Topic: Quadratic functions

Question: Evaluate the integral.

$$\int \frac{x+1}{x^2+2x+2}\ dx$$

Answer choices:

A $\frac{1}{2}\ln\left|x^2+1\right|+C$

B $\frac{1}{2}\ln\left|(x+1)^2+2\right|+C$

C $\frac{1}{2}\ln\left|(x-1)^2+1\right|+C$

D $\frac{1}{2}\ln\left|x^2+2x+2\right|+C$

Solution: D

Quadratic functions are polynomial functions of the specific form

$$f(x) = ax^2 + bx + c$$

Integrals of simple quadratic functions, like

$$\int ax^2 + bx + c \ dx$$

can be easily evaluated using power rule, like any other polynomial function. However, if we start manipulating the quadratic function, we'll likely have to use other techniques to solve the integral. For example, when the quadratic function appears as the denominator of a rational function (fraction), we can very often use trigonometric substitution and/or u-substitution in order to evaluate the integral.

For this particular integral, we need to start by completing the square in the denominator. Taking the coefficient 2 on the first-degree x-term, we'll complete the square by dividing it by 2 and then squaring the result. This will be the number we have to add in (and subtract out) to complete the square.

$$\int \frac{x+1}{x^2+2x+2} \ dx$$

$$\int \frac{x+1}{x^2+2x+\left(\frac{2}{2}\right)^2-\left(\frac{2}{2}\right)^2+2} \ dx$$

$$\int \frac{x+1}{x^2+2x+1-1+2} \ dx$$

$$\int \frac{x+1}{\left(x^2+2x+1\right)+1}\ dx$$

$$\int \frac{x+1}{(x+1)^2+1}\ dx$$

$$\int \frac{x+1}{(x+1)^2+1^2}\ dx$$

Because the denominator is the sum of two squares, we can try trigonometric substitution to evaluate the integral. Setting up trigonometric substitution by comparing u^2+a^2 with $(x+1)^2+1^2$, we get

$$u^2+a^2=(x+1)^2+1^2$$

$$u=x+1$$

$$a=1$$

$$u=a\tan\theta$$

$$x+1=1\tan\theta$$

$$x+1=\tan\theta$$

$$x=\tan\theta-1$$

$$dx=\sec^2\theta\ d\theta$$

In the right triangle,

Adjacent side	1
Opposite side	$x+1$

Hypotenuse $\sqrt{x^2+2x+2}$

Plugging these values into the integral, we get

$$\int \frac{\tan\theta}{(\tan\theta)^2+1}\sec^2\theta\ d\theta$$

$$\int \frac{\tan\theta\sec^2\theta}{\tan^2\theta+1}\ d\theta$$

Since we know that $\tan^2\theta+1=\sec^2\theta$, we can make a substitution into the denominator.

$$\int \frac{\tan\theta\sec^2\theta}{\sec^2\theta}\ d\theta$$

$$\int \tan\theta\ d\theta$$

From here we can use a formula for the integral of $\tan\theta$, or we can solve it without the formula. In case you forget the formula, here's how you can solve it with u-substitution.

$$\int \tan\theta\ d\theta$$

$$\int \frac{\sin\theta}{\cos\theta}\ d\theta$$

$$u=\cos\theta$$

$$du=-\sin\theta\ d\theta$$

$$d\theta = \frac{du}{-\sin\theta}$$

$$\int \frac{\sin\theta}{u}\left(\frac{du}{-\sin\theta}\right)$$

$$-\int \frac{1}{u}\ du$$

$$-\ln|u| + C$$

$$-\ln|\cos\theta| + C$$

$$\ln\left|(\cos\theta)^{-1}\right| + C$$

$$\ln\left|\frac{1}{\cos\theta}\right| + C$$

Since cosine of an angle is equal to the adjacent side over the hypotenuse, we get

$$\ln\left|\frac{1}{\frac{1}{\sqrt{x^2+2x+2}}}\right| + C$$

$$\ln\left|\sqrt{x^2+2x+2}\right| + C$$

$$\ln\left|\left(x^2+2x+2\right)^{\frac{1}{2}}\right| + C$$

$$\frac{1}{2}\ln\left|x^2+2x+2\right| + C$$

Topic: Quadratic functions

Question: Evaluate the integral.

$$\int \sqrt{x^2-2x-8}\ dx$$

Answer choices:

A $\quad \dfrac{(x-1)\sqrt{x^2-2x-8}}{2}-\dfrac{9}{2}\ln\left|\dfrac{x-1+\sqrt{x^2-2x-8}}{3}\right|$

B $\quad \dfrac{(x+1)\sqrt{x^2-2x-8}}{2}-\dfrac{9}{2}\ln\left|\dfrac{x+1+\sqrt{x^2-2x-8}}{3}\right|$

C $\quad \dfrac{(x-1)\sqrt{x^2-2x-8}}{2}+\dfrac{9}{2}\ln\left|\dfrac{x-1+\sqrt{x^2-2x-8}}{3}\right|$

D $\quad \dfrac{(x+1)\sqrt{x^2-2x-8}}{2}+\dfrac{9}{2}\ln\left|\dfrac{x+1+\sqrt{x^2-2x-8}}{3}\right|$

Solution: A

Quadratic functions are polynomial functions of the specific form

$$f(x) = ax^2 + bx + c$$

Integrals of simple quadratic functions, like

$$\int ax^2 + bx + c\ dx$$

can be easily evaluated using power rule, like any other polynomial function. However, if we start manipulating the quadratic function, we'll likely have to use other techniques to solve the integral. For example, when the quadratic function appears as the denominator of a rational function (fraction), we can very often use trigonometric substitution and/or u-substitution in order to evaluate the integral.

For this particular integral, we need to start by completing the square inside the radical. Taking the coefficient -2 on the first-degree x-term, we'll complete the square by dividing it by 2 and then squaring the result. This will be the number we have to add in (and subtract out) to complete the square.

$$\int \sqrt{x^2 - 2x - 8}\ dx$$

$$\int \sqrt{x^2 - 2x + \left(\frac{-2}{2}\right)^2 - \left(\frac{-2}{2}\right)^2 - 8}\ dx$$

$$\int \sqrt{x^2 - 2x + 1 - 1 - 8}\ dx$$

$$\int \sqrt{\left(x^2 - 2x + 1\right) - 1 - 8}\ dx$$

$$\int \sqrt{(x-1)^2 - 3^2}\ dx$$

Because the value inside the radical is the difference of two squares, we can try trigonometric substitution to evaluate the integral. Setting up trigonometric substitution by comparing $u^2 - a^2$ with $(x-1)^2 - 3^2$, we get

$$u^2 - a^2 = (x-1)^2 - 3^2$$

$$u = x - 1$$

$$a = 3$$

$$u = a \sec \theta$$

$$x - 1 = 3 \sec \theta$$

$$\frac{x-1}{3} = \sec \theta$$

$$x = 1 + 3 \sec \theta$$

$$dx = 3 \sec \theta \tan \theta\ d\theta$$

In the right triangle,

Adjacent side	3
Opposite side	$\sqrt{x^2 - 2x - 8}$
Hypotenuse	$x - 1$

Plugging these values into the integral, we get

$$\int 3\sec\theta\tan\theta\sqrt{(3\sec\theta)^2-3^2}\ d\theta$$

$$\int 3\sec\theta\tan\theta\sqrt{9\sec^2\theta-9}\ d\theta$$

$$\int 3\sec\theta\tan\theta\sqrt{9\left(\sec^2\theta-1\right)}\ d\theta$$

$$9\int \sec\theta\tan\theta\sqrt{\sec^2\theta-1}\ d\theta$$

Since we know that $\tan^2\theta=\sec^2\theta-1$, we can make a substitution.

$$9\int \sec\theta\tan\theta\sqrt{\tan^2\theta}\ d\theta$$

$$9\int \sec\theta\tan\theta\tan\theta\ d\theta$$

$$9\int \sec\theta\tan^2\theta\ d\theta$$

$$9\int \sec\theta\left(\sec^2\theta-1\right)\ d\theta$$

$$9\int \sec^3\theta-\sec\theta\ d\theta$$

$$9\int \sec^3\theta\ d\theta-9\int\sec\theta\ d\theta$$

Integrating both integrals, we get

$$\frac{9}{2}\left(\sec\theta\tan\theta+\ln\left|\sec\theta+\tan\theta\right|\right)-9\ln\left|\sec\theta+\tan\theta\right|$$

$$\frac{9}{2}\sec\theta\tan\theta+\frac{9}{2}\ln\left|\sec\theta+\tan\theta\right|-\frac{18}{2}\ln\left|\sec\theta+\tan\theta\right|$$

$$\frac{9}{2}\sec\theta\tan\theta-\frac{9}{2}\ln\left|\sec\theta+\tan\theta\right|$$

With

$$\frac{x-1}{3}=\sec\theta$$

$$\sec^{-1}\left(\frac{x-1}{3}\right)=\theta$$

we can back-substitute for θ.

$$\frac{9}{2}\left(\frac{x-1}{3}\right)\tan\left(\sec^{-1}\left(\frac{x-1}{3}\right)\right)-\frac{9}{2}\ln\left|\frac{x-1}{3}+\tan\left(\sec^{-1}\left(\frac{x-1}{3}\right)\right)\right|$$

We know that

$$\tan(\sec^{-1}x)=x\sqrt{1-\frac{1}{x^2}}$$

In this case, the “x” is $(x-1)/3$, so let’s make that substitution.

$$\tan\left(\sec^{-1}\left(\frac{x-1}{3}\right)\right)=\frac{x-1}{3}\sqrt{1-\frac{1}{\left(\frac{x-1}{3}\right)^2}}$$

$$\tan\left(\sec^{-1}\left(\frac{x-1}{3}\right)\right)=\frac{x-1}{3}\sqrt{1-\frac{1}{\frac{(x-1)^2}{9}}}$$

$$\tan\left(\sec^{-1}\left(\frac{x-1}{3}\right)\right)=\frac{x-1}{3}\sqrt{1-\frac{9}{(x-1)^2}}$$

Find a common denominator.

$$\tan\left(\sec^{-1}\left(\frac{x-1}{3}\right)\right)=\frac{x-1}{3}\sqrt{\frac{(x-1)^2}{(x-1)^2}-\frac{9}{(x-1)^2}}$$

$$\tan\left(\sec^{-1}\left(\frac{x-1}{3}\right)\right)=\frac{x-1}{3}\sqrt{\frac{(x-1)^2-9}{(x-1)^2}}$$

Apply the square root to the numerator and denominator separately.

$$\tan\left(\sec^{-1}\left(\frac{x-1}{3}\right)\right)=\frac{x-1}{3}\cdot\frac{\sqrt{(x-1)^2-9}}{\sqrt{(x-1)^2}}$$

$$\tan\left(\sec^{-1}\left(\frac{x-1}{3}\right)\right)=\frac{x-1}{3}\cdot\frac{\sqrt{(x-1)^2-9}}{x-1}$$

$$\tan\left(\sec^{-1}\left(\frac{x-1}{3}\right)\right)=\frac{\sqrt{(x-1)^2-9}}{3}$$

Simplify the numerator.

$$\tan\left(\sec^{-1}\left(\frac{x-1}{3}\right)\right)=\frac{\sqrt{x^2-2x+1-9}}{3}$$

$$\tan\left(\sec^{-1}\left(\frac{x-1}{3}\right)\right)=\frac{\sqrt{x^2-2x-8}}{3}$$

Then we can simplify the parts of the expression where we're taking the tangent of the inverse secant.

$$\frac{9}{2}\left(\frac{x-1}{3}\right)\frac{\sqrt{x^2-2x-8}}{3}-\frac{9}{2}\ln\left|\frac{x-1}{3}+\frac{\sqrt{x^2-2x-8}}{3}\right|$$

$$\frac{9(x-1)\sqrt{x^2-2x-8}}{18}-\frac{9}{2}\ln\left|\frac{x-1+\sqrt{x^2-2x-8}}{3}\right|$$

$$\frac{(x-1)\sqrt{x^2-2x-8}}{2}-\frac{9}{2}\ln\left|\frac{x-1+\sqrt{x^2-2x-8}}{3}\right|$$

Topic: Quadratic functions

Question: Evaluate the integral.

$$\int \frac{1}{x^2 + 6x + 13} \, dx$$

Answer choices:

A $\quad \frac{1}{2} \tan \left(\frac{x+3}{2} \right)$

B $\quad \frac{1}{2} \tan^{-1} \left(\frac{x+3}{2} \right)$

C $\quad \frac{1}{2} \tan^{-1} \left(\frac{x-3}{2} \right)$

D $\quad \frac{1}{2} \tan \left(\frac{x-3}{2} \right)$

Solution: B

Quadratic functions are polynomial functions of the specific form

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Integrals of simple quadratic functions, like

$$\int ax^2 + bx + c\ dx$$

can be easily evaluated using power rule, like any other polynomial function. However, if we start manipulating the quadratic function, we'll likely have to use other techniques to solve the integral. For example, when the quadratic function appears as the denominator of a rational function (fraction), we can very often use trigonometric substitution and/or u-substitution in order to evaluate the integral.

For this particular integral, we need to start by completing the square in the denominator. Taking the coefficient 6 on the first-degree x-term, we'll complete the square by dividing it by 2 and then squaring the result. This will be the number we have to add in (and subtract out) to complete the square.

$$\int \frac{1}{x^2 + 6x + 13}\ dx$$

$$\int \frac{1}{x^2 + 6x + \left(\frac{6}{2}\right)^2 - \left(\frac{6}{2}\right)^2 + 13}\ dx$$

$$\int \frac{1}{\left(x^2+6x+9\right)-9+13}\ dx$$

$$\int \frac{1}{(x+3)^2+4}\ dx$$

$$\int \frac{1}{(x+3)^2+2^2}\ dx$$

$$u=x+3$$

$$du=dx$$

$$\int \frac{1}{u^2+2^2}\ du$$

We know that

$$\int \frac{1}{u^2+a^2}\ du=\frac{1}{a}\tan^{-1}\left(\frac{u}{a}\right)$$

so we get

$$\frac{1}{2}\tan^{-1}\left(\frac{x+3}{2}\right)$$