Topic: Error or remainder of a series

Question: Estimate the remainder of the series using the first three terms.

$$\sum_{n=1}^{\infty} \frac{1}{3n^2+1}$$

Answer choices:

A $R_3 \leq 0.0333$

B $R_3 \leq 0.3000$

C $R_3 \leq 0.2500$

D $R_3 \leq 0.3333$

Solution: D

To find the remainder of the series, we'll need to

Estimate the total sum by calculating a **partial sum** for the series.

Use the **comparison test** to say whether the series converges or diverges.

Use the **integral test** to solve for the remainder.

The first thing we need to do is to find the sum of the first three terms s_3 of our original series a_n.

$$n = 1 \qquad a_1 = \frac{1}{3(1)^2+1} \qquad a_1 = \frac{1}{4}$$

$$n = 2 \qquad a_2 = \frac{1}{3(2)^2+1} \qquad a_2 = \frac{1}{13}$$

$$n = 3 \qquad a_3 = \frac{1}{3(3)^2+1} \qquad a_3 = \frac{1}{28}$$

The sum of the first three terms of the series a_n is

$$s_3 = \frac{1}{4} + \frac{1}{13} + \frac{1}{28}$$

$$s_3 = 0.2500 + 0.0769 + 0.0357$$

$$s_3 = 0.3626$$

Since we've rounded our decimals, we'll say

$$s_3 \approx 0.3626$$

Next, we need to use the comparison test to figure out whether a_n converges or diverges. We will need to create a similar but simpler comparison series b_n. We can use the same numerator in b_n as the numerator from a_n, since it's already simple. For the denominator, we can use n^2, since it's the element of the denominator that has the most impact on the series. The comparison series b_n will be

$$b_n = \frac{1}{n^2}$$

The comparison series b_n is a p-series where $p = 2$. The p-series test tells us that the series

will converge when $p > 1$

will diverge when $p \leq 1$

Since $p = 2$, we know that b_n converges.

To use the comparison test to show that a_n also converges, we have to show that $0 \leq a_n \leq b_n$. We'll find some of the first few values of the comparison series b_n and compare them to a_n. Let's use $n = 1, 2, 3$.

$$n = 1 \qquad b_1 = \frac{1}{(1)^2} \qquad b_1 = 1$$

$$n = 2 \qquad b_2 = \frac{1}{(2)^2} \qquad b_2 = \frac{1}{4}$$

$$n = 3 \qquad b_3 = \frac{1}{(3)^2} \qquad b_3 = \frac{1}{9}$$

Looking at these three terms, we can see that all of our answers have $b_n > a_n$ as well as $a_n > 0$. Since we have verified $0 \leq a_n \leq b_n$, we can state that a_n converges.

Now that we know that the series converges, we'll use the integral test to find the remainder of the series a_n after the first three terms, R_3. We'll call the remainder of the comparison series b_n after the first three terms, T_3. Since we know that $0 \leq a_n \leq b_n$, and that a_n and b_n converge, we can say that $R_3 \leq T_3$, which will be less than the total area under b_n.

$$R_3 \leq T_3 \leq \int_3^\infty b_n\ dx = \int_3^\infty f(x)\ dx$$

$$R_3 \leq T_3 \leq \int_3^\infty b_n\ dx = \int_3^\infty \frac{1}{x^2}\ dx$$

$$R_3 \leq T_3 \leq \int_3^\infty b_n\ dx = \int_3^\infty x^{-2}\ dx$$

$$R_3 \leq \frac{x^{-1}}{-1}\Big|_3^\infty$$

$$R_3 \leq \lim_{b\to\infty} \frac{x^{-1}}{-1}\Big|_3^b$$

$$R_3 \leq \lim_{b\to\infty} -\frac{1}{x}\Big|_3^b$$

$$R_3 \leq \lim_{b\to\infty} -\frac{1}{b} - \left(-\frac{1}{3}\right)$$

$$R_3 \leq \lim_{b \to \infty} -\frac{1}{b} + \frac{1}{3}$$

$$R_3 \leq -\frac{1}{\infty} + \frac{1}{3}$$

$$R_3 \leq 0 + \frac{1}{3}$$

$$R_3 \leq \frac{1}{3}$$

$$R_3 \leq 0.3333$$

The third partial sum of the series a_n is $s_3 \approx 0.3626$, with error $R_3 \leq 0.3333$.

Topic: Error or remainder of a series

Question: Estimate the remainder of the series using the first five terms.

$$\sum_{n=1}^{\infty} \frac{n}{5n^4 + 2}$$

Answer choices:

A $R_5 \leq 0.0800$

B $R_5 \leq 0.2500$

C $R_5 \leq 0.0200$

D $R_5 \leq 0.2000$

Solution: C

To find the remainder of the series, we'll need to

Estimate the total sum by calculating a **partial sum** for the series.

Use the **comparison test** to say whether the series converges or diverges.

Use the **integral test** to solve for the remainder.

The first thing we need to do is to find the sum of the first five terms s_5 of our original series a_n.

$$n = 1 \qquad a_1 = \frac{(1)}{5(1)^4 + 2} \qquad a_1 = \frac{1}{7}$$

$$n = 2 \qquad a_2 = \frac{(2)}{5(2)^4 + 2} \qquad a_2 = \frac{1}{41}$$

$$n = 3 \qquad a_3 = \frac{(3)}{5(3)^4 + 2} \qquad a_3 = \frac{3}{407}$$

$$n = 4 \qquad a_4 = \frac{(4)}{5(4)^4 + 2} \qquad a_4 = \frac{2}{641}$$

$$n = 5 \qquad a_5 = \frac{(5)}{5(5)^4 + 2} \qquad a_5 = \frac{5}{3{,}127}$$

The sum of the first five terms of the series a_n is

$$s_5 = \frac{1}{7} + \frac{1}{41} + \frac{3}{407} + \frac{2}{641} + \frac{5}{3{,}127}$$

$$s_5 = 0.1429 + 0.0244 + 0.0074 + 0.0031 + 0.0016$$

$$s_5 = 0.1794$$

Since we've rounded our decimals, we'll say

$$s_5 \approx 0.1794$$

Next, we need to use the comparison test to figure out whether a_n converges or diverges. We will need to create a similar but simpler comparison series b_n. We can use the same numerator in b_n as the numerator from a_n, since it's already pretty simple. For the denominator, we can use n^4, since it's the element of the denominator that has the most impact on the series. The comparison series b_n will be

$$b_n = \frac{n}{n^4}$$

$$b_n = \frac{1}{n^3}$$

The comparison series b_n is a p-series where $p = 3$. The p-series test tells us that the series

will converge when $p > 1$

will diverge when $p \le 1$

Since $p = 3$, we know that b_n converges.

To use the comparison test to show that a_n also converges, we have to show that $0 \le a_n \le b_n$. We'll find some of the first few values of the comparison series b_n and compare them to a_n. Let's use $n = 1, 2, 3$.

$n = 1$	$b_1 = \frac{1}{(1)^3}$	$b_1 = 1$
$n = 2$	$b_2 = \frac{1}{(2)^3}$	$b_2 = \frac{1}{8}$
$n = 3$	$b_3 = \frac{1}{(3)^3}$	$b_3 = \frac{1}{27}$

Looking at these three terms, we can see that all of our answers have $b_n > a_n$ as well as $a_n > 0$. Since we have verified $0 \le a_n \le b_n$, we can state that a_n converges.

Now that we know that the series converges, we'll use the integral test to find the remainder of the series a_n after the first five terms, R_5. We'll call the remainder of the comparison series b_n after the first five terms, T_5. Since we know that $0 \le a_n \le b_n$, and that a_n and b_n converge, we can say that $R_5 \le T_5$, which will be less than the total area under b_n.

$$R_5 \le T_5 \le \int_5^\infty b_n \ dx = \int_5^\infty f(x) \ dx$$

$$R_5 \le T_5 \le \int_5^\infty b_n \ dx = \int_5^\infty \frac{1}{x^3} \ dx$$

$$R_5 \le T_5 \le \int_5^\infty b_n \ dx = \int_5^\infty x^{-3} \ dx$$

$$R_5 \le \frac{x^{-2}}{-2} \Big|_5^\infty$$

$$R_5 \leq \lim_{b \to \infty} \frac{x^{-2}}{-2} \Bigg|_5^b$$

$$R_5 \leq \lim_{b \to \infty} -\frac{1}{2x^2} \Bigg|_5^b$$

$$R_5 \leq \lim_{b \to \infty} -\frac{1}{2(b)^2} - \left[-\frac{1}{2(5)^2}\right]$$

$$R_5 \leq \lim_{b \to \infty} -\frac{1}{2b^2} + \frac{1}{50}$$

$$R_5 \leq -\frac{1}{\infty} + \frac{1}{50}$$

$$R_5 \leq 0 + \frac{1}{50}$$

$$R_5 \leq \frac{1}{50}$$

$$R_5 \leq 0.0200$$

The fifth partial sum of the series a_n is $s_5 \approx 0.1794$, with error $R_5 \leq 0.0200$.

Topic: Error or remainder of a series

Question: Estimate the remainder of the series using the first seven terms.

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{2n^4+1}}$$

Answer choices:

A $R_7 \leq 0.1429$

B $R_7 \leq 0.0204$

C $R_7 \leq 0.2858$

D $R_7 \leq 0.0408$

Solution: A

To find the remainder of the series, we'll need to

Estimate the total sum by calculating a **partial sum** for the series.

Use the **comparison test** to say whether the series converges or diverges.

Use the **integral test** to solve for the remainder.

The first thing we need to do is to find the sum of the first seven terms s_7 of our original series a_n.

$$n = 1 \qquad a_1 = \frac{1}{\sqrt{2(1)^4 + 1}} \qquad a_1 = \frac{1}{\sqrt{3}}$$

$$n = 2 \qquad a_2 = \frac{1}{\sqrt{2(2)^4 + 1}} \qquad a_2 = \frac{1}{\sqrt{33}}$$

$$n = 3 \qquad a_3 = \frac{1}{\sqrt{2(3)^4 + 1}} \qquad a_3 = \frac{1}{\sqrt{163}}$$

$$n = 4 \qquad a_4 = \frac{1}{\sqrt{2(4)^4 + 1}} \qquad a_4 = \frac{1}{\sqrt{513}}$$

$$n = 5 \qquad a_5 = \frac{1}{\sqrt{2(5)^4 + 1}} \qquad a_5 = \frac{1}{\sqrt{1,251}}$$

$$n = 6 \qquad a_6 = \frac{1}{\sqrt{2(6)^4 + 1}} \qquad a_6 = \frac{1}{\sqrt{2,593}}$$

$n = 7$ $\quad a_7 = \frac{1}{\sqrt{2(7)^4 + 1}}$ $\quad a_7 = \frac{1}{\sqrt{4{,}803}}$

The sum of the first seven terms of the series a_n is

$$s_7 = \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{33}} + \frac{1}{\sqrt{163}} + \frac{1}{\sqrt{513}} + \frac{1}{\sqrt{1{,}251}} + \frac{1}{\sqrt{2{,}593}} + \frac{1}{\sqrt{4{,}803}}$$

$$s_7 = 0.5774 + 0.1741 + 0.0783 + 0.0442 + 0.0283 + 0.0196 + 0.0144$$

$$s_7 = 0.9363$$

Since we've rounded our decimals, we'll say

$$s_7 \approx 0.9363$$

Next, we need to use the comparison test to figure out whether a_n converges or diverges. We will need to create a similar but simpler comparison series b_n. We can use the same numerator in b_n as the numerator from a_n, since it's already simple. For the denominator, we can use n^2 (the square root of n^4), since it's the element of the denominator that has the most impact on the series. The comparison series b_n will be

$$b_n = \frac{1}{n^2}$$

The comparison series b_n is a p-series where $p = 2$. The p-series test tells us that the series

will converge when $p > 1$

will diverge when $p \le 1$

Since $p = 2$, we know that b_n converges.

To use the comparison test to show that a_n also converges, we have to show that $0 \leq a_n \leq b_n$. We'll find some of the first few values of the comparison series b_n and compare them to a_n. Let's use $n = 1, 2, 3$.

$n = 1$	$b_1 = \frac{1}{1^2}$	$b_1 = 1$
$n = 2$	$b_2 = \frac{1}{2^2}$	$b_2 = \frac{1}{4}$
$n = 3$	$b_3 = \frac{1}{3^2}$	$b_3 = \frac{1}{9}$

Looking at these three terms, we can see that all of our answers have $b_n > a_n$ as well as $a_n > 0$. Since we have verified $0 \leq a_n \leq b_n$, we can state that a_n converges.

Now that we know that the series converges, we'll use the integral test to find the remainder of the series a_n after the first seven terms, R_7. We'll call the remainder of the comparison series b_n after the first seven terms, T_7. Since we know that $0 \leq a_n \leq b_n$, and that a_n and b_n converge, we can say that $R_7 \leq T_7$, which will be less than the total area under b_n.

$$R_7 \leq T_7 \leq \int_7^\infty b_n\ dx = \int_7^\infty f(x)\ dx$$

$$R_7 \leq T_7 \leq \int_7^\infty b_n\ dx = \int_7^\infty \frac{1}{x^2}\ dx$$

$$R_7 \leq T_7 \leq \int_7^\infty b_n\ dx = \int_7^\infty x^{-2}\ dx$$

$$R_7 \leq \frac{x^{-1}}{-1}\Bigg|_7^\infty$$

$$R_7 \leq -\frac{1}{x}\Bigg|_7^\infty$$

$$R_7 \leq \lim_{b\to\infty} -\frac{1}{x}\Bigg|_7^b$$

$$R_7 \leq \lim_{b\to\infty} -\frac{1}{b} - \left(-\frac{1}{7}\right)$$

$$R_7 \leq -\frac{1}{\infty} + \frac{1}{7}$$

$$R_7 \leq 0 + \frac{1}{7}$$

$$R_7 \leq \frac{1}{7}$$

$$R_7 \leq 0.1429$$

The seventh partial sum of the series a_n is $s_7 \approx 0.9363$, with error $R_7 \leq 0.1429$.