, we also have to divide the numerator by $\sqrt{4}$.

$$\int \frac{\frac{1}{\sqrt{4}}}{\sqrt{\frac{4}{4}+\frac{(x+2)^2}{4}}}\ dx$$

Simplify the denominator and include the 4 in the squared binomial, as a 2.

$$\int \frac{\frac{1}{2}}{\sqrt{1+\left(\frac{x+2}{2}\right)^2}}\ dx$$

Now the integral is almost in the form of the $\sinh^{-1}$ integral. The angle is the function

$$u=\frac{x+2}{2}$$

The derivative is $du/dx=1/2$, so $dx=2\ du$. The integral becomes

$$\int \frac{\frac{1}{2}}{\sqrt{1+u^2}}\cdot 2\ du$$

$$\int \frac{1}{\sqrt{1+u^2}}\ du$$

Now we can evaluate the integral.

$$\sinh^{-1}u+C$$

$$\sinh^{-1}\left(\frac{x+2}{2}\right)+C$$

Topic: Inverse hyperbolic integrals

Question: Evaluate the integral using integration of inverse hyperbolic functions.

$$\int_0^1 \frac{x}{4 - x^4}\, dx$$

Answer choices:

A $\quad \frac{1}{4}\tanh^{-1}\left(\frac{1}{2}\right)$

B $\quad \frac{1}{2}\tanh^{-1}\left(\frac{1}{2}\right)$

C $\quad \frac{1}{4}\sinh^{-1}\left(\frac{1}{2}\right)$

D $\quad \frac{1}{2}\sinh^{-1}\left(\frac{1}{2}\right)$

Solution: A

We should use a substitution with $u^2 = x^4$, and therefore $u = x^2$, $du/dx = 2x$, $du = 2x\ dx$, or $dx = du/2x$.

Convert the bounds $x = [0,1]$ into bounds in terms of u, using $u = x^2$.

$$u = 0^2 = 0$$

$$u = 1^2 = 1$$

Then the integral becomes

$$\int_0^1 \frac{x}{4 - u^2}\left(\frac{du}{2x}\right)$$

$$\frac{1}{2}\int_0^1 \frac{1}{4 - u^2}\ du$$

$$\frac{1}{2}\int_0^1 \frac{1}{2^2 - u^2}\ du$$

Whether this particular integrand integrates to inverse hyperbolic tangent or inverse hyperbolic cotangent depends on the relationship between a^2 and u^2.

$$\int \frac{1}{a^2 - u^2}\ du = \frac{1}{a}\text{arctanh}\left(\frac{u}{a}\right) + C \qquad \text{if } u^2 < a^2$$

$$\int \frac{1}{a^2 - u^2}\ du = \frac{1}{a}\text{arccoth}\left(\frac{u}{a}\right) + C \qquad \text{if } u^2 > a^2$$

Because the bounds on the integral are $u=[0,1]$, the largest possible value of u^2 in the interval is $u^2=1^2=1$. Then because the value of a^2 is $a^2=2^2=4$, we know $u^2=1<a^2=4$, which means we'll evaluate this integral to inverse hyperbolic tangent, instead of inverse hyperbolic cotangent.

$$\frac{1}{2}\left(\frac{1}{2}\tanh^{-1}\left(\frac{u}{2}\right)\right)\Big|_0^1$$

$$\frac{1}{4}\tanh^{-1}\left(\frac{u}{2}\right)\Big|_0^1$$

Evaluate over the interval.

$$\frac{1}{4}\tanh^{-1}\left(\frac{1}{2}\right)-\frac{1}{4}\tanh^{-1}\left(\frac{0}{2}\right)$$

$$\frac{1}{4}\tanh^{-1}\left(\frac{1}{2}\right)$$