Convergence of a sequence

If we say that a sequence converges, it means that the limit of the sequence exists as $n \to \infty$. If the limit of the sequence as $n \to \infty$ does not exist, we say that the sequence diverges. A sequence always either converges or diverges, there is no other option. This doesn't mean we'll always be able to tell whether the sequence converges or diverges, sometimes it can be very difficult for us to determine convergence or divergence.

There are many ways to test a sequence to see whether or not it converges.

Sometimes all we have to do is evaluate the limit of the sequence at $n \to \infty$. If the limit exists then the sequence converges, and the answer we found is the value of the limit.

Sometimes it's convenient to use the squeeze theorem to determine convergence because it'll show whether or not the sequence has a limit, and therefore whether or not it converges. Then we'll take the limit of our sequence to get the real value of the limit.

Example

Say whether or not the sequence converges and find the limit of the sequence if it does converge.

$$a_n = \frac{\sin^2(n)}{3^n}$$

Remember, when a sequence converges, its limit exists at $n \to \infty$.

Let's evaluate the sequence using the squeeze theorem. We'll start by evaluating the numerator of a_n, $\sin^2(n)$. We know that the sine function exists between -1 and 1, so we can say that

$$-1 \le \sin(n) \le 1$$

We also know that when the sine function is squared, it only exists between 0 and 1, so we can modify the inequality to say that

$$0 \le \sin^2(n) \le 1$$

Finally, we can multiply the above inequality by $1/3^n$ to make it match our original sequence.

$$\left(0 \le \sin^2(n) \le 1\right)\frac{1}{3^n}$$

$$\frac{0}{3^n} \le \frac{\sin^2(n)}{3^n} \le \frac{1}{3^n}$$

$$0 \le \frac{\sin^2(n)}{3^n} \le \frac{1}{3^n}$$

Now, we have our original sequence bounded by two values. When we take the limit as $n \to \infty$, $1/3^n$ on the right side of the inequality will approach 0.

$$0 \le \lim_{n\to\infty}\frac{\sin^2(n)}{3^n} \le \lim_{n\to\infty}\frac{1}{3^n}$$

$$0 \leq \lim_{n\to\infty} \frac{\sin^2(n)}{3^n} \leq 0$$

Since the limit of the sequence is bounded by two real numbers, this means that our limit exists and our sequence converges. Finally, we can take the limit of our sequence as it approaches infinity.

$$\lim_{n\to\infty} \frac{\sin^2(n)}{3^n} = \frac{k}{\infty}$$

where k represents the constant number from 0 to 1 that we derived from the inequality $0 \leq \sin^2(n) \leq 1$. We get ∞ in the denominator because as $n \to \infty$, 3^n will approach ∞. Since we have a constant in the numerator and an infinity large value in the denominator, we know that

$$\lim_{n\to\infty} \frac{\sin^2(n)}{3^n} = 0$$

We can conclude that the sequence

$$a_n = \frac{\sin^2(n)}{3^n}$$

converges and that its limit as $n \to \infty$ is 0.

$$\lim_{n\to\infty} \frac{\sin^2(n)}{3^n} = \lim_{n\to\infty} a_n = 0$$
