Topic: Trigonometric substitution with tangent

Question: Use trigonometric substitution to evaluate the integral.

$$\int \frac{1}{1+4x^2}\ dx$$

Answer choices:

A $\frac{1}{2}\sin^{-1}(2x)+C$

B $\frac{1}{2}\cos^{-1}(2x)+C$

C $\frac{1}{2}\tan^{-1}(2x)+C$

D $\frac{1}{2}\cot^{-1}(2x)+C$

Solution: C

We can use trigonometric substitution to evaluate the integral. Recognizing that

$$a^2 + u^2 = 1 + 4x^2$$

we get

$$a = 1$$

$$u = 2x$$

Knowing that

$$u = a \tan \theta$$

is the substitution we use for $a^2 + u^2$, we get

$$2x = 1 \tan \theta$$

$$2x = \tan \theta$$

$$x = \frac{1}{2} \tan \theta$$

$$dx = \frac{1}{2} \sec^2 \theta \ d\theta$$

$$\theta = \tan^{-1} 2x$$

Plugging these into the integral we get

$$\int \frac{1}{1 + 4x^2} \ dx$$

$$\int \frac{1}{1+4\left(\frac{1}{2}\tan\theta\right)^2}\frac{1}{2}\sec^2\theta\ d\theta$$

$$\frac{1}{2}\int \frac{\sec^2\theta}{1+4\left(\frac{1}{4}\tan^2\theta\right)}\ d\theta$$

$$\frac{1}{2}\int \frac{\sec^2\theta}{1+\tan^2\theta}\ d\theta$$

We know that $1+\tan^2 x=\sec^2 x$, so we'll make a substitution to simplify the integral.

$$\frac{1}{2}\int \frac{\sec^2\theta}{\sec^2\theta}\ d\theta$$

$$\frac{1}{2}\int\ d\theta$$

$$\frac{1}{2}\theta+C$$

Back-substituting for x, we get

$$\frac{1}{2}\tan^{-1}(2x)+C$$

Topic: Trigonometric substitution with tangent

Question: Use trigonometric substitution to evaluate the integral.

$$\int_{0}^{1} \frac{dx}{\sqrt{x^2 + 2x + 2}}$$

Answer choices:

A 0.65

B 0.74

C 0.56

D 0.99

Solution: C

First, complete the square to rewrite the integral as

$$\int_0^1 \frac{dx}{\sqrt{x^2+2x+2}}$$

$$\int_0^1 \frac{dx}{\sqrt{\left(x^2+2x+1\right)+1}}$$

$$\int_0^1 \frac{dx}{\sqrt{(x+1)^2+1}}$$

We can now use trigonometric substitution to evaluate the integral. Recognizing that

$$u^2+a^2=(x+1)^2+1$$

we get

$$u=x+1$$

$$a=1$$

Knowing that

$$u=a\tan\theta$$

is the substitution we use for u^2+a^2, we get

$$x+1=1\tan\theta$$

$$x+1=\tan\theta$$

$$x=\tan\theta-1$$

$$dx=\sec^2\theta\ d\theta$$

$$\theta=\tan^{-1}(x+1)$$

Plugging these into the integral we get

$$\int_{x=0}^{x=1}\frac{\sec^2\theta\ d\theta}{\sqrt{(\tan\theta-1+1)^2+1}}$$

$$\int_{x=0}^{x=1}\frac{\sec^2\theta}{\sqrt{\tan^2\theta+1}}\ d\theta$$

We know that $\tan^2 x+1=\sec^2 x$, so we'll make a substitution to simplify the integral.

$$\int_{x=0}^{x=1}\frac{\sec^2\theta}{\sqrt{\sec^2\theta}}\ d\theta$$

$$\int_{x=0}^{x=1}\frac{\sec^2\theta}{\sec\theta}\ d\theta$$

$$\int_{x=0}^{x=1}\sec\theta\ d\theta$$

Our formula for the integral of $\sec x$ is

$$\int\sec x\ dx=\ln|\sec x+\tan x|+C$$

so the integral becomes

$$\ln|\sec\theta+\tan\theta|\Big|_{x=0}^{x=1}$$

Using $\theta=\tan^{-1}(x+1)$ to back-substitute, we get

$$\ln\left|\sec\left[\tan^{-1}(x+1)\right]+\tan\left[\tan^{-1}(x+1)\right]\right|\Bigg|_0^1$$

The $\tan$ and $\tan^{-1}$ functions cancel with one another. We also know that

$$\sec\left(\tan^{-1}x\right)=\sqrt{x^2+1}$$

so we can simplify to

$$\ln\left|\sqrt{(x+1)^2+1}+(x+1)\right|\Bigg|_0^1$$

$$\ln\left|\sqrt{x^2+2x+1+1}+x+1\right|\Bigg|_0^1$$

$$\ln\left|x+1+\sqrt{x^2+2x+2}\right|\Bigg|_0^1$$

Evaluate over the interval.

$$\ln\left|1+1+\sqrt{(1)^2+2(1)+2}\right|-\ln\left|0+1+\sqrt{(0)^2+2(0)+2}\right|$$

$$\ln\left|2+\sqrt{1+2+2}\right|-\ln\left|1+\sqrt{2}\right|$$

$$\ln\left|2+\sqrt{5}\right| - \ln\left|1+\sqrt{2}\right|$$

Using laws of logarithms, we get

$$\ln\frac{\left|2+\sqrt{5}\right|}{\left|1+\sqrt{2}\right|}$$

The absolute values are irrelevant, since neither the numerator or denominator could ever be negative.

$$\ln\frac{2+\sqrt{5}}{1+\sqrt{2}}$$

$$0.56$$

Topic: Trigonometric substitution with tangent

Question: Use trigonometric substitution to evaluate the integral.

$$\int \frac{x^3}{\left(x^2+4\right)^{\frac{3}{2}}}\ dx$$

Answer choices:

A $\quad \dfrac{x^2-12}{\sqrt{x^2+4}}+C$

B $\quad \dfrac{x^2-8}{\sqrt{x^2+4}}+C$

C $\quad \dfrac{x^2+12}{\sqrt{x^2+4}}+C$

D $\quad \dfrac{x^2+8}{\sqrt{x^2+4}}+C$

Solution: D

Recognizing that we have

$$u^2 + a^2 = x^2 + 2^2$$

in the integral, we get

$$u = x$$

$$a = 2$$

Knowing that

$$u = a \tan \theta$$

is the substitution we use for $u^2 + a^2$, we get

$$x = 2 \tan \theta$$

$$\frac{x}{2} = \tan \theta$$

$$dx = 2 \sec^2 \theta \ d\theta$$

$$\theta = \tan^{-1} \frac{x}{2}$$

Plugging these into the integral we get

$$\int \frac{x^3}{\left(x^2 + 4\right)^{\frac{3}{2}}} \ dx$$

$$\int \frac{(2\tan\theta)^3}{\left[(2\tan\theta)^2+4\right]^{\frac{3}{2}}} 2\sec^2\theta\ d\theta$$

$$16\int \frac{\tan^3\theta\sec^2\theta}{\left(4\tan^2\theta+4\right)^{\frac{3}{2}}}\ d\theta$$

$$16\int \frac{\tan^3\theta\sec^2\theta}{\left(4^{\frac{1}{2}}\right)^3\left(\tan^2\theta+1\right)^{\frac{3}{2}}}\ d\theta$$

$$2\int \frac{\tan^3\theta\sec^2\theta}{\left(\tan^2\theta+1\right)^{\frac{3}{2}}}\ d\theta$$

We know that $\tan^2 x + 1 = \sec^2 x$, so we'll make a substitution to simplify the integral.

$$2\int \frac{\tan^3\theta\sec^2\theta}{\left[\left(\sec^2\theta\right)^{\frac{1}{2}}\right]^3}\ d\theta$$

$$2\int \frac{\tan^3\theta\sec^2\theta}{\sec^3\theta}\ d\theta$$

$$2\int \frac{\tan^3\theta}{\sec\theta}\ d\theta$$

$$2\int \frac{\frac{\sin^3\theta}{\cos^3\theta}}{\frac{1}{\cos\theta}}\ d\theta$$

$$2\int \frac{\sin^3\theta}{\cos^2\theta}\ d\theta$$

$$2\int \frac{\sin^2\theta\sin\theta}{\cos^2\theta}\ d\theta$$

We know that $\sin^2 x = 1 - \cos^2 x$, so we'll make a substitution to simplify the integral.

$$2\int \frac{\left(1-\cos^2\theta\right)\sin\theta}{\cos^2\theta}\ d\theta$$

$$2\int \frac{\sin\theta - \sin\theta\cos^2\theta}{\cos^2\theta}\ d\theta$$

$$2\int \frac{\sin\theta}{\cos^2\theta} - \frac{\sin\theta\cos^2\theta}{\cos^2\theta}\ d\theta$$

$$2\int \sin\theta\cos^{-2}\theta - \sin\theta\ d\theta$$

$$2\int \sin\theta\cos^{-2}\theta\ d\theta - 2\int \sin\theta\ d\theta$$

$$2\int \sin\theta\cos^{-2}\theta\ d\theta - 2(-\cos\theta) + C$$

$$2\int \sin\theta\cos^{-2}\theta\ d\theta + 2\cos\theta + C$$

For the remaining integral, we'll use substitution and let

$$v = \cos\theta$$

$$dv = -\sin\theta \ d\theta$$

$$-\frac{dv}{\sin\theta} = d\theta$$

Plugging these back into the integral, we'll get

$$2\int \sin\theta v^{-2}\left(-\frac{dv}{\sin\theta}\right) + 2\cos\theta + C$$

$$-2\int v^{-2} \ dv + 2\cos\theta + C$$

$$-2\left(-v^{-1}\right) + 2\cos\theta + C$$

$$\frac{2}{v} + 2\cos\theta + C$$

Back-substituting for θ, we get

$$\frac{2}{\cos\theta} + 2\cos\theta + C$$

Back-substituting for x, we get

$$\frac{2}{\cos\left(\tan^{-1}\frac{x}{2}\right)} + 2\cos\left(\tan^{-1}\frac{x}{2}\right) + C$$

$$\frac{2}{\frac{1}{\sqrt{\left(\frac{x}{2}\right)^2+1}}} + 2\frac{1}{\sqrt{\left(\frac{x}{2}\right)^2+1}} + C$$

$$\frac{2}{\frac{1}{\sqrt{\frac{x^2+4}{4}}}}+\frac{2}{\sqrt{\frac{x^2+4}{4}}}+C$$

$$2\sqrt{\frac{x^2+4}{4}}+\frac{2}{\sqrt{\frac{x^2+4}{4}}}+C$$

$$\sqrt{x^2+4}+\frac{2}{\frac{1}{2}\sqrt{x^2+4}}+C$$

$$\sqrt{x^2+4}+\frac{4}{\sqrt{x^2+4}}+C$$

$$\frac{x^2+4}{\sqrt{x^2+4}}+\frac{4}{\sqrt{x^2+4}}+C$$

$$\frac{x^2+8}{\sqrt{x^2+4}}+C$$