Tabular integration

Tabular integration is an alternative method we can use to deal with problems that would normally be integrated using integration by parts.

Tabular integration is often extremely useful in situations where our integral requires us to use integration by parts multiple times.

To use tabular integration, we create a table with two columns. In the first column, we differentiate u until it goes to 0. In the second column, we integrate dv as many times as we differentiated u to get it to 0.

	1. Differentiate u	2. Integrate dv
a.	$u = 1a$	$dv = 2a$
b.	$u' = 1b$	$v = 2b$
c.	$u'' = 1c$	$\int v dv = 2c$
	...	...
d.	constant	$\dots \int v dv \dots = 2e$
e.	0	$\dots \int v dv \dots = 2f$

Once we've built our table, we can start compiling our answer. Our first term will be the product of the value from the first row of the first column ($1a$), and the from the second row of the second column ($2b$). We continue the same pattern to get the product of $1b$ and $2c$, the product of $1c$ and $2d$, etc. The last term is the one that includes the last non-zero term from the first column (the constant).

It's very important that we alternate the signs of these terms as we add them together. The odd terms (first, third, etc.) are positive; the even terms (second, fourth, etc.) are negative.

Example

Use tabular integration to evaluate the integral. Verify your answer using integration by parts.

$$\int x^4 e^{3x}\ dx$$

First, we need to name our parts. Let

$$u_0 = x^4$$

and

$$dv_0 = e^{3x}\ dx$$

We can then calculate

$$du_0 = 4x^3\ dx$$

and

$$v_0 = \frac{1}{3}e^{3x}$$

Now let's set up our table.

	1. Differentiate u	2. Integrate dv
a.	x^4	e^{3x}
b.	$4x^3$	$\frac{1}{3}e^{3x}$
c.	$12x^2$	$\frac{1}{9}e^{3x}$
d.	$24x$	$\frac{1}{27}e^{3x}$
e.	24	$\frac{1}{81}e^{3x}$
f.	0	$\frac{1}{243}e^{3x}$

Next, we can compile our answer, remembering to alternate signs, starting with +. We get,

$$\int x^4e^{3x}\ dx = (x^4)\left(\frac{1}{3}e^{3x}\right) - (4x^3)\left(\frac{1}{9}e^{3x}\right) + (12x^2)\left(\frac{1}{27}e^{3x}\right)$$

$$-(24x)\left(\frac{1}{81}e^{3x}\right) + (24)\left(\frac{1}{243}e^{3x}\right) + C$$

$$\int x^4e^{3x}\ dx = \frac{1}{3}x^4e^{3x} - \frac{4}{9}x^3e^{3x} + \frac{4}{9}x^2e^{3x} - \frac{8}{27}xe^{3x} + \frac{8}{81}e^{3x} + C$$

We can now verify our answer using integration by parts. Remember, $u_0 = x^4$, $dv_0 = e^{3x}\ dx$, $du_0 = 4x^3\ dx$ and $v_0 = \frac{1}{3}e^{3x}$ and our formula for integration by parts is $\int u\ dv = uv - \int v\ du$.

$$\int x^4 e^{3x}\ dx = (x^4)\left(\frac{1}{3}e^{3x}\right) - \int \left(\frac{1}{3}e^{3x}\right)(4x^3\ dx)$$

$$\int x^4 e^{3x}\ dx = \frac{1}{3}x^4 e^{3x} - \frac{4}{3}\int x^3 e^{3x}\ dx$$

Using integration by parts a second time and letting $u_1 = x^3$, $dv_1 = e^{3x}\ dx$, $du_1 = 3x^2\ dx$, and $v_1 = \frac{1}{3}e^{3x}$, we get

$$\int x^4 e^{3x}\ dx = \frac{1}{3}x^4 e^{3x} - \frac{4}{3}\left[(x^3)\left(\frac{1}{3}e^{3x}\right) - \int \left(\frac{1}{3}e^{3x}\right)(3x^2\ dx)\right]$$

$$\int x^4 e^{3x}\ dx = \frac{1}{3}x^4 e^{3x} - \frac{4}{9}x^3 e^{3x} + \frac{4}{3}\int x^2 e^{3x}\ dx$$

Using integration by parts a third time and letting $u_2 = x^2$, $dv_2 = e^{3x}\ dx$, $du_2 = 2x\ dx$, and $v_2 = \frac{1}{3}e^{3x}$, we get

$$\int x^4 e^{3x}\ dx = \frac{1}{3}x^4 e^{3x} - \frac{4}{9}x^3 e^{3x} + \frac{4}{3}\left[(x^2)\left(\frac{1}{3}e^{3x}\right) - \int \left(\frac{1}{3}e^{3x}\right)(2x\ dx)\right]$$

$$\int x^4 e^{3x}\ dx = \frac{1}{3}x^4 e^{3x} - \frac{4}{9}x^3 e^{3x} + \frac{4}{9}x^2 e^{3x} - \frac{8}{9}\int x e^{3x}\ dx$$

Using integration by parts a fourth time and letting $u_3 = x$, $dv_3 = e^{3x}\ dx$, $du_3 = dx$, and $v_3 = \frac{1}{3}e^{3x}$, we get

$$\int x^4 e^{3x}\ dx = \frac{1}{3}x^4 e^{3x} - \frac{4}{9}x^3 e^{3x} + \frac{4}{9}x^2 e^{3x} - \frac{8}{9}\left[(x)\left(\frac{1}{3}e^{3x}\right) - \int \left(\frac{1}{3}e^{3x}\right)(dx)\right]$$

$$\int x^4 e^{3x}\ dx = \frac{1}{3}x^4 e^{3x} - \frac{4}{9}x^3 e^{3x} + \frac{4}{9}x^2 e^{3x} - \frac{8}{27}xe^{3x} + \frac{8}{27}\int e^{3x}\ dx$$

We're finally at a point where we can easily integrate the remaining integral, so we integrate and get

$$\int x^4 e^{3x}\ dx = \frac{1}{3}x^4 e^{3x} - \frac{4}{9}x^3 e^{3x} + \frac{4}{9}x^2 e^{3x} - \frac{8}{27}xe^{3x} + \frac{8}{81}e^{3x} + C$$

The answer we just got using integration by parts was the same answer we got when we used tabular integration, which tells us that we did the tabular integration correctly.

In this particular example, we can see how much faster and easier it was to use tabular integration than it was to use integration by parts. Tabular integration will often be the faster method when we have to set u equal to a high-degree power function like x^3, x^7 or x^{12}.

That's because when we set u equal to a high-degree power function and then apply integration by parts, we only reduce the degree of the power function by 1. In other words, x^3 would become x^2, x^7 would become x^6, and x^{12} would become x^{11}. We have to continue applying integration by parts over and over again until the degree is reduced all the way to 0, so that we get x^0, which is just 1, and that part of the function drops away, leaving us with only the dv part.

Now that you know how to use tabular integration, try thinking through your integration by parts problems before you start them to see if tabular integration might be an easier way to solve them.