Observer and the airplane

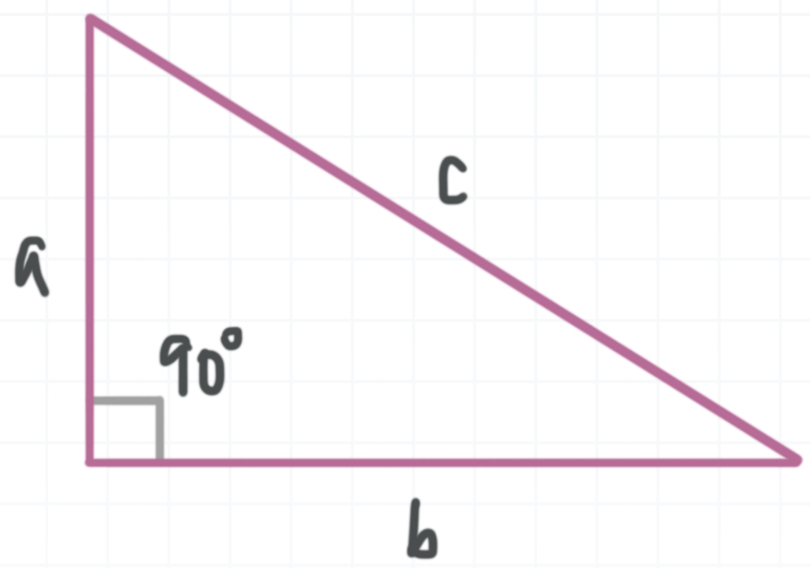
In these kinds of related rates problems, we have two objects, where either one of them is moving toward or away from the other, or where both objects are in motion toward or away from each other.

This could be a stationary observer on the ground, and an airplane flying toward or away from them overhead. It could be two cars in motion, either where both of them are approaching the same intersection, both of them are driving away from the same intersection, or one is driving toward the intersection while the other drives away.

When we tackle these kinds of problems, or really any kind of related rates problem, it's especially helpful to draw a picture of what's happening, and then label the parts of the picture.

Almost always with this type of problem, we'll use the two objects as two vertices of a **right triangle** (a triangle with one 90° angle). Once we have the triangle formed, we'll need the Pythagorean theorem in order to solve for some of the missing values that we need.

Remember that, for a right triangle, the two shorter sides that border the right angle are called a and b and the longest side that's opposite the right angle (the **hypotenuse**) is called c.



Then the **Pythagorean theorem** says that

$$a^2 + b^2 = c^2$$

Let's do a complete example so that we can see what it looks like to apply the Pythagorean theorem as part of a related rates problem.

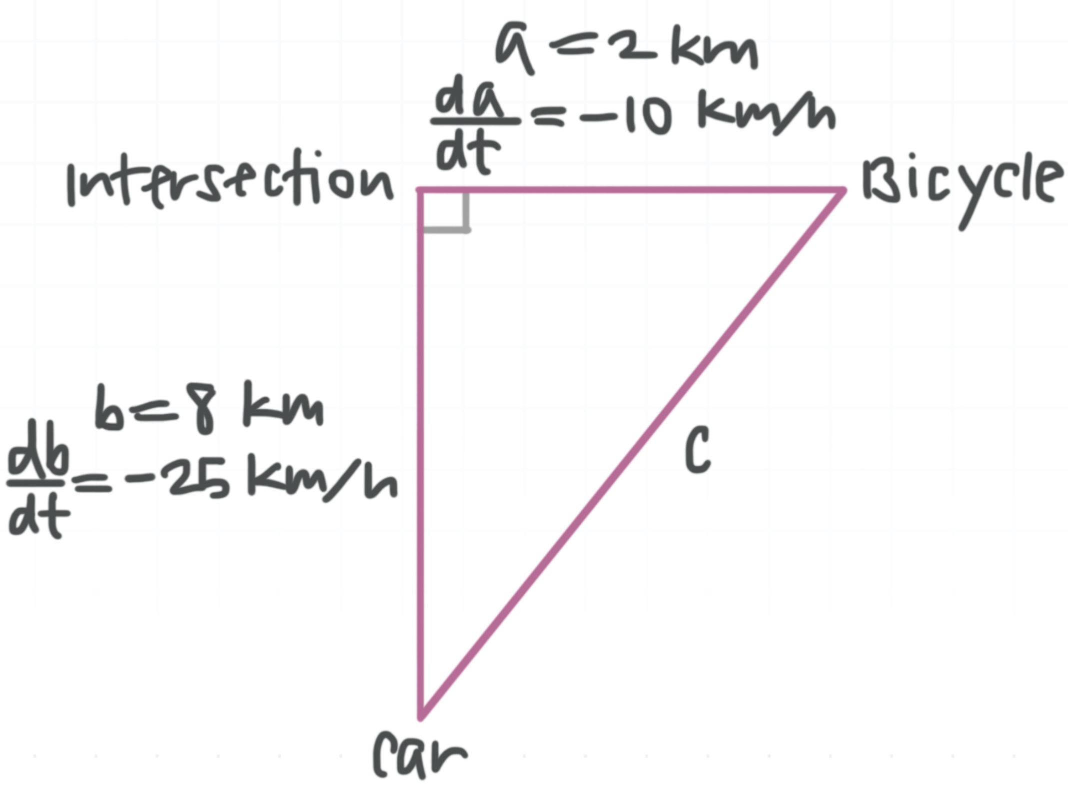
Example

A bicycle is 2 km east of an intersection, traveling toward it at 10 km/h. Meanwhile, a car is 8 km south of the intersection, traveling toward it at 25 km/h. How fast is the distance between them decreasing?

We want to start by sketching the diagram, showing the intersection, the bicycle, and the car.

We can also label the diagram with the values we've been given. The shortest side is the distance between the intersection and the bicycle, so we'll say $a = 2$ km. Because the bicycle is moving toward the intersection, the distance between them is decreasing, which means the length of that side is getting shorter, so we'll give the rate of change of that side length the negative value $da/dt = -10$ km/h.

The other leg is the distance between the intersection and the car, so we'll say $b = 8$ km. Because the car is moving toward the intersection, the distance between them is decreasing, which means the length of that side is getting shorter, so we'll give the rate of change of that side length the negative value $db/dt = -25$ km/h.



Ultimately, we're trying to find the rate of change of the distance between the bicycle and the car, which is the rate at which the length of side c is decreasing, or dc/dt. We can use the Pythagorean theorem to find the distance between the bicycle and the car, c, at the time when $a = 2$ and $b = 8$.

$$a^2 + b^2 = c^2$$

$$2^2 + 8^2 = c^2$$

$$4+64=c^2$$

$$68=c^2$$

$$c=\sqrt{68}$$

$$c=2\sqrt{17}$$

Then, if we start with the Pythagorean theorem, which relates the three sides lengths, $a^2+b^2=c^2$, we can use implicit differentiation to take the derivative of both sides.

$$2a\left(\frac{da}{dt}\right)+2b\left(\frac{db}{dt}\right)=2c\left(\frac{dc}{dt}\right)$$

Substitute what we already know.

$$2(2)(-10)+2(8)(-25)=2(2\sqrt{17})\left(\frac{dc}{dt}\right)$$

$$-40-400=4\sqrt{17}\left(\frac{dc}{dt}\right)$$

Now solve for dc/dt, which is the rate of change between the bicycle and the car.

$$-440=4\sqrt{17}\left(\frac{dc}{dt}\right)$$

$$\frac{dc}{dt}=-\frac{440}{4\sqrt{17}}$$

$$\frac{dc}{dt} = -\frac{110}{\sqrt{17}}$$

$$\frac{dc}{dt} \approx -26.68$$

This result tells us that the distance between the bicycle and the car is decreasing at a rate of about 26.68 km/h, which means that the car and bicycle are getting closer to each other at that rate.