Topic: Distinct linear factors

Question: Use partial fractions to evaluate the integral.

$$\int \frac{5x+3}{x^2-9}\ dx$$

Answer choices:

A $\quad 3\ln|x+3| + 2\ln|x-3| + C$

B $\quad 3\ln|x+3| - 2\ln|x-3| + C$

C $\quad 2\ln|x+3| + 3\ln|x-3| + C$

D $\quad 2\ln|x+3| - 3\ln|x-3| + C$

Solution: C

First, factor the denominator.

$$\int \frac{5x+3}{x^2-9}\ dx = \int \frac{5x+3}{(x+3)(x-3)}\ dx$$

Using partial fractions with distinct linear factors since we have two, unequal, linear factors, the decomposition gives us

$$\frac{5x+3}{(x+3)(x-3)} = \frac{A}{x+3} + \frac{B}{x-3}$$

$$5x+3 = \frac{A}{x+3}(x+3)(x-3) + \frac{B}{x-3}(x+3)(x-3)$$

$$5x+3 = A(x-3) + B(x+3)$$

$$5x+3 = Ax - 3A + Bx + 3B$$

$$5x+3 = (Ax+Bx) + (-3A+3B)$$

$$5x+3 = (A+B)x + (-3A+3B)$$

Equating coefficients on both sides gives

[1] $5 = A + B$

and

$$3 = -3A + 3B$$

$$1 = -A + B$$

[2] $1 + A = B$

Substituting [2] into [1] gives

$$5 = A + (1 + A)$$

$$4 = 2A$$

$$A = 2$$

Plugging this value back into [2] gives

$$1 + A = B$$

$$1 + 2 = B$$

$$B = 3$$

With values for both coefficients, we'll plug into the partial fractions decomposition.

$$\frac{5x + 3}{(x + 3)(x - 3)} = \frac{A}{x + 3} + \frac{B}{x - 3}$$

$$\frac{5x + 3}{(x + 3)(x - 3)} = \frac{2}{x + 3} + \frac{3}{x - 3}$$

Then we'll put the decomposition back into the integral in place of the original function.

$$\int \frac{5x + 3}{x^2 - 9}\ dx = \int \frac{2}{x + 3} + \frac{3}{x - 3}\ dx$$

$$2\int \frac{1}{x + 3}\ dx + 3\int \frac{1}{x - 3}\ dx$$

$$2\ln|x+3|+3\ln|x-3|+C$$

Topic: Distinct linear factors

Question: Rewrite the integral using partial fractions. Do not solve it.

$$\int \frac{2}{(x-1)(x+1)}\ dx$$

Answer choices:

A $\quad \int \frac{1}{x+1} - \frac{1}{x+1}\ dx$

B $\quad \int \frac{1}{x-1} + \frac{1}{x-1}\ dx$

C $\quad \int \frac{1}{x-1} + \frac{1}{x+1}\ dx$

D $\quad \int \frac{1}{x-1} - \frac{1}{x+1}\ dx$

Solution: D

The denominator is already factored as much as it can be, which means it's a product of irreducible factors.

$$\int \frac{2}{(x-1)(x+1)}\ dx$$

Since the factors are linear, we know the numerators are going to be A, B, C, etc. For the partial fractions decomposition, we get

$$\frac{2}{(x-1)(x+1)} = \frac{A}{x-1} + \frac{B}{x+1}$$

Now we'll solve for constants.

$$\left[\frac{2}{(x-1)(x+1)} = \frac{A}{x-1} + \frac{B}{x+1}\right](x-1)(x+1)$$

$$2 = A(x+1) + B(x-1)$$

Now we need to solve for A and B. Let's start by solving for A. The easiest way to do this is to figure out what value of x will make the B term go away. In this case if $x = 1$, we'll get

$$2 = A(1+1) + B(1-1)$$

$$2 = A(2) + B(0)$$

$$A = 1$$

We set x equal to the value that would make the factor with B equal to 0, which made B disappear and allowed us to solve for A.

We'll solve for B the same way. If $x = -1$, we'll get

$$2 = A(-1+1) + B(-1-1)$$

$$2 = A(0) + B(-2)$$

$$B = -1$$

Plugging the values for both constants back into our partial fractions decomposition, and putting the decomposition back into the integral, we get

$$\int \frac{2}{(x-1)(x+1)}\ dx = \int \frac{1}{x-1} + \frac{-1}{x+1}\ dx$$

$$\int \frac{2}{(x-1)(x+1)}\ dx = \int \frac{1}{x-1} - \frac{1}{x+1}\ dx$$

Topic: Distinct linear factors

Question: Rewrite the integral using partial fractions. Do not solve it.

$$\int \frac{4}{(3x-1)(x+1)}\ dx$$

Answer choices:

A $\displaystyle \int \frac{3}{3x-1} + \frac{1}{x+1}\ dx$

B $\displaystyle \int \frac{3}{3x-1} - \frac{1}{x+1}\ dx$

C $\displaystyle \int \frac{3}{x+1} + \frac{1}{3x-1}\ dx$

D $\displaystyle \int \frac{3}{x+1} - \frac{1}{3x-1}\ dx$

Solution: B

The denominator is already factored as much as it can be, which means it's a product of irreducible factors.

$$\int \frac{4}{(3x-1)(x+1)}\ dx$$

Since the factors are linear, we know the numerators are going to be A, B, C, etc. For the partial fractions decomposition, we get

$$\frac{4}{(3x-1)(x+1)} = \frac{A}{3x-1} + \frac{B}{x+1}$$

Now we'll solve for constants.

$$\left[\frac{4}{(3x-1)(x+1)} = \frac{A}{3x-1} + \frac{B}{x+1}\right](3x-1)(x+1)$$

$$4 = A(x+1) + B(3x-1)$$

Now we need to solve for A and B. Let's start by solving for A. The easiest way to do this is to figure out what value of x will make the B term go away. In this case if $x = 1/3$, we'll get

$$4 = A\left(\frac{1}{3}+1\right) + B\left(3\cdot\frac{1}{3}-1\right)$$

$$4 = A\left(\frac{4}{3}\right) + B(0)$$

$$A = 3$$

We set x equal to the value that would make the factor with B equal to 0, which made B disappear and allowed us to solve for A.

We'll solve for B the same way. If $x = -1$, we'll get

$$4 = A(-1+1) + B\left[3(-1) - 1\right]$$

$$4 = A(0) + B(-4)$$

$$B = -1$$

Plugging the values for both constants back into our partial fractions decomposition, and putting the decomposition back into the integral, we get

$$\int \frac{4}{(3x-1)(x+1)}\ dx = \int \frac{3}{3x-1} + \frac{-1}{x+1}\ dx$$

$$\int \frac{4}{(3x-1)(x+1)}\ dx = \int \frac{3}{3x-1} - \frac{1}{x+1}\ dx$$