Power series differentiation

Sometimes we can generate the power series representation of a function by manipulating a standard power series like

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots = \sum_{n=0}^{\infty} x^n$$

and then differentiating the result. The goal is to use differentiation to get the left side of this equation to match exactly the function we've been given. When we differentiate, we have to remember to differentiate all three parts of the equation.

We'll try to simplify the sum on the right as much as possible, and the result will be the power series representation of our function. If we need to, we can then use the power series representation to find the radius and interval of convergence.

Example

Differentiate to find a power series representation for the function, then find the radius of convergence.

$$f(x) = \frac{1}{(1+x)^2}$$

The function we've been given is similar to the known power series

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots = \sum_{n=0}^{\infty} x^n$$

We need to manipulate this known power series so that it matches the function in our problem.

The first thing we'll do is change the denominator of the known power series from $1 - x$ to $1 + (-x)$ by replacing x with $-x$.

$$\frac{1}{1-(-x)} = 1 + (-x) + (-x)^2 + (-x)^3 + \dots = \sum_{n=0}^{\infty} (-x)^n$$

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \dots = \sum_{n=0}^{\infty} (-1)^n x^n$$

If we differentiate this equation, we'll have to use quotient rule to find the derivative of $1/(1+x)$, and the new denominator will be $(1+x)^2$, which is the denominator from our original function. Differentiating every part of the equation with respect to x, we get

$$\frac{(0)(1+x) - (1)(1)}{(1+x)^2} = 0 - 1 + 2x - 3x^2 + \dots = \sum_{n=0}^{\infty} (-1)^n n x^{n-1}$$

$$\frac{-1}{(1+x)^2} = -1 + 2x - 3x^2 + \dots = \sum_{n=0}^{\infty} (-1)^n n x^{n-1}$$

The left-hand side of the equation is now almost identical to the given function. To make them truly identical, we just have to multiply by -1.

$$\frac{1}{(1+x)^2} = 1 - 2x + 3x^2 - \dots = -1^1 \sum_{n=0}^{\infty} (-1)^n n x^{n-1}$$

$$\frac{1}{(1+x)^2} = 1 - 2x + 3x^2 - \ldots = \sum_{n=0}^{\infty} (-1)^{n+1} n x^{n-1}$$

Now that the left side of this equation is identical to the given function, we want to use the right side as a power series representation. If at all possible, we want the exponent of x to be just n. To make that happen, we can substitute $n+1$ for n.

$$\sum_{n=0}^{\infty} (-1)^{n+1} n x^{n-1}$$

$$\sum_{n=0}^{\infty} (-1)^{(n+1)+1} (n+1) x^{(n+1)-1}$$

$$\sum_{n=0}^{\infty} (-1)^{n+2} (n+1) x^{n}$$

$$\sum_{n=0}^{\infty} (-1)^{2} (-1)^{n} (n+1) x^{n}$$

$$\sum_{n=0}^{\infty} (-1)^{n} (n+1) x^{n}$$

This is the power series representation of $f(x)$.

With the power series representation in hand, we can find the radius of convergence using the ratio test. We'll need to identify that a_n is the power series representation, and a_{n+1} is whatever we get when we substitute $n+1$ into the power series representation for n.

$$a_n = (-1)^n (n+1) x^n$$

$$a_{n+1} = (-1)^{n+1}(n+2)x^{n+1}$$

Plugging these into the limit formula from the ratio test, we get

$$L = \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right|$$

$$L = \lim_{n \to \infty} \left| \frac{(-1)^{n+1}(n+2)x^{n+1}}{(-1)^n(n+1)x^n} \right|$$

$$L = \lim_{n \to \infty} \left| \frac{(-1)^{n+1}}{(-1)^n} \cdot \frac{n+2}{n+1} \cdot \frac{x^{n+1}}{x^n} \right|$$

$$L = \lim_{n \to \infty} \left| (-1)^{n+1-n} \cdot \frac{n+2}{n+1} \cdot x^{n+1-n} \right|$$

$$L = \lim_{n \to \infty} \left| -1 \cdot \frac{n+2}{n+1} \cdot x \right|$$

$$L = \lim_{n \to \infty} \left| \frac{n+2}{n+1} \cdot x \right|$$

Since the limit only effects n, we can pull the x out in front.

$$L = |x| \lim_{n \to \infty} \left| \frac{n+2}{n+1} \right|$$

$$L = |x| \lim_{n \to \infty} \left| \frac{(n+2)\left(\frac{1}{n}\right)}{(n+1)\left(\frac{1}{n}\right)} \right|$$

$$L = |x| \lim_{n \to \infty} \left| \frac{\frac{n}{n} + \frac{2}{n}}{\frac{n}{n} + \frac{1}{n}} \right|$$

$$L = |x| \lim_{n \to \infty} \left| \frac{1 + \frac{2}{n}}{1 + \frac{1}{n}} \right|$$

$$L = |x| \left| \frac{1 + \frac{2}{\infty}}{1 + \frac{1}{\infty}} \right|$$

$$L = |x| \left| \frac{1 + 0}{1 + 0} \right|$$

$$L = |x|$$

The ratio test tells us that that series converges when $L < 1$. Since we know $L = |x|$, we'll say that the series converges when

$$|x| < 1$$

This inequality is already in the form $|x - a| < R$ as

$$|x - 0| < 1$$

so we can say that the radius of convergence is $R = 1$.