Calculus 1
Workbook Solutions

Chain rule

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MATH

CHAIN RULE WITH POWER RULE

■ 1. Find $h'(x)$ if $h(x) = (3x^2 - 7)^4$.

Solution:

To find the derivative, we have to apply chain rule. We'll say that the inside function is $3x^2 - 7$, and that the derivative of that inside function is $6x$. Therefore, the derivative is

$$h'(x) = 4(3x^2 - 7)^3(6x)$$

$$h'(x) = 24x(3x^2 - 7)^3$$

■ 2. Find $h'(x)$ if $h(x) = \sqrt{2 - 4x^2}$.

Solution:

To find the derivative, we have to apply chain rule. We'll say that the inside function is $2 - 4x^2$, and that the derivative of that inside function is $-8x$. Therefore, the derivative is

$$h'(x) = \frac{1}{2}(2 - 4x^2)^{-\frac{1}{2}}(-8x)$$

$$h'(x) = -4x(2 - 4x^2)^{-\frac{1}{2}}$$

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$$h'(x) = -\frac{4x}{\sqrt{2-4x^2}}$$

■ 3. Find $h'(x)$ if $h(x) = (2x^2 - 6x + 5)^7$.

Solution:

To find the derivative, we have to apply chain rule. We'll say that the inside function is $2x^2 - 6x + 5$, and that the derivative of that inside function is $4x - 6$. Therefore, the derivative is

$$h'(x) = 7(2x^2 - 6x + 5)^6(4x - 6)$$

$$h'(x) = 7(4x - 6)(2x^2 - 6x + 5)^6$$

■ 4. Find $h'(x)$ if $h(x) = 2(x^3 + 4x^2 - 2x)^{-5}$.

Solution:

To find the derivative, we have to apply chain rule. We'll say that the inside function is $x^3 + 4x^2 - 2x$, and that the derivative of that inside function is $3x^2 + 8x - 2$. Therefore, the derivative is

$$h'(x) = 2(-5)(x^3 + 4x^2 - 2x)^{-6}(3x^2 + 8x - 2)$$

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$$h'(x) = -10(3x^2 + 8x - 2)(x^3 + 4x^2 - 2x)^{-6}$$

$$h'(x) = -\frac{10(3x^2 + 8x - 2)}{(x^3 + 4x^2 - 2x)^6}$$

■ 5. Find $f'(x)$ if $f(x) = 3(5x^2 + \sin x)^4$.

Solution:

To find the derivative, we have to apply chain rule. We'll say that the inside function is $5x^2 + \sin x$, and that the derivative of that inside function is $10x + \cos x$. Therefore, the derivative is

$$f'(x) = 3(4)(5x^2 + \sin x)^3(10x + \cos x)$$

$$f'(x) = 12(10x + \cos x)(5x^2 + \sin x)^3$$

■ 6. Find $g'(y)$ if $g(y) = \sqrt{3y + (2y + y^2)^2}$.

Solution:

To find the derivative, we have to apply chain rule two times. We'll say that the inside function is $3y + (2y + y^2)^2$, and that the derivative of that inside function is $3 + 2(2y + y^2)(2 + 2y)$ or $3 + (4 + 4y)(2y + y^2)$. Therefore, the derivative is

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$$g'(y) = \frac{1}{2}(3y + (2y + y^2)^2)^{-\frac{1}{2}}(3 + (4 + 4y)(2y + y^2))$$

$$g'(y) = \frac{3 + (4 + 4y)(2y + y^2)}{2\sqrt{3y + (2y + y^2)^2}}$$

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CHAIN RULE WITH TRIG, LOG, AND EXPONENTIAL FUNCTIONS

■ 1. Find $f'(x)$.

$$f(x) = \ln(x^2 + 6x + 9)$$

Solution:

Take the derivative, remembering to apply chain rule.

$$f'(x) = \frac{1}{x^2 + 6x + 9} \cdot (2x + 6)$$

$$f'(x) = \frac{2x + 6}{x^2 + 6x + 9}$$

$$f'(x) = \frac{2(x + 3)}{(x + 3)(x + 3)}$$

$$f'(x) = \frac{2}{x + 3}$$

■ 2. Find $g'(x)$ if $g(x) = 3\sin(4x^3) - 4\cos(6x) + 3\sec(2x^4)$.

Solution:

Differentiate one term at a time, remembering to apply chain rule.

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$$g'(x) = 3\cos(4x^3)(12x^2) - 4(-\sin(6x))(6) + 3\sec(2x^4)\tan(2x^4)(8x^3)$$

$$g'(x) = 36x^2\cos(4x^3) + 24\sin(6x) + 24x^3\tan(2x^4)\sec(2x^4)$$

$$g'(x) = 12(2x^3\tan(2x^4)\sec(2x^4) + 3x^2\cos(4x^3) + 2\sin(6x))$$

■ 3. Find $h'(x)$ if $h(x) = \cos(\sin x + 3x^3)$.

Solution:

To find the derivative, we have to apply chain rule. We'll say that the inside function is $\sin x + 3x^3$, and that the derivative of that inside function is $\cos x + 9x^2$. Therefore, the derivative is

$$h'(x) = -\sin(\sin x + 3x^3)(\cos x + 9x^2)$$

$$h'(x) = -(\cos x + 9x^2)\sin(\sin x + 3x^3)$$

■ 4. Find $f'(y)$ if $f(y) = e^{y+\ln y} + 8^{\cos y}$.

Solution:

To find the derivative, we have to apply chain rule two times. In the first term, $e^{y+\ln y}$, the inside function is $y + \ln y$, and the derivative of that inside

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function is $1 + (1/y)$. In the second term, $8^{\cos y}$, the inside function is $\cos y$, and the derivative of that inside function is $-\sin y$. Therefore, the derivative is

$$f'(y) = e^{y+\ln y}\left(1 + \frac{1}{y}\right) + 8^{\cos y}\ln 8(-\sin y)$$

$$f'(y) = e^{y+\ln y}\left(1 + \frac{1}{y}\right) - \ln 8\sin(y)8^{\cos y}$$

Use properties of exponential functions to simplify.

$$f'(y) = ye^{y}\left(1 + \frac{1}{y}\right) - \ln 8\sin(y)8^{\cos y}$$

$$f'(y) = ye^{y}\left(\frac{y+1}{y}\right) - \ln 8\sin(y)8^{\cos y}$$

$$f'(y) = e^{y}(y+1) - \ln 8\sin(y)8^{\cos y}$$

■ 5. Find $f'(x)$ if $f(x) = \tan^5 x + \tan x^5$.

Solution:

To find the derivative, we have to apply chain rule two times. In the first term, $\tan^5 x$, the inside function is $\tan x$, and the derivative of that inside function is $\sec^2 x$. In the second term, $\tan x^5$, the inside function is x^5, and the derivative of that inside function is $5x^4$. Therefore, the derivative is

$$f'(x) = 5\tan^4 x(\sec^2 x) + \sec^2(x^5)(5x^4)$$

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$$f'(x) = 5\tan^4 x \sec^2 x + 5x^4 \sec^2(x^5)$$

■ 6. Find $g'(x)$ if $g(x) = \ln(e^{\sin x} - \sin^2 x)$.

Solution:

To find the derivative, we have to apply chain rule three times. We'll say that the inside function is $e^{\sin x} - \sin^2 x$, and to find the derivative of that inside function we need to use chain rule for each term.

The derivative of that inside function is $e^{\sin x}(\cos x) - 2\sin x(\cos x)$, so the derivative $g'(x)$ is

$$g'(x) = \frac{1}{e^{\sin x} - \sin^2 x}(e^{\sin x}(\cos x) - 2\sin x(\cos x))$$

$$g'(x) = \frac{e^{\sin x}(\cos x) - 2\sin x(\cos x)}{e^{\sin x} - \sin^2 x}$$

$$g'(x) = \frac{\cos x(e^{\sin x} - 2\sin x)}{e^{\sin x} - \sin^2 x}$$

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CHAIN RULE WITH PRODUCT RULE

■ 1. Find $y'(x)$ if $y(x) = (3x - 2)(5x^3)^5$.

Solution:

To find the derivative, we have to apply product rule.

$$y'(x) = \frac{d}{dx}(3x - 2) \cdot (5x^3)^5 + (3x - 2) \cdot \frac{d}{dx}(5x^3)^5$$

To find each derivative, we have to apply chain rule.

$$y'(x) = 3 \cdot (5x^3)^5 + (3x - 2) \cdot 5(5x^3)^4(15x^2)$$

$$y'(x) = 3(5x^3)^5 + 75x^2(3x - 2)(5x^3)^4$$

■ 2. Find $h'(x)$ if $h(x) = (x^2 - 5x)^2(2x^3 - 3x^2)^5$.

Solution:

To find the derivative, we have to apply product rule.

$$h'(x) = \frac{d}{dx}(x^2 - 5x)^2 \cdot (2x^3 - 3x^2)^5 + (x^2 - 5x)^2 \cdot \frac{d}{dx}(2x^3 - 3x^2)^5$$

To find each derivative, we have to apply chain rule.

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$$h'(x) = 2(x^2 - 5x)(2x - 5) \cdot (2x^3 - 3x^2)^5 + (x^2 - 5x)^2 \cdot 5(2x^3 - 3x^2)^4(6x^2 - 6x)$$

$$h'(x) = 2(2x - 5)(x^2 - 5x)(2x^3 - 3x^2)^5 + 5(6x^2 - 6x)(x^2 - 5x)^2(2x^3 - 3x^2)^4$$

■ 3. Find the derivative of the function.

$$y = (\sin(7x))(7e^{4x})(2x^6 + 1)$$

Solution:

The derivative of a function $y = f(x)g(x)h(x)$ using the product rule is

$$y' = f'(x)g(x)h(x) + f(x)g'(x)h(x) + f(x)g(x)h'(x)$$

To find each derivative, we have to apply chain rule. We'll let

$$f(x) = \sin(7x)$$

$$f'(x) = 7\cos(7x)$$

$$g(x) = 7e^{4x}$$

$$g'(x) = 28e^{4x}$$

$$h(x) = 2x^6 + 1$$

$$h'(x) = 12x^5$$

Then by product rule, the derivative is

$$y' = (7\cos(7x))(7e^{4x})(2x^6 + 1) + (\sin(7x))(28e^{4x})(2x^6 + 1) + (\sin(7x))(7e^{4x})(12x^5)$$

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$$y' = 49e^{4x}(2x^6 + 1)\cos(7x) + 28e^{4x}(2x^6 + 1)\sin(7x) + 84x^5e^{4x}\sin(7x)$$

■ 4. Find $h'(x)$ if $h(x) = \sin(4x)e^{3x^2+4}$.

Solution:

Use product rule to take the derivative.

$$h'(x) = \frac{d}{dx}\sin(4x) \cdot e^{3x^2+4} + \sin(4x) \cdot \frac{d}{dx}e^{3x^2+4}$$

To find each derivative, we have to apply chain rule.

$$h'(x) = \cos(4x)(4) \cdot e^{3x^2+4} + \sin(4x) \cdot e^{3x^2+4}(6x)$$

$$h'(x) = 4e^{3x^2+4}\cos(4x) + 6xe^{3x^2+4}\sin(4x)$$

Factor to simplify.

$$h'(x) = 2e^{3x^2+4}(3x\sin(4x) + 2\cos(4x))$$

■ 5. Find the derivative of the function.

$$y = \sin(x^2e^{x^2})$$

Solution:

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If we use substitution with $u = x^2e^{x^2}$, then we can rewrite the function as

$$y = \sin u$$

$$y' = \cos(u)u'$$

We need to plug in for u and u', so let's find u' using product rule and chain rule with

$$f(x) = x^2$$

$$f'(x) = 2x$$

$$g(x) = e^{x^2}$$

$$g'(x) = e^{x^2}(2x) = 2xe^{x^2}$$

Then u' is

$$u' = (x^2)(2xe^{x^2}) + (2x)(e^{x^2})$$

$$u' = 2x^3e^{x^2} + 2xe^{x^2}$$

$$u' = 2xe^{x^2}(x^2 + 1)$$

Now we can back-substitute into the equation we found for y', and then simplify.

$$y' = \cos(x^2e^{x^2}) \cdot 2xe^{x^2}(x^2 + 1)$$

$$y' = 2xe^{x^2}(x^2 + 1)\cos(x^2e^{x^2})$$

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■ 6. Find $h'(x)$ if $h(x) = \ln(x^3\sqrt{3x^4 - 2x^2 + 3})$.

Solution:

If we use substitution with $u = x^3\sqrt{3x^4 - 2x^2 + 3}$, then we can rewrite the function and its derivative as

$$y = \ln u$$

$$y' = \frac{1}{u} \cdot u'$$

We need to plug in for u and u', so let's find u' using product rule and chain rule with

$$f(x) = x^3$$

$$f'(x) = 3x^2$$

$$g(x) = \sqrt{3x^4 - 2x^2 + 3}$$

$$g'(x) = \frac{1}{2}(3x^4 - 2x^2 + 3)^{-\frac{1}{2}}(12x^3 - 4x)$$

$$g'(x) = \frac{6x^3 - 2x}{\sqrt{3x^4 - 2x^2 + 3}}$$

Then

$$u' = (x^3)\left(\frac{6x^3 - 2x}{\sqrt{3x^4 - 2x^2 + 3}}\right) + (3x^2)(\sqrt{3x^4 - 2x^2 + 3})$$

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$$u' = \frac{6x^6 - 2x^4}{\sqrt{3x^4 - 2x^2 + 3}} + 3x^2\sqrt{3x^4 - 2x^2 + 3}$$

Now we can back-substitute into the equation we found for y', and then simplify.

$$y' = \frac{1}{x^3\sqrt{3x^4 - 2x^2 + 3}}\left(\frac{6x^6 - 2x^4}{\sqrt{3x^4 - 2x^2 + 3}} + 3x^2\sqrt{3x^4 - 2x^2 + 3}\right)$$

$$y' = \frac{6x^3 - 2x}{3x^4 - 2x^2 + 3} + \frac{3}{x}$$

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CHAIN RULE WITH QUOTIENT RULE

■ 1. Find $h'(x)$.

$$h(x) = \frac{(2x+1)^3}{(3x-2)^2}$$

Solution:

To find the derivative, we have to apply quotient rule.

$$h'(x) = \frac{\frac{d}{dx}(2x+1)^3 \cdot (3x-2)^2 - (2x+1)^3 \cdot \frac{d}{dx}(3x-2)^2}{((3x-2)^2)^2}$$

To find each derivative, we have to apply chain rule.

$$h'(x) = \frac{3(2x+1)^2(2) \cdot (3x-2)^2 - (2x+1)^3 \cdot 2(3x-2)(3)}{((3x-2)^2)^2}$$

$$h'(x) = \frac{6(2x+1)^2(3x-2)^2 - 6(2x+1)^3(3x-2)}{(3x-2)^4}$$

$$h'(x) = \frac{6(2x+1)^2(3x-2) - 6(2x+1)^3}{(3x-2)^3}$$

Factor the numerator.

$$h'(x) = \frac{6(2x+1)^2(3x-2-(2x+1))}{(3x-2)^3}$$

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$$h'(x) = \frac{6(2x+1)^2(3x-2-2x-1)}{(3x-2)^3}$$

$$h'(x) = \frac{6(2x+1)^2(x-3)}{(3x-2)^3}$$

■ 2. Find $h'(x)$.

$$h(x) = \frac{(4x+5)^5}{(x+3)^2}$$

Solution:

To find the derivative, we have to apply quotient rule.

$$h'(x) = \frac{\frac{d}{dx}(4x+5)^5 \cdot (x+3)^2 - (4x+5)^5 \cdot \frac{d}{dx}(x+3)^2}{((x+3)^2)^2}$$

To find each derivative, we have to apply chain rule.

$$h'(x) = \frac{5(4x+5)^4(4) \cdot (x+3)^2 - (4x+5)^5 \cdot 2(x+3)(1)}{(x+3)^4}$$

$$h'(x) = \frac{20(4x+5)^4(x+3)^2 - 2(4x+5)^5(x+3)}{(x+3)^4}$$

$$h'(x) = \frac{20(4x+5)^4(x+3) - 2(4x+5)^5}{(x+3)^3}$$

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Factor the numerator.

$$h'(x) = \frac{2(4x+5)^4(10(x+3)-(4x+5))}{(x+3)^3}$$

$$h'(x) = \frac{2(4x+5)^4(10x+30-4x-5)}{(x+3)^3}$$

$$h'(x) = \frac{2(4x+5)^4(6x+25)}{(x+3)^3}$$

■ 3. Find $h'(x)$.

$$h(x) = \ln\left(\frac{x^3}{x^2+3}\right)$$

Solution:

If we use substitution with

$$u = \frac{x^3}{x^2+3}$$

then we can rewrite the function and take its derivative.

$$y = \ln u$$

$$y' = \frac{1}{u} \cdot u'$$

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We need to plug in for u and u', so let's find u' using quotient rule, where $f(x)$ is the numerator and $g(x)$ is the denominator.

$$u' = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$$

$$u' = \frac{(3x^2)(x^2+3) - (x^3)(2x)}{(x^2+3)^2}$$

Now we can back-substitute into the equation we found for y'.

$$h'(x) = \frac{1}{\frac{x^3}{x^2+3}} \cdot \frac{3x^2(x^2+3) - x^3(2x)}{(x^2+3)^2}$$

$$h'(x) = \frac{x^2+3}{x^3} \cdot \frac{3x^2(x^2+3) - x^3(2x)}{(x^2+3)^2}$$

$$h'(x) = \frac{1}{x} \cdot \frac{3(x^2+3) - x(2x)}{x^2+3}$$

$$h'(x) = \frac{3x^2+9-2x^2}{x(x^2+3)}$$

$$h'(x) = \frac{x^2+9}{x(x^2+3)}$$

■ 4. Find $h'(x)$.

$$h(x) = \frac{\sec(2-x)}{2x+e^{-x}}$$

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Solution:

To find the derivative, we have to apply quotient rule.

$$h'(x) = \frac{\frac{d}{dx}\sec(2-x)\cdot(2x+e^{-x}) - \sec(2-x)\cdot\frac{d}{dx}(2x+e^{-x})}{(2x+e^{-x})^2}$$

To find each derivative, we have to apply chain rule.

$$h'(x) = \frac{\sec(2-x)\tan(2-x)(-1)\cdot(2x+e^{-x}) - \sec(2-x)\cdot(2+e^{-x}(-1))}{(2x+e^{-x})^2}$$

$$h'(x) = \frac{-(2x+e^{-x})\sec(2-x)\tan(2-x) - (2-e^{-x})\sec(2-x)}{(2x+e^{-x})^2}$$

$$h'(x) = -\frac{(2x+e^{-x})\sec(2-x)\tan(2-x) + (2-e^{-x})\sec(2-x)}{(2x+e^{-x})^2}$$

■ 5. Find $h'(x)$.

$$h(x) = \frac{2+\ln(3x)}{x+\cot(2x)}$$

Solution:

To find the derivative, we have to apply quotient rule.

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$$h'(x) = \frac{\frac{d}{dx}(2 + \ln(3x)) \cdot (x + \cot(2x)) - (2 + \ln(3x)) \cdot \frac{d}{dx}((x + \cot(2x))}{(x + \cot(2x))^2}$$

To find each derivative, we have to apply chain rule.

$$h'(x) = \frac{\frac{1}{3x}(3) \cdot (x + \cot(2x)) - (2 + \ln(3x)) \cdot (1 - \csc^2(2x)(2))}{(x + \cot(2x))^2}$$

$$h'(x) = \frac{\frac{x + \cot(2x)}{x} - (2 + \ln(3x))(1 - 2\csc^2(2x))}{(x + \cot(2x))^2}$$

$$h'(x) = \frac{\frac{x + \cot(2x)}{x}}{(x + \cot(2x))^2} - \frac{(2 + \ln(3x))(1 - 2\csc^2(2x))}{(x + \cot(2x))^2}$$

$$h'(x) = \frac{1}{x(x + \cot(2x))} - \frac{(2 + \ln(3x))(1 - 2\csc^2(2x))}{(x + \cot(2x))^2}$$

■ 6. Find $h'(x)$.

$$h(x) = x^2 \sin\left(\frac{x^3 + 4x}{\sqrt{x^4 - 2}}\right)$$

Solution:

Use product rule with $h'(x) = f(x)g'(x) + f'(x)g(x)$ and let

$$f(x) = x^2$$

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$$f'(x) = 2x$$

and

$$g(x) = \sin\left(\frac{x^3 + 4x}{\sqrt{x^4 - 2}}\right)$$

To find the derivative $g'(x)$, we have to apply chain rule. If we use substitution with

$$u = \frac{x^3 + 4x}{\sqrt{x^4 - 2}}$$

then we can rewrite $g(x)$ and its derivative as

$$g(u) = \sin u$$

$$g'(u) = \cos u \cdot u'$$

We need to plug in for u and u', so let's find u' using quotient rule.

$$u' = \frac{(3x^2 + 4)(\sqrt{x^4 - 2}) - (x^3 + 4x)\left(\frac{1}{2\sqrt{x^4 - 2}}\right)(4x^3)}{(\sqrt{x^4 - 2})^2}$$

$$u' = \frac{(3x^2 + 4)(\sqrt{x^4 - 2}) - 2x^3(x^3 + 4x)\left(\frac{1}{\sqrt{x^4 - 2}}\right)}{x^4 - 2}$$

$$u' = \frac{(3x^2 + 4)(\sqrt{x^4 - 2})}{x^4 - 2} - \frac{2x^3(x^3 + 4x)\left(\frac{1}{\sqrt{x^4 - 2}}\right)}{x^4 - 2}$$

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$$u' = \frac{3x^2+4}{\sqrt{x^4-2}} - \frac{2x^3(x^3+4x)}{\sqrt{(x^4-2)^3}}$$

Now we can back-substitute into $g'(u) = \cos u \cdot u'$, and then simplify.

$$g'(x) = \cos\left(\frac{x^3+4x}{\sqrt{x^4-2}}\right) \cdot \left(\frac{3x^2+4}{\sqrt{x^4-2}} - \frac{2x^3(x^3+4x)}{\sqrt{(x^4-2)^3}}\right)$$

Now plug into the product rule formula.

$$h'(x) = f(x)g'(x) + f'(x)g(x)$$

$$h'(x) = x^2 \cos\left(\frac{x^3+4x}{\sqrt{x^4-2}}\right)\left(\frac{3x^2+4}{\sqrt{x^4-2}} - \frac{2x^3(x^3+4x)}{\sqrt{(x^4-2)^3}}\right) + 2x \sin\left(\frac{x^3+4x}{\sqrt{x^4-2}}\right)$$

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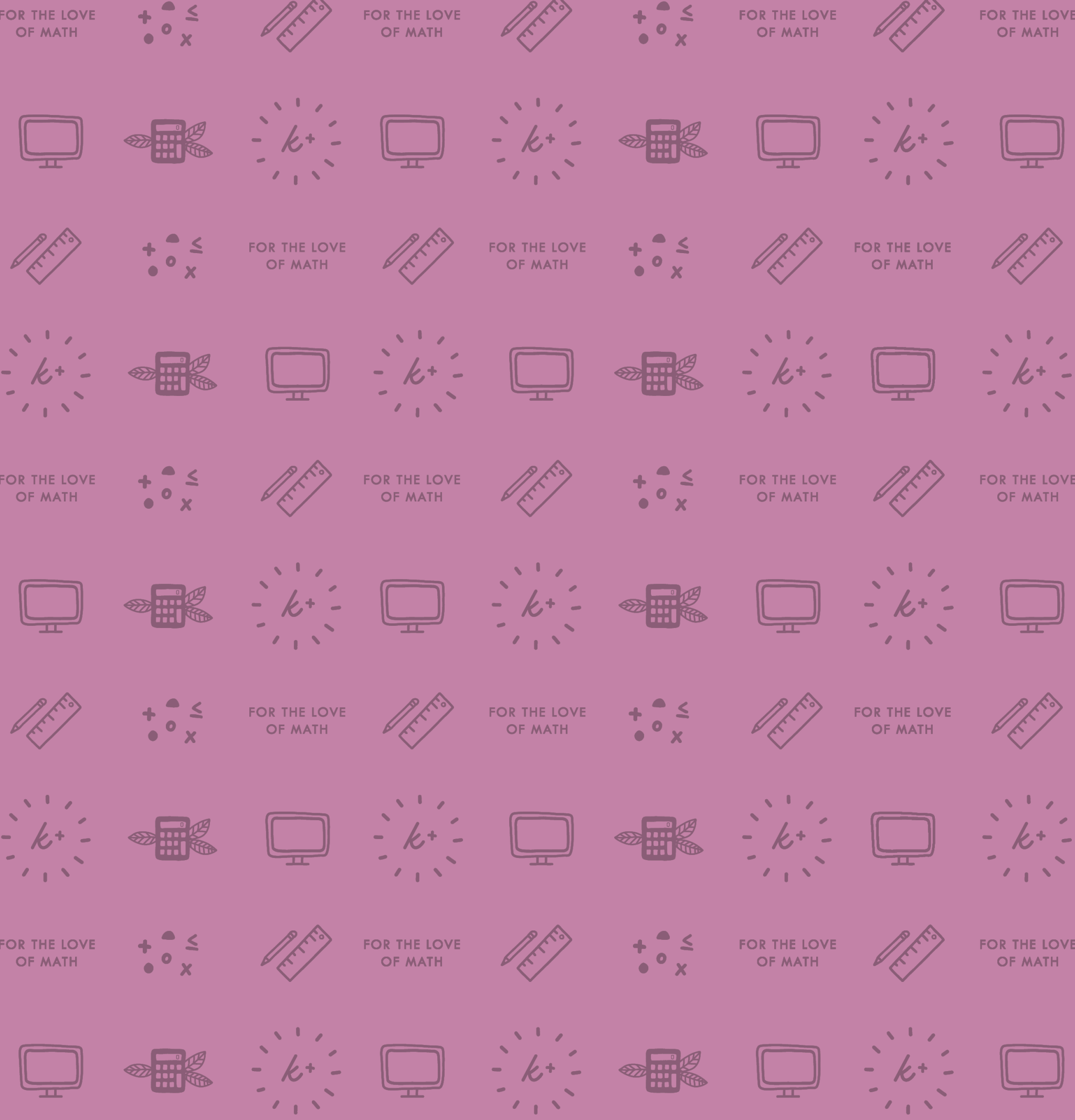

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