Topic: Limit vs. sum of the series

Question: How is the limit of a series different from its sum?

Answer choices:

A They aren't different. They both indicate the size of the series from its start to ∞.

B They aren't different. They both indicate how the series behaves as it approaches ∞.

C The limit indicates how the series is behaving as it approaches ∞ and the sum indicates the size of the series from its start to ∞.

D The sum indicates how the series is behaving as it approaches ∞ and the limit indicates the size of the series from its start to ∞.

Solution: C

The limit of a series (at ∞) indicates how a series is behaving as it approaches ∞.

The sum of a series indicates the size of the series from its start to ∞.

Since these two values are different, it's useful to be able to find both for the same series.

Topic: Limit vs. sum of the series

Question: Find the limit of the series, and if it converges, find its sum.

$$\sum_{n=1}^{\infty} e^{-n}$$

Answer choices:

A $\lim a_n = -1$ and $\sum a_n = \frac{e}{e+1}$

B $\lim a_n = 0$ and $\sum a_n = \frac{1}{e-1}$

C $\lim a_n = 1$ and $\sum a_n = e-1$

D $\lim a_n = \text{DNE}$ and $\sum a_n = \frac{1}{e-1}$

Solution: B

The limit of the series is given by

$$\lim_{n\to\infty} e^{-n} = \lim_{n\to\infty} \frac{1}{e^n}$$

Notice that as $n \to \infty$, the denominator gets bigger and bigger. With a constant numerator, the value of the fraction approaches 0. Therefore,

$$\lim_{n\to\infty} e^{-n} = 0$$

Because the limit is equal to 0, it means that the series is convergent and its graph will approach 0 as the value of n is increased. To evaluate the sum of the series, rewrite it as

$$\sum_{n=1}^{\infty} e^{-n}$$

$$\sum_{n=1}^{\infty} \frac{1}{e^n}$$

$$\sum_{n=1}^{\infty} \left(\frac{1}{e}\right)^n$$

If we let $a_n = \left(\frac{1}{e}\right)^n$ and we write out the first three terms of the series as

$$a_1 = \left(\frac{1}{e}\right)^1 = \frac{1}{e}$$

$$a_2 = \left(\frac{1}{e}\right)^2 = \frac{1}{e^2}$$

$$a_3 = \left(\frac{1}{e}\right)^3 = \frac{1}{e^3}$$

then its sum is equal to

$$s = \frac{1}{e} + \frac{1}{e^2} + \frac{1}{e^3} + \dots + \frac{1}{e^n}$$

$$s = \frac{1}{e}\left(1 + \frac{1}{e} + \frac{1}{e^2} + \dots + \frac{1}{e^n}\right)$$

This is a geometric series with $a = 1/e$ and $r = 1/e$. The sum of a geometric series is given by

$$s = \frac{a}{1 - r}$$

so the sum of the series is

$$s = \frac{\frac{1}{e}}{1 - \frac{1}{e}}$$

$$s = \frac{\frac{1}{e}}{\frac{e}{e} - \frac{1}{e}}$$

$$s = \frac{\frac{1}{e}}{\frac{e-1}{e}}$$

$$s=\frac{1}{e}\cdot\frac{e}{e-1}$$

$$s=\frac{1}{e-1}$$

Topic: Limit vs. sum of the series

Question: Find the limit of the series, and if it converges, find its sum.

$$\sum_{n=1}^{\infty} \frac{3^{2n}}{81^{\frac{n}{2}}}$$

Answer choices:

A	$\lim a_n = 0$	and	$\sum a_n = \frac{1}{3}$
B	$\lim a_n = \frac{1}{3}$	and	$\sum a_n = 1$
C	$\lim a_n = \infty$	and	$\sum a_n = 0$
D	$\lim a_n = 1$	and	$\sum a_n = \infty$

Solution: D

The limit of the series is given by

$$\lim_{n\to\infty} \frac{3^{2n}}{81^{\frac{n}{2}}}$$

$$\lim_{n\to\infty} \frac{\left(3^2\right)^n}{\left(81^{\frac{1}{2}}\right)^n}$$

$$\lim_{n\to\infty} \frac{9^n}{\left(\sqrt{81}\right)^n}$$

$$\lim_{n\to\infty} \frac{9^n}{9^n}$$

$$\lim_{n\to\infty} 1$$

$$1$$

The limit of the series as $n \to \infty$ is 1. Therefore,

$$\lim_{n\to\infty} \frac{3^{2n}}{81^{\frac{n}{2}}} = 1$$

Because the limit is equal to 1, it means that the series is convergent and its graph will approach 1 as the value of n is increased. To evaluate the sum of the series, we'll expand it through its first few terms.

$$\sum_{n=1}^{\infty} \frac{3^{2n}}{81^{\frac{n}{2}}}$$

$$\frac{3^{2(1)}}{81^{\frac{1}{2}}}+\frac{3^{2(2)}}{81^{\frac{2}{2}}}+\frac{3^{2(3)}}{81^{\frac{3}{2}}}+\dots$$

$$\frac{3^2}{\sqrt{81}}+\frac{3^4}{81}+\frac{3^6}{\left(81^{\frac{1}{2}}\right)^3}+\dots$$

$$\frac{9}{9}+\frac{81}{81}+\frac{3^6}{9^3}+\dots$$

$$\frac{9}{9}+\frac{81}{81}+\frac{3\cdot3\cdot3\cdot3\cdot3\cdot3}{9\cdot9\cdot9}+\dots$$

$$\frac{9}{9}+\frac{81}{81}+\frac{3\cdot3\cdot3\cdot3\cdot3\cdot3}{3\cdot3\cdot3\cdot3\cdot3\cdot3}+\dots$$

$$\frac{9}{9}+\frac{81}{81}+\frac{3^6}{3^6}+\dots$$

$$1+1+1+\dots$$

$$\infty$$

The sum of the series is therefore

$$\sum_{n=1}^{\infty}\frac{3^{2n}}{81^{\frac{n}{2}}}=\infty$$