Newton's Method

The **root** of a function is a point at which the graph of the function crosses the x-axis. Because $y = 0$ everywhere along the x-axis, a root is any point where the value of the function is 0.

Newton's Method is a tool that allows us to approximate the point at which the root exists. When we use Newton's Method, the function must be in the form $f(x) = 0$. If it isn't, we'll need to put it in that form before we start.

We'll start with one approximating value, and use it to get a better approximation. Then we'll use this new approximation to get an even better approximation. We'll continue that process, over and over, until we get an approximation we're satisfied with.

Usually, we'll choose to find an approximation to a certain number of decimal places, and that's how we'll know when to be "satisfied" with the approximation.

For example, we might choose to find an approximation to three decimal places. If so, then once we get an answer that's stable to three decimal places, meaning that the first three decimal places don't change as we keep taking better and better approximations, then we know we're done.

If we use a starting approximation x_n, then we can say that the next subsequent approximation is the x_{n+1} approximation, and Newton's Method gives us a formula for that x_{n+1} approximation.

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

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If we don't know an initial approximation to the solution x_0, we can sketch the graph of the function and use that to get an estimate of the solution, which we can then use as x_0. Or if we know the interval where the function has a solution, then we can use the midpoint of the interval as x_0.

Let's work through an example where we use Newton's Method to find an approximation of the root of a function.

Example

Use Newton's Method to find an approximation of the root of the function to four decimal places, with $x_0 = -1$.

$$f(x) = x^2 - x$$

When we use Newton's Method, the function must be in the form $f(x) = 0$.

$$x^2 - x = 0$$

Take the derivative of the function.

$$f'(x) = 2x - 1$$

We're using $x_0 = -1$ as the starting approximation, which means $n = 0$. We can substitute this value into the Newton's Method formula to get a formula for the next approximation.

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

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$$x_{0+1} = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

Substitute the starting approximation $x_0 = -1$.

$$x_1 = -1 - \frac{f(-1)}{f'(-1)} = -1 - \frac{(-1)^2 - (-1)}{2(-1) - 1} = -1 - \frac{1+1}{-2-1}$$

$$= -1 - \frac{2}{-3} = -1 + \frac{2}{3} = -\frac{1}{3} \approx -0.3333$$

Use this value for x_1 to find the next approximation for x_2.

$$x_2 = -\frac{1}{3} - \frac{f\left(-\frac{1}{3}\right)}{f'\left(-\frac{1}{3}\right)} = -\frac{1}{3} - \frac{\left(-\frac{1}{3}\right)^2 - \left(-\frac{1}{3}\right)}{2\left(-\frac{1}{3}\right) - 1} = -\frac{1}{3} - \frac{\frac{1}{9} + \frac{1}{3}}{-\frac{2}{3} - 1}$$

$$= -\frac{1}{3} - \frac{\frac{1}{9} + \frac{3}{9}}{-\frac{2}{3} - \frac{3}{3}} = -\frac{1}{3} - \frac{\frac{4}{9}}{-\frac{5}{3}} = -\frac{1}{3} + \frac{\frac{4}{9}}{\frac{5}{3}} = -\frac{1}{3} + \frac{4}{9}\left(\frac{3}{5}\right)$$

$$= -\frac{1}{3} + \frac{4}{3}\left(\frac{1}{5}\right) = -\frac{1}{3} + \frac{4}{15} = -\frac{5}{15} + \frac{4}{15} = -\frac{1}{15} \approx -0.0667$$

Use this value for x_2 to find the next approximation for x_3.

$$x_3 = -\frac{1}{15} - \frac{f\left(-\frac{1}{15}\right)}{f'\left(-\frac{1}{15}\right)} = -\frac{1}{15} - \frac{\left(-\frac{1}{15}\right)^2 - \left(-\frac{1}{15}\right)}{2\left(-\frac{1}{15}\right) - 1} = -\frac{1}{15} - \frac{\frac{1}{225} + \frac{1}{15}}{-\frac{2}{15} - 1}$$

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$$= -\frac{1}{15} - \frac{\frac{1}{225} + \frac{15}{225}}{-\frac{2}{15} - \frac{15}{15}} = -\frac{1}{15} - \frac{\frac{16}{225}}{-\frac{17}{15}} = -\frac{1}{15} + \frac{\frac{16}{225}}{\frac{17}{15}}$$

$$= -\frac{1}{15} + \frac{16}{225}\left(\frac{15}{17}\right) = -\frac{1}{15} + \frac{16}{15}\left(\frac{1}{17}\right) = -\frac{1}{15} + \frac{16}{255}$$

$$= -\frac{17}{255} + \frac{16}{255} = -\frac{1}{255} \approx -0.0039$$

Use this value for x_3 to find the next approximation for x_4.

$$x_4 = -\frac{1}{255} - \frac{f\left(-\frac{1}{255}\right)}{f'\left(-\frac{1}{255}\right)} = -\frac{1}{255} - \frac{\left(-\frac{1}{255}\right)^2 - \left(-\frac{1}{255}\right)}{2\left(-\frac{1}{255}\right) - 1} = -\frac{1}{255} - \frac{\frac{1}{255^2} + \frac{1}{255}}{-\frac{2}{255} - 1}$$

$$= -\frac{1}{255} - \frac{\frac{1}{255^2} + \frac{255}{255^2}}{-\frac{2}{255} - \frac{255}{255}} = -\frac{1}{255} - \frac{\frac{256}{255^2}}{-\frac{257}{255}} = -\frac{1}{255} + \frac{\frac{256}{255^2}}{\frac{257}{255}}$$

$$= -\frac{1}{255} + \frac{256}{255^2}\left(\frac{255}{257}\right) = -\frac{1}{255} + \frac{256}{255}\left(\frac{1}{257}\right) = -\frac{1}{255} + \frac{256}{255 \cdot 257}$$

$$= -\frac{257}{255 \cdot 257} + \frac{256}{255 \cdot 257} = -\frac{1}{255 \cdot 257} = -\frac{1}{65{,}535} \approx -0.0000$$

If we pull together the approximations we've found so far,

$$x_1 = -\frac{1}{3}$$

$$x_2 = -\frac{1}{15}$$

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$$x_3 = -\frac{1}{255}$$

$$x_4 = -\frac{1}{65,535}$$

we can see that the solution is only getting closer and closer to 0 as the denominator gets larger.

To four decimal places, we've already found $x_4 \approx -0.0000$. Because we're getting closer to 0 with each approximation, the next approximation will also be all zeros to the first four decimal places. Therefore, the approximation of the root of the function to four decimal places is 0.

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