Partial fractions

The method of partial fractions is an extremely useful tool whenever you need to integrate a fraction with polynomials in both the numerator and denominator; something like this:

$$f(x) = \frac{7x+1}{x^2-1}$$

If you were asked to integrate

$$f(x) = \frac{3}{x+1} + \frac{4}{x-1}$$

you shouldn't have too much trouble, because if you don't have a variable in the numerator of your fraction, then your integral is simply the numerator multiplied by the natural log ($\ln$) of the absolute value of the denominator, like this:

$$\int \frac{3}{x+1} + \frac{4}{x-1}\ dx$$

$$3\ln|x+1| + 4\ln|x-1| + C$$

where C is the constant of integration. Not *too* hard, right?

Don't forget to use chain rule and divide by the derivative of your denominator. In the case above, the derivatives of both of our denominators are 1, so this step didn't appear. But if your integral is

$$\int \frac{3}{2x+1}\ dx$$

then your answer will be

$$\frac{3}{2}\ln|2x+1|+C$$

because the derivative of our denominator is 2, which means we have to divide by 2, according to chain rule.

So back to the original example. We said at the beginning of this section that

$$f(x)=\frac{7x+1}{x^2-1}$$

would be difficult to integrate, but that we wouldn't have as much trouble with

$$f(x)=\frac{3}{x+1}+\frac{4}{x-1}$$

In fact, these two are actually the same function. If we try adding $3/(x+1)$ and $4/(x-1)$ together, you'll see that we get back to the original function.

$$f(x)=\frac{3}{x+1}+\frac{4}{x-1}$$

$$f(x)=\frac{3(x-1)+4(x+1)}{(x+1)(x-1)}$$

$$f(x)=\frac{3x-3+4x+4}{x^2-x+x-1}$$

$$f(x)=\frac{7x+1}{x^2-1}$$

Again, attempting to integrate $f(x) = (7x + 1)/\left(x^2 - 1\right)$ is extremely difficult. But if you can express this function as $f(x) = 3/(x + 1) + 4/(x - 1)$, then integrating is much simpler. This method of converting complicated fractions into simpler fractions that are easier to integrate is called decomposition into “partial fractions”.

Let’s start talking about how to perform a partial fractions decomposition. Before we move forward it’s important to remember that you must perform long division with your polynomials whenever the degree (value of the greatest exponent) of your denominator is not greater than the degree of your numerator, as is the case in the following example.

Example

Evaluate the integral.

$$\int \frac{x^3 - 3x^2 + 2}{x + 3}\ dx$$

Because the degree (the value of the highest exponent in the numerator, 3), is greater than the degree of the denominator, 1, we have to perform long division first.

$$
\begin{array}{rl}
 & x^2-6x+18-\frac{52}{x+3} \\
x+3 & \overline{)\, x^3-3x^2+0x+2} \\
 & -(x^3+3x^2) \\
\hline
 & -6x^2 \\
 & -(-6x^2-18x) \\
\hline
 & 18x+2 \\
 & -(18x+54) \\
\hline
 & -52
\end{array}
$$

After performing long division, our fraction has been decomposed into

$$\left(x^2-6x+18\right)-\frac{52}{x+3}$$

Now the function is easy to integrate.

$$\int x^2-6x+18-\frac{52}{x+3}\ dx$$

$$\frac{1}{3}x^3-3x^2+18x-52\ln|x+3|+C$$

Okay. So now that you’ve either performed long division or confirmed that the degree of your denominator is greater than the degree of your numerator (such that you don’t have to perform long division), it’s time for full-blown partial fractions.

The first step is to factor your denominator as much as you can. Your second step will be determining which type of denominator you're dealing with, depending on how it factors. Your denominator will be the product of the following:

1. Distinct linear factors
2. Repeated linear factors
3. Distinct quadratic factors
4. Repeated quadratic factors

Let's take a look at an example of each of these four cases so that you understand the difference between them.

Distinct linear factors

In this first example, we'll look at the first case above, in which the denominator is a product of distinct linear factors.

Example

Evaluate the integral.

$$\int \frac{x^2+2x+1}{x^3-2x^2-x+2}\ dx$$

Since the degree of the denominator is higher than the degree of the numerator, we don't have to perform long division before we start. Instead, we can move straight to factoring the denominator, as follows.

$$\int \frac{x^2+2x+1}{(x-1)(x+1)(x-2)}\ dx$$

We can see that our denominator is a product of distinct linear factors because $(x-1)$, $(x+1)$, and $(x-2)$ are all different first-degree factors.

Once we have it factored, we set our fraction equal to the sum of its component parts, assigning new variables to the numerator of each of our fractions. Since our denominator can be broken down into three different factors, we need three variables A, B and C to go on top of each one of our new fractions, like so:

$$\frac{x^2+2x+1}{(x-1)(x+1)(x-2)} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{C}{x-2}$$

Now that we've separated our original function into its partial fractions, we multiply both sides by the denominator of the left-hand side. The denominator will cancel on the left-hand side, and on the right, each of the three partial fractions will end up multiplied by all the factors other than the one that was previously included in its denominator.

$$x^2+2x+1 = A(x+1)(x-2) + B(x-1)(x-2) + C(x-1)(x+1)$$

The next step is to multiply out all of these terms.

$$x^2+2x+1 = A\left(x^2-x-2\right) + B\left(x^2-3x+2\right) + C\left(x^2-1\right)$$

$$x^2 + 2x + 1 = Ax^2 - Ax - 2A + Bx^2 - 3Bx + 2B + Cx^2 - C$$

Now we collect like terms together, meaning that we re-order them, putting all the x^2 terms next to each other, all the x terms next to each other, and then all the constants next to each other.

$$x^2 + 2x + 1 = \left(Ax^2 + Bx^2 + Cx^2\right) + (-Ax - 3Bx) + (-2A + 2B - C)$$

Finally, we factor out the x terms.

$$x^2 + 2x + 1 = (A + B + C)x^2 + (-A - 3B)x + (-2A + 2B - C)$$

Doing this allows us to equate coefficients on the left and right sides. Do you see how the coefficient on the x^2 term on the left-hand side of the equation is 1? Well, the coefficient on the x^2 term on the right-hand side is $(A + B + C)$, which means those two must be equal. We can do the same for the x term, as well as for the constants. We get the following three equations:

[1] $A + B + C = 1$

[2] $-A - 3B = 2$

[3] $-2A + 2B - C = 1$

Now that we have these equations, we need to solve for our three constants A, B, and C. This can easily get confusing, but with practice, you should get the hang of it. If you have one equation with only two variables instead of all three, like [2], that's a good place to start. Solving [2] for A gives us

[4] $A = -3B - 2$

Now we'll substitute [4] for A into [1] and [3] and then simplify, such that these equations:

$$(-3B-2)+B+C=1$$

$$-2(-3B-2)+2B-C=1$$

become these equations:

[5] $-2B+C=3$

[6] $8B-C=-3$

Now we can add [5] and [6] together to solve for B.

$$-2B+C+8B-C=3-3$$

$$6B=0$$

[7] $B=0$

Plugging [7] back into [4] to find A, we get

$$A=-3(0)-2$$

[8] $A=-2$

Plugging [7] back into [5] to find B, we get

$$-2(0)+C=3$$

[9] $C=3$

Having solved for the values of our three constants in [7], [8] and [9], we're finally ready to plug them back into our partial fractions decomposition. Doing so should produce something that's easier for us to integrate than our original function.

$$\int \frac{x^2+2x+1}{(x-1)(x+1)(x-2)}\ dx = \int \frac{-2}{x-1} + \frac{0}{x+1} + \frac{3}{x-2}\ dx$$

Simplifying the integral on the right side, we get

$$\int \frac{3}{x-2} - \frac{2}{x-1}\ dx$$

Remembering that the integral of $1/x$ is $\ln|x| + C$, we integrate and get

$$3\ln|x-2| - 2\ln|x-1| + C$$

Repeated linear factors

Let's move now to the second of our four case types above, in which the denominator will be a product of linear factors, some of which are repeated.

Example

Evaluate the integral.

$$\int \frac{2x^5 - 3x^4 + 5x^3 + 3x^2 - 9x + 13}{x^4 - 2x^2 + 1}\ dx$$

You'll see that we need to carry out long division before we start factoring, since the degree of the numerator is greater than the degree of the denominator ($5 > 4$).

$$2x - 3 + \frac{9x^3 - 3x^2 - 11x + 16}{x^4 - 2x^2 + 1}$$

$$x^4 - 2x^2 + 1 \ \overline{)\ 2x^5 - 3x^4 + 5x^3 + 3x^2 - 9x + 13}$$

$$\underline{-(2x^5 + 0x^4 - 4x^3 + 0x^2 + 2x)}$$

$$-3x^4 + 9x^3 + 3x^2 - 11x + 13$$

$$\underline{-(-3x^4 + 0x^3 + 6x^2 + 0x - 3)}$$

$$9x^3 - 3x^2 - 11x + 16$$

Now that the degree of the remainder is less than the degree of the original denominator, we can rewrite the problem as

$$\int 2x - 3 + \frac{9x^3 - 3x^2 - 11x + 16}{x^4 - 2x^2 + 1}\ dx$$

Integrating the $2x - 3$ will be simple, so for now, let's focus on the fraction. We'll factor the denominator.

$$\frac{9x^3 - 3x^2 - 11x + 16}{(x^2 - 1)(x^2 - 1)}$$

$$\frac{9x^3 - 3x^2 - 11x + 16}{(x+1)(x-1)(x+1)(x-1)}$$

$$\frac{9x^3 - 3x^2 - 11x + 16}{(x+1)^2(x-1)^2}$$

Given the factors involved in our denominator, you might think that the partial fraction decomposition would look like this:

$$\frac{9x^3 - 3x^2 - 11x + 16}{(x+1)^2(x-1)^2} = \frac{A}{x+1} + \frac{B}{x+1} + \frac{C}{x-1} + \frac{D}{x-1}$$

However, the fact that we're dealing with repeated factors, ($(x+1)$ is a factor twice and $(x-1)$ is a factor twice), the partial fractions decomposition is actually the following:

$$\frac{9x^3 - 3x^2 - 11x + 16}{x^4 - 2x^2 + 1} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+1} + \frac{D}{(x+1)^2}$$

To see why, let's take a simpler example. The partial fractions decomposition of $x^2/\left[(x+1)^4\right]$ is

$$\frac{x^2}{(x+1)^4} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{(x+1)^3} + \frac{D}{(x+1)^4}$$

Notice how we included $(x+1)^4$, our original factor, as well as each factor of lesser degree? We have to do this every time we have a repeated factor.

Let's continue with our original example.

$$\frac{9x^3 - 3x^2 - 11x + 16}{x^4 - 2x^2 + 1} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+1} + \frac{D}{(x+1)^2}$$

We'll multiply both sides of our equation by the denominator from the left side, $(x+1)^2(x-1)^2$, which will cancel the denominator on the left and some of the factors on the right.

$$9x^3 - 3x^2 - 11x + 16 = A(x-1)(x+1)^2 + B(x+1)^2 + C(x-1)^2(x+1) + D(x-1)^2$$

To simplify, we'll start multiplying all terms on the right side together.

$$9x^3 - 3x^2 - 11x + 16 = A\left(x^3 + x^2 - x - 1\right) + B\left(x^2 + 2x + 1\right)$$

$$+C\left(x^3 - x^2 - x + 1\right) + D\left(x^2 - 2x + 1\right)$$

Now we'll group like terms together.

$$9x^3 - 3x^2 - 11x + 16 = (A+C)x^3 + (A+B-C+D)x^2$$

$$+(-A+2B-C-2D)x + (-A+B+C+D)$$

Equating coefficients on both sides of the equation gives us the following equations.

[1] $A + C = 9$

[2] $A + B - C + D = -3$

[3] $-A + 2B - C - 2D = -11$

[4] $-A + B + C + D = 16$

Now we'll start solving for variables. If we subtract A from both sides of [1], we get

[5] $C = 9 - A$

If we plug [5] into [2], [3] and [4], we have

$$A + B - (9 - A) + D = -3$$

$$-A + 2B - (9 - A) - 2D = -11$$

$$-A + B + (9 - A) + D = 16$$

And simplifying, we get the following:

[6] $2A + B + D = 6$

[7] $2B - 2D = -2$

[8] $-2A + B + D = 7$

Let's now solve [7] for B.

$$2B - 2D = -2$$

$$2B = -2 + 2D$$

$$B = -1 + D$$

[9] $B = D - 1$

Plugging [9] into [6] and [8], we get

$$2A + (D - 1) + D = 6$$

$$-2A + (D - 1) + D = 7$$

And simplifying, we get the following:

[10] $2A + 2D = 7$

[11] $-2A + 2D = 8$

We solve [11] for D.

$$-2A + 2D = 8$$

$$2D = 8 + 2A$$

[12] $D = 4 + A$

We plug [12] into [10] to solve for A.

$$2A + 2(4 + A) = 7$$

$$2A + 8 + 2A = 7$$

$$4A = -1$$

[13] $A = -\frac{1}{4}$

At last! We've solved for one variable. Now it's pretty quick to find the other three. With [13], we can use [12] to find D.

$$D = 4 - \frac{1}{4}$$

[14] $D = \frac{15}{4}$

We plug [14] into [9] to find B.

$$B=\frac{15}{4}-1$$

[15] $B=\frac{11}{4}$

Last but not least, we plug [13] into [5] to solve for C.

$$C=9-\left(-\frac{1}{4}\right)$$

$$C=9+\frac{1}{4}$$

[16] $C=\frac{37}{4}$

Taking the values of the constants from [13], [14], [15], [16] and bringing back the $2x-3$ that we put aside following the long division earlier in this example, we'll write out the partial fractions decomposition.

$$\int \frac{2x^5-3x^4+5x^3+3x^2-9x+13}{x^4-2x^2+1}\ dx$$

$$\int 2x-3+\frac{9x^3-3x^2-11x+16}{x^4-2x^2+1}\ dx$$

$$\int 2x-3+\frac{-\frac{1}{4}}{x-1}+\frac{\frac{11}{4}}{(x-1)^2}+\frac{\frac{37}{4}}{x+1}+\frac{\frac{15}{4}}{(x+1)^2}\ dx$$

Now we can integrate. Using the rule from algebra that $1/(x^n)=x^{-n}$, we'll flip the second and fourth fractions so that they are easier to integrate.

$$\int 2x-3\ dx-\frac{1}{4}\int\frac{1}{x-1}\ dx+\frac{11}{4}\int(x-1)^{-2}\ dx+\frac{37}{4}\int\frac{1}{x+1}\ dx+\frac{15}{4}\int(x+1)^{-2}\ dx$$

Now that we've simplified, we'll integrate to get our final answer.

$$x^2-3x-\frac{1}{4}\ln|x-1|-\frac{11}{4(x-1)}+\frac{37}{4}\ln|x+1|-\frac{15}{4(x+1)}+C$$

Distinct quadratic factors

Now let's take a look at an example in which the denominator is a product of distinct quadratic factors.

In order to solve these types of integrals, you'll sometimes need the following formula:

[A] $$\int\frac{m}{x^2+n^2}\ dx=\frac{m}{n}\tan^{-1}\left(\frac{x}{n}\right)+C$$

Example

Evaluate the integral.

$$\int\frac{x^2-2x-5}{x^3-x^2+9x-9}\ dx$$

As always, the first thing to notice is that the degree of the denominator is larger than the degree of the numerator, which means that we don't have to perform long division before we can start factoring the denominator. So let's get right to it and factor the denominator.

$$\int \frac{x^2 - 2x - 5}{(x - 1)(x^2 + 9)}\ dx$$

We have one distinct linear factor, $(x - 1)$, and one distinct quadratic factor, $(x^2 + 9)$.

As we already know, linear factors require one constant in the numerator, like this:

$$\frac{A}{x - 1}$$

The numerators of quadratic factors require a polynomial, like this:

$$\frac{Ax + B}{x^2 + 9}$$

Remember though that when we add these fractions together in the partial fractions decomposition, we never want to repeat the same constant, so the partial fractions decomposition is

$$\frac{x^2 - 2x - 5}{(x - 1)(x^2 + 9)} = \frac{A}{x - 1} + \frac{Bx + C}{x^2 + 9}$$

See how we started the second fraction with B instead of A? If we added a second quadratic factor to this example, its numerator would be $Dx + E$.

Multiplying both sides of our decomposition by the denominator on the left gives

$$x^2 - 2x - 5 = A(x^2 + 9) + (Bx + C)(x - 1)$$

$$x^2 - 2x - 5 = Ax^2 + 9A + Bx^2 - Bx + Cx - C$$

$$x^2 - 2x - 5 = (Ax^2 + Bx^2) + (-Bx + Cx) + (9A - C)$$

$$x^2 - 2x - 5 = (A + B)x^2 + (-B + C)x + (9A - C)$$

Then equating coefficients on the left and right sides gives us the following equations.

[1] $A + B = 1$

[2] $-B + C = -2$

[3] $9A - C = -5$

We solve [1] for A.

[4] $A = 1 - B$

Plugging [4] into [3] leaves us with two equations in terms of B and C.

[2] $-B + C = -2$

[5] $9(1 - B) - C = -5$

Simplifying [5] leaves us with

[2] $-B + C = -2$

[6] $-9B - C = -14$

Solving [2] for C we get

[7] $C = B - 2$

Plugging [7] into [6] gives

$$-9B - (B - 2) = -14$$

$$-10B + 2 = -14$$

$$-10B = -16$$

[8] $B = \frac{8}{5}$

Now that we have a value for B, we'll plug [8] into [7] to solve for C.

$$C = \frac{8}{5} - 2$$

[9] $C = -\frac{2}{5}$

We can also plug [8] into [4] to solve for A.

$$A = 1 - \frac{8}{5}$$

[10] $A = -\frac{3}{5}$

Plugging [8], [9] and [10] into our partial fractions decomposition, we get

$$\int \frac{x^2 - 2x - 5}{(x-1)(x^2+9)}\ dx = \int \frac{-\frac{3}{5}}{x-1} + \frac{\frac{8}{5}x - \frac{2}{5}}{x^2+9}\ dx$$

$$-\frac{3}{5}\int\frac{1}{x-1}\ dx+\frac{8}{5}\int\frac{x}{x^2+9}\ dx-\frac{2}{5}\int\frac{1}{x^2+9}\ dx$$

Integrating the first term only, we get

$$-\frac{3}{5}\ln|x-1|+\frac{8}{5}\int\frac{x}{x^2+9}\ dx-\frac{2}{5}\int\frac{1}{x^2+9}\ dx$$

Using u-substitution to integrate the second integral, letting

$$u=x^2+9$$

$$du=2x\ dx$$

$$dx=\frac{du}{2x}$$

we get

$$-\frac{3}{5}\ln|x-1|+\frac{8}{5}\int\frac{x}{u}\cdot\frac{du}{2x}-\frac{2}{5}\int\frac{1}{x^2+9}\ dx$$

$$-\frac{3}{5}\ln|x-1|+\frac{4}{5}\int\frac{1}{u}\ du-\frac{2}{5}\int\frac{1}{x^2+9}\ dx$$

$$-\frac{3}{5}\ln|x-1|+\frac{4}{5}\ln|u|-\frac{2}{5}\int\frac{1}{x^2+9}\ dx$$

$$-\frac{3}{5}\ln|x-1|+\frac{4}{5}\ln|x^2+9|-\frac{2}{5}\int\frac{1}{x^2+9}\ dx$$

Using [A] to integrate the third term, we get

[A] $$\int\frac{m}{x^2+n^2}\ dx=\frac{m}{n}\tan^{-1}\left(\frac{x}{n}\right)+C$$

$$m = 1$$

$$n = 3$$

$$-\frac{3}{5}\ln|x-1| + \frac{4}{5}\ln|x^2+9| - \frac{2}{5}\left[\frac{1}{3}\tan^{-1}\left(\frac{x}{3}\right)\right] + C$$

$$-\frac{3}{5}\ln|x-1| + \frac{4}{5}\ln|x^2+9| - \frac{2}{15}\tan^{-1}\left(\frac{x}{3}\right) + C$$

$$\frac{1}{5}\left[4\ln|x^2+9| - 3\ln|x-1| - \frac{2}{3}\tan^{-1}\left(\frac{x}{3}\right)\right] + C$$

Repeated quadratic factors

Last but not least, let's take a look at an example in which the denominator is a product of quadratic factors, at least some of which are repeated.

We'll be using formula [A] like we did in the last example.

Example

Evaluate the integral.

$$\int \frac{-x^3+2x^2-x+1}{x(x^2+1)^2}\ dx$$

Remember, when we're dealing with repeated factors, we have to include every lesser degree of that factor in our partial fractions decomposition, which will be

$$\frac{-x^3+2x^2-x+1}{x(x^2+1)^2}=\frac{A}{x}+\frac{Bx+C}{x^2+1}+\frac{Dx+E}{(x^2+1)^2}$$

Multiplying both sides by the denominator of the left-hand side gives us

$$-x^3+2x^2-x+1=A(x^2+1)^2+(Bx+C)x(x^2+1)+(Dx+E)x$$

Simplifying the right-hand side, we get

$$-x^3+2x^2-x+1=A(x^4+2x^2+1)+(Bx+C)(x^3+x)+(Dx+E)x$$

$$-x^3+2x^2-x+1=Ax^4+2Ax^2+A+Bx^4+Bx^2+Cx^3+Cx+Dx^2+Ex$$

Grouping like terms together, we have

$$-x^3+2x^2-x+1=(Ax^4+Bx^4)+(Cx^3)+(2Ax^2+Bx^2+Dx^2)+(Cx+Ex)+(A)$$

And factoring, we get

$$-x^3+2x^2-x+1=(A+B)x^4+(C)x^3+(2A+B+D)x^2+(C+E)x+(A)$$

Now we equate coefficients and write down the equations we'll use to solve for each of our constants.

[1] $A+B=0$

[2] $C=-1$

[3] $2A+B+D=2$

[4] $C+E=-1$

[5] $A=1$

We already have values for A and C. Plugging [5] into [1] to solve for B gives us

$$1+B=0$$

[6] $B=-1$

Plugging [2] into [4] to solve for E, we get

$$-1+E=-1$$

[7] $E=0$

Plugging [5] and [6] into [3] to solve for D gives us

$$2(1)-1+D=2$$

[8] $D=1$

Plugging our constants from [2], [5], [6], [7] and [8] back into the decomposition, we get

$$\int \frac{(1)}{x}+\frac{(-1)x+(-1)}{x^2+1}+\frac{(1)x+(0)}{(x^2+1)^2}\ dx$$

$$\int \frac{1}{x}-\frac{x+1}{x^2+1}+\frac{x}{(x^2+1)^2}\ dx$$

$$\int \frac{1}{x}\ dx-\int \frac{x+1}{x^2+1}\ dx+\int \frac{x}{(x^2+1)^2}\ dx$$

$$\int \frac{1}{x}\ dx - \int \frac{x}{x^2+1}\ dx - \int \frac{1}{x^2+1}\ dx + \int \frac{x}{(x^2+1)^2}\ dx$$

Evaluating the first integral only, we get

$$\ln|x| - \int \frac{x}{x^2+1}\ dx - \int \frac{1}{x^2+1}\ dx + \int \frac{x}{(x^2+1)^2}\ dx$$

Using u-substitution to evaluate the second integral, letting

$$u = x^2 + 1$$

$$du = 2x\ dx$$

$$dx = \frac{du}{2x}$$

we get

$$\ln|x| - \int \frac{x}{u} \cdot \frac{du}{2x} - \int \frac{1}{x^2+1}\ dx + \int \frac{x}{(x^2+1)^2}\ dx$$

$$\ln|x| - \frac{1}{2}\int \frac{1}{u}\ du - \int \frac{1}{x^2+1}\ dx + \int \frac{x}{(x^2+1)^2}\ dx$$

$$\ln|x| - \frac{1}{2}\ln|u| - \int \frac{1}{x^2+1}\ dx + \int \frac{x}{(x^2+1)^2}\ dx$$

$$\ln|x| - \frac{1}{2}\ln|x^2+1| - \int \frac{1}{x^2+1}\ dx + \int \frac{x}{(x^2+1)^2}\ dx$$

Using formula [A] to evaluate the third integral, we get

[A] $$\int \frac{m}{x^2+n^2}\ dx = \frac{m}{n}\tan^{-1}\left(\frac{x}{n}\right) + C$$

$$m = 1$$

$$n = 1$$

$$\ln|x| - \frac{1}{2}\ln|x^2+1| - \frac{1}{1}\tan^{-1}\left(\frac{x}{1}\right) + \int \frac{x}{(x^2+1)^2}\ dx$$

$$\ln|x| - \frac{1}{2}\ln|x^2+1| - \tan^{-1}x + \int \frac{x}{(x^2+1)^2}\ dx$$

Using u-substitution to evaluate the fourth integral, letting

$$u = x^2 + 1$$

$$du = 2x\ dx$$

$$dx = \frac{du}{2x}$$

we get

$$\ln|x| - \frac{1}{2}\ln|x^2+1| - \tan^{-1}x + \int \frac{x}{u^2} \cdot \frac{du}{2x}$$

$$\ln|x| - \frac{1}{2}\ln|x^2+1| - \tan^{-1}x + \frac{1}{2}\int \frac{1}{u^2}\ du$$

$$\ln|x| - \frac{1}{2}\ln|x^2+1| - \tan^{-1}x + \frac{1}{2}\int u^{-2}\ du$$

$$\ln|x| - \frac{1}{2}\ln|x^2+1| - \tan^{-1}x - \frac{1}{2u} + C$$

And plugging back in for u gives us the final answer.

$$\ln|x| - \frac{1}{2}\ln|x^2+1| - \tan^{-1}x - \frac{1}{2(x^2+1)} + C$$

In summary, in order to integrate by expressing rational functions (fractions) in terms of their partial fractions decomposition, you should follow these steps:

1. Ensure that the rational function is "proper", such that the degree (greatest exponent) of the numerator is less than the degree of the denominator. If necessary, use long division to make it proper.

2. Perform the partial fractions decomposition by factoring the denominator, which will always be expressible as the product of either linear or quadratic factors, some of which may be repeated.

    a. If the denominator is a product of distinct linear factors: This is the simplest kind of partial fractions decomposition. Nothing fancy here.

    b. If the denominator is a product of linear factors, some of which are repeated: Remember to include factors of lesser degree than your repeated factors.

    c. If the denominator is a product of distinct quadratic factors: You'll need the following equation:

[A] $$\int \frac{m}{x^2+n^2}\,dx = \frac{m}{n}\tan^{-1}\left(\frac{x}{n}\right) + C$$

d. If the denominator is a product of quadratic factors, some of which are repeated: Use the two formulas above and remember to include factors of lesser degree than your repeated factors.