Arc length

We can use integration to calculate the arc length of a function, which is the length the function would be if we took the line of its graph and stretched it out straight and measured it.

Sometimes we'll need to find arc length of a function in the form $y = f(x)$, but other times the function will be in the form $x = g(y)$.

1. If the equation is in the form $y = f(x)$, the interval will be $a \leq x \leq b$ and the equation for arc length is

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\ dx$$

2. If the equation is in the form $x = g(y)$, the interval will be $c \leq y \leq d$ and the equation for arc length is

$$L = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\ dy$$

Example

Calculate the arc length of the curve over the interval.

$$y = \ln(\sec x)$$

on $0 \leq x \leq \frac{\pi}{3}$

Since the function we're given is in the form $y = f(x)$, we have to use the formula

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\ dx$$

First, we'll calculate dy/dx and then plug it back into the arc length formula.

$$\frac{dy}{dx} = \frac{\sec x \tan x}{\sec x}$$

$$\frac{dy}{dx} = \tan x$$

Plugging the derivative into the arc length formula, we get

$$L = \int_0^{\frac{\pi}{3}} \sqrt{1 + (\tan x)^2}\ dx$$

$$L = \int_0^{\frac{\pi}{3}} \sqrt{1 + \tan^2 x}\ dx$$

Remembering that $\sec^2(x) = 1 + \tan^2(x)$, we get

$$L = \int_0^{\frac{\pi}{3}} \sqrt{\sec^2 x}\ dx$$

$$L = \int_0^{\frac{\pi}{3}} \sec x\ dx$$

Integrate.

$$L = \ln\left|\sec(x) + \tan(x)\right|\Big|_0^{\frac{\pi}{3}}$$

Now we evaluate over the interval and get

$$L = \ln\left|\sec\left(\frac{\pi}{3}\right) + \tan\left(\frac{\pi}{3}\right)\right| - \ln\left|\sec 0 + \tan 0\right|$$

$$L = 1.32$$

The arc length of $y = \ln(\sec x)$ over the interval $0 \le x \le \frac{\pi}{3}$ is $L = 1.32$.

Now let's try an example where the curve is defined for x in terms of y.

Example

Calculate the arc length of the curve over the interval.

$$x = \frac{2}{3}(y-1)^{\frac{3}{2}}$$

on $2 \le y \le 5$

Since the function we're given is in the form $x = g(y)$, we have to use the formula

$$L = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\ dy$$

First, we'll calculate dx/dy and then plug it back into the arc length formula.

$$\frac{dx}{dy} = (y-1)^{\frac{1}{2}}$$

Plugging the derivative into the arc length formula, we get

$$L = \int_2^5 \sqrt{1 + \left[(y-1)^{\frac{1}{2}}\right]^2}\ dy$$

$$L = \int_2^5 \sqrt{y}\ dy$$

Integrate.

$$L = \frac{2}{3}y^{\frac{3}{2}}\Big|_2^5$$

Now we evaluate over the interval and get

$$L = \frac{2}{3}(5)^{\frac{3}{2}} - \frac{2}{3}(2)^{\frac{3}{2}}$$

$$L = 5.6$$

The arc length of $x = \frac{2}{3}(y-1)^{\frac{3}{2}}$ over the interval $2 \le y \le 5$ is $L = 5.6$.