Calculus 2 Workbook Solutions

Sequences

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MATH

SEQUENCES VS. SERIES

■ 1. Determine whether the expression is a sequence or a series.

$$5, 10, 15, 20 , 25, 30$$

Solution:

A sequence is a list of terms. A series is the sum of a sequence of terms. The question provides a list, not a sum, so it's a sequence.

■ 2. Determine whether the expression is a sequence or a series.

$$\sum_{n=1}^{15} 5n - 2$$

Solution:

A sequence is a list of terms. A series is the sum of a sequence of terms. The question provides a sum of a sequence, so it's a series.

■ 3. Determine whether the expression is a sequence or a series.

$$3 + 6 + 9 + 12 + 15 + 18 + 21$$

Solution:

A sequence is a list of terms. A series is the sum of a sequence of terms. The question provides a sum of a sequence, so it’s a series.

LISTING THE FIRST TERMS

■ 1. Write the first five terms of the sequence.

$$a_{n+1} = 3a_n + 4$$

$$a_1 = 4$$

Solution:

Since $a_1 = 4$, the first term of the sequence is 4. Use the rule for a_{n+1} to find the rest of the first five terms.

$$a_1 = 4$$

$$a_2 = 3a_1 + 4 = 3(4) + 4 = 16$$

$$a_3 = 3a_2 + 4 = 3(16) + 4 = 52$$

$$a_4 = 3a_3 + 4 = 3(52) + 4 = 160$$

$$a_5 = 3a_4 + 4 = 3(160) + 4 = 484$$

■ 2. Write the first five terms of the sequence.

$$a_{n+1} = 4a_n - 5$$

$$a_1 = 3$$

Solution:

Since $a_1 = 3$, the first term of the sequence is 3. Use the rule $a_{n+1} = 4a_n - 5$ to find the rest of the first five terms.

$$a_1 = 3$$

$$a_2 = 4a_1 - 5 = 4(3) - 5 = 7$$

$$a_3 = 4a_2 - 5 = 4(7) - 5 = 23$$

$$a_4 = 4a_3 - 5 = 4(23) - 5 = 87$$

$$a_5 = 4a_4 - 5 = 3(87) - 5 = 343$$

■ 3. Write the first five terms of the sequence.

$$a_{n+1} = a_n + 9$$

$$a_1 = 24$$

Solution:

Since $a_1 = 24$, the first term of the sequence is 24. Use the rule $a_{n+1} = a_n + 9$ to find the rest of the first five terms.

$$a_1 = 24$$

$$a_2 = a_1 + 9 = 24 + 9 = 33$$

$$a_3 = a_2 + 9 = 33 + 9 = 42$$

$$a_4 = a_3 + 9 = 42 + 9 = 51$$

$$a_5 = a_4 + 9 = 51 + 9 = 60$$

CALCULATING THE FIRST TERMS

1. Write the first five terms of the sequence and find the limit of the sequence a_n as $n \to \infty$.

$$a_n = \frac{5n^2 - 2}{n^2 + 3n - 2}$$

Solution:

To get the first five terms of the sequence, plug $n = 1, 2, 3, 4, 5$ into the formula for a_n.

n	$a_n = \dfrac{5n^2 - 2}{n^2 + 3n - 2}$	a_n
1	$a_1 = \dfrac{5(1)^2 - 2}{1^2 + 3(1) - 2}$	$a_1 = \dfrac{3}{2}$
2	$a_2 = \dfrac{5(2)^2 - 2}{2^2 + 3(2) - 2}$	$a_2 = \dfrac{9}{4}$
3	$a_3 = \dfrac{5(3)^2 - 2}{3^2 + 3(3) - 2}$	$a_3 = \dfrac{43}{16}$
4	$a_4 = \dfrac{5(4)^2 - 2}{4^2 + 3(4) - 2}$	$a_4 = 3$

$5 \qquad a_5 = \dfrac{5(5)^2 - 2}{5^2 + 3(5) - 2} \qquad a_5 = \dfrac{123}{38}$

Find the limit.

$$\lim_{n \to \infty} a_n$$

$$\lim_{n \to \infty} \frac{5n^2 - 2}{n^2 + 3n - 2}$$

$$\lim_{n \to \infty} \frac{5n^2 - 2}{n^2 + 3n - 2} \cdot \left(\frac{\frac{1}{n^2}}{\frac{1}{n^2}} \right)$$

$$\lim_{n \to \infty} \frac{\frac{5n^2}{n^2} - \frac{2}{n^2}}{\frac{n^2}{n^2} + \frac{3n}{n^2} - \frac{2}{n^2}}$$

$$\lim_{n \to \infty} \frac{5 - 0}{1 + 0 - 0}$$

$$\lim_{n \to \infty} \frac{5}{1}$$

$$5$$

■ 2. Write the first five terms of the sequence and find the limit of the sequence a_n as $n \to \infty$.

$$a_n = \frac{6n}{e^{2n}}$$

Solution:

To get the first five terms of the sequence, plug $n = 1, 2, 3, 4, 5$ into the formula for a_n.

n	$a_n = \dfrac{6n}{e^{2n}}$	a_n
1	$a_1 = \dfrac{6(1)}{e^{2(1)}}$	$a_1 = \dfrac{6}{e^2}$
2	$a_2 = \dfrac{6(2)}{e^{2(2)}}$	$a_2 = \dfrac{12}{e^4}$
3	$a_3 = \dfrac{6(3)}{e^{2(3)}}$	$a_3 = \dfrac{18}{e^6}$
4	$a_4 = \dfrac{6(4)}{e^{2(4)}}$	$a_4 = \dfrac{24}{e^8}$
5	$a_4 = \dfrac{6(5)}{e^{2(5)}}$	$a_4 = \dfrac{30}{e^{10}}$

To find the limit, apply L'Hospital's Rule.

$$\lim_{n\to\infty} a_n$$

$$\lim_{n\to\infty} \frac{6n}{e^{2n}}$$

$$\lim_{n\to\infty} \frac{6}{2e^{2n}}$$

$$\lim_{n\to\infty} \frac{3}{e^{2n}}$$

$$\lim_{n\to\infty} \frac{0}{\infty}$$

$$\lim_{n\to\infty} 0$$

$$0$$

■ 3. Write the first five terms of the sequence and find the limit of the sequence a_n as $n \to \infty$.

$$a_n = \frac{n^2+1}{n^2+8n}$$

Solution:

To get the first five terms of the sequence, plug $n = 1, 2, 3, 4, 5$ into the formula for a_n.

n	$a_n = \dfrac{n^2+1}{n^2+8n}$	a_n
1	$a_1 = \dfrac{1^2+1}{1^2+8(1)}$	$a_1 = \dfrac{2}{9}$
2	$a_2 = \dfrac{2^2+1}{2^2+8(2)}$	$a_2 = \dfrac{1}{4}$

3	$a_3 = \dfrac{3^2+1}{3^2+8(3)}$	$a_3 = \dfrac{10}{33}$
4	$a_4 = \dfrac{4^2+1}{4^2+8(4)}$	$a_4 = \dfrac{17}{48}$
5	$a_5 = \dfrac{5^2+1}{5^2+8(5)}$	$a_5 = \dfrac{2}{5}$

Find the limit.

$$\lim_{n\to\infty} a_n$$

$$\lim_{n\to\infty} \frac{n^2+1}{n^2+8n}$$

$$\lim_{n\to\infty} \frac{n^2+1}{n^2+8n} \cdot \left(\frac{\frac{1}{n^2}}{\frac{1}{n^2}}\right)$$

$$\lim_{n\to\infty} \frac{\frac{n^2}{n^2}+\frac{1}{n^2}}{\frac{n^2}{n^2}+\frac{8n}{n^2}}$$

$$\lim_{n\to\infty} \frac{\frac{n^2}{n^2}+\frac{1}{n^2}}{\frac{n^2}{n^2}+\frac{8}{n}}$$

$$\lim_{n\to\infty} \frac{1+0}{1+0}$$

$$\lim_{n\to\infty} \frac{1}{1}$$

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FORMULA FOR THE GENERAL TERM

■ 1. What is a formula for the general term of the sequence?

$$\frac{3}{4}, \frac{5}{8}, \frac{7}{12}, \frac{9}{16}, \frac{11}{20}$$

Solution:

In a sequence of fractions, consider the sequence of the numerators and the sequence of the denominators separately.

The sequence of the numerators is

n	1	2	3	4	5
a_n	3	5	7	9	11

$$a_n = 2n + 1$$

The sequence of the denominators is

n	1	2	3	4	5
a_n	4	8	12	16	20

$$a_n = 4n$$

So the rule for the sequence is

$$a_n = \frac{2n+1}{4n}$$

2. What is a formula for the general term of the sequence?

$$5, 8, 13, 20, 29, 40$$

Solution:

The sequence of the terms is

n	1	2	3	4	5	6
a_n	5	8	13	20	29	40

We can build upon the chart, to see that the pattern is

n	1	2	3	4	5	6
n^2	1	4	9	16	25	36
Add 4	+4	+4	+4	+4	+4	+4
a_n	5	8	13	20	29	40

$$a_n = n^2 + 4$$

3. What is a formula for the general term of the sequence?

$$-\frac{1}{6}, \frac{2}{7}, -\frac{3}{8}, \frac{4}{9}, -\frac{1}{2}, \frac{6}{11}$$

Solution:

In a sequence of fractions, consider the sequence of the numerators and the sequence of the denominators separately.

The sequence of the numerators is

n	1	2	3	4	5	6
a_n	-1	2	-3	4	-1	6

Notice that the 5th term seems out of sequence, but if the term is changed to match the pattern of the other terms, the sequence becomes

$$-\frac{1}{6}, \frac{2}{7}, -\frac{3}{8}, \frac{4}{9}, -\frac{5}{10}, \frac{6}{11}$$

So the new sequence of the numerators is

n	1	2	3	4	5	6
a_n	-1	2	-3	4	-5	6

The sign of the numerator is negative when n is odd and is positive when n is even. So the term value for the numerator is found using the formula $a_n = (-1)^n(n)$.

The sequence of the denominators is

n	1	2	3	4	5	6
a_n	6	7	8	9	10	11

The value of the denominator is consistently 5 more than the term number. So the formula for the denominator is $n+5$.

Therefore, the rule for the sequence is the rule for the numerator divided by the rule for the denominator.

$$a_n = \frac{(-1)^n(n)}{n+5}$$

CONVERGENCE OF A SEQUENCE

■ 1. If the sequence converges, find its limit.

$$a_n = \frac{5n}{n^2 + 2n - 1}$$

Solution:

The sequence converges if the limit of the sequence as $n \to \infty$ exists and is finite. The sequence diverges if the limit does not exist or is infinite. Because

$$\lim_{n \to \infty} \frac{5n}{n^2 + 2n - 1} = \frac{\infty}{\infty}$$

is indeterminate, use L'Hospital's Rule to find the limit.

$$\lim_{n \to \infty} \frac{5n}{n^2 + 2n - 1} = \lim_{n \to \infty} \frac{5}{2n^2 + 2} = \frac{5}{\infty} = 0$$

■ 2. If the sequence converges, find its limit.

$$a_n = \frac{9n^3 - 27n^2 + 5n}{3n^3 + 12n^2 - n}$$

Solution:

The sequence converges if the limit of the sequence as $n \to \infty$ exists and is finite. The sequence diverges if the limit does not exist or is infinite. Because

$$\lim_{n\to\infty} \frac{9n^3 - 27n^2 + 5n}{3n^3 + 12n^2 - n} = \frac{\infty}{\infty}$$

is indeterminate, use L'Hospital's Rule to find the limit.

$$\lim_{n\to\infty} \frac{9n^3 - 27n^2 + 5n}{3n^3 + 12n^2 - n} = \lim_{n\to\infty} \frac{27n^2 - 54n + 5}{9n^2 + 24n - 1} = \lim_{n\to\infty} \frac{54n - 54}{18n + 24} = \lim_{n\to\infty} \frac{54}{18} = 3$$

■ 3. If the sequence converges, find its limit.

$$a_n = \left(\frac{n^2 + 3}{n^3}\right)^2$$

Solution:

The sequence converges if the limit of the sequence as $n \to \infty$ exists and is finite. The sequence diverges if the limit does not exist or is infinite. Because

$$\lim_{n\to\infty} \left(\frac{n^2 + 3}{n^3}\right)^2 = \lim_{n\to\infty} \frac{(n^2 + 3)^2}{(n^3)^2} = \lim_{n\to\infty} \frac{n^4 + 6n^2 + 9}{n^6} = \frac{\infty}{\infty}$$

is indeterminate, use L'Hospital's Rule to find the limit.

$$\lim_{n\to\infty}\frac{n^4+6n^2+9}{n^6}=\lim_{n\to\infty}\frac{4n^3+12n}{6n^5}=\lim_{n\to\infty}\frac{12n^2+12}{30n^4}$$

$$=\lim_{n\to\infty}\frac{24n}{120n^3}=\lim_{n\to\infty}\frac{24}{120n^2}=\frac{24}{\infty}=0$$

LIMIT OF A CONVERGENT SEQUENCE

■ 1. Find the limit of the convergent sequence.

$$a_n = \frac{3n^2 - 6}{9n^2 + 3n - 12}$$

Solution:

Since the sequence converges, the limit of the sequence as $n \to \infty$ exists and is finite. Find the limit. Because

$$\lim_{n\to\infty} \frac{3n^2 - 6}{9n^2 + 3n - 12} = \lim_{n\to\infty} \frac{n^2 - 2}{3n^2 + n - 4} = \frac{\infty}{\infty}$$

is indeterminate, use L'Hospital's Rule to find the limit.

$$\lim_{n\to\infty} \frac{2n}{6n + 1} = \lim_{n\to\infty} \frac{2}{6} = \frac{1}{3}$$

■ 2. Find the limit of the convergent sequence.

$$a_n = \frac{n^3}{3^n}$$

Solution:

Since the sequence converges, the limit of the sequence as $n \to \infty$ exists and is finite. Find the limit. Because

$$\lim_{n\to\infty} \frac{n^3}{3^n} = \frac{\infty}{\infty}$$

is indeterminate, use L'Hospital's Rule to find the limit.

$$\lim_{n\to\infty} \frac{n^3}{3^n} = \lim_{n\to\infty} \frac{3n^2}{3^n \cdot \ln 3} = \frac{1}{\ln 3} \lim_{n\to\infty} \frac{3n^2}{3^n} = \frac{1}{\ln 3} \lim_{n\to\infty} \frac{6n}{3^n \cdot \ln 3}$$

$$= \frac{1}{(\ln 3)^2} \lim_{n\to\infty} \frac{6n}{3^n} = \frac{1}{(\ln 3)^2} \lim_{n\to\infty} \frac{6}{3^n \cdot \ln 3} = \frac{1}{(\ln 3)^3} \lim_{n\to\infty} \frac{6}{3^n}$$

$$= \frac{1}{(\ln 3)^3} \cdot \frac{6}{\infty} = \frac{6}{\infty} = 0$$

■ 3. Find the limit of the convergent sequence.

$$a_n = n^5 e^{-2n}$$

Solution:

Since the sequence converges, the limit of the sequence as $n \to \infty$ exists and is finite. Find the limit. Because

$$\lim_{n\to\infty} n^5 e^{-2n} = \lim_{n\to\infty} \frac{n^5}{e^{2n}} = \frac{\infty}{\infty}$$

is indeterminate, use L'Hospital's Rule to find the limit.

$$\lim_{n\to\infty} \frac{n^5}{e^{2n}} = \lim_{n\to\infty} \frac{5n^4}{2e^{2n}} = \lim_{n\to\infty} \frac{20n^3}{4e^{2n}} = \lim_{n\to\infty} \frac{60n^2}{8e^{2n}}$$

$$= \lim_{n\to\infty} \frac{120n}{16e^{2n}} = \lim_{n\to\infty} \frac{120}{32e^{2n}} = \frac{120}{\infty} = 0$$

INCREASING, DECREASING, AND NOT MONOTONIC

1. State whether the sequence is increasing, decreasing, and monotonic or not monotonic.

$$a_n = \frac{17}{4n^2 + 6n + 3}$$

Solution:

Calculate the value of the first few terms.

$$n = 1 \qquad a_1 = \frac{17}{4(1)^2 + 6(1) + 3} = \frac{17}{13}$$

$$n = 2 \qquad a_2 = \frac{17}{4(2)^2 + 6(2) + 3} = \frac{17}{31}$$

$$n = 3 \qquad a_3 = \frac{17}{4(3)^2 + 6(3) + 3} = \frac{17}{57}$$

$$n = 4 \qquad a_4 = \frac{17}{4(4)^2 + 6(4) + 3} = \frac{17}{91}$$

$$n = 5 \qquad a_5 = \frac{17}{4(5)^2 + 6(5) + 3} = \frac{17}{133}$$

Based on the first five terms, the value of the terms get consistently smaller as n gets larger, which means the sequence is decreasing, and also monotonic.

■ 2. State whether the sequence is increasing, decreasing, and monotonic or not monotonic.

$$a_n = \frac{3n^2 - 5}{4n + 2}$$

Solution:

Calculate the value of the first few terms.

$$n = 1 \qquad a_1 = \frac{3(1)^2 - 5}{4(1) + 2} = -\frac{1}{3}$$

$$n = 2 \qquad a_2 = \frac{3(2)^2 - 5}{4(2) + 2} = \frac{7}{10}$$

$$n = 3 \qquad a_3 = \frac{3(3)^2 - 5}{4(3) + 2} = \frac{11}{7}$$

$$n = 4 \qquad a_4 = \frac{3(4)^2 - 5}{4(4) + 2} = \frac{43}{18}$$

$$n = 5 \qquad a_5 = \frac{3(5)^2 - 5}{4(5) + 2} = \frac{35}{11}$$

Based on the first five terms, the value of the terms get consistently larger as n gets larger, which means the sequence is increasing, and also monotonic.

■ 3. State whether the sequence is increasing, decreasing, and monotonic or not monotonic.

$$a_n = n^5 + 1$$

Solution:

Calculate the value of the first few terms.

$$n = 1 \qquad a_1 = (1)^5 + 1 = 2$$

$$n = 2 \qquad a_2 = (2)^5 + 1 = 33$$

$$n = 3 \qquad a_3 = (3)^5 + 1 = 244$$

$$n = 4 \qquad a_4 = (4)^5 + 1 = 1{,}025$$

$$n = 5 \qquad a_5 = (5)^5 + 1 = 3{,}126$$

Based on the first five terms, the value of the terms get consistently larger as n gets larger, which means the sequence is increasing, and also monotonic.

BOUNDED SEQUENCES

1. Describe how the sequence is bounded by indicating the upper and lower bounds, or say whether there is no upper bound or now lower bound.

$$a_n = \frac{2n+5}{n^2}$$

Solution:

Find the first few terms of the sequence.

$$n = 1 \qquad a_1 = \frac{2(1)+5}{1^2} = 7$$

$$n = 2 \qquad a_2 = \frac{2(2)+5}{2^2} = \frac{9}{4}$$

$$n = 3 \qquad a_3 = \frac{2(3)+5}{3^2} = \frac{11}{9}$$

$$n = 4 \qquad a_4 = \frac{2(4)+5}{4^2} = \frac{13}{16}$$

$$n = 5 \qquad a_5 = \frac{2(5)+5}{5^2} = \frac{15}{25}$$

The sequence is bounded above at $a_1 = 7$. Now find the limit as $n \to \infty$. Because

$$\lim_{n\to\infty} \frac{2n+5}{n^2} = \frac{\infty}{\infty}$$

is indeterminate, use L'Hospital's Rule to find the limit.

$$\lim_{n\to\infty} \frac{2n+5}{n^2} = \lim_{n\to\infty} \frac{2}{2n} = \frac{2}{\infty} = 0$$

Therefore, the sequence is bounded below at 0, and above at $a_1 = 7$.

■ 2. Describe how the sequence is bounded by indicating the upper and lower bounds, or say whether there is no upper bound or now lower bound.

$$a_n = \frac{3n^3 + 2}{n^4}$$

Solution:

Find the first few terms of the sequence.

$n = 1$ $\qquad a_1 = \frac{3(1)^3 + 2}{1^4} = 5$

$n = 2$ $\qquad a_2 = \frac{3(2)^3 + 2}{2^4} = \frac{13}{8}$

$n = 3$ $\qquad a_3 = \frac{3(3)^3 + 2}{3^4} = \frac{83}{81}$

$$n = 4 \qquad a_4 = \frac{3(4)^3 + 2}{4^4} = \frac{97}{128}$$

$$n = 5 \qquad a_5 = \frac{3(5)^3 + 2}{5^4} = \frac{377}{625}$$

The sequence is bounded above at $a_1 = 5$. Now find the limit as $n \to \infty$. Because

$$\lim_{n \to \infty} \frac{3n^3 + 2}{n^4} = \frac{\infty}{\infty}$$

is indeterminate, use L'Hospital's Rule to find the limit.

$$\lim_{n \to \infty} \frac{3n^3 + 2}{n^4} = \lim_{n \to \infty} \frac{9n^2}{4n^3} = \lim_{n \to \infty} \frac{18n}{12n^2} = \lim_{n \to \infty} \frac{18}{24n} = \frac{18}{\infty} = 0$$

Therefore, the sequence is bounded below at 0, and above at $a_1 = 5$.

■ 3. Describe how the sequence is bounded by indicating the upper and lower bounds, or say whether there is no upper bound or now lower bound.

$$a_n = \frac{7n^3 + 15}{2n^3}$$

Solution:

Find the first few terms of the sequence.

$$n = 1 \qquad a_1 = \frac{7(1)^3 + 15}{2(1)^3} = 11$$

$$n = 2 \qquad a_2 = \frac{7(2)^3 + 15}{2(2)^3} = \frac{71}{16}$$

$$n = 3 \qquad a_3 = \frac{7(3)^3 + 15}{2(3)^3} = \frac{34}{9}$$

$$n = 4 \qquad a_4 = \frac{7(4)^3 + 15}{2(4)^3} = \frac{463}{128}$$

$$n = 5 \qquad a_5 = \frac{7(5)^3 + 15}{2(5)^3} = \frac{89}{25}$$

The sequence is bounded above at $a_1 = 11$. Now find the limit as $n \to \infty$. Because

$$\lim_{n\to\infty} \frac{7n^3 + 15}{2n^3} = \frac{\infty}{\infty}$$

is indeterminate, use L'Hospital's Rule to find the limit.

$$\lim_{n\to\infty} \frac{21n^2}{6n^2} = \lim_{n\to\infty} \frac{21}{6} = \frac{7}{2}$$

Therefore, the sequence is bounded below at $7/2$ and above at $a_1 = 11$.

■ 4. Describe how the sequence is bounded by indicating the upper and lower bounds, or say whether there is no upper bound or now lower bound.

$$a_n = \frac{3n^4 + 9}{4n^3}$$

Solution:

Find the first few terms of the sequence.

$n = 1$ $$a_1 = \frac{3(1)^4 + 9}{4(1)^3} = 3$$

$n = 2$ $$a_2 = \frac{3(2)^4 + 9}{4(2)^3} = \frac{57}{32}$$

$n = 3$ $$a_3 = \frac{3(3)^4 + 9}{4(3)^3} = \frac{7}{3}$$

$n = 4$ $$a_4 = \frac{3(4)^4 + 9}{4(4)^3} = \frac{777}{256}$$

$n = 5$ $$a_5 = \frac{3(5)^4 + 9}{4(5)^3} = \frac{471}{125}$$

The sequence is bounded below at $a_2 = 57/32$. Because

$$\lim_{n\to\infty} \frac{3n^4 + 9}{4n^3} = \frac{\infty}{\infty}$$

is indeterminate, use L'Hospital's Rule to find the limit.

$$\lim_{n \to \infty} \frac{12n^3}{12n^2} = \lim_{n \to \infty} \frac{n}{1} = \infty$$

Therefore, the sequence is bounded below at $a_2 = 57/32$ and has no upper bound.

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