Calculus 2 Workbook Solutions

krista king MATH

INDEFINITE INTEGRALS

■ 1. Evaluate the indefinite integral.

$$\int 5x^4 - 4x^3 + 6x^2 - 2x + 1 \ dx$$

Solution:

Take the integral one term at a time using the integration rule for basic power functions.

$$\int x^a \ dx = \frac{x^{a+1}}{a+1} + C$$

$$\int 5x^4 - 4x^3 + 6x^2 - 2x + 1 \ dx$$

$$\frac{5x^{4+1}}{4+1} - \frac{4x^{3+1}}{3+1} + \frac{6x^{2+1}}{2+1} - \frac{2x^{1+1}}{1+1} + \frac{x^{0+1}}{0+1} + C$$

$$\frac{5x^5}{5} - \frac{4x^4}{4} + \frac{6x^3}{3} - \frac{2x^2}{2} + \frac{x}{1} + C$$

$$x^5 - x^4 + 2x^3 - x^2 + x + C$$

■ 2. Evaluate the indefinite integral.

$$\int \frac{3x^3 + x^2 - 12x - 4}{x^2 - 4}\ dx$$

Solution:

Take the integral one term at a time using the integration rule for basic power functions.

$$\int x^a\ dx = \frac{x^{a+1}}{a+1} + C$$

$$\int \frac{3x^3 + x^2 - 12x - 4}{x^2 - 4}\ dx$$

$$\int \frac{(3x+1)(x+2)(x-2)}{(x+2)(x-2)}\ dx$$

$$\int 3x + 1\ dx$$

$$\frac{3x^{1+1}}{1+1} + \frac{x^{0+1}}{0+1} + C$$

$$\frac{3}{2}x^2 + x + C$$

3. Evaluate the indefinite integral.

$$\int (5x-7)(3x+2)\ dx$$

Solution:

Take the integral one term at a time using the integration rule for basic power functions. First, rewrite the function by multiplying the binomials.

$$\int (5x-7)(3x+2)\ dx$$

$$\int 15x^2 - 11x - 14\ dx$$

$$\frac{15x^{2+1}}{2+1} - \frac{11x^{1+1}}{1+1} - \frac{14x^{0+1}}{0+1} + C$$

$$\frac{15x^3}{3} - \frac{11x^2}{2} - \frac{14x}{1} + C$$

$$5x^3 - \frac{11}{2}x^2 - 14x + C$$

■ 4. Evaluate the indefinite integral.

$$\int \frac{x^3 - 3x + 2}{x^3}\ dx$$

Solution:

Take the integral one term at a time using the integration rule for basic power functions. First, rewrite the function by separating the fraction and then bringing the power functions to the numerator.

$$\int \frac{x^3 - 3x + 2}{x^3} \ dx$$

$$\int \frac{x^3}{x^3} - \frac{3x}{x^3} + \frac{2}{x^3} \ dx$$

$$\int 1 - \frac{3}{x^2} + \frac{2}{x^3} \ dx$$

$$\int 1 - 3x^{-2} + 2x^{-3} \ dx$$

Then the integrated value is

$$\frac{x^{0+1}}{0+1} - \frac{3x^{-2+1}}{-2+1} + \frac{2x^{-3+1}}{-3+1} + C$$

$$\frac{x^1}{1} - \frac{3x^{-1}}{-1} + \frac{2x^{-2}}{-2} + C$$

$$x + \frac{3}{x} - \frac{1}{x^2} + C$$

PROPERTIES OF INTEGRALS

1. Given the value of each of these integrals,

$$\int_0^3 f(x)\ dx = 7 \qquad \int_3^6 f(x)\ dx = 9 \qquad \int_0^3 g(x)\ dx = 2 \qquad \int_3^6 g(x)\ dx = 5$$

what is the value of of the following integral?

$$\int_0^6 \left[2f(x) + 3g(x)\right]\ dx$$

Solution:

If

$$\int_0^3 f(x)\ dx = 7 \text{ and } \int_3^6 f(x)\ dx = 9,$$

then

$$\int_0^6 f(x)\ dx = 7 + 9 = 16$$

Then,

$$\int_0^6 f(x)\ dx = 16 \int_0^6 2f(x)\ dx = 2\int_0^6 f(x)\ dx = 2 \cdot 16 = 32$$

Similarly, if

$$\int_0^3 g(x)\ dx = 2 \text{ and } \int_3^6 g(x)\ dx = 5$$

then

$$\int_0^6 g(x)\ dx = 2 + 5 = 7$$

Then, if

$$\int_0^6 g(x)\ dx = 7 \int_0^6 3g(x)\ dx = 3\int_0^6 g(x)\ dx = 3 \cdot 7 = 21$$

Therefore,

$$\int_0^6 \left[2f(x) + 3g(x)\right]\ dx = 32 + 21 = 53$$

FIND F GIVEN F''

1. Find $f(x)$ from its second derivative.

$$f''(x) = 3x^2 + 4x - 7$$

Solution:

Given the second derivative, the first derivative is

$$f'(x) = \int 3x^2 + 4x - 7 \ dx$$

$$f'(x) = \frac{3x^3}{3} + \frac{4x^2}{2} - 7x + C_1$$

$$f'(x) = x^3 + 2x^2 - 7x + C_1$$

Then $f(x)$ is

$$f(x) = \int x^3 + 2x^2 - 7x + C_1 \ dx$$

$$f(x) = \frac{x^4}{4} + \frac{2x^3}{3} - \frac{7x^2}{2} + C_1x + C_2$$

2. Find $g(x)$ from its second derivative.

$$g''(x) = \frac{x^4 - 4x^2 + 4}{x^2 - 2}$$

Solution:

Given the second derivative, the first derivative is

$$g'(x) = \int \frac{x^4 - 4x^2 + 4}{x^2 - 2}\ dx$$

$$g'(x) = \int \frac{(x^2 - 2)(x^2 - 2)}{x^2 - 2}\ dx$$

$$g'(x) = \int x^2 - 2\ dx$$

$$g'(x) = \frac{x^3}{3} - 2x + C_1$$

Then $g(x)$ is

$$g(x) = \int \frac{x^3}{3} - 2x + C_1\ dx$$

$$g(x) = \frac{x^4}{12} - x^2 + C_1x + C_2$$

■ 3. Find $h(x)$ from its second derivative.

$$h''(x) = \frac{8x^3 - 9x^2 + 6x}{x^7}$$

Solution:

First, simplify the second derivative function.

$$h''(x) = \frac{8x^3 - 9x^2 + 6x}{x^7}$$

$$h''(x) = \frac{8x^2 - 9x + 6}{x^6}$$

Given the second derivative, the first derivative is

$$h'(x) = \int \frac{8x^2 - 9x + 6}{x^6}\ dx$$

$$h'(x) = \int \frac{8x^2}{x^6} - \frac{9x}{x^6} + \frac{6}{x^6}\ dx$$

$$h'(x) = \int 8x^{-4} - 9x^{-5} + 6x^{-6}\ dx$$

$$h'(x) = \frac{8x^{-4+1}}{-4+1} - \frac{9x^{-5+1}}{-5+1} + \frac{6x^{-6+1}}{-6+1} + C_1$$

$$h'(x) = -\frac{8}{3}x^{-3} + \frac{9}{4}x^{-4} - \frac{6}{5}x^{-5} + C_1$$

Then $h(x)$ is

$$h(x) = \int -\frac{8}{3}x^{-3} + \frac{9}{4}x^{-4} - \frac{6}{5}x^{-5} + C_1\ dx$$

$$h(x) = \frac{-8x^{-2}}{3 \cdot -2} + \frac{9x^{-3}}{4 \cdot -3} - \frac{6x^{-4}}{5 \cdot -4} + C_1x + C_2$$

$$h(x) = \frac{4x^{-2}}{3} - \frac{3x^{-3}}{4} + \frac{3x^{-4}}{10} + C_1x + C_2$$

$$h(x) = \frac{4}{3x^2} - \frac{3}{4x^3} + \frac{3}{10x^4} + C_1x + C_2$$

FIND F GIVEN F'''

1. Find $f(x)$ given its third derivative.

$$f'''(x) = 2x + 3$$

Solution:

Given the third derivative, the second derivative is

$$f''(x) = \int 2x + 3 \ dx$$

$$f''(x) = x^2 + 3x + C_1$$

From the second derivative, the first derivative is

$$f'(x) = \int x^2 + 3x + C_1 \ dx$$

$$f'(x) = \frac{x^3}{3} + \frac{3x^2}{2} + C_1x + C_2$$

Then $f(x)$ is

$$f(x) = \int \frac{x^3}{3} + \frac{3x^2}{2} + C_1x + C_2 \ dx$$

$$f(x) = \frac{x^4}{3 \cdot 4} + \frac{3x^3}{2 \cdot 3} + \frac{C_1x^2}{2} + C_2x + C_3$$

$$f(x) = \frac{1}{12}x^4 + \frac{1}{2}x^3 + C_1x^2 + C_2x + C_3$$

■ 2. Find $g(x)$ given its third derivative.

$$g'''(x) = 4x^3 + x^2 - 3$$

Solution:

Given the third derivative, the second derivative is

$$g''(x) = \int 4x^3 + x^2 - 3\ dx$$

$$g''(x) = x^4 + \frac{x^3}{3} - 3x + C_1$$

From the second derivative, the first derivative is

$$g'(x) = \int x^4 + \frac{x^3}{3} - 3x + C_1\ dx$$

$$g'(x) = \frac{x^5}{5} + \frac{x^4}{12} - \frac{3x^2}{2} + C_1x + C_2$$

Then $g(x)$ is

$$g(x) = \int \frac{x^5}{5} + \frac{x^4}{12} - \frac{3x^2}{2} + C_1x + C_2\ dx$$

$$g(x)=\frac{x^6}{5\cdot 6}+\frac{x^5}{12\cdot 5}-\frac{3x^3}{2\cdot 3}+\frac{C_1x^2}{2}+C_2x+C_3$$

$$g(x)=\frac{1}{30}x^6+\frac{1}{60}x^5-\frac{1}{2}x^3+C_1x^2+C_2x+C_3$$

■ 3. Find $h(x)$ given its third derivative.

$$h'''(x)=\frac{3}{x^5}-\frac{2}{x^4}+4$$

Solution:

Given the third derivative, the second derivative is

$$h''(x)=\int\frac{3}{x^5}-\frac{2}{x^4}+4\ dx$$

$$h''(x)=\int 3x^{-5}-2x^{-4}+4\ dx$$

$$h''(x)=\frac{3x^{-4}}{-4}-\frac{2x^{-3}}{-3}+4x+C_1$$

$$h''(x)=-\frac{3x^{-4}}{4}+\frac{2x^{-3}}{3}+4x+C_1$$

From the second derivative, the first derivative is

$$h'(x)=\int-\frac{3x^{-4}}{4}+\frac{2x^{-3}}{3}+4x+C_1\ dx$$

$$h'(x) = -\frac{3x^{-3}}{4 \cdot -3} + \frac{2x^{-2}}{3 \cdot -2} + \frac{4x^2}{2} + C_1x + C_2$$

$$h'(x) = \frac{x^{-3}}{4} - \frac{x^{-2}}{3} + 2x^2 + C_1x + C_2$$

Then $h(x)$ is

$$h(x) = \int \frac{x^{-3}}{4} - \frac{x^{-2}}{3} + 2x^2 + C_1x + C_2 \ dx$$

$$h(x) = \frac{x^{-2}}{4 \cdot -2} - \frac{x^{-1}}{3 \cdot -1} + \frac{2x^3}{3} + \frac{C_1x^2}{2} + C_2x + C_3$$

$$h(x) = -\frac{1}{8x^2} + \frac{1}{3x} + \frac{2}{3}x^3 + C_1x^2 + C_2x + C_3$$

INITIAL VALUE PROBLEMS

■ 1. Find $f(x)$ if $f'(x)=7x-5$ and $f(4)=24$.

Solution:

Given $f'(x)=7x-5$, then

$$f(x)=\int 7x-5\ dx$$

$$f(x)=\frac{7x^2}{2}-5x+C$$

If $f(4)=24$, then

$$24=\frac{7(4)^2}{2}-5(4)+C$$

$$48=7(4)^2-2(5)(4)+2C$$

$$48=112-40+2C$$

$$-24=2C$$

$$C=-12$$

Therefore,

$$f(x)=\frac{7}{2}x^2-5x-12$$

2. Find $g(x)$ if $g'(x) = 2x^2 + 5x - 9$ and $g(-4) = 34$.

Solution:

Given $g'(x) = 2x^2 + 5x - 9$, then

$$g(x) = \int 2x^2 + 5x - 9 \ dx$$

$$g(x) = \frac{2x^3}{3} + \frac{5x^2}{2} - 9x + C$$

If $g(-4) = 34$, then

$$34 = \frac{2(-4)^3}{3} + \frac{5(-4)^2}{2} - 9(-4) + C$$

$$34 = -\frac{128}{3} + 40 + 36 + C$$

$$102 = -128 + 120 + 108 + 3C$$

$$2 = 3C$$

$$C = \frac{2}{3}$$

Therefore,

$$g(x) = \frac{2}{3}x^3 + \frac{5}{2}x^2 - 9x + \frac{2}{3}$$

■ 3. Find $h(x)$ if $h'(x) = 3x^2 + 8x + 1$ and $h(2) = 31$.

Solution:

Given $h'(x) = 3x^2 + 8x + 1$, then

$$h(x) = \int 3x^2 + 8x + 1 \ dx$$

$$h(x) = x^3 + 4x^2 + x + C$$

If $h(2) = 31$, then

$$31 = 2^3 + 4(2)^2 + 2 + C$$

$$31 = 8 + 16 + 2 + C$$

$$C = 5$$

Therefore,

$$h(x) = x^3 + 4x^2 + x + 5$$

■ 4. Find $f(x)$ if $f'(x) = x^3 + 4x + 3$ and $f(-2) = 15$.

Solution:

Given $f'(x) = x^3 + 4x + 3$, then

$$f(x) = \int x^3 + 4x + 3 \ dx$$

$$f(x) = \frac{x^4}{4} + \frac{4x^2}{2} + 3x + C$$

$$f(x) = \frac{x^4}{4} + 2x^2 + 3x + C$$

If $f(-2) = 15$, then

$$15 = \frac{(-2)^4}{4} + 2(-2)^2 + 3(-2) + C$$

$$15 = 4 + 8 - 6 + C$$

$$15 = 6 + C$$

$$C = 9$$

Therefore,

$$g(x) = \frac{1}{4}x^4 + 2x^2 + 3x + 9$$

FIND F GIVEN F'' AND INITIAL CONDITIONS

■ 1. Find $g(x)$ if $g''(x) = 2x + 1$, $g'(1) = 5$, and $g(1) = 4$.

Solution:

Given $g''(x) = 2x + 1$, then

$$g'(x) = \int 2x + 1 \ dx$$

$$g'(x) = x^2 + x + C$$

If $g'(1) = 5$, then

$$5 = 1^2 + 1 + C$$

$$5 = 2 + C$$

$$C = 3$$

and $g'(x) = x^2 + x + 3$. Then $g(x)$ is

$$g(x) = \int x^2 + x + 3 \ dx$$

$$g(x) = \frac{x^3}{3} + \frac{x^2}{2} + 3x + C$$

If $g(1) = 4$, then

$$4 = \frac{1^3}{3} + \frac{1^2}{2} + 3(1) + C$$

$$4 = \frac{1}{3} + \frac{1}{2} + 3 + C$$

$$24 = 2 + 3 + 18 + 6C$$

$$1 = 6C$$

$$C = \frac{1}{6}$$

Therefore,

$$g(x) = \frac{x^3}{3} + \frac{x^2}{2} + 3x + \frac{1}{6}$$

■ 2. Find $h(x)$ if $h''(x) = 2x - 7$, $h'(3) = -20$, and $h(6) = -98$.

Solution:

Given $h''(x) = 2x - 7$, then

$$h'(x) = \int 2x - 7 \ dx$$

$$h'(x) = x^2 - 7x + C$$

If $h'(3) = -20$, then

$$-20 = 3^2 - 7(3) + C$$

$$-20 = 9 - 21 + C$$

$$C = -8$$

and $h'(x) = x^2 - 7x - 8$. Then $h(x)$ is

$$h(x) = \int x^2 - 7x - 8 \ dx$$

$$h(x) = \frac{x^3}{3} - \frac{7x^2}{2} - 8x + C$$

If $h(6) = -98$, then

$$-98 = \frac{6^3}{3} - \frac{7(6)^2}{2} - 8(6) + C$$

$$-98 = 72 - 7(18) - 48 + C$$

$$C = 4$$

Therefore,

$$h(x) = \frac{1}{3}x^3 - \frac{7}{2}x^2 - 8x + 4$$

■ 3. Find $f(x)$ if $f''(x) = 3x - 6$, $f'(2) = 2$, and $f(2) = 15$.

Solution:

Given $f''(x) = 3x - 6$, then

$$f'(x) = \int 3x - 6 \ dx$$

$$f'(x) = \frac{3x^2}{2} - 6x + C$$

If $f'(2) = 2$, then

$$2 = \frac{3(2)^2}{2} - 6(2) + C$$

$$2 = 6 - 12 + C$$

$$C = 8$$

and $f'(x) = (3/2)x^2 - 6x + 8$. Then $f(x)$ is

$$f(x) = \int \frac{3}{2}x^2 - 6x + 8 \ dx$$

$$f(x) = \frac{1}{2}x^3 - 3x^2 + 8x + C$$

If $f(2) = 15$, then

$$15 = \frac{1}{2}(2)^3 - 3(2)^2 + 8(2) + C$$

$$15 = 4 - 12 + 16 + C$$

$$C = 7$$

Therefore,

$$f(x) = \frac{1}{2}x^3 - 3x^2 + 8x + 7$$

DEFINITE INTEGRALS

1. Evaluate the definite integral.

$$\int_0^3 x^3 + x^2 + x + 1 \ dx$$

Solution:

$$\int_0^3 x^3 + x^2 + x + 1 \ dx$$

$$\frac{x^4}{4} + \frac{x^3}{3} + \frac{x^2}{2} + x \Big|_0^3$$

$$\left(\frac{3^4}{4} + \frac{3^3}{3} + \frac{3^2}{2} + 3\right) - \left(\frac{0^4}{4} + \frac{0^3}{3} + \frac{0^2}{2} + 0\right)$$

$$\frac{81}{4} + 9 + \frac{9}{2} + 3$$

$$\frac{81}{4} + \frac{36}{4} + \frac{18}{4} + \frac{12}{4}$$

$$\frac{147}{4}$$

2. Evaluate the definite integral.

$$\int_{-\frac{\pi}{4}}^{\frac{\pi}{2}} 2\sin x + 3\cos x \ dx$$

Solution:

$$\int_{-\frac{\pi}{4}}^{\frac{\pi}{2}} 2\sin x + 3\cos x \ dx$$

$$-2\cos x + 3\sin x \Big|_{-\frac{\pi}{4}}^{\frac{\pi}{2}}$$

$$\left(-2\cos\frac{\pi}{2} + 3\sin\frac{\pi}{2}\right) - \left(-2\cos\left(-\frac{\pi}{4}\right) + 3\sin\left(-\frac{\pi}{4}\right)\right)$$

$$(-2(0) + 3(1)) - \left(-2\left(\frac{\sqrt{2}}{2}\right) + 3\left(-\frac{\sqrt{2}}{2}\right)\right)$$

$$3 + \sqrt{2} + \frac{3\sqrt{2}}{2}$$

3. Evaluate the definite integral.

$$\int_{-4}^{4} 2x^3 - 4x^2 + 25 \ dx$$

Solution:

$$\int_{-4}^{4} 2x^3 - 4x^2 + 25\ dx$$

$$\frac{x^4}{2} - \frac{4x^3}{3} + 25x \Big|_{-4}^{4}$$

$$\left(\frac{4^4}{2} - \frac{4(4)^3}{3} + 25(4)\right) - \left(\frac{(-4)^4}{2} - \frac{4(-4)^3}{3} + 25(-4)\right)$$

$$\left(128 - \frac{256}{3} + 100\right) - \left(128 + \frac{256}{3} - 100\right)$$

$$-\frac{512}{3} + 200$$

$$-\frac{512}{3} + \frac{600}{3}$$

$$\frac{88}{3}$$

■ 4. Evaluate the definite integral.

$$\int_{1}^{2} 6x^5 - 8x^3 + 4x + 3\ dx$$

Solution:

$$\int_{1}^{2} 6x^5 - 8x^3 + 4x + 3 \ dx$$

$$x^6 - 2x^4 + 2x^2 + 3x \Big|_{1}^{2}$$

$$\left(2^6 - 2(2)^4 + 2(2)^2 + 3(2)\right) - \left(1^6 - 2(1)^4 + 2(1)^2 + 3(1)\right)$$

$$(64 - 32 + 8 + 6) - (1 - 2 + 2 + 3)$$

$$46 - 4$$

$$42$$

5. Evaluate the definite integral.

$$\int_{0}^{\pi} 5 \sin x \ dx$$

Solution:

$$\int_{0}^{\pi} 5 \sin x \ dx$$

$$-5 \cos x \Big|_{0}^{\pi}$$

$$-5 \cos \pi - (-5 \cos 0)$$

$5-(-5)$

10

AREA UNDER OR ENCLOSED BY THE CURVE

■ 1. Find the area under the graph of $f(x)=2x^2-3x+5$ over the interval $[-2,6]$.

Solution:

Because we were asked for area "under" the graph, we're looking for net area, which we find by evaluating the integral of the function over the given interval.

$$A=\int_{-2}^{6} 2x^2-3x+5\ dx$$

$$A=\frac{2x^3}{3}-\frac{3x^2}{2}+5x\Big|_{-2}^{6}$$

$$A=\left(\frac{2(6)^3}{3}-\frac{3(6)^2}{2}+5(6)\right)-\left(\frac{2(-2)^3}{3}-\frac{3(-2)^2}{2}+5(-2)\right)$$

$$A=(144-54+30)-\left(-\frac{16}{3}-6-10\right)$$

$$A=120+\frac{64}{3}$$

$$A=141\frac{1}{3}$$

2. Find the area enclosed by the graph of $g(x) = 2x(x+4)(x-2)$ over the interval $[-4,2]$.

Solution:

Because we were asked for area "enclosed by" the graph, we're looking for gross area, which means we need to start by finding the zeros of the function.

The graph crosses the x-axis at $x = -4$, $x = 0$, and $x = 2$. Since we're looking for enclosed area over the interval $[-4,2]$, we'll take the absolute value of the area on the interval $[-4,0]$, and the absolute value of the area on the interval $[0,2]$, and then add the areas together.

$$A = \left| \int_{-4}^{0} 2x(x+4)(x-2)\ dx \right| + \left| \int_{0}^{2} 2x(x+4)(x-2)\ dx \right|$$

$$A = \left| \int_{-4}^{0} 2x^3 + 4x^2 - 16x\ dx \right| + \left| \int_{0}^{2} 2x^3 + 4x^2 - 16x\ dx \right|$$

$$A = \left| \left(\frac{x^4}{2} + \frac{4x^3}{3} - 8x^2 \right) \Big|_{-4}^{0} \right| + \left| \left(\frac{x^4}{2} + \frac{4x^3}{3} - 8x^2 \right) \Big|_{0}^{2} \right|$$

$$A = \left| \left(\frac{0^4}{2} + \frac{4(0)^3}{3} - 8(0)^2 \right) - \left(\frac{(-4)^4}{2} + \frac{4(-4)^3}{3} - 8(-4)^2 \right) \right|$$

$$+\left|\left(\frac{2^4}{2}+\frac{4(2)^3}{3}-8(2)^2\right)-\left(\frac{0^4}{2}+\frac{4(0)^3}{3}-8(0)^2\right)\right|$$

$$A=\left|-\left(\frac{256}{2}+\frac{4(-64)}{3}-8(16)\right)\right|+\left|\left(\frac{16}{2}+\frac{32}{3}-8(4)\right)\right|$$

$$A=\left|-128+\frac{256}{3}+128\right|+\left|8+\frac{32}{3}-32\right|$$

$$A=\left|\frac{256}{3}\right|+\left|\frac{32}{3}-24\right|$$

$$A=\frac{256}{3}+\left|\frac{32}{3}-\frac{72}{3}\right|$$

$$A=\frac{256}{3}+\left|-\frac{40}{3}\right|$$

$$A=\frac{256}{3}+\frac{40}{3}$$

$$A=\frac{296}{3}$$

■ 3. Find the area under the graph of $h(x)=3\sqrt{x}$ over the interval $[4,16]$.

Solution:

Because we were asked for area "under" the graph, we're looking for net area, which we find by evaluating the integral of the function over the given interval.

$$A=\int_4^{16} 3\sqrt{x}\ dx$$

$$A=\int_4^{16} 3x^{\frac{1}{2}}\ dx$$

$$A=3\left(\frac{2}{3}x^{\frac{3}{2}}\right)\Big|_4^{16}$$

$$A=2x^{\frac{3}{2}}\Big|_4^{16}$$

$$A=(2(16)^{\frac{3}{2}})-(2(4)^{\frac{3}{2}})$$

$$A=(2(4)^3)-(2(2)^3)$$

$$A=(2(64))-(2(8))$$

$$A=128-16$$

$$A=112$$

DEFINITE INTEGRALS OF EVEN AND ODD FUNCTIONS

1. Evaluate the definite integral.

$$\int_{-3}^{3} -x^4 + 19\ dx$$

Solution:

In an even function, $f(x) = f(-x)$ and the graph is symmetric about the y-axis. The integral contains an even function.

$$x^4 + 19 = (-x)^4 + 19 = x^4 + 19$$

Because of the symmetry across the y-axis, we can rewrite the integral as

$$2\int_{0}^{3} -x^4 + 19\ dx$$

$$2\left(-\frac{x^5}{5} + 19x\right)\Bigg|_0^3$$

$$2\left(-\frac{3^5}{5} + 19(3)\right) - 2\left(-\frac{0^5}{5} + 19(0)\right)$$

$$2\left(-\frac{243}{5} + 57\right)$$

$$\frac{84}{5} \approx 16.8$$

■ 2. Evaluate the definite integral.

$$\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} 7\cos x \ dx$$

Solution:

In an even function, $f(x) = f(-x)$ and the graph is symmetric about the y-axis. The integral contains an even function.

$$7\cos x = 7\cos(-x) = 7\cos x$$

Because of the symmetry across the y-axis, we can rewrite the integral as

$$2\int_{0}^{\frac{\pi}{4}} 7\cos x \ dx$$

$$14\sin x\Big|_{0}^{\frac{\pi}{4}}$$

$$14\sin\left(\frac{\pi}{4}\right) - 14\sin(0)$$

$$14\left(\frac{\sqrt{2}}{2}\right)$$

$$7\sqrt{2}$$

■ 3. Evaluate the definite integral.

$$\int_{-2}^{2} \frac{3}{4}x^2 + 5\ dx$$

Solution:

In an even function, $f(x) = f(-x)$ and the graph is symmetric about the y-axis. The integral contains an even function.

$$\frac{3}{4}x^2 + 5 = \frac{3}{4}(-x)^2 + 5 = \frac{3}{4}x^2 + 5$$

Because of the symmetry across the y-axis, we can rewrite the integral as

$$2\int_{0}^{2} \frac{3}{4}x^2 + 5\ dx$$

$$\left(\frac{x^3}{2} + 10x\right)\Bigg|_0^2$$

$$\left(\frac{2^3}{2} + 10(2)\right) - \left(\frac{0^3}{2} + 10(0)\right)$$

$$4 + 20$$

$$24$$

■ 4. Evaluate the definite integral.

$$\int_{-2}^{2} 3x^5 - 4x^3 + 8x \ dx$$

Solution:

In an odd function, $f(-x) = -f(x)$ and the graph is symmetric about the origin. Check to see if the function is odd.

$$3(-x)^5 - 4(-x)^3 + 8(-x) = -3x^5 + 4x^3 - 8x = -\left(3x^5 - 4x^3 + 8x\right)$$

The function is odd, and the interval $[-2,2]$ fits the format $[-a, a]$, so

$$\int_{-2}^{2} 3x^5 - 4x^3 + 8x \ dx = 0$$

■ 5. Evaluate the definite integral.

$$\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} 9 \sin x \ dx$$

Solution:

In an odd function, $f(-x) = -f(x)$ and the graph is symmetric about the origin. Check to see if the function is odd.

$$9\sin(-x) = -9\sin x = -(9\sin x)$$

The function is odd, and the interval $[-\pi/3, \pi/3]$ fits the format $[-a, a]$, so

$$\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} 9\sin x \ dx = 0$$

6. Evaluate the definite integral.

$$\int_{-2}^{2} 2x^3 - 4x \ dx$$

Solution:

In an odd function, $f(-x) = -f(x)$ and the graph is symmetric about the origin. Check to see if the function is odd.

$$2(-x)^3 - 4(-x) = -2x^3 + 4x = -\left(2x^3 - 4x\right)$$

The function is odd, and the interval $[-2, 2]$ fits the format $[-a, a]$, so

$$\int_{-2}^{2} 2x^3 - 4x \ dx = 0$$

SUMMATION NOTATION, FINDING THE SUM

1. Calculate the exact sum.

$$\sum_{n=1}^{6} \frac{2n^2}{3^n}$$

Solution:

To find the sum, write each term with the value of n plugged in.

$$n = 1 \qquad \frac{2n^2}{3^n} = \frac{2(1)^2}{3^1} = \frac{2}{3}$$

$$n = 2 \qquad \frac{2n^2}{3^n} = \frac{2(2)^2}{3^2} = \frac{8}{9}$$

$$n = 3 \qquad \frac{2n^2}{3^n} = \frac{2(3)^2}{3^3} = \frac{2}{3}$$

$$n = 4 \qquad \frac{2n^2}{3^n} = \frac{2(4)^2}{3^4} = \frac{32}{81}$$

$$n = 5 \qquad \frac{2n^2}{3^n} = \frac{2(5)^2}{3^5} = \frac{50}{243}$$

$$n = 6 \qquad \frac{2n^2}{3^n} = \frac{2(6)^2}{3^6} = \frac{8}{81}$$

So the sum is

$$\sum_{n=1}^{6} \frac{2n^2}{3^n} = \frac{2}{3} + \frac{8}{9} + \frac{2}{3} + \frac{32}{81} + \frac{50}{243} + \frac{8}{81}$$

$$\sum_{n=1}^{6} \frac{2n^2}{3^n} = \frac{162}{243} + \frac{216}{243} + \frac{162}{243} + \frac{96}{243} + \frac{50}{243} + \frac{24}{243}$$

$$\sum_{n=1}^{6} \frac{2n^2}{3^n} = \frac{710}{243}$$

■ 2. Calculate the exact sum.

$$\sum_{n=1}^{5} \frac{2n}{3n+1}$$

Solution:

To find the sum, write each term with the value of n plugged in.

$$n = 1 \qquad \frac{2(1)}{3(1)+1} = \frac{2}{4} = \frac{1}{2}$$

$$n = 2 \qquad \frac{2(2)}{3(2)+1} = \frac{4}{7}$$

$$n = 3 \qquad \frac{2(3)}{3(3)+1} = \frac{6}{10} = \frac{3}{5}$$

$$n = 4 \qquad \frac{2(4)}{3(4)+1} = \frac{8}{13}$$

$$n = 5 \qquad \frac{2(5)}{3(5)+1} = \frac{10}{16} = \frac{5}{8}$$

So the sum is

$$\sum_{n=1}^{5} \frac{2n}{3n+1} = \frac{1}{2} + \frac{4}{7} + \frac{3}{5} + \frac{8}{13} + \frac{5}{8}$$

$$\sum_{n=1}^{5} \frac{2n}{3n+1} = \frac{1{,}820}{3{,}640} + \frac{2{,}080}{3{,}640} + \frac{2{,}184}{3{,}640} + \frac{2{,}240}{3{,}640} + \frac{2{,}275}{3{,}640}$$

$$\sum_{n=1}^{5} \frac{2n}{3n+1} = \frac{10{,}599}{3{,}640}$$

■ 3. Calculate the exact sum.

$$\sum_{n=0}^{6} 3n^2 - 5n + 7$$

Solution:

To find the sum, write each term with the value of n plugged in.

$$n = 0 \qquad 3(0)^2 - 5(0) + 7 = 7$$

$$n = 1 \qquad 3(1)^2 - 5(1) + 7 = 5$$

$$n = 2 \qquad 3(2)^2 - 5(2) + 7 = 9$$

$n = 3 \qquad 3(3)^2 - 5(3) + 7 = 19$

$n = 4 \qquad 3(4)^2 - 5(4) + 7 = 35$

$n = 5 \qquad 3(5)^2 - 5(5) + 7 = 57$

$n = 6 \qquad 3(6)^2 - 5(6) + 7 = 85$

So the sum is

$$\sum_{n=0}^{5} 3n^2 - 5n + 7 = 7 + 5 + 9 + 19 + 35 + 57 + 85$$

$$\sum_{n=0}^{5} 3n^2 - 5n + 7 = 217$$

SUMMATION NOTATION, EXPANDING

1. Expand the sum.

$$\sum_{n=1}^{6} \frac{5n+3}{2n-1}$$

Solution:

To find the sum, write each term with the value of n plugged in.

$$n=1 \qquad \frac{5(1)+3}{2(1)-1} = \frac{8}{1} = 8$$

$$n=2 \qquad \frac{5(2)+3}{2(2)-1} = \frac{13}{3}$$

$$n=3 \qquad \frac{5(3)+3}{2(3)-1} = \frac{18}{5}$$

$$n=4 \qquad \frac{5(4)+3}{2(4)-1} = \frac{23}{7}$$

$$n=5 \qquad \frac{5(5)+3}{2(5)-1} = \frac{28}{9}$$

$$n=6 \qquad \frac{5(6)+3}{2(6)-1} = \frac{33}{11} = 3$$

So the sum is

$$\sum_{n=1}^{6} \frac{5n+3}{2n-1} = 8 + \frac{13}{3} + \frac{18}{5} + \frac{23}{7} + \frac{28}{9} + 3$$

■ 2. Expand the sum.

$$\sum_{n=0}^{7} 2x^3 - 5x^2 + 9x + 3$$

Solution:

To find the sum, write each term with the value of n plugged in.

$n = 0$ $\quad 2(0)^3 - 5(0)^2 + 9(0) + 3 = 3$

$n = 1$ $\quad 2(1)^3 - 5(1)^2 + 9(1) + 3 = 9$

$n = 2$ $\quad 2(2)^3 - 5(2)^2 + 9(2) + 3 = 17$

$n = 3$ $\quad 2(3)^3 - 5(3)^2 + 9(3) + 3 = 39$

$n = 4$ $\quad 2(4)^3 - 5(4)^2 + 9(4) + 3 = 87$

$n = 5$ $\quad 2(5)^3 - 5(5)^2 + 9(5) + 3 = 173$

$n = 6$ $\quad 2(6)^3 - 5(6)^2 + 9(6) + 3 = 309$

$n = 7$ $\quad 2(7)^3 - 5(7)^2 + 9(7) + 3 = 507$

So the sum is

$$\sum_{n=0}^{7} 2x^3 - 5x^2 + 9x + 3 = 3 + 9 + 17 + 39 + 87 + 173 + 309 + 507$$

■ 3. Expand the sum.

$$\sum_{n=0}^{8} \frac{2n-8}{n+1}$$

Solution:

To find the sum, write each term with the value of n plugged in.

$$n = 0 \qquad \frac{2(0)-8}{0+1} = -\frac{8}{1} = -8$$

$$n = 1 \qquad \frac{2(1)-8}{1+1} = -\frac{6}{2} = -3$$

$$n = 2 \qquad \frac{2(2)-8}{2+1} = -\frac{4}{3}$$

$$n = 3 \qquad \frac{2(3)-8}{3+1} = -\frac{2}{4} = -\frac{1}{2}$$

$$n = 4 \qquad \frac{2(4)-8}{4+1} = \frac{0}{5} = 0$$

$$n = 5 \qquad \frac{2(5)-8}{5+1} = \frac{2}{6} = \frac{1}{3}$$

$$n = 6 \qquad \frac{2(6)-8}{6+1} = \frac{4}{7}$$

$$n = 7 \qquad \frac{2(7)-8}{7+1} = \frac{6}{8} = \frac{3}{4}$$

$$n = 8 \qquad \frac{2(8)-8}{8+1} = \frac{8}{9}$$

So the sum is

$$\sum_{n=0}^{8} \frac{2n-8}{n+1} = -8 - 3 - \frac{4}{3} - \frac{1}{2} + 0 + \frac{1}{3} + \frac{4}{7} + \frac{3}{4} + \frac{8}{9}$$

SUMMATION NOTATION, COLLAPSING

1. Use summation notation to rewrite the sum.

$$\frac{(x+3)^2}{3-1}+\frac{(x+3)^4}{9-2}+\frac{(x+3)^6}{27-3}+\frac{(x+3)^8}{81-4}+\frac{(x+3)^{10}}{243-5}+\frac{(x+3)^{12}}{729-6}$$

Solution:

Find patterns in the sum.

$$\frac{(x+3)^2}{3-1}+\frac{(x+3)^4}{9-2}+\frac{(x+3)^6}{27-3}+\frac{(x+3)^8}{81-4}+\frac{(x+3)^{10}}{243-5}+\frac{(x+3)^{12}}{729-6}$$

$$\frac{(x+3)^{2(1)}}{3-1}+\frac{(x+3)^{2(2)}}{9-2}+\frac{(x+3)^{2(3)}}{27-3}+\frac{(x+3)^{2(4)}}{81-4}+\frac{(x+3)^{2(5)}}{243-5}+\frac{(x+3)^{2(6)}}{729-6}$$

$$\frac{(x+3)^{2(1)}}{3^1-1}+\frac{(x+3)^{2(2)}}{3^2-2}+\frac{(x+3)^{2(3)}}{3^3-3}+\frac{(x+3)^{2(4)}}{3^4-4}+\frac{(x+3)^{2(5)}}{3^5-5}+\frac{(x+3)^{2(6)}}{3^6-6}$$

$$\frac{(x+3)^{2(1)}}{3^1-1}+\frac{(x+3)^{2(2)}}{3^2-2}+\frac{(x+3)^{2(3)}}{3^3-3}+\frac{(x+3)^{2(4)}}{3^4-4}+\frac{(x+3)^{2(5)}}{3^5-5}+\frac{(x+3)^{2(6)}}{3^6-6}$$

The pattern that emerges is the sum from $n=1$ to $n=6$.

$$\sum_{n=1}^{6}\frac{(x+3)^{2n}}{3^n-n}$$

■ 2. Use summation notation to rewrite the sum.

$$\frac{3x+1}{7x}+\frac{6x+2}{14x^2}+\frac{9x+3}{21x^3}+\frac{12x+4}{28x^4}+\frac{15x+5}{35x^5}+\frac{18x+6}{42x^6}$$

Solution:

Find patterns in the sum.

$$\frac{3x+1}{7x}+\frac{6x+2}{14x^2}+\frac{9x+3}{21x^3}+\frac{12x+4}{28x^4}+\frac{15x+5}{35x^5}+\frac{18x+6}{42x^6}$$

$$\frac{3(1)x+1}{7x}+\frac{3(2)x+2}{14x^2}+\frac{3(3)x+3}{21x^3}+\frac{3(4)x+4}{28x^4}+\frac{3(5)x+5}{35x^5}+\frac{3(6)x+6}{42x^6}$$

$$\frac{3(1)x+1}{7(1)x}+\frac{3(2)x+2}{7(2)x^2}+\frac{3(3)x+3}{7(3)x^3}+\frac{3(4)x+4}{7(4)x^4}+\frac{3(5)x+5}{7(5)x^5}+\frac{3(6)x+6}{7(6)x^6}$$

$$\frac{3(1)x+1}{7(1)x^1}+\frac{3(2)x+2}{7(2)x^2}+\frac{3(3)x+3}{7(3)x^3}+\frac{3(4)x+4}{7(4)x^4}+\frac{3(5)x+5}{7(5)x^5}+\frac{3(6)x+6}{7(6)x^6}$$

The pattern that emerges is the sum from $n=1$ to $n=6$.

$$\sum_{n=1}^{6}\frac{3nx+n}{7nx^n}$$

■ 3. Use summation notation to rewrite the sum.

$$\frac{x^2-3x+1}{4x}+\frac{x^3-6x+2}{8x}+\frac{x^4-9x+3}{12x}+\frac{x^5-12x+4}{16x}$$

$$+\frac{x^6-15x+5}{20x}+\frac{x^7-18x+6}{24x}+\frac{x^8-21x+7}{28x}$$

Solution:

Find patterns in the sum.

$$\frac{x^2-3x+1}{4x}+\frac{x^3-6x+2}{8x}+\frac{x^4-9x+3}{12x}+\frac{x^5-12x+4}{16x}$$

$$+\frac{x^6-15x+5}{20x}+\frac{x^7-18x+6}{24x}+\frac{x^8-21x+7}{28x}$$

$$\frac{x^2-3(1)x+1}{4x}+\frac{x^3-3(2)x+2}{8x}+\frac{x^4-3(3)x+3}{12x}+\frac{x^5-3(4)x+4}{16x}$$

$$+\frac{x^6-3(5)x+5}{20x}+\frac{x^7-3(6)x+6}{24x}+\frac{x^8-3(7)x+7}{28x}$$

$$\frac{x^{1+1}-3(1)x+1}{4x}+\frac{x^{2+1}-3(2)x+2}{8x}+\frac{x^{3+1}-3(3)x+3}{12x}+\frac{x^{4+1}-3(4)x+4}{16x}$$

$$+\frac{x^{5+1}-3(5)x+5}{20x}+\frac{x^{6+1}-3(6)x+6}{24x}+\frac{x^{7+1}-3(7)x+7}{28x}$$

$$\frac{x^{1+1}-3(1)x+1}{4(1)x}+\frac{x^{2+1}-3(2)x+2}{4(2)x}+\frac{x^{3+1}-3(3)x+3}{4(3)x}+\frac{x^{4+1}-3(4)x+4}{4(4)x}$$

$$+\frac{x^{5+1}-3(5)x+5}{4(5)x}+\frac{x^{6+1}-3(6)x+6}{4(6)x}+\frac{x^{7+1}-3(7)x+7}{4(7)x}$$

The pattern that emerges is the sum from $n=1$ to $n=7$.

$$\sum_{n=1}^{7} \frac{x^{n+1} - 3nx + n}{4nx}$$

RIEMANN SUMS, LEFT ENDPOINTS

■ 1. Use a left endpoint Riemann Sum with $n = 5$ to find the area under $f(x)$ on the interval $[0,10]$.

x	0	1	2	3	4	5	6	7	8	9	10
f(x)	3	2	3	6	11	18	27	38	51	66	83

Solution:

Five equal subdivisions of $[0,10]$ gives five subintervals.

$[0,2]$, $[2,4]$, $[4,6]$, $[6,8]$, and $[8,10]$

Choose the value of the function for the left endpoint of each subinterval. The width of each subinterval is 2 units, so the estimation of the area under the curve is

$$2(3) + 2(3) + 2(11) + 2(27) + 2(51)$$

$$190$$

■ 2. Use a left endpoint Riemann Sum with $n = 5$ to find the area under $g(x)$ on the interval $[0,20]$. Round the final answer to 2 decimal places.

$$g(x) = 2\sqrt{x} + 5$$

Solution:

Five equal subdivisions of $[0,20]$ gives five subintervals.

$$[0,4], [4,8], [8,12], [12,16], \text{ and } [16,20]$$

Find the value of the function for the left endpoint of each subinterval.

$$g(0) = 2\sqrt{0} + 5 = 5$$

$$g(4) = 2\sqrt{4} + 5 = 9$$

$$g(8) = 2\sqrt{8} + 5 \approx 10.6569$$

$$g(12) = 2\sqrt{12} + 5 \approx 11.9282$$

$$g(16) = 2\sqrt{16} + 5 = 13$$

The width of each subinterval is 4 units, so the estimation of the area under the curve is

$$4(5) + 4(9) + 4(10.6569) + 4(11.9282) + 4(13)$$

$$198.3404$$

The sum rounds to 198.34.

■ 3. Use a left endpoint Riemann Sum with $n = 3$ to find the area under $h(x)$ on the interval $[-2,4]$.

$$h(x) = \frac{1}{3}x^3 - \frac{1}{2}x^2 - x + 3$$

Solution:

Three equal subdivisions of $[-2,4]$ gives three subintervals.

$[-2,0]$, $[0,2]$, and $[2,4]$

Find the value of the function for the left endpoint of each subinterval.

$$h(-2) = \frac{1}{3}(-2)^3 - \frac{1}{2}(-2)^2 - (-2) + 3 = \frac{1}{3}$$

$$h(0) = \frac{1}{3}(0)^3 - \frac{1}{2}(0)^2 - 0 + 3 = 3$$

$$h(2) = \frac{1}{3}(2)^3 - \frac{1}{2}(2)^2 - 2 + 3 = \frac{5}{3}$$

The width of each subinterval is 2 units, so the estimation of the area under the curve is

$$2\left(\frac{1}{3}\right) + 2(3) + 2\left(\frac{5}{3}\right)$$

$$\frac{2}{3} + 6 + \frac{10}{3}$$

$$\frac{2}{3} + \frac{18}{3} + \frac{10}{3}$$

$$\frac{30}{3}$$

10

■ 4. Use a left endpoint Riemann Sum with $n = 4$ to find the area under $k(x)$ on the interval $[0,28]$. Round the final answer to 2 decimal places.

$$k(x) = \frac{x^2 + 4x + 4}{x^2 + 4}$$

Solution:

Four equal subdivisions of $[0,28]$ gives four subintervals.

$[0,7]$, $[7,14]$, $[14,21]$, and $[21,28]$

Find the value of the function for the left endpoint of each subinterval.

$$k(0) = \frac{0^2 + 4(0) + 4}{0^2 + 4} = 1$$

$$k(7) = \frac{7^2 + 4(7) + 4}{7^2 + 4} = \frac{81}{53} \approx 1.5283$$

$$k(14) = \frac{14^2 + 4(14) + 4}{14^2 + 4} = \frac{32}{25} \approx 1.28$$

$$k(21) = \frac{21^2 + 4(21) + 4}{21^2 + 4} = \frac{529}{445} \approx 1.1888$$

The width of each subinterval is 7 units, so the estimation of the area under the curve is

$$7(1) + 7(1.5283) + 7(1.28) + 7(1.1888)$$

$$34.9795$$

The sum rounds to 34.98.

■ 5. Use a left endpoint Riemann Sum with $n = 4$ to find the area under $f(x)$ on the interval $[0,2]$. Round the final answer to 2 decimal places.

$$f(x) = 2\ln(x + 3) + 6$$

Solution:

Four equal subdivisions of $[0,2]$ gives four subintervals.

$[0,0.5]$, $[0.5,1]$, $[1,1.5]$, and $[1.5,2]$

Find the value of the function for the left endpoint of each subinterval.

$$f(0) = 2\ln(0 + 3) + 6 \approx 8.1972$$

$$f(0.5) = 2\ln(0.5 + 3) + 6 \approx 8.5055$$

$$f(1) = 2\ln(1 + 3) + 6 \approx 8.7726$$

$$f(1.5) = 2\ln(1.5 + 3) + 6 \approx 9.0082$$

The width of each subinterval is 0.5 units, so the estimation of the area under the curve is

$$0.5(8.1972) + 0.5(8.5055) + 1.5(8.7726) + 0.5(9.0082)$$

$$17.24175$$

The sum rounds to 17.24.

■ 6. Use a left endpoint Riemann Sum with $n = 5$ to find the area under $g(x)$ on the interval $[0,1]$. Round the final answer to 2 decimal places.

$$g(x) = x^4 + 2x^3 - 3x^2 + 4x + 5$$

Solution:

Five equal subdivisions of $[0,1]$ gives five subintervals.

$[0,0.2]$, $[0.2,0.4]$, $[0.4,0.6]$, $[0.6,0.8]$, and $[0.8,1]$

Find the value of the function for the left endpoint of each subinterval.

$$g(0) = 0^4 + 2(0)^3 - 3(0)^2 + 4(0) + 5 = 5$$

$$g(0.2) = 0.2^4 + 2(0.2)^3 - 3(0.2)^2 + 4(0.2) + 5 \approx 5.6976$$

$$g(0.4) = 0.4^4 + 2(0.4)^3 - 3(0.4)^2 + 4(0.4) + 5 \approx 6.2736$$

$$g(0.6) = 0.6^4 + 2(0.6)^3 - 3(0.6)^2 + 4(0.6) + 5 \approx 6.8816$$

$$g(0.8) = 0.8^4 + 2(0.8)^3 - 3(0.8)^2 + 4(0.8) + 5 \approx 7.7136$$

The width of each subinterval is 0.2 units, so the estimation of the area under the curve is

$$0.2(5) + 0.2(5.6976) + 0.2(6.2736) + 0.2(6.8816) + 0.2(7.7136)$$

$$6.31328$$

The sum rounds to 6.31.

RIEMANN SUMS, RIGHT ENDPOINTS

■ 1. Use a right endpoint Riemann Sum with $n = 5$ to find the area under $g(x)$ on the interval $[1,11]$.

x	1	2	3	4	5	6	7	8	9	10	11
g(x)	5	4	5	8	13	20	29	40	53	68	85

Solution:

Five equal subdivisions of $[1,11]$ gives five subintervals.

$[1,3]$, $[3,5]$, $[5,7]$, $[7,9]$, and $[9,11]$

Choose the value of the function for the right endpoint of each subinterval. The width of each subinterval is 2 units, so the estimation of the area under the curve is

$$2(5) + 2(13) + 2(29) + 2(53) + 2(85)$$

$$370$$

■ 2. Use a right endpoint Riemann Sum with $n = 5$ to find the area under $f(x)$ on the interval $[5,25]$. Round the final answer to 2 decimal places.

$$f(x) = \sqrt{2x} - 1$$

Solution:

Five equal subdivisions of $[5,25]$ gives five subintervals.

$[5,9]$, $[9,13]$, $[13,17]$, $[17,21]$, and $[21,25]$

Find the value of the function for the right endpoint of each subinterval.

$$f(9) = \sqrt{2(9)} - 1 \approx 3.2426$$

$$f(13) = \sqrt{2(13)} - 1 \approx 4.0990$$

$$f(17) = \sqrt{2(17)} - 1 \approx 4.8310$$

$$f(21) = \sqrt{2(21)} - 1 \approx 5.4807$$

$$f(25) = \sqrt{2(25)} - 1 \approx 6.0711$$

The width of each subinterval is 4 units, so the estimation of the area under the curve is

$$4(3.2426) + 4(4.0990) + 4(4.8310) + 4(5.4807) + 4(6.0711)$$

$$94.8976$$

The sum rounds to 94.90.

■ 3. Use a right endpoint Riemann Sum with $n = 3$ to find the area under $h(x)$ on the interval $[-2,4]$.

$$h(x) = \frac{1}{3}x^3 - \frac{1}{2}x^2 - x + 3$$

Solution:

Five equal subdivisions of $[-2,4]$ gives five subintervals.

$[-2,0]$, $[0,2]$, and $[2,4]$

Find the value of the function for the right endpoint of each subinterval.

$$h(0) = \frac{1}{3}(0)^3 - \frac{1}{2}(0)^2 - 0 + 3 = 3$$

$$h(2) = \frac{1}{3}(2)^3 - \frac{1}{2}(2)^2 - 2 + 3 = \frac{5}{3}$$

$$h(4) = \frac{1}{3}(4)^3 - \frac{1}{2}(4)^2 - 4 + 3 = \frac{37}{3}$$

The width of each subinterval is 2 units, so the estimation of the area under the curve is

$$2(3) + 2\left(\frac{5}{3}\right) + 2\left(\frac{37}{3}\right)$$

$$6 + \frac{10}{3} + \frac{74}{3}$$

$$\frac{18}{3} + \frac{10}{3} + \frac{74}{3}$$

$$\frac{102}{3}$$

34

■ 4. Use a right endpoint Riemann Sum with $n=4$ to find the area under $k(x)$ on the interval $[0,28]$. Round the final answer to 2 decimal places.

$$k(x)=\frac{x^2+4x+4}{x^2+4}$$

Solution:

Four equal subdivisions of $[0,28]$ gives four subintervals.

$[0,7]$, $[7,14]$, $[14,21]$, and $[21,28]$

Find the value of the function for the right endpoint of each subinterval.

$$k(7)=\frac{7^2+4(7)+4}{7^2+4}=\frac{81}{53}\approx 1.5283$$

$$k(14)=\frac{14^2+4(14)+4}{14^2+4}=\frac{32}{25}\approx 1.28$$

$$k(21)=\frac{21^2+4(21)+4}{21^2+4}=\frac{529}{445}\approx 1.1888$$

$$k(28)=\frac{28^2+4(28)+4}{28^2+4}=\frac{225}{197}\approx 1.1421$$

The width of each subinterval is 7 units, so the estimation of the area under the curve is

$$7(1.5283)+7(1.28)+7(1.1888)+7(1.1421)$$

$$35.9744$$

The sum rounds to 35.97.

■ 5. Use a right endpoint Riemann Sum with $n=4$ to find the area under $f(x)$ on the interval $[0,2]$. Round the final answer to 2 decimal places.

$$f(x)=2\ln{(x+3)}+6$$

Solution:

Four equal subdivisions of $[0,2]$ gives four subintervals.

$[0,0.5]$, $[0.5,1]$, $[1,1.5]$, and $[1.5,2]$

Find the value of the function for the right endpoint of each subinterval.

$$f(0.5)=2\ln(0.5+3)+6\approx 8.5055$$

$$f(1)=2\ln(1+3)+6\approx 8.7726$$

$$f(1.5)=2\ln(1.5+3)+6\approx 9.0082$$

$$f(2)=2\ln(2+3)+6\approx 9.2189$$

The width of each subinterval is 0.5 units, so the estimation of the area under the curve is

$$0.5(8.5055) + 1.5(8.7726) + 0.5(9.0082) + 0.5(9.2189)$$

$$17.7526$$

The sum rounds to 17.75.

■ 6. Use a right endpoint Riemann Sum with $n = 5$ to find the area under $h(x)$ on the interval $[0,1]$. Round the final answer to 2 decimal places.

$$h(x) = x^4 + 2x^3 - 3x^2 + 4x + 5$$

Solution:

Four equal subdivisions of $[0,1]$ gives four subintervals.

$[0,0.2]$, $[0.2,0.4]$, $[0.4,0.6]$, $[0.6,0.8]$, and $[0.8,1]$

Find the value of the function for the right endpoint of each subinterval.

$$g(0.2) = 0.2^4 + 2(0.2)^3 - 3(0.2)^2 + 4(0.2) + 5 \approx 5.6976$$

$$g(0.4) = 0.4^4 + 2(0.4)^3 - 3(0.4)^2 + 4(0.4) + 5 \approx 6.2736$$

$$g(0.6) = 0.6^4 + 2(0.6)^3 - 3(0.6)^2 + 4(0.6) + 5 \approx 6.8816$$

$$g(0.8) = 0.8^4 + 2(0.8)^3 - 3(0.8)^2 + 4(0.8) + 5 \approx 7.7136$$

$$g(1) = 1^4 + 2(1)^3 - 3(1)^2 + 4(1) + 5 = 9$$

The width of each subinterval is 0.2 units, so the estimation of the area under the curve is

$$0.2(5.6976) + 0.2(6.2736) + 0.2(6.8816) + 0.2(7.7136) + 0.2(9)$$

$$7.11328$$

The sum rounds to 7.11.

RIEMANN SUMS, MIDPOINTS

■ 1. Use a midpoint Riemann Sum with $n = 5$ to find the area under $h(x)$ on the interval $[6,16]$.

x	6	7	8	9	10	11	12	13	14	15	16
h(x)	84	67	52	39	26	17	10	7	4	3	4

Solution:

Five equal subdivisions of $[6,16]$ gives five subintervals.

$[6,8]$, $[8,10]$, $[10,12]$, $[12,14]$, and $[14,16]$

The midpoints of the subintervals are

$x = 7, 9, 11, 13,$ and 15

Choose the value of the function for the midpoint of each subinterval. The width of each subinterval is 2 units, so the estimation of the area under the curve is

$$2(67) + 2(39) + 2(17) + 2(7) + 2(3)$$

$$266$$

■ 2. Use a midpoint Riemann Sum with $n = 5$ to find the area under $k(x)$ on the interval $[2,22]$. Round the final answer to 2 decimal places.

$$k(x) = 3\sqrt{7x} - 8$$

Solution:

Five equal subdivisions of $[2,22]$ gives five subintervals.

$[2,6]$, $[6,10]$, $[10,14]$, $[14,18]$, and $[18,22]$

The midpoints of the subintervals are

$x = 4,\ 8,\ 12,\ 16,$ and 20

Find the value of the function for the midpoint of each subinterval.

$$k(4) = 3\sqrt{7(4)} - 8 \approx 7.8745$$

$$k(8) = 3\sqrt{7(8)} - 8 \approx 14.4499$$

$$k(12) = 3\sqrt{7(12)} - 8 \approx 19.4955$$

$$k(16) = 3\sqrt{7(16)} - 8 \approx 23.7490$$

$$k(20) = 3\sqrt{7(20)} - 8 \approx 27.4965$$

The width of each subinterval is 4 units, so the estimation of the area under the curve is

$$4(7.8745) + 4(14.4499) + 4(19.4955) + 4(23.7490) + 4(27.4965)$$

$$372.2616$$

The sum rounds to 372.26.

■ 3. Use a midpoint Riemann Sum with $n = 3$ to find the area under $h(x)$ on the interval $[-2,4]$.

$$h(x) = \frac{1}{3}x^3 - \frac{1}{2}x^2 - x + 3$$

Solution:

Three equal subdivisions of $[-2,4]$ gives three subintervals.

$[-2,0]$, $[0,2]$, and $[2,4]$

The midpoints of the subintervals are

$x = -1$, 1, and 3

Find the value of the function for the midpoint of each subinterval.

$$h(-1) = \frac{1}{3}(-1)^3 - \frac{1}{2}(-1)^2 - (-1) + 3 = \frac{19}{6}$$

$$h(1) = \frac{1}{3}(1)^3 - \frac{1}{2}(1)^2 - 2 + 3 = \frac{11}{6}$$

$$h(3) = \frac{1}{3}(3)^3 - \frac{1}{2}(3)^2 - 3 + 3 = \frac{9}{2}$$

The width of each subinterval is 2 units, so the estimation of the area under the curve is

$$2\left(\frac{19}{6}\right)+2\left(\frac{11}{6}\right)+2\left(\frac{9}{2}\right)$$

$$\frac{19}{3}+\frac{11}{3}+9$$

$$\frac{19}{3}+\frac{11}{3}+\frac{27}{3}$$

$$\frac{57}{3}$$

$$19$$

■ 4. Use a midpoint Riemann Sum with $n=4$ to find the area under $k(x)$ on the interval $[0,28]$. Round the final answer to 2 decimal places.

$$k(x)=\frac{x^2+4x+4}{x^2+4}$$

Solution:

Four equal subdivisions of $[0,28]$ gives four subintervals.

$[0,7]$, $[7,14]$, $[14,21]$, and $[21,28]$

The midpoints of the subintervals are

$x = 3.5, 10.5, 17.5,$ and 24.5

Find the value of the function for the midpoint endpoint of each subinterval.

$$k(3.5) = \frac{3.5^2 + 4(3.5) + 4}{3.5^2 + 4} \approx 1.8615$$

$$k(10.5) = \frac{10.5^2 + 4(10.5) + 4}{10.5^2 + 4} \approx 1.3676$$

$$k(17.5) = \frac{17.5^2 + 4(17.5) + 4}{17.5^2 + 4} \approx 1.2256$$

$$k(24.5) = \frac{24.5^2 + 4(24.5) + 4}{24.5^2 + 4} \approx 1.1622$$

The width of each subinterval is 7 units, so the estimation of the area under the curve is

$$7(1.8615) + 7(1.3676) + 7(1.2256) + 7(1.1622)$$

$$39.3183$$

The sum rounds to 39.32.

5. Use a midpoint Riemann Sum with $n = 4$ to find the area under $f(x)$ on the interval $[0,2]$. Round the final answer to 2 decimal places.

$$f(x) = 2\ln(x + 3) + 6$$

Solution:

Five equal subdivisions of $[0,2]$ gives five subintervals.

$[0,0.5]$, $[0.5,1]$, $[1,1.5]$, and $[1.5,2]$

The midpoints of the subintervals are

$x = 0.25, 0.75, 1.25$, and 1.75

Find the value of the function for the midpoint endpoint of each subinterval.

$$f(0.25) = 2\ln(0.25 + 3) + 6 \approx 8.3573$$

$$f(0.75) = 2\ln(0.75 + 3) + 6 \approx 8.6435$$

$$f(1.25) = 2\ln(1.25 + 3) + 6 \approx 8.8938$$

$$f(1.75) = 2\ln(1.75 + 3) + 6 = 9.1163$$

The width of each subinterval is 0.5 units, so the estimation of the area under the curve is

$$0.5(8.3573) + 1.5(8.6435) + 0.5(8.8938) + 0.5(9.1163)$$

$$17.50545$$

The sum rounds to 17.51.

■ 6. Use a midpoint Riemann Sum with $n = 5$ to find the area under $g(x)$ on the interval $[0,1]$. Round the final answer to 2 decimal places.

$$g(x) = x^4 + 2x^3 - 3x^2 + 4x + 5$$

Solution:

Five equal subdivisions of $[0,1]$ gives five subintervals.

$[0,0.2]$, $[0.2,0.4]$, $[0.4,0.6]$, $[0.6,0.8]$, and $[0.8,1]$

The midpoints of the subintervals are

$x = 0.1$, 0.3, 0.5, 0.7, and 0.9

Find the value of the function for the midpoint endpoint of each subinterval.

$$g(0.1) = 0.1^4 + 2(0.1)^3 - 3(0.1)^2 + 4(0.1) + 5 \approx 5.3721$$

$$g(0.3) = 0.3^4 + 2(0.3)^3 - 3(0.3)^2 + 4(0.3) + 5 \approx 5.9921$$

$$g(0.5) = 0.5^4 + 2(0.5)^3 - 3(0.5)^2 + 4(0.5) + 5 \approx 6.5625$$

$$g(0.7) = 0.7^4 + 2(0.7)^3 - 3(0.7)^2 + 4(0.7) + 5 \approx 7.2561$$

$$g(0.9) = 0.9^4 + 2(0.9)^3 - 3(0.9)^2 + 4(0.9) + 5 \approx 8.2841$$

The width of each subinterval is 0.2 units, so the estimation of the area under the curve is

$$0.2(5.3721) + 0.2(5.9921) + 0.2(6.5625) + 0.2(7.2561) + 0.2(8.2841)$$

$$6.69338$$

The sum rounds to 6.69.

MOVING FROM SUMMATION NOTATION TO THE INTEGRAL

■ 1. Convert the Riemann sum to a definite integral over the interval $[1,8]$.

$$\sum_{i=1}^{n}\left(6x_i^5-4x_i^{\frac{4}{3}}+2x_i^{-3}\right)\Delta x$$

Solution:

Convert the sum to a limit.

$$\lim_{n\to\infty}\sum_{i=1}^{n}\left(6x_i^5-4x_i^{\frac{4}{3}}+2x_i^{-3}\right)\Delta x=\int f(x_i)\ dx$$

Then the integral over the interval $[1,8]$ is

$$\int_1^8 6x^5-4x^{\frac{4}{3}}+2x^{-3}\ dx$$

■ 2. Convert the Riemann sum to a definite integral over the interval $[-2,4]$.

$$\sum_{i=1}^{n}\left((5x_i+3)(2x_i^2+x_i)^5\right)\Delta x$$

Solution:

Convert the sum to a limit.

$$\sum_{i=1}^{n} \left((5x_i + 3)(2x_i^2 + x_i)^5 \right) \Delta x = \int f(x_i)\ dx$$

Then the integral over the interval $[-2,4]$ is

$$\int_{-2}^{4} (5x + 3)(2x^2 + x)^5\ dx$$

■ 3. Convert the Riemann sum to a definite integral over the interval $[5,11]$.

$$\sum_{i=1}^{n} \left((4 - x_i)\sqrt{x_i - 5} \right) \Delta x$$

Solution:

Convert the sum to a limit.

$$\sum_{i=1}^{n} \left((4 - x_i)\sqrt{x_i - 5} \right) \Delta x = \int f(x_i)\ dx$$

Then the integral over the interval $[5,11]$ is

$$\int_{5}^{11} (4 - x)\sqrt{x - 5}\ dx$$

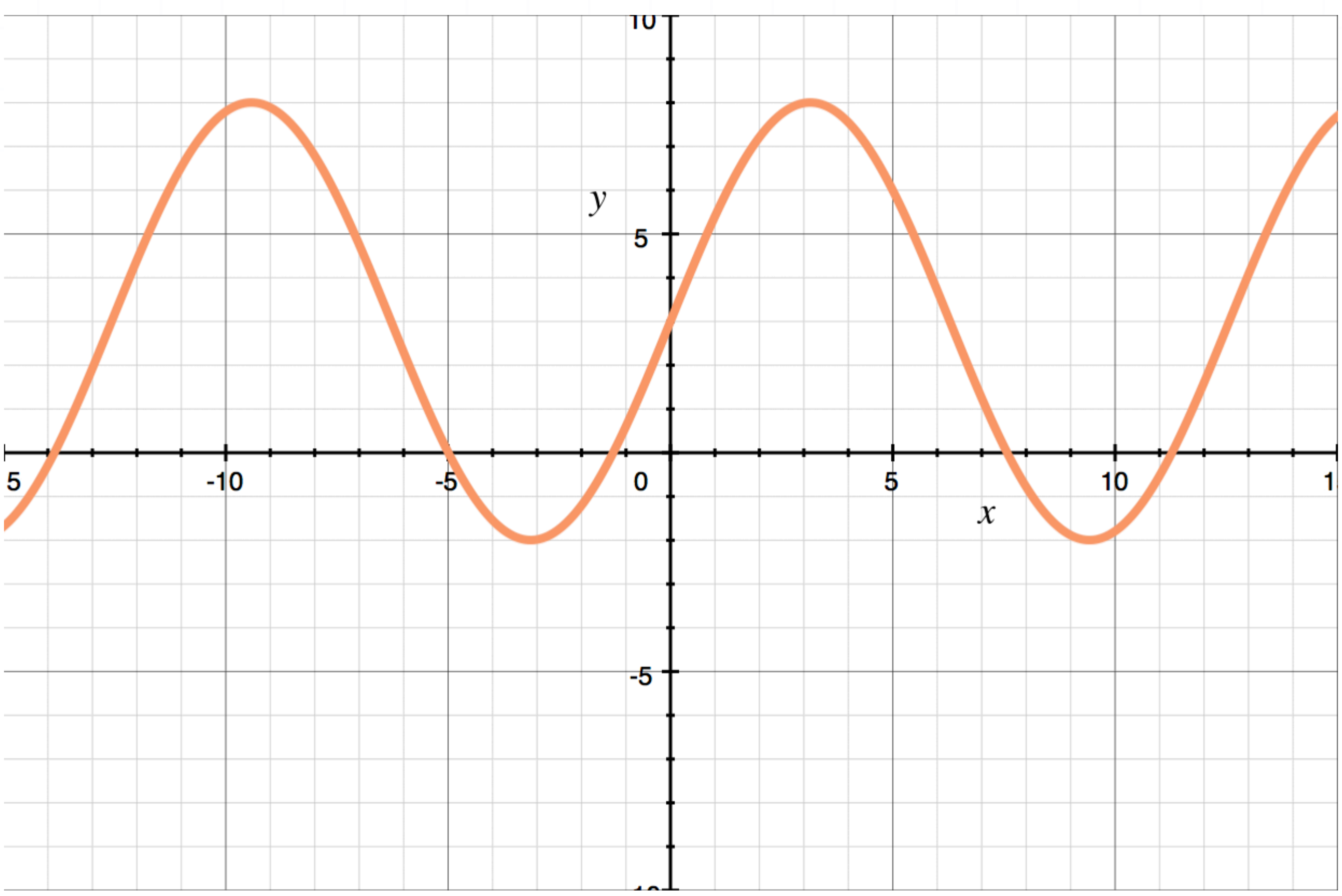
OVER AND UNDERESTIMATION

1. Use a Riemann sum to estimate the maximum and minimum area under the curve on $[0,\pi]$. Use rectangular approximation methods with 4 equal subintervals. Round the answer to 2 decimal places.

$$f(x) = 5\sin\frac{x}{2} + 3$$

Solution:

The graph of $f(x)$ is increasing on $[0,\pi]$.



So LRAM will underestimate the area and RRAM will overestimate the area. With 4 equal subintervals, calculate the value of $f(x)$ at

$x = 0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}$, and π.

The function's values are

$$f(0) = 5\sin\left(\frac{0}{2}\right) + 3 = 3$$

$$f\left(\frac{\pi}{4}\right) = 5\sin\left(\frac{\pi}{8}\right) + 3 \approx 4.9134$$

$$f\left(\frac{\pi}{2}\right) = 5\sin\left(\frac{\pi}{4}\right) + 3 \approx 6.5355$$

$$f\left(\frac{3\pi}{4}\right) = 5\sin\left(\frac{3\pi}{8}\right) + 3 \approx 7.6194$$

$$f(\pi) = 5\sin\left(\frac{\pi}{2}\right) + 3 = 8$$

Then the LRAM and RRAM are

$$LRAM = \frac{\pi}{4}(3 + 4.9134 + 6.5355 + 7.6194) \approx 17.3324$$

$$RRAM = \frac{\pi}{4}(4.9134 + 6.5355 + 7.6194 + 8) \approx 21.2594$$

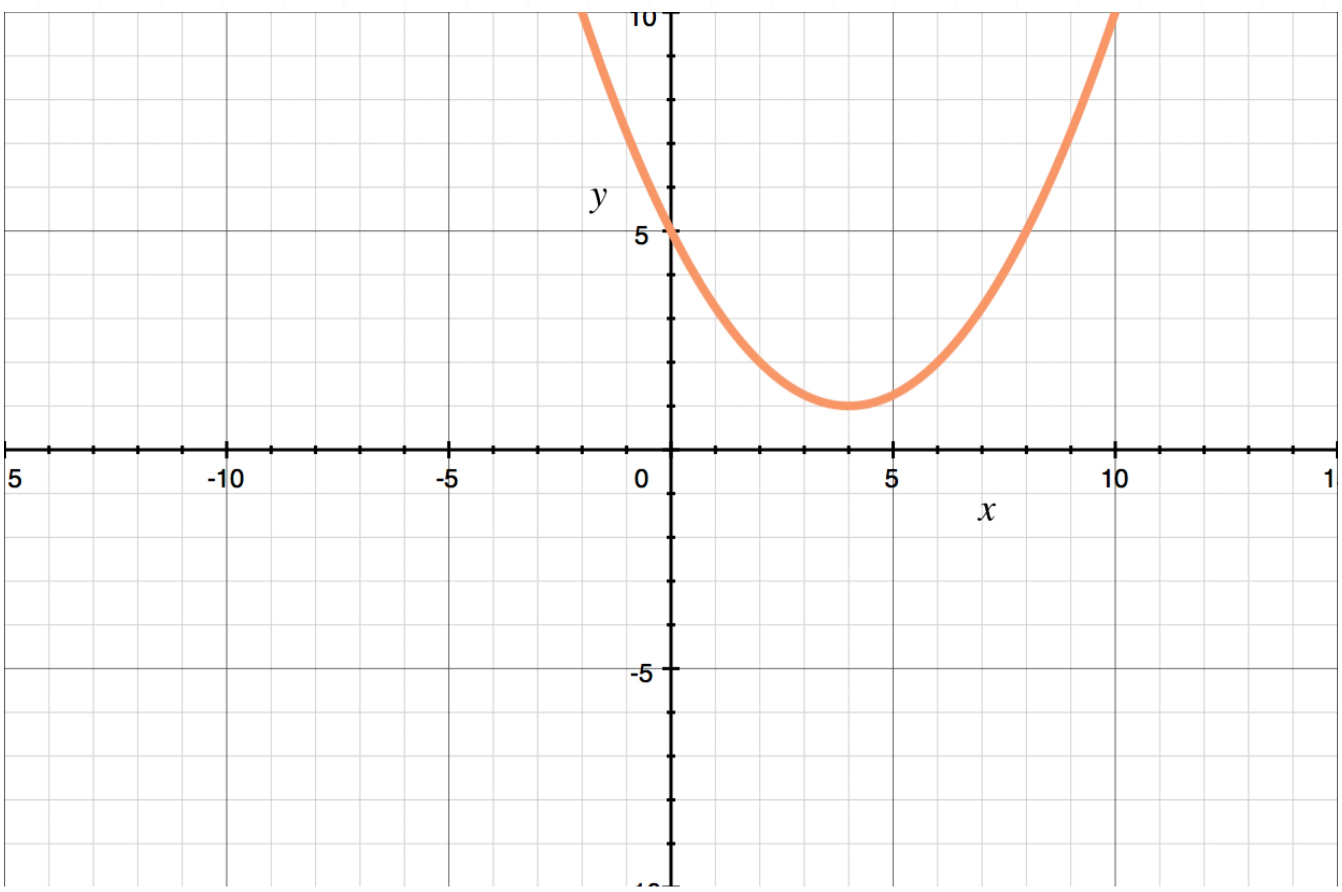
So the minimum area under the curve is 17.33 and the maximum area under the curve is 21.26.

■ 2. Use a Riemann sum to estimate the maximum and minimum area under the curve on $[0,4]$. Use rectangular approximation methods with 4 equal subintervals.

$$g(x) = \frac{1}{4}(x-4)^2 + 1$$

Solution:

The graph of $g(x)$ is decreasing on $[0,4]$.



So RRAM will underestimate the area and LRAM will overestimate the area. With 4 equal subintervals, calculate the value of $g(x)$ at

$$x = 0,\ 1,\ 2,\ 3, \text{ and } 4$$

The function's values are

$$g(0) = \frac{1}{4}(0-4)^2 + 1 = 5$$

$$g(1) = \frac{1}{4}(1-4)^2 + 1 = \frac{13}{4}$$

$$g(2) = \frac{1}{4}(2-4)^2 + 1 = 2$$

$$g(3) = \frac{1}{4}(3-4)^2 + 1 = \frac{5}{4}$$

$$g(4) = \frac{1}{4}(3-4)^2 + 1 = 1$$

Then the LRAM and RRAM are

$$LRAM = 1\left(5 + \frac{13}{4} + 2 + \frac{5}{4}\right) = 11.5$$

$$RRAM = 1\left(\frac{13}{4} + 2 + \frac{5}{4} + 1\right) = 7.5$$

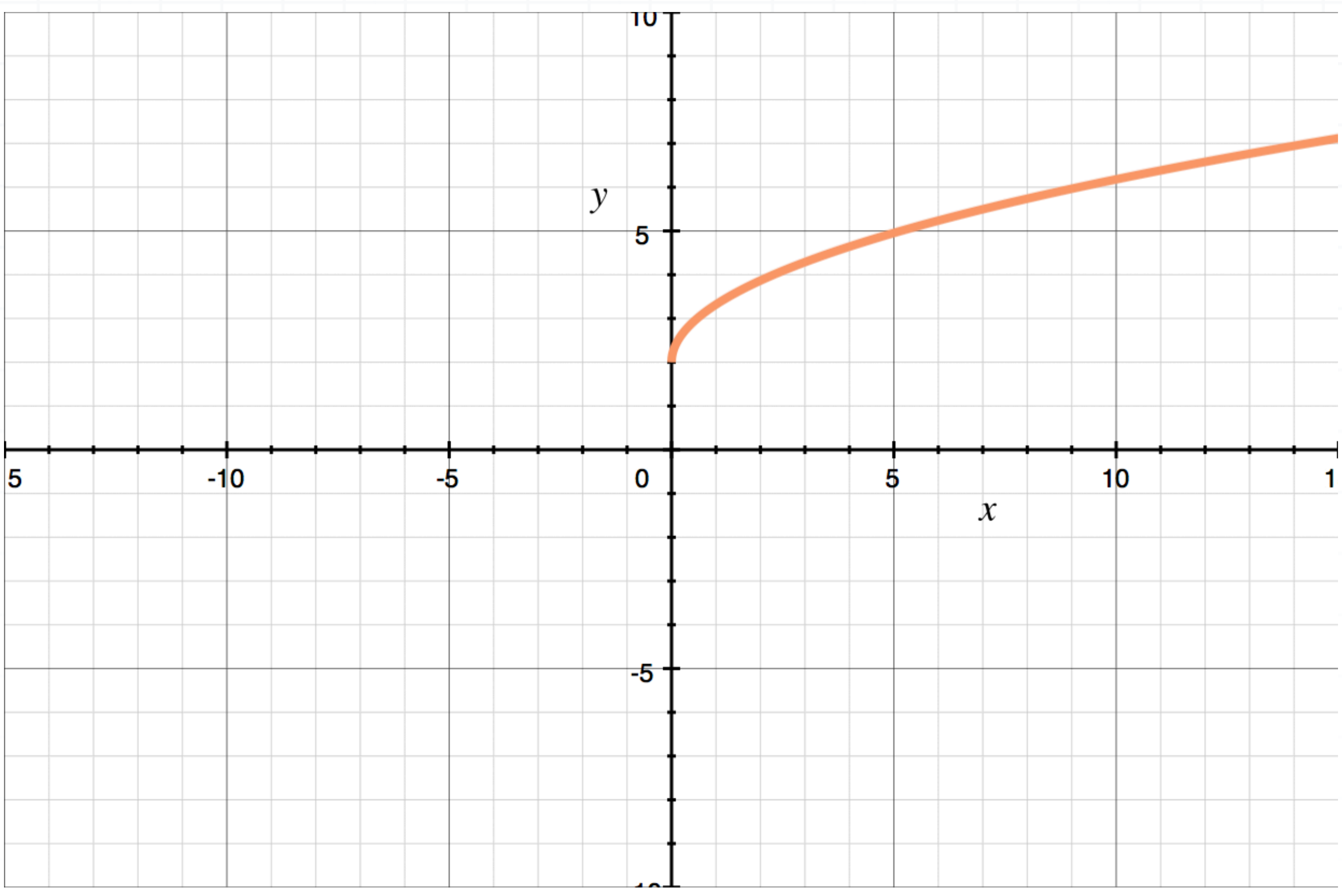
So the minimum area under the curve is 7.5 and the maximum area under the curve is 11.5.

■ 3. Use a Riemann sum to estimate the maximum and minimum area under the curve on $[0,9]$. Use rectangular approximation methods with 3 equal subintervals. Round the answer to 2 decimal places.

$$h(x) = \frac{1}{2}\sqrt{7x} + 2$$

Solution:

The graph of $h(x)$ is increasing on $[0,9]$.



So LRAM will underestimate the area and RRAM will overestimate the area. With 3 equal subintervals, calculate the value of $h(x)$ at

$x = 0$, 3, 6, and 9

The function's values are

$$h(0) = \frac{1}{2}\sqrt{7(0)} + 2 = 2$$

$$h(3) = \frac{1}{2}\sqrt{7(3)} + 2 \approx 4.2913$$

$$h(6) = \frac{1}{2}\sqrt{7(6)} + 2 \approx 5.2404$$

$$h(9) = \frac{1}{2}\sqrt{7(9)} + 2 \approx 5.9686$$

Then the LRAM and RRAM are

$$LRAM = 3(2 + 4.2913 + 5.2404) \approx 34.5951$$

$$RRAM = 3(4.2913 + 5.2404 + 5.9686) \approx 46.5009$$

So the minimum area under the curve is 34.60 and the maximum area under the curve is 46.50.

LIMIT PROCESS TO FIND AREA ON [A,B]

■ 1. Use the limit process to find the area of the region between the graph of $f(x)$ and the x-axis on the interval $[3,7]$.

$$f(x) = x^2 + 2$$

Solution:

Find Δx.

$$\Delta x = \frac{b-a}{n} = \frac{7-3}{n} = \frac{4}{n}$$

With Δx, find x_i.

$$x_i = 3 + i\Delta x = 3 + i \cdot \frac{4}{n} = 3 + \frac{4i}{n}$$

Then you get

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$$

$$\lim_{n\to\infty} \sum_{i=1}^{n} f\left(3 + \frac{4i}{n}\right)\frac{4}{n}$$

$$\lim_{n\to\infty} \frac{4}{n} \sum_{i=1}^{n} \left(3 + \frac{4i}{n}\right)^2 + 2$$

$$\lim_{n\to\infty}\frac{4}{n}\sum_{i=1}^{n}9+\frac{24i}{n}+\frac{16i^2}{n^2}+2$$

$$\lim_{n\to\infty}\frac{4}{n}\sum_{i=1}^{n}11+\frac{24i}{n}+\frac{16i^2}{n^2}$$

Making substitutions for i and i^2 gives

$$\lim_{n\to\infty}\frac{4}{n}\left[11n+\frac{24}{n}\cdot\frac{n(n+1)}{2}+\frac{16}{n^2}\cdot\frac{n(n+1)(2n+1)}{6}\right]$$

$$\lim_{n\to\infty}\frac{4}{n}\left[11n+12(n+1)+\frac{8(2n^2+3n+1)}{3n}\right]$$

$$\lim_{n\to\infty}\frac{4}{n}\left[11n+12n+12+\frac{16n^2}{3n}+\frac{24n}{3n}+\frac{8}{3n}\right]$$

$$\lim_{n\to\infty}\frac{4}{n}\left[23n+12+\frac{16n}{3}+8+\frac{8}{3n}\right]$$

$$\lim_{n\to\infty}\left[92+\frac{48}{n}+\frac{64n}{3n}+\frac{32}{n}+\frac{32}{3n^2}\right]$$

$$\lim_{n\to\infty}\left[92+\frac{48}{n}+\frac{64}{3}+\frac{32}{n}+\frac{32}{3n^2}\right]$$

Evaluate the limit.

$$92+0+\frac{64}{3}+0+0$$

$$\frac{276}{3}+\frac{64}{3}$$

$$\frac{340}{3}$$

■ 2. Use the limit process to find the area of the region between the graph of $g(x)$ and the x-axis on the interval $[2,6]$.

$$f(x)=x^2-x+3$$

Solution:

Find Δx.

$$\Delta x=\frac{b-a}{n}=\frac{6-2}{n}=\frac{4}{n}$$

With Δx, find x_i.

$$x_i=2+i\Delta x=2+i\cdot\frac{4}{n}=2+\frac{4i}{n}$$

Then you get

$$\lim_{n\to\infty}\sum_{i=1}^{n}g(x_i)\Delta x$$

$$\lim_{n\to\infty}\sum_{i=1}^{n}g\left(2+\frac{4i}{n}\right)\frac{4}{n}$$

$$\lim_{n\to\infty} \frac{4}{n} \sum_{i=1}^{n} \left(2 + \frac{4i}{n}\right)^2 - \left(2 + \frac{4i}{n}\right) + 3$$

$$\lim_{n\to\infty} \frac{4}{n} \sum_{i=1}^{n} 4 + \frac{16i}{n} + \frac{16i^2}{n^2} - 2 - \frac{4i}{n} + 3$$

$$\lim_{n\to\infty} \frac{4}{n} \sum_{i=1}^{n} 5 + \frac{12i}{n} + \frac{16i^2}{n^2}$$

Making substitutions for i and i^2 gives

$$\lim_{n\to\infty} \frac{4}{n} \left[5n + \frac{12}{n} \cdot \frac{n(n+1)}{2} + \frac{16}{n^2} \cdot \frac{n(n+1)(2n+1)}{6}\right]$$

$$\lim_{n\to\infty} \frac{4}{n} \left[5n + 6(n+1) + \frac{8(2n^2+3n+1)}{3n}\right]$$

$$\lim_{n\to\infty} \frac{4}{n} \left[5n + 6n + 6 + \frac{16n^2}{3n} + \frac{24n}{3n} + \frac{8}{3n}\right]$$

$$\lim_{n\to\infty} \frac{4}{n} \left[11n + 6 + \frac{16n}{3} + 8 + \frac{8}{3n}\right]$$

$$\lim_{n\to\infty} \left[44 + \frac{24}{n} + \frac{64n}{3n} + \frac{32}{n} + \frac{32}{3n^2}\right]$$

$$\lim_{n\to\infty} \left[44 + \frac{24}{n} + \frac{64}{3} + \frac{32}{n} + \frac{32}{3n^2}\right]$$

Evaluate the limit.

$$44 + 0 + \frac{64}{3} + 0 + 0$$

$$\frac{132}{3} + \frac{64}{3}$$

$$\frac{196}{3}$$

■ 3. Use the limit process to find the area of the region between the graph of $h(x)$ and the x-axis on the interval $[2,5]$.

$$h(x) = x^2 - 3x + 7$$

Solution:

Find Δx.

$$\Delta x = \frac{b-a}{n} = \frac{5-2}{n} = \frac{3}{n}$$

With Δx, find x_i.

$$x_i = 2 + i\Delta x = 2 + i \cdot \frac{3}{n} = 2 + \frac{3i}{n}$$

Then you get

$$\lim_{n\to\infty} \sum_{i=1}^{n} h(x_i)\Delta x$$

$$\lim_{n\to\infty}\sum_{i=1}^{n} h\left(2+\frac{3i}{n}\right)\frac{3}{n}$$

$$\lim_{n\to\infty}\frac{3}{n}\sum_{i=1}^{n}\left(2+\frac{3i}{n}\right)^2-3\left(2+\frac{3i}{n}\right)+7$$

$$\lim_{n\to\infty}\frac{3}{n}\sum_{i=1}^{n}4+\frac{12i}{n}+\frac{9i^2}{n^2}-6-\frac{9i}{n}+7$$

$$\lim_{n\to\infty}\frac{3}{n}\sum_{i=1}^{n}5+\frac{3i}{n}+\frac{9i^2}{n^2}$$

Making substitutions for i and i^2 gives

$$\lim_{n\to\infty}\frac{3}{n}\left[5n+\frac{3}{n}\cdot\frac{n(n+1)}{2}+\frac{9}{n^2}\cdot\frac{n(n+1)(2n+1)}{6}\right]$$

$$\lim_{n\to\infty}\frac{3}{n}\left[5n+\frac{3}{2}(n+1)+\frac{3(2n^2+3n+1)}{2n}\right]$$

$$\lim_{n\to\infty}\frac{3}{n}\left[5n+\frac{3}{2}n+\frac{3}{2}+\frac{6n^2}{2n}+\frac{9n}{2n}+\frac{3}{2n}\right]$$

$$\lim_{n\to\infty}\frac{3}{n}\left[\frac{13}{2}n+\frac{3}{2}+3n+\frac{9}{2}+\frac{3}{2n}\right]$$

$$\lim_{n\to\infty}\left[\frac{39}{2}+\frac{9}{2n}+9+\frac{27}{2n}+\frac{9}{2n^2}\right]$$

Evaluate the limit.

$$\frac{39}{2}+0+9+0+0$$

$$\frac{39}{2}+\frac{18}{2}$$

$$\frac{57}{2}$$

LIMIT PROCESS TO FIND AREA ON [-A,A]

■ 1. Use the limit process to find the area of the region between the graph of $f(x)$ and the x-axis on the interval $[-5,5]$.

$$f(x) = x^2 + 1$$

Solution:

Find Δx.

$$\Delta x = \frac{b-a}{n} = \frac{5-(-5)}{n} = \frac{10}{n}$$

With Δx, find x_i.

$$x_i = -5 + i\Delta x = -5 + i \cdot \frac{10}{n} = -5 + \frac{10i}{n}$$

Then you get

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$$

$$\lim_{n\to\infty} \sum_{i=1}^{n} f\left(-5 + \frac{10i}{n}\right)\frac{10}{n}$$

$$\lim_{n\to\infty} \frac{10}{n} \sum_{i=1}^{n} \left(-5 + \frac{10i}{n}\right)^2 + 1$$

$$\lim_{n\to\infty} \frac{10}{n} \sum_{i=1}^{n} 25 - \frac{100i}{n} + \frac{100i^2}{n^2} + 1$$

$$\lim_{n\to\infty} \frac{10}{n} \sum_{i=1}^{n} 26 - \frac{100i}{n} + \frac{100i^2}{n^2}$$

Making substitutions for i and i^2 gives

$$\lim_{n\to\infty} \frac{10}{n} \left[26n - \frac{100}{n} \cdot \frac{n(n+1)}{2} + \frac{100}{n^2} \cdot \frac{n(n+1)(2n+1)}{6}\right]$$

$$\lim_{n\to\infty} \frac{10}{n} \left[26n - 50(n+1) + \frac{50(2n^2+3n+1)}{3n}\right]$$

$$\lim_{n\to\infty} \frac{10}{n} \left[26n - 50n - 50 + \frac{100n^2}{3n} + \frac{150n}{3n} + \frac{50}{3n}\right]$$

$$\lim_{n\to\infty} \frac{10}{n} \left[-24n - 50 + \frac{100n}{3} + 50 + \frac{50}{3n}\right]$$

$$\lim_{n\to\infty} \frac{10}{n} \left[-24n + \frac{100n}{3} + \frac{50}{3n}\right]$$

$$\lim_{n\to\infty} \left[-240 + \frac{1{,}000}{3} + \frac{500}{3n^2}\right]$$

Evaluate the limit.

$$-240 + \frac{1{,}000}{3} + 0$$

$$\frac{280}{3}$$

2. Use the limit process to find the area of the region between the graph of $g(x)$ and the x-axis on the interval $[-3,3]$.

$$g(x) = 3x^2 - 4$$

Solution:

Find Δx.

$$\Delta x = \frac{b-a}{n} = \frac{3-(-3)}{n} = \frac{6}{n}$$

With Δx, find x_i.

$$x_i = -3 + i\Delta x = -3 + i \cdot \frac{6}{n} = -3 + \frac{6i}{n}$$

Then you get

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$$

$$\lim_{n\to\infty} \sum_{i=1}^{n} g\left(-3 + \frac{6i}{n}\right)\frac{6}{n}$$

$$\lim_{n\to\infty} \frac{6}{n}\sum_{i=1}^{n} 3\left(-3 + \frac{6i}{n}\right)^2 - 4$$

$$\lim_{n\to\infty} \frac{6}{n} \sum_{i=1}^{n} 3\left(9 - \frac{36i}{n} + \frac{36i^2}{n^2}\right) - 4$$

$$\lim_{n\to\infty} \frac{6}{n} \sum_{i=1}^{n} 23 - \frac{108i}{n} + \frac{108i^2}{n^2}$$

Making substitutions for i and i^2 gives

$$\lim_{n\to\infty} \frac{6}{n} \left[23n - \frac{108}{n} \cdot \frac{n(n+1)}{2} + \frac{108}{n^2} \cdot \frac{n(n+1)(2n+1)}{6}\right]$$

$$\lim_{n\to\infty} \frac{6}{n} \left[23n - 54(n+1) + \frac{18(2n^2+3n+1)}{n}\right]$$

$$\lim_{n\to\infty} \frac{6}{n} \left[23n - 54n - 54 + \frac{36n^2}{n} + \frac{54n}{n} + \frac{18}{n}\right]$$

$$\lim_{n\to\infty} \frac{6}{n} \left[-31n - 54 + 36n + 54 + \frac{18}{n}\right]$$

$$\lim_{n\to\infty} \frac{6}{n} \left[5n + \frac{18}{n}\right]$$

$$\lim_{n\to\infty} \left[30 + \frac{108}{n^2}\right]$$

Evaluate the limit.

$$30 + 0$$

$$30$$

■ 3. Use the limit process to find the area of the region between the graph of $h(x)$ and the x-axis on the interval $[-1,1]$.

$$h(x) = 4x^2 - x + 1$$

Solution:

Find Δx.

$$\Delta x = \frac{b-a}{n} = \frac{1-(-1)}{n} = \frac{2}{n}$$

With Δx, find x_i.

$$x_i = -1 + i\Delta x = -1 + i \cdot \frac{2}{n} = -1 + \frac{2i}{n}$$

Then you get

$$\lim_{n\to\infty} \sum_{i=1}^{n} h(x_i)\Delta x$$

$$\lim_{n\to\infty} \sum_{i=1}^{n} h\left(-1 + \frac{2i}{n}\right)\frac{2}{n}$$

$$\lim_{n\to\infty} \frac{2}{n} \sum_{i=1}^{n} 4\left(-1 + \frac{2i}{n}\right)^2 - \left(-1 + \frac{2i}{n}\right) + 1$$

$$\lim_{n\to\infty} \frac{2}{n} \sum_{i=1}^{n} 4\left(1 - \frac{4i}{n} + \frac{4i^2}{n^2}\right) - \left(-1 + \frac{2i}{n}\right) + 1$$

$$\lim_{n\to\infty} \frac{2}{n} \sum_{i=1}^{n} 4 - \frac{16i}{n} + \frac{16i^2}{n^2} + 1 - \frac{2i}{n} + 1$$

$$\lim_{n\to\infty} \frac{2}{n} \sum_{i=1}^{n} 6 - \frac{18i}{n} + \frac{16i^2}{n^2}$$

Making substitutions for i and i^2 gives

$$\lim_{n\to\infty} \frac{2}{n} \left[6n - \frac{18}{n} \cdot \frac{n(n+1)}{2} + \frac{16}{n^2} \cdot \frac{n(n+1)(2n+1)}{6}\right]$$

$$\lim_{n\to\infty} \frac{2}{n} \left[6n - 9(n+1) + \frac{8(2n^2+3n+1)}{3n}\right]$$

$$\lim_{n\to\infty} \frac{2}{n} \left[6n - 9n - 9 + \frac{16n^2}{3n} + \frac{24n}{3n} + \frac{8}{3n}\right]$$

$$\lim_{n\to\infty} \frac{2}{n} \left[-3n - 9 + \frac{16n}{3} + 8 + \frac{8}{3n}\right]$$

$$\lim_{n\to\infty} \frac{2}{n} \left[-3n - 1 + \frac{16n}{3} + \frac{8}{3n}\right]$$

$$\lim_{n\to\infty} \left[-6 - \frac{2}{n} + \frac{32}{3} + \frac{16}{3n^2}\right]$$

Evaluate the limit.

$$-6-0+\frac{32}{3}+0$$

$$-\frac{18}{3}+\frac{32}{3}$$

$$\frac{14}{3}$$

TRAPEZOIDAL RULE

1. Using $n=6$ and the Trapezoidal rule, approximate the value of the integral. Round the answer to 2 decimal places.

$$\int_4^{16} 2\sqrt[3]{x}+3\ dx$$

Solution:

Evaluating the integral with $n=6$ means the interval of $[4,16]$ is split into 6 subintervals.

$$[4,6],\ [6,8],\ [8,10],\ [10,12],\ [12,14],\text{ and }[14,16]$$

Evaluate the integrand at the endpoints of each subinterval.

$$f(4)=2\sqrt[3]{4}+3\approx 6.1748$$

$$f(6)=2\sqrt[3]{6}+3\approx 6.6342$$

$$f(8)=2\sqrt[3]{8}+3=7$$

$$f(10)=2\sqrt[3]{10}+3\approx 7.3089$$

$$f(12)=2\sqrt[3]{12}+3\approx 7.5789$$

$$f(14)=2\sqrt[3]{14}+3\approx 7.8203$$

$$f(16)=2\sqrt[3]{16}+3\approx 8.0397$$

Use these values in the Trapezoidal rule with $\Delta x = 2$.

$$\frac{2}{2}\left[6.1748 + 2(6.6342) + 2(7) + 2(7.3089) + 2(7.5789) + 2(7.8203) + 8.0397\right]$$

$$6.1748 + 13.2684 + 14 + 14.6178 + 15.1578 + 15.6406 + 8.0397$$

$$86.8991$$

This answer rounds to 86.90.

■ 2. Using $n = 6$ and the Trapezoidal rule, approximate the value of the integral.

$$\int_0^6 \frac{1}{4}x^4 - \frac{1}{2}x^3 + 2x^2 - 5x + 8\ dx$$

Solution:

Evaluating the integral with $n = 6$ means the interval of $[0,6]$ is split into 6 subintervals.

$[0,1]$, $[1,2]$, $[2,3]$, $[3,4]$, $[4,5]$, and $[5,6]$

Evaluate the integrand at the endpoints of each subinterval.

$$f(0) = \frac{1}{4}(0)^4 - \frac{1}{2}(0)^3 + 2(0)^2 - 5(0) + 8 = 8$$

$$f(1)=\frac{1}{4}(1)^4-\frac{1}{2}(1)^3+2(1)^2-5(1)+8=\frac{19}{4}$$

$$f(2)=\frac{1}{4}(2)^4-\frac{1}{2}(2)^3+2(2)^2-5(2)+8=6$$

$$f(3)=\frac{1}{4}(3)^4-\frac{1}{2}(3)^3+2(3)^2-5(3)+8=\frac{71}{4}$$

$$f(4)=\frac{1}{4}(4)^4-\frac{1}{2}(4)^3+2(4)^2-5(4)+8=52$$

$$f(5)=\frac{1}{4}(5)^4-\frac{1}{2}(5)^3+2(5)^2-5(5)+8=\frac{507}{4}$$

$$f(6)=\frac{1}{4}(6)^4-\frac{1}{2}(6)^3+2(6)^2-5(6)+8=266$$

Use these values in the Trapezoidal rule with $\Delta x=1$.

$$\frac{1}{2}\left[8+2\left(\frac{19}{4}\right)+2\left(6\right)+2\left(\frac{71}{4}\right)+2\left(52\right)+2\left(\frac{507}{4}\right)+266\right]$$

$$\frac{1}{2}\left(8+\frac{19}{2}+12+\frac{71}{2}+104+\frac{507}{2}+266\right)$$

$$\frac{1}{2}\left(\frac{597}{2}+\frac{780}{2}\right)$$

$$\frac{1{,}377}{4}$$

3. Using $n = 4$ and the Trapezoidal rule, approximate the value of the integral.

$$\int_0^8 \frac{1}{2}x^2 - 3x + 6\ dx$$

Solution:

Evaluating the integral with $n = 4$ means the interval of $[0,8]$ is split into 4 subintervals.

$[0,2]$, $[2,4]$, $[4,6]$, and $[6,8]$

Evaluate the integrand at the endpoints of each subinterval.

$$f(0) = \frac{1}{2}(0)^2 - 3(0) + 6 = 6$$

$$f(2) = \frac{1}{2}(2)^2 - 3(2) + 6 = 2$$

$$f(4) = \frac{1}{2}(4)^2 - 3(4) + 6 = 2$$

$$f(6) = \frac{1}{2}(6)^2 - 3(6) + 6 = 6$$

$$f(8) = \frac{1}{2}(8)^2 - 3(8) + 6 = 14$$

Use these values in the Trapezoidal rule with $\Delta x = 2$.

$$\frac{2}{2}\left[6+2(2)+2(2)+2(6)+14\right]$$

$$6+4+4+12+14$$

$$40$$

4. Using $n=4$ and the Trapezoidal rule, approximate the value of the integral.

$$\int_0^{16} \frac{1}{16}x^4-\frac{1}{2}x^3-x^2-x+1\ dx$$

Solution:

Evaluating the integral with $n=4$ means the interval of $[0,16]$ is split into 4 subintervals.

$[0,4]$, $[4,8]$, $[8,12]$, and $[12,16]$

Evaluate the integrand at the endpoints of each subinterval.

$$f(0)=\frac{1}{16}(0)^4-\frac{1}{2}(0)^3-(0)^2-(0)+1=1$$

$$f(4)=\frac{1}{16}(4)^4-\frac{1}{2}(4)^3-(4)^2-(4)+1=-35$$

$$f(8)=\frac{1}{16}(8)^4-\frac{1}{2}(8)^3-(8)^2-(8)+1=-71$$

$$f(12) = \frac{1}{16}(12)^4 - \frac{1}{2}(12)^3 - (12)^2 - (12) + 1 = 277$$

$$f(16) = \frac{1}{16}(16)^4 - \frac{1}{2}(16)^3 - (16)^2 - (16) + 1 = 1{,}777$$

Use these values in the Trapezoidal rule with $\Delta x = 4$.

$$\frac{4}{2}\left[1 + 2(-35) + 2(-71) + 2(277) + 1{,}777\right]$$

$$2(1 - 70 - 142 + 554 + 1{,}777)$$

$$4{,}240$$

SIMPSON'S RULE

1. Use Simpson's Rule with $n=6$ to approximate the value of the integral. Round the answer to 2 decimal places.

$$\int_2^8 6\sqrt{3x}+5\ dx$$

Solution:

Since $n=6$ and the interval is $[2,8]$, that means $\Delta x=1$. So

$$x_1=2,\ x_2=3,\ x_3=4,\ x_3=5,\ x_4=6,\ x_5=7,\text{ and } x_6=8$$

Evaluate the integrand at each of these values.

$$f(2)=6\sqrt{3(2)}+5\approx 19.6969$$

$$f(3)=6\sqrt{3(3)}+5=23$$

$$f(4)=6\sqrt{3(4)}+5\approx 25.7846$$

$$f(5)=6\sqrt{3(5)}+5\approx 28.2379$$

$$f(6)=6\sqrt{3(6)}+5\approx 30.4558$$

$$f(7)=6\sqrt{3(7)}+5\approx 32.4955$$

$$f(8)=6\sqrt{3(8)}+5\approx 34.3939$$

Use these values in Simpson's rule with $\Delta x = 1$.

$$\frac{1}{3}\left[19.6969 + 4(23) + 2(25.7846) + 4(28.2379) + 2(30.4558) + 4(32.4955) + 34.3939\right]$$

$$\frac{1}{3}(19.6969 + 92 + 51.5692 + 112.9516 + 60.9116 + 129.9820 + 34.3939)$$

$$167.1684$$

This answer rounds to 167.17.

■ 2. Use Simpson's Rule with $n = 8$ to approximate the value of the integral. Round the answer to 2 decimal places.

$$\int_4^{28} 120(0.95)^x\ dx$$

Solution:

Since $n = 6$ and the interval is $[4,25]$, that means $\Delta x = 3$. So

$x_1 = 4$, $x_2 = 7$, $x_3 = 10$, $x_4 = 13$, $x_5 = 16$, $x_6 = 19$, $x_7 = 22$, $x_8 = 25$, and $x_9 = 28$

Evaluate the integrand at each of these values.

$$f(4) = 120(0.95)^4 \approx 97.7408$$

$$f(7) = 120(0.95)^7 \approx 83.8005$$

$$f(10) = 120(0.95)^{10} \approx 71.8484$$

$$f(13) = 120(0.95)^{13} \approx 61.6010$$

$$f(16) = 120(0.95)^{16} \approx 52.8152$$

$$f(19) = 120(0.95)^{19} \approx 45.2824$$

$$f(22) = 120(0.95)^{22} \approx 38.8240$$

$$f(25) = 120(0.95)^{25} \approx 33.2867$$

$$f(28) = 120(0.95)^{28} \approx 28.5392$$

Use these values in Simpson's rule with $\Delta x = 3$.

$$\frac{3}{3}\Big[97.7408 + 4(83.8005) + 2(71.8484) + 4(61.6010) + 2(52.8152)$$

$$+4(45.2824) + 2(38.8240) + 4(33.2867) + 28.5392\Big]$$

$$97.7408 + 335.2020 + 142.6968 + 246.4040 + 105.6304$$

$$+181.1296 + 77.6480 + 133.1468 + 28.5392$$

$$1{,}349.1376$$

This answer rounds to $1{,}349.14$.

■ 3. Use Simpson's Rule with $n = 4$ to approximate the value of the integral. Round the answer to 2 decimal places.

$$\int_5^7 3\ln(x+5) - 2\ dx$$

Solution:

Since $n = 4$ and the interval is $[5,7]$, that means $\Delta x = 0.5$. So

$x_1 = 5$, $x_2 = 5.5$, $x_3 = 6$, $x_4 = 6.5$, and $x_5 = 7$

Evaluate the integrand at each of these values.

$$f(5) = 3\ln(5+5) - 2 \approx 4.9078$$

$$f(5.5) = 3\ln(5.5+5) - 2 \approx 5.0541$$

$$f(6) = 3\ln(6+5) - 2 \approx 5.1937$$

$$f(6.5) = 3\ln(6.5+5) - 2 \approx 5.3270$$

$$f(7) = 3\ln(7+5) - 2 \approx 5.4547$$

Use these values in Simpson's rule with $\Delta x = 0.5$.

$$\frac{0.5}{3}\left[4.9078 + 4(5.0541) + 2(5.1937) + 4(5.3270) + 5.4547\right]$$

$$\frac{0.5}{3}\left(4.9078 + 20.2164 + 10.3874 + 21.3080 + 5.4547\right)$$

$$10.37905$$

This answer rounds to 10.38.

■ 4. Use Simpson's Rule with $n = 4$ to approximate the value of the integral.

$$\int_{-3}^{9} x^2 + 3x + 2 \ dx$$

Solution:

Since $n = 4$ and the interval is $[-3,9]$, that means $\Delta x = 3$. So

$$x_1 = -3, x_2 = 0, x_3 = 3, x_4 = 6, \text{ and } x_5 = 9$$

Evaluate the integrand at each of these values.

$$f(-3) = (-3)^2 + 3(-3) + 2 = 2$$

$$f(0) = (0)^2 + 3(0) + 2 = 2$$

$$f(3) = (3)^2 + 3(3) + 2 = 20$$

$$f(6) = (6)^2 + 3(6) + 2 = 56$$

$$f(9) = (9)^2 + 3(9) + 2 = 110$$

Use these values in Simpson's rule with $\Delta x = 3$.

$$\frac{3}{3}\left[2 + 4(2) + 2(20) + 4(56) + 110\right]$$

$$2 + 8 + 40 + 224 + 110$$

384

■ 5. Use Simpson's Rule with $n=6$ to approximate the value of the integral. Round the answer to 2 decimal places.

$$\int_{0.4}^{1.6}\frac{1}{3}x^3-x^2+5x+4\ dx$$

Solution:

Since $n=6$ and the interval is $[0.4,1.6]$, that means $\Delta x=0.2$. So

$x_1=0.4$, $x_2=0.6$, $x_3=0.8$, $x_4=1.0$, $x_5=1.2$, $x_6=1.4$, and $x_7=1.6$

Evaluate the integrand at each of these values.

$$f(0.4)=\frac{1}{3}(0.4)^3-(0.4)^2+5(0.4)+4\approx 5.8613$$

$$f(0.6)=\frac{1}{3}(0.6)^3-(0.6)^2+5(0.6)+4\approx 6.712$$

$$f(0.8)=\frac{1}{3}(0.8)^3-(0.8)^2+5(0.8)+4\approx 7.5307$$

$$f(1)=\frac{1}{3}(1)^3-(1)^2+5(1)+4\approx 8.3333$$

$$f(1.2)=\frac{1}{3}(1.2)^3-(1.2)^2+5(1.2)+4\approx 9.136$$

$$f(1.4) = \frac{1}{3}(1.4)^3 - (1.4)^2 + 5(1.4) + 4 \approx 9.9547$$

$$f(1.6) = \frac{1}{3}(1.6)^3 - (1.6)^2 + 5(1.6) + 4 \approx 10.8053$$

Use these values in Simpson's rule with $\Delta x = 0.2$.

$$\frac{0.2}{3}\Big[5.8613 + 4(6.712) + 2(7.5307) + 4(8.3333)$$

$$+2(9.136) + 4(9.9547) + 10.8053\Big]$$

$$\frac{0.2}{3}(5.8613 + 26.848 + 15.0614 + 33.3332 + 18.272 + 39.8188 + 10.8053)$$

$$10$$

MIDPOINT RULE ERROR BOUND

1. Calculate the area under the curve. Then use the Midpoint Rule with $n = 3$ to approximate the same area. Compare the actual area to the result to determine the error of the of the Midpoint Rule approximation.

$$\int_0^6 3x^2 - 2x + 5 \ dx$$

Solution:

The area under the curve is

$$A = \frac{3x^3}{3} - \frac{2x^2}{2} + 5x \Big|_0^6 = x^3 - x^2 + 5x \Big|_0^6$$

$$A = 6^3 - 6^2 + 5(6) - \left(0^3 - 0^2 + 5(0)\right)$$

$$A = 210$$

With $n = 3$, the integration interval of $[0,6]$ is split into the three subintervals $[0,2]$, $[2,4]$, and $[4,6]$. The midpoints of those subintervals are 1, 3, and 5. Evaluate $3x^2 - 2x + 5$ at each of these values.

At $x = 1$, $3(1)^2 - 2(1) + 5 = 6$

At $x = 3$, $3(3)^2 - 2(3) + 5 = 26$

At $x = 5$, $3(5)^2 - 2(5) + 5 = 70$

Since the original interval is $[0,6]$ and $n = 3$, each subinterval is 2 units wide. So the Midpoint Rule gives

$$A_M = 6(2) + 26(2) + 70(2)$$

$$A_M = 204$$

Compared to the actual area under the curve, the Midpoint Rule gives an error of $210 - 204 = 6$.

2. Calculate the area under the curve. Then use the Midpoint Rule with $n = 4$ to approximate the same area. Compare the actual area to the result to determine the error of the of the Midpoint Rule approximation. Round your answer to the nearest 3 decimal places.

$$\int_5^{13} 4\sqrt{x-2}\ dx$$

Solution:

The area under the curve is

$$A = \frac{8}{3}(x-2)^{\frac{3}{2}}\Big|_5^{13}$$

$$A = \frac{8}{3}(13-2)^{\frac{3}{2}} - \frac{8}{3}(5-2)^{\frac{3}{2}}$$

$$A = 97.287661 - 13.856406 = 83.431255$$

With $n = 4$, the integration interval of $[5,13]$ is split into the four subintervals $[5,7]$, $[7,9]$, $[9,11]$, and $[11,13]$. The midpoints of those subintervals are 6, 8, 10, and 12. Evaluate $4\sqrt{x-2}$ at each of these values.

At $x = 6$, $4\sqrt{6-2} = 8$

At $x = 8$, $4\sqrt{8-2} = 9.797959$

At $x = 10$, $4\sqrt{10-2} = 11.313708$

At $x = 12$, $4\sqrt{12-2} = 12.649111$

Since the original interval is $[5,13]$ and $n = 4$, each subinterval is 2 units wide. So the Midpoint Rule gives

$$A_M \approx 8(2) + 9.797959(2) + 11.313708\,(2) + 12.649111(2)$$

$$A_M \approx 83.521556$$

Compared to the actual area under the curve, the Midpoint Rule gives an error of $|83.431255 - 83.521556| \approx 0.090301 \approx 0.090$.

■ 3. Calculate the area under the curve. Then use the Midpoint Rule with $n = 4$ to approximate the same area. Compare the actual area to the result to determine the error of the of the Midpoint Rule approximation.

$$\int_2^{10} 4x^3 - 3x^2 + 2x - 1\ dx$$

Solution:

The area under the curve is

$$A = \frac{4x^4}{4} - \frac{3x^3}{3} + \frac{2x^2}{2} - x \Big|_2^{10} = x^4 - x^3 + x^2 - x \Big|_2^{10}$$

$$A = 10^4 - 10^3 + 10^2 - 10 - \left(2^4 - 2^3 + 2^2 - 2\right) = 9{,}090 - 10$$

$$A = 9{,}080$$

With $n = 4$, the integration interval of $[2,10]$ is split into the four subintervals $[2,4]$, $[4,6]$, $[6,8]$, and $[8,10]$. The midpoints of those subintervals are 3, 5, 7, and 9. Evaluate $4x^3 - 3x^2 + 2x - 1$ at each of these values.

At $x = 3$, $4(3)^3 - 3(3)^2 + 2(3) - 1 = 86$

At $x = 5$, $4(5)^3 - 3(5)^2 + 2(5) - 1 = 434$

At $x = 7$, $4(7)^3 - 3(7)^2 + 2(7) - 1 = 1{,}238$

At $x = 9$, $4(9)^3 - 3(9)^2 + 2(9) - 1 = 2{,}690$

Since the original interval is $[2,10]$ and $n = 4$, each subinterval is 2 units wide. So the Midpoint Rule gives

$$A_M = 86(2) + 434(2) + 1{,}238(2) + 2{,}690(2)$$

$$A_M = 8{,}896$$

Compared to the actual area under the curve, the Midpoint Rule gives an error of $9{,}080 - 8{,}896 = 184$.

TRAPEZOIDAL RULE ERROR BOUND

1. Calculate the area under the curve. Then use the Trapezoidal Rule with $n = 4$ to approximate the same area. Compare the actual area to the result to determine the error of the of the Trapezoidal Rule approximation.

$$\int_1^5 6x^2 - 8x + 5\ dx$$

Solution:

The area under the curve is

$$A = \frac{6x^3}{3} - \frac{8x^2}{2} + 5x \Big|_1^5 = 2x^3 - 4x^2 + 5x \Big|_1^5$$

$$A = 2(5)^3 - 4(5)^2 + 5(5) - \left(2(1)^3 - 4(1)^2 + 5(1)\right) = 175 - 3$$

$$A = 172$$

With $n = 4$, the integration interval of $[1,5]$ is split into the four subintervals $[1,2]$, $[2,3]$, $[3,4]$, and $[4,5]$. Evaluate the integrand at the endpoints of each subinterval.

At $x = 1$, $6(1)^2 - 8(1) + 5 = 3$

At $x = 2$, $6(2)^2 - 8(2) + 5 = 13$

At $x = 3$, $6(3)^2 - 8(3) + 5 = 35$

At $x = 4$, $6(4)^2 - 8(4) + 5 = 69$

At $x = 5$, $6(5)^2 - 8(5) + 5 = 115$

Use these values in the Trapezoidal Rule with $\Delta x = 1$.

$$A_T = \frac{1}{2}\left[3 + 2(13) + 2(35) + 2(69) + 115\right]$$

$$A_T = 176$$

Compared to the actual area under the curve, the Trapezoidal Rule gives an error of $|172 - 176| = 4$.

■ 2. Calculate the area under the curve. Then use the Trapezoidal Rule with $n = 5$ to approximate the same area. Compare the actual area to the result to determine the error of the of the Trapezoidal Rule approximation. Round your answer to the nearest 3 decimal places.

$$\int_2^{12} e^{-x} + 3\ dx$$

Solution:

The area under the curve is

$$A = -e^{-x} + 3x\Big|_2^{12}$$

$$A = -e^{-12} + 3(12) - \left(-e^{-2} + 3(2)\right)$$

$$A = 30.135329$$

With $n = 4$, the integration interval of $[2,12]$ is split into the five subintervals $[2,4]$, $[4,6]$, $[6,8]$, $[8,10]$, and $[10,12]$. Evaluate the integrand at the endpoints of each subinterval.

At $x = 2$, $e^{-2} + 3 = 3.135335$

At $x = 4$, $e^{-4} + 3 = 3.018316$

At $x = 6$, $e^{-6} + 3 = 3.002479$

At $x = 8$, $e^{-8} + 3 = 3.000335$

At $x = 10$, $e^{-10} + 3 = 3.000045$

At $x = 12$, $e^{-12} + 3 = 3.000006$

Use these values in the Trapezoidal Rule with $\Delta x = 2$.

$$A_T = \frac{2}{2}\Big[3.135335 + 2(3.018316) + 2(3.002479)$$

$$+2(3.000335) + 2(3.000045) + 3.000006\Big]$$

$$A_T = 30.177691$$

Compared to the actual area under the curve, the Trapezoidal Rule gives an error of $|30.135329 - 30.177691| = 0.042362$.

3. Calculate the area under the curve. Then use the Trapezoidal Rule with $n = 4$ to approximate the same area. Compare the actual area to the result to determine the error of the of the Trapezoidal Rule approximation. Round your answer to the nearest three decimal places.

$$\int_0^2 4\sqrt{x} + 1 \ dx$$

Solution:

The area under the curve is

$$A = \frac{8}{3}x^{\frac{3}{2}} + x \Big|_0^2$$

$$A = \frac{8}{3}(2)^{\frac{3}{2}} + 2 - \left(\frac{8}{3}(0)^{\frac{3}{2}} + 0\right)$$

$$A = 9.542472$$

With $n = 4$, the integration interval of $[0,2]$ is split into the four subintervals $[0,0.5]$, $[0.5,1]$, $[1,1.5]$, and $[1.5,2]$. Evaluate the integrand at the endpoints of each subinterval.

At $x = 0$, $4\sqrt{0} + 1 = 1$

At $x = 0.5$, $4\sqrt{0.5} + 1 = 3.828427$

At $x = 1$, $4\sqrt{1} + 1 = 5$

At $x=1.5$, $4\sqrt{1.5}+1=5.898979$

At $x=2$, $4\sqrt{2}+1=6.656854$

Use these values in the Trapezoidal Rule with $\Delta x=0.5$.

$$A_T=\frac{0.5}{2}\left[1+2(3.828427)+2(5)+2(5.898979)+6.656854\right]$$

$$A_T=9.277917$$

Compared to the actual area under the curve, the Trapezoidal Rule gives an error of $|9.542472-9.277917|=0.264826\approx0.265$.

SIMPSON'S RULE ERROR BOUND

■ 1. Calculate the area under the curve. Then use Simpson's Rule with $n = 6$ to approximate the same area. Compare the actual area to the result to determine the error of the of Simpson's Rule approximation. Round your answer to the nearest three decimal places.

$$\int_{2.2}^{3.4} x^2 - x + 2\ dx$$

Solution:

The area under the curve is

$$A = \frac{x^3}{3} - \frac{x^2}{2} + 2x \Big|_{2.2}^{3.4}$$

$$A = \frac{(3.4)^3}{3} - \frac{(3.4)^2}{2} + 2(3.4) - \left(\frac{(2.2)^3}{3} - \frac{(2.2)^2}{2} + 2(2.2)\right)$$

$$A = 8.592$$

Since $n = 6$, the interval of integration is $[2.2,3.4]$, and $\Delta x = 0.2$,

$$x_1 = 2.2$$

$$x_2 = 2.4$$

$x_3 = 2.6$

$x_4 = 2.8$

$x_5 = 3.0$

$x_6 = 3.2$

$x_7 = 3.4$

Evaluate the integrand at each of these values.

At 2.2, $(2.2)^2 - 2.2 + 2 = 4.64$

At 2.4, $(2.4)^2 - 2.4 + 2 = 5.36$

At 2.6, $(2.6)^2 - 2.6 + 2 = 6.16$

At 2.8, $(2.8)^2 - 2.8 + 2 = 7.04$

At 3.0, $(3.0)^2 - 3.0 + 2 = 8.00$

At 3.2, $(3.2)^2 - 3.2 + 2 = 9.04$

At 3.4, $(3.4)^2 - 3.4 + 2 = 10.16$

Use these values in Simpson's Rule with $\Delta x = 0.2$.

$$A_S \approx \frac{0.2}{3}\left[4.64 + 4(5.36) + 2(6.16) + 4(7.04) + 2(8) + 4(9.04) + 10.16\right]$$

$$A_S \approx 8.592$$

Simpson's Rule gives an error of $|8.592 - 8.592| = 0$. There is no error in this problem.

■ 2. Calculate the area under the curve. Then use Simpson's Rule with $n = 4$ to approximate the same area. Compare the actual area to the result to determine the error of the of Simpson's Rule approximation. Round your answer to the nearest four decimal places.

$$\int_0^{1.2} e^x - 2x + 3 \ dx$$

Solution:

The area under the curve is

$$A = e^x - \frac{2x^2}{2} + 3x \Big|_0^{1.2}$$

$$A = e^{1.2} - (1.2)^2 + 3(1.2) - \left(e^0 - (0)^2 + 3(0)\right)$$

$$A = 4.480117$$

Since $n = 4$, the interval of integration is $[0,1.2]$, and $\Delta x = 0.3$,

$$x_1 = 0$$

$$x_2 = 0.3$$

$$x_3 = 0.6$$

$x_4 = 0.9$

$x_4 = 1.2$

Evaluate the integrand at each of these values.

At 0, $e^0 - 2(0) + 3 = 4$

At 0.3, $e^{0.3} - 2(0.3) + 3 = 3.749859$

At 0.6, $e^{0.6} - 2(0.6) + 3 = 3.622119$

At 0.9, $e^{0.9} - 2(0.9) + 3 = 3.659603$

At 1.2, $e^{1.2} - 2(1.2) + 3 = 3.920117$

Use these values in Simpson's Rule with $\Delta x = 0.3$.

$$A_S \approx \frac{0.3}{3}\left[4 + 4(3.749859) + 2(3.622119) + 4(3.659603) + 3.920117\right]$$

$$A_S \approx 4.480220$$

Simpson's Rule gives an error of $|4.480117 - 4.480220| = 0.0001033$.

■ 3. Calculate the area under the curve. Then use Simpson's Rule with $n = 4$ to approximate the same area. Compare the actual area to the result to determine the error of the of Simpson's Rule approximation. Round your answer to the nearest three decimal places.

$$\int_{-4}^{4} 2x^2 + 3x + 4 \ dx$$

Solution:

The area under the curve is

$$A = \frac{2x^3}{3} + \frac{3x^2}{2} + 4x \Big|_{-4}^{4}$$

$$A = \frac{2(4)^3}{3} + \frac{3(4)^2}{2} + 4(4) - \left(\frac{2(-4)^3}{3} + \frac{3(-4)^2}{2} + 4(-4)\right)$$

$$A = 117\frac{1}{3}$$

Since $n = 4$, the interval of integration is $[-4,4]$, and $\Delta x = 2$,

$$x_1 = -4$$

$$x_2 = -2$$

$$x_3 = 0$$

$$x_4 = 2$$

$$x_4 = 4$$

Evaluate the integrand at each of these values.

At -4, $2(-4)^2 + 3(-4) + 4 = 24$

At -2, $2(-2)^2 + 3(-2) + 4 = 6$

At 0, $(0)^2 + 3(0) + 4 = 4$

At 2, $2(2)^2+3(2)+4=18$

At 4, $(4)^2+3(4)+4=4$

Use these values in Simpson's Rule with $\Delta x=2$.

$$A_S=\frac{2}{3}\left[24+4(6)+2(4)+4(18)+48\right]$$

$$A_S=117\frac{1}{3}$$

Simpson's Rule gives an error of $|117.33-117.33|=0$.

PART 1 OF THE FTC

1. Use Part 1 of the Fundamental Theorem of Calculus to find the derivative of $f(x)$.

$$f(x) = \int_0^{x^2} 7t\cos(2t)\ dt$$

Solution:

Evaluate the integrand at the upper bound, x^2, multiplying by the derivative of x^2. That will give the derivative of the function $f(x)$.

$$f'(x) = 7x^2\cos(2x^2) \cdot \frac{d}{dx}(x^2)$$

$$f'(x) = 7x^2\cos(2x^2) \cdot 2x$$

$$f'(x) = 14x^3\cos(2x^2)$$

2. Use Part 1 of the Fundamental Theorem of Calculus to find the derivative of $g(x)$.

$$g(x) = \int_2^{x^3} \frac{5}{3+e^t}\ dt$$

Solution:

Evaluate the integrand at the upper bound, x^3, multiplying by the derivative of x^3. That will give the derivative of the function $g(x)$.

$$g'(x) = \frac{5}{3+e^{x^3}} \cdot \frac{d}{dx}(x^3)$$

$$g'(x) = \frac{5}{3+e^{x^3}} \cdot 3x^2$$

$$g'(x) = \frac{15x^2}{3+e^{x^3}}$$

■ 3. Use Part 1 of the Fundamental Theorem of Calculus to find the derivative of $h(x)$.

$$h(x) = \int_{\cos(3x)}^{7} 8t + 1 \ dt$$

Solution:

Flip the upper and lower bound, multiplying the integral by -1.

$$h(x) = -\int_{7}^{\cos(3x)} 8t + 1 \ dt$$

Evaluate the integrand at the upper bound, $\cos(3x)$, multiplying by the derivative of $\cos(3x)$. That will give the derivative of the function $h(x)$.

$$h'(x) = -(8\cos(3x)+1)\cdot\frac{d}{dx}(\cos(3x))$$

$$h'(x) = -(8\cos(3x)+1)(-3\sin(3x))$$

$$h'(x) = (8\cos(3x)+1)(3\sin(3x))$$

$$h'(x) = 24\sin(3x)\cos(3x)+3\sin(3x)$$

■ 4. Use Part 1 of the Fundamental Theorem of Calculus to find the derivative of $f(x)$.

$$f(x) = \int_1^{3x^2} \frac{\sin t}{t^3+5}\,dt$$

Solution:

Evaluate the integrand at the upper bound, $3x^2$, multiplying by the derivative of $3x^2$. That will give the derivative of the function $f(x)$.

$$f'(x) = \frac{\sin(3x^2)}{(3x^2)^3+5}\cdot\frac{d}{dx}(3x^2)$$

$$f'(x) = \frac{\sin(3x^2)}{27x^6+5}\cdot 6x$$

$$f'(x) = \frac{6x\sin(3x^2)}{27x^6+5}$$

5. Use Part 1 of the Fundamental Theorem of Calculus to find the derivative of $g(x)$.

$$g(x) = \int_{3x}^{2x^2} t^2 - 5t + 4 \ dt$$

Solution:

Split the interval at 0.

$$g(x) = \int_{3x}^{0} t^2 - 5t + 4 \ dt + \int_{0}^{2x^2} t^2 - 5t + 4 \ dt$$

Flip the bounds on the first integral, multiplying by -1.

$$g(x) = -\int_{0}^{3x} t^2 - 5t + 4 \ dt + \int_{0}^{2x^2} t^2 - 5t + 4 \ dt$$

For the first integral, evaluate the integrand at the upper bound, $3x$, multiplying by the derivative of $3x$. For the second integral, evaluate at the upper bound, $2x^2$, multiplying by the derivative of $2x^2$. That will give the derivative of the function $g(x)$.

$$g'(x) = -\left((3x)^2 - 5(3x) + 4\right) \cdot \frac{d}{dx}(3x) + \left((2x^2)^2 - 5(2x^2) + 4\right) \cdot \frac{d}{dx}\left(2x^2\right)$$

$$g'(x) = -(9x^2 - 15x + 4) \cdot 3 + (4x^4 - 10x^2 + 4) \cdot 4x$$

$$g'(x) = -(27x^2 - 45x + 12) + (16x^5 - 40x^3 + 16x)$$

$$g'(x) = -27x^2 + 45x - 12 + 16x^5 - 40x^3 + 16x$$

$$g'(x) = 16x^5 - 40x^3 - 27x^2 + 61x - 12$$

PART 2 OF THE FTC

1. Use Part 2 of the Fundamental Theorem of Calculus to evaluate the integral.

$$\int_2^5 5-\frac{3}{x}\ dx$$

Solution:

Integrate, then evaluate over the interval.

$$5x-3\ln x\Big|_2^5$$

$$5(5)-3\ln 5-(5(2)-3\ln 2)$$

$$25-3\ln 5-10+3\ln 2$$

$$15+3\ln 2-3\ln 5$$

$$15+3(\ln 2-\ln 5)$$

$$15+3\ln\frac{2}{5}$$

2. Use Part 2 of the Fundamental Theorem of Calculus to evaluate the integral.

$$\int_4^9 4x^3 - \sqrt{x}\ dx$$

Solution:

Integrate, then evaluate over the interval.

$$x^4 - \frac{2x^{\frac{3}{2}}}{3}\Bigg|_4^9$$

$$9^4 - \frac{2(9)^{\frac{3}{2}}}{3} - \left(4^4 - \frac{2(4)^{\frac{3}{2}}}{3}\right)$$

$$6{,}561 - \frac{2(27)}{3} - 256 + \frac{2(8)}{3}$$

$$6{,}305 - \frac{54}{3} + \frac{16}{3}$$

$$\frac{18{,}915}{3} - \frac{38}{3}$$

$$\frac{18{,}877}{3}$$

■ 3. Use Part 2 of the Fundamental Theorem of Calculus to evaluate the integral.

$$\int_{-3}^{-1} \frac{3}{x^3}\ dx$$

Solution:

Integrate, then evaluate over the interval.

$$-\frac{3}{2}x^{-2}\Big|_{-3}^{-1}$$

$$-\frac{3}{2x^2}\Big|_{-3}^{-1}$$

$$-\frac{3}{2(-1)^2}-\left(-\frac{3}{2(-3)^2}\right)$$

$$-\frac{3}{2}+\frac{3}{2(9)}$$

$$\frac{3}{18}-\frac{3}{2}$$

$$\frac{3}{18}-\frac{27}{18}$$

$$-\frac{24}{18}$$

$$-\frac{4}{3}$$

■ 4. Use Part 2 of the Fundamental Theorem of Calculus to evaluate the integral.

$$\int_{25}^{36} \frac{2-\sqrt{x}}{\sqrt{x}}\ dx$$

Solution:

Rewrite the integral.

$$\int_{25}^{36} \frac{2}{\sqrt{x}} - \frac{\sqrt{x}}{\sqrt{x}}\ dx$$

$$\int_{25}^{36} 2x^{-\frac{1}{2}} - 1\ dx$$

Integrate, then evaluate over the interval.

$$4x^{\frac{1}{2}} - x \Big|_{25}^{36}$$

$$4(36)^{\frac{1}{2}} - 36 - \left(4(25)^{\frac{1}{2}} - 25\right)$$

$$4(6) - 36 - 4(5) + 25$$

$$24 - 36 - 20 + 25$$

$$-7$$

NET CHANGE THEOREM

1. Suppose the position of a particle moving along the horizontal s-axis is at $s = -2$ when $t = 0$. The velocity of the particle is given by $v(t)$ with $0 \le t \le 10$, where t is time in seconds since the particle began moving. Use the Net Change Theorem to determine the position of the particle on the s-axis after the particle has been moving for 5 seconds.

$$v(t) = \frac{1}{4}t^2 - \frac{9}{(t+1)^2}$$

Solution:

The interval of time from the time the particle starts moving until 5 seconds later is $0 \le t \le 5$, and that will be the bounds for the integral of $v(t)$. Because the particle starts at $s = -2$, we need to subtract 2 from the value of the integral. So the particle's position is given by

$$-2 + \int_0^5 \frac{1}{4}t^2 - \frac{9}{(t+1)^2}\ dt$$

$$-2 + \int_0^5 \frac{1}{4}t^2 - 9(t+1)^{-2}\ dt$$

Integrate, then evaluate over the interval.

$$-2 + \left(\frac{1}{12}t^3 + 9(t+1)^{-1}\right)\Bigg|_0^5$$

$$-2+\left(\frac{1}{12}(5)^3+9(5+1)^{-1}\right)-\left(\frac{1}{12}(0)^3+9(0+1)^{-1}\right)$$

$$-2+\frac{1}{12}(125)+\frac{9}{6}-9$$

$$-11+\frac{125}{12}+\frac{3}{2}$$

$$-\frac{132}{12}+\frac{125}{12}+\frac{18}{12}$$

$$\frac{11}{12}$$

■ 2. Water is being pumped from a tank at a rate (in gallons per minute) given by $w(t)=80-4\sqrt{t+3}$, with $0\le t\le 60$, where t is the time in minutes since the pumping began. The tank had $5{,}000$ gallons of water in it when pumping began. Use the Net Change Theorem to determine how many gallons of water will be in the tank after 30 minutes of pumping.

Solution:

The interval of time from the time the pumping begins until 30 minutes after pumping starts is $0\le t\le 30$, and that will be the bounds for the integral of $w(t)$. Because the tank starts with $5{,}000$ gallons, we need to add $5{,}000$ to the value of the integral. So the gallons in the tank is given by

$$5{,}000 - \int_0^{30} 80 - 4\sqrt{t+3}\ dt$$

$$5{,}000 - \left(80t - \frac{8}{3}(t+3)^{\frac{3}{2}}\right)\Bigg|_0^{30}$$

$$= 5{,}000 - \left(80(30) - \frac{8}{3}(30+3)^{\frac{3}{2}}\right) + \left(80(0) - \frac{8}{3}(0+3)^{\frac{3}{2}}\right)$$

$$5{,}000 - 2{,}400 + \frac{8}{3}(33)^{\frac{3}{2}} - \frac{8}{3}(3)^{\frac{3}{2}}$$

$$2{,}600 + \frac{264\sqrt{33}}{3} - \frac{24\sqrt{3}}{3}$$

$$\frac{7{,}800 + 264\sqrt{33} - 24\sqrt{3}}{3}$$

■ 3. From 1990 to 2010, the rate of rice consumption in a particular country was $R(t) = 5.8 + 1.07^t$ million pounds per year, with t being years since the beginning of the year 1990. The country had 7.2 million pounds of rice on hand at the beginning of 1994 and produced 7.5 million pounds of rice every year. Use the Net Change Theorem to determine how many millions of pounds of rice were on hand in that country at the end of 1998.

Solution:

Since 1990 is the beginning of the time frame, that's when $t=0$. Therefore, the interval of time from 1994 to 1998 is $4 \le t \le 8$, and that will be the bounds for the integral of $R(t)$. Because the country starts with 7.2 million pounds of rice, we need to add 7.2 to the value of the integral. Because the country produces 7.5 million pounds of rice per year, but consumes $R(t)$ pounds per year, the amount of rice at the end of 1998 is

$$7.2+\int_4^8 7.5-(5.8+1.07^t)\ dt$$

$$7.2+\int_4^8 1.7-1.07^t\ dt$$

$$7.2+\left(1.7t-\frac{1.07^t}{\ln 1.07}\right)\Bigg|_4^8$$

$$7.2+\left(1.7(8)-\frac{1.07^8}{\ln 1.07}\right)-\left(1.7(4)-\frac{1.07^4}{\ln 1.07}\right)$$

$$7.2-11.794923+12.573665$$

$$7.978742$$

The country has slightly less than 8 million pounds of rice on hand at the end of 1998.

■ 4. A cooling pump connected to a power plant operates at a varying rate, depending on how much cooling is needed by the power plant. The rate (in gallons per second) at which the pump is operated is modeled by

$r(t) = 0.003t^3 - 0.02t^2 + 0.29t + 59.81$, where t is defined in seconds for $0 \le t \le 120$. The pump has already pumped $1{,}508$ gallons during the first 25 seconds. Use the Net Change Theorem to determine how many gallons the pump will have pumped after 2 minutes.

Solution:

The interval of time from 25 seconds after the pump starts until 2 minutes after pumping starts is $25 \le t \le 120$ (since 2 minutes is 120 seconds), and that will be the bounds for the integral of $r(t)$. Because the pump had already pumped $1{,}508$ gallons, we need to add $1{,}508$ to the value of the integral. So the gallons pumped after 2 minutes is given by

$$1{,}508 + \int_{25}^{120} 0.003t^3 - 0.02t^2 + 0.29t + 59.81 \ dt$$

$$1{,}508 + \left(\frac{0.003}{4}t^4 - \frac{0.02}{3}t^3 + \frac{0.29}{2}t^2 + 59.81t \right) \Bigg|_{25}^{120}$$

$$1{,}508 + \left(\frac{0.003}{4}(120)^4 - \frac{0.02}{3}(120)^3 + \frac{0.29}{2}(120)^2 + 59.81(120) \right)$$

$$- \left(\frac{0.003}{4}(25)^4 - \frac{0.02}{3}(25)^3 + \frac{0.29}{2}(25)^2 + 59.81(25) \right)$$

$$152{,}998.5229$$

After 2 minutes, the pump has pumped just about $153{,}000$ gallons.

5. A rocket is launched upward from a cliff that's 86 feet above ground level. The velocity of the rocket is modeled by $v(t) = -32t + 88$, in feet per second, where t is seconds after the launch. Use the Net Change Theorem to determine the height in feet of the rocket 2 seconds after it's launched.

Solution:

The interval of time from the time the rocket is launched until 2 seconds later is $0 \le t \le 2$, and that will be the bounds for the integral of $v(t)$. Because the rocket starts at 86 feet above the ground, we need to add 86 to the value of the integral. So the height of the rocket is given by

$$86 + \int_0^2 -32t + 88\ dt$$

$$86 + (-16t^2 + 88t)\Big|_0^2$$

$$86 + (-16(2)^2 + 88(2)) - (-16(0)^2 + 88(0))$$

$$86 + (-64 + 176)$$

$$86 - 64 + 176$$

$$198$$

U-SUBSTITUTION IN DEFINITE INTEGRALS

1. Use u-substitution to evaluate the integral.

$$\int_2^4 8x^3\sqrt{7+x^4}\ dx$$

Solution:

Let

$$u=7+x^4$$

$$du=4x^3\ dx,\text{ so } dx=\frac{du}{4x^3}$$

Substitute.

$$\int_{x=2}^{x=4} 8x^3\sqrt{u}\left(\frac{du}{4x^3}\right)$$

$$\frac{8}{4}\int_{x=2}^{x=4}\sqrt{u}\ du$$

$$2\int_{x=2}^{x=4}u^{\frac{1}{2}}\ du$$

Integrate and back-substitute.

$$2 \cdot \frac{2}{3}u^{\frac{3}{2}}\Bigg|_{x=2}^{x=4}$$

$$\frac{4}{3}u^{\frac{3}{2}}\Bigg|_{x=2}^{x=4}$$

$$\frac{4}{3}(7+x^4)^{\frac{3}{2}}\Bigg|_{2}^{4}$$

Evaluate over the interval.

$$\frac{4}{3}(7+4^4)^{\frac{3}{2}} - \left(\frac{4}{3}(7+2^4)^{\frac{3}{2}}\right)$$

$$\frac{4}{3}(263)^{\frac{3}{2}} - \frac{4}{3}(23)^{\frac{3}{2}}$$

$$\frac{1{,}052}{3}\sqrt{263} - \frac{92}{3}\sqrt{23}$$

$$\frac{1{,}052\sqrt{263} - 92\sqrt{23}}{3}$$

INTEGRATION BY PARTS

1. Use integration by parts to evaluate the integral.

$$\int 9x\sin x\ dx$$

Solution:

Pick

$u=9x$	differentiating	$du=9\ dx$
$dv=\sin x\ dx$	integrating	$v=-\cos x$

Plug into the integration by parts formula.

$$\int 9x\sin x\ dx=(9x)(-\cos x)-\int(-\cos x)(9\ dx)$$

$$\int 9x\sin x\ dx=-9x\cos x+9\int\cos x\ dx$$

$$\int 9x\sin x\ dx=-9x\cos x+9\sin x+C$$

2. Use integration by parts to evaluate the integral.

$$\int 5xe^x\ dx$$

Solution:

Pick

$u = 5x$	differentiating	$du = 5\ dx$
$dv = e^x\ dx$	integrating	$v = e^x$

Plug into the integration by parts formula.

$$\int 5xe^x\ dx = (5x)(e^x) - \int (e^x)(5\ dx)$$

$$\int 5xe^x\ dx = 5xe^x - 5\int e^x\ dx$$

$$\int 5xe^x\ dx = 5xe^x - 5e^x + C$$

You could leave the answer this way, or factor it as

$$\int 5xe^x\ dx = 5e^x(x-1) + C$$

■ 3. Use integration by parts to evaluate the integral.

$$\int 7x \ln x\ dx$$

Solution:

Pick

$$u = \ln x \qquad \text{differentiating} \qquad du = \frac{1}{x}\ dx$$

$$dv = 7x\ dx \qquad \text{integrating} \qquad v = \frac{7}{2}x^2$$

Plug into the integration by parts formula.

$$\int 7x \ln x\ dx = (\ln x)\left(\frac{7}{2}x^2\right) - \int \left(\frac{7}{2}x^2\right)\left(\frac{1}{x}\ dx\right)$$

$$\int 7x \ln x\ dx = \frac{7}{2}x^2 \ln x - \frac{7}{2}\int x\ dx$$

$$\int 7x \ln x\ dx = \frac{7}{2}x^2 \ln x - \frac{7}{2}\left(\frac{1}{2}x^2\right) + C$$

$$\int 7x \ln x\ dx = \frac{7}{2}x^2 \ln x - \frac{7}{4}x^2 + C$$

You could leave the answer this way, or factor it as

$$\int 7x \ln x\ dx = \frac{7}{2}x^2\left(\ln x - \frac{1}{2}\right) + C$$

■ 4. Use integration by parts to evaluate the integral.

$$\int 2x \cos x \ dx$$

Solution:

Pick

$u = 2x$ differentiating $du = 2 \ dx$

$dv = \cos x \ dx$ integrating $v = \sin x$

Plug into the integration by parts formula.

$$\int 2x \cos x \ dx = (2x)(\sin x) - \int (\sin x)(2 \ dx)$$

$$\int 2x \cos x \ dx = 2x \sin x - 2 \int \sin x \ dx$$

$$\int 2x \cos x \ dx = 2x \sin x - 2(-\cos x) + C$$

$$\int 2x \cos x \ dx = 2x \sin x + 2 \cos x + C$$

■ 5. Use integration by parts to evaluate the integral.

$$\int 3\sqrt{x} \ln x \ dx$$

Solution:

Pick

$$u = \ln x \qquad \text{differentiating} \qquad du = \frac{1}{x}\ dx$$

$$dv = 3\sqrt{x}\ dx \qquad \text{integrating} \qquad v = 3\left(\frac{2}{3}x^{\frac{3}{2}}\right) = 2x^{\frac{3}{2}}$$

Plug into the integration by parts formula.

$$\int 3\sqrt{x}\ln x\ dx = (\ln x)(2x^{\frac{3}{2}}) - \int (2x^{\frac{3}{2}})\left(\frac{1}{x}\ dx\right)$$

$$\int 3\sqrt{x}\ln x\ dx = 2x^{\frac{3}{2}}\ln x - 2\int \frac{x^{\frac{3}{2}}}{x}\ dx$$

$$\int 3\sqrt{x}\ln x\ dx = 2x^{\frac{3}{2}}\ln x - 2\int x^{\frac{1}{2}}\ dx$$

$$\int 3\sqrt{x}\ln x\ dx = 2x^{\frac{3}{2}}\ln x - 2\left(\frac{2}{3}x^{\frac{3}{2}}\right) + C$$

$$\int 3\sqrt{x}\ln x\ dx = 2x^{\frac{3}{2}}\ln x - \frac{4}{3}x^{\frac{3}{2}} + C$$

You could leave the answer this way, or factor it as

$$\int 3\sqrt{x}\ln x\ dx = 2x^{\frac{3}{2}}\left(\ln x - \frac{2}{3}\right) + C$$

INTEGRATION BY PARTS TWO TIMES

1. Apply integration by parts two times to evaluate the integral.

$$\int 3x^2e^x\ dx$$

Solution:

Pick

$u = 3x^2$ differentiating $du = 6x\ dx$

$dv = e^x\ dx$ integrating $v = e^x$

Plug into the integration by parts formula.

$$\int 3x^2e^x\ dx = (3x^2)(e^x) - \int (e^x)(6x\ dx)$$

$$\int 3x^2e^x\ dx = 3x^2e^x - 6\int xe^x\ dx$$

Apply integration by parts again to replace the integral on the right side. Pick

$u = x$ differentiating $du = 1\ dx$

$dv = e^x\ dx$ integrating $v = e^x$

Plug into the integration by parts formula.

$$\int xe^x\ dx = (x)(e^x) - \int (e^x)(1\ dx)$$

$$\int xe^x\ dx = xe^x - \int e^x\ dx$$

The integral on the right is now simple enough to evaluate directly.

$$\int xe^x\ dx = xe^x - e^x + C$$

Take the right side of this equation, and plug it into the equation from earlier.

$$\int 3x^2e^x\ dx = 3x^2e^x - 6\int xe^x\ dx$$

$$\int 3x^2e^x\ dx = 3x^2e^x - 6\left(xe^x - e^x + C\right)$$

$$\int 3x^2e^x\ dx = 3x^2e^x - 6xe^x + 6e^x - 6C$$

If C is a constant, then $-6C$ is also a constant, so we can simplify.

$$\int 3x^2e^x\ dx = 3x^2e^x - 6xe^x + 6e^x + C$$

You could leave the answer this way, or factor it as

$$\int 3x^2e^x\ dx = 3e^x(x^2 - 2x + 2) + C$$

2. Use integration by parts to evaluate the integral.

$$\int e^{3x}\cos(5x)\ dx$$

Solution:

First, break down the given integral into suitable expressions for u and dv as follows:

$$u = \cos(5x)$$

$$dv = e^{3x}\ dx$$

Differentiating u and integrating dv, we get

$$du = -5\sin(5x)\ dx$$

$$v = \int e^{3x}\ dx = \frac{1}{3}e^{3x}$$

Plug the values into the formula for integration by parts.

$$\int u\ dv = uv - \int v\ du$$

$$\int e^{3x}\cos(5x)\ dx = \frac{1}{3}e^{3x}\cos(5x) - \int \frac{1}{3}e^{3x}\left[-5\sin(5x)\right]\ dx$$

$$\int e^{3x}\cos(5x)\ dx = \frac{1}{3}e^{3x}\cos(5x) + \frac{5}{3}\int e^{3x}\sin(5x)\ dx$$

Notice that the resulting integral on the right side of the equal sign is still not readily integrable. We again use integration by parts and define a new set of u and dv.

$$u = \sin(5x)$$

$$dv = e^{3x}\ dx$$

and

$$du = 5\cos(5x)\ dx$$

$$v = \int e^{3x}\ dx = \frac{1}{3}e^{3x}$$

Replacing the integral on the right with the integration by parts formula and the new values we found, we get

$$\int e^{3x}\cos(5x)\ dx = \frac{1}{3}e^{3x}\cos(5x) + \frac{5}{3}\left[uv - \int v\ du\right]$$

$$\int e^{3x}\cos(5x)\ dx = \frac{1}{3}e^{3x}\cos(5x) + \frac{5}{3}\left[(\sin(5x))\left(\frac{1}{3}e^{3x}\right) - \int \frac{1}{3}e^{3x}(5\cos(5x)\ dx)\right]$$

$$\int e^{3x}\cos(5x)\ dx = \frac{1}{3}e^{3x}\cos(5x) + \frac{5}{3}\left[\frac{1}{3}e^{3x}\sin(5x) - \frac{5}{3}\int e^{3x}\cos(5x)\ dx\right]$$

$$\int e^{3x}\cos(5x)\ dx = \frac{1}{3}e^{3x}\cos(5x) + \frac{5}{9}e^{3x}\sin(5x) - \frac{25}{9}\int e^{3x}\cos(5x)\ dx$$

Notice that the resulting integral on the right side of the equal sign is exactly the same as the given integral. So we can use a little algebra and move it to the left-hand side to combine it with the given integral.

$$\int e^{3x}\cos(5x)\ dx+\frac{25}{9}\int e^{3x}\cos(5x)\ dx=\frac{1}{3}e^{3x}\cos(5x)+\frac{5}{9}e^{3x}\sin(5x)+C$$

$$\frac{9}{9}\int e^{3x}\cos(5x)\ dx+\frac{25}{9}\int e^{3x}\cos(5x)\ dx=\frac{1}{3}e^{3x}\cos(5x)+\frac{5}{9}e^{3x}\sin(5x)+C$$

$$\frac{34}{9}\int e^{3x}\cos(5x)\ dx=\frac{1}{3}e^{3x}\cos(5x)+\frac{5}{9}e^{3x}\sin(5x)+C$$

Now multiply both sides by $9/34$ to solve for the given integral, keeping in mind that the $9/34$ gets absorbed into the constant C.

$$\int e^{3x}\cos(5x)\ dx=\frac{9}{34}\left[\frac{1}{3}e^{3x}\cos(5x)+\frac{5}{9}e^{3x}\sin(5x)+C\right]$$

$$\int e^{3x}\cos(5x)\ dx=\frac{3}{34}e^{3x}\cos(5x)+\frac{5}{34}e^{3x}\sin(5x)+C$$

$$\int e^{3x}\cos(5x)\ dx=\frac{1}{34}e^{3x}\left[3\cos(5x)+5\sin(5x)\right]+C$$

INTEGRATION BY PARTS THREE TIMES

1. Apply integration by parts three times to evaluate the integral.

$$\int 7x^3e^x\ dx$$

Solution:

Pick

$u=7x^3$	differentiating	$du=21x^2\ dx$
$dv=e^x\ dx$	integrating	$v=e^x$

Plug into the integration by parts formula.

$$\int 7x^3e^x\ dx=(7x^3)(e^x)-\int (e^x)(21x^2\ dx)$$

$$\int 7x^3e^x\ dx=7x^3e^x-21\int x^2e^x\ dx$$

Apply integration by parts again to replace the integral on the right side. Pick

$u=x^2$	differentiating	$du=2x\ dx$
$dv=e^x\ dx$	integrating	$v=e^x$

Plug into the integration by parts formula.

$$\int x^2e^x\ dx = (x^2)(e^x) - \int (e^x)(2x\ dx)$$

$$\int x^2e^x\ dx = x^2e^x - 2\int xe^x\ dx$$

Apply integration by parts again to replace the integral on the right side. Pick

$u = x$	differentiating	$du = 1\ dx$
$dv = e^x\ dx$	integrating	$v = e^x$

Plug into the integration by parts formula.

$$\int xe^x\ dx = (x)(e^x) - \int (e^x)(1\ dx)$$

$$\int xe^x\ dx = xe^x - \int e^x\ dx$$

The integral on the right is now simple enough to evaluate directly.

$$\int xe^x\ dx = xe^x - e^x + C$$

Take the right side of this equation, and plug it into the equation from earlier.

$$\int x^2e^x\ dx = x^2e^x - 2\int xe^x\ dx$$

$$\int x^2e^x\ dx = x^2e^x - 2\left(xe^x - e^x + C\right)$$

$$\int x^2e^x\ dx = x^2e^x - 2xe^x + 2e^x - 2C$$

If C is a constant, then $-2C$ is also a constant, so we can simplify.

$$\int x^2e^x\ dx = x^2e^x - 2xe^x + 2e^x + C$$

Take the right side of this equation, and plug it into the equation from earlier.

$$\int 7x^3e^x\ dx = 7x^3e^x - 21\int x^2e^x\ dx$$

$$\int 7x^3e^x\ dx = 7x^3e^x - 21\left(x^2e^x - 2xe^x + 2e^x + C\right)$$

$$\int 7x^3e^x\ dx = 7x^3e^x - 21x^2e^x + 42xe^x - 42e^x - 21C$$

If C is a constant, then $-21C$ is also a constant, so we can simplify.

$$\int 7x^3e^x\ dx = 7x^3e^x - 21x^2e^x + 42xe^x - 42e^x + C$$

You could leave the answer this way, or factor it as

$$\int 7x^3e^x\ dx = 7e^x\left(x^3 - 3x^2 + 6x - 6\right) + C$$

■ 2. Apply integration by parts three times to evaluate the integral.

$$\int (2x^3 + x^2) e^x \ dx$$

Solution:

Pick

$u = 2x^3 + x^2$ differentiating $du = 6x^2 + 2x \ dx$

$dv = e^x \ dx$ integrating $v = e^x$

Plug into the integration by parts formula.

$$\int (2x^3 + x^2) e^x \ dx = (2x^3 + x^2)(e^x) - \int (e^x)(6x^2 + 2x \ dx)$$

$$\int (2x^3 + x^2) e^x \ dx = 2x^3e^x + x^2e^x - \int (6x^2 + 2x)(e^x) \ dx$$

$$\int (2x^3 + x^2) e^x \ dx = 2x^3e^x + x^2e^x - 2\int (3x^2 + x)(e^x) \ dx$$

Apply integration by parts again to replace the integral on the right side. Pick

$u = 3x^2 + x$ differentiating $du = 6x + 1 \ dx$

$dv = e^x \ dx$ integrating $v = e^x$

Plug into the integration by parts formula.

$$\int (3x^2 + x)(e^x) \ dx = (3x^2 + x)(e^x) - \int (e^x)(6x + 1 \ dx)$$

$$\int (3x^2 + x)(e^x)\ dx = 3x^2e^x + xe^x - \int (6x+1)(e^x)\ dx$$

Apply integration by parts again to replace the integral on the right side. Pick

$u = 6x + 1$	differentiating	$du = 6\ dx$
$dv = e^x\ dx$	integrating	$v = e^x$

Plug into the integration by parts formula.

$$\int (6x+1)(e^x)\ dx = (6x+1)(e^x) - \int (e^x)(6\ dx)$$

$$\int (6x+1)(e^x)\ dx = 6xe^x + e^x - 6\int e^x\ dx$$

The integral on the right is now simple enough to evaluate directly.

$$\int (6x+1)(e^x)\ dx = 6xe^x + e^x - 6e^x + C$$

Take the right side of this equation, and plug it into the equation from earlier.

$$\int (3x^2 + x)(e^x)\ dx = 3x^2e^x + xe^x - \int (6x+1)(e^x)\ dx$$

$$\int (3x^2 + x)(e^x)\ dx = 3x^2e^x + xe^x - \left(6xe^x + e^x - 6e^x + C\right)$$

$$\int (3x^2 + x)(e^x)\ dx = 3x^2e^x + xe^x - 6xe^x - e^x + 6e^x - C$$

$$\int (3x^2+x)(e^x)\ dx = 3x^2e^x - 5xe^x + 5e^x - C$$

If C is a constant, then $-C$ is also a constant, so we can simplify.

$$\int (3x^2+x)(e^x)\ dx = 3x^2e^x - 5xe^x + 5e^x + C$$

Take the right side of this equation, and plug it into the equation from earlier.

$$\int \left(2x^3+x^2\right) e^x\ dx = 2x^3e^x + x^2e^x - 2\int (3x^2+x)(e^x)\ dx$$

$$\int \left(2x^3+x^2\right) e^x\ dx = 2x^3e^x + x^2e^x - 2\left(3x^2e^x - 5xe^x + 5e^x + C\right)$$

$$\int \left(2x^3+x^2\right) e^x\ dx = 2x^3e^x + x^2e^x - 6x^2e^x + 10xe^x - 10e^x - 2C$$

$$\int \left(2x^3+x^2\right) e^x\ dx = 2x^3e^x - 5x^2e^x + 10xe^x - 10e^x - 2C$$

If C is a constant, then $-2C$ is also a constant, so we can simplify.

$$\int \left(2x^3+x^2\right) e^x\ dx = 2x^3e^x - 5x^2e^x + 10xe^x - 10e^x + C$$

You could leave the answer this way, or factor it as

$$\int \left(2x^3+x^2\right) e^x\ dx = e^x\left(2x^3 - 5x^2 + 10x - 10\right) + C$$

3. Use integration by parts three times to evaluate the integral.

$$\int (\ln x)^3 \ dx$$

Solution:

Pick

$u = (\ln x)^3$ differentiating $du = 3(\ln x)^2 \left(\frac{1}{x}\right) \ dx$

$dv = dx$ integrating $v = x$

Plug into the integration by parts formula.

$$\int (\ln x)^3 \ dx = ((\ln x)^3)(x) - \int (x)\left(3(\ln x)^2 \left(\frac{1}{x}\right) \ dx\right)$$

$$\int (\ln x)^3 \ dx = x(\ln x)^3 - 3\int (\ln x)^2 \ dx$$

Apply integration by parts again to replace the integral on the right side.

Pick

$u = (\ln x)^2$ differentiating $du = 2(\ln x)\left(\frac{1}{x}\right) \ dx$

$dv = dx$ integrating $v = x$

Plug into the integration by parts formula.

$$\int (\ln x)^2 \ dx = ((\ln x)^2)(x) - \int (x)\left(2(\ln x)\left(\frac{1}{x}\right) \ dx \right)$$

$$\int (\ln x)^2 \ dx = x(\ln x)^2 - 2\int \ln x \ dx$$

Apply integration by parts again to replace the integral on the right side. Pick

$u = \ln x$	differentiating	$du = \frac{1}{x} \ dx$
$dv = dx$	integrating	$v = x$

Plug into the integration by parts formula.

$$\int \ln x \ dx = (\ln x)(x) - \int (x)\left(\frac{1}{x} \ dx\right)$$

$$\int \ln x \ dx = x \ln x - \int \ dx$$

The integral on the right is now simple enough to evaluate directly.

$$\int \ln x \ dx = x \ln x - x + C$$

Take the right side of this equation, and plug it into the equation from earlier.

$$\int (\ln x)^2 \ dx = x(\ln x)^2 - 2\int \ln x \ dx$$

$$\int (\ln x)^2\, dx = x(\ln x)^2 - 2\left(x \ln x - x + C\right)$$

$$\int (\ln x)^2\, dx = x(\ln x)^2 - 2x \ln x + 2x - 2C$$

If C is a constant, then $-2C$ is also a constant, so we can simplify.

$$\int (\ln x)^2\, dx = x(\ln x)^2 - 2x \ln x + 2x + C$$

Take the right side of this equation, and plug it into the equation from earlier.

$$\int (\ln x)^3\, dx = x(\ln x)^3 - 3\int (\ln x)^2\, dx$$

$$\int (\ln x)^3\, dx = x(\ln x)^3 - 3\left(x(\ln x)^2 - 2x \ln x + 2x + C\right)$$

$$\int (\ln x)^3\, dx = x(\ln x)^3 - 3x(\ln x)^2 + 6x \ln x - 6x - 3C$$

If C is a constant, then $-3C$ is also a constant, so we can simplify.

$$\int (\ln x)^3\, dx = x(\ln x)^3 - 3x(\ln x)^2 + 6x \ln x - 6x + C$$

You could leave the answer this way, or factor it as

$$\int (\ln x)^3\, dx = x\left[(\ln x)^3 - 3(\ln x)^2 + 6 \ln x - 6\right] + C$$

INTEGRATION BY PARTS WITH U-SUBSTITUTION

■ 1. Use integration by parts and substitution to evaluate the integral.

$$\int \tan^{-1} x \ dx$$

Solution:

Use integration by parts first. Pick

$u = \tan^{-1} x$ differentiating $du = \frac{1}{x^2+1} \ dx$

$dv = dx$ integrating $v = x$

Plug into the integration by parts formula.

$$\int \tan^{-1} x \ dx = (\tan^{-1} x)(x) - \int (x)\left(\frac{1}{x^2+1} \ dx\right)$$

$$\int \tan^{-1} x \ dx = x \tan^{-1} x - \int \frac{x}{x^2+1} \ dx$$

Use substitution to evaluate the integral that remains. Let

$$k = x^2 + 1$$

$dk = 2x \ dx$ so $dx = \dfrac{dk}{2x}$

Substitute into the integral on the right.

$$\int \tan^{-1} x \ dx = x \tan^{-1} x - \int \frac{x}{x^2+1} \ dx$$

$$\int \tan^{-1} x \ dx = x \tan^{-1} x - \int \frac{x}{k} \left(\frac{dk}{2x} \right)$$

$$\int \tan^{-1} x \ dx = x \tan^{-1} x - \frac{1}{2} \int \frac{1}{k} \ dk$$

Integrate, then back-substitute.

$$\int \tan^{-1} x \ dx = x \tan^{-1} x - \frac{1}{2} \ln |k| + C$$

$$\int \tan^{-1} x \ dx = x \tan^{-1} x - \frac{1}{2} \ln |x^2+1| + C$$

$$\int \tan^{-1} x \ dx = x \tan^{-1} x - \frac{1}{2} \ln (x^2+1) + C$$

■ 2. Use integration by parts and substitution to evaluate the integral.

$$\int 7x \cos(9x) \ dx$$

Solution:

Use integration by parts first. Pick

$u = 7x$ differentiating $du = 7\ dx$

$dv = \cos(9x)\ dx$ integrating $v = \frac{1}{9}\sin(9x)$

Plug into the integration by parts formula.

$$\int 7x\cos(9x)\ dx = (7x)\left(\frac{1}{9}\sin(9x)\right) - \int\left(\frac{1}{9}\sin(9x)\right)(7\ dx)$$

$$\int 7x\cos(9x)\ dx = \frac{7}{9}x\sin(9x) - \frac{7}{9}\int\sin(9x)\ dx$$

Use substitution to evaluate the integral that remains. Let

$k = 9x$

$dk = 9\ dx$ so $dx = \frac{dk}{9}$

Substitute into the integral on the right.

$$\int 7x\cos(9x)\ dx = \frac{7}{9}x\sin(9x) - \frac{7}{9}\int\sin k\left(\frac{dk}{9}\right)$$

$$\int 7x\cos(9x)\ dx = \frac{7}{9}x\sin(9x) - \frac{7}{81}\int\sin k\ dk$$

Integrate, then back-substitute.

$$\int 7x\cos(9x)\ dx = \frac{7}{9}x\sin(9x) - \frac{7}{81}(-\cos k) + C$$

$$\int 7x\cos(9x)\ dx = \frac{7}{9}x\sin(9x) + \frac{7}{81}\cos k + C$$

$$\int 7x\cos(9x)\ dx = \frac{7}{9}x\sin(9x) + \frac{7}{81}\cos(9x) + C$$

3. Use integration by parts and substitution to evaluate the integral.

$$\int \ln(3x+5)\ dx$$

Solution:

Use substitution first. Let

$$k = 3x + 5$$

$$dk = 3\ dx \text{ so } dx = \frac{dk}{3}$$

Substitute into the integral.

$$\int \ln(3x+5)\ dx$$

$$\int \ln k \left(\frac{dk}{3}\right)$$

$$\frac{1}{3}\int \ln k\ dk$$

Now use integration by parts. Pick

$u = \ln k$ differentiating $du = \frac{1}{k}\ dk$

$dv = dk$ integrating $v = k$

Plug into the integration by parts formula.

$$\int \ln k\ dk = (\ln k)(k) - \int (k)\left(\frac{1}{k}\ dk\right)$$

$$\int \ln k\ dk = k \ln k - \int\ dk$$

Integrate.

$$\int \ln k\ dk = k \ln k - k + C$$

Plug the value from the right side of this equation into the equation from earlier.

$$\frac{1}{3}\int \ln k\ dk$$

$$\frac{1}{3}(k \ln k - k + C)$$

Now back substitute.

$$\frac{1}{3}((3x+5)\ln(3x+5) - (3x+5) + C)$$

$$\frac{1}{3}\left[(3x+5)\ln(3x+5) - (3x+5)\right] + \frac{1}{3}C$$

If C is a constant, then $(1/3)C$ is also a constant, so we can simplify.

$$\frac{1}{3}\left[(3x+5)\ln(3x+5)-(3x+5)\right]+C$$

PROVE THE REDUCTION FORMULA

1. Use integration by parts, and $n=8$, to prove the reduction formula for the integral.

$$\int x^n \sin x \ dx = -x^n \cos x + n \int x^{n-1} \cos x \ dx$$

Solution:

If $n=8$, then

$$\int x^n \sin x \ dx = \int x^8 \sin x \ dx$$

If we're going to apply integration by parts to the integral on the right side of the equation, then pick

$u = x^8$ differentiating $du = 8x^7 \ dx$

$dv = \sin x \ dx$ integrating $v = -\cos x$

Plugging these values into the integration by parts formula gives

$$\int x^8 \sin x \ dx = (x^8)(-\cos x) - \int (-\cos x)(8x^7 \ dx)$$

$$\int x^8 \sin x \ dx = -x^8 \cos x + 8 \int x^7 \cos x \ dx$$

$$\int x^8 \sin x \ dx = -x^8 \cos x + 8\int x^{8-1} \cos x \ dx$$

The format of this equation now matches the format of the reduction formula.

■ 2. Use integration by parts, and $n = 11$, to prove the reduction formula for the integral.

$$\int x^n \cos x \ dx = x^n \sin x - n\int x^{n-1} \sin x \ dx$$

Solution:

If $n = 11$, then

$$\int x^n \cos x \ dx = \int x^{11} \cos x \ dx$$

If we're going to apply integration by parts to the integral on the right side of the equation, then pick

$u = x^{11}$ differentiating $du = 11x^{10} \ dx$

$dv = \cos x \ dx$ integrating $v = \sin x$

Plugging these values into the integration by parts formula gives

$$\int x^{11} \cos x \ dx = (x^{11})(\sin x) - \int (\sin x)(11x^{10} \ dx)$$

$$\int x^{11} \cos x \ dx = x^{11} \sin x - 11 \int x^{10} \sin x \ dx$$

$$\int x^{11} \cos x \ dx = x^{11} \sin x - 11 \int x^{11-1} \sin x \ dx$$

The format of this equation now matches the format of the reduction formula.

■ 3. Use integration by parts, $a = 5$, and $n = 9$, to prove the reduction formula for the integral.

$$\int x^n a^x \ dx = \frac{x^n a^x}{\ln a} - \frac{n}{\ln a} \int x^{n-1} a^x \ dx$$

Solution:

If $a = 5$, and $n = 9$, then

$$\int x^n a^x \ dx = \int x^9 5^x \ dx$$

If we're going to apply integration by parts to the integral on the right side of the equation, then pick

$u = x^9$	differentiating	$du = 9x^8 \ dx$
$dv = 5^x \ dx$	integrating	$v = \dfrac{5^x}{\ln 5}$

Plugging these values into the integration by parts formula gives

$$\int x^9 5^x\ dx = (x^9)\left(\frac{5^x}{\ln 5}\right) - \int \left(\frac{5^x}{\ln 5}\right)(9x^8\ dx)$$

$$\int x^9 5^x\ dx = \frac{x^9 5^x}{\ln 5} - \frac{9}{\ln 5}\int x^8 5^x\ dx$$

$$\int x^9 5^x\ dx = \frac{x^9 5^x}{\ln 5} - \frac{9}{\ln 5}\int x^{9-1} 5^x\ dx$$

The format of this equation now matches the format of the reduction formula.

TABULAR INTEGRATION

1. Use tabular integration to evaluate the integral.

$$\int \left(5x^2 + 4x - 3\right) e^{2x}\ dx$$

Solution:

Let $f(x) = 5x^2 + 4x - 3$ and $g(x) = e^{2x}$.

Derivatives of $f(x)$	**Antiderivatives of $g(x)$**
$5x^2 + 4x - 3$	e^{2x}
$10x + 4$	$\frac{1}{2}e^{2x}$
10	$\frac{1}{4}e^{2x}$
0	$\frac{1}{8}e^{2x}$

Evaluate the integral by multiplying the entry in the first line, first column, by the entry in the second line, second column, beginning with a positive product. Then continue to pattern going down the table using opposite signs. The value of the integral will be

$$(5x^2+4x-3)\left(\frac{e^{2x}}{2}\right)-(10x+4)\left(\frac{e^{2x}}{4}\right)+10\left(\frac{e^{2x}}{8}\right)+C$$

Factor.

$$\frac{e^{2x}}{2}\left[(5x^2+4x-3)-(10x+4)\left(\frac{1}{2}\right)+10\left(\frac{1}{4}\right)\right]+C$$

$$\frac{e^{2x}}{2}\left(5x^2+4x-3-5x-2+\frac{5}{2}\right)+C$$

$$\frac{e^{2x}}{2}\left(5x^2-x-\frac{5}{2}\right)+C$$

■ 2. Use tabular integration to evaluate the integral.

$$\int x^3\cos(3x)\ dx$$

Solution:

Let $f(x)=x^3$ and $g(x)=\cos x$.

Derivatives of $f(x)$	**Antiderivatives of $g(x)$**
x^3	$\cos(3x)$
$3x^2$	$\frac{1}{3}\sin(3x)$

$6x$	$-\frac{1}{9}\cos(3x)$
6	$-\frac{1}{27}\sin(3x)$
0	$\frac{1}{81}\cos(3x)$

Evaluate the integral by multiplying the entry in the first line, first column, by the entry in the second line, second column, beginning with a positive product. Then continue to pattern going down the table using opposite signs. The value of the integral will be

$$\frac{x^3 \sin 3x}{3} + \frac{3x^2 \cos 3x}{9} - \frac{6x \sin 3x}{27} - \frac{6 \cos 3x}{81} + C$$

$$\frac{x^3 \sin 3x}{3} + \frac{x^2 \cos 3x}{3} - \frac{2x \sin 3x}{9} - \frac{2 \cos 3x}{27} + C$$

3. Use tabular integration to evaluate the integral.

$$\int \frac{x^4 e^x}{6}\ dx$$

Solution:

Let $f(x) = x^4$ and $g(x) = e^x/6$.

Derivatives of $f(x)$ **Antiderivatives of $g(x)$**

x^4	$\frac{1}{6}e^x$
$4x^3$	$\frac{1}{6}e^x$
$12x^2$	$\frac{1}{6}e^x$
$24x$	$\frac{1}{6}e^x$
24	$\frac{1}{6}e^x$
0	$\frac{1}{6}e^x$

Evaluate the integral by multiplying the entry in the first line, first column, by the entry in the second line, second column, beginning with a positive product. Then continue to pattern going down the table using opposite signs. The value of the integral will be

$$x^4 \cdot \frac{e^x}{6} - 4x^3 \cdot \frac{e^x}{6} + 12x^2 \cdot \frac{e^x}{6} - 24x \cdot \frac{e^x}{6} + 24 \cdot \frac{e^x}{6} + C$$

$$\frac{e^x}{6}\left(x^4 - 4x^3 + 12x^2 - 24x + 24\right) + C$$

DISTINCT LINEAR FACTORS

1. Use partial fractions to evaluate the integral.

$$\int \frac{4x+5}{x^2+5x+6}\ dx$$

Solution:

Factor the denominator, then do the partial fractions decomposition.

$$\frac{4x+5}{(x+2)(x+3)} = \frac{A}{x+2} + \frac{B}{x+3}$$

$$4x+5 = A(x+3) + B(x+2)$$

$$4x+5 = Ax + 3A + Bx + 2B$$

$$4x+5 = (A+B)x + (3A+2B)$$

Then the system of equations is

$$A + B = 4$$

$$3A + 2B = 5$$

Solve $A + B = 4$ for A.

$$A = 4 - B$$

Substitute $A = 4 - B$ into $3A + 2B = 5$.

$$3(4-B)+2B=5$$

$$12-3B+2B=5$$

$$12-B=5$$

$$12=5+B$$

$$7=B$$

Then plugging this back into $A=4-B$ gives

$$A=4-7$$

$$A=-3$$

Then the integral becomes

$$\int \frac{4x+5}{x^2+5x+6}\ dx$$

$$\int -\frac{3}{x+2}+\frac{7}{x+3}\ dx$$

$$-3\ln|x+2|+7\ln|x+3|+C$$

DISTINCT QUADRATIC FACTORS

1. Use partial fractions to evaluate the integral.

$$\int \frac{3x+6}{(x^2+2)(x^2+1)}\ dx$$

Solution:

Factor the denominator, then do the partial fractions decomposition.

$$\frac{3x+6}{(x^2+2)(x^2+1)} = \frac{Ax+B}{x^2+2} + \frac{Cx+D}{x^2+1}$$

$$3x+6 = (Ax+B)(x^2+1) + (Cx+D)(x^2+2)$$

$$3x+6 = Ax^3 + Ax + Bx^2 + B + Cx^3 + 2Cx + Dx^2 + 2D$$

$$3x+6 = (A+C)x^3 + (B+D)x^2 + (A+2C)x + (B+2D)$$

Then the system of equations is

$$A+C=0$$

$$B+D=0$$

$$A+2C=3$$

$$B+2D=6$$

Solve the system

$$A + C = 0$$

$$A + 2C = 3$$

Solve $A + C = 0$ for A to get $A = -C$. Plug this into $A + 2C = 3$ to get

$$-C + 2C = 3$$

$$C = 3$$

Then $A = -3$. Now solve the system

$$B + D = 0$$

$$B + 2D = 6$$

Solve $B + D = 0$ for B to get $B = -D$. Plug this into $B + 2D = 6$ to get

$$-D + 2D = 6$$

$$D = 6$$

Then $B = -6$. Then the integral becomes

$$\int \frac{3x + 6}{(x^2 + 2)(x^2 + 1)} \ dx$$

$$\int \frac{-3x - 6}{x^2 + 2} + \frac{3x + 6}{x^2 + 1} \ dx$$

$$\int -\frac{3x}{x^2 + 2} - \frac{6}{x^2 + 2} + \frac{3x}{x^2 + 1} + \frac{6}{x^2 + 1} \ dx$$

$$-\int \frac{3x}{x^2 + 2} \ dx - \int \frac{6}{x^2 + 2} \ dx + \int \frac{3x}{x^2 + 1} \ dx + \int \frac{6}{x^2 + 1} \ dx$$

Use u-substitution.

$$u = x^2 + 2$$

$$\frac{du}{dx} = 2x, \text{ so } du = 2x\ dx, \text{ so } dx = \frac{du}{2x}$$

and

$$u = x^2 + 1$$

$$\frac{du}{dx} = 2x, \text{ so } du = 2x\ dx, \text{ so } dx = \frac{du}{2x}$$

Substituting into the integral gives

$$-\int \frac{3x}{u}\left(\frac{du}{2x}\right) - \int \frac{6}{x^2+2}\ dx + \int \frac{3x}{u}\left(\frac{du}{2x}\right) + \int \frac{6}{x^2+1}\ dx$$

$$-\frac{3}{2}\int \frac{1}{u}\ du - \int \frac{6}{x^2+2}\ dx + \frac{3}{2}\int \frac{1}{u}\ du + \int \frac{6}{x^2+1}\ dx$$

$$-\frac{3}{2}\ln u - \int \frac{6}{x^2+2}\ dx + \frac{3}{2}\ln u + \int \frac{6}{x^2+1}\ dx$$

$$-\frac{3}{2}\ln(x^2+2) - \int \frac{6}{x^2+2}\ dx + \frac{3}{2}\ln(x^2+1) + \int \frac{6}{x^2+1}\ dx$$

Rewrite the integral.

$$-\frac{3}{2}\ln(x^2+2) - \int \frac{3}{\frac{x^2}{2}+1}\ dx + \frac{3}{2}\ln(x^2+1) + \int \frac{6}{x^2+1}\ dx$$

$$-\frac{3}{2}\ln(x^2+2)-3\int\frac{1}{\frac{x^2}{2}+1}\ dx+\frac{3}{2}\ln(x^2+1)+\int\frac{6}{x^2+1}\ dx$$

$$-\frac{3}{2}\ln(x^2+2)-3\int\frac{1}{\left(\frac{x}{\sqrt{2}}\right)^2+1}\ dx+\frac{3}{2}\ln(x^2+1)+6\int\frac{1}{x^2+1}\ dx$$

Use inverse tangent rules to integrate.

$$-\frac{3}{2}\ln(x^2+2)-3\sqrt{2}\arctan\frac{x}{\sqrt{2}}+\frac{3}{2}\ln(x^2+1)+6\arctan x+C$$

REPEATED LINEAR FACTORS

1. Use partial fractions to evaluate the integral.

$$\int \frac{5x-3}{(x+2)^2}\,dx$$

Solution:

Factor the denominator, then do the partial fractions decomposition.

$$\frac{5x-3}{(x+2)(x+2)} = \frac{A}{(x+2)^2} + \frac{B}{x+2}$$

$$5x - 3 = A + B(x+2)$$

$$5x - 3 = A + Bx + 2B$$

$$5x - 3 = (B)x + (A + 2B)$$

Then the system of equations is

$$B = 5$$

$$-3 = A + 2B$$

Substitute $B = 5$ into $-3 = A + 2B$.

$$-3 = A + 2(5)$$

$$-3 = A + 10$$

$$-13 = A$$

Then the integral becomes

$$\int \frac{5x-3}{(x+2)(x+2)}\ dx$$

$$\int \frac{-13}{(x+2)^2} + \frac{5}{x+2}\ dx$$

$$-13\int \frac{1}{(x+2)^2}\ dx + 5\int \frac{1}{x+2}\ dx$$

$$-13\int (x+2)^{-2}\ dx + 5\int \frac{1}{x+2}\ dx$$

Integrate.

$$13(x+2)^{-1} + 5\ln|x+2| + C$$

$$\frac{13}{x+2} + 5\ln|x+2| + C$$

■ 2. Use partial fractions to evaluate the integral.

$$\int \frac{x+12}{(3x-2)^2}\ dx$$

Solution:

Factor the denominator, then do the partial fractions decomposition.

$$\frac{x+12}{(3x-2)(3x-2)} = \frac{A}{(3x-2)^2} + \frac{B}{3x-2}$$

$$x + 12 = A + B(3x - 2)$$

$$x + 12 = A + 3Bx - 2B$$

$$x + 12 = (3B)x + (A - 2B)$$

Then the system of equations is

$$3B = 1$$

$$A - 2B = 12$$

Then $B = 1/3$, and

$$A - 2 \cdot \frac{1}{3} = 12$$

$$A - \frac{2}{3} = 12$$

$$A = \frac{36}{3} + \frac{2}{3}$$

$$A = \frac{38}{3}$$

Then the integral becomes

$$\int \frac{x+12}{(3x-2)(3x-2)}\ dx$$

$$\int \frac{\frac{38}{3}}{(3x-2)^2} + \frac{\frac{1}{3}}{3x-2}\ dx$$

$$\frac{38}{3}\int \frac{1}{(3x-2)^2}\ dx + \frac{1}{3}\int \frac{1}{3x-2}\ dx$$

$$\frac{38}{3}\int (3x-2)^{-2}\ dx + \frac{1}{3}\int \frac{1}{3x-2}\ dx$$

Integrate.

$$-\frac{38}{9}(3x-2)^{-1} + \frac{1}{9}\ln|3x-2| + C$$

$$-\frac{38}{9(3x-2)} + \frac{1}{9}\ln|3x-2| + C$$

■ 3. Use partial fractions to evaluate the integral.

$$\int \frac{7x-4}{(5x+1)^2}\ dx$$

Solution:

Factor the denominator, then do the partial fractions decomposition.

$$\frac{7x-4}{(5x+1)(5x+1)} = \frac{A}{(5x+1)^2} + \frac{B}{5x+1}$$

$$7x-4 = A + B(5x+1)$$

$$7x - 4 = A + 5Bx + B$$

$$7x - 4 = (5B)x + (A + B)$$

Then the system of equations is

$$5B = 7$$

$$A + B = -4$$

Then $B = 7/5$, and we can substitute $B = 7/5$ into $A + B = -4$

$$A + \frac{7}{5} = -4$$

$$A = -\frac{20}{5} - \frac{7}{5}$$

$$A = -\frac{27}{5}$$

Then the integral becomes

$$\int \frac{7x - 4}{(5x + 1)(5x + 1)}\ dx$$

$$\int \frac{-\frac{27}{5}}{(5x + 1)^2} + \frac{\frac{7}{5}}{5x + 1}\ dx$$

$$-\frac{27}{5}\int \frac{1}{(5x + 1)^2}\ dx + \frac{7}{5}\int \frac{1}{5x + 1}\ dx$$

$$-\frac{27}{5}\int (5x + 1)^{-2}\ dx + \frac{7}{5}\int \frac{1}{5x + 1}\ dx$$

Integrate.

$$\frac{27}{25}(5x+1)^{-1}+\frac{7}{25}\ln|5x+1|+C$$

$$\frac{27}{25(5x+1)}+\frac{7}{25}\ln|5x+1|+C$$

■ 4. Use partial fractions to evaluate the integral.

$$\int \frac{12x+9}{(2x+7)^2}\ dx$$

Solution:

Factor the denominator, then do the partial fractions decomposition.

$$\frac{12x+9}{(2x+7)(2x+7)}=\frac{A}{(2x+7)^2}+\frac{B}{2x+7}$$

$$12x+9=A+B(2x+7)$$

$$12x+9=A+2Bx+7B$$

$$12x+9=(2B)x+(A+7B)$$

Then the system of equations is

$$2B=12$$

$$A+7B=9$$

Then $B = 6$, and we can substitute $B = 6$ into $A + 7B = 9$.

$$A + 7(6) = 9$$

$$A + 42 = 9$$

$$A = -33$$

Then the integral becomes

$$\int \frac{12x + 9}{(2x + 7)(2x + 7)}\ dx$$

$$\int \frac{-33}{(2x + 7)^2} + \frac{6}{2x + 7}\ dx$$

$$-33\int \frac{1}{(2x + 7)^2}\ dx + 6\int \frac{1}{2x + 7}\ dx$$

$$-33\int (2x + 7)^{-2}\ dx + 6\int \frac{1}{2x + 7}\ dx$$

Integrate.

$$\frac{33}{2}(2x + 7)^{-1} + 3\ln|2x + 7| + C$$

$$\frac{33}{2(2x + 7)} + 3\ln|2x + 7| + C$$

5. Use partial fractions to evaluate the integral.

$$\int \frac{24x+41}{(3x+4)^2}\ dx$$

Solution:

Factor the denominator, then do the partial fractions decomposition.

$$\frac{24x+41}{(3x+4)(3x+4)} = \frac{A}{(3x+4)^2} + \frac{B}{3x+4}$$

$$24x+41 = A + B(3x+4)$$

$$24x+41 = A + 3Bx + 4B$$

$$24x+41 = (3B)x + (A+4B)$$

Then the system of equations is

$$3B = 24$$

$$A + 4B = 41$$

Then $B = 8$ and we can substitute $B = 8$ into $A + 4B = 41$.

$$A + 4(8) = 41$$

$$A + 32 = 41$$

$$A = 9$$

Then the integral becomes

$$\int \frac{24x+41}{(3x+4)(3x+4)}\ dx$$

$$\int \frac{9}{(3x+4)^2} + \frac{8}{3x+4}\ dx$$

$$9\int (3x+4)^{-2}\ dx + 8\int \frac{1}{3x+4}\ dx$$

Integrate.

$$-3(3x+4)^{-1} + \frac{8}{3}\ln|3x+4| + C$$

$$-\frac{3}{3x+4} + \frac{8}{3}\ln|3x+4| + C$$

REPEATED QUADRATIC FACTORS

1. Rewrite the integral using partial fractions, but do not evaluate it.

$$\int \frac{x^2-3x+2}{(x^2+2)^2}\ dx$$

Solution:

Factor the denominator, then do the partial fractions decomposition.

$$\frac{x^2-3x+2}{(x^2+2)(x^2+2)}=\frac{Ax+B}{(x^2+2)^2}+\frac{Cx+D}{x^2+2}$$

$$x^2-3x+2=Ax+B+(Cx+D)(x^2+2)$$

$$x^2-3x+2=Ax+B+Cx^3+2Cx+Dx^2+2D$$

$$x^2-3x+2=(C)x^3+(D)x^2+(A+2C)x+(B+2D)$$

Then the system of equations is

$$C=0$$

$$D=1$$

$$A+2C=-3$$

$$B+2D=2$$

Substituting $C=0$ into $A+2C=-3$ gives

$$A+2(0)=-3$$

$$A=-3$$

Substituting $D=1$ into $B+2D=2$ gives

$$B+2(1)=2$$

$$B+2=2$$

$$B=0$$

Then the integral becomes

$$\int \frac{x^2-3x+2}{(x^2+2)(x^2+2)}\ dx$$

$$\int \frac{Ax+B}{(x^2+2)^2}+\frac{Cx+D}{x^2+2}\ dx$$

$$\int \frac{-3x+0}{(x^2+2)^2}+\frac{0x+1}{x^2+2}\ dx$$

$$-3\int \frac{x}{(x^2+2)^2}\ dx+\int \frac{1}{x^2+2}\ dx$$

■ 2. Rewrite the integral using partial fractions, but do not evaluate it.

$$\int \frac{x^2-4x+6}{(x^2+3)^2}\ dx$$

Solution:

Factor the denominator, then do the partial fractions decomposition.

$$\frac{x^2-4x+6}{(x^2+3)(x^2+3)}=\frac{Ax+B}{(x^2+3)^2}+\frac{Cx+D}{x^2+3}$$

$$x^2-4x+6=Ax+B+(Cx+D)(x^2+3)$$

$$x^2-4x+6=Ax+B+Cx^3+3Cx+Dx^2+3D$$

$$x^2-4x+6=(C)x^3+(D)x^2+(A+3C)x+(B+3D)$$

Then the system of equations is

$$C=0$$

$$D=1$$

$$A+3C=-4$$

$$B+3D=6$$

Substituting $C=0$ into $A+3C=-4$ gives

$$A+3(0)=-4$$

$$A=-4$$

Substituting $D=1$ into $B+3D=6$ gives

$$B+3(1)=6$$

$$B + 3 = 6$$

$$B = 3$$

Then the integral becomes

$$\int \frac{x^2 - 4x + 6}{(x^2 + 3)(x^2 + 3)} \ dx$$

$$\int \frac{-4x + 3}{(x^2 + 3)^2} + \frac{0x + 1}{x^2 + 3} \ dx$$

$$-4 \int \frac{x}{(x^2 + 3)^2} \ dx + 3 \int \frac{1}{(x^2 + 3)^2} \ dx + \int \frac{1}{x^2 + 3} \ dx$$

■ 3. Rewrite the integral using partial fractions, but do not evaluate it.

$$\int \frac{4x^3 - 2x^2 + x + 1}{(2x^2 + 1)^2} \ dx$$

Solution:

Factor the denominator, then do the partial fractions decomposition.

$$\frac{4x^3 - 2x^2 + x + 1}{(2x^2 + 1)(2x^2 + 1)} = \frac{Ax + B}{(2x^2 + 1)^2} + \frac{Cx + D}{2x^2 + 1}$$

$$4x^3 - 2x^2 + x + 1 = Ax + B + (Cx + D)(2x^2 + 1)$$

$$4x^3 - 2x^2 + x + 1 = Ax + B + 2Cx^3 + Cx + 2Dx^2 + D$$

$$4x^3 - 2x^2 + x + 1 = (2C)x^3 + (2D)x^2 + (A + C)x + (B + D)$$

Then the system of equations is

$$2C = 4$$

$$2D = -2$$

$$A + C = 1$$

$$B + D = 1$$

Then $C = 2$ and $D = -1$. Substitute $C = 2$ into $A + C = 1$.

$$A + 2 = 1$$

$$A = -1$$

Substitute $D = -1$ into $B + D = 1$.

$$B - 1 = 1$$

$$B = 2$$

Then the integral becomes

$$\int \frac{4x^3 - 2x^2 + x + 1}{(2x^2 + 1)(2x^2 + 1)} \ dx$$

$$\int \frac{-1x + 2}{(2x^2 + 1)^2} + \frac{2x - 1}{2x^2 + 1} \ dx$$

$$-\int \frac{x}{(2x^2 + 1)^2} \ dx + 2\int \frac{1}{(2x^2 + 1)^2} \ dx + 2\int \frac{x}{2x^2 + 1} \ dx - \int \frac{1}{2x^2 + 1} \ dx$$

4. Rewrite the integral using partial fractions, but do not evaluate it.

$$\int \frac{x^3 - 2x^2 + 3x + 5}{(x^2 + 1)^3}\ dx$$

Solution:

Factor the denominator, then do the partial fractions decomposition.

$$\frac{x^3 - 2x^2 + 3x + 5}{(x^2 + 1)(x^2 + 1)(x^2 + 1)} = \frac{Ax + B}{(x^2 + 1)^3} + \frac{Cx + D}{(x^2 + 1)^2} + \frac{Ex + F}{x^2 + 1}$$

$$x^3 - 2x^2 + 3x + 5 = Ax + B + (Cx + D)(x^2 + 1) + (Ex + F)(x^2 + 1)^2$$

$$x^3 - 2x^2 + 3x + 5 = Ax + B + Cx^3 + Cx + Dx^2 + D + (Ex + F)(x^4 + 2x^2 + 1)$$

$$x^3 - 2x^2 + 3x + 5 = Ax + B + Cx^3 + Cx + Dx^2 + D$$

$$+Ex^5 + 2Ex^3 + Ex + Fx^4 + 2Fx^2 + F$$

$$x^3 - 2x^2 + 3x + 5 = (E)x^5 + (F)x^4 + (C + 2E)x^3 + (D + 2F)x^2$$

$$+(A + C + E)x + (B + D + F)$$

Then the system of equations is

$$E = 0$$

$$F = 0$$

$$C + 2E = 1$$

$$D + 2F = -2$$

$$A + C + E = 3$$

$$B + D + F = 5$$

Substitute $E = 0$ into $C + 2E = 1$.

$$C + 2(0) = 1$$

$$C = 1$$

Substitute $F = 0$ into $D + 2F = -2$.

$$D + 2(0) = -2$$

$$D = -2$$

Substitute $C = 1$ and $E = 0$ into $A + C + E = 3$.

$$A + 1 + 0 = 3$$

$$A = 2$$

Substitute $D = -2$ and $F = 0$ into $B + D + F = 5$.

$$B - 2 + 0 = 5$$

$$B = 7$$

Then the integral becomes

$$\int \frac{x^3 - 2x^2 + 3x + 5}{(x^2+1)(x^2+1)(x^2+1)}\ dx$$

$$\int \frac{2x+7}{(x^2+1)^3} + \frac{1x-2}{(x^2+1)^2} + \frac{0x+0}{x^2+1}\ dx$$

$$\int \frac{2x+7}{(x^2+1)^3}\ dx + \int \frac{x-2}{(x^2+1)^2}\ dx$$

$$2\int \frac{x}{(x^2+1)^3}\ dx + 7\int \frac{1}{(x^2+1)^3}\ dx + \int \frac{x}{(x^2+1)^2}\ dx - 2\int \frac{1}{(x^2+1)^2}\ dx$$

RATIONALIZING SUBSTITUTIONS

1. Use a rationalizing substitution to rewrite the integral in terms of u, but don't integrate it.

$$\int \frac{\sqrt{x+16}}{x}\ dx$$

Solution:

Set up the rationalizing substitution.

$u=\sqrt{x+16}$, so $u^2=x+16$, so $x=u^2-16$

$du=\dfrac{1}{2\sqrt{x+16}}\ dx$, so $dx=2\sqrt{x+16}\ du$

Substitute into the integral.

$$\int \frac{u}{x}\cdot 2\sqrt{x+16}\ du$$

$$2\int \frac{u}{u^2-16}\cdot u\ du$$

$$2\int \frac{u^2}{u^2-16}\ du$$

■ 2. Use a rationalizing substitution to rewrite the integral in terms of u, but don't integrate it.

$$\int \frac{\sqrt{3x+5}}{x}\ dx$$

Solution:

Set up the rationalizing substitution.

$u = \sqrt{3x+5}$, so $u^2 = 3x + 5$, so $3x = u^2 - 5$ and $x = (u^2 - 5)/3$

$du = \dfrac{3}{2\sqrt{3x+5}}\ dx$, so $dx = \dfrac{2}{3}\sqrt{3x+5}\ du$

Substitute into the integral.

$$\int \frac{u}{x} \cdot \frac{2}{3}\sqrt{3x+5}\ du$$

$$\frac{2}{3}\int \frac{u}{\frac{u^2-5}{3}} \cdot u\ du$$

$$\frac{2}{3}\int \frac{3u^2}{u^2-5}\ du$$

$$2\int \frac{u^2}{u^2-5}\ du$$

3. Use a rationalizing substitution to rewrite the integral in terms of u, but don't integrate it.

$$\int \frac{\sqrt{7x-2}}{x}\ dx$$

Solution:

Set up the rationalizing substitution.

$u = \sqrt{7x-2}$, so $u^2 = 7x - 2$, so $7x = u^2 + 2$ and $x = (u^2+2)/7$

$$du = \frac{7}{2\sqrt{7x-2}}\ dx\text{, so } dx = \frac{2}{7}\sqrt{7x-2}\ du$$

Substitute into the integral.

$$\int \frac{u}{\frac{u^2+2}{7}} \cdot \frac{2}{7}\sqrt{7x-2}\ du$$

$$\frac{2}{7}\int u \cdot \frac{7}{u^2+2} \cdot u\ du$$

$$2\int \frac{u^2}{u^2+2}\ du$$

HOW TO FACTOR DIFFICULT DENOMINATORS

1. Use partial fractions to factor the denominator and evaluate the integral.

$$\int \frac{2x^2 - 5x + 4}{4x^3 - x^2 - 4x + 1}\ dx$$

Solution:

Factor the denominator by grouping terms.

$$4x^3 - x^2 - 4x + 1$$

$$(4x^3 - x^2) + (-4x + 1)$$

$$x^2(4x - 1) - (4x - 1)$$

$$(x^2 - 1)(4x - 1)$$

$$(x + 1)(x - 1)(4x - 1)$$

Then the partial fractions decomposition will be

$$\frac{2x^2 - 5x + 4}{(x + 1)(x - 1)(4x - 1)} = \frac{A}{4x - 1} + \frac{B}{x + 1} + \frac{C}{x - 1}$$

$$2x^2 - 5x + 4 = A(x + 1)(x - 1) + B(x - 1)(4x - 1) + C(x + 1)(4x - 1)$$

$$2x^2 - 5x + 4 = A(x^2 - 1) + B(4x^2 - 5x + 1) + C(4x^2 + 3x - 1)$$

$$2x^2 - 5x + 4 = Ax^2 - A + 4Bx^2 - 5Bx + B + 4Cx^2 + 3Cx - C$$

$$2x^2 - 5x + 4 = (A + 4B + 4C)x^2 + (-5B + 3C)x + (-A + B - C)$$

Then the system of equations is

$$A + 4B + 4C = 2$$

$$-5B + 3C = -5$$

$$-A + B - C = 4$$

Solve the system using any method. Using Gaussian elimination and working on the first column gives

$$\begin{bmatrix} 1 & 4 & 4 & | & 2 \\ 0 & -5 & 3 & | & -5 \\ -1 & 1 & -1 & | & 4 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 4 & 4 & | & 2 \\ 0 & -5 & 3 & | & -5 \\ 0 & 5 & 3 & | & 6 \end{bmatrix}$$

Working on the second column gives

$$\begin{bmatrix} 1 & 0 & \frac{32}{5} & | & -2 \\ 0 & 1 & -\frac{3}{5} & | & 1 \\ 0 & 0 & 6 & | & 1 \end{bmatrix}$$

Working on the third column gives

$$\begin{bmatrix} 1 & 0 & 0 & | & -\frac{46}{15} \\ 0 & 1 & 0 & | & \frac{11}{10} \\ 0 & 0 & 1 & | & \frac{1}{6} \end{bmatrix}$$

So the integral is

$$\int -\frac{46}{15(4x-1)} + \frac{11}{10(x+1)} + \frac{1}{6(x-1)}\ dx$$

$$-\frac{23}{30}\ln|4x-1| + \frac{11}{10}\ln|x+1| + \frac{1}{6}\ln|x-1| + C$$

■ 2. Use partial fractions to factor the denominator and evaluate the integral.

$$\int \frac{3x^2+7x-2}{x^5-34x^3+225x}\ dx$$

Solution:

Factor the denominator.

$$x^5 - 34x^3 + 225x$$

$$x(x^4 - 34x^2 + 225)$$

$$x(x^2-9)(x^2-25)$$

$$x(x+3)(x-3)(x+5)(x-5)$$

Then the partial fractions decomposition will be

$$\frac{3x^2+7x-2}{x(x+3)(x-3)(x+5)(x-5)}=\frac{A}{x}+\frac{B}{x+3}+\frac{C}{x-3}+\frac{D}{x+5}+\frac{E}{x-5}$$

$$3x^2+7x-2=A(x+3)(x-3)(x+5)(x-5)+Bx(x-3)(x+5)(x-5)$$

$$+Cx(x+3)(x+5)(x-5)+Dx(x+3)(x-3)(x-5)+Ex(x+3)(x-3)(x+5)$$

$$3x^2+7x-2=A(x^2-9)(x^2-25)+B(x^2-3x)(x^2-25)$$

$$+C(x^2+3x)(x^2-25)+D(x^2-9)(x^2-5x)+E(x^2-9)(x^2+5x)$$

$$3x^2+7x-2=A(x^4-34x^2+225)+B(x^4-25x^2-3x^3+75x)$$

$$+C(x^4-25x^2+3x^3-75x)+D(x^4-5x^3-9x^2+45x)$$

$$+E(x^4+5x^3-9x^2-45x)$$

$$3x^2+7x-2=Ax^4-34Ax^2+225A+Bx^4-25Bx^2-3Bx^3+75Bx$$

$$+Cx^4-25Cx^2+3Cx^3-75Cx+Dx^4-5Dx^3-9Dx^2+45Dx$$

$$+Ex^4+5Ex^3-9Ex^2-45Ex$$

$$3x^2+7x-2=Ax^4+Bx^4+Cx^4+Dx^4+Ex^4-3Bx^3+3Cx^3-5Dx^3+5Ex^3$$

$$-34Ax^2-25Bx^2-25Cx^2-9Dx^2-9Ex^2$$

$$+75Bx-75Cx+45Dx-45Ex+225A$$

$$3x^2+7x-2=(A+B+C+D+E)x^4+(-3B+3C-5D+5E)x^3$$

$$+(-34A - 25B - 25C - 9D - 9E)x^2$$

$$+(75B - 75C + 45D - 45E)x + (225A)$$

Then the system of equations is

$$A + B + C + D + E = 0$$

$$-3B + 3C - 5D + 5E = 0$$

$$-34A - 25B - 25C - 9D - 9E = 3$$

$$75B - 75C + 45D - 45E = 7$$

$$225A = -2$$

So $A = -2/225$, and the remaining system is

$$B + C + D + E = \frac{2}{225}$$

$$-3B + 3C - 5D + 5E = 0$$

$$-25B - 25C - 9D - 9E = \frac{607}{225}$$

$$75B - 75C + 45D - 45E = 7$$

Solve the system using any method. Using Gaussian elimination and working on the first column gives

$$\begin{bmatrix} 1 & 1 & 1 & 1 & | & \frac{2}{225} \\ -3 & 3 & -5 & 5 & | & 0 \\ -25 & -25 & -9 & -9 & | & \frac{607}{225} \\ 75 & -75 & 45 & -45 & | & 7 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 & | & \frac{2}{225} \\ 0 & 6 & -2 & 8 & | & \frac{6}{225} \\ 0 & 0 & 16 & 16 & | & \frac{657}{225} \\ 0 & -150 & -30 & -120 & | & \frac{1,425}{225} \end{bmatrix}$$

Working on the second column gives

$$\begin{bmatrix} 1 & 0 & \frac{4}{3} & -\frac{1}{3} & | & \frac{1}{225} \\ 0 & 1 & -\frac{1}{3} & \frac{4}{3} & | & \frac{1}{225} \\ 0 & 0 & 16 & 16 & | & \frac{657}{225} \\ 0 & 0 & -80 & 80 & | & \frac{1,575}{225} \end{bmatrix}$$

Working on the third column gives

$$\begin{bmatrix} 1 & 0 & 0 & -\frac{5}{3} & | & -\frac{645}{2,700} \\ 0 & 1 & 0 & \frac{5}{3} & | & \frac{705}{10,800} \\ 0 & 0 & 1 & 1 & | & \frac{657}{3,600} \\ 0 & 0 & 0 & 160 & | & \frac{4,860}{225} \end{bmatrix}$$

Working on the fourth column gives

$$\begin{bmatrix} 1 & 0 & 0 & 0 & | & -\frac{15}{1,080} \\ 0 & 1 & 0 & 0 & | & -\frac{1,725}{10,800} \\ 0 & 0 & 1 & 0 & | & \frac{171}{3,600} \\ 0 & 0 & 0 & 1 & | & \frac{27}{200} \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & | & -\frac{1}{72} \\ 0 & 1 & 0 & 0 & | & -\frac{23}{144} \\ 0 & 0 & 1 & 0 & | & \frac{19}{400} \\ 0 & 0 & 0 & 1 & | & \frac{27}{200} \end{bmatrix}$$

So the integral is

$$\int \frac{-2}{225x} - \frac{1}{72(x+3)} - \frac{23}{144(x-3)} + \frac{19}{400(x+5)} + \frac{27}{200(x-5)} \ dx$$

$$-\frac{2}{225} \ln|x| - \frac{1}{72} \ln|x+3| - \frac{23}{144} \ln|x-3|$$

$$+\frac{19}{400} \ln|x+5| + \frac{27}{200} \ln|x-5| + C$$

■ 3. Use partial fractions to factor the denominator and evaluate the integral.

$$\int \frac{4x^2+3x+1}{x^5+x^4-13x^3-13x^2+36x+36}\,dx$$

Solution:

Use polynomial long division to factor out $x+1$ from the denominator.

$$x^5+x^4-13x^3-13x^2+36x+36$$

$$(x+1)(x^4-13x^2+36)$$

$$(x+1)(x^2-4)(x^2-9)$$

$$(x+1)(x+2)(x-2)(x+3)(x-3)$$

Then the partial fractions decomposition will be

$$\frac{4x^2+3x+1}{(x+1)(x+2)(x-2)(x+3)(x-3)}=\frac{A}{x+1}+\frac{B}{x+2}+\frac{C}{x-2}+\frac{D}{x+3}+\frac{E}{x-3}$$

$$4x^2+3x+1=A(x+2)(x-2)(x+3)(x-3)+B(x+1)(x-2)(x+3)(x-3)$$

$$+C(x+1)(x+2)(x+3)(x-3)+D(x+1)(x+2)(x-2)(x-3)$$

$$+E(x+1)(x+2)(x-2)(x+3)$$

$$4x^2+3x+1=A(x^2-4)(x^2-9)+B(x^2-x-2)(x^2-9)$$

$$+C(x^2+3x+2)(x^2-9)+D(x^2-2x-3)(x^2-4)$$

$$+E(x^2-4)(x^2+4x+3)$$

$$4x^2+3x+1=A(x^4-13x^2+36)+B(x^4-x^3-11x^2+9x+18)$$

$$+C(x^4-9x^2+3x^3-27x+2x^2-18)+D(x^4-4x^2-2x^3+8x-3x^2+12)$$

$$+E(x^4+4x^3-x^2-16x-12)$$

$$4x^2+3x+1=Ax^4-13Ax^2+36A+Bx^4-Bx^3-11Bx^2+9Bx+18B$$

$$+Cx^4-9Cx^2+3Cx^3-27Cx+2Cx^2-18C$$

$$+Dx^4-4Dx^2-2Dx^3+8Dx-3Dx^2+12D$$

$$+Ex^4+4Ex^3-Ex^2-16Ex-12E$$

$$4x^2+3x+1=(A+B+C+D+E)x^4+(-B+3C-2D+4E)x^3$$

$$+(-13A-11B-7C-7D-E)x^2+(9B-27C+8D-16E)x$$

$$+(36A+18B-18C+12D-12E)$$

Then the system of equations is

$$A+B+C+D+E=0$$

$$-B+3C-2D+4E=0$$

$$-13A-11B-7C-7D-E=4$$

$$9B-27C+8D-16E=3$$

$$36A+18B-18C+12D-12E=1$$

Solve the system using any method. Using Gaussian elimination and working on the first column gives

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & | & 0 \\ 0 & -1 & 3 & -2 & 4 & | & 0 \\ -13 & -11 & -7 & -7 & -1 & | & 4 \\ 0 & 9 & -27 & 8 & -16 & | & 3 \\ 36 & 18 & -18 & 12 & -12 & | & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & | & 0 \\ 0 & -1 & 3 & -2 & 4 & | & 0 \\ 0 & 2 & 6 & 6 & 12 & | & 4 \\ 0 & 9 & -27 & 8 & -16 & | & 3 \\ 0 & -18 & -54 & -24 & -48 & | & 1 \end{bmatrix}$$

Working on the second column gives

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & | & 0 \\ 0 & 1 & -3 & 2 & -4 & | & 0 \\ 0 & 2 & 6 & 6 & 12 & | & 4 \\ 0 & 9 & -27 & 8 & -16 & | & 3 \\ 0 & -18 & -54 & -24 & -48 & | & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 4 & -1 & 5 & | & 0 \\ 0 & 1 & -3 & 2 & -4 & | & 0 \\ 0 & 0 & 12 & 2 & 20 & | & 4 \\ 0 & 0 & 0 & -10 & 20 & | & 3 \\ 0 & 0 & -108 & 12 & -120 & | & 1 \end{bmatrix}$$

Working on the third column gives

$$\begin{bmatrix} 1 & 0 & 4 & -1 & 5 & | & 0 \\ 0 & 1 & -3 & 2 & -4 & | & 0 \\ 0 & 0 & 1 & \frac{1}{6} & \frac{5}{3} & | & \frac{1}{3} \\ 0 & 0 & 0 & -10 & 20 & | & 3 \\ 0 & 0 & -108 & 12 & -120 & | & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & -\frac{5}{3} & -\frac{5}{3} & | & -\frac{4}{3} \\ 0 & 1 & 0 & \frac{5}{2} & 1 & | & 1 \\ 0 & 0 & 1 & \frac{1}{6} & \frac{5}{3} & | & \frac{1}{3} \\ 0 & 0 & 0 & -10 & 20 & | & 3 \\ 0 & 0 & 0 & 30 & 60 & | & 37 \end{bmatrix}$$

Working on the fourth column gives

$$\begin{bmatrix} 1 & 0 & 0 & -\frac{5}{3} & -\frac{5}{3} & | & -\frac{4}{3} \\ 0 & 1 & 0 & \frac{5}{2} & 1 & | & 1 \\ 0 & 0 & 1 & \frac{1}{6} & \frac{5}{3} & | & \frac{1}{3} \\ 0 & 0 & 0 & 1 & -2 & | & -\frac{3}{10} \\ 0 & 0 & 0 & 30 & 60 & | & 37 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & -5 & | & -\frac{11}{6} \\ 0 & 1 & 0 & 0 & 6 & | & \frac{7}{4} \\ 0 & 0 & 1 & 0 & 2 & | & \frac{23}{60} \\ 0 & 0 & 0 & 1 & -2 & | & -\frac{3}{10} \\ 0 & 0 & 0 & 0 & 120 & | & 46 \end{bmatrix}$$

Working on the fifth column gives

$$\begin{bmatrix} 1 & 0 & 0 & 0 & -5 & | & -\frac{11}{6} \\ 0 & 1 & 0 & 0 & 6 & | & \frac{7}{4} \\ 0 & 0 & 1 & 0 & 2 & | & \frac{23}{60} \\ 0 & 0 & 0 & 1 & -2 & | & -\frac{3}{10} \\ 0 & 0 & 0 & 0 & 1 & | & \frac{23}{60} \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & | & \frac{1}{12} \\ 0 & 1 & 0 & 0 & 0 & | & -\frac{11}{20} \\ 0 & 0 & 1 & 0 & 0 & | & -\frac{23}{60} \\ 0 & 0 & 0 & 1 & 0 & | & \frac{7}{15} \\ 0 & 0 & 0 & 0 & 1 & | & \frac{23}{60} \end{bmatrix}$$

So the integral is

$$\int \frac{1}{12(x+1)} - \frac{11}{20(x+2)} - \frac{23}{60(x-2)} + \frac{7}{15(x+3)} + \frac{23}{60(x-3)}\ dx$$

$$\frac{1}{12}\ln|x+1| - \frac{11}{20}\ln|x+2| - \frac{23}{60}\ln|x-2| + \frac{7}{15}\ln|x+3| + \frac{23}{60}\ln|x-3| + C$$

TWO WAYS TO FIND THE CONSTANTS

1. Use an alternative approach for partial fractions to evaluate the integral by setting the factors of the constants equal to 0 to find the other constants.

$$\int \frac{3x-2}{x^2+9x+18}\ dx$$

Solution:

Use a partial fractions decomposition on the integrand.

$$\frac{3x-2}{(x+3)(x+6)}=\frac{A}{x+3}+\frac{B}{x+6}$$

$$3x-2=A(x+6)+B(x+3)$$

Because $x+6=0$ gives $x=-6$,

$$3(-6)-2=A(-6+6)+B(-6+3)$$

$$-20=-3B$$

$$B=\frac{20}{3}$$

Because $x+3=0$ gives $x=-3$,

$$3(-3)-2=A(-3+6)+B(-3+3)$$

$$-11 = 3A$$

$$A = -\frac{11}{3}$$

Then the integral becomes

$$\int -\frac{11}{3(x+3)} + \frac{20}{3(x+6)}\ dx$$

$$-\frac{11}{3}\ln|x+3| + \frac{20}{3}\ln|x+6| + C$$

■ 2. Use an alternative approach for partial fractions to evaluate the integral by setting the factors of the constants equal to 0 to find the other constants.

$$\int \frac{8x+13}{x^2+x-12}\ dx$$

Solution:

Use a partial fractions decomposition on the integrand.

$$\frac{8x+13}{(x+4)(x-3)} = \frac{A}{x+4} + \frac{B}{x-3}$$

$$8x + 13 = A(x-3) + B(x+4)$$

Because $x - 3 = 0$ gives $x = 3$,

$$8(3) + 13 = A(3 - 3) + B(3 + 4)$$

$$37 = 7B$$

$$B = \frac{37}{7}$$

Because $x + 4 = 0$ gives $x = -4$,

$$8(-4) + 13 = A(-4 - 3) + B(-4 + 4)$$

$$-19 = -7A$$

$$A = \frac{19}{7}$$

Then the integral becomes

$$\int \frac{19}{7(x + 4)} + \frac{37}{7(x - 3)}\ dx$$

$$\frac{19}{7} \ln|x + 4| + \frac{37}{7} \ln|x - 3| + C$$

■ 3. Use an alternative approach for partial fractions to evaluate the integral by setting the factors of the constants equal to 0 to find the other constants.

$$\int \frac{x - 21}{9x^2 + 9x + 2}\ dx$$

Solution:

Use a partial fractions decomposition on the integrand.

$$\frac{x-21}{(3x+1)(3x+2)}=\frac{A}{3x+1}+\frac{B}{3x+2}$$

$$x-21=A(3x+2)+B(3x+1)$$

Because $3x+2=0$ gives $x=-2/3$,

$$-\frac{2}{3}-21=A\left(3\left(-\frac{2}{3}\right)+2\right)+B\left(3\left(-\frac{2}{3}\right)+1\right)$$

$$-\frac{2}{3}-21=-B$$

$$-\frac{65}{3}=-B$$

$$\frac{65}{3}=B$$

Because $3x+1=0$ gives $x=-1/3$,

$$-\frac{1}{3}-21=A\left(3\left(-\frac{1}{3}\right)+2\right)+B\left(3\left(-\frac{1}{3}\right)+1\right)$$

$$-\frac{1}{3}-21=A$$

$$-\frac{64}{3}=A$$

Then the integral becomes

$$\int -\frac{64}{3(3x+1)} + \frac{65}{3(3x+2)}\ dx$$

$$-\frac{64}{3} \cdot \frac{\ln|3x+1|}{3} + \frac{65}{3} \cdot \frac{\ln|3x+2|}{3} + C$$

$$\frac{65\ln|3x+2|}{9} - \frac{64\ln|3x+1|}{9} + C$$

$$\frac{65\ln|3x+2| - 64\ln|3x+1|}{9} + C$$

SIN^M COS^N, ODD M

■ 1. Evaluate the trigonometric integral.

$$\int \sin^5(3x^2+2x+1)\cos(3x^2+2x+1)(6x+2)\ dx$$

Solution:

Use u-substitution.

$$u = \sin(3x^2+2x+1)$$

$$\frac{du}{dx} = \cos(3x^2+2x+1)(6x+2)$$

$$du = \cos(3x^2+2x+1)(6x+2)\ dx$$

$$dx = \frac{du}{\cos(3x^2+2x+1)(6x+2)}$$

Substitute.

$$\int u^5 \cos(3x^2+2x+1)(6x+2) \cdot \frac{du}{\cos(3x^2+2x+1)(6x+2)}$$

$$\int u^5\ du$$

Integrate and back-substitute.

$$\frac{1}{6}u^6 + C$$

$$\frac{1}{6}(\sin(3x^2 + 2x + 1))^6 + C$$

$$\frac{1}{6}\sin^6(3x^2 + 2x + 1) + C$$

SIN^M COS^N, ODD N

1. Evaluate the trigonometric integral.

$$\int_{-\frac{\pi}{6}}^{\frac{\pi}{3}} (4+\cos x)\sin x \ dx$$

Solution:

Use u-substitution.

$$u = 4 + \cos x$$

$$\frac{du}{dx} = -\sin x, \text{ so } du = -\sin x \ dx, \text{ so } dx = -\frac{du}{\sin x}$$

Change the limits of integration.

$$u\left(\frac{\pi}{3}\right) = 4 + \cos\left(\frac{\pi}{3}\right) = 4 + \frac{1}{2} = \frac{9}{2}$$

$$u\left(-\frac{\pi}{6}\right) = 4 + \cos\left(-\frac{\pi}{6}\right) = 4 + \frac{\sqrt{3}}{2}$$

Substitute into the integral.

$$\int_{4+\frac{\sqrt{3}}{2}}^{\frac{9}{2}} u \sin x \left(-\frac{du}{\sin x}\right)$$

$$-\int_{4+\frac{\sqrt{3}}{2}}^{\frac{9}{2}} u \ du$$

Integrate and evaluate over the interval.

$$-\frac{1}{2}u^2\Big|_{4+\frac{\sqrt{3}}{2}}^{\frac{9}{2}}$$

$$-\frac{1}{2}\left(\frac{9}{2}\right)^2+\frac{1}{2}\left(4+\frac{\sqrt{3}}{2}\right)^2$$

$$-\frac{1}{2}\left(\frac{81}{4}\right)+\frac{1}{2}\left(16+4\sqrt{3}+\frac{3}{4}\right)$$

$$-\frac{81}{8}+8+2\sqrt{3}+\frac{3}{8}$$

$$-\frac{7}{4}+2\sqrt{3}$$

■ 2. Evaluate the trigonometric integral.

$$\int \sin(2x)\cos^3(2x) \ dx$$

Solution:

Use u-substitution.

$$u = \cos(2x)$$

$$\frac{du}{dx} = -2\sin(2x), \text{ so } du = -2\sin(2x)\ dx, \text{ so } dx = -\frac{du}{2\sin(2x)}$$

Substitute into the integral.

$$\int \sin(2x) \cdot u^3 \cdot -\frac{du}{2\sin(2x)}$$

$$-\frac{1}{2}\int u^3\ du$$

Integrate, then back-substitute.

$$-\frac{1}{2}\left(\frac{1}{4}u^4\right) + C$$

$$-\frac{1}{8}u^4 + C$$

$$-\frac{1}{8}\cos^4(2x) + C$$

SIN^M COS^N, M AND N EVEN

1. Evaluate the trigonometric integral.

$$\int \sin^2(2x+3)\cos^2(2x+3)\ dx$$

Solution:

Use the trig identity $\sin\theta\cos\theta = (1/2)\sin(2\theta)$ to rewrite the integrand.

$$\sin^2(2x+3)\cos^2(2x+3)$$

$$\left[\sin(2x+3)\cos(2x+3)\right]\left[\sin(2x+3)\cos(2x+3)\right]$$

$$\left[\frac{1}{2}\sin(2(2x+3))\right]^2$$

$$\frac{1}{4}\sin^2(4x+6)$$

Use the trig identity $\sin^2\theta = (1/2)(1-\cos(2\theta))$ to rewrite the integrand.

$$\frac{1}{4}\left[\frac{1}{2}(1-\cos(2(4x+6)))\right]$$

$$\frac{1}{8}(1-\cos(8x+12))$$

$$\frac{1}{8}-\frac{1}{8}\cos(8x+12)$$

Integrate.

$$\int \frac{1}{8} - \frac{1}{8}\cos(8x+12)\ dx$$

$$\frac{1}{8}x - \frac{1}{64}\sin(8x+12) + C$$

■ 2. Evaluate the trigonometric integral.

$$\int \sin^4(2x)\cos^2(2x)\ dx$$

Solution:

Use the trig identity $\sin\theta\cos\theta = (1/2)\sin(2\theta)$ to rewrite the integrand.

$$\sin^4(2x)\cos^2(2x)$$

$$\sin^2(2x)\sin^2(2x)\cos^2(2x)$$

$$\sin^2(2x)(\sin(2x)\cos(2x))(\sin(2x)\cos(2x))$$

$$\sin^2(2x)\left[\frac{1}{2}\sin(2(2x))\right]^2$$

$$\frac{1}{4}\sin^2(2x)\sin^2(4x)$$

Use the trig identity $\sin^2\theta = (1/2)(1-\cos(2\theta))$ to rewrite the integrand.

$$\frac{1}{4}\left(\frac{1}{2}(1-\cos(2(2x)))\right)\left(\frac{1}{2}(1-\cos(2(4x)))\right)$$

$$\frac{1}{16}(1-\cos(4x))(1-\cos(8x))$$

$$\frac{1}{16}\left[1-\cos(8x)-\cos(4x)+\cos(8x)\cos(4x)\right]$$

Use the trig identity $\cos a\cos b=(1/2)\left[\cos(a-b)+\cos(a+b)\right]$ to rewrite the integrand.

$$\frac{1}{16}\left[1-\cos(8x)-\cos(4x)+\frac{1}{2}\left[\cos(8x-4x)+\cos(8x+4x)\right]\right]$$

$$\frac{1}{16}\left[1-\cos(8x)-\cos(4x)+\frac{1}{2}\cos(4x)+\frac{1}{2}\cos(12x)\right]$$

$$\frac{1}{16}\left[1-\cos(8x)-\frac{1}{2}\cos(4x)+\frac{1}{2}\cos(12x)\right]$$

Integrate.

$$\frac{1}{16}\int 1-\cos(8x)-\frac{1}{2}\cos(4x)+\frac{1}{2}\cos(12x)\ dx$$

$$\frac{1}{16}\left(x-\frac{1}{8}\sin(8x)-\frac{1}{8}\sin(4x)+\frac{1}{24}\sin(12x)\right)+C$$

$$\frac{1}{16}x-\frac{1}{128}\sin(8x)-\frac{1}{128}\sin(4x)+\frac{1}{384}\sin(12x)+C$$

3. Evaluate the trigonometric integral.

$$\int \sin^6(3x)\cos^4(3x)\ dx$$

Solution:

Use the trig identity $\sin\theta\cos\theta = (1/2)\sin(2\theta)$ to rewrite the integrand.

$$\int \sin^2(3x)\sin^4(3x)\cos^4(3x)\ dx$$

$$\int \sin^2(3x)\left[\sin(3x)\cos(3x)\right]^4\ dx$$

$$\int \sin^2(3x)\left[\frac{1}{2}\sin(2(3x))\right]^4\ dx$$

$$\frac{1}{16}\int \sin^2(3x)\sin^4(6x)\ dx$$

Use the trig identity $\sin^2\theta = (1/2)(1-\cos(2\theta))$ to rewrite the integrand.

$$\frac{1}{16}\int \frac{1}{2}(1-\cos(2(3x)))\sin^4(6x)\ dx$$

$$\frac{1}{32}\int (1-\cos(6x))\sin^4(6x)\ dx$$

Use the trig identity $\sin^2\theta = (1/2)(1-\cos(2\theta))$ to rewrite the integrand.

$$\frac{1}{32}\int (1-\cos(6x))(\sin^2(6x))^2\ dx$$

$$\frac{1}{32}\int(1-\cos(6x))\left[\frac{1}{2}(1-\cos(2(6x)))\right]^2\ dx$$

$$\frac{1}{32}\int(1-\cos(6x))\frac{1}{4}(1-\cos(12x))^2\ dx$$

$$\frac{1}{128}\int(1-\cos(6x))(1-\cos(12x))^2\ dx$$

$$\frac{1}{128}\int(1-\cos(6x))(1-2\cos(12x)+\cos^2(12x))\ dx$$

$$\frac{1}{128}\int 1-2\cos(12x)+\cos^2(12x)-\cos(6x)$$

$$+2\cos(6x)\cos(12x)-\cos(6x)\cos^2(12x)\ dx$$

Use the trig identity $\cos^2\theta=(1/2)(1+\cos(2\theta))$ to rewrite the integrand.

$$\frac{1}{128}\int 1-2\cos(12x)+\frac{1}{2}(1+\cos(2(12x)))-\cos(6x)$$

$$+2\cos(6x)\cos(12x)-\cos(6x)\cos^2(12x)\ dx$$

$$\frac{1}{128}\int 1-2\cos(12x)+\frac{1}{2}+\frac{1}{2}\cos(24x)-\cos(6x)$$

$$+2\cos(6x)\cos(12x)-\cos(6x)\cos^2(12x)\ dx$$

$$\frac{1}{128}\int\frac{3}{2}-\cos(6x)-2\cos(12x)+\frac{1}{2}\cos(24x)$$

$$+2\cos(6x)\cos(12x)-\cos(6x)\cos^2(12x)\ dx$$

Use the trig identity $\cos a \cos b = (1/2)\left[\cos(a-b) + \cos(a+b)\right]$ to rewrite the integrand.

$$\frac{1}{128}\int \frac{3}{2} - \cos(6x) - 2\cos(12x) + \frac{1}{2}\cos(24x)$$

$$+2\left[\frac{1}{2}\left[\cos(12x-6x) + \cos(12x+6x)\right]\right] - \cos(6x)\cos^2(12x)\ dx$$

$$\frac{1}{128}\int \frac{3}{2} - \cos(6x) - 2\cos(12x) + \frac{1}{2}\cos(24x)$$

$$+\cos(6x) + \cos(18x) - \cos(6x)\cos^2(12x)\ dx$$

Use the trig identity $\cos a \cos b = (1/2)\left[\cos(a-b) + \cos(a+b)\right]$ to rewrite the integrand.

$$\frac{1}{128}\int \frac{3}{2} - \cos(6x) - 2\cos(12x) + \frac{1}{2}\cos(24x)$$

$$+\cos(6x) + \cos(18x) - \cos(6x)\cos(12x)\cos(12x)\ dx$$

$$\frac{1}{128}\int \frac{3}{2} - \cos(6x) - 2\cos(12x) + \frac{1}{2}\cos(24x)$$

$$+\cos(6x) + \cos(18x) - \frac{1}{2}\left[\cos(12x-6x) + \cos(12x+6x)\right]\cos(12x)\ dx$$

$$\frac{1}{128}\int \frac{3}{2} - \cos(6x) - 2\cos(12x) + \frac{1}{2}\cos(24x)$$

$$+\cos(6x) + \cos(18x) + \left[-\frac{1}{2}\cos(6x) - \frac{1}{2}\cos(18x)\right]\cos(12x)\ dx$$

$$\frac{1}{128}\int \frac{3}{2}-\cos(6x)-2\cos(12x)+\frac{1}{2}\cos(24x)+\cos(6x)+\cos(18x)$$

$$-\frac{1}{2}\left[\cos(6x)\cos(12x)+\cos(18x)\cos(12x)\right]\ dx$$

$$\frac{1}{128}\int \frac{3}{2}-\cos(6x)-2\cos(12x)+\frac{1}{2}\cos(24x)+\cos(6x)+\cos(18x)$$

$$-\frac{1}{2}\left[\frac{1}{2}\left[\cos(12x-6x)+\cos(12x+6x)\right]+\cos(18x)\cos(12x)\right]\ dx$$

$$\frac{1}{128}\int \frac{3}{2}-\cos(6x)-2\cos(12x)+\frac{1}{2}\cos(24x)+\cos(6x)+\cos(18x)$$

$$-\frac{1}{2}\left[\frac{1}{2}\cos(6x)+\frac{1}{2}\cos(18x)+\cos(18x)\cos(12x)\right]\ dx$$

$$\frac{1}{128}\int \frac{3}{2}-\cos(6x)-2\cos(12x)+\frac{1}{2}\cos(24x)+\cos(6x)+\cos(18x)$$

$$-\frac{1}{4}\cos(6x)-\frac{1}{4}\cos(18x)-\frac{1}{2}\cos(18x)\cos(12x)\ dx$$

$$\frac{1}{128}\int \frac{3}{2}-\cos(6x)-2\cos(12x)+\frac{1}{2}\cos(24x)+\cos(6x)+\cos(18x)$$

$$-\frac{1}{4}\cos(6x)-\frac{1}{4}\cos(18x)-\frac{1}{2}\left[\frac{1}{2}\left[\cos(18x-12x)+\cos(18x+12x)\right]\right]\ dx$$

$$\frac{1}{128}\int \frac{3}{2}-\cos(6x)-2\cos(12x)+\frac{1}{2}\cos(24x)+\cos(6x)+\cos(18x)$$

$$-\frac{1}{4}\cos(6x)-\frac{1}{4}\cos(18x)-\frac{1}{4}\left[\cos(6x)+\cos(30x)\right]\ dx$$

$$\frac{1}{128}\int \frac{3}{2}-\cos(6x)-2\cos(12x)+\frac{1}{2}\cos(24x)+\cos(6x)+\cos(18x)$$

$$-\frac{1}{4}\cos(6x)-\frac{1}{4}\cos(18x)-\frac{1}{4}\cos(6x)-\frac{1}{4}\cos(30x)\ dx$$

$$\frac{1}{128}\int \frac{3}{2}-\cos(6x)+\cos(6x)-\frac{1}{4}\cos(6x)-\frac{1}{4}\cos(6x)-2\cos(12x)$$

$$+\cos(18x)-\frac{1}{4}\cos(18x)+\frac{1}{2}\cos(24x)-\frac{1}{4}\cos(30x)\ dx$$

$$\frac{1}{128}\int \frac{3}{2}-\frac{1}{2}\cos(6x)-2\cos(12x)$$

$$+\frac{3}{4}\cos(18x)+\frac{1}{2}\cos(24x)-\frac{1}{4}\cos(30x)\ dx$$

Integrate.

$$\frac{1}{128}\left(\frac{3}{2}x-\frac{1}{12}\sin(6x)-\frac{1}{6}\sin(12x)+\frac{1}{24}\sin(18x)+\frac{1}{48}\sin(24x)-\frac{1}{120}\sin(30x)\right)+C$$

$$\frac{1}{256}\left(3x-\frac{1}{6}\sin(6x)-\frac{1}{3}\sin(12x)+\frac{1}{12}\sin(18x)+\frac{1}{24}\sin(24x)-\frac{1}{60}\sin(30x)\right)+C$$

TAN^M SEC^N, ODD M

1. Evaluate the trigonometric integral.

$$\int \tan^3(2x)\sec(2x)\ dx$$

Solution:

Use the trig identity $\tan^2\theta = \sec^2\theta - 1$ to rewrite the integrand.

$$\int \tan^2(2x)\tan(2x)\sec(2x)\ dx$$

$$\int (\sec^2(2x) - 1)\tan(2x)\sec(2x)\ dx$$

$$\int \sec^2(2x)\tan(2x)\sec(2x) - \tan(2x)\sec(2x)\ dx$$

$$\int \sec^2(2x)\tan(2x)\sec(2x)\ dx - \int \tan(2x)\sec(2x)\ dx$$

Use u-substitution.

$$u = \sec(2x)$$

$\dfrac{du}{dx} = 2\sec(2x)\tan(2x)$, so $du = 2\sec(2x)\tan(2x)\ dx$, so $dx = \dfrac{du}{2\sec(2x)\tan(2x)}$

Substitute and integrate the first integral.

$$\int u^2 \tan(2x)\sec(2x) \cdot \frac{du}{2\sec(2x)\tan(2x)} - \int \tan(2x)\sec(2x)\ dx$$

$$\frac{1}{2}\int u^2\ du - \int \tan(2x)\sec(2x)\ dx$$

$$\frac{1}{2}\left(\frac{1}{3}u^3\right) - \int \tan(2x)\sec(2x)\ dx$$

$$\frac{1}{6}\sec^3(2x) - \int \tan(2x)\sec(2x)\ dx$$

Integrate the second integral.

$$\frac{1}{6}\sec^3(2x) - \frac{1}{2}\sec(2x) + C$$

■ 2. Evaluate the trigonometric integral.

$$\int \tan^5(3x)\sec(3x)\ dx$$

Solution:

Use the trig identity $\tan^2\theta = \sec^2\theta - 1$ to rewrite the integrand.

$$\int \tan^4(3x)\tan(3x)\sec(3x)\ dx$$

$$\int (\tan^2(3x))^2\tan(3x)\sec(3x)\ dx$$

$$\int (\sec^2(3x) - 1)^2 \tan(3x)\sec(3x)\ dx$$

Use u-substitution.

$$u = \sec(3x)$$

$$\frac{du}{dx} = 3\sec(3x)\tan(3x), \text{ so } du = 3\sec(3x)\tan(3x)\ dx, \text{ so } dx = \frac{du}{3\sec(3x)\tan(3x)}$$

Substitute.

$$\int (u^2 - 1)^2 \tan(3x)\sec(3x) \cdot \frac{du}{3\sec(3x)\tan(3x)}$$

$$\frac{1}{3}\int (u^2 - 1)^2\ du$$

$$\frac{1}{3}\int u^4 - 2u^2 + 1\ du$$

Integrate and back-substitute.

$$\frac{1}{3}\left(\frac{1}{5}u^5 - \frac{2}{3}u^3 + u\right) + C$$

$$\frac{1}{3}\left(\frac{1}{5}\sec^5(3x) - \frac{2}{3}\sec^3(3x) + \sec(3x)\right) + C$$

TAN^M SEC^N, EVEN N

1. Evaluate the trigonometric integral.

$$\int \tan^2(4x)\sec^4(4x)\ dx$$

Solution:

Use the trig identity $\sec^2\theta = 1 + \tan^2\theta$ to rewrite the integrand.

$$\int \tan^2(4x)\sec^2(4x)\sec^2(4x)\ dx$$

$$\int \tan^2(4x)\sec^2(4x)(1 + \tan^2(4x))\ dx$$

Use u-substitution.

$$u = \tan(4x)$$

$\dfrac{du}{dx} = 4\sec^2(4x)$, so $du = 4\sec^2(4x)\ dx$, so $dx = \dfrac{du}{4\sec^2(4x)}$

Substitute.

$$\int u^2\sec^2(4x)(1 + u^2)\left(\frac{du}{4\sec^2(4x)}\right)$$

$$\frac{1}{4}\int u^2(1 + u^2)\ du$$

$$\frac{1}{4}\int u^2+u^4\ du$$

Integrate and back-substitute.

$$\frac{1}{4}\left(\frac{1}{3}u^3+\frac{1}{5}u^5\right)+C$$

$$\frac{1}{4}\left(\frac{1}{3}\tan^3(4x)+\frac{1}{5}\tan^5(4x)\right)+C$$

■ 2. Evaluate the trigonometric integral.

$$\int \tan^4(2x)\sec^4(2x)\ dx$$

Solution:

Use the trig identity $\sec^2\theta=1+\tan^2\theta$ to rewrite the integrand.

$$\int \tan^4(2x)\sec^2(2x)\sec^2(2x)\ dx$$

$$\int \tan^4(2x)\sec^2(2x)(1+\tan^2\theta)\ dx$$

Use u-substitution.

$$u=\tan(2x)$$

$$\frac{du}{dx} = 2\sec^2(2x), \text{ so } du = 2\sec^2(2x)\ dx, \text{ so } dx = \frac{du}{2\sec^2(2x)}$$

Substitute.

$$\int u^4 \sec^2(2x)(1+u^2)\left(\frac{du}{2\sec^2(2x)}\right)$$

$$\frac{1}{2}\int u^4(1+u^2)\ du$$

$$\frac{1}{2}\int u^4 + u^6\ du$$

Integrate and back-substitute.

$$\frac{1}{2}\left(\frac{1}{5}u^5 + \frac{1}{7}u^7\right) + C$$

$$\frac{1}{2}\left(\frac{1}{5}\tan^5(2x) + \frac{1}{7}\tan^7(2x)\right) + C$$

■ 3. Evaluate the trigonometric integral.

$$\int \tan^4(3x-1)\sec^4(3x-1)\ dx$$

Solution:

Use the trig identity $\sec^2\theta = 1 + \tan^2\theta$ to rewrite the integrand.

$$\int \tan^4(3x-1)\sec^2(3x-1)(1+\tan^2\theta)\ dx$$

Use u-substitution.

$$u=\tan(3x-1)$$

$$\frac{du}{dx}=3\sec^2(3x-1)\text{, so } du=3\sec^2(3x-1)\ dx\text{, so } dx=\frac{du}{3\sec^2(3x-1)}$$

Substitute.

$$\int u^4\sec^2(3x-1)(1+u^2)\cdot\frac{du}{3\sec^2(3x-1)}$$

$$\frac{1}{3}\int u^4(1+u^2)\ du$$

$$\frac{1}{3}\int u^4+u^6\ du$$

Integrate and back-substitute.

$$\frac{1}{3}\left(\frac{1}{5}u^5+\frac{1}{7}u^7\right)+C$$

$$\frac{1}{3}\left(\frac{1}{5}\tan^5(3x-1)+\frac{1}{7}\tan^7(3x-1)\right)+C$$

SIN(MX) COS(NX)

■ 1. Evaluate the trigonometric integral.

$$\int 5\sin(6x)\cos(3x)\ dx$$

Solution:

Use the trig identity $\sin a\cos b = (1/2)(\sin(a-b)+\sin(a+b))$ to rewrite the integrand.

$$\int 5\cdot\frac{1}{2}(\sin(6x-3x)+\sin(6x+3x))\ dx$$

$$\frac{5}{2}\int \sin(3x)+\sin(9x)\ dx$$

Integrate.

$$\frac{5}{2}\left(-\frac{1}{3}\cos(3x)-\frac{1}{9}\cos(9x)\right)+C$$

$$-\frac{5}{6}\left(\cos(3x)+\frac{1}{3}\cos(9x)\right)+C$$

■ 2. Evaluate the trigonometric integral.

$$\int 2\sin(9x)\cos(4x)\ dx$$

Solution:

Use the trig identity $\sin a \cos b = (1/2)(\sin(a-b)+\sin(a+b))$ to rewrite the integrand.

$$\int 2 \cdot \frac{1}{2}(\sin(9x-4x)+\sin(9x+4x))\ dx$$

$$\int \sin(5x)+\sin(13x)\ dx$$

Integrate.

$$-\frac{1}{5}\cos(5x)-\frac{1}{13}\cos(13x)+C$$

■ 3. Evaluate the trigonometric integral.

$$\int \frac{1}{3}\sin(12x)\cos(7x)\ dx$$

Solution:

Use the trig identity $\sin a \cos b = (1/2)(\sin(a-b)+\sin(a+b))$ to rewrite the integrand.

$$\int \frac{1}{3} \cdot \frac{1}{2}(\sin(12x - 7x) + \sin(12x + 7x)) \ dx$$

$$\frac{1}{6} \int \sin(5x) + \sin(19x) \ dx$$

Integrate.

$$\frac{1}{6}\left(-\frac{1}{5}\cos(5x) - \frac{1}{19}\cos(19x)\right) + C$$

SIN(MX) SIN(NX)

1. Evaluate the trigonometric integral.

$$\int 6\sin(9x)\sin(2x)\ dx$$

Solution:

Use the trig identity $\sin a \sin b = (1/2)(\cos(a-b) - \cos(a+b))$ to rewrite the integrand.

$$\int 6 \cdot \frac{1}{2}(\cos(9x-2x) - \cos(9x+2x))\ dx$$

$$3\int \cos(7x) - \cos(11x)\ dx$$

Integrate.

$$3\left(\frac{1}{7}\sin(7x) - \frac{1}{11}\sin(11x)\right) + C$$

2. Evaluate the trigonometric integral.

$$\int \frac{1}{2}\sin(8x)\sin(4x)\ dx$$

Solution:

Use the trig identity $\sin a \sin b = (1/2)(\cos(a-b) - \cos(a+b))$ to rewrite the integrand.

$$\int \frac{1}{2} \cdot \frac{1}{2}(\cos(8x-4x) - \cos(8x+4x))\ dx$$

$$\frac{1}{4}\int \cos(4x) - \cos(12x)\ dx$$

Integrate.

$$\frac{1}{4}\left(\frac{1}{4}\sin(4x) - \frac{1}{12}\sin(12x)\right) + C$$

$$\frac{1}{16}\left(\sin(4x) - \frac{1}{3}\sin(12x)\right) + C$$

3. Evaluate the trigonometric integral.

$$\int 8\sin(14x)\sin(7x)\ dx$$

Solution:

Use the trig identity $\sin a \sin b = (1/2)(\cos(a-b) - \cos(a+b))$ to rewrite the integrand.

$$\int 8 \cdot \frac{1}{2}(\cos(14x - 7x) - \cos(14x + 7x)) \ dx$$

$$4\int \cos(7x) - \cos(21x) \ dx$$

Integrate.

$$4\left(\frac{1}{7}\sin(7x) - \frac{1}{21}\sin(21x)\right) + C$$

$$\frac{4}{7}\left(\sin(7x) - \frac{1}{3}\sin(21x)\right) + C$$

COS(MX) COS(NX)

■ 1. Evaluate the trigonometric integral.

$$\int 7\cos(8x)\cos(3x)\ dx$$

Solution:

Use the trig identity $\cos a \cos b = (1/2)(\cos(a-b)+\cos(a+b))$ to rewrite the integrand.

$$\int 7\cdot\frac{1}{2}(\cos(8x-3x)+\cos(8x+3x))\ dx$$

$$\frac{7}{2}\int \cos(5x)+\cos(11x)\ dx$$

Integrate.

$$\frac{7}{2}\left(\frac{1}{5}\sin(5x)+\frac{1}{11}\sin(11x)\right)+C$$

■ 2. Evaluate the trigonometric integral.

$$\int 5\cos(15x)\cos(5x)\ dx$$

Solution:

Use the trig identity $\cos a \cos b = (1/2)(\cos(a-b) + \cos(a+b))$ to rewrite the integrand.

$$\int 5 \cdot \frac{1}{2}(\cos(15x - 5x) + \cos(15x + 5x))\ dx$$

$$\frac{5}{2}\int \cos(10x) + \cos(20x)\ dx$$

Integrate.

$$\frac{5}{2}\left(\frac{1}{10}\sin(10x) + \frac{1}{20}\sin(20x)\right) + C$$

$$\frac{1}{4}\left(\sin(10x) + \frac{1}{2}\sin(20x)\right) + C$$

■ 3. Evaluate the trigonometric integral.

$$\int 49\cos(21x)\cos(14x)\ dx$$

Solution:

Use the trig identity $\cos a \cos b = (1/2)(\cos(a-b) + \cos(a+b))$ to rewrite the integrand.

$$\int 49 \cdot \frac{1}{2}(\cos(21x - 14x) + \cos(21x + 14x))\ dx$$

$$\frac{49}{2} \int \cos(7x) + \cos(35x)\ dx$$

Integrate.

$$\frac{49}{2}\left(\frac{1}{7}\sin(7x) + \frac{1}{35}\sin(35x)\right) + C$$

$$\frac{7}{2}\left(\sin(7x) + \frac{1}{5}\sin(35x)\right) + C$$

INVERSE HYPERBOLIC INTEGRALS

1. Evaluate the hyperbolic integral.

$$\int x \sinh(3x^2+7)\ dx$$

Solution:

Use u-substitution.

$$u = 3x^2 + 7$$

$$\frac{du}{dx} = 6x, \text{ so } du = 6x\ dx, \text{ so } dx = \frac{du}{6x}$$

Substitute.

$$\int x \sinh u \cdot \frac{du}{6x}$$

$$\frac{1}{6}\int \sinh u\ du$$

Integrate and back-substitute.

$$\frac{1}{6}\cosh u + C$$

$$\frac{1}{6}\cosh(3x^2+7) + C$$

2. Evaluate the hyperbolic integral using the substitution $x = \sqrt{3} \cosh u$.

$$\int \frac{\sqrt{x^2 - 3}}{x^2} \ dx$$

Solution:

Starting with the substitution $x = \sqrt{3} \cosh u$, we get

$$x = \sqrt{3} \cosh u$$

$$x^2 = 3 \cosh^2 u$$

$$x^2 - 3 = 3 \cosh^2 u - 3$$

$$\sqrt{x^2 - 3} = \sqrt{3 \cosh^2 u - 3}$$

$$\sqrt{x^2 - 3} = \sqrt{3(\cosh^2 u - 1)}$$

$$\sqrt{x^2 - 3} = \sqrt{3}\sqrt{\cosh^2 u - 1}$$

Using the substitution $\sinh^2 u = \cosh^2 u - 1$ gives

$$\sqrt{x^2 - 3} = \sqrt{3}\sqrt{\sinh^2 u}$$

$$\sqrt{x^2 - 3} = \sqrt{3} \sinh u$$

And

$$dx = \sqrt{3}\sinh u \ du$$

Substituting into the integral gives

$$\int \frac{\sqrt{3}\sinh u}{\left(\sqrt{3}\cosh u\right)^2} \cdot \sqrt{3}\sinh u \ du$$

$$\int \frac{3\sinh u}{3\cosh^2 u} \cdot \sinh u \ du$$

$$\int \frac{\sinh^2 u}{\cosh^2 u} \ du$$

Substitute using $\sinh^2 u = \cosh^2 u - 1$.

$$\int \frac{\cosh^2 u - 1}{\cosh^2 u} \ du$$

$$\int \frac{\cosh^2 u}{\cosh^2 u} - \frac{1}{\cosh^2 u} \ du$$

$$\int 1 - \text{sech}^2 u \ du$$

Integrate, then back-substitute using

$$u = \cosh^{-1}\left(\frac{x}{\sqrt{3}}\right)$$

You get

$$u - \tanh u + C$$

$$\cosh^{-1}\left(\frac{x}{\sqrt{3}}\right) - \tanh\left(\cosh^{-1}\left(\frac{x}{\sqrt{3}}\right)\right) + C$$

TRIGONOMETRIC SUBSTITUTION WITH SECANT

■ 1. Set up and simplify the integral for trig substitution, but don't integrate.

$$\int \frac{3}{\sqrt{9x^2+6x}}\ dx$$

Solution:

Rewrite the integrand.

$$\int \frac{3}{\sqrt{9\left(x^2+\frac{2}{3}x\right)}}\ dx$$

$$\int \frac{1}{\sqrt{x^2+\frac{2}{3}x}}\ dx$$

$$\int \frac{1}{\sqrt{\left(x^2+\frac{2}{3}x+\frac{1}{9}\right)-\frac{1}{9}}}\ dx$$

$$\int \frac{1}{\sqrt{\left(x+\frac{1}{3}\right)^2-\frac{1}{9}}}\ dx$$

Set up the trig substitution.

$$u^2 - a^2 = \left(x + \frac{1}{3}\right)^2 - \frac{1}{9}$$

$u = x + \frac{1}{3}$ and $a = \frac{1}{3}$

$x + \frac{1}{3} = \frac{1}{3}\sec\theta$ so $x = -\frac{1}{3} + \frac{1}{3}\sec\theta$

$$dx = \frac{1}{3}\sec\theta\tan\theta\ d\theta$$

Substitute.

$$\int \frac{1}{\sqrt{\left(-\frac{1}{3} + \frac{1}{3}\sec\theta + \frac{1}{3}\right)^2 - \frac{1}{9}}} \cdot \frac{1}{3}\sec\theta\tan\theta\ d\theta$$

$$\frac{1}{3}\int \frac{\sec\theta\tan\theta}{\sqrt{\left(\frac{1}{3}\sec\theta\right)^2 - \frac{1}{9}}}\ d\theta$$

$$\frac{1}{3}\int \frac{\sec\theta\tan\theta}{\sqrt{\frac{1}{9}\sec^2\theta - \frac{1}{9}}}\ d\theta$$

$$\frac{1}{3}\int \frac{\sec\theta\tan\theta}{\sqrt{\frac{1}{9}(\sec^2\theta - 1)}}\ d\theta$$

Simplify with the trig identity $\sec^2\theta - 1 = \tan^2\theta$.

$$\frac{1}{3}\int \frac{\sec\theta\tan\theta}{\sqrt{\frac{1}{9}\tan^2\theta}}\ d\theta$$

$$\frac{1}{3}\int \frac{\sec\theta\tan\theta}{\frac{1}{3}\tan\theta}\ d\theta$$

$$\int \sec\theta\ d\theta$$

■ 2. Set up and simplify the integral for trig substitution, but don't integrate.

$$\int \frac{5}{\sqrt{4x^2+4x}}\ dx$$

Solution:

Rewrite the integrand.

$$\int \frac{5}{\sqrt{4(x^2+x)}}\ dx$$

$$\frac{5}{2}\int \frac{1}{\sqrt{x^2+x}}\ dx$$

$$\frac{5}{2}\int \frac{1}{\sqrt{x^2+x+\frac{1}{4}-\frac{1}{4}}}\ dx$$

$$\frac{5}{2}\int \frac{1}{\sqrt{\left(x+\frac{1}{2}\right)^2-\frac{1}{4}}}\ dx$$

Set up the trig substitution.

$$u^2-a^2=\left(x+\frac{1}{2}\right)^2-\frac{1}{4}$$

$u=x+\dfrac{1}{2}$ and $a=\dfrac{1}{2}$

$x+\dfrac{1}{2}=\dfrac{1}{2}\sec\theta$ so $x=-\dfrac{1}{2}+\dfrac{1}{2}\sec\theta$

$$dx=\frac{1}{2}\sec\theta\tan\theta\ d\theta$$

Substitute.

$$\frac{5}{2}\int \frac{1}{\sqrt{\left(-\frac{1}{2}+\frac{1}{2}\sec\theta+\frac{1}{2}\right)^2-\frac{1}{4}}}\cdot\frac{1}{2}\sec\theta\tan\theta\ d\theta$$

$$\frac{5}{4}\int \frac{\sec\theta\tan\theta}{\sqrt{\left(\frac{1}{2}\sec\theta\right)^2-\frac{1}{4}}}\ d\theta$$

$$\frac{5}{4}\int \frac{\sec\theta\tan\theta}{\sqrt{\frac{1}{4}\sec^2\theta-\frac{1}{4}}}\ d\theta$$

$$\frac{5}{4}\int \frac{\sec\theta\tan\theta}{\sqrt{\frac{1}{4}(\sec^2\theta-1)}}\ d\theta$$

Simplify with the trig identity $\sec^2\theta - 1 = \tan^2\theta$.

$$\frac{5}{4}\int \frac{\sec\theta\tan\theta}{\sqrt{\frac{1}{4}\tan^2\theta}}\ d\theta$$

$$\frac{5}{4}\int \frac{\sec\theta\tan\theta}{\frac{1}{2}\tan\theta}\ d\theta$$

$$\frac{5}{2}\int \sec\theta\ d\theta$$

■ 3. Set up and simplify the integral for trig substitution, but don't integrate.

$$\int \frac{dx}{x^2\sqrt{x^2-9}}$$

Solution:

Set up the trig substitution.

$$u^2 - a^2 = x^2 - 9$$

$u = x$ and $a = 3$

$$x = 3\sec\theta$$

$$dx = 3\sec\theta\tan\theta\ d\theta$$

Substitute.

$$\int \frac{3\sec\theta\tan\theta\ d\theta}{(3\sec\theta)^2\sqrt{(3\sec\theta)^2 - 9}}$$

$$\int \frac{3\sec\theta\tan\theta\ d\theta}{9\sec^2\theta\sqrt{9\sec^2\theta - 9}}$$

$$\int \frac{\tan\theta}{3\sec\theta\sqrt{9(\sec^2\theta - 1)}}\ d\theta$$

Simplify with the trig identity $\sec^2\theta - 1 = \tan^2\theta$.

$$\int \frac{\tan\theta}{3\sec\theta\sqrt{9\tan^2\theta}}\ d\theta$$

$$\int \frac{\tan\theta}{3\sec\theta(3\tan\theta)}\ d\theta$$

$$\frac{1}{9}\int \frac{1}{\sec\theta}\ d\theta$$

$$\frac{1}{9}\int \cos\theta\ d\theta$$

■ 4. Set up and simplify the integral for trig substitution, but don't integrate.

$$\int \frac{4\ dx}{x^2\sqrt{x^2-25}}$$

Solution:

Set up the trig substitution.

$$u^2 - a^2 = x^2 - 25$$

$u = x$ and $a = 5$

$$x = 5\sec\theta$$

$$dx = 5\sec\theta\tan\theta\ d\theta$$

Substitute.

$$\int \frac{4(5\sec\theta\tan\theta\ d\theta)}{(5\sec\theta)^2\sqrt{(5\sec\theta)^2-25}}$$

$$\int \frac{20\sec\theta\tan\theta\ d\theta}{25\sec^2\theta\sqrt{25\sec^2\theta-25}}$$

$$\int \frac{4\tan\theta}{5\sec\theta\sqrt{25(\sec^2\theta-1)}}\ d\theta$$

Simplify with the trig identity $\sec^2\theta - 1 = \tan^2\theta$.

$$\int \frac{4\tan\theta}{5\sec\theta\sqrt{25\tan^2\theta}}\ d\theta$$

$$\int \frac{4\tan\theta}{5\sec\theta(5\tan\theta)}\ d\theta$$

$$\frac{4}{25}\int \frac{1}{\sec\theta}\ d\theta$$

$$\frac{4}{25}\int \cos\theta\ d\theta$$

TRIGONOMETRIC SUBSTITUTION WITH SINE

■ 1. Set up and simplify the integral for trig substitution, but don't integrate.

$$\int \frac{3x}{\sqrt{64-49x^2}}\ dx$$

Solution:

Set up the trig substitution.

$$a^2-u^2=64-49x^2$$

$u=7x$ and $a=8$

$7x=8\sin\theta$ so $x=\frac{8}{7}\sin\theta$

$$dx=\frac{8}{7}\cos\theta\ d\theta$$

Substitute.

$$\int \frac{3\cdot\frac{8}{7}\sin\theta}{\sqrt{64-49\left(\frac{8}{7}\sin\theta\right)^2}}\cdot\frac{8}{7}\cos\theta\ d\theta$$

$$\frac{192}{49}\int \frac{\sin\theta\cos\theta}{\sqrt{64-49\left(\frac{64}{49}\sin^2\theta\right)}}\ d\theta$$

$$\frac{192}{49}\int \frac{\sin\theta\cos\theta}{\sqrt{64-64\sin^2\theta}}\ d\theta$$

$$\frac{192}{49}\int \frac{\sin\theta\cos\theta}{\sqrt{64(1-\sin^2\theta)}}\ d\theta$$

Simplify with the trig identity $1-\sin^2\theta=\cos^2\theta$.

$$\frac{192}{49}\int \frac{\sin\theta\cos\theta}{\sqrt{64\cos^2\theta}}\ d\theta$$

$$\frac{192}{49}\int \frac{\sin\theta\cos\theta}{8\cos\theta}\ d\theta$$

$$\frac{24}{49}\int \sin\theta\ d\theta$$

■ 2. Set up and simplify the integral for trig substitution, but don't integrate.

$$\int \frac{2x}{\sqrt{121-144x^2}}\ dx$$

Solution:

Set up the trig substitution.

$$a^2 - u^2 = 121 - 144x^2$$

$u = 12x$ and $a = 11$

$12x = 11\sin\theta$ so $x = \dfrac{11}{12}\sin\theta$

$$dx = \frac{11}{12}\cos\theta\ d\theta$$

Substitute.

$$\int \frac{2 \cdot \frac{11}{12}\sin\theta}{\sqrt{121 - 144\left(\frac{11}{12}\sin\theta\right)^2}} \cdot \frac{11}{12}\cos\theta\ d\theta$$

$$\frac{121}{72}\int \frac{\sin\theta\cos\theta}{\sqrt{121 - 144\left(\frac{121}{144}\sin^2\theta\right)}}\ d\theta$$

$$\frac{121}{72}\int \frac{\sin\theta\cos\theta}{\sqrt{121 - 121\sin^2\theta}}\ d\theta$$

$$\frac{121}{72}\int \frac{\sin\theta\cos\theta}{\sqrt{121(1 - \sin^2\theta)}}\ d\theta$$

Simplify with the trig identity $1 - \sin^2\theta = \cos^2\theta$.

$$\frac{121}{72}\int \frac{\sin\theta\cos\theta}{11\cos\theta}\ d\theta$$

$$\frac{11}{72}\int \sin\theta\ d\theta$$

■ 3. Set up and simplify the integral for trig substitution, but don't integrate.

$$\int \frac{6x}{\sqrt{81-36x^2}}\ dx$$

Solution:

Rewrite the integrand.

$$\int \frac{6x}{\sqrt{9(9-4x^2)}}\ dx$$

$$\int \frac{6x}{3\sqrt{9-4x^2}}\ dx$$

$$\int \frac{2x}{\sqrt{9-4x^2}}\ dx$$

Set up the trig substitution.

$$a^2 - u^2 = 9 - 4x^2$$

$u = 2x$ and $a = 3$

$$2x = 3\sin\theta \text{ so } x = \frac{3}{2}\sin\theta$$

$$dx = \frac{3}{2}\cos\theta\ d\theta$$

Substitute.

$$\int \frac{2 \cdot \frac{3}{2}\sin\theta}{\sqrt{9 - 4\left(\frac{3}{2}\sin\theta\right)^2}} \cdot \frac{3}{2}\cos\theta\ d\theta$$

$$\frac{9}{2}\int \frac{\sin\theta\cos\theta}{\sqrt{9 - 4\left(\frac{9}{4}\sin^2\theta\right)}}\ d\theta$$

$$\frac{9}{2}\int \frac{\sin\theta\cos\theta}{\sqrt{9 - 9\sin^2\theta}}\ d\theta$$

$$\frac{9}{2}\int \frac{\sin\theta\cos\theta}{\sqrt{9(1 - \sin^2\theta)}}\ d\theta$$

$$\frac{3}{2}\int \frac{\sin\theta\cos\theta}{\sqrt{1 - \sin^2\theta}}\ d\theta$$

Simplify with the trig identity $1 - \sin^2\theta = \cos^2\theta$.

$$\frac{3}{2}\int \frac{\sin\theta\cos\theta}{\sqrt{\cos^2\theta}}\ d\theta$$

$$\frac{3}{2}\int \frac{\sin\theta\cos\theta}{\cos\theta}\ d\theta$$

$$\frac{3}{2}\int \sin\theta \ d\theta$$

■ 4. Set up and simplify the integral for trig substitution, but don't integrate.

$$\int \frac{35x}{\sqrt{25-100x^2}} \ dx$$

Solution:

Rewrite the integrand.

$$\int \frac{35x}{\sqrt{25(1-4x^2)}} \ dx$$

$$\int \frac{35x}{5\sqrt{1-4x^2}} \ dx$$

$$\int \frac{7x}{\sqrt{1-4x^2}} \ dx$$

Set up the trig substitution.

$$a^2-u^2=1-4x^2$$

$u=2x$ and $a=1$

$$2x = \sin\theta \text{ so } x = \frac{1}{2}\sin\theta$$

$$dx = \frac{1}{2}\cos\theta\ d\theta$$

Substitute.

$$\int \frac{7 \cdot \frac{1}{2}\sin\theta}{\sqrt{1 - 4\left(\frac{1}{2}\sin\theta\right)^2}} \cdot \frac{1}{2}\cos\theta\ d\theta$$

$$\frac{7}{4}\int \frac{\sin\theta\cos\theta}{\sqrt{1 - 4\left(\frac{1}{4}\sin^2\theta\right)}}\ d\theta$$

$$\frac{7}{4}\int \frac{\sin\theta\cos\theta}{\sqrt{1 - \sin^2\theta}}\ d\theta$$

Simplify with the trig identity $1 - \sin^2\theta = \cos^2\theta$.

$$\frac{7}{4}\int \frac{\sin\theta\cos\theta}{\sqrt{\cos^2\theta}}\ d\theta$$

$$\frac{7}{4}\int \frac{\sin\theta\cos\theta}{\cos\theta}\ d\theta$$

$$\frac{7}{4}\int \sin\theta\ d\theta$$

TRIGONOMETRIC SUBSTITUTION WITH TANGENT

■ 1. Set up and simplify the integral for trig substitution, but don't integrate.

$$\int \sqrt{36x^2 + 25}\ dx$$

Solution:

Set up the trig substitution.

$$u^2 + a^2 = 36x^2 + 25$$

$u = 6x$ and $a = 5$

$6x = 5\tan\theta$ so $x = \frac{5}{6}\tan\theta$

$$dx = \frac{5}{6}\sec^2\theta\ d\theta$$

Substitute.

$$\int \sqrt{36\left(\frac{5}{6}\tan\theta\right)^2 + 25} \cdot \frac{5}{6}\sec^2\theta\ d\theta$$

$$\frac{5}{6}\int \sec^2\theta \sqrt{36\left(\frac{25}{36}\tan^2\theta\right) + 25}\ d\theta$$

$$\frac{5}{6}\int \sec^2\theta\sqrt{25\tan^2\theta+25}\ d\theta$$

$$\frac{5}{6}\int \sec^2\theta\sqrt{25(\tan^2\theta+1)}\ d\theta$$

$$\frac{25}{6}\int \sec^2\theta\sqrt{\tan^2\theta+1}\ d\theta$$

Simplify with the trig identity $\tan^2\theta+1=\sec^2\theta$.

$$\frac{25}{6}\int \sec^2\theta\sqrt{\sec^2\theta}\ d\theta$$

$$\frac{25}{6}\int \sec^2\theta\sec\theta\ d\theta$$

$$\frac{25}{6}\int \sec^3\theta\ d\theta$$

■ 2. Set up and simplify the integral for trig substitution, but don't integrate.

$$\int\sqrt{4x^2+81}\ dx$$

Solution:

Set up the trig substitution.

$$u^2+a^2=4x^2+81$$

$u = 2x$ and $a = 9$

$2x = 9\tan\theta$ so $x = \dfrac{9}{2}\tan\theta$

$$dx = \frac{9}{2}\sec^2\theta\ d\theta$$

Substitute.

$$\int \sqrt{4\left(\frac{9}{2}\tan\theta\right)^2 + 81} \cdot \frac{9}{2}\sec^2\theta\ d\theta$$

$$\frac{9}{2}\int \sec^2\theta\sqrt{4\left(\frac{81}{4}\tan^2\theta\right) + 81}\ d\theta$$

$$\frac{9}{2}\int \sec^2\theta\sqrt{81\tan^2\theta + 81}\ d\theta$$

$$\frac{9}{2}\int \sec^2\theta\sqrt{81(\tan^2\theta + 1)}\ d\theta$$

$$\frac{81}{2}\int \sec^2\theta\sqrt{\tan^2\theta + 1}\ d\theta$$

Simplify with the trig identity $\tan^2\theta + 1 = \sec^2\theta$.

$$\frac{81}{2}\int \sec^2\theta\sqrt{\sec^2\theta}\ d\theta$$

$$\frac{81}{2}\int \sec^2\theta\sec\theta\ d\theta$$

$$\frac{81}{2}\int \sec^3\theta\ d\theta$$

■ 3. Set up and simplify the integral for trig substitution, but don't integrate.

$$\int \frac{7}{\sqrt{x^2+4x+8}}\ dx$$

Solution:

Rewrite the integrand by completing the square.

$$\int \frac{7}{\sqrt{(x^2+4x+4)+4}}\ dx$$

$$\int \frac{7}{\sqrt{(x+2)^2+4}}\ dx$$

Set up the trig substitution.

$$u^2+a^2=(x+2)^2+4$$

$u=x+2$ and $a=2$

$x+2=2\tan\theta$ so $x=-2+2\tan\theta$

$$dx=2\sec^2\theta\ d\theta$$

Substitute.

$$\int \frac{7}{\sqrt{(-2+2\tan\theta+2)^2+4}} \cdot 2\sec^2\theta \ d\theta$$

$$14\int \frac{\sec^2\theta}{\sqrt{(2\tan\theta)^2+4}} \ d\theta$$

$$14\int \frac{\sec^2\theta}{\sqrt{4\tan^2\theta+4}} \ d\theta$$

$$14\int \frac{\sec^2\theta}{\sqrt{4(\tan^2\theta+1)}} \ d\theta$$

$$7\int \frac{\sec^2\theta}{\sqrt{\tan^2\theta+1}} \ d\theta$$

Simplify with the trig identity $\tan^2\theta+1=\sec^2\theta$.

$$7\int \frac{\sec^2\theta}{\sqrt{\sec^2\theta}} \ d\theta$$

$$7\int \frac{\sec^2\theta}{\sec\theta} \ d\theta$$

$$7\int \sec\theta \ d\theta$$

IMPROPER INTEGRALS, CASE 1

1. Evaluate the improper integral.

$$\int_1^{\infty} \frac{5}{x^3}\ dx$$

Solution:

Replace the upper limit, rewriting the integral as

$$\lim_{b \to \infty} \int_1^b \frac{5}{x^3}\ dx$$

$$\lim_{b \to \infty} \int_1^b 5x^{-3}\ dx$$

Integrate, then evaluate over the interval.

$$\lim_{b \to \infty} \frac{5x^{-2}}{-2}\Bigg|_1^b$$

$$-\frac{5}{2} \lim_{b \to \infty} \frac{1}{x^2}\Bigg|_1^b$$

$$-\frac{5}{2} \lim_{b \to \infty} \frac{1}{b^2} - \frac{1}{1^2}$$

$$-\frac{5}{2}\left(\frac{1}{\infty^2}-1\right)$$

$$-\frac{5}{2}(0-1)=\frac{5}{2}$$

■ 2. Evaluate the improper integral.

$$\int_3^{\infty} \frac{7}{(x-2)^2}\ dx$$

Solution:

Replace the upper limit, rewriting the integral as

$$\lim_{b\to\infty}\int_3^b \frac{7}{(x-2)^2}\ dx$$

$$\lim_{b\to\infty}\int_3^b 7(x-2)^{-2}\ dx$$

Integrate, then evaluate over the interval.

$$\lim_{b\to\infty}\frac{7(x-2)^{-1}}{-1}\Bigg|_3^b$$

$$-7\lim_{b\to\infty}\frac{1}{x-2}\Bigg|_3^b$$

$$-7 \lim_{b \to \infty} \frac{1}{b-2} - \frac{1}{3-2}$$

$$-7\left(\frac{1}{\infty - 2} - \frac{1}{1}\right)$$

$$-7(0-1)$$

$$7$$

3. Evaluate the improper integral.

$$\int_0^\infty 2e^{-2x}\ dx$$

Solution:

Replace the upper limit, rewriting the integral as

$$\lim_{b \to \infty} \int_0^b 2e^{-2x}\ dx$$

Integrate, then evaluate over the interval.

$$\lim_{b \to \infty} -e^{-2x} \Big|_0^b$$

$$-\lim_{b \to \infty} \frac{1}{e^{2x}} \Big|_0^b$$

$$-\lim_{b\to\infty}\frac{1}{e^{2b}}-\frac{1}{e^{2(0)}}$$

$$-\left(\frac{1}{e^{2(\infty)}}-\frac{1}{1}\right)$$

$$-(0-1)$$

$$1$$

■ 4. Evaluate the improper integral.

$$\int_0^\infty \frac{3x}{2+2x^2}\ dx$$

Solution:

Simplify the integrand.

$$\int_0^\infty \frac{3x}{2+2x^2}\ dx=\int_0^\infty \frac{3x}{2(1+x^2)}\ dx=\frac{3}{2}\int_0^\infty \frac{x}{1+x^2}\ dx$$

Replace the upper limit, rewriting the integral as

$$\lim_{b\to\infty}\frac{3}{2}\int_0^b \frac{x}{1+x^2}\ dx$$

Let

$$u=1+x^2$$

$$du = 2x\ dx, \text{ so } dx = \frac{du}{2x}$$

Substitute, then integrate.

$$\lim_{b \to \infty} \frac{3}{2} \int_{x=0}^{x=b} \frac{x}{u} \left(\frac{du}{2x} \right)$$

$$\lim_{b \to \infty} \frac{3}{4} \int_{x=0}^{x=b} \frac{1}{u}\ du$$

$$\lim_{b \to \infty} \frac{3}{4} \ln|u| \Big|_{x=0}^{x=b}$$

Back-substitute.

$$\lim_{b \to \infty} \frac{3}{4} \ln|1 + x^2| \Big|_{x=0}^{x=b}$$

$$\lim_{b \to \infty} \frac{3}{4} \ln(1 + x^2) \Big|_0^b$$

Evaluate over the interval.

$$\lim_{b \to \infty} \frac{3}{4} \ln(1 + b^2) - \frac{3}{4} \ln(1 + 0^2)$$

$$\lim_{b \to \infty} \frac{3}{4} \ln(1 + b^2) - \frac{3}{4} \ln(1)$$

$$\lim_{b \to \infty} \frac{3}{4} \ln(1 + b^2) - \frac{3}{4}(0)$$

$$\lim_{b\to\infty} \frac{3}{4}\ln(1+b^2)$$

$$\frac{3}{4}\ln(1+\infty^2)$$

$$\frac{3}{4}\ln(\infty)$$

$$\frac{3}{4}(\infty)$$

$$\infty$$

IMPROPER INTEGRALS, CASE 2

1. Evaluate the improper integral.

$$\int_{-\infty}^{0} e^{3x}\ dx$$

Solution:

Replace the lower limit, rewriting the integral as

$$\lim_{a\to-\infty}\int_{a}^{0} e^{3x}\ dx$$

Integrate, then evaluate over the interval.

$$\lim_{a\to-\infty}\frac{e^{3x}}{3}\Big|_{a}^{0}$$

$$\lim_{a\to-\infty}\frac{e^{3(0)}}{3}-\frac{e^{3a}}{3}$$

$$\lim_{a\to-\infty}\frac{1}{3}-\frac{e^{3a}}{3}$$

$$\frac{1}{3}-\frac{e^{3(-\infty)}}{3}$$

$$\frac{1}{3}-\frac{1}{3e^{\infty}}$$

$$\frac{1}{3}-0$$

$$\frac{1}{3}$$

■ 2. Evaluate the improper integral.

$$\int_{-\infty}^{1} xe^{x^2}\ dx$$

Solution:

Replace the lower limit, rewriting the integral as

$$\lim_{a\to-\infty}\int_{a}^{1} xe^{x^2}\ dx$$

Use u-substitution.

$$u=x^2$$

$du=2x\ dx$, so $dx=\dfrac{du}{2x}$

Substitute.

$$\lim_{a\to-\infty}\int_{x=a}^{x=1} xe^{u}\left(\frac{du}{2x}\right)$$

$$\frac{1}{2}\lim_{a\to-\infty}\int_{x=a}^{x=1} e^u\ du$$

Integrate, back-substitute, then evaluate over the interval.

$$\frac{1}{2}\lim_{a\to-\infty} e^u\Big|_{x=a}^{x=1}$$

$$\frac{1}{2}\lim_{a\to-\infty} e^{x^2}\Big|_{a}^{1}$$

$$\frac{1}{2}\lim_{a\to-\infty} e^{1^2}-e^{a^2}$$

$$\frac{1}{2}\lim_{a\to-\infty} e-e^{a^2}$$

$$\frac{1}{2}\left(e-e^{(-\infty)^2}\right)$$

$$\frac{1}{2}(e-\infty)$$

$$\frac{1}{2}(-\infty)$$

$$-\infty$$

■ 3. Evaluate the improper integral.

$$\int_{-\infty}^{-2}\frac{2}{x-1}-\frac{2}{x+1}\ dx$$

Solution:

Replace the lower limit, rewriting the integral as

$$\lim_{a\to-\infty}\int_a^{-2}\frac{2}{x-1}-\frac{2}{x+1}\ dx$$

Integrate, then evaluate over the interval.

$$\lim_{a\to-\infty} 2\ln|x-1|-2\ln|x+1|\Big|_a^{-2}$$

$$2\lim_{a\to-\infty} \ln|x-1|-\ln|x+1|\Big|_a^{-2}$$

$$2\lim_{a\to-\infty} \ln|-2-1|-\ln|-2+1|-\left(\ln|a-1|-\ln|a+1|\right)$$

$$2\lim_{a\to-\infty} \ln|-3|-\ln|-1|-\ln|a-1|+\ln|a+1|$$

$$2\lim_{a\to-\infty} \ln 3-\ln 1-\ln|a-1|+\ln|a+1|$$

$$2\lim_{a\to-\infty} \ln 3-\ln|a-1|+\ln|a+1|$$

$$2\lim_{a\to-\infty} \ln 3+\ln\frac{|a+1|}{|a-1|}$$

$$2\ln 3+2\ln\left(\lim_{a\to-\infty}\frac{|a+1|}{|a-1|}\right)$$

$$2\ln 3 + 2\ln\left(\lim_{a\to-\infty}\frac{|1+\frac{1}{a}|}{|1-\frac{1}{a}|}\right)$$

$$2\ln 3 + 2\ln\left(\frac{1}{1}\right)$$

$$2\ln 3 + 2(0)$$

$$2\ln 3$$

$$\ln 9$$

■ 4. Evaluate the improper integral.

$$\int_{-\infty}^{3}\frac{3}{x^2+9}\ dx$$

Solution:

Replace the lower limit, rewriting the integral as

$$\lim_{a\to-\infty}\int_{a}^{3}\frac{3}{x^2+9}\ dx$$

Integrate, then evaluate over the interval.

$$\lim_{a\to-\infty}\arctan\frac{x}{3}\Big|_{a}^{3}$$

$$\lim_{a \to -\infty} \arctan \frac{3}{3} - \arctan \frac{a}{3}$$

$$\lim_{a \to -\infty} \arctan 1 - \arctan \frac{a}{3}$$

$$\lim_{a \to -\infty} \frac{\pi}{4} - \arctan \frac{a}{3}$$

$$\frac{\pi}{4} - \arctan \frac{-\infty}{3}$$

$$\frac{\pi}{4} - \arctan(-\infty)$$

$$\frac{\pi}{4} - \left(-\frac{\pi}{2}\right)$$

$$\frac{\pi}{4} + \frac{\pi}{2}$$

$$\frac{3\pi}{4}$$

5. Evaluate the improper integral.

$$\int_{-\infty}^{0} \frac{2\ dx}{e^x}$$

Solution:

Replace the lower limit, rewriting the integral as

$$\lim_{a \to -\infty} \int_a^0 \frac{2\ dx}{e^x}$$

$$2 \lim_{a \to -\infty} \int_a^0 \frac{1}{e^x}\ dx$$

$$2 \lim_{a \to -\infty} \int_a^0 e^{-x}\ dx$$

Integrate, then evaluate over the interval.

$$2 \lim_{a \to -\infty} -e^{-x} \Big|_a^0$$

$$2 \lim_{a \to -\infty} -e^{-0} - (-e^{-a})$$

$$2 \lim_{a \to -\infty} -e^{-0} + e^{-a}$$

$$2 \lim_{a \to -\infty} -1 + e^{-a}$$

$$2 \lim_{a \to -\infty} e^{-a} - 1$$

$$2(e^{-(-\infty)} - 1)$$

$$2(e^{\infty} - 1)$$

$$2(\infty - 1)$$

$$2(\infty)$$

$$\infty$$

■ 6. Evaluate the improper integral.

$$\int_{-\infty}^{0} 4e^{-4x}\ dx$$

Solution:

Replace the lower limit, rewriting the integral as

$$\lim_{a \to -\infty} \int_{a}^{0} 4e^{-4x}\ dx$$

Integrate, then evaluate over the interval.

$$\lim_{a \to -\infty} \frac{4}{-4} e^{-4x} \Big|_{a}^{0}$$

$$\lim_{a \to -\infty} -e^{-4x} \Big|_{a}^{0}$$

$$\lim_{a \to -\infty} -e^{-4(0)} - (-e^{-4(a)})$$

$$\lim_{a \to -\infty} -1 + e^{-4a}$$

$$\lim_{a \to -\infty} e^{-4a} - 1$$

$$e^{-4(-\infty)} - 1$$

$$e^{4\infty} - 1$$

$$e^{\infty} - 1$$

$$\infty - 1$$

$$\infty$$

IMPROPER INTEGRALS, CASE 3

1. Evaluate the improper integral.

$$\int_{-\infty}^{\infty} 2xe^{-x^2}\ dx$$

Solution:

Separate the integral in two, splitting the interval at 0.

$$\int_{-\infty}^{0} 2xe^{-x^2}\ dx + \int_{0}^{\infty} 2xe^{-x^2}\ dx$$

Replace the infinite limits, rewriting the integral as

$$\lim_{a\to-\infty}\int_{a}^{0} 2xe^{-x^2}\ dx + \lim_{b\to\infty}\int_{0}^{b} 2xe^{-x^2}\ dx$$

Use u-substitution.

$$u = -x^2$$

$du = -2x\ dx$, so $dx = \dfrac{du}{-2x}$

Substitute.

$$\lim_{a\to-\infty}\int_{x=a}^{x=0} 2xe^{u}\left(\frac{du}{-2x}\right) + \lim_{b\to\infty}\int_{x=0}^{x=b} 2xe^{u}\left(\frac{du}{-2x}\right)$$

$$\lim_{a \to -\infty} \int_{x=a}^{x=0} -e^u \ du + \lim_{b \to \infty} \int_{x=0}^{x=b} -e^u \ du$$

Integrate, then evaluate over the interval.

$$\lim_{a \to -\infty} -e^u \Big|_{x=a}^{x=0} + \lim_{b \to \infty} -e^u \Big|_{x=0}^{x=b}$$

$$\lim_{a \to -\infty} -e^{-x^2} \Big|_a^0 + \lim_{b \to \infty} -e^{-x^2} \Big|_0^b$$

$$\lim_{a \to -\infty} -e^{-0^2} - (-e^{-a^2}) + \lim_{b \to \infty} -e^{-b^2} - (-e^{-0^2})$$

$$\lim_{a \to -\infty} -e^0 + e^{-a^2} + \lim_{b \to \infty} -e^{-b^2} + e^0$$

$$\lim_{a \to -\infty} e^{-a^2} - 1 + \lim_{b \to \infty} 1 - e^{-b^2}$$

$$e^{-(-\infty)^2} - 1 + 1 - e^{-(\infty)^2}$$

$$e^{-\infty} - e^{-\infty}$$

$$0$$

2. Evaluate the improper integral.

$$\int_{-\infty}^{\infty} \frac{3 \ dx}{x^2 + 1}$$

Solution:

Separate the integral in two, splitting the interval at 0.

$$\int_{-\infty}^{0} \frac{3\ dx}{x^2+1} + \int_{0}^{\infty} \frac{3\ dx}{x^2+1}$$

Replace the infinite limits, rewriting the integral as

$$\lim_{a\to-\infty} \int_{a}^{0} \frac{3\ dx}{x^2+1} + \lim_{b\to\infty} \int_{0}^{b} \frac{3\ dx}{x^2+1}$$

$$3 \lim_{a\to-\infty} \int_{a}^{0} \frac{1}{x^2+1}\ dx + 3 \lim_{b\to\infty} \int_{0}^{b} \frac{1}{x^2+1}\ dx$$

Integrate, then evaluate over the interval.

$$3 \lim_{a\to-\infty} \arctan x \Big|_{a}^{0} + 3 \lim_{b\to\infty} \arctan x \Big|_{0}^{b}$$

$$3 \lim_{a\to-\infty} \arctan 0 - \arctan a + 3 \lim_{b\to\infty} \arctan b - \arctan 0$$

$$3(\arctan 0 - \arctan(-\infty)) + 3(\arctan(\infty) - \arctan 0)$$

$$3\left(0 - \left(-\frac{\pi}{2}\right)\right) + 3\left(\frac{\pi}{2} - 0\right)$$

$$\frac{3\pi}{2} + \frac{3\pi}{2}$$

$$3\pi$$

■ 3. Evaluate the improper integral.

$$\int_{-\infty}^{\infty} x^2 + 7x + 1 \ dx$$

Solution:

Separate the integral in two, splitting the interval at 0.

$$\int_{-\infty}^{0} x^2 + 7x + 1 \ dx + \int_{0}^{\infty} x^2 + 7x + 1 \ dx$$

Replace the infinite limits, rewriting the integral as

$$\lim_{a \to -\infty} \int_{a}^{0} x^2 + 7x + 1 \ dx + \lim_{b \to \infty} \int_{0}^{b} x^2 + 7x + 1 \ dx$$

Integrate, then evaluate over the interval.

$$\lim_{a \to -\infty} \frac{1}{3}x^3 + \frac{7}{2}x^2 + x \Big|_{a}^{0} + \lim_{b \to \infty} \frac{1}{3}x^3 + \frac{7}{2}x^2 + x \Big|_{0}^{b}$$

$$\lim_{a \to -\infty} \frac{1}{3}(0)^3 + \frac{7}{2}(0)^2 + 0 - \left(\frac{1}{3}(a)^3 + \frac{7}{2}(a)^2 + a\right) + \lim_{b \to \infty} \frac{1}{3}(b)^3 + \frac{7}{2}(b)^2 + b - \left(\frac{1}{3}(0)^3 + \frac{7}{2}(0)^2 + 0\right)$$

$$\lim_{a \to -\infty} -\frac{1}{3}a^3 - \frac{7}{2}a^2 - a + \lim_{b \to \infty} \frac{1}{3}b^3 + \frac{7}{2}b^2 + b$$

$$-\frac{1}{3}(-\infty)^3 - \frac{7}{2}(-\infty)^2 - (-\infty) + \frac{1}{3}(\infty)^3 + \frac{7}{2}(\infty)^2 + (\infty)$$

$$\frac{1}{3}\infty - \frac{7}{2}\infty + \infty + \frac{1}{3}\infty + \frac{7}{2}\infty + \infty$$

$$\frac{2}{3}\infty + \infty$$

The value diverges to ∞.

■ 4. Evaluate the improper integral.

$$\int_{-\infty}^{\infty} 3x^2e^{-x^3}\ dx$$

Solution:

Separate the integral in two, splitting the interval at 0.

$$\int_{-\infty}^{0} 3x^2e^{-x^3}\ dx + \int_{0}^{\infty} 3x^2e^{-x^3}\ dx$$

Replace the infinite limits, rewriting the integral as

$$\lim_{a\to-\infty}\int_{a}^{0} 3x^2e^{-x^3}\ dx + \lim_{b\to\infty}\int_{0}^{b} 3x^2e^{-x^3}\ dx$$

Use u-substitution.

$$u = -x^3$$

$$du = -3x^2\ dx, \text{ so } dx = \frac{du}{-3x^2}$$

Substitute.

$$\lim_{a\to-\infty}\int_{x=a}^{x=0} 3x^2e^u\left(\frac{du}{-3x^2}\right) + \lim_{b\to\infty}\int_{x=0}^{x=b} 3x^2e^u\left(\frac{du}{-3x^2}\right)$$

$$\lim_{a\to-\infty}\int_{x=a}^{x=0} -e^u\ du + \lim_{b\to\infty}\int_{x=0}^{x=b} -e^u\ du$$

Integrate, then evaluate over the interval.

$$\lim_{a\to-\infty} -e^u\Big|_{x=a}^{x=0} + \lim_{b\to\infty} -e^u\Big|_{x=0}^{x=b}$$

$$\lim_{a\to-\infty} -e^{-x^3}\Big|_a^0 + \lim_{b\to\infty} -e^{-x^3}\Big|_0^b$$

$$\lim_{a\to-\infty} -e^{-0^3} - (-e^{-a^3}) + \lim_{b\to\infty} -e^{-b^3} - (-e^{-0^3})$$

$$\lim_{a\to-\infty} -e^{-0^3} + e^{-a^3} + \lim_{b\to\infty} -e^{-b^3} + e^{-0^3}$$

$$\lim_{a\to-\infty} e^{-a^3} - 1 + \lim_{b\to\infty} 1 - e^{-b^3}$$

$$e^{-(-\infty)^3} - 1 + 1 - e^{-(\infty)^3}$$

$$e^{-(-\infty)^3} - e^{-(\infty)^3}$$

$$e^{\infty} - e^{-\infty}$$

$$\infty - 0$$

∞

IMPROPER INTEGRALS, CASE 4

1. Evaluate the improper integral.

$$\int_{-\frac{\pi}{2}}^{0} \frac{3\cos x}{2\sin x}\ dx$$

Solution:

The integrand is undefined at the upper bound, $x = 0$. Therefore, rewrite the integral as

$$\lim_{b\to 0^-} \int_{-\frac{\pi}{2}}^{b} \frac{3\cos x}{2\sin x}\ dx$$

$$\frac{3}{2}\lim_{b\to 0^-} \int_{-\frac{\pi}{2}}^{b} \frac{\cos x}{\sin x}\ dx$$

Use u-substitution.

$$u = \sin x$$

$$du = \cos x\ dx\text{, so } dx = \frac{du}{\cos x}$$

Substitute.

$$\frac{3}{2}\lim_{b\to 0^-} \int_{x=-\frac{\pi}{2}}^{x=b} \frac{\cos x}{u}\left(\frac{du}{\cos x}\right)$$

$$\frac{3}{2}\lim_{b\to 0^-}\int_{x=-\frac{\pi}{2}}^{x=b}\frac{1}{u}\ du$$

Integrate, then evaluate over the interval.

$$\frac{3}{2}\lim_{b\to 0^-}\ln|u|\Big|_{x=-\frac{\pi}{2}}^{x=b}$$

$$\frac{3}{2}\lim_{b\to 0^-}\ln|\sin x|\Big|_{-\frac{\pi}{2}}^{b}$$

$$\frac{3}{2}\lim_{b\to 0^-}\ln|\sin b|-\ln\left|\sin\left(-\frac{\pi}{2}\right)\right|$$

$$\frac{3}{2}\lim_{b\to 0^-}\ln|\sin b|-\ln|-1|$$

$$\frac{3}{2}\lim_{b\to 0^-}\ln|\sin b|-\ln 1$$

$$\frac{3}{2}\lim_{b\to 0^-}\ln|\sin b|$$

$$\frac{3}{2}(-\infty)$$

$$-\infty$$

■ 2. Evaluate the improper integral.

$$\int_{-8}^{0} \frac{e^x\ dx}{e^x - 1}$$

Solution:

The integrand is undefined at the upper bound, $x = 0$. Therefore, rewrite the integral as

$$\lim_{b \to 0^-} \int_{-8}^{b} \frac{e^x\ dx}{e^x - 1}$$

Use u-substitution.

$$u = e^x - 1$$

$$du = e^x\ dx, \text{ so } dx = \frac{du}{e^x}$$

Substitute.

$$\lim_{b \to 0^-} \int_{x=-8}^{x=b} \frac{e^x}{u} \left(\frac{du}{e^x} \right)$$

$$\lim_{b \to 0^-} \int_{x=-8}^{x=b} \frac{1}{u}\ du$$

Integrate, then evaluate over the interval.

$$\lim_{b \to 0^-} \ln|u| \Big|_{x=-8}^{x=b}$$

$$\lim_{b\to 0^-} \ln|e^x - 1| \Big|_{-8}^{b}$$

$$\lim_{b\to 0^-} \ln|e^b - 1| - \ln|e^{-8} - 1|$$

$$\ln|1 - 1| - \ln|e^{-8} - 1|$$

$$\ln|0| - \ln|e^{-8} - 1|$$

$$-\infty - \ln|e^{-8} - 1|$$

$$-\infty$$

■ 3. Evaluate the improper integral.

$$\int_{-9}^{0} \frac{e^{\sqrt{-x}}\ dx}{\sqrt{-x}}$$

Solution:

The integrand is undefined at the upper bound, $x = 0$. Therefore, rewrite the integral as

$$\lim_{b\to 0^-} \int_{-9}^{b} \frac{e^{\sqrt{-x}}\ dx}{\sqrt{-x}}$$

Use u-substitution.

$$u = \sqrt{-x}$$

$$du = -\frac{1}{2\sqrt{-x}}\ dx, \text{ so } dx = -2\sqrt{-x}\ du$$

Substitute.

$$\lim_{b \to 0^-} \int_{x=-9}^{x=b} \frac{e^u}{u} \left(-2\sqrt{-x}\ du\right)$$

$$-2 \lim_{b \to 0^-} \int_{x=-9}^{x=b} \frac{e^u}{u} \left(u\ du\right)$$

$$-2 \lim_{b \to 0^-} \int_{x=-9}^{x=b} e^u\ du$$

Integrate, then evaluate over the interval.

$$-2 \lim_{b \to 0^-} e^u \Big|_{x=-9}^{x=b}$$

$$-2 \lim_{b \to 0^-} e^{\sqrt{-x}} \Big|_{-9}^{b}$$

$$-2 \lim_{b \to 0^-} e^{\sqrt{-b}} - e^{\sqrt{-(-9)}}$$

$$-2 \lim_{b \to 0^-} e^{\sqrt{-b}} - e^{\sqrt{9}}$$

$$-2(e^{\sqrt{-0}} - e^3)$$

$$-2(1 - e^3)$$

$$-2+2e^3$$

$$2e^3-2$$

■ 4. Evaluate the improper integral.

$$\int_1^3 \frac{2x-3}{\sqrt{3x-x^2}}\ dx$$

Solution:

The integrand is undefined at the upper bound, $x=3$. Therefore, rewrite the integral as

$$\lim_{b\to 3^-}\int_1^b \frac{2x-3}{\sqrt{3x-x^2}}\ dx$$

Use u-substitution.

$$u=3x-x^2$$

$du=(3-2x)\ dx$, so $dx=\dfrac{du}{3-2x}$

Substitute.

$$\lim_{b\to 3^-}\int_{x=1}^{x=b} \frac{2x-3}{\sqrt{u}}\left(\frac{du}{3-2x}\right)$$

$$-\lim_{b\to3^-}\int_{x=1}^{x=b}\frac{2x-3}{\sqrt{u}}\left(\frac{du}{-(3-2x)}\right)$$

$$-\lim_{b\to3^-}\int_{x=1}^{x=b}\frac{2x-3}{\sqrt{u}}\left(\frac{du}{2x-3}\right)$$

$$-\lim_{b\to3^-}\int_{x=1}^{x=b}\frac{1}{\sqrt{u}}\ du$$

$$-\lim_{b\to3^-}\int_{x=1}^{x=b}u^{-\frac{1}{2}}\ du$$

Integrate, then evaluate over the interval.

$$-\lim_{b\to3^-}2u^{\frac{1}{2}}\Big|_{x=1}^{x=b}$$

$$-\lim_{b\to3^-}2\sqrt{u}\Big|_{x=1}^{x=b}$$

$$-\lim_{b\to3^-}2\sqrt{3x-x^2}\Big|_{1}^{b}$$

$$-\lim_{b\to3^-}2\sqrt{3b-b^2}-2\sqrt{3(1)-1^2}$$

$$-\lim_{b\to3^-}2\sqrt{3b-b^2}-2\sqrt{2}$$

$$\lim_{b\to3^-}2\sqrt{2}-2\sqrt{3b-b^2}$$

$$2\sqrt{2} - 2\sqrt{3(3) - 3^2}$$

$$2\sqrt{2} - 2\sqrt{9 - 9}$$

$$2\sqrt{2} - 2(0)$$

$$2\sqrt{2}$$

■ 5. Evaluate the improper integral.

$$\int_0^{2\sqrt{2}} \frac{x}{\sqrt{8 - x^2}}\ dx$$

Solution:

The integrand is undefined at the upper bound, $x = 2\sqrt{2}$. Therefore, rewrite the integral as

$$\lim_{b \to 2\sqrt{2}^-} \int_0^b \frac{x}{\sqrt{8 - x^2}}\ dx$$

Use u-substitution.

$$u = 8 - x^2$$

$du = -2x\ dx$, so $dx = \dfrac{du}{-2x}$

Substitute.

$$\lim_{b \to 2\sqrt{2}^-} \int_{x=0}^{x=b} \frac{x}{\sqrt{u}} \left(\frac{du}{-2x} \right)$$

$$-\frac{1}{2} \lim_{b \to 2\sqrt{2}^-} \int_{x=0}^{x=b} \frac{1}{\sqrt{u}}\ du$$

$$-\frac{1}{2} \lim_{b \to 2\sqrt{2}^-} \int_{x=0}^{x=b} u^{-\frac{1}{2}}\ du$$

Integrate, then evaluate over the interval.

$$-\frac{1}{2} \lim_{b \to 2\sqrt{2}^-} 2u^{\frac{1}{2}} \Big|_{x=0}^{x=b}$$

$$-\lim_{b \to 2\sqrt{2}^-} \sqrt{u} \Big|_{x=0}^{x=b}$$

$$-\lim_{b \to 2\sqrt{2}^-} \sqrt{8 - x^2} \Big|_{0}^{b}$$

$$-\lim_{b \to 2\sqrt{2}^-} \sqrt{8 - b^2} - \sqrt{8 - 0^2}$$

$$-\left(\sqrt{8 - (2\sqrt{2})^2} - \sqrt{8} \right)$$

$$-\left(\sqrt{8 - 4(2)} - \sqrt{8} \right)$$

$$-\sqrt{8 - 8} + \sqrt{8}$$

$$\sqrt{8} - \sqrt{0}$$

$$2\sqrt{2}$$

6. Evaluate the improper integral.

$$\int_1^3 \frac{x-1}{x^2-4x+3}\ dx$$

Solution:

Simplify the integrand by factoring the denominator.

$$\int_1^3 \frac{x-1}{(x-1)(x-3)}\ dx$$

$$\int_1^3 \frac{1}{x-3}\ dx$$

The integrand is undefined at the upper bound, $x=3$. Therefore, rewrite the integral as

$$\lim_{b\to 3^-} \int_1^b \frac{1}{x-3}\ dx$$

Integrate, then evaluate over the interval.

$$\lim_{b\to 3^-} \ln|x-3|\,\Big|_1^b$$

$$\lim_{b\to 3^-} \ln|b-3| - \ln|1-3|$$

$$\lim_{b\to 3^-} \ln|b-3| - \ln|-2|$$

$$\lim_{b\to 3^-} \ln|b-3| - \ln 2$$

$$\ln|3-3| - \ln 2$$

$$\ln 0 - \ln 2$$

$$-\infty - \ln 2$$

$$-\infty$$

IMPROPER INTEGRALS, CASE 5

1. Evaluate the improper integral.

$$\int_0^2 \frac{3}{\sqrt[3]{x}}\ dx$$

Solution:

The integrand is undefined at the lower bound, $x = 0$. Therefore, rewrite the integral as

$$\lim_{a\to 0^+}\int_a^2 \frac{3}{\sqrt[3]{x}}\ dx$$

$$\lim_{a\to 0^+}\int_a^2 3x^{-\frac{1}{3}}\ dx$$

Integrate, then evaluate over the interval.

$$\lim_{a\to 0^+}\frac{9}{2}x^{\frac{2}{3}}\Big|_a^2$$

$$\lim_{a\to 0^+}\frac{9}{2}(2)^{\frac{2}{3}}-\frac{9}{2}(a)^{\frac{2}{3}}$$

$$\frac{9}{2}(2)^{\frac{2}{3}}-\frac{9}{2}(0)^{\frac{2}{3}}$$

$$\frac{9}{2}(2)^{\frac{2}{3}}$$

$$\frac{9}{2}\sqrt[3]{4}$$

■ 2. Evaluate the improper integral.

$$\int_{-1}^{5} \frac{3}{\sqrt{x+1}}\ dx$$

Solution:

The integrand is undefined at the lower bound, $x = -1$. Therefore, rewrite the integral as

$$\lim_{a \to -1^+} \int_{a}^{5} \frac{3}{\sqrt{x+1}}\ dx$$

$$3 \lim_{a \to -1^+} \int_{a}^{5} (x+1)^{-\frac{1}{2}}\ dx$$

Integrate, then evaluate over the interval.

$$3 \lim_{a \to -1^+} 2(x+1)^{\frac{1}{2}} \Big|_{a}^{5}$$

$$6 \lim_{a \to -1^+} \sqrt{x+1} \Big|_{a}^{5}$$

$$6 \lim_{a \to -1^+} \sqrt{5+1} - \sqrt{a+1}$$

$$6\left(\sqrt{6} - \sqrt{-1+1}\right)$$

$$6\sqrt{6}$$

■ 3. Evaluate the improper integral.

$$\int_3^7 \frac{5}{x-3}\ dx$$

Solution:

The integrand is undefined at the lower bound, $x = 3$. Therefore, rewrite the integral as

$$\lim_{a \to 3^+} \int_a^7 \frac{5}{x-3}\ dx$$

Integrate, then evaluate over the interval.

$$\lim_{a \to 3^+} 5 \ln|x-3| \Big|_a^7$$

$$\lim_{a \to 3^+} 5 \ln|7-3| - 5 \ln|a-3|$$

$$5 \ln 4 - 5 \ln|3-3|$$

$$5\ln 4 - 5\ln 0$$

$$5\ln 4 - (-\infty)$$

$$5\ln 4 + \infty$$

$$\infty$$

■ 4. Evaluate the improper integral.

$$\int_0^6 \frac{9}{5\sqrt[4]{x^3}}\ dx$$

Solution:

The integrand is undefined at the lower bound, $x = 0$. Therefore, rewrite the integral as

$$\lim_{a\to 0^+}\int_a^6 \frac{9}{5\sqrt[4]{x^3}}\ dx$$

$$\frac{9}{5}\lim_{a\to 0^+}\int_a^6 x^{-\frac{3}{4}}\ dx$$

Integrate, then evaluate over the interval.

$$\frac{9}{5}\lim_{a\to 0^+} 4x^{\frac{1}{4}}\Big|_a^6$$

$$\frac{36}{5}\lim_{a\to 0^+} x^{\frac{1}{4}}\Big|_a^6$$

$$\frac{36}{5}\lim_{a\to 0^+} 6^{\frac{1}{4}} - a^{\frac{1}{4}}$$

$$\frac{36}{5}\left(6^{\frac{1}{4}} - 0^{\frac{1}{4}}\right)$$

$$\frac{36}{5}\sqrt[4]{6}$$

5. Evaluate the improper integral.

$$\int_{-1}^{7} \frac{x^2}{x^3+1}\ dx$$

Solution:

The integrand is undefined at the lower bound, $x = -1$. Therefore, rewrite the integral as

$$\lim_{a\to -1^+}\int_a^7 \frac{x^2}{x^3+1}\ dx$$

Use u-substitution.

$$u = x^3 + 1$$

$$du = 3x^2\ dx, \text{ so } dx = \frac{du}{3x^2}$$

Substitute.

$$\lim_{a\to-1^+}\int_{x=a}^{x=7}\frac{x^2}{u}\left(\frac{du}{3x^2}\right)$$

$$\frac{1}{3}\lim_{a\to-1^+}\int_{x=a}^{x=7}\frac{1}{u}\ du$$

Integrate, then evaluate over the interval.

$$\frac{1}{3}\lim_{a\to-1^+}\ln|u|\Big|_{x=a}^{x=7}$$

$$\frac{1}{3}\lim_{a\to-1^+}\ln|x^3+1|\Big|_{a}^{7}$$

$$\frac{1}{3}\lim_{a\to-1^+}\ln|7^3+1|-\ln|a^3+1|$$

$$\frac{1}{3}\lim_{a\to-1^+}\ln 344-\ln|a^3+1|$$

$$\frac{1}{3}(\ln 344-\ln 0)$$

$$\frac{1}{3}(\ln 344-(-\infty))$$

$$\frac{1}{3}(\ln 344+\infty)$$

$$\frac{1}{3}(\infty)$$

$$\infty$$

6. Evaluate the improper integral.

$$\int_{-4}^{4} \frac{x+4}{x^2+8x+16}\ dx$$

Solution:

Simplify the integrand by factoring the denominator.

$$\int_{-4}^{4} \frac{x+4}{(x+4)(x+4)}\ dx$$

$$\int_{-4}^{4} \frac{1}{x+4}\ dx$$

The integrand is undefined at the lower bound, $x = -4$. Therefore, rewrite the integral as

$$\lim_{a \to -4^+} \int_{a}^{4} \frac{1}{x+4}\ dx$$

Integrate, then evaluate over the interval.

$$\lim_{a\to -4^+} \ln|x+4|\Big|_a^4$$

$$\lim_{a\to -4^+} \ln|4+4| - \ln|a+4|$$

$$\lim_{a\to -4^+} \ln 8 - \ln|a+4|$$

$$\ln 8 - \ln|-4+4|$$

$$\ln 8 - \ln 0$$

$$\ln 8 - (-\infty)$$

$$\ln 8 + \infty$$

$$\infty$$

IMPROPER INTEGRALS, CASE 6

1. Evaluate the improper integral.

$$\int_{-2}^{2} \frac{3}{2\sqrt[5]{x^3}}\ dx$$

Solution:

The integrand is undefined between the lower and upper bounds, at $x = 0$. So we'll split the integral in two at $x = 0$.

$$\int_{-2}^{0} \frac{3}{2\sqrt[5]{x^3}}\ dx + \int_{0}^{2} \frac{3}{2\sqrt[5]{x^3}}\ dx$$

The first integral is undefined at the upper bound, and the second integral is undefined at the lower bound, so rewrite the expression as

$$\lim_{b \to 0^-} \int_{-2}^{b} \frac{3}{2\sqrt[5]{x^3}}\ dx + \lim_{a \to 0^+} \int_{a}^{2} \frac{3}{2\sqrt[5]{x^3}}\ dx$$

$$\frac{3}{2} \lim_{b \to 0^-} \int_{-2}^{b} \frac{1}{\sqrt[5]{x^3}}\ dx + \frac{3}{2} \lim_{a \to 0^+} \int_{a}^{2} \frac{1}{\sqrt[5]{x^3}}\ dx$$

$$\frac{3}{2} \lim_{b \to 0^-} \int_{-2}^{b} x^{-\frac{3}{5}}\ dx + \frac{3}{2} \lim_{a \to 0^+} \int_{a}^{2} x^{-\frac{3}{5}}\ dx$$

Integrate, then evaluate over the interval.

$$\frac{3}{2}\lim_{b\to 0^-}\frac{5}{2}x^{\frac{2}{5}}\Big|_{-2}^{b}+\frac{3}{2}\lim_{a\to 0^+}\frac{5}{2}x^{\frac{2}{5}}\Big|_{a}^{2}$$

$$\frac{3}{2}\lim_{b\to 0^-}\frac{5}{2}b^{\frac{2}{5}}-\frac{5}{2}(-2)^{\frac{2}{5}}+\frac{3}{2}\lim_{a\to 0^+}\frac{5}{2}(2)^{\frac{2}{5}}-\frac{5}{2}a^{\frac{2}{5}}$$

$$\frac{3}{2}\left(\frac{5}{2}(0)^{\frac{2}{5}}-\frac{5}{2}(-2)^{\frac{2}{5}}\right)+\frac{3}{2}\left(\frac{5}{2}(2)^{\frac{2}{5}}-\frac{5}{2}(0)^{\frac{2}{5}}\right)$$

$$\frac{3}{2}\left(-\frac{5}{2}(-2)^{\frac{2}{5}}\right)+\frac{3}{2}\left(\frac{5}{2}(2)^{\frac{2}{5}}\right)$$

$$-\frac{15}{4}(-2)^{\frac{2}{5}}+\frac{15}{4}(2)^{\frac{2}{5}}$$

$$-\frac{15}{4}\sqrt[5]{4}+\frac{15}{4}\sqrt[5]{4}$$

$$\frac{15}{4}\sqrt[5]{4}-\frac{15}{4}\sqrt[5]{4}$$

$$0$$

2. Evaluate the improper integral.

$$\int_0^4 \frac{7\ dx}{2(x-2)^2}$$

Solution:

The integrand is undefined between the lower and upper bounds, at $x=2$. So we'll split the integral in two at $x=2$.

$$\int_0^2 \frac{7\ dx}{2(x-2)^2}+\int_2^4 \frac{7\ dx}{2(x-2)^2}$$

The first integral is undefined at the upper bound, and the second integral is undefined at the lower bound, so rewrite the expression as

$$\lim_{a\to 2^-}\int_0^a \frac{7\ dx}{2(x-2)^2}+\lim_{a\to 2^+}\int_a^4 \frac{7\ dx}{2(x-2)^2}$$

$$\frac{7}{2}\lim_{a\to 2^-}\int_0^a \frac{1}{(x-2)^2}\ dx+\frac{7}{2}\lim_{a\to 2^+}\int_a^4 \frac{1}{(x-2)^2}\ dx$$

$$\frac{7}{2}\lim_{a\to 2^-}\int_0^a (x-2)^{-2}\ dx+\frac{7}{2}\lim_{a\to 2^+}\int_a^4 (x-2)^{-2}\ dx$$

Integrate, then evaluate over the interval.

$$\frac{7}{2}\lim_{a\to 2^-}-(x-2)^{-1}\Big|_0^a+\frac{7}{2}\lim_{a\to 2^+}-(x-2)^{-1}\Big|_a^4$$

$$\frac{7}{2}\lim_{a\to 2^-}-\frac{1}{(x-2)}\Big|_0^a+\frac{7}{2}\lim_{a\to 2^+}-\frac{1}{(x-2)}\Big|_a^4$$

$$\frac{7}{2}\lim_{a\to 2^-}-\frac{1}{(a-2)}-\left(-\frac{1}{(0-2)}\right)+\frac{7}{2}\lim_{a\to 2^+}-\frac{1}{(4-2)}-\left(-\frac{1}{(a-2)}\right)$$

$$\frac{7}{2}\lim_{a\to 2^-}-\frac{1}{a-2}+\frac{1}{0-2}+\frac{7}{2}\lim_{a\to 2^+}-\frac{1}{4-2}+\frac{1}{a-2}$$

$$\frac{7}{2}\lim_{a\to 2^-} -\frac{1}{a-2} - \frac{1}{2} + \frac{7}{2}\lim_{a\to 2^+} \frac{1}{a-2} - \frac{1}{2}$$

$$\frac{7}{2}(\infty) + \frac{7}{2}(\infty)$$

$$\infty$$

3. Evaluate the improper integral.

$$\int_{-27}^{8} \frac{3\ dx}{\sqrt[3]{x}}$$

Solution:

The integrand is undefined between the lower and upper bounds, at $x = 0$. So we'll split the integral in two at $x = 0$.

$$\int_{-27}^{0} \frac{3\ dx}{\sqrt[3]{x}} + \int_{0}^{8} \frac{3\ dx}{\sqrt[3]{x}}$$

The first integral is undefined at the upper bound, and the second integral is undefined at the lower bound, so rewrite the expression as

$$\lim_{b\to 0^-} \int_{-27}^{b} \frac{3\ dx}{\sqrt[3]{x}} + \lim_{a\to 0^+} \int_{a}^{8} \frac{3\ dx}{\sqrt[3]{x}}$$

$$3\lim_{b\to 0^-} \int_{-27}^{b} x^{-\frac{1}{3}}\ dx + 3\lim_{a\to 0^+} \int_{a}^{8} x^{-\frac{1}{3}}\ dx$$

Integrate, then evaluate over the interval.

$$3\left(\frac{3}{2}\right)\lim_{b\to 0^-} x^{\frac{2}{3}}\Big|_{-27}^{b} + 3\left(\frac{3}{2}\right)\lim_{a\to 0^+} x^{\frac{2}{3}}\Big|_{a}^{8}$$

$$\frac{9}{2}\lim_{b\to 0^-} x^{\frac{2}{3}}\Big|_{-27}^{b} + \frac{9}{2}\lim_{a\to 0^+} x^{\frac{2}{3}}\Big|_{a}^{8}$$

$$\frac{9}{2}\lim_{b\to 0^-} b^{\frac{2}{3}} - (-27)^{\frac{2}{3}} + \frac{9}{2}\lim_{a\to 0^+} 8^{\frac{2}{3}} - a^{\frac{2}{3}}$$

$$\frac{9}{2}\lim_{b\to 0^-} b^{\frac{2}{3}} - (-3)^{2} + \frac{9}{2}\lim_{a\to 0^+} 2^{2} - a^{\frac{2}{3}}$$

$$\frac{9}{2}\lim_{b\to 0^-} b^{\frac{2}{3}} - (-3)^{2} + \frac{9}{2}\lim_{a\to 0^+} 2^{2} - a^{\frac{2}{3}}$$

$$\frac{9}{2}\lim_{b\to 0^-} b^{\frac{2}{3}} - 9 + \frac{9}{2}\lim_{a\to 0^+} 4 - a^{\frac{2}{3}}$$

$$\frac{9}{2}(0^{\frac{2}{3}} - 9) + \frac{9}{2}(4 - 0^{\frac{2}{3}})$$

$$\frac{9}{2}(-9) + \frac{9}{2}(4)$$

$$-\frac{81}{2} + \frac{36}{2}$$

$$-\frac{45}{2}$$

■ 4. Evaluate the improper integral.

$$\int_{-3}^{3} \frac{x+2}{x^2-4}\ dx$$

Solution:

The integrand is undefined between the lower and upper bounds, at $x = 2$. So we'll split the integral in two at $x = 2$.

$$\int_{-3}^{2} \frac{x+2}{x^2-4}\ dx + \int_{2}^{3} \frac{x+2}{x^2-4}\ dx$$

The first integral is undefined at the upper bound, and the second integral is undefined at the lower bound, so rewrite the expression as

$$\lim_{b\to 2^-} \int_{-3}^{b} \frac{x+2}{x^2-4}\ dx + \lim_{a\to 2^+} \int_{a}^{3} \frac{x+2}{x^2-4}\ dx$$

$$\lim_{b\to 2^-} \int_{-3}^{b} \frac{x+2}{(x+2)(x-2)}\ dx + \lim_{a\to 2^+} \int_{a}^{3} \frac{x+2}{(x+2)(x-2)}\ dx$$

$$\lim_{b\to 2^-} \int_{-3}^{b} \frac{1}{x-2}\ dx + \lim_{a\to 2^+} \int_{a}^{3} \frac{1}{x-2}\ dx$$

Integrate, then evaluate over the interval.

$$\lim_{b\to 2^-} \ln|x-2| \Big|_{-3}^{b} + \lim_{a\to 2^+} \ln|x-2| \Big|_{a}^{3}$$

$$\lim_{b\to 2^-} \ln|b-2| - \ln|-3-2| + \lim_{a\to 2^+} \ln|3-2| - \ln|a-2|$$

$$\lim_{b \to 2^-} \ln|b-2| - \ln 5 + \lim_{a \to 2^+} \ln 1 - \ln|a-2|$$

$$\lim_{b \to 2^-} \ln|b-2| - \ln 5 + \lim_{a \to 2^+} -\ln|a-2|$$

$$\ln|2-2| - \ln 5 + (-\ln|2-2|)$$

$$\ln 0 - \ln 5 - \ln 0$$

$$-\infty - \ln 5 - (-\infty)$$

$$-\infty - \ln 5 + \infty$$

This value is indeterminate, which means that the integral does not converge.

5. Evaluate the improper integral.

$$\int_0^6 \frac{4}{x-3} - \frac{4}{x+3}\ dx$$

Solution:

The integrand is undefined between the lower and upper bounds, at $x = 3$. So we'll split the integral in two at $x = 3$.

$$\int_0^3 \frac{4}{x-3} - \frac{4}{x+3}\ dx + \int_3^6 \frac{4}{x-3} - \frac{4}{x+3}\ dx$$

The first integral is undefined at the upper bound, and the second integral is undefined at the lower bound, so rewrite the expression as

$$\lim_{b \to 3^-} \int_0^b \frac{4}{x-3} - \frac{4}{x+3}\ dx + \lim_{a \to 3^+} \int_a^6 \frac{4}{x-3} - \frac{4}{x+3}\ dx$$

Integrate, then evaluate over the interval.

$$\lim_{b \to 3^-} 4 \ln |x-3| - 4 \ln |x+3| \Big|_0^b + \lim_{a \to 3^+} 4 \ln |x-3| - 4 \ln |x+3| \Big|_a^6$$

$$\lim_{b \to 3^-} 4 \ln |b-3| - 4 \ln |b+3| - (4 \ln |0-3| - 4 \ln |0+3|)$$

$$+ \lim_{a \to 3^+} 4 \ln |6-3| - 4 \ln |6+3| - (4 \ln |a-3| - 4 \ln |a+3|)$$

$$\lim_{b \to 3^-} 4 \ln |b-3| - 4 \ln |b+3| - 4 \ln 3 + 4 \ln 3$$

$$+ \lim_{a \to 3^+} 4 \ln 3 - 4 \ln 9 - 4 \ln |a-3| + 4 \ln |a+3|$$

$$\lim_{b \to 3^-} 4(\ln |b-3| - \ln |b+3|) + \lim_{a \to 3^+} 4 \ln 3 - 4 \ln 9 + 4(\ln |a+3| - \ln |a-3|)$$

$$4 \ln 3 + 4 \ln 6 - 4 \ln 6 - 4 \ln 9 + \lim_{b \to 3^-} 4 \ln |b-3| - 4 \lim_{a \to 3^+} \ln |a-3|$$

$$4 \ln \frac{1}{3} + \lim_{b \to 3^-} 4 \ln |b-3| - 4 \lim_{a \to 3^+} \ln |a-3|$$

$$4 \ln \frac{1}{3} + \infty - \infty$$

The integral does not converge.

COMPARISON THEOREM

1. Use the Comparison Theorem to say whether the integral converges or diverges.

$$\int_1^{\infty} \frac{1}{2+2x^2}\ dx$$

Solution:

Let

$$f(x)=\frac{1}{x^2}$$

$$g(x)=\frac{1}{2+2x^2}$$

Compare the two functions using limits and L'Hospital's Rule.

$$\lim_{x\to\infty}\frac{g(x)}{f(x)}=\lim_{x\to\infty}\frac{\frac{1}{2+2x^2}}{\frac{1}{x^2}}=\lim_{x\to\infty}\frac{x^2}{2+2x^2}=\lim_{x\to\infty}\frac{2x}{4x}=\lim_{x\to\infty}\frac{2}{4}=\frac{1}{2}$$

So $g(x)\le f(x)$ on $[1,\infty)$. Now compare the integral of both functions on $[1,\infty)$.

$$\int_1^{\infty}\frac{1}{x^2}\ dx=\lim_{b\to\infty}\int_1^b\frac{1}{x^2}\ dx=\lim_{b\to\infty}\int_1^b x^{-2}\ dx$$

Integrate and evaluate over the interval.

$$\lim_{b\to\infty} \frac{x^{-1}}{-1}\Big|_1^b$$

$$\lim_{b\to\infty} -\frac{1}{x}\Big|_1^b$$

$$\lim_{b\to\infty} -\frac{1}{b} - \left(-\frac{1}{1}\right)$$

$$\lim_{b\to\infty} 1 - \frac{1}{b}$$

$$1 - \frac{1}{\infty}$$

$$1 - 0$$

$$1$$

So $f(x)$ converges on $[1,\infty)$. Since $g(x) \le f(x)$ on $[1,\infty)$, and $f(x)$ converges on $[1,\infty)$, $g(x)$ also converges on $[1,\infty)$.

■ 2. Use the Comparison Theorem to say whether the integral converges or diverges.

$$\int_1^\infty \frac{1}{5x + e^x}\ dx$$

Solution:

Let

$$f(x) = \frac{1}{e^x}$$

$$g(x) = \frac{1}{5x + e^x}$$

Compare the two functions.

$$\frac{g(x)}{f(x)} = \frac{\frac{1}{5x + e^x}}{\frac{1}{e^x}} = \frac{e^x}{5x + e^x}$$

So

$$0 < \frac{e^x}{5x + e^x} < 1$$

on $[1,\infty)$. Now compare the integral of both functions on $[1,\infty)$.

$$\int_1^\infty \frac{1}{e^x}\ dx = \lim_{b \to \infty} \int_1^b \frac{1}{e^x}\ dx = \lim_{b \to \infty} \int_1^b e^{-x}\ dx$$

Integrate and evaluate over the interval.

$$\lim_{b \to \infty} -e^{-x} \Big|_1^b$$

$$\lim_{b \to \infty} -\frac{1}{e^x} \Big|_1^b$$

$$\lim_{b\to\infty} -\frac{1}{e^b} - \left(-\frac{1}{e}\right)$$

$$\lim_{b\to\infty} \frac{1}{e} - \frac{1}{e^b}$$

$$\frac{1}{e} - \frac{1}{e^\infty}$$

$$\frac{1}{e}$$

So $f(x)$ converges on $[1,\infty)$. Since $g(x) \le f(x)$ on $[1,\infty)$, and $f(x)$ converges on $[1,\infty)$, $g(x)$ also converges on $[1,\infty)$.

3. Can we use the harmonic series $1/x$ as a comparison series to say whether or not the integral converges?

$$\int_1^\infty \frac{x}{x^2+1}\ dx$$

Solution:

We'll call the original function

$$g(x) = \frac{x}{x^2+1}$$

If we take just the leading terms from the numerator and denominator, we get

$$f(x) = \frac{x}{x^2} = \frac{1}{x}$$

This comparison series is the harmonic series, which we know diverges. Therefore, as long as $g(x) \geq f(x)$, we know the original series will diverge as well.

$$\frac{x}{x^2+1} \geq \frac{1}{x}$$

$$x \geq \frac{x^2+1}{x}$$

$$x^2 \geq x^2 + 1$$

This inequality will always be false, which means the harmonic series is useless to us as a comparison series, and we therefore can't use the harmonic series to as a comparison series to say whether or not the integral converges.

INTEGRALS USING REDUCTION FORMULAS

1. Use a reduction formula to evaluate the integral.

$$\int \cot^4 x \ dx$$

Solution:

Split up the integrand and make a substitution.

$$\int \cot^4 x \ dx$$

$$\int \cot^2 x \cot^2 x \ dx$$

$$\int \cot^2 x \left(\csc^2 x - 1\right) \ dx$$

$$\int \cot^2 x \csc^2 x \ dx - \int \cot^2 x \ dx$$

For the first integral, use substitution.

$$u = \cot x$$

$$du = -\csc^2 x \ dx$$

$$dx = \frac{du}{-\csc^2 x}$$

Make the substitution.

$$\int u^2 \csc^2 x \cdot \frac{du}{-\csc^2 x} - \int \cot^2 x \ dx$$

$$-\int u^2 \ du - \int \cot^2 x \ dx$$

$$-\frac{1}{3}u^3 + C - \int \cot^2 x \ dx$$

$$-\frac{1}{3}\cot^3 x + C - \int \cot^2 x \ dx$$

Work on the second integral.

$$-\frac{1}{3}\cot^3 x + C - \int 1 + \cot^2 x - 1 \ dx$$

$$-\frac{1}{3}\cot^3 x + C - \int \csc^2 x - 1 \ dx$$

$$-\frac{1}{3}\cot^3 x + C - (-\cot x - x)$$

$$-\frac{1}{3}\cot^3 x + \cot x + x + C$$

■ 2. Use a reduction formula to evaluate the integral.

$$\int \sec^4 x \ dx$$

Solution:

Split up the integrand and make a substitution.

$$\int \sec^4 x \ dx$$

$$\int \sec^2 x \sec^2 x \ dx$$

$$\int \sec^2 x \left(\tan^2 x + 1\right) \ dx$$

$$\int \sec^2 x \tan^2 x \ dx + \int \sec^2 x \ dx$$

For the first integral, use substitution.

$$u = \tan x$$

$$du = \sec^2 x \ dx$$

$$dx = \frac{du}{\sec^2 x}$$

Make the substitution.

$$\int \sec^2 x \cdot u^2 \frac{du}{\sec^2 x} + \int \sec^2 x \ dx$$

$$\int u^2 \ du + \int \sec^2 x \ dx$$

$$\frac{1}{3}u^3+C+\int \sec^2 x\ dx$$

$$\frac{1}{3}\tan^3 x+C+\int \sec^2 x\ dx$$

Work on the second integral.

$$\frac{1}{3}\tan^3 x+C+\tan x$$

$$\frac{1}{3}\tan^3 x+\tan x+C$$

■ 3. Use a reduction formula to evaluate the integral.

$$\int \csc^4 x\ dx$$

Solution:

Split up the integrand and make a substitution.

$$\int \csc^4 x\ dx$$

$$\int \csc^2 x\csc^2 x\ dx$$

$$\int \csc^2 x\left(\cot^2 x+1\right)\ dx$$

$$\int \csc^2 x \cot^2 x \ dx + \int \csc^2 x \ dx$$

For the first integral, use substitution.

$$u = \cot x$$

$$du = -\csc^2 x \ dx$$

$$dx = \frac{du}{-\csc^2 x}$$

Make the substitution.

$$\int \csc^2 x \cdot u^2 \frac{du}{-\csc^2 x} + \int \csc^2 x \ dx$$

$$-\int u^2 \ du + \int \csc^2 x \ dx$$

$$-\frac{1}{3}u^3 + C + \int \csc^2 x \ dx$$

$$-\frac{1}{3}\cot^3 x + C + \int \csc^2 x \ dx$$

Work on the second integral.

$$-\frac{1}{3}\cot^3 x + C + (-\cot x)$$

$$-\frac{1}{3}\cot^3 x - \cot x + C$$

AREA BETWEEN UPPER AND LOWER CURVES

1. Find the area, in square units, between the two curves. Round your answer to two decimal places.

$$f(x) = -2x^2 + 7$$

$$g(x) = -x + 3$$

Solution:

Find the intersection points of the curves.

$$-2x^2 + 7 = -x + 3$$

$$2x^2 - x - 4 = 0$$

Use the quadratic formula.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(2)(-4)}}{2(2)} = \frac{1 \pm \sqrt{1 + 32}}{4} = \frac{1 \pm \sqrt{33}}{4}$$

Between these two points, $f(x) > g(x)$. Therefore, the area between the curves is

$$A=\int_{\frac{1-\sqrt{33}}{4}}^{\frac{1+\sqrt{33}}{4}}-2x^2+7-(-x+3)\ dx$$

$$A=\int_{\frac{1-\sqrt{33}}{4}}^{\frac{1+\sqrt{33}}{4}}-2x^2+7+x-3\ dx$$

$$A=\int_{\frac{1-\sqrt{33}}{4}}^{\frac{1+\sqrt{33}}{4}}-2x^2+x+4\ dx$$

Integrate and evaluate over the interval.

$$A=-\frac{2}{3}x^3+\frac{1}{2}x^2+4x\Big|_{\frac{1-\sqrt{33}}{4}}^{\frac{1+\sqrt{33}}{4}}$$

$$A=-\frac{2}{3}\left(\frac{1+\sqrt{33}}{4}\right)^3+\frac{1}{2}\left(\frac{1+\sqrt{33}}{4}\right)^2+4\left(\frac{1+\sqrt{33}}{4}\right)$$

$$-\left[-\frac{2}{3}\left(\frac{1-\sqrt{33}}{4}\right)^3+\frac{1}{2}\left(\frac{1-\sqrt{33}}{4}\right)^2+4\left(\frac{1-\sqrt{33}}{4}\right)\right]$$

$$A=-\frac{2}{3}\left(\frac{1+\sqrt{33}}{4}\right)^3+\frac{1}{2}\left(\frac{1+\sqrt{33}}{4}\right)^2+4\left(\frac{1+\sqrt{33}}{4}\right)$$

$$+\frac{2}{3}\left(\frac{1-\sqrt{33}}{4}\right)^3-\frac{1}{2}\left(\frac{1-\sqrt{33}}{4}\right)^2-4\left(\frac{1-\sqrt{33}}{4}\right)$$

$$A\approx 7.90$$

2. Find the area, in square units, between the two curves.

$$f(x) = -3x^2 + 9x$$

$$g(x) = 3x^2 - 9x$$

Solution:

Find the intersection points of the curves.

$$-3x^2 + 9x = 3x^2 - 9x$$

$$6x^2 - 18x = 0$$

$$x^2 - 3x = 0$$

$$x(x-3) = 0$$

$$x = 0,\ 3$$

Between these two points, $f(x) > g(x)$. Therefore, the area between the curves is

$$A = \int_0^3 f(x) - g(x)\ dx$$

$$A = \int_0^3 -3x^2 + 9x - \left(3x^2 - 9x\right)\ dx$$

$$A = \int_0^3 -3x^2 + 9x - 3x^2 + 9x \ dx$$

$$A = \int_0^3 -6x^2 + 18x \ dx$$

Integrate and evaluate over the interval.

$$A = -\frac{6}{3}x^3 + \frac{18}{2}x^2 \Big|_0^3$$

$$A = -2x^3 + 9x^2 \Big|_0^3$$

$$A = 9x^2 - 2x^3 \Big|_0^3$$

$$A = 9(3)^2 - 2(3)^3 - \left(9(0)^2 - 2(0)^3\right)$$

$$A = 9(9) - 2(27)$$

$$A = 81 - 54$$

$$A = 27$$

AREA BETWEEN LEFT AND RIGHT CURVES

■ 1. Find the area, in square units, between the two curves. Round your answer to two decimal places.

$$f(y) = 2y^2 + 12y + 15$$

$$g(y) = -2y^2 - 12y - 15$$

Solution:

Find the intersection points of the curves.

$$2y^2 + 12y + 15 = -2y^2 - 12y - 15$$

$$4y^2 + 24y + 30 = 0$$

$$2y^2 + 12y + 15 = 0$$

Use the quadratic formula.

$$y = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$y = \frac{-12 \pm \sqrt{12^2 - 4(2)(15)}}{2(2)} = \frac{-12 \pm \sqrt{144 - 120}}{4} = \frac{-12 \pm \sqrt{24}}{4}$$

$$= \frac{-12 \pm 2\sqrt{6}}{4} = \frac{-6 \pm \sqrt{6}}{2}$$

Between these two points, $g(y) > f(y)$. Therefore, the area between the curves is

$$A = \int_{\frac{-6-\sqrt{6}}{2}}^{\frac{-6+\sqrt{6}}{2}} -2y^2 - 12y - 15 - \left(2y^2 + 12y + 15\right)\ dy$$

$$A = \int_{\frac{-6-\sqrt{6}}{2}}^{\frac{-6+\sqrt{6}}{2}} -2y^2 - 12y - 15 - 2y^2 - 12y - 15\ dy$$

$$A = \int_{\frac{-6-\sqrt{6}}{2}}^{\frac{-6+\sqrt{6}}{2}} -4y^2 - 24y - 30\ dy$$

Integrate and evaluate over the interval.

$$A = -\frac{4}{3}y^3 - \frac{24}{2}y^2 - 30y\Bigg|_{\frac{-6-\sqrt{6}}{2}}^{\frac{-6+\sqrt{6}}{2}}$$

$$A = -\frac{4}{3}y^3 - 12y^2 - 30y\Bigg|_{\frac{-6-\sqrt{6}}{2}}^{\frac{-6+\sqrt{6}}{2}}$$

$$A = -\frac{4}{3}\left(\frac{-6+\sqrt{6}}{2}\right)^3 - 12\left(\frac{-6+\sqrt{6}}{2}\right)^2 - 30\left(\frac{-6+\sqrt{6}}{2}\right)$$

$$-\left[-\frac{4}{3}\left(\frac{-6-\sqrt{6}}{2}\right)^3 - 12\left(\frac{-6-\sqrt{6}}{2}\right)^2 - 30\left(\frac{-6-\sqrt{6}}{2}\right)\right]$$

$$A=-\frac{4}{3}\left(\frac{-6+\sqrt{6}}{2}\right)^3-12\left(\frac{-6+\sqrt{6}}{2}\right)^2-30\left(\frac{-6+\sqrt{6}}{2}\right)$$

$$+\frac{4}{3}\left(\frac{-6-\sqrt{6}}{2}\right)^3+12\left(\frac{-6-\sqrt{6}}{2}\right)^2+30\left(\frac{-6-\sqrt{6}}{2}\right)$$

$$A\approx 9.80$$

■ 2. Find the area, in square units, between the two curves, and between $y=-2$ and $y=-5$.

$$f(y)=2y^2+12y+19$$

$$g(y)=-\frac{y^2}{2}-4y-10$$

Solution:

Find the intersection points of the curves.

$$2y^2+12y+19=-\frac{y^2}{2}-4y-10$$

$$4y^2+24y+38=-y^2-8y-20$$

$$5y^2+32y+58=0$$

Use the quadratic formula.

$$y = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$y = \frac{-32 \pm \sqrt{32^2 - 4(5)(58)}}{2(5)} = \frac{-32 \pm \sqrt{1{,}024 - 1{,}160}}{10} = \frac{-32 \pm \sqrt{-136}}{10}$$

Because we can't take the square root of a negative number, this means that the curves do not intersect. Which means only $y = -2$ and $y = -5$ provide the limits of integration. $f(y)$ is to the right of $g(y)$, so

$$A = \int_{-5}^{-2} 2y^2 + 12y + 19 - \left(-\frac{y^2}{2} - 4y - 10\right)\ dy$$

$$A = \int_{-5}^{-2} 2y^2 + 12y + 19 + \frac{y^2}{2} + 4y + 10\ dy$$

$$A = \int_{-5}^{-2} \frac{5}{2}y^2 + 16y + 29\ dy$$

Integrate and evaluate over the interval.

$$A = \frac{5y^3}{6} + \frac{16y^2}{2} + 29y \Big|_{-5}^{-2}$$

$$A = \frac{5y^3}{6} + 8y^2 + 29y \Big|_{-5}^{-2}$$

$$A = \frac{5(-2)^3}{6} + 8(-2)^2 + 29(-2) - \left(\frac{5(-5)^3}{6} + 8(-5)^2 + 29(-5)\right)$$

$$A = \frac{5(-8)}{6} + 8(4) - 58 - \left(\frac{5(-125)}{6} + 8(25) - 145\right)$$

$$A = -\frac{40}{6} + 32 - 58 - \left(-\frac{625}{6} + 200 - 145\right)$$

$$A = -\frac{40}{6} + 32 - 58 + \frac{625}{6} - 200 + 145$$

$$A = \frac{585}{6} - 81$$

$$A = 16.5$$

■ 3. Find the area, in square units, between the two curves.

$$f(y) = -y^3 + 6y$$

$$g(y) = -y^2$$

Solution:

Find the intersection points of the curves.

$$-y^3 + 6y = -y^2$$

$$y^3 - y^2 - 6y = 0$$

$$y(y^2 - y - 6) = 0$$

$$y(y-3)(y+2) = 0$$

$$y = -2,\ 0,\ 3$$

Between $y = -2$ and $y = 0$, $g(y)$ is to the right of $f(y)$. And between $y = 0$ and $y = 3$, $f(y)$ is to the right of $g(y)$. Therefore, the area between the curves is

$$A = \int_{-2}^{0} -y^2 - (-y^3 + 6y)\ dy + \int_{0}^{3} -y^3 + 6y - (-y^2)\ dy$$

$$A = \int_{-2}^{0} -y^2 + y^3 - 6y\ dy + \int_{0}^{3} -y^3 + 6y + y^2\ dy$$

$$A = \int_{-2}^{0} y^3 - y^2 - 6y\ dy + \int_{0}^{3} -y^3 + y^2 + 6y\ dy$$

Integrate and evaluate over the interval.

$$A = \left(\frac{1}{4}y^4 - \frac{1}{3}y^3 - 3y^2\right)\Bigg|_{-2}^{0} + \left(-\frac{1}{4}y^4 + \frac{1}{3}y^3 + \frac{6}{2}y^2\right)\Bigg|_{0}^{3}$$

$$A = \frac{1}{4}(0)^4 - \frac{1}{3}(0)^3 - 3(0)^2 - \left(\frac{1}{4}(-2)^4 - \frac{1}{3}(-2)^3 - 3(-2)^2\right)$$

$$+\left[-\frac{1}{4}(3)^4 + \frac{1}{3}(3)^3 + \frac{6}{2}(3)^2 - \left(-\frac{1}{4}(0)^4 + \frac{1}{3}(0)^3 + \frac{6}{2}(0)^2\right)\right]$$

$$A = -\frac{1}{4}(-2)^4 + \frac{1}{3}(-2)^3 + 3(-2)^2 - \frac{1}{4}(3)^4 + \frac{1}{3}(3)^3 + \frac{6}{2}(3)^2$$

$$A = -\frac{1}{4}(16) + \frac{1}{3}(-8) + 3(4) - \frac{1}{4}(81) + \frac{1}{3}(27) + \frac{6}{2}(9)$$

$$A = -4 - \frac{8}{3} + 12 - \frac{81}{4} + 9 + 27$$

$$A = -\frac{8}{3} - \frac{81}{4} + 44$$

$$A = -\frac{32}{12} - \frac{243}{12} + \frac{528}{12}$$

$$A = \frac{253}{12}$$

■ 4. Find the area, in square units, between the two curves.

$$f(y) = \frac{y^2}{2} - 3y - \frac{1}{2}$$

$$g(y) = 3$$

Solution:

Find the intersection points of the curves.

$$\frac{y^2}{2} - 3y - \frac{1}{2} = 3$$

$$y^2 - 6y - 1 = 6$$

$$y^2 - 6y - 7 = 0$$

$$(y-7)(y+1) = 0$$

$$y = -1, 7$$

Between these two points, $g(y)$ is to the right of $f(y)$. Therefore, the area between the curves is

$$A=\int_{-1}^{7} 3-\left(\frac{y^2}{2}-3y-\frac{1}{2}\right)\ dy$$

$$A=\int_{-1}^{7} 3-\frac{y^2}{2}+3y+\frac{1}{2}\ dy$$

$$A=\int_{-1}^{7} -\frac{y^2}{2}+3y+\frac{7}{2}\ dy$$

Integrate and evaluate over the interval.

$$A=-\frac{y^3}{6}+\frac{3}{2}y^2+\frac{7}{2}y\Big|_{-1}^{7}$$

$$A=-\frac{(7)^3}{6}+\frac{3}{2}(7)^2+\frac{7}{2}(7)-\left(-\frac{(-1)^3}{6}+\frac{3}{2}(-1)^2+\frac{7}{2}(-1)\right)$$

$$A=-\frac{343}{6}+\frac{147}{2}+\frac{49}{2}-\frac{1}{6}-\frac{3}{2}+\frac{7}{2}$$

$$A=-\frac{344}{6}+\frac{200}{2}$$

$$A=\frac{300}{3}-\frac{172}{3}$$

$$A=\frac{128}{3}$$

■ 5. Find the area, in square units, between the two curves, and between $y=0$ and $y=4$.

$$f(y)=2y^2-8y+9$$

$$g(y)=\frac{y^2}{2}-2y-1$$

Solution:

Find the intersection points of the curves.

$$2y^2-8y+9=\frac{y^2}{2}-2y-1$$

$$4y^2-16y+18=y^2-4y-2$$

$$3y^2-12y+20=0$$

Use the quadratic formula.

$$y=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

$$y=\frac{-(-12)\pm\sqrt{(-12)^2-4(3)(20)}}{2(3)}=\frac{12\pm\sqrt{144-240}}{6}=\frac{12\pm\sqrt{-96}}{6}$$

Because we can't take the square root of a negative number, this means that the curves do not intersect. Which means only $y=0$ and $y=4$ provide the limits of integration. $f(y)$ is to the right of $g(y)$, so

$$A = \int_0^4 2y^2 - 8y + 9 - \left(\frac{y^2}{2} - 2y - 1\right)\ dy$$

$$A = \int_0^4 2y^2 - 8y + 9 - \frac{y^2}{2} + 2y + 1\ dy$$

$$A = \int_0^4 \frac{3}{2}y^2 - 6y + 10\ dy$$

Integrate and evaluate over the interval.

$$A = \frac{3}{2(3)}y^3 - \frac{6}{2}y^2 + 10y\Big|_0^4$$

$$A = \frac{1}{2}y^3 - 3y^2 + 10y\Big|_0^4$$

$$A = \frac{1}{2}(4)^3 - 3(4)^2 + 10(4) - \left(\frac{1}{2}(0)^3 - 3(0)^2 + 10(0)\right)$$

$$A = \frac{1}{2}(64) - 3(16) + 40$$

$$A = 32 - 48 + 40$$

$$A = 24$$

SKETCHING THE AREA BETWEEN CURVES

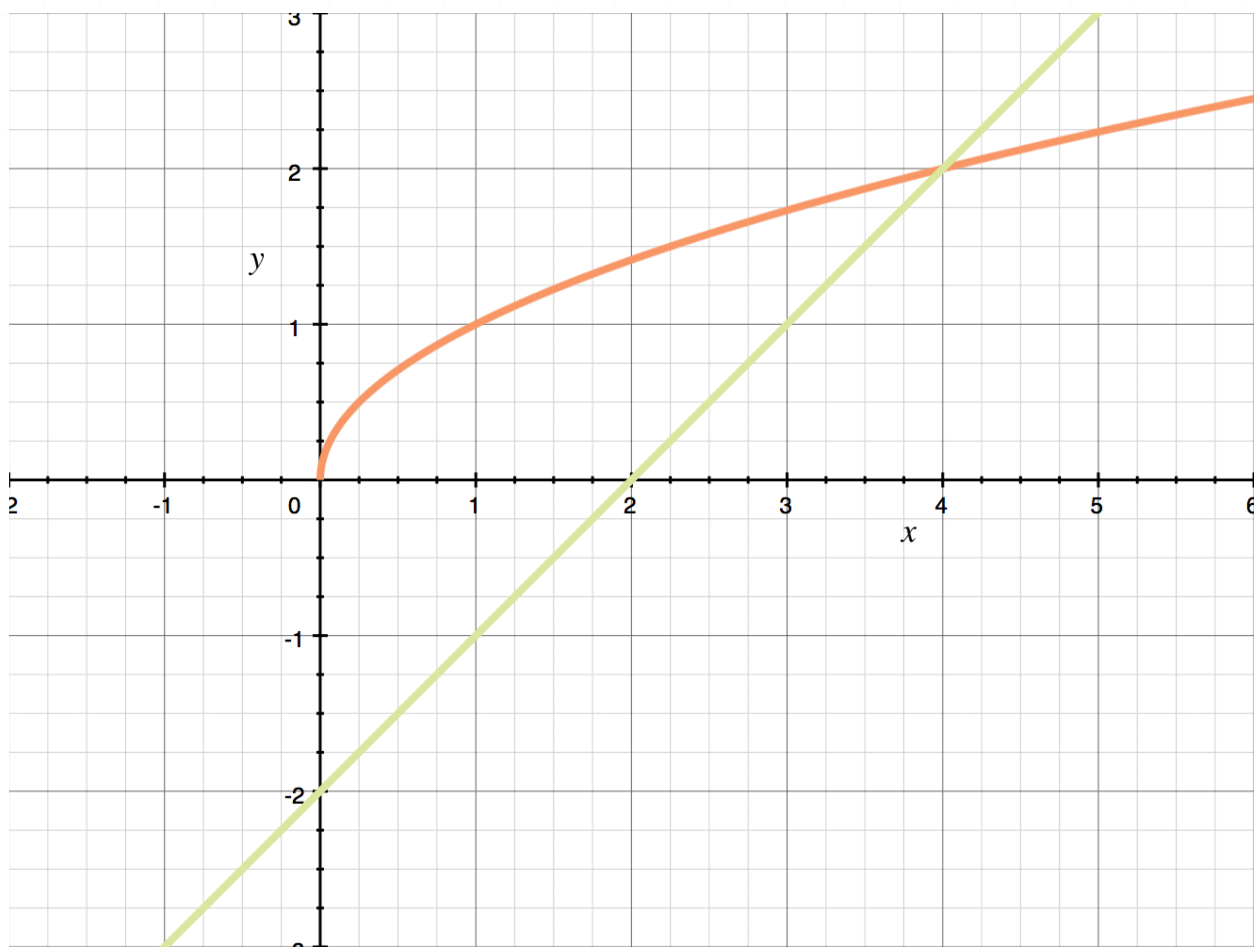
1. Find the area of the region in the first quadrant that's enclosed by the graphs of the curves.

$$y = \sqrt{x}$$

$$y = x - 2$$

Solution:

The graph of the region is



Since these are more left-right curves, we should integrate with respect to y, which means we need to solve both equations for x.

$y=\sqrt{x}$ becomes $x=y^2$

$y=x-2$ becomes $x=y+2$

Find the intersection points of the curves.

$$y^2=y+2$$

$$y^2-y-2=0$$

$$(y-2)(y+1)=0$$

$$y=-1,\ 2$$

So

$$x=y+2$$

$$x=-1+2$$

$$x=1$$

and

$$x=y+2$$

$$x=2+2$$

$$x=4$$

The curves intersect at $(1,-1)$ and $(4,2)$, but only $(4,2)$ is in the first quadrant. With respect to y, that means the region is bounded below by $y=0$ and bounded above by $y=2$.

So the area enclosed by the curves in the first quadrant is

$$A=\int_0^2 (y+2)-y^2\ dy$$

Integrate and evaluate over the interval.

$$A=\frac{y^2}{2}+2y-\frac{y^3}{3}\Big|_0^2$$

$$A=\left(\frac{2^2}{2}+2(2)-\frac{2^3}{3}\right)-\left(\frac{0^2}{2}+2(0)-\frac{0^3}{3}\right)$$

$$A=\left(2+4-\frac{8}{3}\right)-(0+0-0)$$

$$A=\frac{10}{3}$$

2. Find the area of the region that's enclosed by the graphs of the curves.

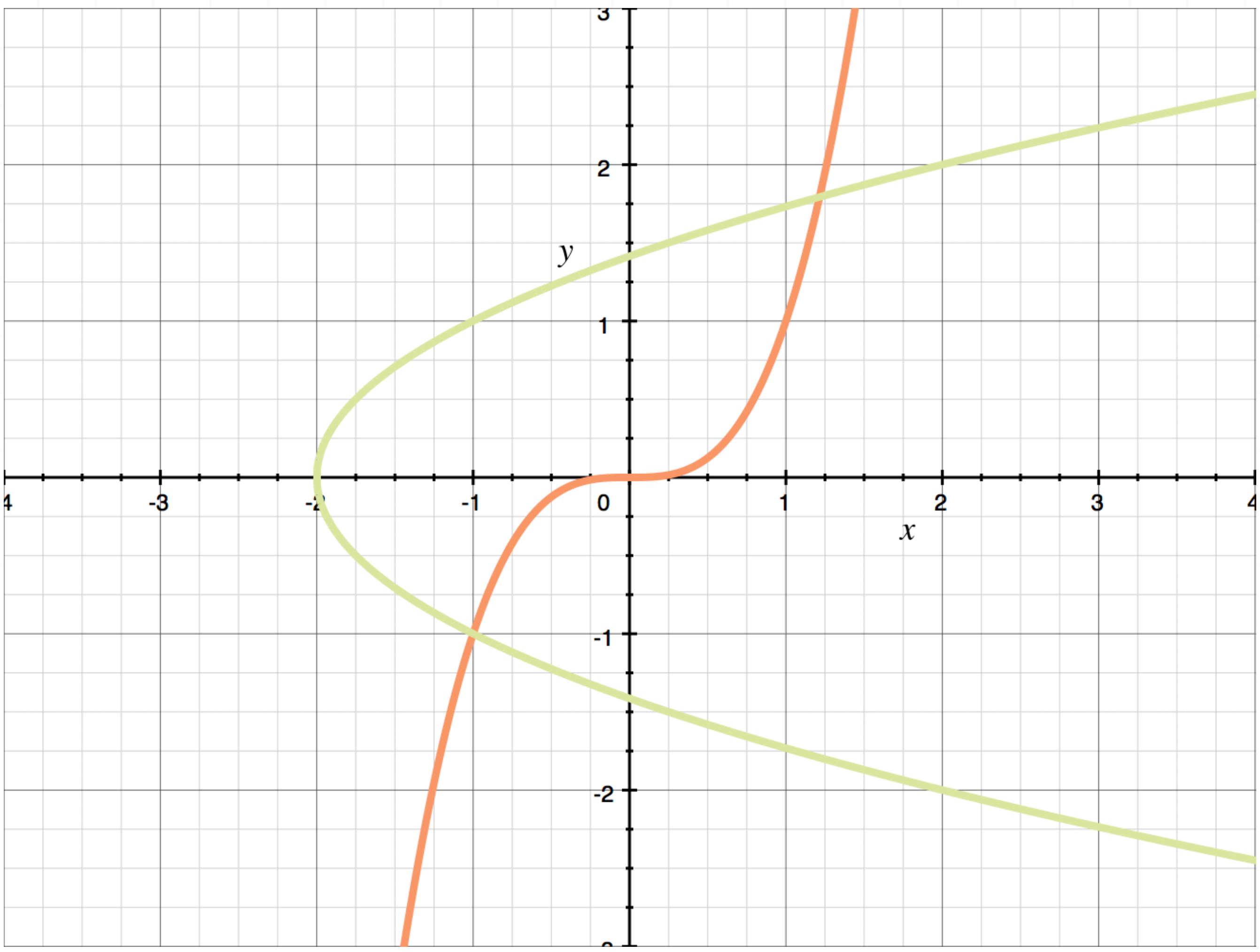
$$y=x^3$$

$$y=\sqrt{x+2}$$

$$y = -\sqrt{x+2}$$

Solution:

The graph of the region is



Since these are more left-right curves, we should integrate with respect to y, which means we need to solve both equations for x.

$y = x^3$ becomes $x = y^{\frac{1}{3}}$

$y = \sqrt{x+2}$ and $y = -\sqrt{x+2}$ both become $x = y^2 - 2$

Find the intersection points of the curves.

$$y^{\frac{1}{3}} = y^2 - 2$$

$$y = (y^2 - 2)^3$$

$$y = (y^4 - 4y^2 + 4)(y^2 - 2)$$

$$y = y^6 - 6y^4 + 12y^2 - 8$$

The roots of this polynomial are $y = -1$ and $y \approx 1.79$

So

$$x = y^2 - 2$$

$$x = (-1)^2 - 2$$

$$x = -1$$

and

$$x = y^2 - 2$$

$$x = 1.79^2 - 2$$

$$x \approx 1.20$$

The curves intersect at $(-1, -1)$ and $(1.20, 1.79)$. With respect to y, that means the region is bounded below by $y = -1$ and bounded above by $y = 1.79$.

So the area enclosed by the curves in the first quadrant is

$$A = \int_{-1}^{1.79} y^{\frac{1}{3}} - (y^2 - 2) \ dy$$

Integrate and evaluate over the interval.

$$A = \frac{3}{4}y^{\frac{4}{3}} - \frac{1}{3}y^3 + 2y\Big|_{-1}^{1.79}$$

$$A = \frac{3}{4}(1.79)^{\frac{4}{3}} - \frac{1}{3}(1.79)^3 + 2(1.79) - \left(\frac{3}{4}(-1)^{\frac{4}{3}} - \frac{1}{3}(-1)^3 + 2(-1)\right)$$

$$A = \frac{3}{4}(1.79)^{\frac{4}{3}} - \frac{1}{3}(1.79)^3 + 2(1.79) - \frac{3}{4} - \frac{1}{3} + 2$$

$$A \approx 4.215$$

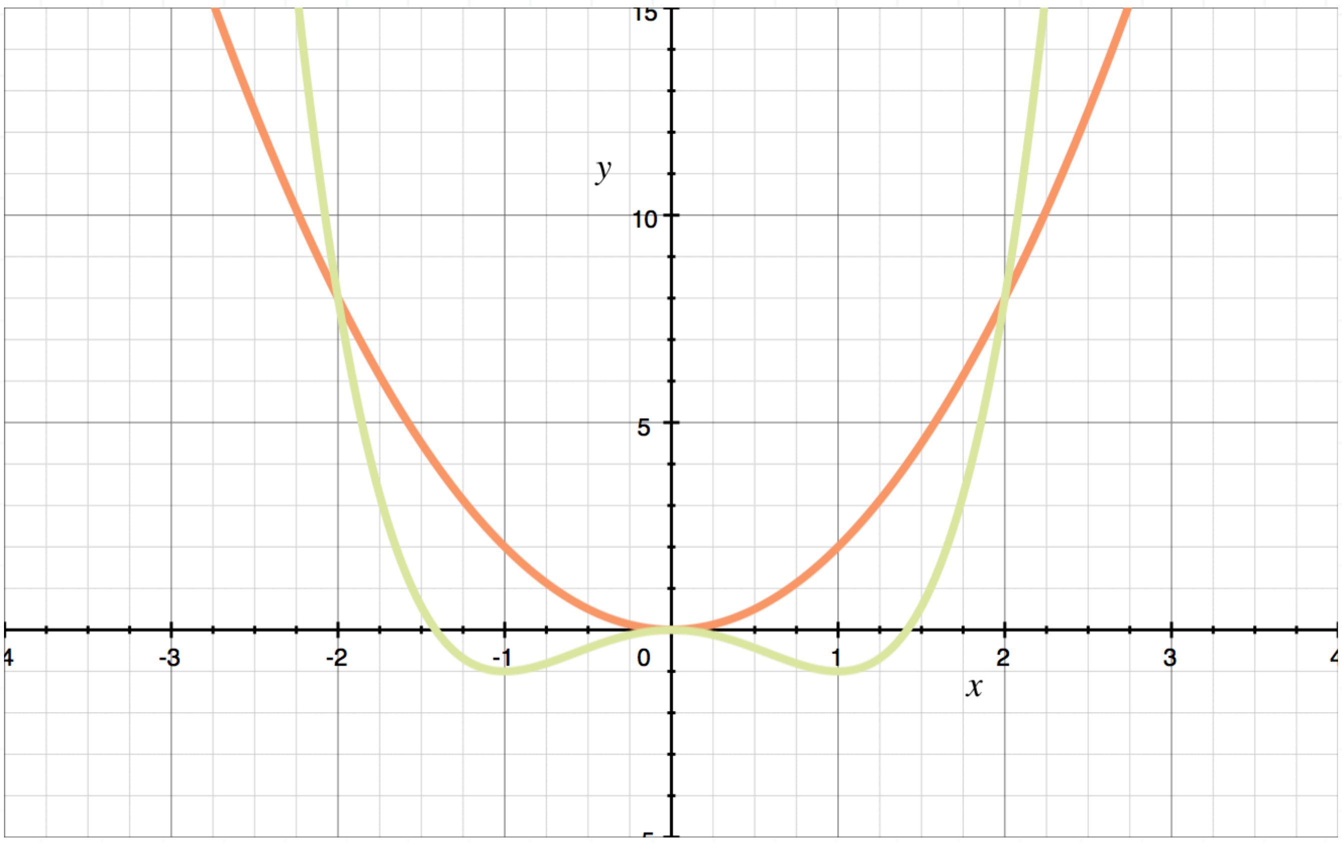
■ 3. Find the area of the region that's enclosed by the graphs of the curves.

$$y = 2x^2$$

$$y = x^4 - 2x^2$$

Solution:

The graph of the region is



The region is symmetric about the y-axis, so the best way to calculate the area is by integrating half the region with respect to x and then doubling the answer.

Find the intersection points of the curves.

$$2x^2 = x^4 - 2x^2$$

$$x^4 - 4x^2 = 0$$

$$x^2(x^2 - 4) = 0$$

$$x^2(x + 2)(x - 2) = 0$$

$$x = -2, 0, 2$$

So

$$y = 2x^2$$

$$y = 2(-2)^2$$

$$y = 8$$

and

$$y = 2x^2$$

$$y = 2(0)^2$$

$$y = 0$$

and

$$y = 2x^2$$

$$y = 2(-2)^2$$

$$y = 8$$

The curves intersect at $(-2,8)$, $(0,0)$, and $(2,8)$. With respect to x, that means we'll integrate from $x = -2$ to $x = 0$, and then double the result.

So the area enclosed by the curves in the first quadrant is

$$A = 2\int_0^2 2x^2 - (x^4 - 2x^2)\ dx$$

$$A = 2\int_0^2 2x^2 - x^4 + 2x^2\ dx$$

$$A = 2\int_0^2 4x^2 - x^4\ dx$$

$$A = \int_0^2 8x^2 - 2x^4 \ dx$$

Integrate and evaluate over the interval.

$$A = \frac{8}{3}x^3 - \frac{2}{5}x^5 \Big|_0^2$$

$$A = \frac{8}{3}(2)^3 - \frac{2}{5}(2)^5 - \left(\frac{8}{3}(0)^3 - \frac{2}{5}(0)^5\right)$$

$$A = \frac{8}{3}(8) - \frac{2}{5}(32)$$

$$A = \frac{64}{3} - \frac{64}{5}$$

$$A = \frac{320}{15} - \frac{192}{15}$$

$$A = \frac{128}{15}$$

DIVIDING THE AREA BETWEEN CURVES INTO EQUAL PARTS

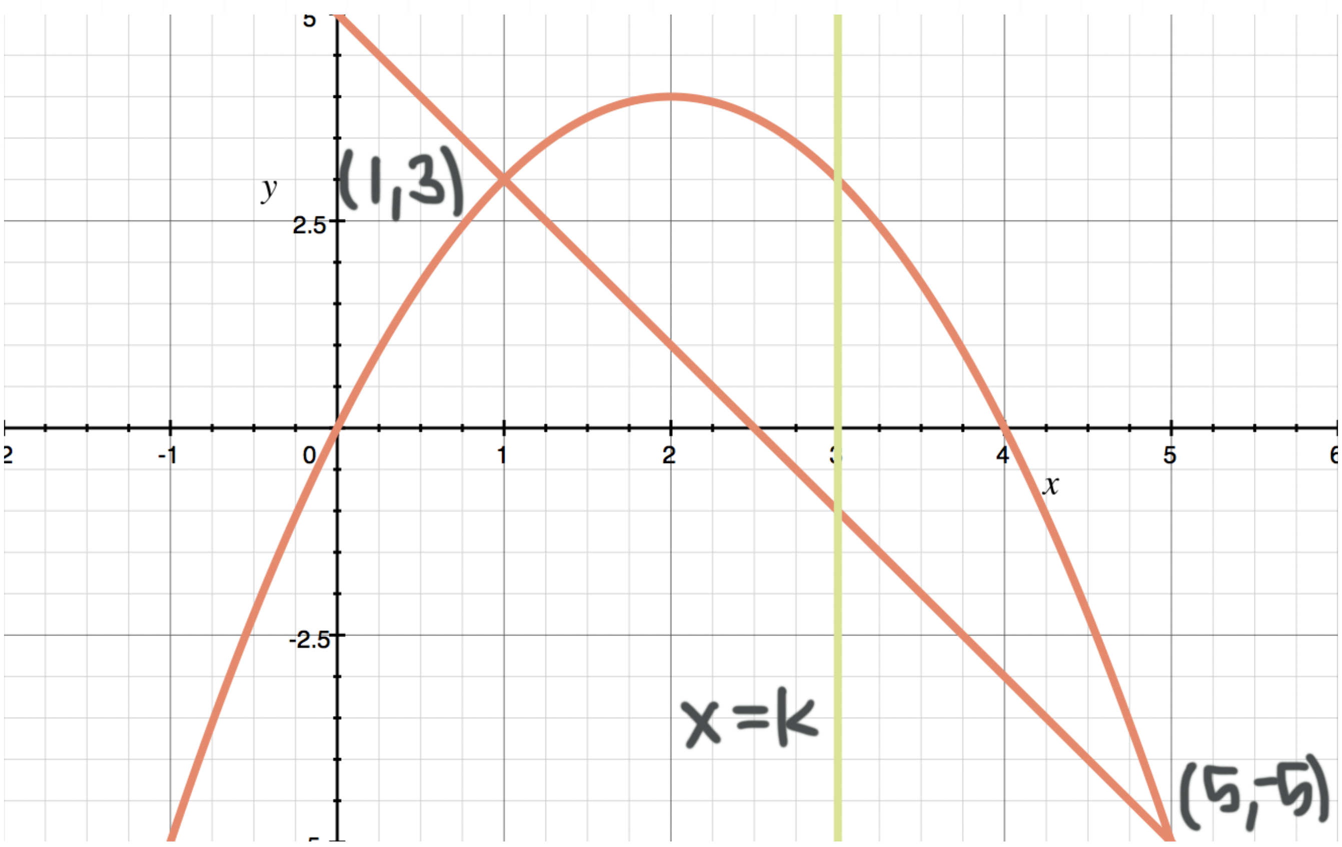
■ 1. The line $x = k$ divides the area bounded by the curves into two equal parts. Find k.

$$f(x) = 4x - x^2$$

$$g(x) = 5 - 2x$$

Solution:

The graph of the area, with the line $x = k$, bounded by the two functions is:



Find the intersection points of the curves.

$$4x - x^2 = 5 - 2x$$

$$x^2 - 6x + 5 = 0$$

$$(x - 5)(x - 1) = 0$$

$$x = 1, 5$$

So

$$y = 5 - 2x$$

$$y = 5 - 2(1)$$

$$y = 3$$

and

$$y = 5 - 2x$$

$$y = 5 - 2(5)$$

$$y = -5$$

The curves intersect at $(1,3)$ and $(5, -5)$. With respect to x, that means the region is bounded below by $x = 1$ and bounded above by $x = 5$.

$$\int_1^5 (4x - x^2) - (5 - 2x)\ dx$$

$$\int_1^5 4x - x^2 - 5 + 2x\ dx$$

$$\int_1^5 -x^2 + 6x - 5\ dx$$

Integrate and evaluate over the interval.

$$-\frac{1}{3}x^3+3x^2-5x\Big|_1^5$$

$$-\frac{1}{3}(5)^3+3(5)^2-5(5)-\left(-\frac{1}{3}(1)^3+3(1)^2-5(1)\right)$$

$$-\frac{125}{3}+75-25+\frac{1}{3}-3+5$$

$$-\frac{124}{3}+52$$

$$-\frac{124}{3}+\frac{156}{3}$$

$$\frac{32}{3}$$

Half of this area is $16/3$, which means we can set up an integral on $[1,k]$ that's equal to $16/3$.

$$\int_1^k (4x-x^2)-(5-2x)\ dx=\frac{16}{3}$$

$$\int_1^k 4x-x^2-5+2x\ dx=\frac{16}{3}$$

$$\int_1^k -x^2+6x-5\ dx=\frac{16}{3}$$

$$-\frac{1}{3}x^3+3x^2-5x\Big|_1^k=\frac{16}{3}$$

$$-\frac{1}{3}k^3 + 3k^2 - 5k - \left(-\frac{1}{3}(1)^3 + 3(1)^2 - 5(1)\right) = \frac{16}{3}$$

$$-\frac{1}{3}k^3 + 3k^2 - 5k + \frac{1}{3} - 3 + 5 = \frac{16}{3}$$

$$-k^3 + 9k^2 - 15k + 1 - 9 + 15 = 16$$

$$-k^3 + 9k^2 - 15k = 9$$

The roots of the polynomial are

$$k = 3$$

$$k = 3 \pm 2\sqrt{3}$$

but $x = 3 - 2\sqrt{3}$ and $x = 3 + 2\sqrt{3}$ are both outside the interval of the bounded region. Which means $x = 3$ must be the line that divides the area of the region in half.

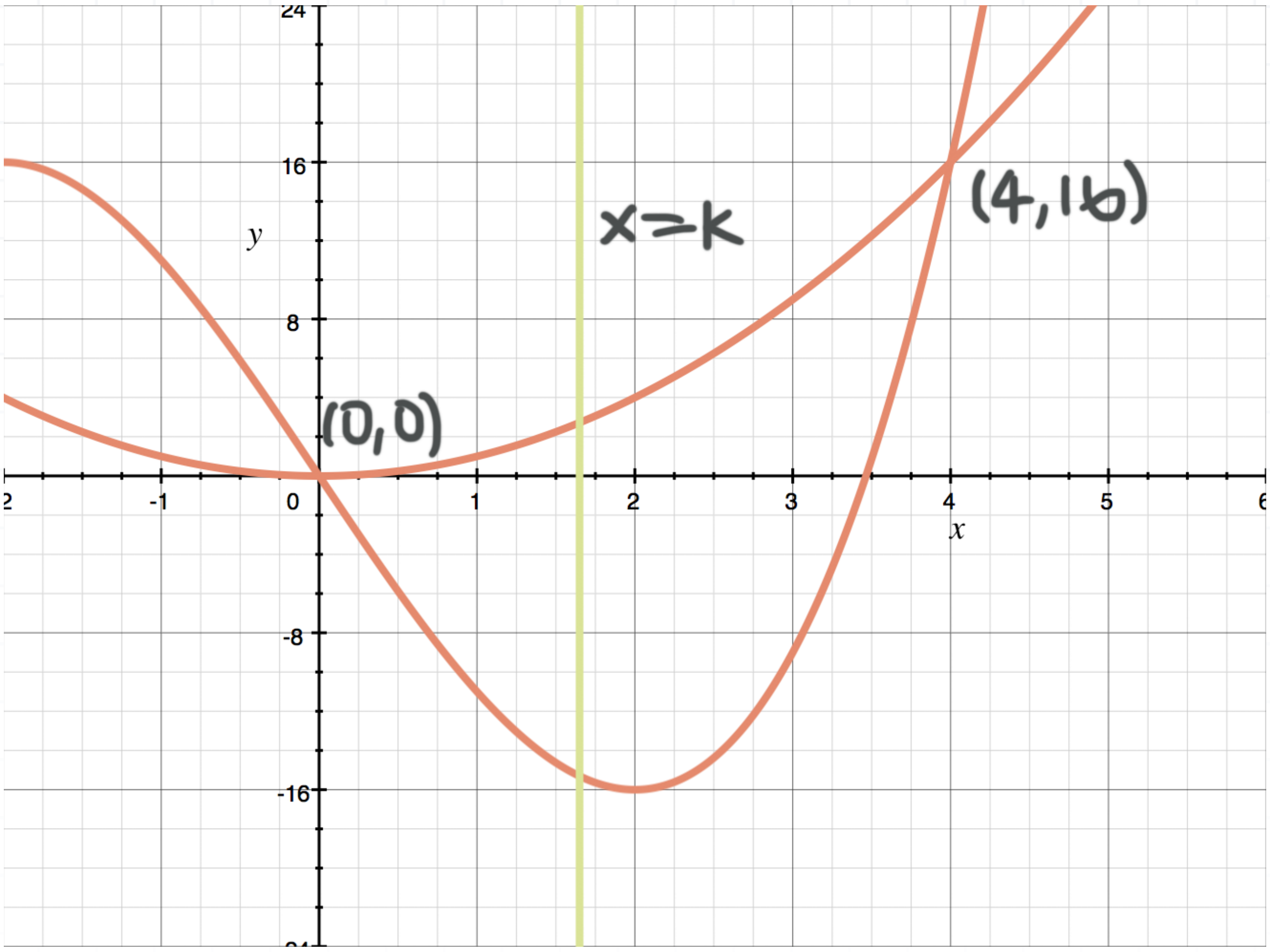
■ 2. The line $x = k$ divides the area bounded by the curves into two equal parts, for $x > 0$. Find k. Round your answer to the nearest three decimal places.

$$f(x) = x^3 - 12x$$

$$g(x) = x^2$$

Solution:

The graph of the area, with the line $x = k$, bounded by the two functions is:



Find the intersection points of the curves.

$$x^3 - 12x = x^2$$

$$x^3 - x^2 - 12x = 0$$

$$x(x^2 - x - 12) = 0$$

$$x(x - 4)(x + 3) = 0$$

$$x = -3,\ 0,\ 4$$

So

$$y = x^2$$

$$y = (-3)^2$$

$$y = 9$$

and

$$y = x^2$$

$$y = 0^2$$

$$y = 0$$

and

$$y = x^2$$

$$y = 4^2$$

$$y = 16$$

The two curves intersect at $(-3,9)$, $(0,0)$ and $(4,16)$. But we're only interested in $x > 0$, so we can ignore $(-3,9)$. So the area of the enclosed region is:

$$\int_0^4 (x^2) - (x^3 - 12x)\ dx$$

$$\int_0^4 x^2 - x^3 + 12x\ dx$$

$$\frac{1}{3}x^3 - \frac{1}{4}x^4 + 6x^2 \Big|_0^4$$

$$\frac{1}{3}(4)^3-\frac{1}{4}(4)^4+6(4)^2-\left(\frac{1}{3}(0)^3-\frac{1}{4}(0)^4+6(0)^2\right)$$

$$\frac{1}{3}(64)-\frac{1}{4}(256)+6(16)$$

$$\frac{64}{3}-64+96$$

$$\frac{64}{3}+32$$

$$\frac{64}{3}+\frac{96}{3}$$

$$\frac{160}{3}$$

Half of this area is $80/3$, which means we can set up an integral on $[0,k]$ that's equal to $80/3$.

$$\int_0^k (x^2)-(x^3-12x)\ dx=\frac{80}{3}$$

$$\int_0^k x^2-x^3+12x\ dx=\frac{80}{3}$$

$$\frac{1}{3}x^3-\frac{1}{4}x^4+6x^2\Big|_0^k=\frac{80}{3}$$

$$\frac{1}{3}k^3-\frac{1}{4}k^4+6k^2-\left(\frac{1}{3}(0)^3-\frac{1}{4}(0)^4+6(0)^2\right)=\frac{80}{3}$$

$$\frac{1}{3}k^3 - \frac{1}{4}k^4 + 6k^2 = \frac{80}{3}$$

$$(12)\frac{1}{3}k^3 - (12)\frac{1}{4}k^4 + (12)6k^2 = (12)\frac{80}{3}$$

$$4k^3 - 3k^4 + 72k^2 = 320$$

$$3k^4 - 4k^3 - 72k^2 + 320 = 0$$

The roots of the polynomial are

$$k \approx 2.20$$

$$k \approx 5.19$$

but only $k \approx 2.20$ is inside the interval $x = [0,4]$. Which means $x \approx 2.20$ must be the line that divides the area of the region in half.

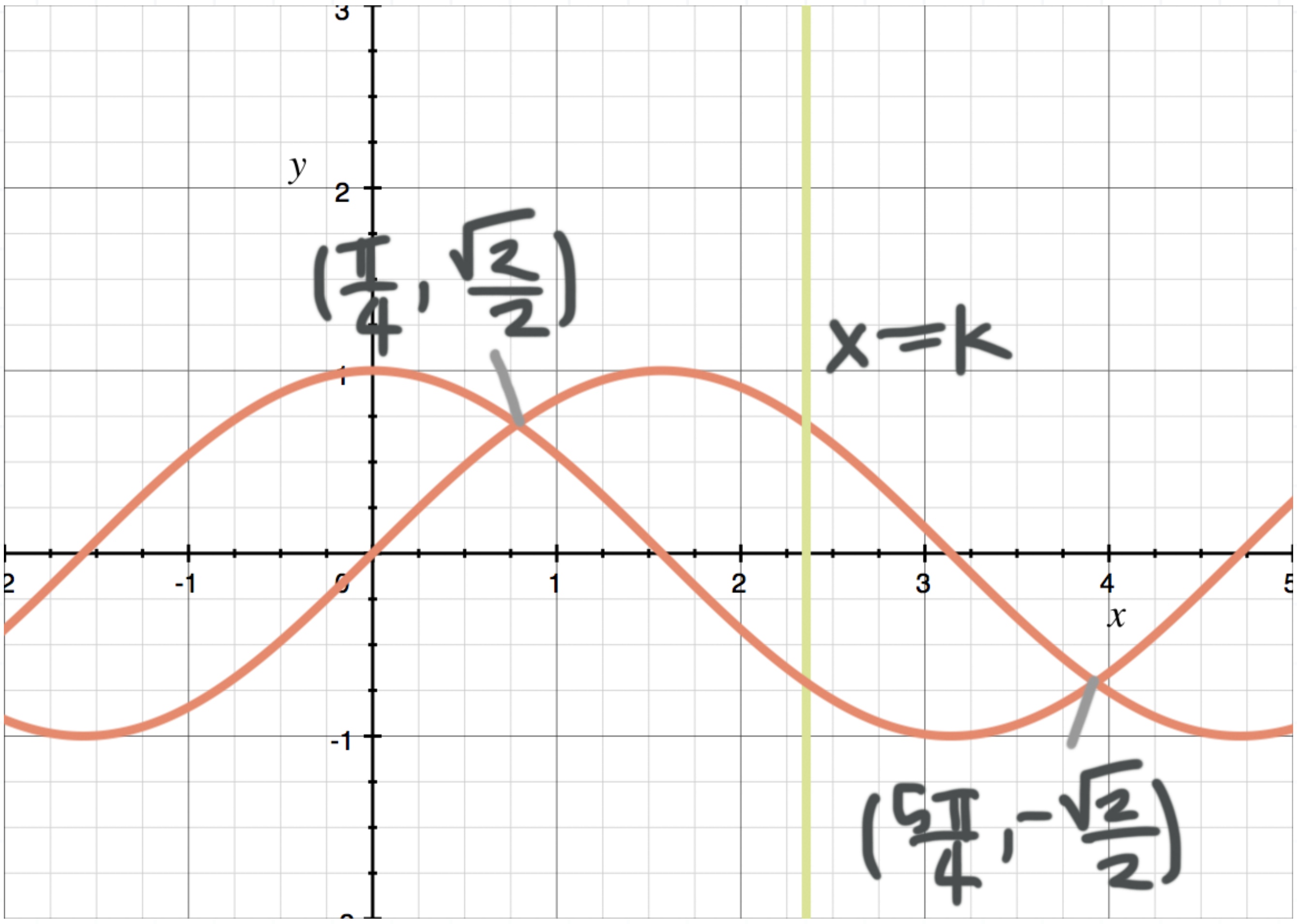
■ 3. The line $x = k$ divides the area bounded by the curves on $\pi/4 \le x \le 5\pi/4$ into two equal parts. Find k.

$$f(x) = \sin x$$

$$g(x) = \cos x$$

Solution:

The graph of the area, with the line $x = k$, bounded by the two functions is:



Find the intersection points of the curves.

$$\sin x = \cos x$$

$$x = \frac{\pi}{4}, \frac{5\pi}{4}$$

So

$$y = \sin x$$

$$y = \sin \frac{\pi}{4}$$

$$y = \frac{\sqrt{2}}{2}$$

and

$$y = \sin x$$

$$y = \sin\frac{5\pi}{4}$$

$$y = -\frac{\sqrt{2}}{2}$$

The two curves intersect at $(\pi/4, \sqrt{2}/2)$ and $(5\pi/4, -\sqrt{2}/2)$. So the area of the enclosed region is:

$$\int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} \sin x - \cos x \ dx$$

$$-\cos x - \sin x \Big|_{\frac{\pi}{4}}^{\frac{5\pi}{4}}$$

$$-\cos\frac{5\pi}{4} - \sin\frac{5\pi}{4} - \left(-\cos\frac{\pi}{4} - \sin\frac{\pi}{4}\right)$$

$$-\cos\frac{5\pi}{4} - \sin\frac{5\pi}{4} + \cos\frac{\pi}{4} + \sin\frac{\pi}{4}$$

$$-\left(-\frac{\sqrt{2}}{2}\right) - \left(-\frac{\sqrt{2}}{2}\right) + \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}$$

$$\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}$$

$$\frac{4\sqrt{2}}{2}$$

$$2\sqrt{2}$$

Half of this area is $\sqrt{2}$, which means we can set up an integral on $[\pi/4,k]$ that's equal to $\sqrt{2}$.

$$\int_{\frac{\pi}{4}}^{k} \sin x - \cos x \ dx = \sqrt{2}$$

$$-\cos x - \sin x \Big|_{\frac{\pi}{4}}^{k} = \sqrt{2}$$

$$-\cos k - \sin k - \left(-\cos\frac{\pi}{4} - \sin\frac{\pi}{4}\right) = \sqrt{2}$$

$$-\cos k - \sin k + \cos\frac{\pi}{4} + \sin\frac{\pi}{4} = \sqrt{2}$$

$$-\cos k - \sin k + \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} = \sqrt{2}$$

$$-\cos k - \sin k + \sqrt{2} = \sqrt{2}$$

$$-\cos k - \sin k = 0$$

$$-\cos k = \sin k$$

$$k = \frac{3\pi}{4}$$

Which means $x = 3\pi/4$ must be the line that divides the area of the region in half.

ARC LENTH OF Y=F(X)

1. Find the arc length of the curve over $[0,2]$.

$$y=\frac{4\sqrt{2}}{3}x^{\frac{3}{2}}+6$$

Solution:

The derivative of the function is

$$f'(x)=\frac{3}{2}\cdot\frac{4\sqrt{2}}{3}x^{\frac{3}{2}-1}$$

$$f'(x)=2\sqrt{2}x^{\frac{1}{2}}$$

$$f'(x)=2\sqrt{2x}$$

Then the arc length over the interval is

$$L=\int_a^b\sqrt{1+\left[f'(x)\right]^2}\ dx$$

$$L=\int_0^2\sqrt{1+\left[2\sqrt{2x}\right]^2}\ dx$$

$$L=\int_0^2\sqrt{1+4(2x)}\ dx$$

$$L = \int_0^2 \sqrt{1+8x}\ dx$$

Use substitution.

$$u = 1 + 8x$$

$$\frac{du}{dx} = 8, \text{ so } du = 8\ dx, \text{ so } \ dx = \frac{du}{8}$$

Substitute, integrate, then back-substitute and evaluate over the interval.

$$L = \int_{x=0}^{x=2} \sqrt{u} \left(\frac{du}{8}\right)$$

$$L = \frac{1}{8}\int_{x=0}^{x=2} u^{\frac{1}{2}}\ du$$

$$L = \frac{1}{8}\left(\frac{2}{3}u^{\frac{3}{2}}\right)\Bigg|_{x=0}^{x=2}$$

$$L = \frac{1}{12}u^{\frac{3}{2}}\Bigg|_{x=0}^{x=2}$$

$$L = \frac{1}{12}(1+8x)^{\frac{3}{2}}\Bigg|_0^2$$

$$L = \frac{1}{12}(1+8(2))^{\frac{3}{2}} - \frac{1}{12}(1+8(0))^{\frac{3}{2}}$$

$$L = \frac{1}{12}(17)^{\frac{3}{2}} - \frac{1}{12}(1)^{\frac{3}{2}}$$

$$L=\frac{1}{12}(17)^{\frac{3}{2}}-\frac{1}{12}$$

$$L=\frac{17\sqrt{17}-1}{12}$$

■ 2. Find the arc length of the curve over $[-3,3]$. Round your answer to the nearest three decimal places.

$$y=x^2-3$$

Solution:

The derivative of the function is

$$f'(x)=2x$$

Then the arc length over the interval is

$$L=\int_a^b\sqrt{1+\left[f'(x)\right]^2}\ dx$$

$$L=\int_{-3}^{3}\sqrt{1+\left[2x\right]^2}\ dx$$

$$L=\int_{-3}^{3}\sqrt{1+4x^2}\ dx$$

Use trigonometric substitution.

$$a = 1$$

$$u = 2x$$

$2x = \tan\theta$, so $\theta = \arctan(2x)$

$$x = \frac{1}{2}\tan\theta$$

$$dx = \frac{1}{2}\sec^2\theta\ d\theta$$

Substitute.

$$L = \int_{x=-3}^{x=3} \sqrt{1+\tan^2\theta}\left(\frac{1}{2}\sec^2\theta\ d\theta\right)$$

$$L = \frac{1}{2}\int_{x=-3}^{x=3} \sec^2\theta\sqrt{1+\tan^2\theta}\ d\theta$$

$$L = \frac{1}{2}\int_{x=-3}^{x=3} \sec^2\theta\sqrt{\sec^2\theta}\ d\theta$$

$$L = \frac{1}{2}\int_{x=-3}^{x=3} \sec^3\theta\ d\theta$$

Integrate.

$$L = \frac{1}{2}\left(\frac{1}{2}\sec\theta\tan\theta + \frac{1}{2}\ln|\sec\theta+\tan\theta|\right)\Bigg|_{x=-3}^{x=3}$$

$$L = \frac{1}{4}\left(\sec\theta\tan\theta + \ln|\sec\theta+\tan\theta|\right)\Bigg|_{x=-3}^{x=3}$$

Back-substitute, then evaluate over the interval.

$$L=\frac{1}{4}\Big(\sec(\arctan(2x))\tan(\arctan(2x))$$

$$+\ln|\sec(\arctan(2x))+\tan(\arctan(2x))|\Big)\Big|_{-3}^{3}$$

$$L=\frac{1}{4}\left(\sqrt{(2x)^2+1}\cdot(2x)+\ln\left|\sqrt{(2x)^2+1}+(2x)\right|\right)\Bigg|_{-3}^{3}$$

$$L=\frac{1}{4}\left(2x\sqrt{4x^2+1}+\ln\left|\sqrt{4x^2+1}+2x\right|\right)\Bigg|_{-3}^{3}$$

$$L=\frac{1}{4}\left(2(3)\sqrt{4(3)^2+1}+\ln\left|\sqrt{4(3)^2+1}+2(3)\right|\right)$$

$$-\frac{1}{4}\left(2(-3)\sqrt{4(-3)^2+1}+\ln\left|\sqrt{4(-3)^2+1}+2(-3)\right|\right)$$

$$L=\frac{1}{4}\left(6\sqrt{37}+\ln\left|\sqrt{37}+6\right|\right)-\frac{1}{4}\left(-6\sqrt{37}+\ln\left|\sqrt{37}-6\right|\right)$$

$$L=\frac{3}{2}\sqrt{37}+\frac{1}{4}\ln(\sqrt{37}+6)+\frac{3}{2}\sqrt{37}-\frac{1}{4}\ln(\sqrt{37}-6)$$

$$L=3\sqrt{37}+\frac{1}{4}\ln(\sqrt{37}+6)-\frac{1}{4}\ln(\sqrt{37}-6)$$

$$L=3\sqrt{37}+\frac{1}{4}\left[\ln(\sqrt{37}+6)-\ln(\sqrt{37}-6)\right]$$

$$L = 3\sqrt{37} + \frac{1}{4}\ln\frac{\sqrt{37}+6}{\sqrt{37}-6}$$

$$L \approx 19.494$$

3. Set up the arc length integral of the curve over $[-1,2]$. Do not evaluate the integral.

$$y = \frac{x^3}{3} + x^2 + 5$$

Solution:

The derivative of the function is

$$f'(x) = x^2 + 2x$$

Then the arc length over the interval is

$$L = \int_a^b \sqrt{1 + \left[f'(x)\right]^2}\ dx$$

$$L = \int_{-1}^2 \sqrt{1 + \left[x^2 + 2x\right]^2}\ dx$$

$$L = \int_{-1}^2 \sqrt{1 + x^4 + 4x^3 + 4x^2}\ dx$$

■ 4. Set up the arc length integral of the curve over $[-\pi, \pi]$. Do not evaluate the integral.

$$y = \sin x - 5$$

Solution:

The derivative of the function is

$$f'(x) = \cos x$$

Then the arc length over the interval is

$$L = \int_a^b \sqrt{1 + \left[f'(x)\right]^2}\ dx$$

$$L = \int_{-\pi}^{\pi} \sqrt{1 + [\cos x]^2}\ dx$$

$$L = \int_{-\pi}^{\pi} \sqrt{1 + \cos^2 x}\ dx$$

■ 5. Set up the arc length integral of the curve over $[-\pi/4, \pi/4]$. Do not evaluate the integral.

$$y = \tan x \sec x + 2$$

Solution:

The derivative of the function is

$$f'(x) = \sec^2 x \sec x + \tan x \sec x \tan x$$

$$f'(x) = \sec^3 x + \tan^2 x \sec x$$

Then the arc length over the interval is

$$L = \int_a^b \sqrt{1 + \left[f'(x)\right]^2}\ dx$$

$$L = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \sqrt{1 + (\sec^3 x + \tan^2 x \sec x)^2}\ dx$$

$$L = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \sqrt{1 + \tan^4 x \sec^2 x + 2\tan^2 x \sec^4 x + \sec^6 x}\ dx$$

ARC LENTH OF X=G(Y)

1. Find the arc length of the curve on the interval $1 \le y \le 6$.

$$x = \frac{y^2}{2} - \frac{\ln y}{4} - 8$$

Solution:

The derivative of the function is

$$g'(y) = y - \frac{1}{4y}$$

Then the arc length over the interval is

$$L = \int_c^d \sqrt{1 + \left[g'(y)\right]^2}\ dy$$

$$L = \int_1^6 \sqrt{1 + \left[y - \frac{1}{4y}\right]^2}\ dy$$

$$L = \int_1^6 \sqrt{1 + y^2 - \frac{1}{2} + \frac{1}{16y^2}}\ dy$$

$$L = \int_1^6 \sqrt{y^2 + \frac{1}{16y^2} + \frac{1}{2}}\ dy$$

$$L = \int_1^6 \sqrt{\left(y + \frac{1}{4y}\right)^2}\ dy$$

$$L = \int_1^6 y + \frac{1}{4y}\ dy$$

Integrate, then evaluate over the interval.

$$L = \frac{1}{2}y^2 + \frac{1}{4}\ln|y|\ \Big|_1^6$$

$$L = \frac{1}{2}(6)^2 + \frac{1}{4}\ln|6| - \left(\frac{1}{2}(1)^2 + \frac{1}{4}\ln|1|\right)$$

$$L = 18 + \frac{1}{4}\ln 6 - \frac{1}{2} - \frac{1}{4}(0)$$

$$L = \frac{35}{2} + \frac{1}{4}\ln 6$$

■ 2. Find the arc length of the curve on the interval $0 \le y \le 4$.

$$x = \frac{1}{3}(y^2 + 2)^{\frac{3}{2}} + 5$$

Solution:

The derivative of the function is

$$g'(y) = \frac{1}{2}(y^2 + 2)^{\frac{1}{2}}(2y)$$

$$g'(y) = y(y^2 + 2)^{\frac{1}{2}}$$

$$g'(y) = y\sqrt{y^2 + 2}$$

Then the arc length over the interval is

$$L = \int_c^d \sqrt{1 + [g'(y)]^2}\ dy$$

$$L = \int_0^4 \sqrt{1 + \left[y\sqrt{y^2 + 2}\right]^2}\ dy$$

$$L = \int_0^4 \sqrt{1 + y^2(y^2 + 2)}\ dy$$

$$L = \int_0^4 \sqrt{y^4 + 2y^2 + 1}\ dy$$

$$L = \int_0^4 \sqrt{(y^2 + 1)^2}\ dy$$

$$L = \int_0^4 y^2 + 1\ dy$$

Integrate, then evaluate over the interval.

$$L = \frac{1}{3}y^3 + y\Big|_0^4$$

$$L=\frac{1}{3}(4)^3+4-\left(\frac{1}{3}(0)^3+0\right)$$

$$L=\frac{64}{3}+4$$

$$L=\frac{64}{3}+\frac{12}{3}$$

$$L=\frac{76}{3}$$

3. Find the arc length of the curve on the interval $4 \le y \le 16$.

$$x=y^{\frac{3}{2}}+15$$

Solution:

The derivative of the function is

$$g'(y)=\frac{3}{2}y^{\frac{1}{2}}$$

$$g'(y)=\frac{3}{2}\sqrt{y}$$

Then the arc length over the interval is

$$L=\int_c^d \sqrt{1+\left[g'(y)\right]^2}\ dy$$

$$L=\int_4^{16}\sqrt{1+\left[\frac{3}{2}\sqrt{y}\right]^2}\ dy$$

$$L=\int_4^{16}\sqrt{1+\frac{9}{4}y}\ dy$$

Integrate, then evaluate over the interval.

$$L=\frac{2}{3}\cdot\frac{4}{9}\left(1+\frac{9}{4}y\right)^{\frac{3}{2}}\Bigg|_4^{16}$$

$$L=\frac{8}{27}\left(1+\frac{9}{4}y\right)^{\frac{3}{2}}\Bigg|_4^{16}$$

$$L=\frac{8}{27}\left(1+\frac{9}{4}(16)\right)^{\frac{3}{2}}-\frac{8}{27}\left(1+\frac{9}{4}(4)\right)^{\frac{3}{2}}$$

$$L=\frac{8}{27}(1+36)^{\frac{3}{2}}-\frac{8}{27}(1+9)^{\frac{3}{2}}$$

$$L=\frac{8}{27}(37)^{\frac{3}{2}}-\frac{8}{27}(10)^{\frac{3}{2}}$$

$$L=\frac{296\sqrt{37}}{27}-\frac{80\sqrt{10}}{27}$$

$$L=\frac{296\sqrt{37}-80\sqrt{10}}{27}$$

■ 4. Find the arc length of the curve on the interval $1\le y\le 8$.

$$x = \left(1 - y^{\frac{2}{3}}\right)^{\frac{3}{2}}$$

Solution:

The derivative of the function is

$$g'(y) = \frac{3}{2}\left(1 - y^{\frac{2}{3}}\right)^{\frac{1}{2}}\left(-\frac{2}{3}y^{-\frac{1}{3}}\right)$$

$$g'(y) = -y^{-\frac{1}{3}}\left(1 - y^{\frac{2}{3}}\right)^{\frac{1}{2}}$$

Then the arc length over the interval is

$$L = \int_c^d \sqrt{1 + \left[g'(y)\right]^2}\ dy$$

$$L = \int_1^8 \sqrt{1 + \left[-y^{-\frac{1}{3}}\left(1 - y^{\frac{2}{3}}\right)^{\frac{1}{2}}\right]^2}\ dy$$

$$L = \int_1^8 \sqrt{1 + y^{-\frac{2}{3}}\left(1 - y^{\frac{2}{3}}\right)}\ dy$$

$$L = \int_1^8 \sqrt{1 + \left(y^{-\frac{2}{3}} - y^0\right)}\ dy$$

$$L = \int_1^8 \sqrt{1 + y^{-\frac{2}{3}} - 1}\ dy$$

$$L = \int_{1}^{8} \sqrt{y^{-\frac{2}{3}}}\ dy$$

$$L = \int_{1}^{8} \left(y^{-\frac{2}{3}}\right)^{\frac{1}{2}}\ dy$$

$$L = \int_{1}^{8} y^{-\frac{1}{3}}\ dy$$

Integrate, then evaluate over the interval.

$$L = \frac{3}{2}y^{\frac{2}{3}}\Big|_{1}^{8}$$

$$L = \frac{3}{2}(8)^{\frac{2}{3}} - \frac{3}{2}(1)^{\frac{2}{3}}$$

$$L = \frac{3}{2}(4) - \frac{3}{2}(1)$$

$$L = \frac{3}{2}(4 - 1)$$

$$L = \frac{3}{2}(3)$$

$$L = \frac{9}{2}$$

■ 5. Find the arc length of the curve on the interval $1 \le y \le 5$.

$$x = \frac{y^2}{8} - \ln y$$

Solution:

The derivative of the function is

$$g'(y) = \frac{y}{4} - \frac{1}{y}$$

Then the arc length over the interval is

$$L = \int_c^d \sqrt{1 + \left[g'(y)\right]^2}\ dy$$

$$L = \int_1^5 \sqrt{1 + \left[\frac{y}{4} - \frac{1}{y}\right]^2}\ dy$$

$$L = \int_1^5 \sqrt{1 + \frac{y^2}{16} - \frac{1}{2} + \frac{1}{y^2}}\ dy$$

$$L = \int_1^5 \sqrt{\frac{y^2}{16} + \frac{1}{2} + \frac{1}{y^2}}\ dy$$

$$L = \int_1^5 \sqrt{\left(\frac{y}{4} + \frac{1}{y}\right)^2}\ dy$$

$$L = \int_1^5 \frac{y}{4} + \frac{1}{y}\ dy$$

Integrate, then evaluate over the interval.

$$L = \frac{y^2}{8} + \ln|y| \Big|_1^5$$

$$L = \frac{5^2}{8} + \ln|5| - \frac{1^2}{8} - \ln|1|$$

$$L = \frac{25}{8} + \ln 5 - \frac{1}{8} - 0$$

$$L = \frac{24}{8} + \ln 5$$

$$L = 3 + \ln 5$$

AVERAGE VALUE

1. Find the average value of $f(x)$ over the interval $[-3,5]$.

$$f(x) = -3x^3 - 5x^2 + x + 4$$

Solution:

Plug the interval and the function into the average value integral formula.

$$f_{avg} = \frac{1}{b-a}\int_a^b f(x)\ dx$$

$$f_{avg} = \frac{1}{5-(-3)}\int_{-3}^{5} -3x^3 - 5x^2 + x + 4\ dx$$

$$f_{avg} = \frac{1}{8}\left(-\frac{3}{4}x^4 - \frac{5}{3}x^3 + \frac{1}{2}x^2 + 4x\right)\Bigg|_{-3}^{5}$$

$$f_{avg} = \frac{1}{8}\left(-\frac{3}{4}(5)^4 - \frac{5}{3}(5)^3 + \frac{1}{2}(5)^2 + 4(5)\right)$$

$$-\frac{1}{8}\left(-\frac{3}{4}(-3)^4 - \frac{5}{3}(-3)^3 + \frac{1}{2}(-3)^2 + 4(-3)\right)$$

$$f_{avg} = \frac{1}{8}\left(-\frac{1{,}875}{4} - \frac{625}{3} + \frac{25}{2} + 20\right) - \frac{1}{8}\left(-\frac{243}{4} + \frac{135}{3} + \frac{9}{2} - 12\right)$$

$$f_{avg} = -\frac{1{,}875}{32} - \frac{625}{24} + \frac{25}{16} + \frac{20}{8} + \frac{243}{32} - \frac{135}{24} - \frac{9}{16} + \frac{12}{8}$$

$$f_{avg} = -\frac{1{,}632}{32} - \frac{760}{24} + \frac{16}{16} + \frac{32}{8}$$

$$f_{avg} = -51 - \frac{95}{3} + 1 + 4$$

$$f_{avg} = -46 - \frac{95}{3}$$

$$f_{avg} = -\frac{138}{3} - \frac{95}{3}$$

$$f_{avg} = -\frac{233}{3}$$

■ 2. Find the average value of $g(x)$ over the interval $[-4,3]$.

$$g(x) = \frac{1}{3}x^3 + \frac{3}{2}x^2 + \frac{2}{5}x - 2$$

Solution:

Plug the interval and the function into the average value integral formula.

$$g_{avg} = \frac{1}{b-a}\int_a^b g(x)\ dx$$

$$g_{avg} = \frac{1}{3-(-4)}\int_{-4}^{3} \frac{1}{3}x^3 + \frac{3}{2}x^2 + \frac{2}{5}x - 2\ dx$$

$$g_{avg} = \frac{1}{7}\left(\frac{1}{12}x^4 + \frac{1}{2}x^3 + \frac{1}{5}x^2 - 2x\right)\Bigg|_{-4}^{3}$$

$$g_{avg} = \frac{1}{7}\left(\frac{1}{12}(3)^4 + \frac{1}{2}(3)^3 + \frac{1}{5}(3)^2 - 2(3)\right) - \frac{1}{7}\left(\frac{1}{12}(-4)^4 + \frac{1}{2}(-4)^3 + \frac{1}{5}(-4)^2 - 2(-4)\right)$$

$$g_{avg} = \frac{27}{28} + \frac{27}{14} + \frac{9}{35} - \frac{6}{7} - \frac{64}{21} + \frac{24}{7} - \frac{16}{35}$$

$$g_{avg} = -\frac{7}{35} + \frac{27}{28} - \frac{64}{21} + \frac{27}{14} + \frac{18}{7}$$

$$g_{avg} = -\frac{84}{420} + \frac{405}{420} - \frac{1,280}{420} + \frac{810}{420} + \frac{1,080}{420}$$

$$g_{avg} = \frac{931}{420}$$

$$g_{avg} = \frac{133}{60}$$

■ 3. Find the average value of $h(x)$ over the interval $[-2,3]$.

$$h(x) = 3(2x-5)^2$$

Solution:

Plug the interval and the function into the average value integral formula.

$$h_{avg} = \frac{1}{b-a}\int_a^b h(x)\ dx$$

$$h_{avg} = \frac{1}{3-(-2)} \int_{-2}^{3} 3(2x-5)^2 \ dx$$

$$h_{avg} = \frac{3}{5} \int_{-2}^{3} 4x^2 - 20x + 25 \ dx$$

$$h_{avg} = \frac{3}{5} \left(\frac{4}{3}x^3 - 10x^2 + 25x \right) \Big|_{-2}^{3}$$

$$h_{avg} = \frac{4}{5}x^3 - 6x^2 + 15x \Big|_{-2}^{3}$$

$$h_{avg} = \frac{4}{5}(3)^3 - 6(3)^2 + 15(3) - \left(\frac{4}{5}(-2)^3 - 6(-2)^2 + 15(-2) \right)$$

$$h_{avg} = \frac{108}{5} - 54 + 45 + \frac{32}{5} + 24 + 30$$

$$h_{avg} = \frac{140}{5} + 45$$

$$h_{avg} = 28 + 45$$

$$h_{avg} = 73$$

4. Set up the average value formula for $f(x)$ over the interval $[-4,4]$. Do not evaluate the integral.

$$f(x) = \sqrt{16 - x^2}$$

Solution:

Plug the interval and the function into the average value integral formula.

$$f_{avg} = \frac{1}{b-a}\int_a^b f(x)\ dx$$

$$f_{avg} = \frac{1}{4-(-4)}\int_{-4}^4 \sqrt{16-x^2}\ dx$$

$$f_{avg} = \frac{1}{8}\int_{-4}^4 \sqrt{16-x^2}\ dx$$

MEAN VALUE THEOREM FOR INTEGRALS

■ 1. Use the Mean Value Theorem for integrals to find a value for $f(c)$.

$$\int_4^{20} f(x)\ dx = 26$$

Solution:

Comparing the integral to the Mean Value Theorem formula,

$$\int_a^b f(x)\ dx = f(c)(b - a)$$

we have $a = 4$ and $b = 20$. So we can set up the equation for $f(c)$.

$$f(c)(20 - 4) = 26$$

$$16f(c) = 26$$

$$f(c) = \frac{26}{16} = \frac{13}{8}$$

■ 2. Use the Mean Value Theorem for integrals to find a value for $g(c)$.

$$\int_{-15}^{35} g(x)\ dx = -20$$

Solution:

Comparing the integral to the Mean Value Theorem formula,

$$\int_a^b g(x)\ dx = g(c)(b-a)$$

we have $a = -15$ and $b = 35$. So we can set up the equation for $g(c)$.

$$g(c)(35-(-15)) = -20$$

$$50g(c) = -20$$

$$g(c) = -\frac{20}{50} = -\frac{2}{5}$$

■ 3. Use the Mean Value Theorem for integrals to find a value for $h(c)$.

$$\int_{-1}^5 h(x)\ dx = 48$$

Solution:

Comparing the integral to the Mean Value Theorem formula,

$$\int_a^b h(x)\ dx = h(c)(b-a)$$

we have $a = -1$ and $b = 5$. So we can set up the equation for $h(c)$.

$$h(c)(5 - (-1)) = 48$$

$$6h(c) = 48$$

$$h(c) = \frac{48}{6} = 8$$

SURFACE AREA OF REVOLUTION

1. Find the surface area of the object generated by revolving the curve around the x-axis on the interval $2 \le x \le 7$.

$$f(x) = \frac{1}{3}x + 4$$

Solution:

The surface area of an object formed by rotating the graph of a function $y = f(x)$ on the interval $[a, b]$ is given by

$$A = \int_a^b 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\ dx$$

If we plug in what we've been given, we get

$$A = \int_2^7 2\pi \left(\frac{1}{3}x + 4\right) \sqrt{1 + \left(\frac{1}{3}\right)^2}\ dx$$

$$A = 2\pi \int_2^7 \left(\frac{1}{3}x + 4\right) \sqrt{1 + \frac{1}{9}}\ dx$$

$$A = 2\pi \int_2^7 \left(\frac{1}{3}x + 4\right) \sqrt{\frac{10}{9}}\ dx$$

$$A = \frac{2\sqrt{10}\pi}{3} \int_2^7 \frac{1}{3}x + 4\ dx$$

Integrate, then evaluate over the interval.

$$A = \frac{2\sqrt{10}\pi}{3} \left(\frac{x^2}{6} + 4x \right) \Bigg|_2^7$$

$$A = \frac{2\sqrt{10}\pi}{3} \left(\frac{7^2}{6} + 4(7) \right) - \frac{2\sqrt{10}\pi}{3} \left(\frac{2^2}{6} + 4(2) \right)$$

$$A = \frac{2\sqrt{10}\pi}{3} \left(\frac{49}{6} + 28 \right) - \frac{2\sqrt{10}\pi}{3} \left(\frac{4}{6} + 8 \right)$$

$$A = \frac{2\sqrt{10}\pi}{3} \left(\frac{49}{6} + \frac{168}{6} \right) - \frac{2\sqrt{10}\pi}{3} \left(\frac{4}{6} + \frac{48}{6} \right)$$

$$A = \frac{2\sqrt{10}\pi}{3} \left(\frac{217}{6} \right) - \frac{2\sqrt{10}\pi}{3} \left(\frac{52}{6} \right)$$

$$A = \frac{434\sqrt{10}\pi}{18} - \frac{104\sqrt{10}\pi}{18}$$

Combine into one fraction.

$$A = \frac{434\sqrt{10}\pi - 104\sqrt{10}\pi}{18}$$

$$A = \frac{330\sqrt{10}\pi}{18}$$

$$A = \frac{55\sqrt{10}\pi}{3}$$

2. Find the surface area of the object generated by revolving the curve around the x-axis on the interval $1 \leq x \leq 5$.

$$g(x) = \frac{2}{3}x + 5$$

Solution:

The surface area of an object formed by rotating the graph of a function $y = g(x)$ on the interval $[a, b]$ is given by

$$A = \int_a^b 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \ dx$$

If we plug in what we've been given, we get

$$A = \int_1^5 2\pi \left(\frac{2}{3}x + 5\right) \sqrt{1 + \left(\frac{2}{3}\right)^2} \ dx$$

$$A = 2\pi \int_1^5 \left(\frac{2}{3}x + 5\right) \sqrt{1 + \frac{4}{9}} \ dx$$

$$A = 2\pi \int_1^5 \left(\frac{2}{3}x + 5\right) \sqrt{\frac{13}{9}} \ dx$$

$$A = \frac{2\sqrt{13}\pi}{3} \int_1^5 \frac{2}{3}x + 5\ dx$$

Integrate, then evaluate over the interval.

$$A = \frac{2\sqrt{13}\pi}{3} \left(\frac{1}{3}x^2 + 5x \right) \Bigg|_1^5$$

$$A = \frac{2\sqrt{13}\pi}{3} \left(\frac{1}{3}(5)^2 + 5(5) \right) - \frac{2\sqrt{13}\pi}{3} \left(\frac{1}{3}(1)^2 + 5(1) \right)$$

$$A = \frac{2\sqrt{13}\pi}{3} \left(\frac{25}{3} + 25 \right) - \frac{2\sqrt{13}\pi}{3} \left(\frac{1}{3} + 5 \right)$$

$$A = \frac{2\sqrt{13}\pi}{3} \left(\frac{25}{3} + \frac{75}{3} \right) - \frac{2\sqrt{13}\pi}{3} \left(\frac{1}{3} + \frac{15}{3} \right)$$

$$A = \frac{2\sqrt{13}\pi}{3} \left(\frac{100}{3} \right) - \frac{2\sqrt{13}\pi}{3} \left(\frac{16}{3} \right)$$

$$A = \frac{200\sqrt{13}\pi}{9} - \frac{32\sqrt{13}\pi}{9}$$

Combine into one fraction.

$$A = \frac{168\sqrt{13}\pi}{9}$$

$$A = \frac{56\sqrt{13}\pi}{3}$$

■ 3. Set up the integral that approximates the surface area of the object generated by revolving the curve around the x-axis on the interval $-3 \leq x \leq 3$. Do not evaluate the integral.

$$h(x) = x^2 + 3$$

Solution:

The surface area of an object formed by rotating the graph of a function $y = h(x)$ on the interval $[a, b]$ is given by

$$A = \int_a^b 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\ dx$$

If we plug in what we've been given, we get

$$A = \int_{-3}^{3} 2\pi(x^2 + 3)\sqrt{1 + (2x)^2}\ dx$$

$$A = 2\pi \int_{-3}^{3} (x^2 + 3)\sqrt{1 + 4x^2}\ dx$$

■ 4. Find the surface area of the object generated by revolving the curve around the line $y = -1$ on the interval $3 \leq x \leq 9$.

$$g(x) = 2\sqrt{2x + 7}$$

Solution:

The surface area of an object formed by rotating the graph of a function $y = g(x)$ on the interval $[a, b]$ is given by

$$A = \int_a^b 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\ dx$$

Since the curve is rotated around the line $y = -1$, which is 1 unit below the x-axis, add 1 to the function to get $g(x) = 2\sqrt{2}x + 8$ in the integral.

$$A = \int_3^9 2\pi \left(2\sqrt{2}x + 8\right)\sqrt{1 + (2\sqrt{2})^2}\ dx$$

$$A = 2\pi \int_3^9 \left(2\sqrt{2}x + 8\right)\sqrt{1 + 4(2)}\ dx$$

$$A = 2\sqrt{9}\pi \int_3^9 2\sqrt{2}x + 8\ dx$$

$$A = 6\pi \int_3^9 2\sqrt{2}x + 8\ dx$$

Integrate, then evaluate over the interval.

$$A = 6\pi\left(\sqrt{2}x^2 + 8x\right)\Big|_3^9$$

$$A = 6\pi\left(\sqrt{2}(9)^2 + 8(9)\right) - 6\pi\left(\sqrt{2}(3)^2 + 8(3)\right)$$

$$A = 6\pi\left(81\sqrt{2} + 72\right) - 6\pi\left(9\sqrt{2} + 24\right)$$

$$A = 6\pi\left(81\sqrt{2} + 72 - 9\sqrt{2} - 24\right)$$

$$A = 6\pi\left(72\sqrt{2} + 48\right)$$

$$A = 144\pi\left(3\sqrt{2} + 2\right)$$

SURFACE OF REVOLUTION EQUATION

1. Find an equation for the surface generated by revolving the curve around the x-axis.

$$3x^2 + 2y^2 = 8$$

Solution:

Pick a point $P(x, y, z)$ on the surface of the rotation. Then pick another point $Q(x, y_1, 0)$ with the same x-coordinate as point P.

Then for point Q, the equation is $3x^2 + 2y_1{}^2 = 8$. Since the distance from the x-axis to point P is the same as the distance from the x-axis to point Q, the square of the distances are also equal.

$$d_P = \sqrt{y^2 + z^2} \qquad d_P{}^2 = y^2 + z^2$$

$$d_Q = \sqrt{y_1{}^2 + 0^2} \qquad d_Q{}^2 = y_1{}^2$$

So

$$y_1{}^2 = y^2 + z^2$$

Substitute this expression into the original equation, simplify, and get an equation for the surface.

$$3x^2 + 2\left(y^2 + z^2\right) = 8$$

$$3x^2 + 2y^2 + 2z^2 = 8$$

■ 2. Find an equation for the surface generated by revolving the curve around the y-axis.

$$5x^2 = 8y^2$$

Solution:

Pick a point $P(x, y, z)$ on the surface of the rotation. Then pick another point $Q(x, y_1, 0)$ with the same y-coordinate as point P.

Then for point Q, the equation is $5x_1{}^2 = 8y^2$. Since the distance from the y-axis to point P is the same as the distance from the y-axis to point Q, the square of the distances are also equal.

$$d_P = \sqrt{y^2 + z^2} \qquad d_P{}^2 = y^2 + z^2$$

$$d_Q = \sqrt{x_1{}^2 + 0^2} \qquad d_Q{}^2 = x_1{}^2$$

So

$$x_1{}^2 = x^2 + z^2$$

Substitute this expression into the original equation, simplify, and get an equation for the surface.

$$5(x^2 + z^2) = 8y^2$$

$$5x^2 + 5z^2 = 8y^2$$

■ 3. Find an equation for the surface generated by revolving the curve around the x-axis.

$$9x^2 + 25y^2 = 36$$

Solution:

Pick a point $P(x, y, z)$ on the surface of the rotation. Then pick another point $Q(x, y_1, 0)$ with the same x-coordinate as point P.

Then for point Q, the equation is $9x^2 + 25{y_1}^2 = 36$. Since the distance from the x-axis to point P is the same as the distance from the x-axis to point Q, the square of the distances are also equal.

$$d_P = \sqrt{y^2 + z^2} \qquad {d_P}^2 = y^2 + z^2$$

$$d_Q = \sqrt{{y_1}^2 + 0^2} \qquad {d_Q}^2 = {y_1}^2$$

So

$${y_1}^2 = y^2 + z^2$$

Substitute this expression into the original equation, simplify, and get an equation for the surface.

$$9x^2 + 25(y^2 + z^2) = 36$$

$$9x^2 + 25y^2 + 25z^2 = 36$$

DISKS, HORIZONTAL AXIS

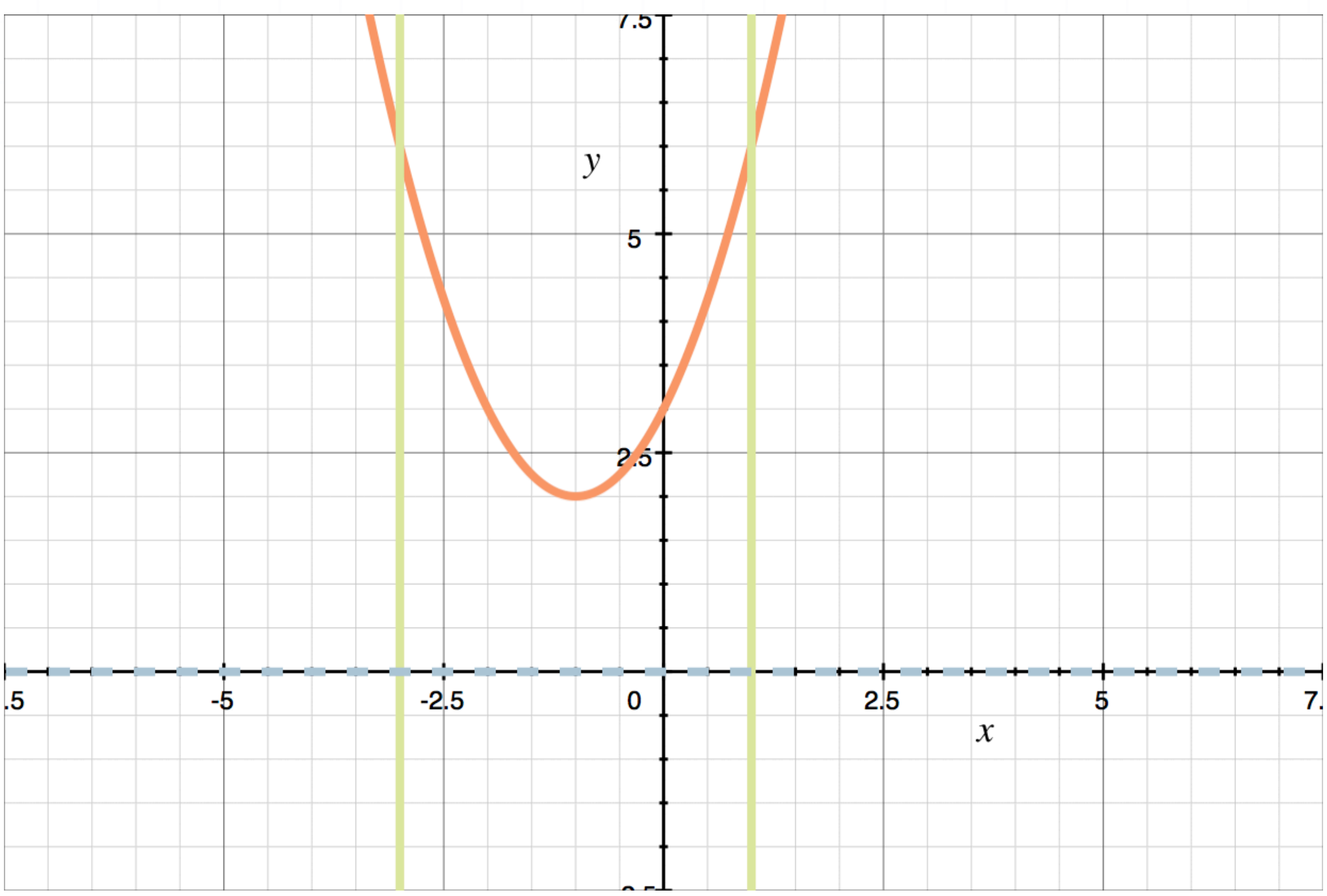
■ 1. Use disks to find the volume of the solid that's formed by rotating the region enclosed by the curves about the x-axis.

$$y = x^2 + 2x + 3$$

$x = -3$ and $x = 1$

Solution:

A sketch of the region and the axis of revolution $y = 0$ is



The volume given by disks is

$$V=\int_a^b \pi \left[f(x)\right]^2\ dx$$

$$V=\int_{-3}^{1} \pi(x^2+2x+3)^2\ dx$$

$$V=\int_{-3}^{1} \pi(x^4+2x^3+3x^2+2x^3+4x^2+6x+3x^2+6x+9)\ dx$$

$$V=\pi\int_{-3}^{1} x^4+4x^3+10x^2+12x+9\ dx$$

Integrate, then evaluate over the interval.

$$V=\pi\left(\frac{1}{5}x^5+x^4+\frac{10}{3}x^3+6x^2+9x\right)\Bigg|_{-3}^{1}$$

$$V=\pi\left(\frac{1}{5}(1)^5+1^4+\frac{10}{3}(1)^3+6(1)^2+9(1)\right)$$

$$-\pi\left(\frac{1}{5}(-3)^5+(-3)^4+\frac{10}{3}(-3)^3+6(-3)^2+9(-3)\right)$$

$$V=\pi\left(\frac{1}{5}+1+\frac{10}{3}+6+9\right)$$

$$-\pi\left(\frac{1}{5}(-243)+81+\frac{10}{3}(-27)+6(9)-27\right)$$

$$V=\pi\left(\frac{1}{5}+\frac{10}{3}+16\right)-\pi\left(-\frac{243}{5}+81-90+54-27\right)$$

$$V=\pi\left(\frac{1}{5}+\frac{10}{3}+16\right)-\pi\left(-\frac{243}{5}+18\right)$$

$$V=\pi\left(\frac{3}{15}+\frac{50}{15}+\frac{240}{15}\right)-\pi\left(-\frac{729}{15}+\frac{270}{15}\right)$$

$$V=\frac{293}{15}\pi+\frac{459}{15}\pi$$

$$V=\frac{752\pi}{15}$$

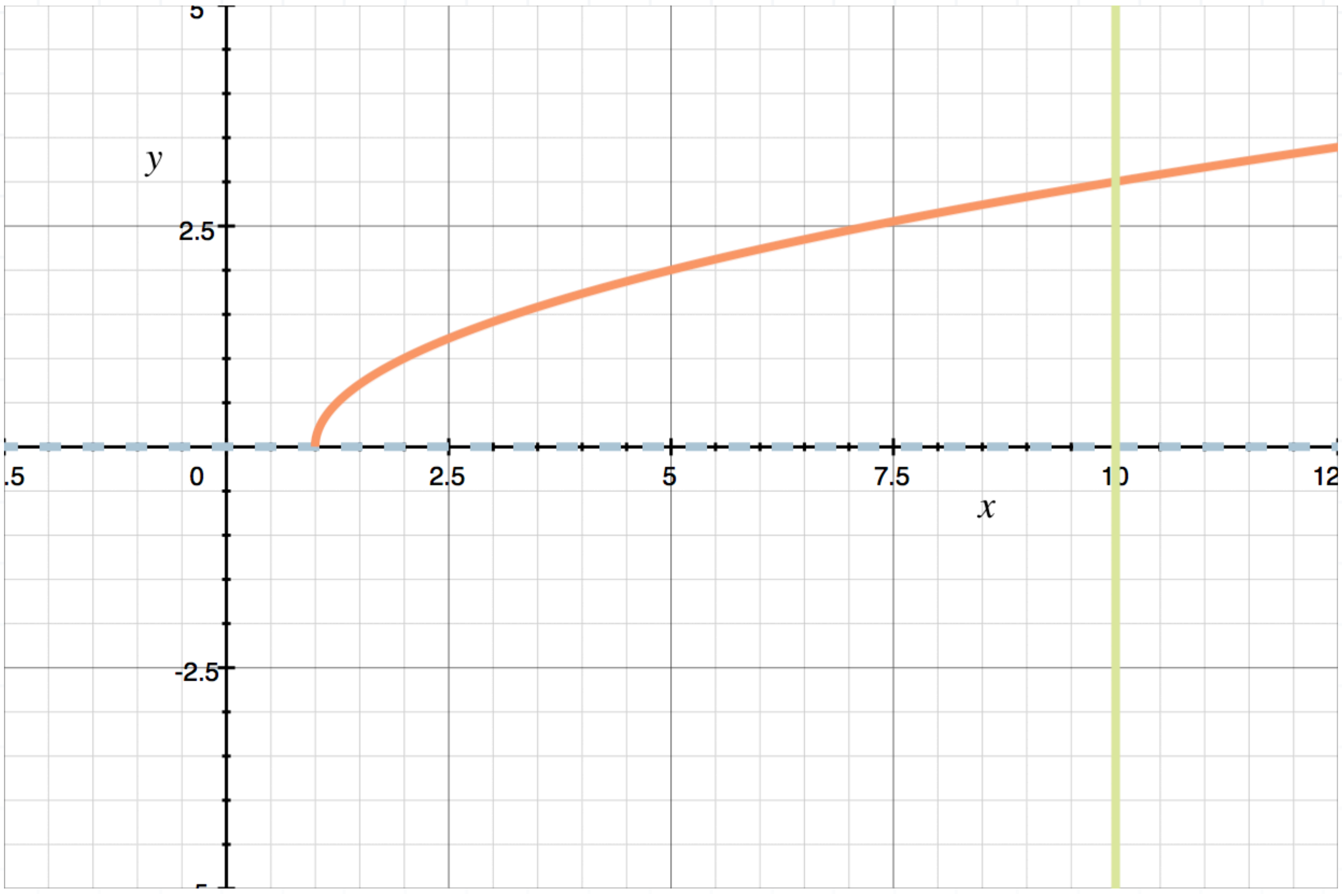
■ 2. Use disks to find the volume of the solid that's formed by rotating the region enclosed by the curves about the x-axis.

$$y=\sqrt{x-1}$$

$x=1$ and $x=10$

Solution:

A sketch of the region and the axis of revolution $y=0$ is



The volume given by disks is

$$V = \int_a^b \pi \left[f(x)\right]^2 \ dx$$

$$V = \int_1^{10} \pi \left(\sqrt{x-1}\right)^2 \ dx$$

$$V = \pi \int_1^{10} x - 1 \ dx$$

Integrate, then evaluate over the interval.

$$V = \pi \left(\frac{1}{2}x^2 - x\right)\Big|_1^{10}$$

$$V = \pi \left(\frac{1}{2}(10)^2 - 10\right) - \pi \left(\frac{1}{2}(1)^2 - 1\right)$$

$$V = \pi(50 - 10) - \pi\left(\frac{1}{2} - 1\right)$$

$$V = 40\pi + \frac{1}{2}\pi$$

$$V = \frac{81\pi}{2}$$

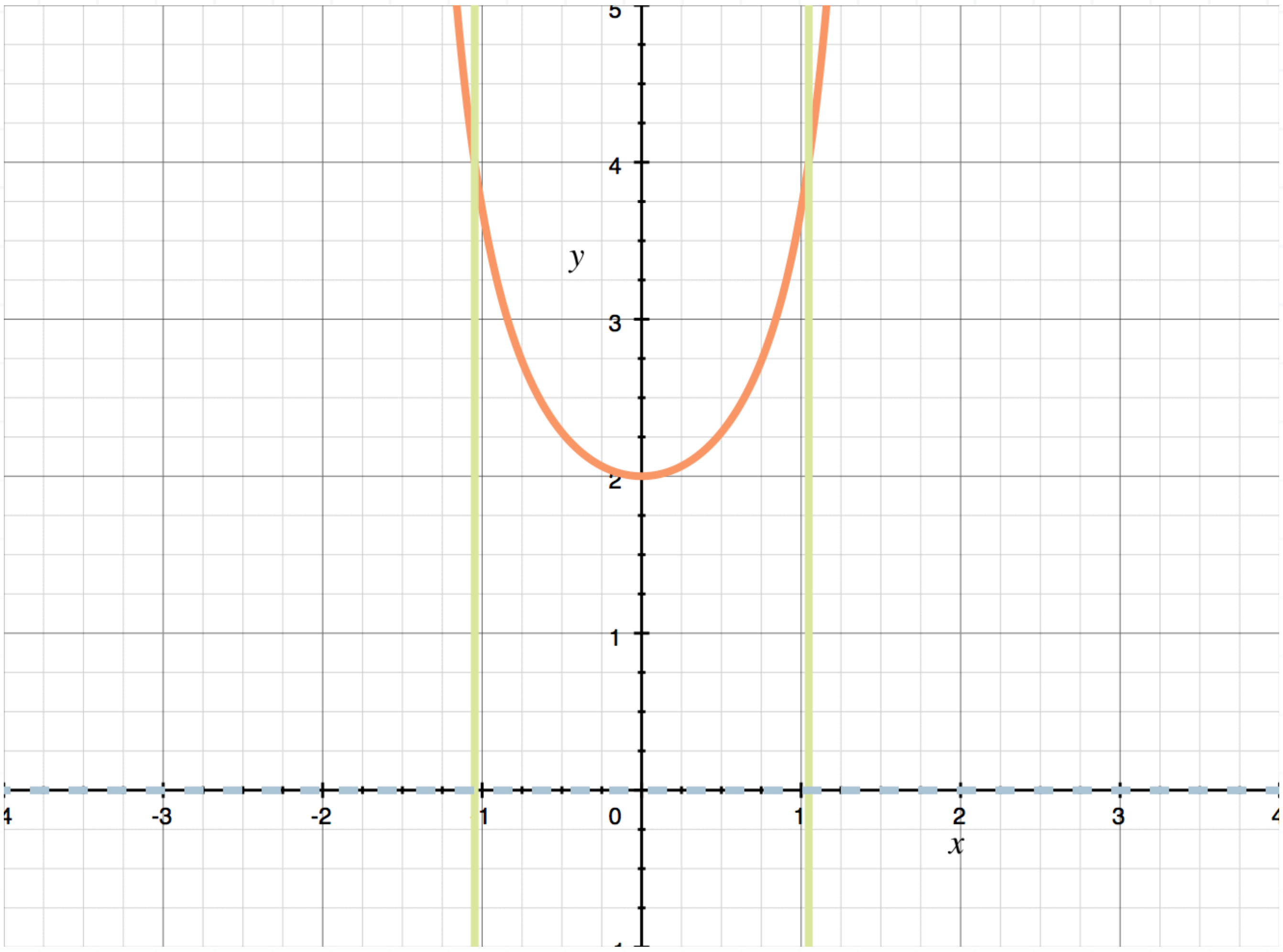
3. Use disks to find the volume of the solid that's formed by rotating the region enclosed by the curves about the x-axis.

$$y = 2\sec x$$

$$x = -\frac{\pi}{3} \text{ and } x = \frac{\pi}{3}$$

Solution:

A sketch of the region and the axis of revolution $y = 0$ is



The volume given by disks is

$$V = \int_a^b \pi \left[f(x)\right]^2 \ dx$$

$$V = \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \pi(2\sec x)^2 \ dx$$

$$V = 4\pi \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \sec^2 x \ dx$$

Integrate, then evaluate over the interval.

$$V = 4\pi \tan x \Big|_{-\frac{\pi}{3}}^{\frac{\pi}{3}}$$

$$V = 4\pi \tan \frac{\pi}{3} - 4\pi \tan \left(-\frac{\pi}{3}\right)$$

$$V = 4\pi\sqrt{3} - 4\pi\left(-\sqrt{3}\right)$$

$$V = 4\sqrt{3}\pi + 4\sqrt{3}\pi$$

$$V = 8\sqrt{3}\pi$$

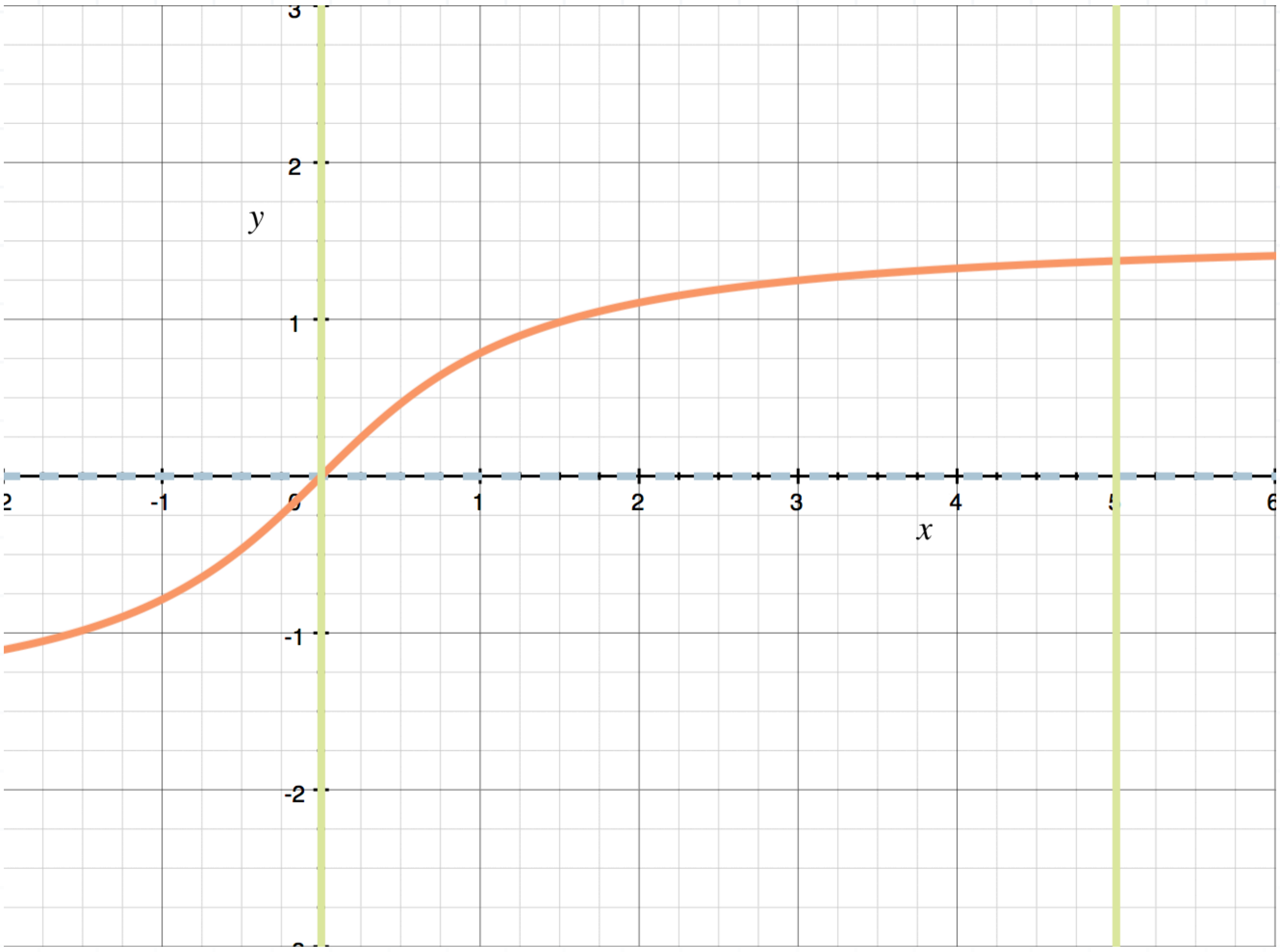
■ 4. Set up the integral that approximates the volume of the solid that's formed by rotating the region enclosed by the curves about the x-axis. Do not evaluate the integral.

$$y = \arctan x$$

$x = 0$ and $x = 5$

Solution:

A sketch of the region and the axis of revolution $y = 0$ is



The volume given by disks is

$$V = \int_a^b \pi \left[f(x)\right]^2 \ dx$$

$$V = \int_0^5 \pi \arctan^2 x \ dx$$

$$V = \pi \int_0^5 \arctan^2 x \ dx$$

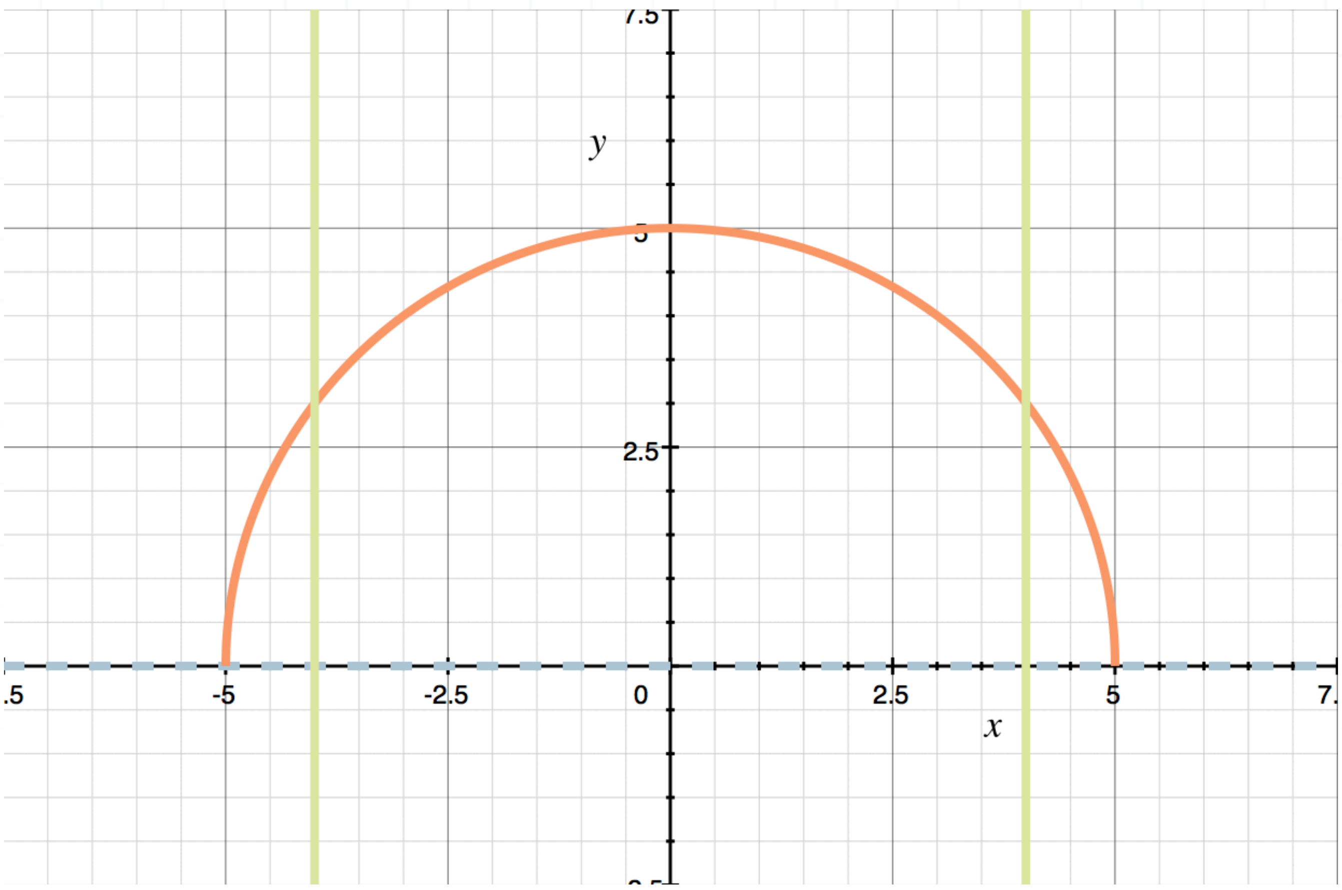
■ 5. Use disks to find the volume of the solid that's formed by rotating the region enclosed by the curves about the x-axis.

$$y = \sqrt{25 - x^2}$$

$x = -4$ and $x = 4$

Solution:

A sketch of the region and the axis of revolution $y = 0$ is



The volume given by disks is

$$V = \int_a^b \pi \left[f(x)\right]^2 \, dx$$

$$V = \int_{-4}^{4} \pi \left[\sqrt{25 - x^2}\right]^2 \, dx$$

$$V = \pi \int_{-4}^{4} 25 - x^2 \, dx$$

Integrate, then evaluate over the interval.

$$V = \pi \left(25x - \frac{1}{3}x^3 \right) \Big|_{-4}^{4}$$

$$V = \pi \left(25(4) - \frac{1}{3}(4)^3 \right) - \pi \left(25(-4) - \frac{1}{3}(-4)^3 \right)$$

$$V = \pi \left(100 - \frac{64}{3} \right) - \pi \left(-100 + \frac{64}{3} \right)$$

$$V = \pi \left(\frac{300}{3} - \frac{64}{3} + \frac{300}{3} - \frac{64}{3} \right)$$

$$V = \frac{472\pi}{3}$$

DISKS, VERTICAL AXIS

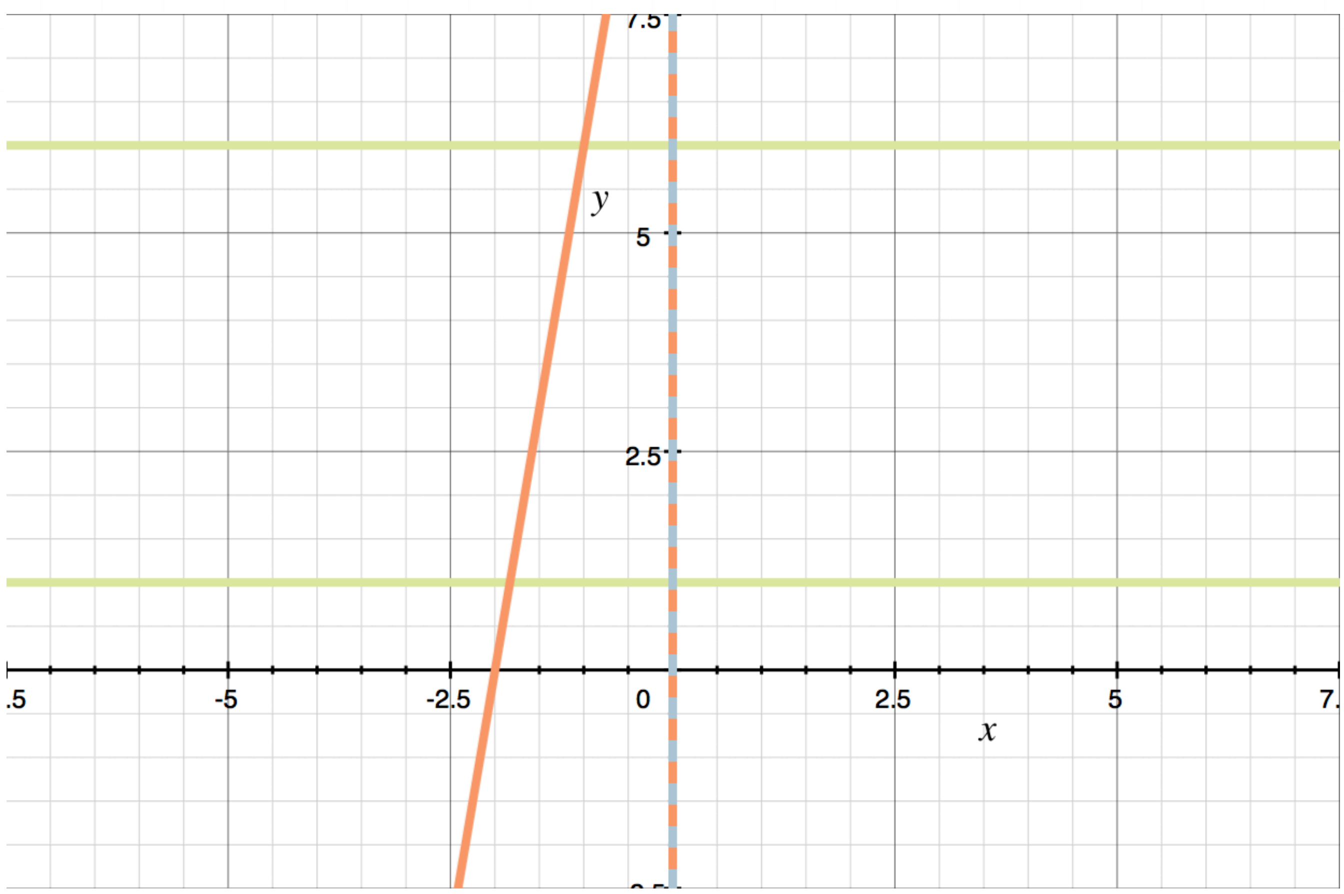
1. Use disks to find the volume of the solid that's formed by rotating the region enclosed by the curves about the y-axis.

$$x = \frac{1}{6}y - 2 \text{ and } x = 0$$

$$y = 1 \text{ and } y = 6$$

Solution:

A sketch of the region and the axis of revolution $x = 0$ is



The volume given by disks is

$$V=\int_c^d \pi \left[f(y)\right]^2\ dy$$

$$V=\int_1^6 \pi \left[\frac{1}{6}y-2\right]^2\ dy$$

$$V=\pi\int_1^6 \frac{1}{36}y^2-\frac{2}{3}y+4\ dy$$

Integrate, then evaluate over the interval.

$$V=\pi\left(\frac{1}{108}y^3-\frac{1}{3}y^2+4y\right)\Bigg|_1^6$$

$$V=\pi\left(\frac{1}{108}(6)^3-\frac{1}{3}(6)^2+4(6)\right)-\pi\left(\frac{1}{108}(1)^3-\frac{1}{3}(1)^2+4(1)\right)$$

$$V=\pi\,(2-12+24)-\pi\left(\frac{1}{108}-\frac{1}{3}+4\right)$$

$$V=\pi\left(2-12+24-\frac{1}{108}+\frac{1}{3}-4\right)$$

$$V=\pi\left(10-\frac{1}{108}+\frac{1}{3}\right)$$

$$V=\pi\left(\frac{1{,}080}{108}-\frac{1}{108}+\frac{36}{108}\right)$$

$$V=\frac{1{,}115\pi}{108}$$

2. Use disks to find the volume of the solid that's formed by rotating the region enclosed by the curves about the y-axis.

$x = \frac{3}{7}\sqrt{y} + 2$ and $x = 0$

$y = 2$ and $y = 5$

Solution:

A sketch of the region and the axis of revolution $y = 0$ is

The volume given by disks is

$$V = \int_c^d \pi \left[f(y)\right]^2 \, dy$$

$$V=\int_{2}^{5}\pi\left[\frac{3}{7}\sqrt{y}+2\right]^{2}\ dy$$

$$V=\pi\int_{2}^{5}\frac{9}{49}y+\frac{12}{7}\sqrt{y}+4\ dy$$

Integrate, then evaluate over the interval.

$$V=\pi\left(\frac{9}{98}y^{2}+\frac{8}{7}y^{\frac{3}{2}}+4y\right)\Big|_{2}^{5}$$

$$V=\pi\left(\frac{9}{98}(5)^{2}+\frac{8}{7}(5)^{\frac{3}{2}}+4(5)\right)-\pi\left(\frac{9}{98}(2)^{2}+\frac{8}{7}(2)^{\frac{3}{2}}+4(2)\right)$$

$$V=\pi\left(\frac{225}{98}+\frac{8\sqrt{125}}{7}+20-\frac{36}{98}-\frac{8\sqrt{8}}{7}-8\right)$$

$$V=\pi\left(\frac{27}{14}+\frac{8\sqrt{125}-8\sqrt{8}}{7}+12\right)$$

$$V=\pi\left(\frac{27}{14}+\frac{16\sqrt{125}-16\sqrt{8}}{14}+\frac{168}{14}\right)$$

$$V=\frac{16\sqrt{125}-16\sqrt{8}+195}{14}\pi$$

$$V=\frac{80\sqrt{5}-32\sqrt{2}+195}{14}\pi$$

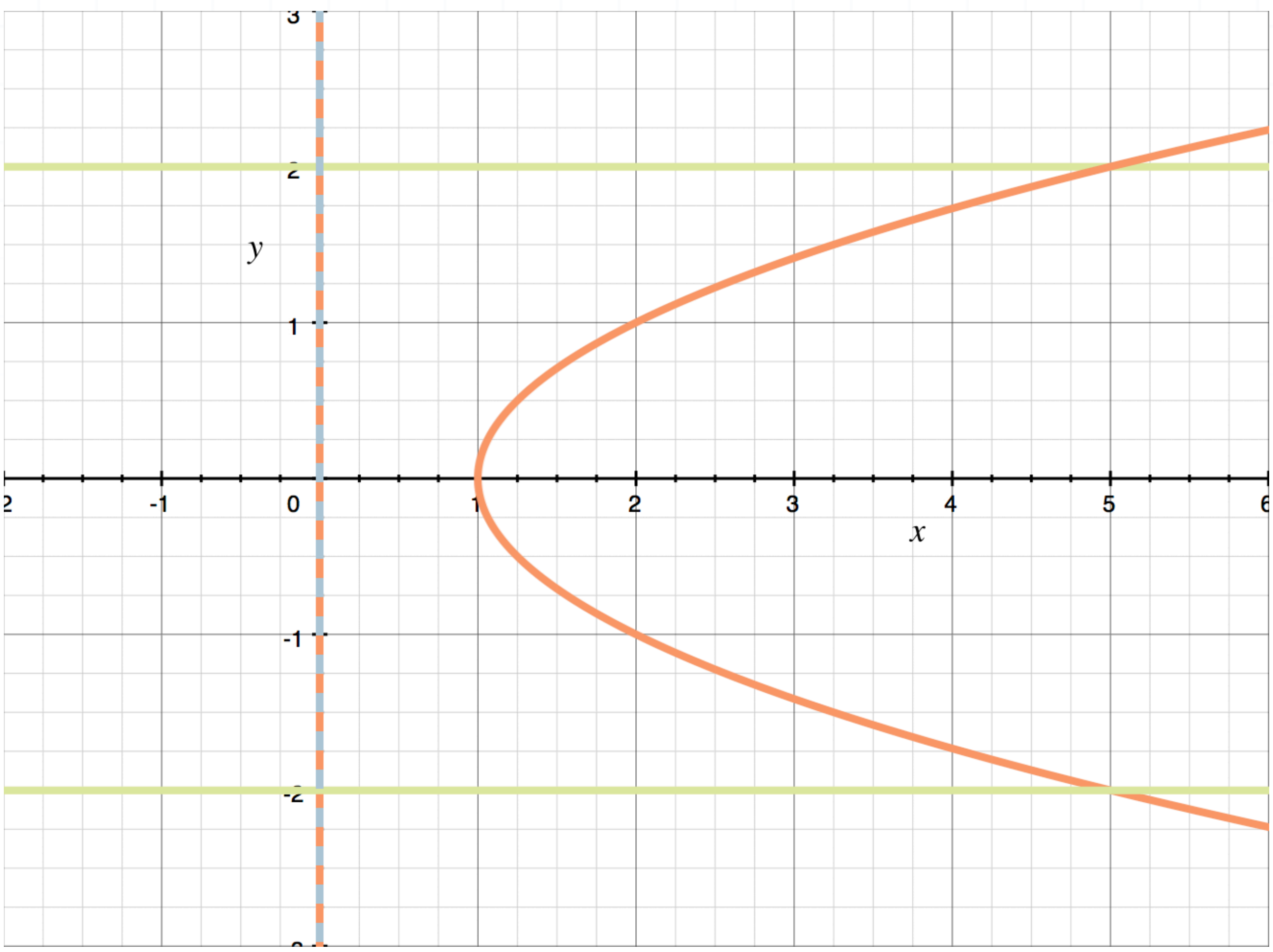
■ 3. Use disks to find the volume of the solid that's formed by rotating the region enclosed by the curves about the y-axis.

$x = y^2 + 1$ and $x = 0$

$y = -2$ and $y = 2$

Solution:

A sketch of the region and the axis of revolution $x = 0$ is



The volume given by disks is

$$V = \int_c^d \pi \left[f(y)\right]^2 \, dy$$

$$V=\int_{-2}^{2}\pi\left[y^2+1\right]^2\ dy$$

$$V=\pi\int_{-2}^{2}y^4+2y^2+1\ dy$$

Integrate, then evaluate over the interval.

$$V=\pi\left(\frac{1}{5}y^5+\frac{2}{3}y^3+y\right)\Big|_{-2}^{2}$$

$$V=\pi\left(\frac{1}{5}(2)^5+\frac{2}{3}(2)^3+2\right)-\pi\left(\frac{1}{5}(-2)^5+\frac{2}{3}(-2)^3-2\right)$$

$$V=\pi\left(\frac{32}{5}+\frac{16}{3}+2\right)-\pi\left(-\frac{32}{5}-\frac{16}{3}-2\right)$$

$$V=\pi\left(\frac{32}{5}+\frac{16}{3}+2+\frac{32}{5}+\frac{16}{3}+2\right)$$

$$V=\pi\left(\frac{64}{5}+\frac{32}{3}+4\right)$$

$$V=\pi\left(\frac{192}{15}+\frac{160}{15}+\frac{60}{15}\right)$$

$$V=\frac{412\pi}{15}$$

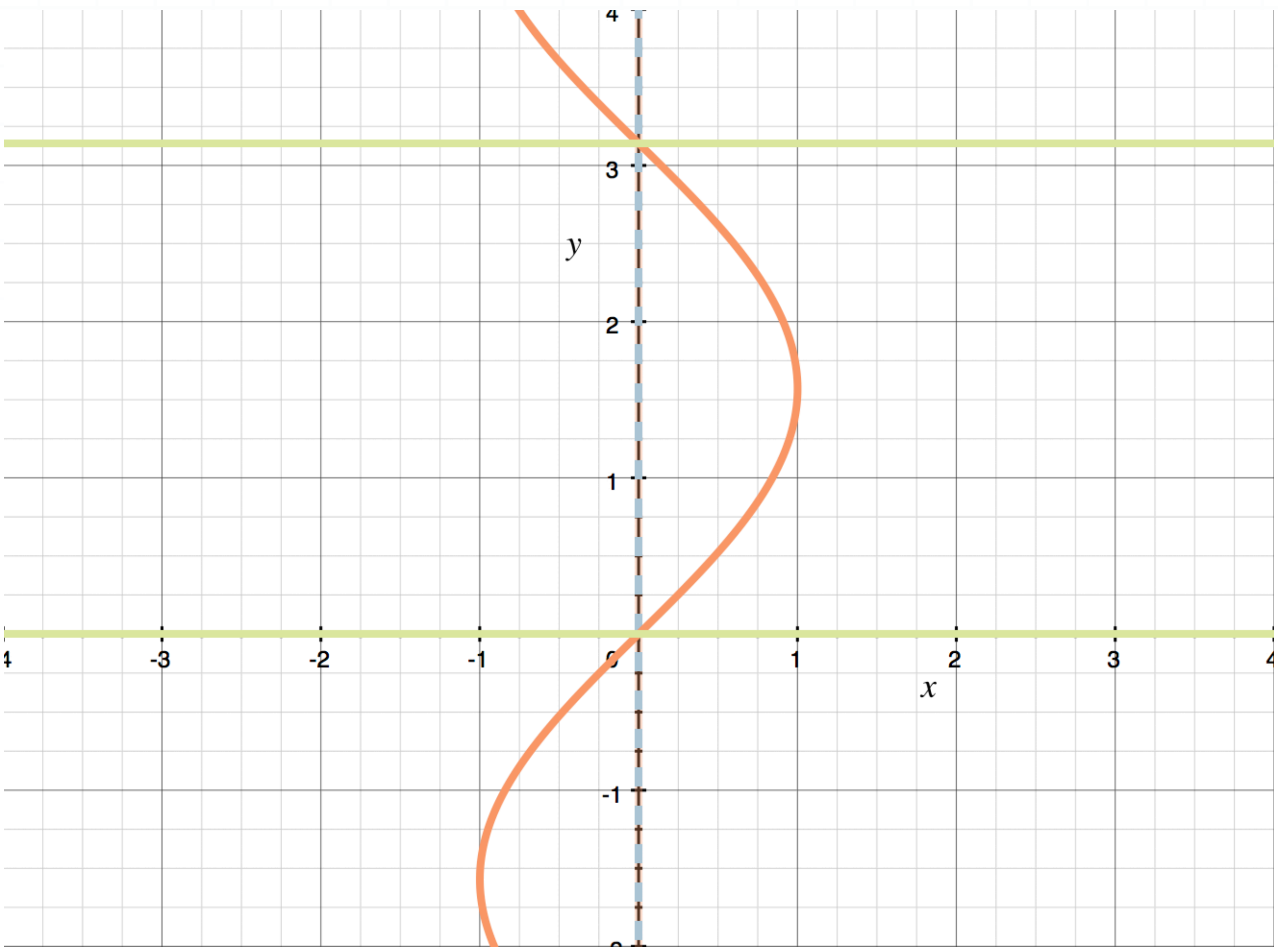
4. Use disks to find the volume of the solid that's formed by rotating the region enclosed by the curves about the y-axis. Set up the integral, but do not evaluate it.

$$x = \sin y$$

$y = 0$ and $y = \pi$

Solution:

A sketch of the region and the axis of revolution $x = 0$ is



The volume given by disks is

$$V=\int_c^d \pi \left[f(y)\right]^2 \ dy$$

$$V=\int_0^\pi \pi \left[\sin y\right]^2 \ dy$$

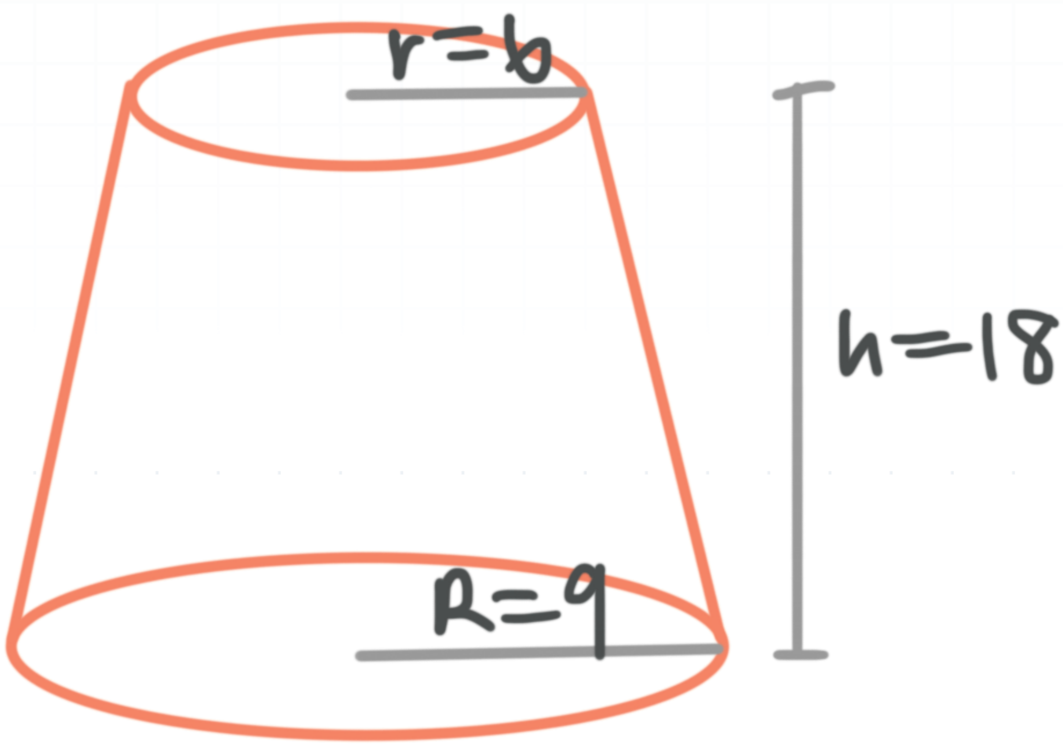
$$V=\pi\int_0^\pi \sin^2 y \ dy$$

DISKS, VOLUME OF THE FRUSTUM

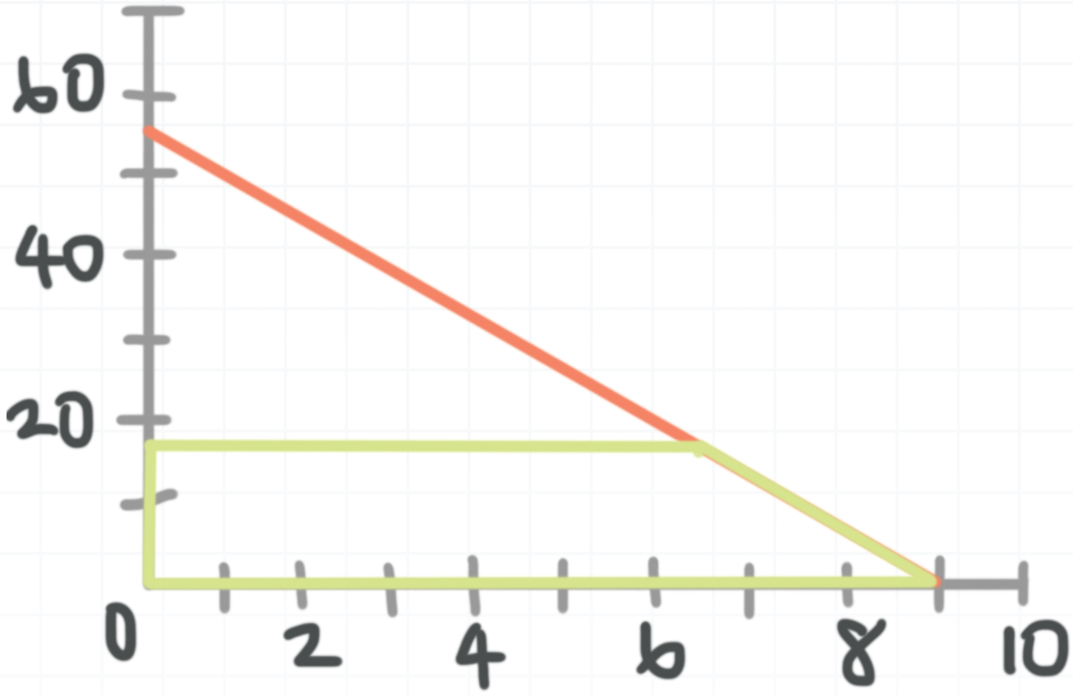
■ 1. Use disks to find the volume of the frustum of a right circular cone with height $h = 18$ inches, a lower base radius $R = 9$ inches, and an upper radius of $r = 6$ inches.

Solution:

A sketch of the frustum is



We could create this frustum by rotating this green region about the y-axis.



The slope of the line contains $(9,0)$ and $(6,18)$. The slope that connects the points is

$$m=\frac{18-0}{6-9}=-\frac{18}{3}=-6$$

Then the line that gives the slant height is

$$y=-6x+54$$

$$y-54=-6x$$

$$x=\frac{y-54}{-6}$$

$$x=9-\frac{1}{6}y$$

Then the volume of the frustum, using disks, is given by

$$V=\int_{c}^{d}\pi\left[f(y)\right]^{2}\ dy$$

$$V = \int_0^{18} \pi \left[9 - \frac{1}{6}y\right]^2 \ dy$$

$$V = \pi \int_0^{18} 81 - 3y + \frac{1}{36}y^2 \ dy$$

Integrate, then evaluate over the interval.

$$V = \pi \left(81y - \frac{3}{2}y^2 + \frac{1}{108}y^3\right)\Big|_0^{18}$$

$$V = \pi \left(81(18) - \frac{3}{2}(18)^2 + \frac{1}{108}(18)^3\right) - \pi \left(81(0) - \frac{3}{2}(0)^2 + \frac{1}{108}(0)^3\right)$$

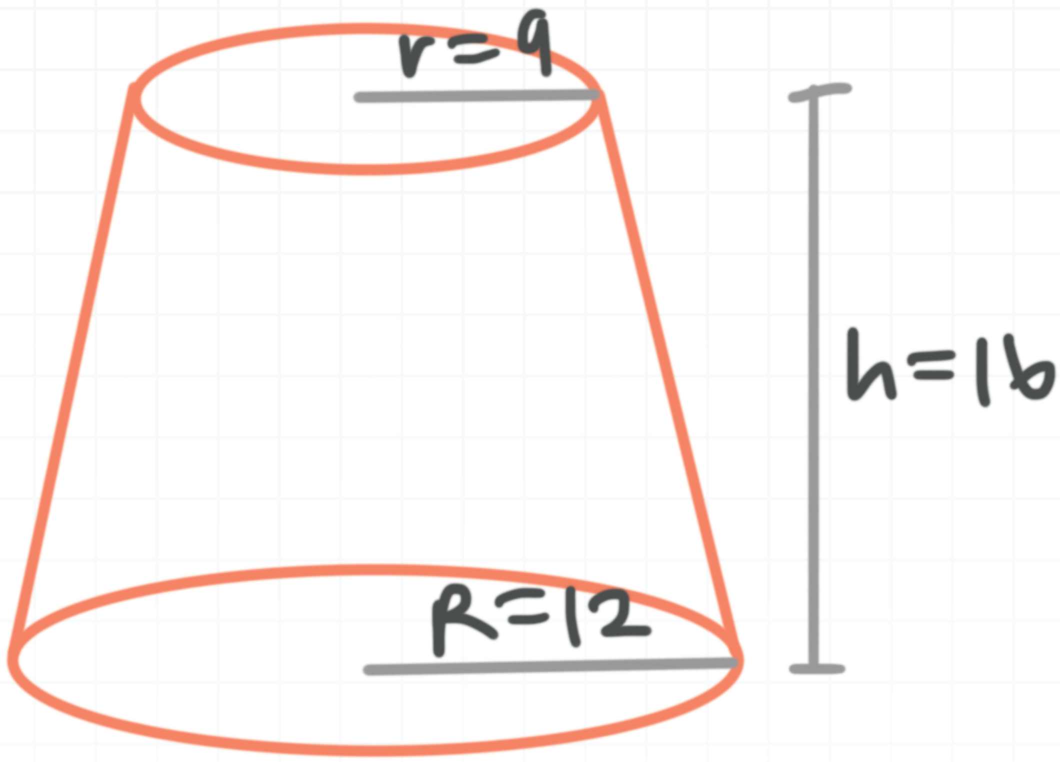
$$V = \pi(1{,}458 - 486 + 54)$$

$$V = 1{,}026\pi$$

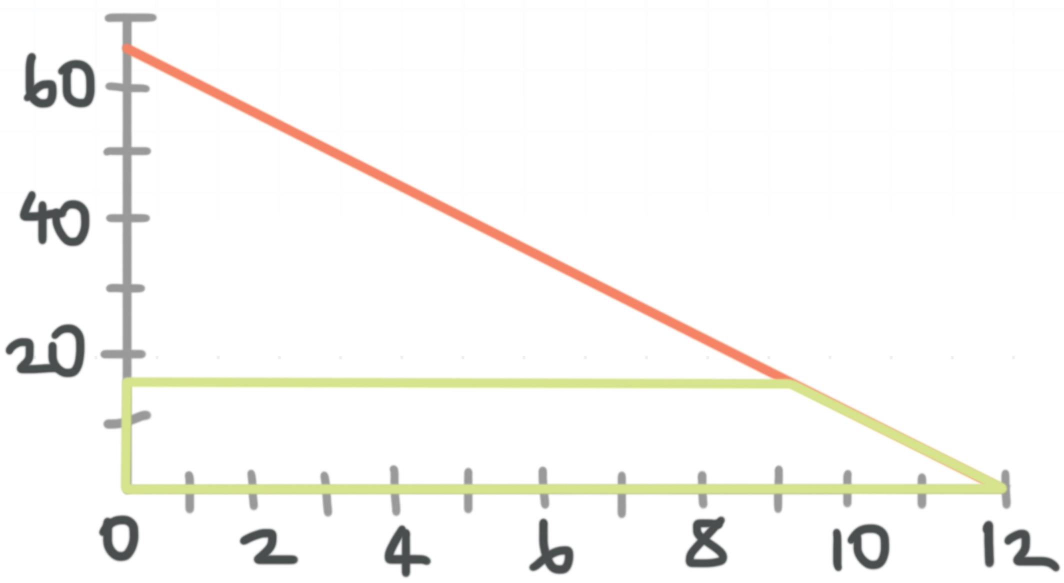
■ 2. Use disks to find the volume of the frustum of a right circular cone with height $h = 16$ inches, a lower base radius $R = 12$ inches, and an upper radius of $r = 9$ inches.

Solution:

A sketch of the frustum is



We could create this frustum by rotating this green region about the y -axis.



The slope of the line contains $(12,0)$ and $(9,16)$. The slope that connects the points is

$$m = \frac{16 - 0}{9 - 12} = -\frac{16}{3}$$

Then the line that gives the slant height is

$$y = -\frac{16}{3}x + 64$$

$$y - 64 = -\frac{16}{3}x$$

$$3y - 192 = -16x$$

$$x = -\frac{3}{16}y + 12$$

Then the volume of the frustum, using disks, is given by

$$V = \int_c^d \pi \left[f(y)\right]^2 \ dy$$

$$V = \int_0^{16} \pi \left[-\frac{3}{16}y + 12\right]^2 \ dy$$

$$V = \pi \int_0^{16} \frac{9}{256}y^2 - \frac{9}{2}y + 144 \ dy$$

Integrate, then evaluate over the interval.

$$V = \pi \left(\frac{3}{256}y^3 - \frac{9}{4}y^2 + 144y \right) \Big|_0^{16}$$

$$V = \pi \left(\frac{3}{256}(16)^3 - \frac{9}{4}(16)^2 + 144(16) \right) - \pi \left(\frac{3}{256}(0)^3 - \frac{9}{4}(0)^2 + 144(0) \right)$$

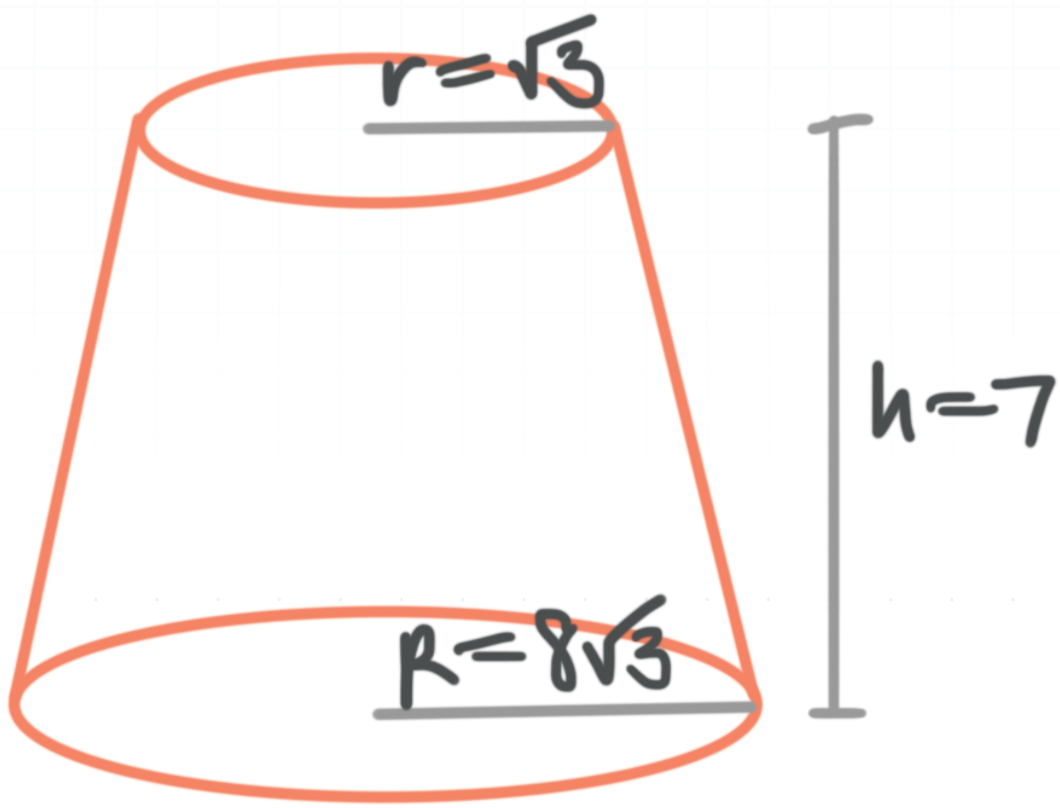
$$V = \pi(48 - 576 + 2{,}304)$$

$$V = 1{,}776\pi$$

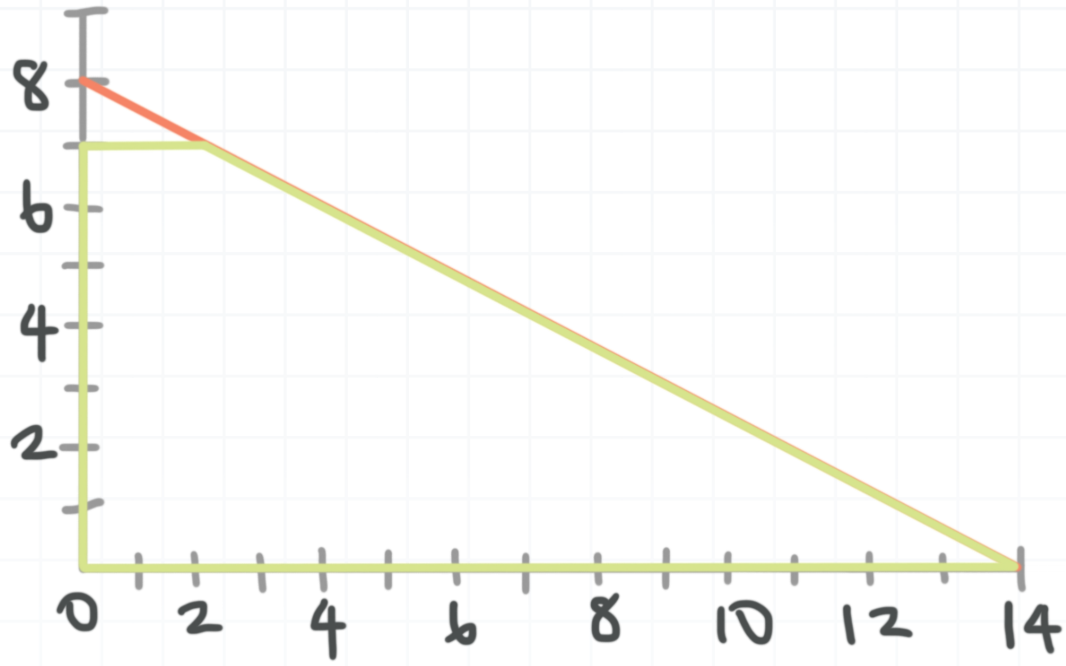
3. Use disks to find the volume of the frustum of a right circular cone with height $h = 7$ inches, a lower base radius $R = 8\sqrt{3}$ inches, and an upper radius of $r = \sqrt{3}$ inches.

Solution:

A sketch of the frustum is



We could create this frustum by rotating this green region about the y-axis.



The slope of the line contains $\left(8\sqrt{3},0\right)$ and $\left(\sqrt{3},7\right)$. The slope that connects the points is

$$m=\frac{7-0}{\sqrt{3}-8\sqrt{3}}=-\frac{7}{7\sqrt{3}}=-\frac{1}{\sqrt{3}}$$

Then the line that gives the slant height is

$$y=-\frac{1}{\sqrt{3}}x+8$$

$$y-8=-\frac{1}{\sqrt{3}}x$$

$$8-y=\frac{1}{\sqrt{3}}x$$

$$x=8\sqrt{3}-\sqrt{3}y$$

Then the volume of the frustum, using disks, is given by

$$V=\int_{c}^{d}\pi\left[f(y)\right]^2\ dy$$

$$V=\int_{0}^{7}\pi\left[8\sqrt{3}-\sqrt{3}y\right]^2\ dy$$

$$V=\pi\int_{0}^{7}192-48y+3y^2\ dy$$

Integrate, then evaluate over the interval.

$$V=\pi(192y-24y^2+y^3)\Big|_0^7$$

$$V=\pi(192(7)-24(7)^2+7^3)-\pi(192(0)-24(0)^2+0^3)$$

$$V=\pi(1{,}344-1{,}176+343)$$

$$V=\pi(1{,}344-1{,}176+343)$$

$$V=511\pi$$

WASHERS, HORIZONTAL AXIS

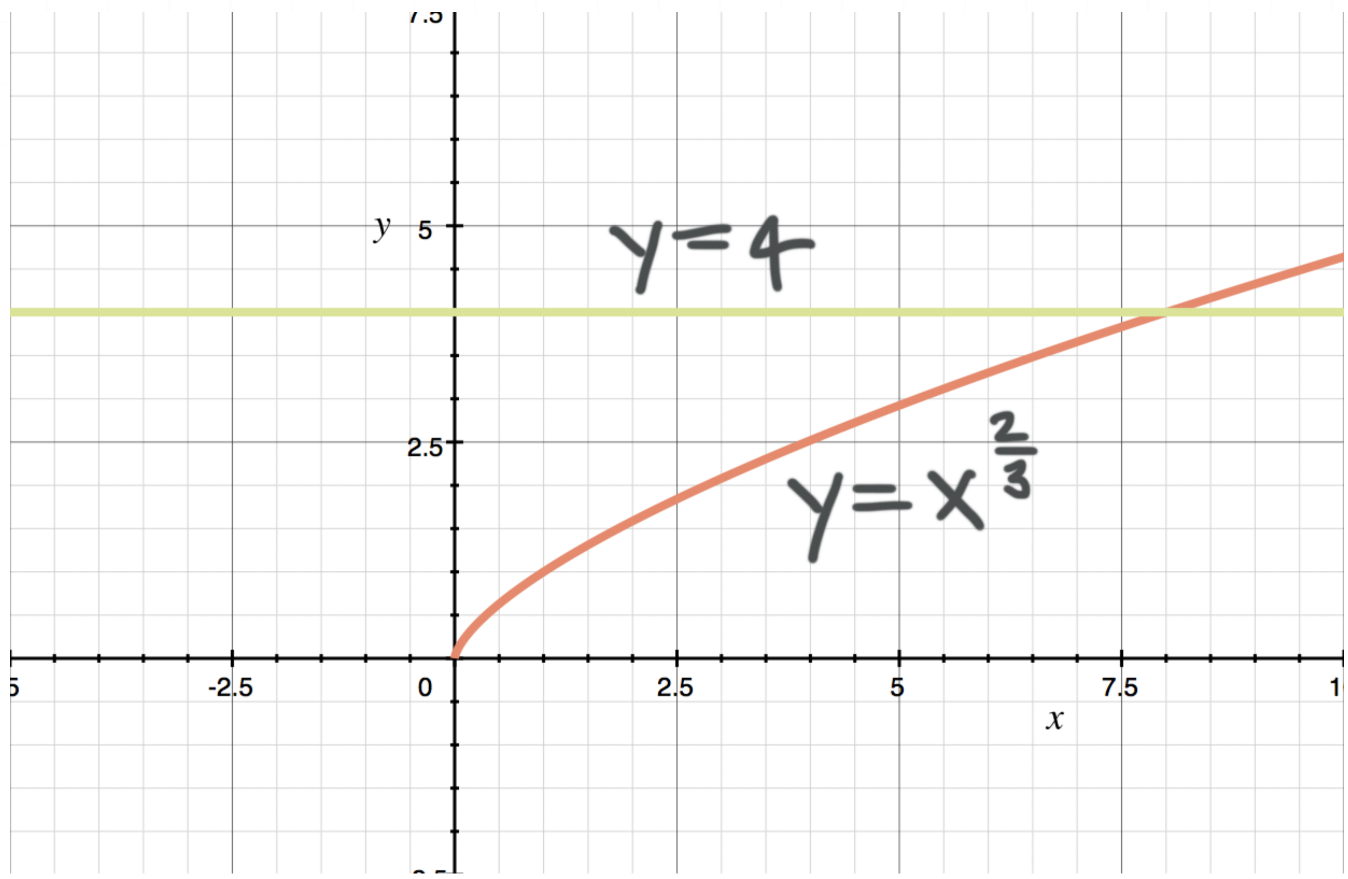
1. Use washers to find the volume of the solid that's formed by rotating the region enclosed by the curves about the x-axis.

$$y = x^{\frac{2}{3}} \text{ and } y = 4$$

$$x = 0 \text{ and } x = 8$$

Solution:

A sketch of the region is



The volume given by washers is

$$\int_a^b \pi \left[f(x)\right]^2 - \pi \left[g(x)\right]^2 \ dx$$

$$\int_0^8 \pi \left[4\right]^2 - \pi \left[x^{\frac{2}{3}}\right]^2 \ dx$$

$$\pi \int_0^8 16 - x^{\frac{4}{3}} \ dx$$

Integrate, then evaluate over the interval.

$$\pi \left(16x - \frac{3}{7}x^{\frac{7}{3}}\right) \Big|_0^8$$

$$\pi \left(16(8) - \frac{3}{7}(8)^{\frac{7}{3}}\right) - \pi \left(16(0) - \frac{3}{7}(0)^{\frac{7}{3}}\right)$$

$$\pi \left(128 - \frac{3}{7}(2)^7\right)$$

$$\frac{896\pi}{7} - \frac{384\pi}{7}$$

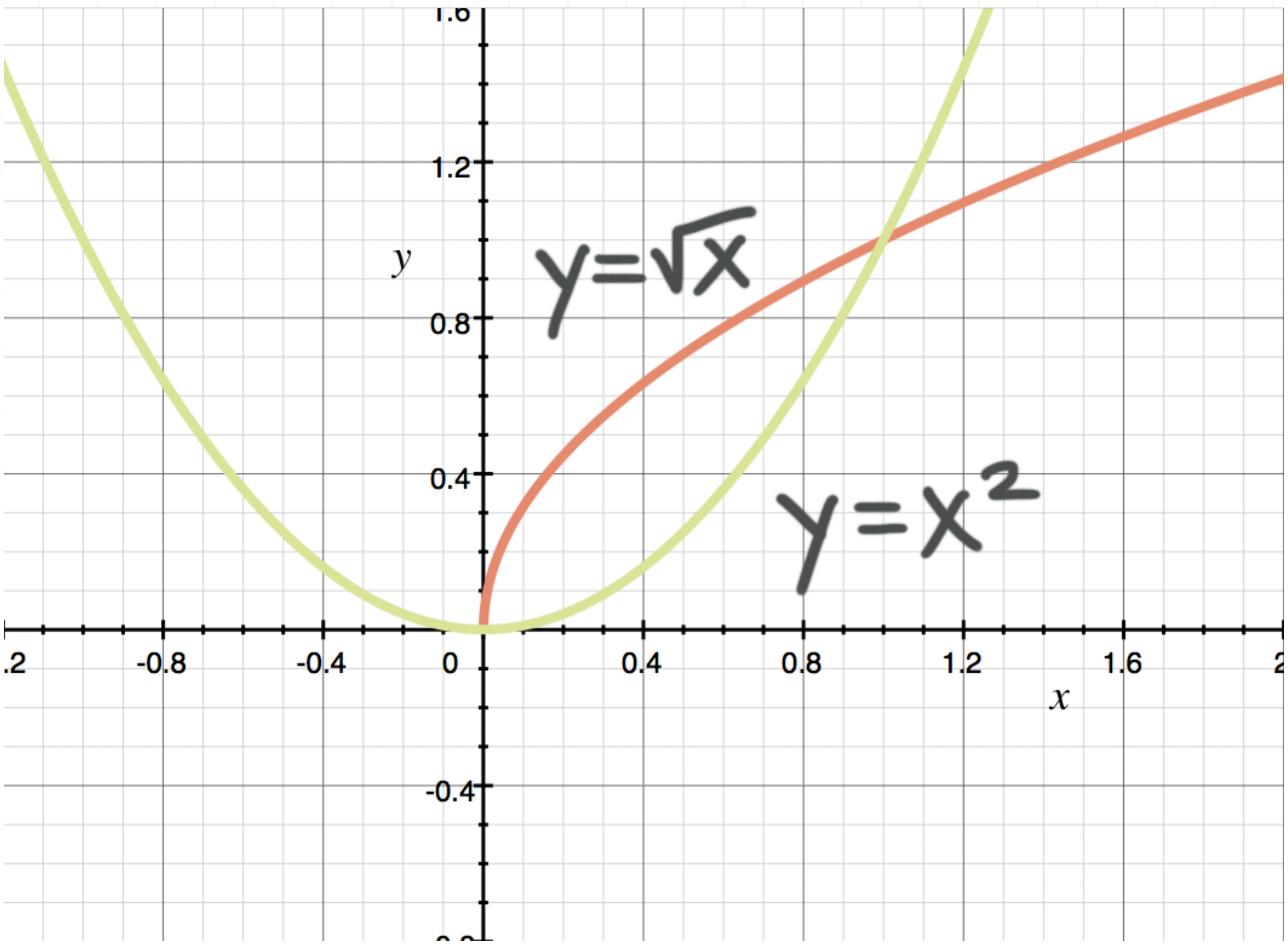
$$\frac{512\pi}{7}$$

■ 2. Use washers to find the volume of the solid that's formed by rotating the region enclosed by the curves about the x-axis.

$y = x^2$ and $y = \sqrt{x}$

Solution:

A sketch of the region is



The volume given by washers is

$$\int_a^b \pi \left[f(x)\right]^2 - \pi \left[g(x)\right]^2 \ dx$$

$$\int_0^1 \pi \left[\sqrt{x}\right]^2 - \pi \left[x^2\right]^2 \ dx$$

$$\pi \int_0^1 x - x^4 \ dx$$

Integrate, then evaluate over the interval.

$$\pi\left(\frac{1}{2}x^2-\frac{1}{5}x^5\right)\Big|_0^1$$

$$\pi\left(\frac{1}{2}(1)^2-\frac{1}{5}(1)^5\right)-\pi\left(\frac{1}{2}(0)^2-\frac{1}{5}(0)^5\right)$$

$$\pi\left(\frac{1}{2}-\frac{1}{5}\right)$$

$$\pi\left(\frac{5}{10}-\frac{2}{10}\right)$$

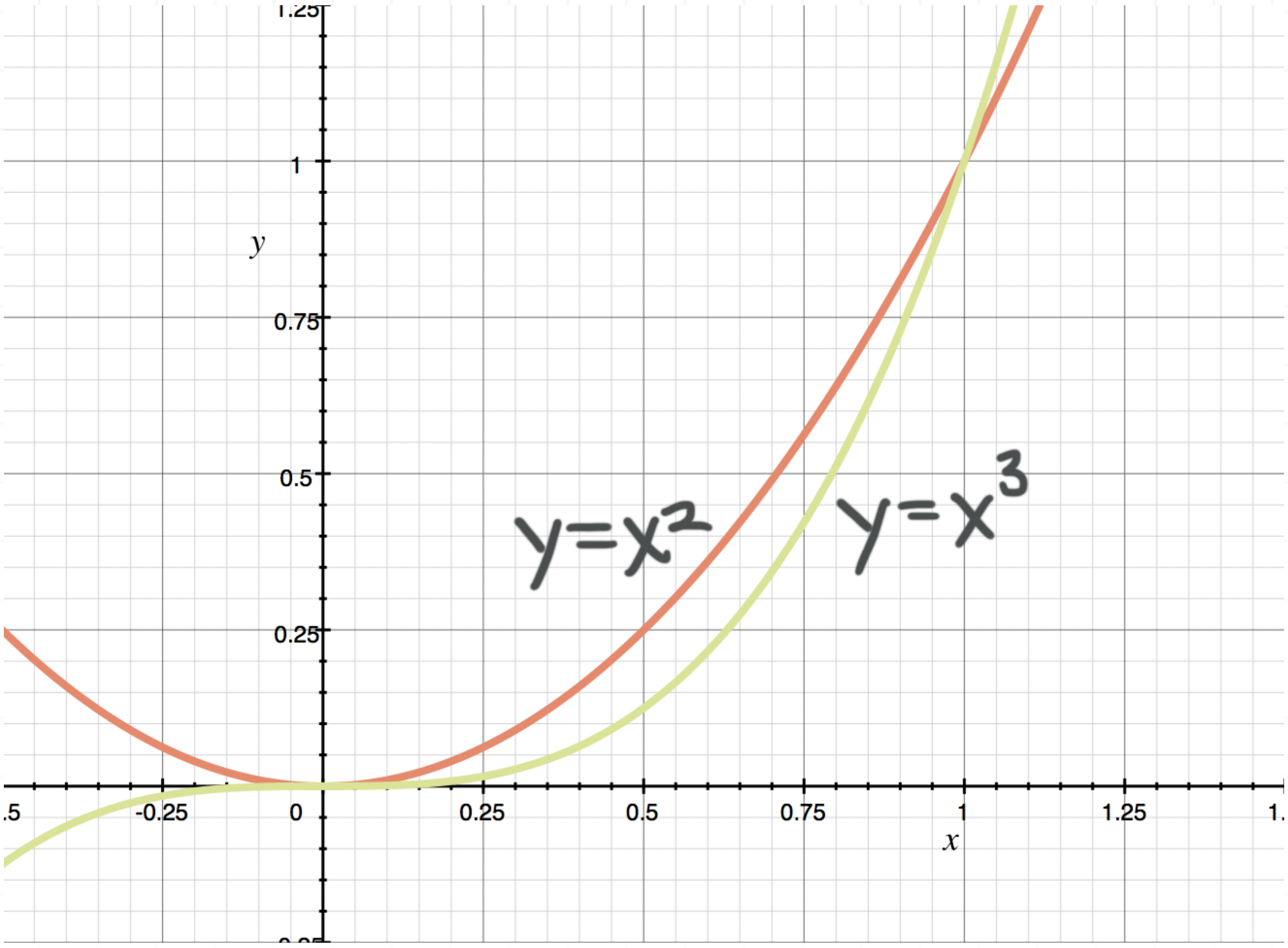
$$\frac{3\pi}{10}$$

■ 3. Use washers to find the volume of the solid that's formed by rotating the region enclosed by the curves about the x-axis.

$y=x^2$ and $y=x^3$

Solution:

A sketch of the region is



The volume given by washers is

$$\int_a^b \pi\left[f(x)\right]^2 - \pi\left[g(x)\right]^2\ dx$$

$$\int_0^1 \pi\left[x^2\right]^2 - \pi\left[x^3\right]^2\ dx$$

$$\pi\int_0^1 x^4 - x^6\ dx$$

Integrate, then evaluate over the interval.

$$\pi\left(\frac{1}{5}x^5 - \frac{1}{7}x^7\right)\Big|_0^1$$

$$\pi\left(\frac{1}{5}(1)^5-\frac{1}{7}(1)^7\right)-\pi\left(\frac{1}{5}(0)^5-\frac{1}{7}(0)^7\right)$$

$$\pi\left(\frac{1}{5}-\frac{1}{7}\right)$$

$$\pi\left(\frac{7}{35}-\frac{5}{35}\right)$$

$$\frac{2\pi}{35}$$

WASHERS, VERTICAL AXIS

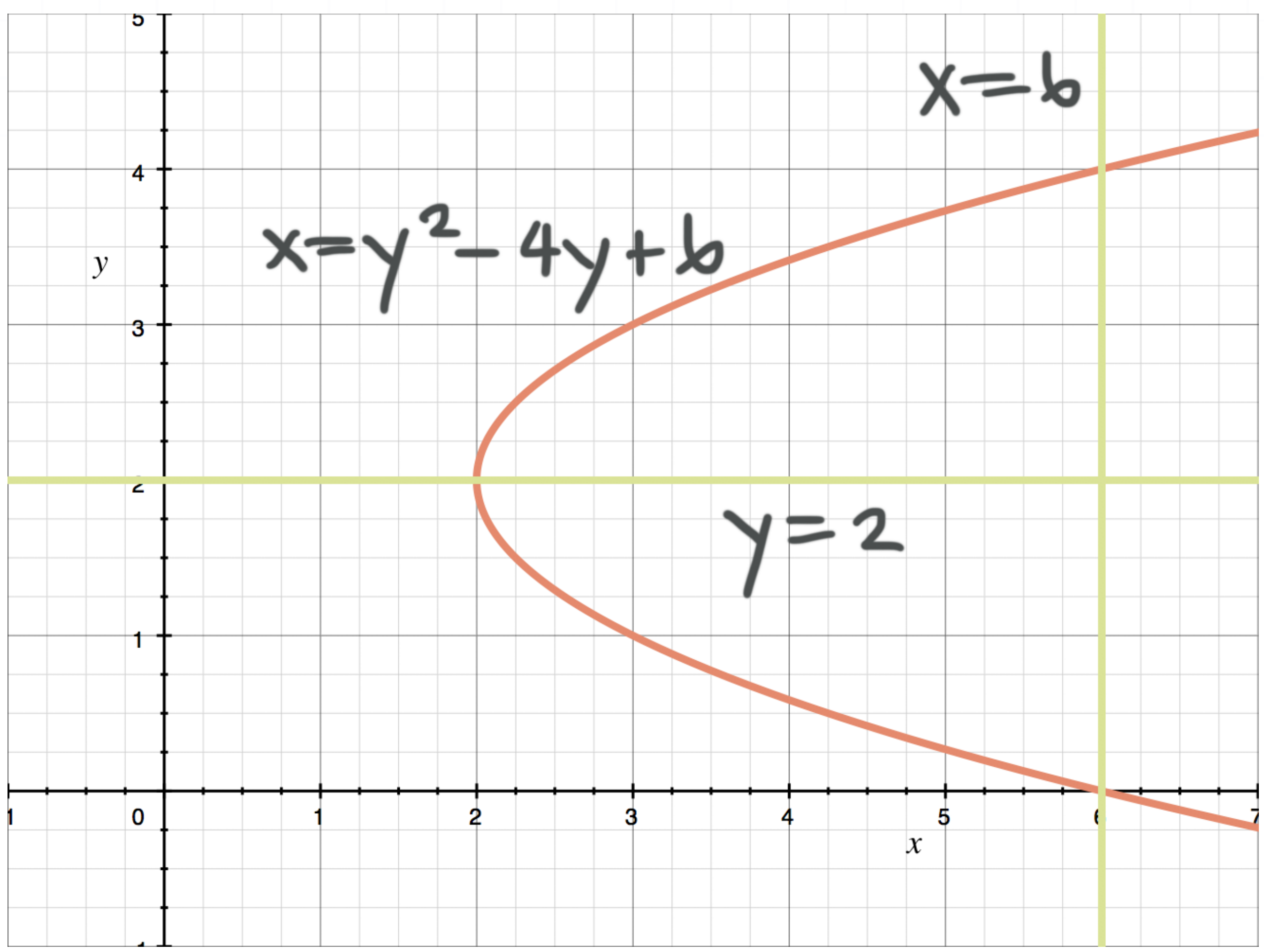
■ 1. Use washers to find the volume of the solid that's formed by rotating the region enclosed by the curves about the y-axis.

$x = y^2 - 4y + 6$ and $x = 6$

$y = 2$ and $y = 4$

Solution:

A sketch of the region is



The volume given by washers is

$$\int_c^d \pi \left[f(y)\right]^2 - \pi \left[g(y)\right]^2 \ dy$$

$$\int_2^4 \pi \left[6\right]^2 - \pi \left[y^2 - 4y + 6\right]^2 \ dy$$

$$\int_2^4 36\pi - \pi(y^4 - 4y^3 + 6y^2 - 4y^3 + 16y^2 - 24y + 6y^2 - 24y + 36) \ dy$$

$$\pi \int_2^4 36 - y^4 + 8y^3 - 28y^2 + 48y - 36 \ dy$$

$$-\pi \int_2^4 y^4 - 8y^3 + 28y^2 - 48y \ dy$$

Integrate, then evaluate over the interval.

$$-\pi \left(\frac{1}{5}y^5 - 2y^4 + \frac{28}{3}y^3 - 24y^2 \right) \Bigg|_2^4$$

$$-\pi \left(\frac{1}{5}(4)^5 - 2(4)^4 + \frac{28}{3}(4)^3 - 24(4)^2 \right) + \pi \left(\frac{1}{5}(2)^5 - 2(2)^4 + \frac{28}{3}(2)^3 - 24(2)^2 \right)$$

$$-\pi \left(\frac{1}{5}(1{,}024) - 2(256) + \frac{28}{3}(64) - 24(16) \right) + \pi \left(\frac{1}{5}(32) - 2(16) + \frac{28}{3}(8) - 24(4) \right)$$

$$-\pi \left(\frac{1{,}024}{5} - 512 + \frac{1{,}792}{3} - 384 \right) + \pi \left(\frac{32}{5} - 32 + \frac{224}{3} - 96 \right)$$

$$-\frac{1{,}024\pi}{5} - \frac{1{,}792\pi}{3} + 896\pi + \frac{32\pi}{5} + \frac{224\pi}{3} - 128\pi$$

$$-\frac{992\pi}{5}-\frac{1{,}568\pi}{3}+768\pi$$

$$-\frac{2{,}976\pi}{15}-\frac{7{,}840\pi}{15}+\frac{11{,}520\pi}{15}$$

$$\frac{704\pi}{15}$$

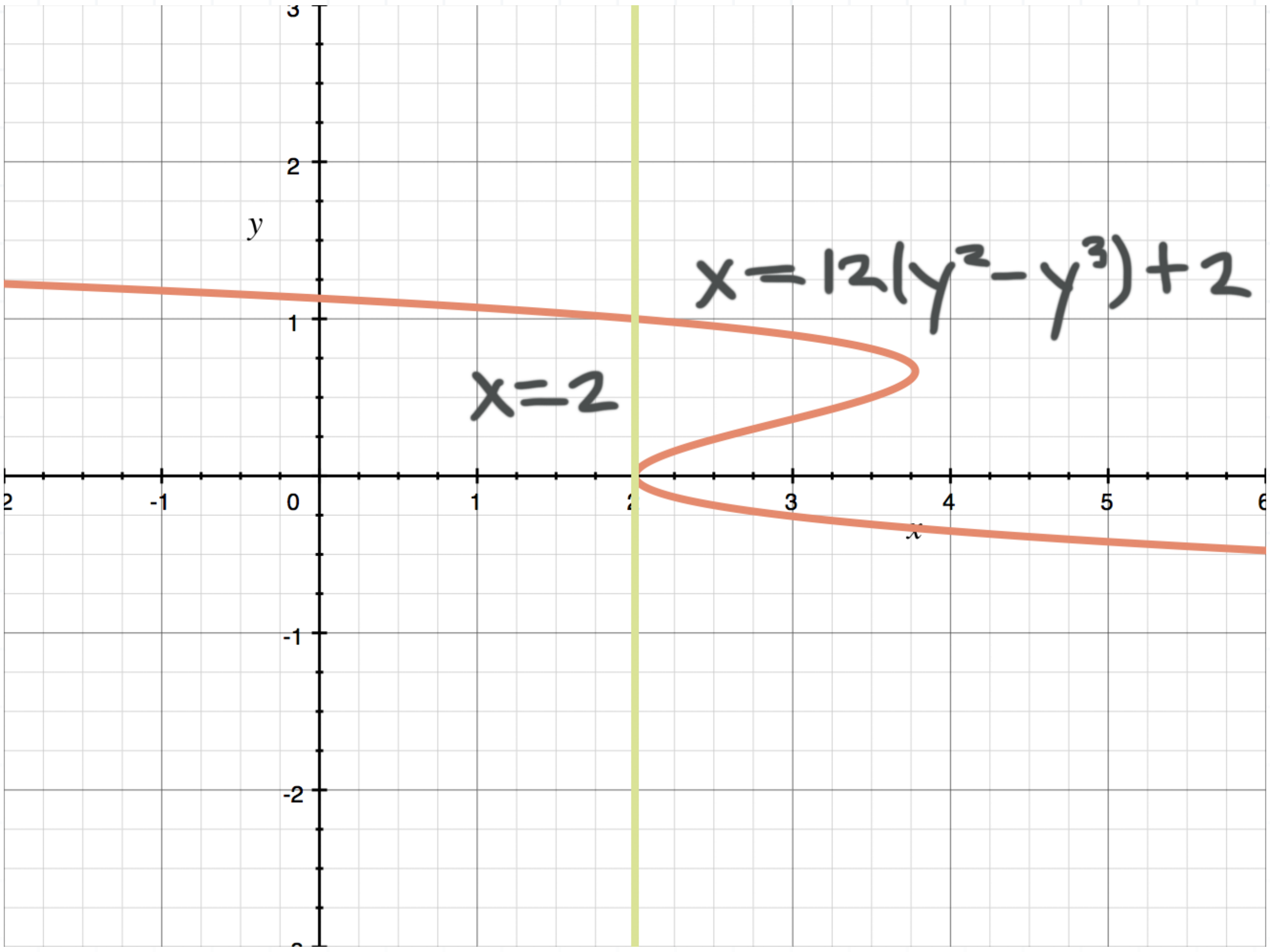
■ 2. Use washers to find the volume of the solid that's formed by rotating the region enclosed by the curves about the y-axis.

$x=12(y^2-y^3)+2$ and $x=2$

$y=0$ and $y=1$

Solution:

A sketch of the region is



The volume given by washers is

$$\int_c^d \pi\left[f(y)\right]^2-\pi\left[g(y)\right]^2\ dy$$

$$\int_0^1 \pi\left[12(y^2-y^3)+2\right]^2-\pi\left[2\right]^2\ dy$$

$$\int_0^1 \pi\left[12y^2-12y^3+2\right]^2-4\pi\ dy$$

$$\int_0^1 \pi(144y^4-144y^5+24y^2-144y^5+144y^6-24y^3+24y^2-24y^3+4)-4\pi\ dy$$

$$48\pi\int_0^1 3y^6-6y^5+3y^4-y^3+y^2\ dy$$

Integrate, then evaluate over the interval.

$$48\pi\left(\frac{3}{7}y^7 - y^6 + \frac{3}{5}y^5 - \frac{1}{4}y^4 + \frac{1}{3}y^3\right)\Big|_0^1$$

$$48\pi\left(\frac{3}{7}(1)^7 - (1)^6 + \frac{3}{5}(1)^5 - \frac{1}{4}(1)^4 + \frac{1}{3}(1)^3\right)$$

$$-48\pi\left(\frac{3}{7}(0)^7 - (0)^6 + \frac{3}{5}(0)^5 - \frac{1}{4}(0)^4 + \frac{1}{3}(0)^3\right)$$

$$48\pi\left(\frac{3}{7} - 1 + \frac{3}{5} - \frac{1}{4} + \frac{1}{3}\right)$$

$$48\pi\left(\frac{180}{420} - \frac{420}{420} + \frac{252}{420} - \frac{105}{420} + \frac{140}{420}\right)$$

$$4\pi\left(\frac{47}{35}\right)$$

$$\frac{188\pi}{35}$$

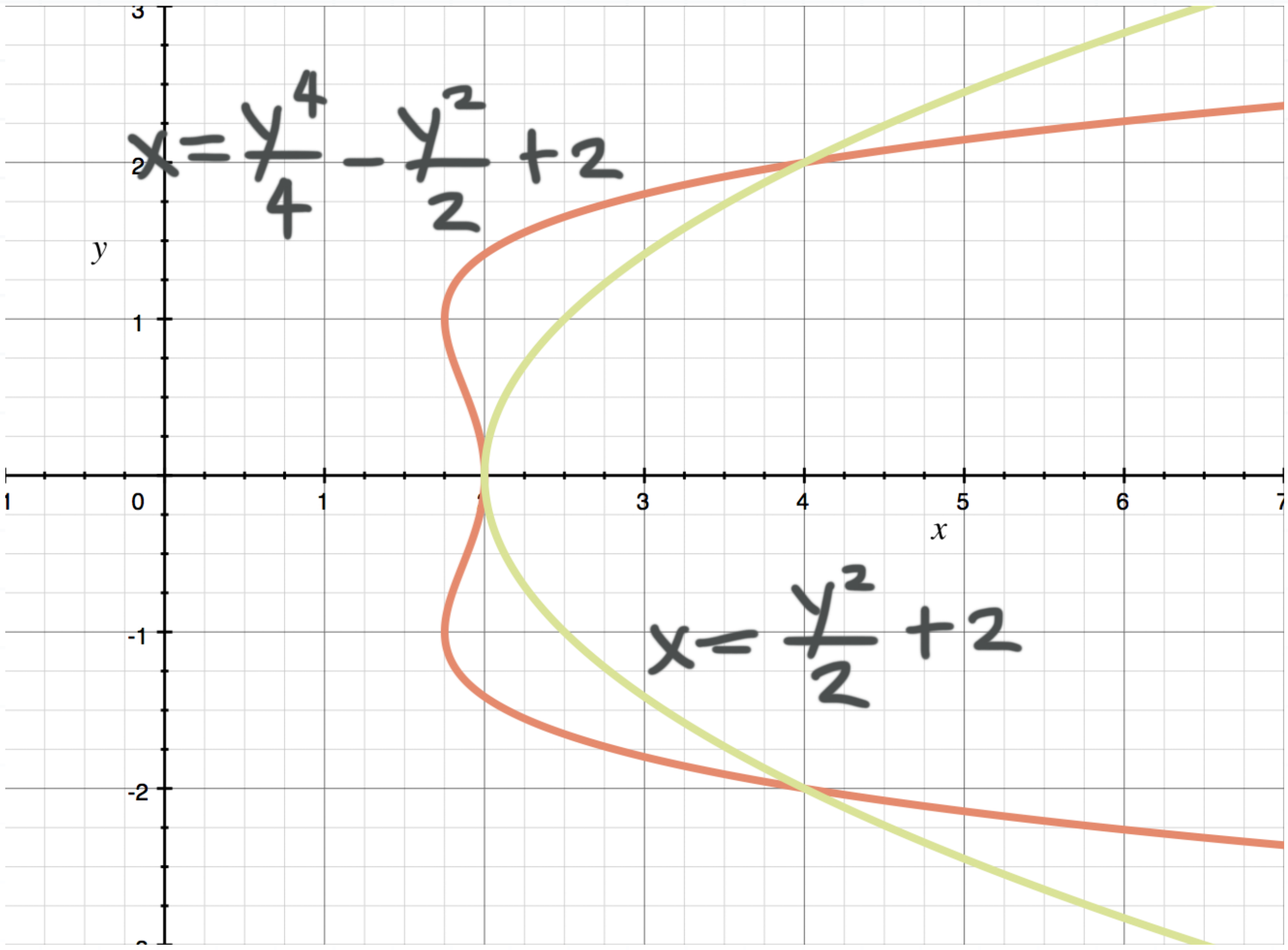
■ 3. Use washers to find the volume of the solid that's formed by rotating the region enclosed by the curves about the y-axis.

$x = \frac{y^4}{4} - \frac{y^2}{2} + 2$ and $x = \frac{y^2}{2} + 2$

$y = -2$ and $y = 2$

Solution:

A sketch of the region is



The volume given by washers is

$$\int_c^d \pi\left[f(y)\right]^2-\pi\left[g(y)\right]^2\ dy$$

$$\int_{-2}^{2}\pi\left[\frac{y^2}{2}+2\right]^2-\pi\left[\frac{y^4}{4}-\frac{y^2}{2}+2\right]^2\ dy$$

$$\pi\int_{-2}^{2}\frac{y^4}{4}+2y^2+4-\left(\frac{y^8}{16}-\frac{y^6}{8}+\frac{y^4}{2}-\frac{y^6}{8}+\frac{y^4}{4}-y^2+\frac{y^4}{2}-y^2+4\right)\ dy$$

$$\pi \int_{-2}^{2} \frac{y^4}{4} + 2y^2 + 4 - \frac{y^8}{16} + \frac{y^6}{8} - \frac{y^4}{2} + \frac{y^6}{8} - \frac{y^4}{4} + y^2 - \frac{y^4}{2} + y^2 - 4 \ dy$$

$$\pi \int_{-2}^{2} -\frac{y^8}{16} + \frac{y^6}{4} - y^4 + 4y^2 \ dy$$

$$-\pi \int_{-2}^{2} \frac{y^8}{16} - \frac{y^6}{4} + y^4 - 4y^2 \ dy$$

Integrate, then evaluate over the interval.

$$-\pi \left(\frac{y^9}{144} - \frac{y^7}{28} + \frac{y^5}{5} - \frac{4y^3}{3} \right) \Bigg|_{-2}^{2}$$

$$-\pi \left(\frac{(2)^9}{144} - \frac{(2)^7}{28} + \frac{(2)^5}{5} - \frac{4(2)^3}{3} \right) + \pi \left(\frac{(-2)^9}{144} - \frac{(-2)^7}{28} + \frac{(-2)^5}{5} - \frac{4(-2)^3}{3} \right)$$

$$\pi \left(-\frac{512}{144} + \frac{128}{28} - \frac{32}{5} + \frac{32}{3} - \frac{512}{144} + \frac{128}{28} - \frac{32}{5} + \frac{32}{3} \right)$$

$$\pi \left(-\frac{64}{9} + \frac{64}{7} - \frac{64}{5} + \frac{64}{3} \right)$$

$$\pi \left(-\frac{6{,}720}{945} + \frac{8{,}640}{945} - \frac{12{,}096}{945} + \frac{20{,}160}{945} \right)$$

$$\frac{9{,}984\pi}{945}$$

$$\frac{3{,}328\pi}{315}$$

CYLINDRICAL SHELLS, HORIZONTAL AXIS

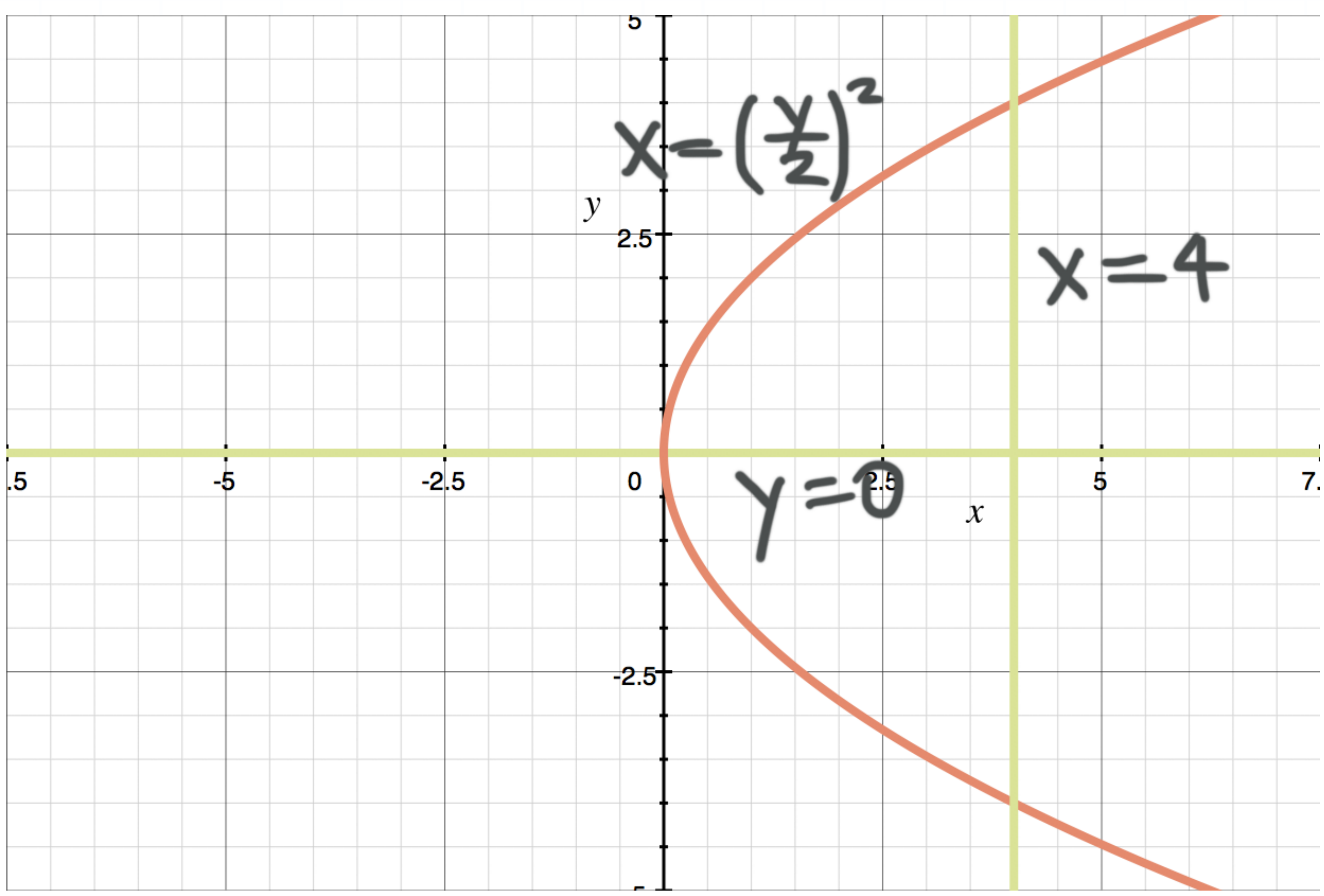
1. Use cylindrical shells to find the volume of the solid that's formed by rotating the region enclosed by the curves about the x-axis.

$$x = \left(\frac{y}{2}\right)^2 \text{ and } x = 4$$

$$y = 0$$

Solution:

A sketch of the region is



The volume given by cylindrical shells is

$$\int_c^d 2\pi y \left[f(y) - g(y) \right] \ dy$$

$$\int_0^4 2\pi y \left[4 - \left(\frac{y}{2} \right)^2 \right] \ dy$$

$$2\pi \int_0^4 y \left(4 - \frac{y^2}{4} \right) \ dy$$

$$2\pi \int_0^4 4y - \frac{y^3}{4} \ dy$$

Integrate, then evaluate over the interval.

$$2\pi \left(2y^2 - \frac{y^4}{16} \right) \Bigg|_0^4$$

$$2\pi \left(2(4)^2 - \frac{4^4}{16} \right) - 2\pi \left(2(0)^2 - \frac{0^4}{16} \right)$$

$$2\pi(32 - 16)$$

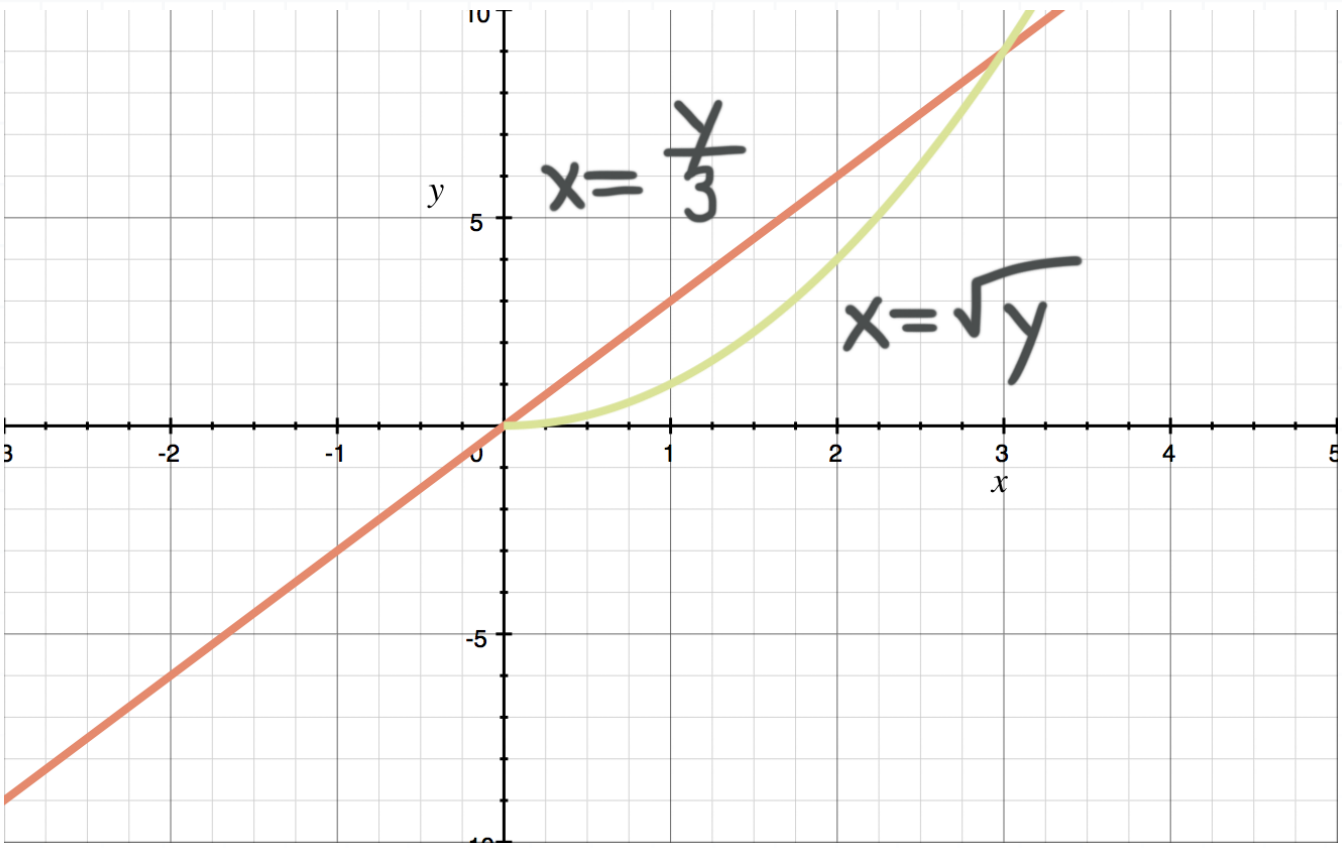
$$32\pi$$

■ 2. Use cylindrical shells to find the volume of the solid that's formed by rotating the region enclosed by the curves about the x-axis.

$$x = \frac{y}{3} \text{ and } x = \sqrt{y}$$

Solution:

A sketch of the region is



The volume given by cylindrical shells is

$$\int_c^d 2\pi y \left[f(y) - g(y) \right] \ dy$$

$$\int_0^9 2\pi y \left[\sqrt{y} - \frac{y}{3} \right] \ dy$$

$$2\pi \int_0^9 y^{\frac{3}{2}} - \frac{y^2}{3} \ dy$$

Integrate, then evaluate over the interval.

$$2\pi\left(\frac{2}{5}y^{\frac{5}{2}}-\frac{y^3}{9}\right)\Bigg|_0^9$$

$$2\pi\left(\frac{2}{5}(9)^{\frac{5}{2}}-\frac{9^3}{9}\right)-2\pi\left(\frac{2}{5}(0)^{\frac{5}{2}}-\frac{0^3}{9}\right)$$

$$2\pi\left(\frac{486}{5}-81\right)$$

$$2\pi\left(\frac{486}{5}-\frac{405}{5}\right)$$

$$\frac{162\pi}{5}$$

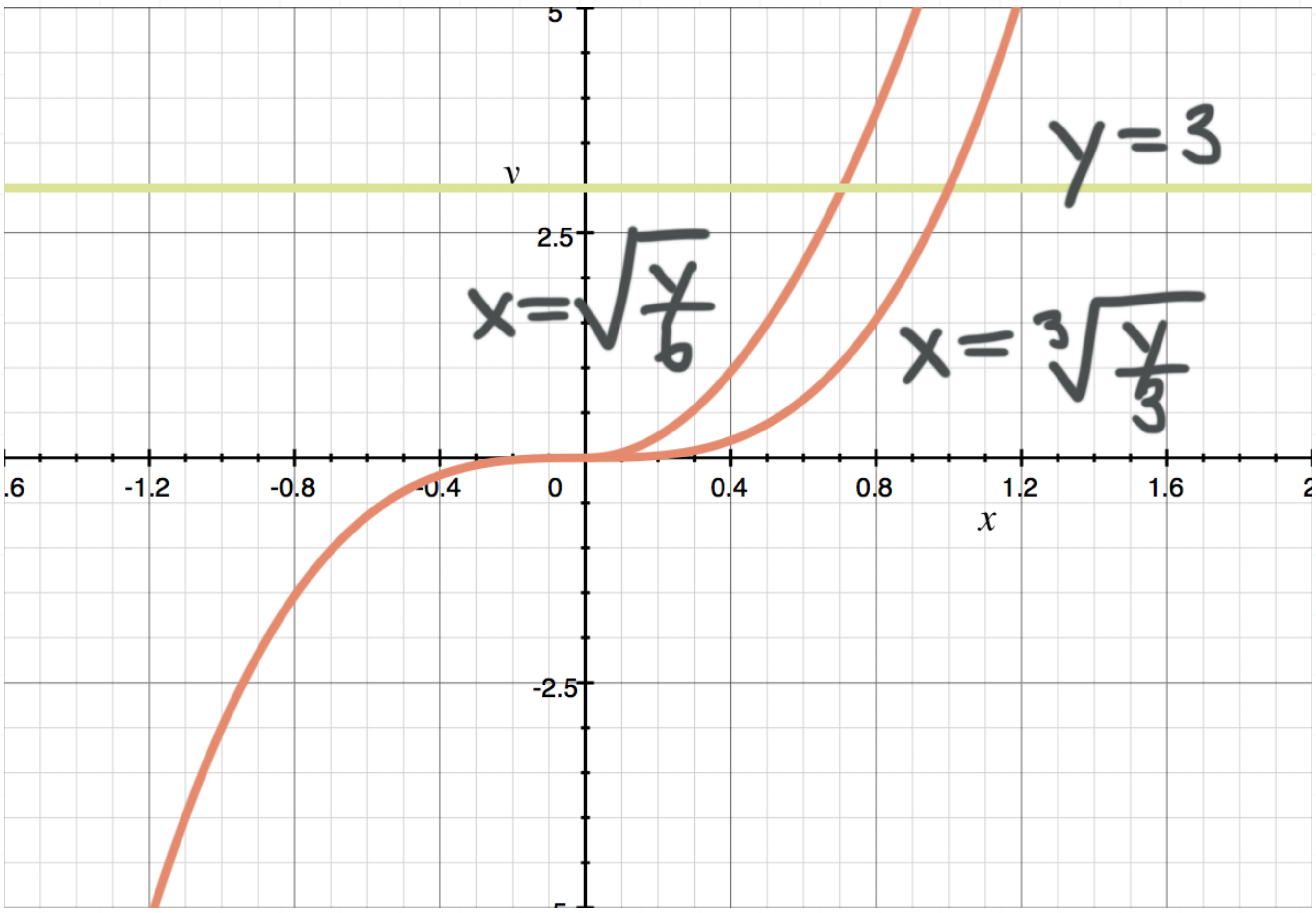
■ 3. Use cylindrical shells to find the volume of the solid that's formed by rotating the region enclosed by the curves about the x-axis.

$x=\sqrt[3]{\frac{y}{3}}$ and $x=\sqrt{\frac{y}{6}}$

$y=3$

Solution:

A sketch of the region is



The volume given by cylindrical shells is

$$\int_c^d 2\pi y\left[f(y)-g(y)\right]\ dy$$

$$\int_0^3 2\pi y\left(\sqrt[3]{\frac{y}{3}}-\sqrt{\frac{y}{6}}\right)\ dy$$

$$2\pi\int_0^3 \frac{\sqrt[3]{9}}{3}y^{\frac{4}{3}}-\frac{\sqrt{6}}{6}y^{\frac{3}{2}}\ dy$$

Integrate, then evaluate over the interval.

$$2\pi\left(\frac{\sqrt[3]{9}}{7}y^{\frac{7}{3}}-\frac{\sqrt{6}}{15}y^{\frac{5}{2}}\right)\Big|_0^3$$

$$2\pi\left(\frac{\sqrt[3]{9}}{7}(3)^{\frac{7}{3}}-\frac{\sqrt{6}}{15}(3)^{\frac{5}{2}}\right)-2\pi\left(\frac{\sqrt[3]{9}}{7}(0)^{\frac{7}{3}}-\frac{\sqrt{6}}{15}(0)^{\frac{5}{2}}\right)$$

$$2\pi\left(\frac{\sqrt[3]{9}}{7}(3)^{\frac{7}{3}}-\frac{\sqrt{6}}{15}(3)^{\frac{5}{2}}\right)$$

$$2\pi(1.3115584)$$

$$8.241$$

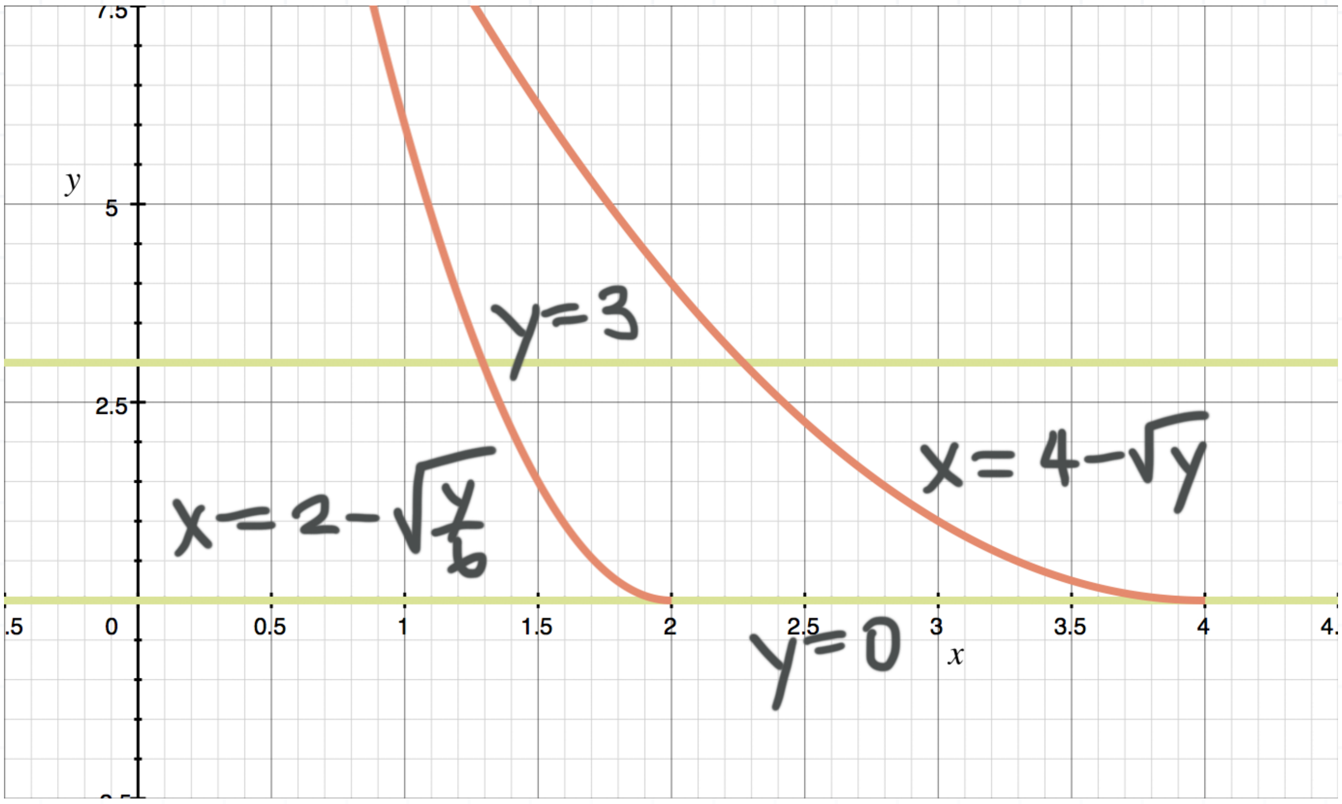
■ 4. Use cylindrical shells to find the volume of the solid that's formed by rotating the region enclosed by the curves about the x-axis.

$x=4-\sqrt{y}$ and $x=2-\sqrt{\frac{y}{6}}$

$y=0$ and $y=3$

Solution:

A sketch of the region is



The volume given by cylindrical shells is

$$\int_c^d 2\pi y\left[f(y)-g(y)\right]\ dy$$

$$\int_0^3 2\pi y\left[\left(4-\sqrt{y}\right)-\left(2-\sqrt{\frac{y}{6}}\right)\right]\ dy$$

$$2\pi\int_0^3 y\left(2-\sqrt{y}+\sqrt{\frac{y}{6}}\right)\ dy$$

$$2\pi\int_0^3 2y-y^{\frac{3}{2}}+y\frac{\sqrt{y}}{\sqrt{6}}\ dy$$

$$2\pi\int_0^3 2y-y^{\frac{3}{2}}+\frac{\sqrt{6}}{6}y^{\frac{3}{2}}\ dy$$

Integrate, then evaluate over the interval.

$$2\pi\left(y^2-\frac{2}{5}y^{\frac{5}{2}}+\frac{\sqrt{6}}{15}y^{\frac{5}{2}}\right)\Bigg|_0^3$$

$$2\pi\left(3^2-\frac{2}{5}(3)^{\frac{5}{2}}+\frac{\sqrt{6}}{15}(3)^{\frac{5}{2}}\right)-2\pi\left(0^2-\frac{2}{5}(0)^{\frac{5}{2}}+\frac{\sqrt{6}}{15}(0)^{\frac{5}{2}}\right)$$

$$2\pi\left(9-\frac{18\sqrt{3}}{5}+\frac{27\sqrt{2}}{15}\right)$$

$$18\pi\left(1-\frac{2\sqrt{3}}{5}+\frac{3\sqrt{2}}{15}\right)$$

$$2\pi(5.31020)$$

$$33.365$$

WORK DONE TO LIFE A WEIGHT OR MASS

■ 1. Find the work required to lift a 50-pound load from ground level up into a tree house that's 60 feet above the ground, if the chain being used to lift the weight itself weighs 1 pound per foot.

Solution:

The work required to lift the load is

$$50 \text{ lbs} \cdot 60 \text{ ft} = 3{,}000 \text{ ft-lbs}$$

The work required to lift the chain is

$$\int_0^{60} x\ dx = \frac{x^2}{2}\Big|_0^{60} = \frac{60^2}{2} - \frac{0^2}{2} = 1{,}800 \text{ ft-lbs}$$

So the total work required to lift the load is

$$3{,}000 + 1{,}800 = 4{,}800 \text{ ft-lbs}$$

■ 2. Find the work required to lift a 40-pound box of roofing nails from ground level up onto a roof that's 35 feet above the ground, if the rope being used to lift the weight itself weighs 2 ounces per foot.

Solution:

The work required to lift the box is

$$40 \text{ lbs} \cdot 35 \text{ ft} = 1{,}400 \text{ ft-lbs}$$

2 ounces is equivalent to $1/8$ pounds, which means the work required to lift the rope is

$$\int_0^{35} \frac{1}{8}x\ dx = \frac{x^2}{16}\Big|_0^{35} = \frac{35^2}{16} - \frac{0^2}{16} = 76.5625 \text{ ft-lbs}$$

So the total work required to lift the load is

$$1{,}400 + 76.5625 = 1{,}476.5625 \approx 1{,}477 \text{ ft-lbs}$$

■ 3. Find the work required to lift a $5{,}500$-pound load of concrete from ground level up onto a construction platform that's 75 feet above the ground, if the cable being used to lift the weight itself weighs 8 pounds per foot.

Solution:

The work required to lift the box is

$$5{,}500 \text{ lbs} \cdot 75 \text{ ft} = 412{,}500 \text{ ft-lbs}$$

The work required to lift the cable is

$$\int_0^{75} 8x \ dx = 4x^2 \Big|_0^{75} = 4(75)^2 - 4(0)^2 = 22{,}500 \text{ ft-lbs}$$

So the total work required to lift the load is

$$412{,}500 + 22{,}500 = 435{,}000 \text{ ft-lbs}$$

■ 4. Find the work required to lift a 5-gallon bucket of water, with each gallon of water weighing 6.75 pounds and the bucket weighing 2 pounds, from ground level up onto a scaffold that's 14 feet above the ground, if the rope being used to lift the weight itself weighs 8 ounces per foot.

Solution:

The weight of the bucket of water is

$$2 + 5 \cdot 6.75 = 35.75 \text{ lbs}$$

The work required to lift the box is

$$35.75 \text{ lbs} \cdot 14 \text{ ft} = 500.5 \text{ ft-lbs}$$

8 ounces is $8/16$, or $1/2$ pounds, which means the work required to lift the rope is

$$\int_0^{14} \frac{1}{2}x \ dx = \frac{x^2}{4}\Big|_0^{14} = \frac{14^2}{4} - \frac{0^2}{4} = 49 \text{ ft-lbs}$$

So the total work required to lift the load is

$$500.5 + 49 = 549.5 \text{ ft-lbs}$$

■ 5. Find the work required to lift a $7{,}200$-pound load of rocks from ground level up into a dump truck that's 13 feet above the ground, if the chain being used to lift the weight itself weighs 12 pounds per foot.

Solution:

The work required to lift the load of concrete is

$$7{,}200 \text{ lbs} \cdot 13 \text{ ft} = 93{,}600 \text{ ft-lbs}$$

The work required to lift the chain is

$$\int_0^{13} 12x\ dx = 6x^2\Big|_0^{13} = 6(13)^2 - 6(0)^2 = 1{,}014 \text{ ft-lbs}$$

So the total work required to lift the load is

$$93{,}600 + 1{,}014 = 94{,}614 \text{ ft-lbs}$$

WORK DONE ON ELASTIC SPRINGS

■ 1. Find the work required to stretch a spring 3 feet beyond its normal length, if a force of $5s$ lbs is required to stretch the spring s feet beyond its normal length.

Solution:

The work needed to stretch a spring a feet is

$$W = \int_0^a F(s)\ ds$$

Using Hooke's Law, $F(s) = ks$ where k is a constant, and s is the distance, we have $ks = 5s$, so $k = 5$ and $F(s) = 5s$. So the work to stretch the spring 3 feet beyond its normal length is

$$W = \int_0^3 5s\ ds = \frac{5s^2}{2}\Bigg|_0^3 = \frac{5(3)^2}{2} - \frac{5(0)^2}{2} = \frac{45}{2} = 22.5 \text{ ft-lbs}$$

■ 2. Find the work required to stretch a spring 7 inches beyond its normal length, if a force of $9s$ lbs is required to stretch the spring s inches beyond its normal length.

Solution:

The work needed to stretch a spring a inches is

$$W = \int_0^a F(s)\ ds$$

Using Hooke's Law, $F(s) = ks$ where k is a constant, and s is the distance, we have $ks = 9s$, so $k = 9$ and $F(s) = 9s$. So the work to stretch the spring 7 inches beyond its normal length is

$$W = \int_0^7 9s\ ds = \frac{9s^2}{2}\Big|_0^7 = \frac{9(7)^2}{2} - \frac{9(0)^2}{2} = \frac{441}{2} = 220.5 \text{ in-lbs}$$

3. Find the work required to stretch a spring 6 feet beyond its normal length, if a force of $15s$ lbs is required to stretch the spring s feet beyond its normal length.

Solution:

The work needed to stretch a spring a feet is

$$W = \int_0^a F(s)\ ds$$

Using Hooke's Law, $F(s) = ks$ where k is a constant, and s is the distance, we have $ks = 15s$, so $k = 15$ and $F(s) = 15s$. So the work to stretch the spring 6 feet beyond its normal length is

$$W = \int_0^6 15s\ ds = \frac{15s^2}{2}\Big|_0^6 = \frac{15(6)^2}{2} - \frac{15(0)^2}{2} = \frac{540}{2} = 270 \text{ ft-lbs}$$

■ 4. Find the work required to stretch a spring 1 foot beyond its normal length, if a force of $3.5s$ lbs is required to stretch the spring s feet beyond its normal length.

Solution:

The work needed to stretch a spring a feet is

$$W = \int_0^a F(s)\ ds$$

Using Hooke's Law, $F(s) = ks$ where k is a constant, and s is the distance, we have $ks = 0.5s$, so $k = 3.5$ and $F(s) = 3.5s$. So the work to stretch the spring 1 foot beyond its normal length is

$$W = \int_0^1 3.5s\ ds = \frac{3.5s^2}{2}\Big|_0^1 = \frac{3.5(1)^2}{2} - \frac{3.5(0)^2}{2} = \frac{3.5}{2} = 1.75 \text{ ft-lbs}$$

■ 5. Find the work required, in foot pounds, to stretch a spring 58 inches beyond its normal length, if a force of $4s$ lbs is required to stretch the spring s feet beyond its normal length.

Solution:

The work needed to stretch a spring a feet is

$$W=\int_0^a F(s)\ ds$$

Using Hooke's Law, $F(s)=ks$ where k is a constant, and s is the distance, we have $ks=4s$, so $k=4$ and $F(s)=4s$. But 58 inches is equivalent to $58/12$, or $29/6$ feet. So the work to stretch the spring 58 inches beyond its normal length is

$$W=\int_0^{\frac{29}{6}} 4s\ ds=\frac{4s^2}{2}\Big|_0^{\frac{29}{6}}=2s^2\Big|_0^{\frac{29}{6}}=2\left(\frac{29}{6}\right)^2-2(0)^2=\frac{841}{18}\text{ ft-lbs}$$

WORK DONE TO EMPTY A TANK

1. Find the work required to empty a tank that is 6 feet wide, 8 feet tall, 12 feet long, and completely full. The tank will be emptied by pumping the liquid in the tank through a hose to a height of 2 feet above the top of the tank. The liquid in the tank has a density of 58.9 lbs/ft^3.

Solution:

The volume of a slice of the liquid is

$$6 \cdot 12 \cdot dy \text{ ft}^3$$

$$72\ dy \text{ ft}^3$$

The force needed to pump a slice of the liquid, which is weight times volume, is

$$58.9 \cdot 72\ dy \text{ ft}^3$$

The distance the liquid will be pumped is $10 - y$ feet. The liquid will be pumped from an original height of 0 to 8 feet. So the work required is

$$W = \int_0^8 (10 - y)(58.9 \cdot 72)\ dy$$

$$W = 4{,}240.8 \int_0^8 10 - y\ dy$$

$$W = 4{,}240.8\left(10y - \frac{y^2}{2}\right)\Big|_0^8$$

$$W = 4{,}240.8\left[\left(10(8) - \frac{8^2}{2}\right) - \left(10(0) - \frac{0^2}{2}\right)\right]$$

$$W = 4{,}240.8(48 - 0)$$

$$W = 203{,}558.4 \text{ ft-lbs}$$

■ 2. Find the work required to empty an in-ground swimming pool that is 20 feet wide, 4 feet deep, 18 feet long, and completely full. The pool will be emptied by pumping the water in the pool through a hose over the top of the pool. The water in the pool has a density of 62.43 lbs/ft^3.

Solution:

The volume of a slice of the water is

$$20 \cdot 18 \cdot dy \text{ ft}^3$$

$$360\ dy \text{ ft}^3$$

The force needed to pump a slice of the water, which is weight times volume, is

$$62.43 \cdot 360\ dy \text{ ft}^3$$

The distance the water will be pumped is $4-y$ feet. The water will be pumped from an original height of 0 to 4 feet. So the work required is

$$W=\int_0^4(4-y)(62.43\cdot 360)\ dy$$

$$W=22{,}474.8\int_0^4 4-y\ dy$$

$$W=22{,}474.8\left(4y-\frac{y^2}{2}\right)\Big|_0^4$$

$$W=22{,}474.8\left[\left(4(4)-\frac{4^2}{2}\right)-\left(4(0)-\frac{0^2}{2}\right)\right]$$

$$W=22{,}474.8(8-0)$$

$$W=179{,}798.4 \text{ ft-lbs}$$

■ 3. Find the work required to empty a cylindrical tank that is 12 feet tall, has a radius of 6 feet, and is half full of diesel fuel. The tank will be emptied by pumping the fuel in the tank through a hose to a height of 6 feet above the top of the tank. The diesel fuel in the tank has a density of 53.5 lbs/ft^3.

Solution:

The volume of a slice of the fuel is

$$\pi \cdot 6^2 \cdot dy \text{ ft}^3$$

$$36\pi \ dy \text{ ft}^3$$

The force needed to pump a slice of the fuel, which is weight times volume, is

$$53.5 \cdot 36\pi \ dy \text{ ft}^3$$

The distance the fuel will be pumped is $18 - y$ feet. The fuel will be pumped from an original height of 0 to 6 feet. So the work required is

$$W = \int_0^6 (18 - y)(53.5 \cdot 36\pi) \ dy$$

$$W = 1{,}926\pi \int_0^6 18 - y \ dy$$

$$W = 1{,}926\pi \left(18y - \frac{y^2}{2} \right) \Bigg|_0^6$$

$$W = 1{,}926\pi \left[\left(18(6) - \frac{6^2}{2} \right) - \left(18(0) - \frac{0^2}{2} \right) \right]$$

$$W = 1{,}926\pi(90 - 0)$$

$$W = 173{,}340\pi \text{ ft-lbs}$$

■ 4. Find the work required to empty an above-ground child's pool that is 2 feet tall, has a diameter of 8 feet, and is three-fourths full. The pool will be emptied by pumping the water in the pool through a hose over the top of the pool. The water in the pool has a density of 62.4 lbs/ft^3.

Solution:

The volume of a slice of the water is

$$\pi \cdot 4^2 \cdot dy \text{ ft}^3$$

$$16\pi \ dy \text{ ft}^3$$

The force needed to pump a slice of the water, which is weight times volume, is

$$62.4 \cdot 16\pi \ dy \text{ ft}^3$$

The distance the water will be pumped is $2 - y$ feet. The water will be pumped from an original height of 0 to 1.5 feet. So the work required is

$$W = \int_0^{1.5} (2 - y)(62.4 \cdot 16\pi) \ dy$$

$$W = 998.4\pi \int_0^{1.5} 2 - y \ dy$$

$$W = 998.4\pi \left(2y - \frac{y^2}{2}\right)\Bigg|_0^{1.5}$$

$$W = 998.4\pi \left[\left(2(1.5) - \frac{1.5^2}{2}\right) - \left(2(0) - \frac{0^2}{2}\right)\right]$$

$$W = 998.4\pi(1.875 - 0)$$

$$W = 5{,}881.061448 \text{ ft-lbs}$$

■ 5. Find the work required to empty a cylindrical tank that is 8 feet tall, has a radius of 9 feet, and is three-fourths full of gasoline. The tank will be emptied by pumping the gas in the tank through a hose into a truck that's 8 feet above the top of the tank. The gasoline in the tank has a density of 54.5 lbs/ft^3.

Solution:

The volume of a slice of the water is

$$\pi \cdot 9^2 \cdot dy \text{ ft}^3$$

$$81\pi\ dy \text{ ft}^3$$

The force needed to pump a slice of the gasoline, which is weight times volume, is

$$54.5 \cdot 81\pi\ dy \text{ ft}^3$$

The distance the gas will be pumped is $16 - y$ feet. The gas will be pumped from an original height of 0 to 6 feet. So the work required is

$$W = \int_0^6 (16 - y)(54.5 \cdot 81\pi)\ dy$$

$$W = 4{,}414.5\pi \int_0^6 16 - y \ dy$$

$$W = 4{,}414.5\pi \left(16y - \frac{y^2}{2} \right) \Big|_0^6$$

$$W = 4{,}414.5\pi \left[\left(16(6) - \frac{6^2}{2} \right) - \left(16(0) - \frac{0^2}{2} \right) \right]$$

$$W = 4{,}414.5\pi(78 - 0)$$

$W = 344{,}331\pi$ ft-lbs

WORK DONE BY A VARIABLE FORCE

■ 1. Calculate the variable force on the interval $[0,2]$.

$$F(x) = 3x^2 + 2x$$

Solution:

Plugging the force equation and the interval into the integral formula for work done by a variable force, we get

$$W = \int_a^b F(x)\ dx$$

$$W = \int_0^2 3x^2 + 2x\ dx$$

$$W = x^3 + x^2 \Big|_0^2$$

$$W = 2^3 + 2^2 - (0^3 + 0^2)$$

$$W = 8 + 4$$

$$W = 12$$

■ 2. Calculate the variable force on the interval $[0,\pi/2]$.

$$F(x) = 3\sin(2x) + x$$

Solution:

Plugging the force equation and the interval into the integral formula for work done by a variable force, we get

$$W = \int_a^b F(x)\ dx$$

$$W = \int_0^{\frac{\pi}{2}} 3\sin(2x) + x\ dx$$

$$W = -\frac{3\cos(2x)}{2} + \frac{x^2}{2}\Big|_0^{\frac{\pi}{2}}$$

$$W = -\frac{3\cos\left(2 \cdot \frac{\pi}{2}\right)}{2} + \frac{\left(\frac{\pi}{2}\right)^2}{2} - \left(-\frac{3\cos(2(0))}{2} + \frac{(0)^2}{2}\right)$$

$$W = -\frac{3\cos\pi}{2} + \frac{\frac{\pi^2}{4}}{2} + \frac{3\cos 0}{2}$$

$$W = -\frac{3(-1)}{2} + \frac{\pi^2}{8} + \frac{3(1)}{2}$$

$$W = 3 + \frac{\pi^2}{8}$$

■ 3. Calculate the variable force on the interval $[1,6]$.

$$F(x) = x^2 + x + 1$$

Solution:

Plugging the force equation and the interval into the integral formula for work done by a variable force, we get

$$W = \int_a^b F(x)\ dx$$

$$W = \int_1^6 x^2 + x + 1\ dx$$

$$W = \frac{x^3}{3} + \frac{x^2}{2} + x \Big|_1^6$$

$$W = \frac{6^3}{3} + \frac{6^2}{2} + 6 - \left(\frac{1^3}{3} + \frac{1^2}{2} + 1\right)$$

$$W = \frac{216}{3} + \frac{36}{2} + 6 - \frac{1}{3} - \frac{1}{2} - 1$$

$$W = \frac{432}{6} + \frac{108}{6} + \frac{36}{6} - \frac{2}{6} - \frac{3}{6} - \frac{6}{6}$$

$$W = \frac{565}{6}$$

4. Calculate the variable force on the interval $[0,\pi/3]$.

$$F(x) = 2\tan^2 x$$

Solution:

Plugging the force equation and the interval into the integral formula for work done by a variable force, we get

$$W = \int_a^b F(x)\ dx$$

$$W = \int_0^{\frac{\pi}{3}} 2\tan^2 x\ dx$$

$$W = 2\int_0^{\frac{\pi}{3}} \sec^2 x - 1\ dx$$

$$W = 2(\tan x - x)\Big|_0^{\frac{\pi}{3}}$$

$$W = 2\left(\tan\frac{\pi}{3} - \frac{\pi}{3}\right) - 2(\tan 0 - 0)$$

$$W = 2\left(\sqrt{3} - \frac{\pi}{3}\right) - 2(0 - 0)$$

$$W = 2\sqrt{3} - \frac{2\pi}{3}$$

5. Calculate the variable force on the interval $[1.2,3.5]$.

$$F(x) = 4(x-2)^3 - 2(x-2) + 1$$

Solution:

Plugging the force equation and the interval into the integral formula for work done by a variable force, we get

$$W = \int_a^b F(x)\ dx$$

$$W = \int_{1.2}^{3.5} 4(x-2)^3 - 2(x-2) + 1\ dx$$

$$W = (x-2)^4 - (x-2)^2 + x \Big|_{1.2}^{3.5}$$

$$W = (3.5-2)^4 - (3.5-2)^2 + 3.5 - \left((1.2-2)^4 - (1.2-2)^2 + 1.2\right)$$

$$W = 1.5^4 - 1.5^2 + 3.5 - 0.8^4 + 0.8^2 - 1.2$$

$$W = 5.0625 - 2.25 + 3.5 - 0.4096 + 0.64 - 1.2$$

$$W = 5.0625 - 2.25 + 3.5 - 0.4096 + 0.64 - 1.2$$

$$W = 5.3429$$

MOMENTS OF THE SYSTEM

■ 1. Calculate the moments of the system.

$$m_1 = 3;\ P_1(2,5)$$

$$m_2 = 4;\ P_2(-2,6)$$

$$m_3 = 6;\ P_3(4,-5)$$

Solution:

If we plug the given points and masses into the formulas for the moments of a system, we get

$$M_y = m_1(x_1) + m_2(x_2) + m_3(x_3)$$

$$M_y = 3(2) + 4(-2) + 6(4) = 22$$

and

$$M_y = m_1(x_1) + m_2(x_2) + m_3(x_3)$$

$$M_x = 3(5) + 4(6) + 6(-5) = 9$$

■ 2. Calculate the moments of the system.

$$m_1 = 7;\ P_1(5,2)$$

$$m_2 = 3; \ P_2(-4,3)$$

$$m_3 = 5; \ P_3(-3,4)$$

Solution:

If we plug the given points and masses into the formulas for the moments of a system, we get

$$M_y = m_1(x_1) + m_2(x_2) + m_3(x_3)$$

$$M_y = 7(5) + 3(-4) + 5(-3) = 8$$

and

$$M_x = m_1(y_1) + m_2(y_2) + m_3(y_3)$$

$$M_x = 7(2) + 3(3) + 5(4) = 43$$

■ 3. Calculate the moments of the system.

$$m_1 = 9; \ P_1(7,5)$$

$$m_2 = -5; \ P_2(3,8)$$

$$m_3 = 4; \ P_3(5,4)$$

Solution:

If we plug the given points and masses into the formulas for the moments of a system, we get

$$M_y = m_1(x_1) + m_2(x_2) + m_3(x_3)$$

$$M_y = 9(7) + (-5)(3) + 4(5) = 68$$

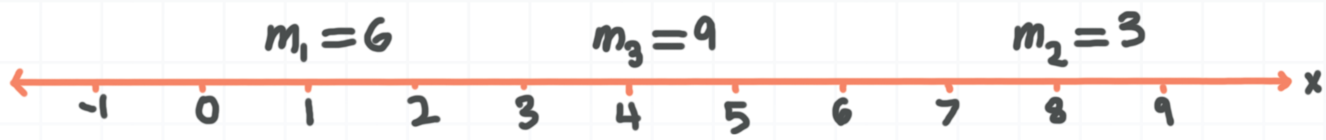
and

$$M_x = m_1(y_1) + m_2(y_2) + m_3(y_3)$$

$$M_x = 9(5) + (-5)(8) + 4(4) = 21$$

MOMENTS OF THE SYSTEM, X-AXIS

1. Calculate the moments of the system.



Solution:

The moments of the system are

$$M_y = m_1(x_1) + m_2(x_2) + m_3(x_3)$$

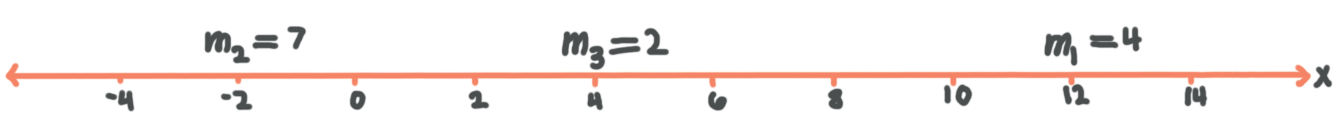
$$M_y = 6(1) + 3(8) + 9(4) = 66$$

and

$$M_x = m_1(y_1) + m_2(y_2) + m_3(y_3)$$

$$M_x = 6(0) + 3(0) + 9(0) = 0$$

2. Calculate the moments of the system.



Solution:

The moments of the system are

$$M_y = m_1(x_1) + m_2(x_2) + m_3(x_3)$$

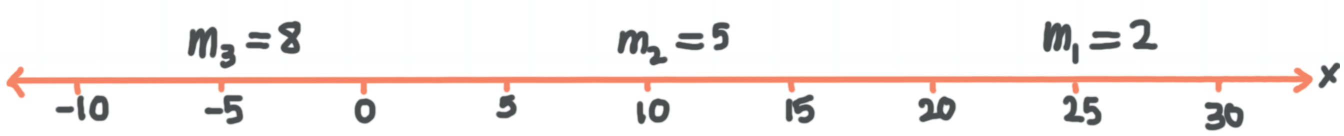
$$M_y = 4(12) + 7(-2) + 2(4) = 42$$

and

$$M_x = m_1(y_1) + m_2(y_2) + m_3(y_3)$$

$$M_x = 4(0) + 7(0) + 2(0) = 0$$

■ 3. Calculate the moments of the system.



Solution:

The moments of the system are

$$M_y = m_1(x_1) + m_2(x_2) + m_3(x_3)$$

$$M_y = 2(25) + 5(10) + 8(-5) = 60$$

and

$$M_x = m_1(y_1) + m_2(y_2) + m_3(y_3)$$

$$M_x = 2(0) + 5(0) + 8(0) = 0$$

CENTER OF MASS OF THE SYSTEM

■ 1. Find the center of mass of the system if $M_y = 16$ and $M_x = 22$ and the total mass is $m_T = 14$.

Solution:

The center of mass is the point

$$(\bar{x}, \bar{y}) = \left(\frac{M_y}{m_T}, \frac{M_x}{m_T}\right) = \left(\frac{16}{14}, \frac{22}{14}\right) = \left(\frac{8}{7}, \frac{11}{7}\right)$$

■ 2. Find the center of mass of the system if $M_y = 32.5$ and $M_x = 28.5$ and the total mass is $m_T = 7.5$.

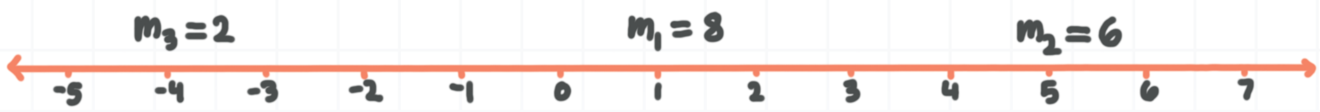
Solution:

The center of mass is the point

$$(\bar{x}, \bar{y}) = \left(\frac{M_y}{m_T}, \frac{M_x}{m_T}\right) = \left(\frac{32.5}{7.5}, \frac{28.5}{7.5}\right) = \left(\frac{13}{3}, \frac{19}{5}\right)$$

CENTER OF MASS OF THE SYSTEM, X-AXIS

■ 1. Find the center of mass of the system.



Solution:

The moments of the system are

$$M_y = m_1(x_1) + m_2(x_2) + m_3(x_3)$$

$$M_y = 8(1) + 6(5) + 2(-4) = 30$$

and

$$M_x = m_1(y_1) + m_2(y_2) + m_3(y_3)$$

$$M_x = 8(0) + 6(0) + 2(0) = 0$$

The total mass in the system is

$$m_T = m_1 + m_2 + m_3$$

$$m_T = 8 + 6 + 2 = 16$$

So the center of mass of the system is

$$(\bar{x}, \bar{y}) = \left(\frac{M_y}{m_T}, \frac{M_x}{m_T}\right) = \left(\frac{30}{16}, \frac{0}{16}\right) = \left(\frac{15}{8}, 0\right)$$

■ 2. Find the center of mass of the system.

Solution:

The moments of the system are

$$M_y = m_1(x_1) + m_2(x_2) + m_3(x_3)$$

$$M_y = 3(6) + 8(-6) + 5(10) = 20$$

and

$$M_x = m_1(y_1) + m_2(y_2) + m_3(y_3)$$

$$M_x = 3(0) + 8(0) + 5(0) = 0$$

The total mass in the system is

$$m_T = m_1 + m_2 + m_3$$

$$m_T = 3 + 8 + 5 = 16$$

So the center of mass of the system is

$$(\bar{x}, \bar{y}) = \left(\frac{M_y}{m_T}, \frac{M_x}{m_T}\right) = \left(\frac{20}{16}, \frac{0}{16}\right) = \left(\frac{5}{4}, 0\right)$$

■ 3. Find the center of mass of the system.

Solution:

The moments of the system are

$$M_y = m_1(x_1) + m_2(x_2) + m_3(x_3)$$

$$M_y = 6(-4) + 5(1) + 7(4) = 9$$

and

$$M_x = m_1(y_1) + m_2(y_2) + m_3(y_3)$$

$$M_x = 6(0) + 5(0) + 7(0) = 0$$

The total mass in the system is

$$m_T = m_1 + m_2 + m_3$$

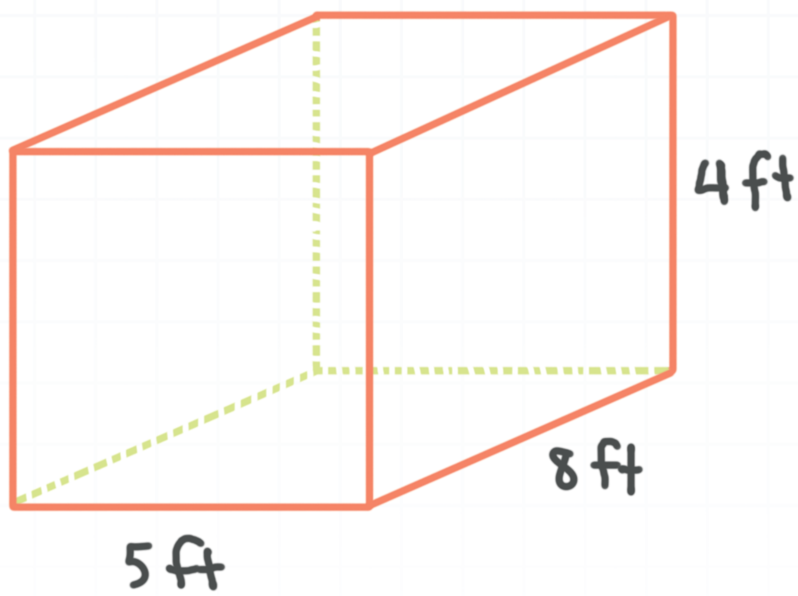
$$m_T = 6 + 5 + 7 = 18$$

So the center of mass of the system is

$$(\bar{x},\bar{y})=\left(\frac{M_y}{m_T},\frac{M_x}{m_T}\right)=\left(\frac{9}{18},\frac{0}{18}\right)=\left(\frac{1}{2},0\right)$$

HYDROSTATIC PRESSURE

1. Find the hydrostatic pressure per square foot on the bottom of the tank, which is filled to the top with gasoline. Assume the weight of a gallon of gasoline is 6.073 pounds per gallon.



Solution:

A gallon of gasoline weighs approximately 6.073 pounds. A cubic foot of the tank holds approximately 7.4805 gallons. So the density of a cubic foot of gasoline is

$$\delta = 6.073 \times 7.4805 = 45.4291$$

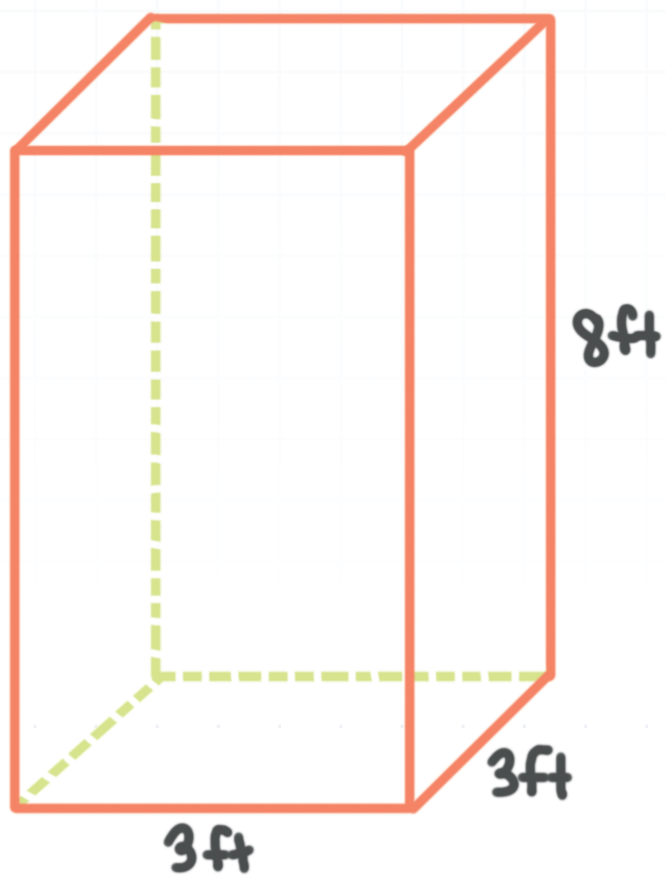
The depth of the gasoline in the tank is 4 feet, and pressure is the product of density and depth, so

$$P = \delta d$$

$$P = 45.4291 \times 4$$

$$P = 181.7164 \text{ lbs/ft}^2$$

2. Find the hydrostatic pressure per square foot on the bottom of the tank, which is filled to the top with water. Assume the weight of a gallon of water is 8.3454 pounds per gallon.



Solution:

A gallon of water weighs approximately 8.3454 pounds. A cubic foot of the tank holds approximately 7.4805 gallons. So the density of a cubic foot of water is

$$\delta = 8.3454 \times 7.4805 = 62.4278$$

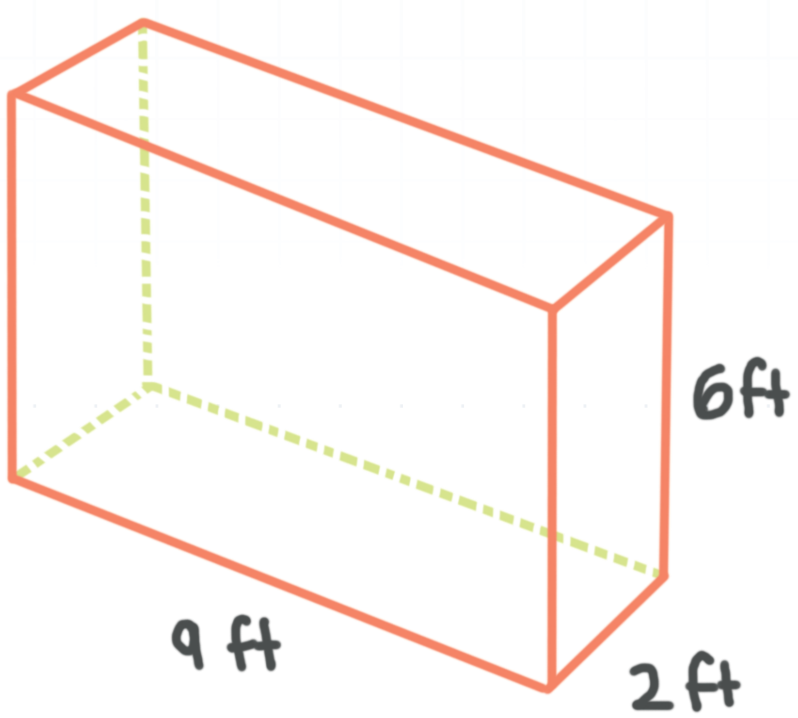
The depth of the water in the tank is 8 feet, and pressure is the product of density and depth, so

$$P = \delta d$$

$$P = 62.4278 \times 8$$

$$P = 499.4224 \text{ lbs/ft}^2$$

■ 3. Find the hydrostatic pressure per square foot on the bottom of the tank, which is filled to the top with diesel fuel. Assume the weight of a gallon of diesel is 7.1089 pounds per gallon.



Solution:

A gallon of diesel fuel weighs approximately 7.1089 pounds. A cubic foot of the tank holds approximately 7.4805 gallons. So the density of a cubic foot of diesel is

$$\delta = 7.1089 \times 7.4805 = 53.1781$$

The depth of the fuel in the tank is 6 feet, and pressure is the product of density and depth, so

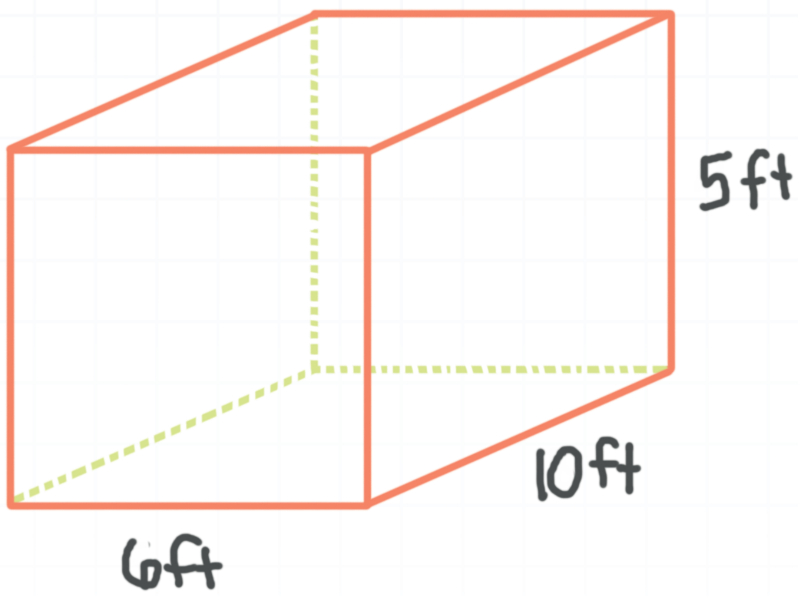
$$P = \delta d$$

$$P = 53.1781 \times 6$$

$$P = 319.0686 \text{ lbs/ft}^2$$

HYDROSTATIC FORCE

1. Find the hydrostatic force on the bottom of the tank, which is filled to the top with gasoline. Assume the weight of a gallon of gasoline is 6.073 pounds per gallon.



Solution:

A gallon of gasoline weighs approximately 6.073 pounds. A cubic foot of the tank holds approximately 7.4805 gallons. So the density of a cubic foot of gasoline is

$$\delta = 6.073 \times 7.4805 = 45.4291$$

The depth of the gasoline in the tank is 5 feet, and pressure is the product of density and depth, so

$$P = \delta d$$

$$P = 45.4291 \times 5$$

$$P = 227.1455 \text{ lbs/ft}^2$$

The area of the bottom of the tank is

$$A = L \cdot W = 6 \cdot 10 = 60$$

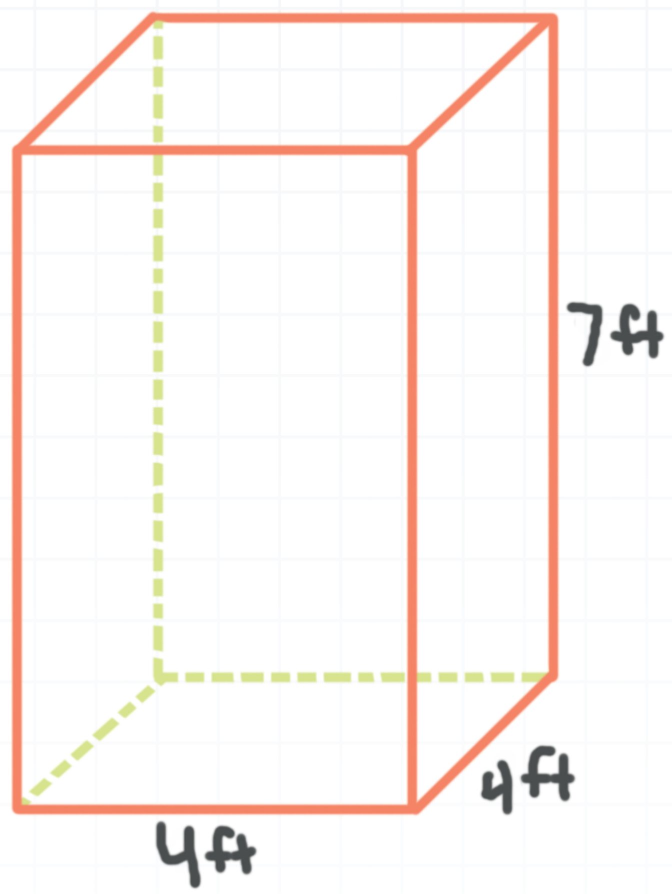
So the force on the bottom of the tank is

$$F = PA$$

$$F = 227.1455 \cdot 60$$

$$F = 13{,}628.73 \text{ pounds}$$

■ 2. Find the hydrostatic force on the bottom of the tank, which is filled to the top with water. Assume the weight of a gallon of water is 8.3454 pounds per gallon.



Solution:

A gallon of water weighs approximately 8.3454 pounds. A cubic foot of the tank holds approximately 7.4805 gallons. So the density of a cubic foot of water is

$$8.3454 \times 7.4805 = 62.4278$$

The depth of the water in the tank is 7 feet, and pressure is the product of density and depth, so

$$P = \delta d$$

$$P = 62.4278 \times 7$$

$$P = 436.9946 \text{ lbs/ft}^2$$

The area of the bottom of the tank is

$$A = L \cdot W = 4 \cdot 4 = 16$$

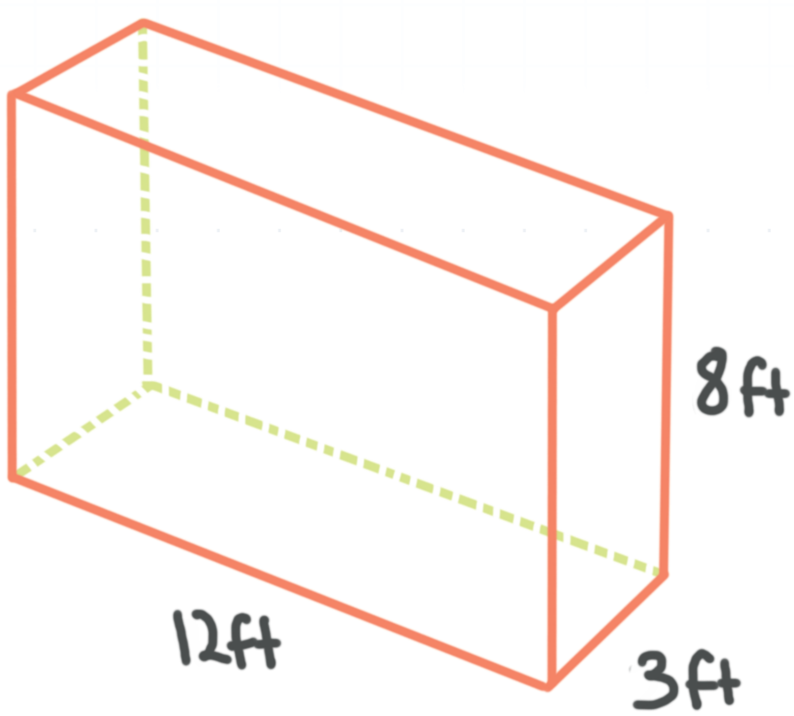
So the force on the bottom of the tank is

$$F = PA$$

$$F = 436.9946 \cdot 16$$

$$F = 6{,}991.9136 \text{ pounds}$$

■ 3. Find the hydrostatic force on the bottom of the tank, which is filled to the top with diesel fuel. Assume the weight of a gallon of diesel is 7.1089 pounds per gallon.



Solution:

A gallon of diesel weighs approximately 7.1089 pounds. A cubic foot of the tank holds approximately 7.4805 gallons. So the density of a cubic foot of water is

$$7.1089 \times 7.4805 = 53.1781$$

The depth of the diesel in the tank is 8 feet, and pressure is the product of density and depth, so

$$P = \delta d$$

$$P = 53.1781 \times 8$$

$$P = 425.4250 \text{ lbs/ft}^2$$

The area of the bottom of the tank is

$$A = L \cdot W = 12 \cdot 3 = 36$$

So the force on the bottom of the tank is

$$F = PA$$

$$F = 425.4250 \cdot 36$$

$$F = 15{,}315.3 \text{ pounds}$$

VERTICAL MOTION

1. What is the maximum height of a baseball that's thrown straight up from a position 6 feet above the ground with an initial velocity of $v(t) = -32t + 88$ ft/sec?

Solution:

The baseball will reach its maximum height when the velocity is 0, so we'll need to find t when $v(t) = 0$.

$$-32t + 88 = 0$$

$$-32t = -88$$

$$t = \frac{-88}{-32}$$

$$t = 2.75$$

The baseball will reach its maximum height at $t = 2.75$ seconds. To find a function for height, integrate velocity.

$$h(t) = \int v(t)\ dt = \int -32t + 88\ dt$$

$$h(t) = -16t^2 + 88t + C$$

The fact that the baseball was thrown from an initial height of 6 feet means we have the initial condition $h(0)=6$. Substitute the initial condition into the height function.

$$6=-16(0)^2+88(0)+C$$

$$C=6$$

So the height function is

$$h(t)=-16t^2+88t+6$$

Then at $t=2.75$ seconds, the height of the baseball is

$$h(2.75)=-16(2.75)^2+88(2.75)+6=127 \text{ feet}$$

■ 2. What is the maximum height of a football that's thrown straight up from 1.67 yards above the ground with an initial velocity of $v(t)=-10.67t+40$ yards/sec?

Solution:

The football will reach its maximum height when the velocity is 0, so we'll need to find t when $v(t)=0$.

$$-10.67t+40=0$$

$$-10.67t=-40$$

$$t=\frac{-40}{-10.67}$$

$$t=3.75$$

The football will reach its maximum height at $t=3.75$ seconds. To find a function for height, integrate velocity.

$$h(t)=\int v(t)\ dt=\int -10.67t+40\ dt$$

$$h(t)=-\frac{10.67t^2}{2}+40t+C$$

The fact that the football was thrown from an initial height of 1.67 yards means we have the initial condition $h(0)=1.67$. Substitute the initial condition into the height function.

$$1.67=-\frac{10.67(0)^2}{2}+40(0)+C$$

$$C=1.67$$

So the height function is

$$h(t)=-\frac{10.67t^2}{2}+40t+1.67$$

Then at $t=3.75$ seconds, the height of the football is

$$h(3.75)=-5.33(3.75)^2+40(3.75)+1.67=76.716875\approx 77 \text{ yards}$$

3. What is the maximum height of a model rocket that's launched straight up from the ground with an initial velocity of $v(t) = -32t + 200$ ft/sec?

Solution:

The rocket will reach its maximum height when the velocity is 0, so we'll need to find t when $v(t) = 0$.

$$-32t + 200 = 0$$

$$-32t = -200$$

$$t = \frac{-200}{-32}$$

$$t = 6.25$$

The rocket will reach its maximum height at $t = 6.25$ seconds. To find a function for height, integrate velocity.

$$h(t) = \int v(t)\ dt = \int -32t + 200\ dt$$

$$h(t) = -16t^2 + 200t + C$$

The fact that the rocket was launched from ground level means we have the initial condition $h(0) = 0$. Substitute the initial condition into the height function.

$$0 = -16(0)^2 + 200(0) + C$$

$$C = 0$$

So the height function is

$$h(t) = -16t^2 + 200t$$

Then at $t = 6.25$ seconds, the height of the rocket is

$$h(6.25) = -16(6.25)^2 + 200(6.25) = 625 \text{ feet}$$

■ 4. What is the maximum height of a bottle rocket that's launched straight up from the ground with an initial velocity of $v(t) = -19.6t + 29.4$ m/sec?

Solution:

The rocket will reach its maximum height when the velocity is 0, so we'll need to find t when $v(t) = 0$.

$$-19.6t + 29.4 = 0$$

$$-19.6t = -29.4$$

$$t = \frac{-29.4}{-19.6}$$

$$t = 1.5$$

The rocket will reach its maximum height at $t = 1.5$ seconds. To find a function for height, integrate velocity.

$$h(t) = \int v(t)\ dt = \int -19.6t + 29.4\ dt$$

$$h(t) = -\frac{19.6t^2}{2} + 29.4t + C$$

The fact that the rocket was launched from ground level means we have the initial condition $h(0) = 0$. Substitute the initial condition into the height function.

$$0 = -\frac{19.6(0)^2}{2} + 29.4(0) + C$$

$$C = 0$$

So the height function is

$$h(t) = -\frac{19.6t^2}{2} + 29.4t$$

Then at $t = 1.5$ seconds, the height of the rocket is

$$h(1.5) = -9.8(1.5)^2 + 29.4(1.5) = 22.05 \approx 22 \text{ meters}$$

■ 5. What is the maximum height of a golf ball that's hit straight up from the ground with an initial velocity of $v(t) = -19.6t + 68.208$ m/sec?

Solution:

The golf ball will reach its maximum height when the velocity is 0, so we'll need to find t when $v(t) = 0$.

$$-19.6t + 68.208 = 0$$

$$-19.6t = -68.208$$

$$t = \frac{-68.208}{-19.6}$$

$$t = 3.48$$

The golf ball will reach its maximum height at $t = 3.48$ seconds. To find a function for height, integrate velocity.

$$h(t) = \int v(t)\ dt = \int -19.6t + 68.208\ dt$$

$$h(t) = -\frac{19.6t^2}{2} + 68.208t + C$$

The fact that the golf ball was hit from ground level means we have the initial condition $h(0) = 0$. Substitute the initial condition into the height function.

$$0 = -\frac{19.6(0)^2}{2} + 68.208(0) + C$$

$$C = 0$$

So the height function is

$$h(t) = -\frac{19.6t^2}{2} + 68.208t$$

Then at $t = 3.48$ seconds, the height of the golf ball is

$$h(3.48) = -9.8(3.48)^2 + 68.208(3.48) = 118.68192 \approx 119 \text{ meters}$$

RECTILINEAR MOTION

■ 1. Find the position function $x(t)$ that models the rectilinear motion of a particle moving along the x-axis.

$$a(t) = 10 - t$$

$$v(0) = -1$$

$$x(0) = 6$$

Solution:

Integrate the acceleration function to find the velocity function.

$$v(t) = \int a(t)\ dt = \int 10 - t\ dt$$

$$v(t) = 10t - \frac{t^2}{2} + C$$

Substitute the initial condition $v(0) = -1$ to find C.

$$-1 = 10(0) - \frac{0^2}{2} + C$$

$$C = -1$$

So the velocity function is

$$v(t) = -\frac{t^2}{2} + 10t - 1$$

Then the position function is the integral of the velocity function.

$$x(t) = \int v(t)\ dt = \int -\frac{t^2}{2} + 10t - 1\ dt$$

$$x(t) = -\frac{t^3}{6} + \frac{10t^2}{2} - t + C$$

$$x(t) = -\frac{t^3}{6} + 5t^2 - t + C$$

Substitute the initial condition $x(0) = 6$ to find C.

$$6 = -\frac{0^3}{6} + 5(0)^2 - 0 + C$$

$$C = 6$$

So the position function is

$$x(t) = -\frac{t^3}{6} + 5t^2 - t + 6$$

■ 2. Find the position function $x(t)$ that models the rectilinear motion of a particle moving along the x-axis.

$$a(t) = 9t^2 - 4t + 6$$

$$v(-1) = 0$$

$$x(0) = 2$$

Solution:

Integrate the acceleration function to find the velocity function.

$$v(t) = \int a(t)\ dt = \int 9t^2 - 4t + 6\ dt$$

$$v(t) = 3t^3 - 2t^2 + 6t + C$$

Substitute the initial condition $v(-1) = 0$ to find C.

$$0 = 3(-1)^3 - 2(-1)^2 + 6(-1) + C$$

$$0 = -3 - 2 - 6 + C$$

$$C = 11$$

So the velocity function is

$$v(t) = 3t^3 - 2t^2 + 6t + 11$$

Then the position function is the integral of the velocity function.

$$x(t) = \int v(t)\ dt = \int 3t^3 - 2t^2 + 6t + 11\ dt$$

$$x(t) = \frac{3t^4}{4} - \frac{2t^3}{3} + 3t^2 + 11t + C$$

Substitute the initial condition $x(0) = 2$ to find C.

$$2 = \frac{3(0)^4}{4} - \frac{2(0)^3}{3} + 3(0)^2 + 11(0) + C$$

$$C = 2$$

So the position function is

$$x(t) = \frac{3t^4}{4} - \frac{2t^3}{3} + 3t^2 + 11t + 2$$

■ 3. Find the position function $x(t)$ that models the rectilinear motion of a particle moving along the x-axis.

$$a(t) = 2 - 6t$$

$$v(0) = 4$$

$$x(0) = 3$$

Solution:

Integrate the acceleration function to find the velocity function.

$$v(t) = \int a(t)\ dt = \int 2 - 6t\ dt$$

$$v(t) = 2t - 3t^2 + C$$

Substitute the initial condition $v(0) = 4$ to find C.

$$4 = -3(0)^2 + 2(0) + C$$

$$C = 4$$

So the velocity function is

$$v(t) = 2t - 3t^2 + 4$$

Then the position function is the integral of the velocity function.

$$x(t) = \int v(t)\ dt = \int 2t - 3t^2 + 4\ dt$$

$$x(t) = -t^3 + t^2 + 4t + C$$

Substitute the initial condition $x(0) = 3$ to find C.

$$3 = -(0)^3 + (0)^2 + 4(0) + C$$

$$C = 3$$

So the position function is

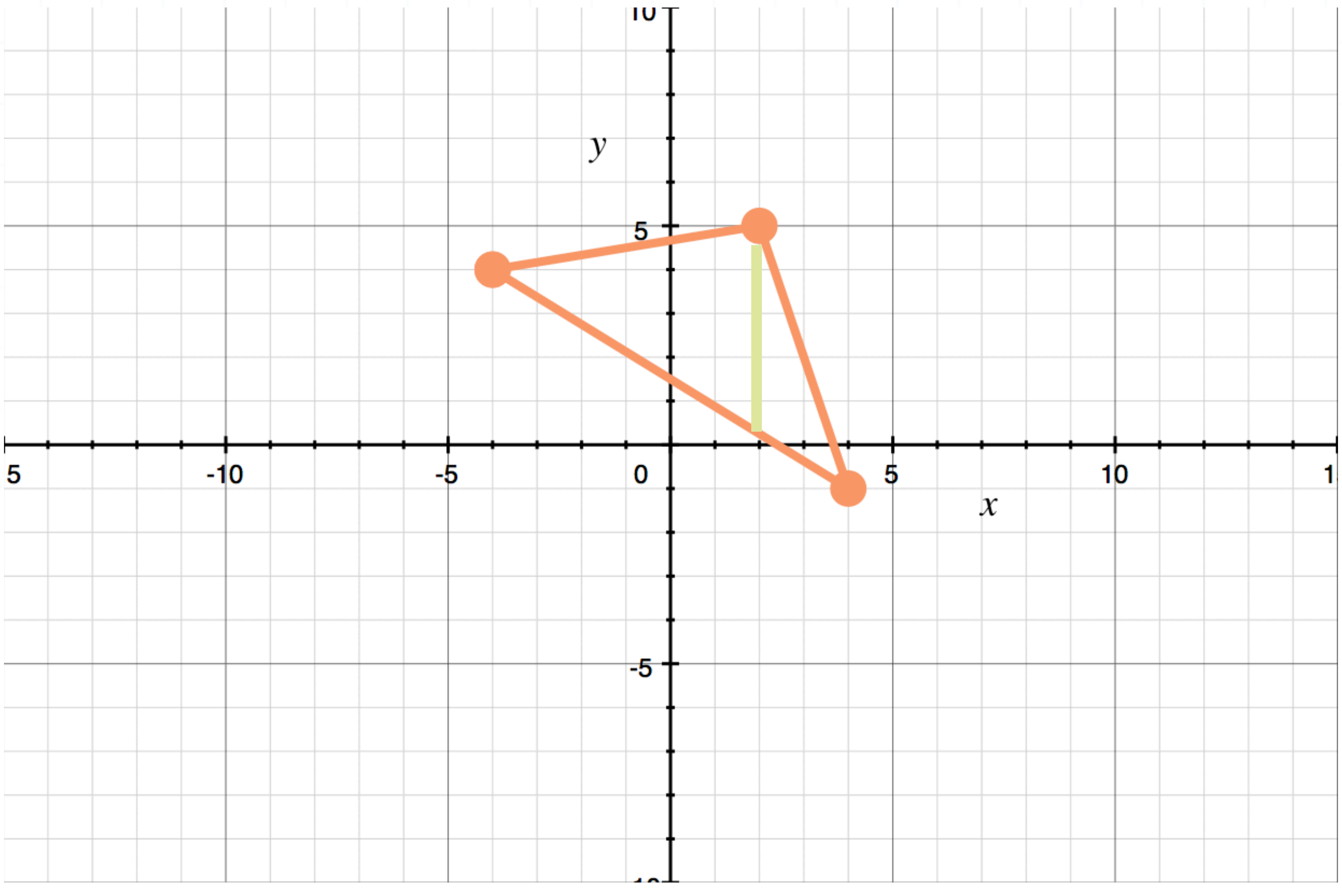
$$x(t) = -t^3 + t^2 + 4t + 3$$

AREA OF A TRIANGLE WITH GIVEN VERTICES

1. Find the area of the triangle with vertices $A(-4,4)$, $B(2,5)$, and $C(4,-1)$.

Solution:

A sketch of the region, separated by a vertical line from B is



The slope of the line connecting $A(-4,4)$ and $B(2,5)$ is

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{5-4}{2-(-4)} = \frac{1}{6}$$

Then using $B(2,5)$ and the slope $m = 1/6$, the equation of that line is

$$y = \frac{1}{6}(x - 2) + 5$$

$$y = \frac{1}{6}x - \frac{2}{6} + 5$$

$$y = \frac{1}{6}x + \frac{14}{3}$$

The slope of the line connecting $A(-4,4)$ and $C(4,-1)$ is

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-1 - 4}{4 - (-4)} = -\frac{5}{8}$$

Then using $A(-4,4)$ and the slope $m = -5/8$, the equation of that line is

$$y = -\frac{5}{8}(x + 4) + 4$$

$$y = -\frac{5}{8}x - \frac{5}{2} + 4$$

$$y = -\frac{5}{8}x + \frac{3}{2}$$

The slope of the line connecting $B(2,5)$ and $C(4,-1)$ is

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-1 - 5}{4 - 2} = -3$$

Then using $B(2,5)$ and the slope $m = -3$, the equation of that line is

$$y = -3(x - 2) + 5$$

$$y = -3x + 6 + 5$$

$$y = -3x + 11$$

Then the area to the left of the vertical line A_L is

$$A_L = \int_{-4}^{2} \left(\frac{1}{6}x + \frac{14}{3}\right) - \left(-\frac{5}{8}x + \frac{3}{2}\right)\ dx$$

$$A_L = \int_{-4}^{2} \frac{1}{6}x + \frac{14}{3} + \frac{5}{8}x - \frac{3}{2}\ dx$$

$$A_L = \int_{-4}^{2} \frac{4}{24}x + \frac{15}{24}x + \frac{28}{6} - \frac{9}{6}\ dx$$

$$A_L = \int_{-4}^{2} \frac{19}{24}x + \frac{19}{6}\ dx$$

Integrate, then evaluate over the interval.

$$A_L = \frac{19}{48}x^2 + \frac{19}{6}x\Big|_{-4}^{2}$$

$$A_L = \frac{19}{48}(2)^2 + \frac{19}{6}(2) - \left(\frac{19}{48}(-4)^2 + \frac{19}{6}(-4)\right)$$

$$A_L = \frac{76}{48} + \frac{38}{6} - \frac{304}{48} + \frac{76}{6}$$

$$A_L = -\frac{228}{48} + \frac{114}{6}$$

$$A_L = -\frac{57}{12} + \frac{228}{12}$$

$$A_L = \frac{171}{12}$$

$$A_L = \frac{57}{4}$$

The area to the right of the vertical line A_R is

$$A_R = \int_2^4 (-3x + 11) - \left(-\frac{5}{8}x + \frac{3}{2}\right)\ dx$$

$$A_R = \int_2^4 -3x + 11 + \frac{5}{8}x - \frac{3}{2}\ dx$$

$$A_R = \int_2^4 -\frac{24}{8}x + \frac{5}{8}x + \frac{22}{2} - \frac{3}{2}\ dx$$

$$A_R = \int_2^4 -\frac{19}{8}x + \frac{19}{2}\ dx$$

Integrate, then evaluate over the interval.

$$A_R = -\frac{19}{16}x^2 + \frac{19}{2}x\Big|_2^4$$

$$A_R = -\frac{19}{16}(4)^2 + \frac{19}{2}(4) - \left(-\frac{19}{16}(2)^2 + \frac{19}{2}(2)\right)$$

$$A_R = -19 + 38 + \frac{19}{4} - 19$$

$$A_R = \frac{19}{4}$$

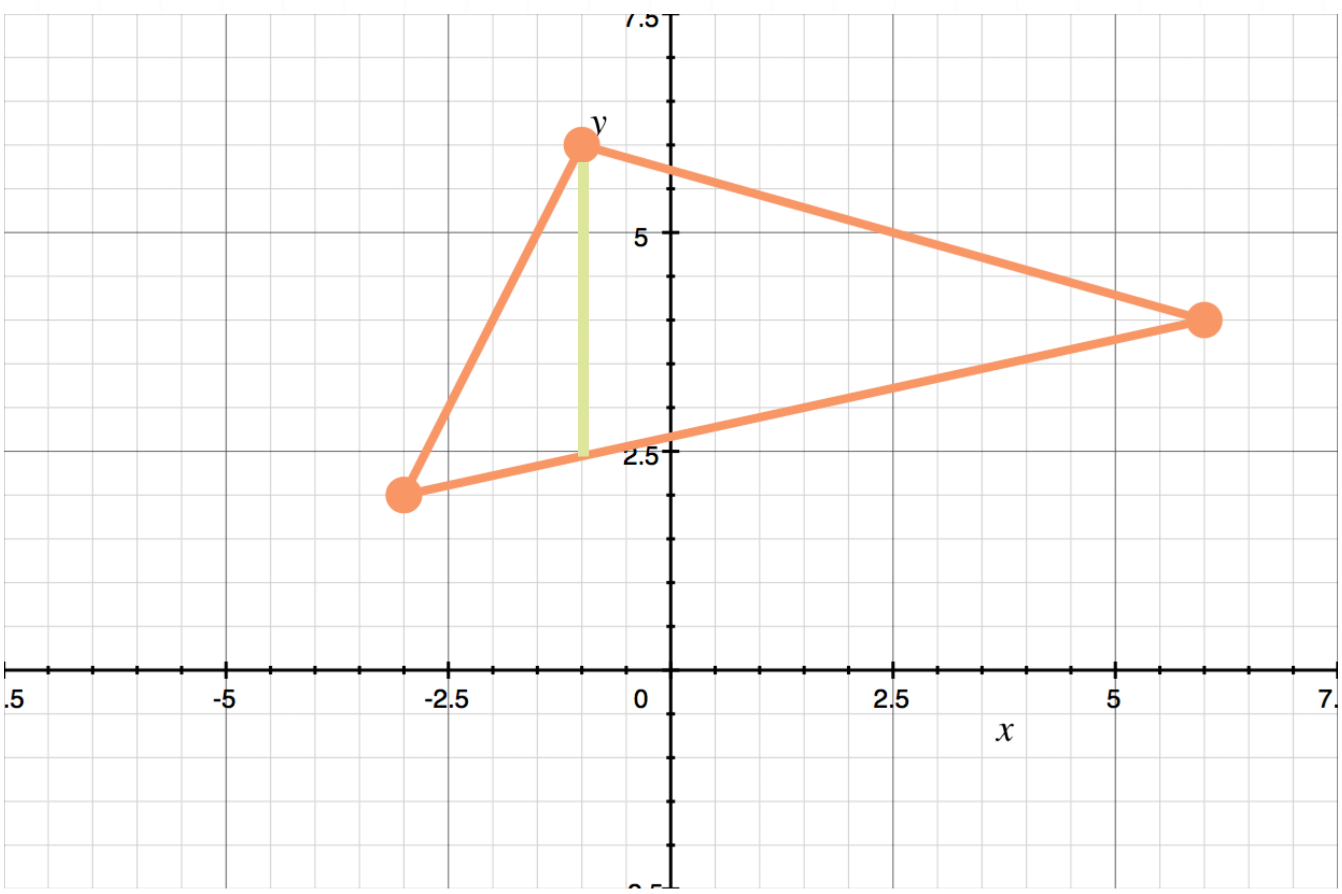
The area of the triangle is

$$A_L + A_R = \frac{57}{4} + \frac{19}{4} = \frac{76}{4} = 19$$

2. Find the area of the triangle with vertices $D(-3,2)$, $E(-1,6)$, and $F(6,4)$.

Solution:

A sketch of the region, separated by a vertical line from E is



The slope of the line connecting $D(-3,2)$ and $E(-1,6)$ is

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{6-2}{-1-(-3)} = \frac{4}{2} = 2$$

Then using $D(-3,2)$ and the slope $m=2$, the equation of that line is

$$y=2(x+3)+2$$

$$y=2x+6+2$$

$$y=2x+8$$

The slope of the line connecting $E(-1,6)$ and $F(6,4)$ is

$$m=\frac{y_2-y_1}{x_2-x_1}=\frac{4-6}{6-(-1)}=-\frac{2}{7}$$

Then using $E(-1,6)$ and the slope $m=-2/7$, the equation of that line is

$$y=-\frac{2}{7}(x+1)+6$$

$$y=-\frac{2}{7}x-\frac{2}{7}+6$$

$$y=-\frac{2}{7}x+\frac{40}{7}$$

The slope of the line connecting $D(-3,2)$ and $F(6,4)$ is

$$m=\frac{y_2-y_1}{x_2-x_1}=\frac{4-2}{6-(-3)}=\frac{2}{9}$$

Then using $F(6,4)$ and the slope $m=2/9$, the equation of that line is

$$y=\frac{2}{9}(x-6)+4$$

$$y = \frac{2}{9}x - \frac{4}{3} + 4$$

$$y = \frac{2}{9}x + \frac{8}{3}$$

Then the area to the left of the vertical line A_L is

$$A_L = \int_{-3}^{-1} (2x+8) - \left(\frac{2}{9}x + \frac{8}{3}\right)\ dx$$

$$A_L = \int_{-3}^{-1} 2x + 8 - \frac{2}{9}x - \frac{8}{3}\ dx$$

$$A_L = \int_{-3}^{-1} \frac{18}{9}x - \frac{2}{9}x + \frac{24}{3} - \frac{8}{3}\ dx$$

$$A_L = \int_{-3}^{-1} \frac{16}{9}x + \frac{16}{3}\ dx$$

Integrate, then evaluate over the interval.

$$A_L = \frac{16}{18}x^2 + \frac{16}{3}x \Big|_{-3}^{-1}$$

$$A_L = \frac{16}{18}(-1)^2 + \frac{16}{3}(-1) - \left(\frac{16}{18}(-3)^2 + \frac{16}{3}(-3)\right)$$

$$A_L = \frac{16}{18} - \frac{16}{3} - (8 - 16)$$

$$A_L = \frac{8}{9} - \frac{48}{9} - \frac{72}{9} + \frac{144}{9}$$

$$A_L = \frac{32}{9}$$

The area to the right of the vertical line A_R is

$$A_R = \int_{-1}^{6} \left(-\frac{2}{7}x + \frac{40}{7}\right) - \left(\frac{2}{9}x + \frac{8}{3}\right)\ dx$$

$$A_R = \int_{-1}^{6} -\frac{2}{7}x + \frac{40}{7} - \frac{2}{9}x - \frac{8}{3}\ dx$$

$$A_R = \int_{-1}^{6} -\frac{18}{63}x - \frac{14}{63}x + \frac{120}{21} - \frac{56}{21}\ dx$$

$$A_R = \int_{-1}^{6} -\frac{32}{63}x + \frac{64}{21}\ dx$$

Integrate, then evaluate over the interval.

$$A_R = -\frac{16}{63}x^2 + \frac{64}{21}x\Big|_{-1}^{6}$$

$$A_R = -\frac{16}{63}(6)^2 + \frac{64}{21}(6) - \left(-\frac{16}{63}(-1)^2 + \frac{64}{21}(-1)\right)$$

$$A_R = -\frac{576}{63} + \frac{384}{21} + \frac{16}{63} + \frac{64}{21}$$

$$A_R = -\frac{560}{63} + \frac{448}{21}$$

$$A_R = -\frac{80}{9} + \frac{64}{3}$$

$$A_R = -\frac{240}{27} + \frac{576}{27}$$

$$A_R = \frac{336}{27}$$

$$A_R = \frac{112}{9}$$

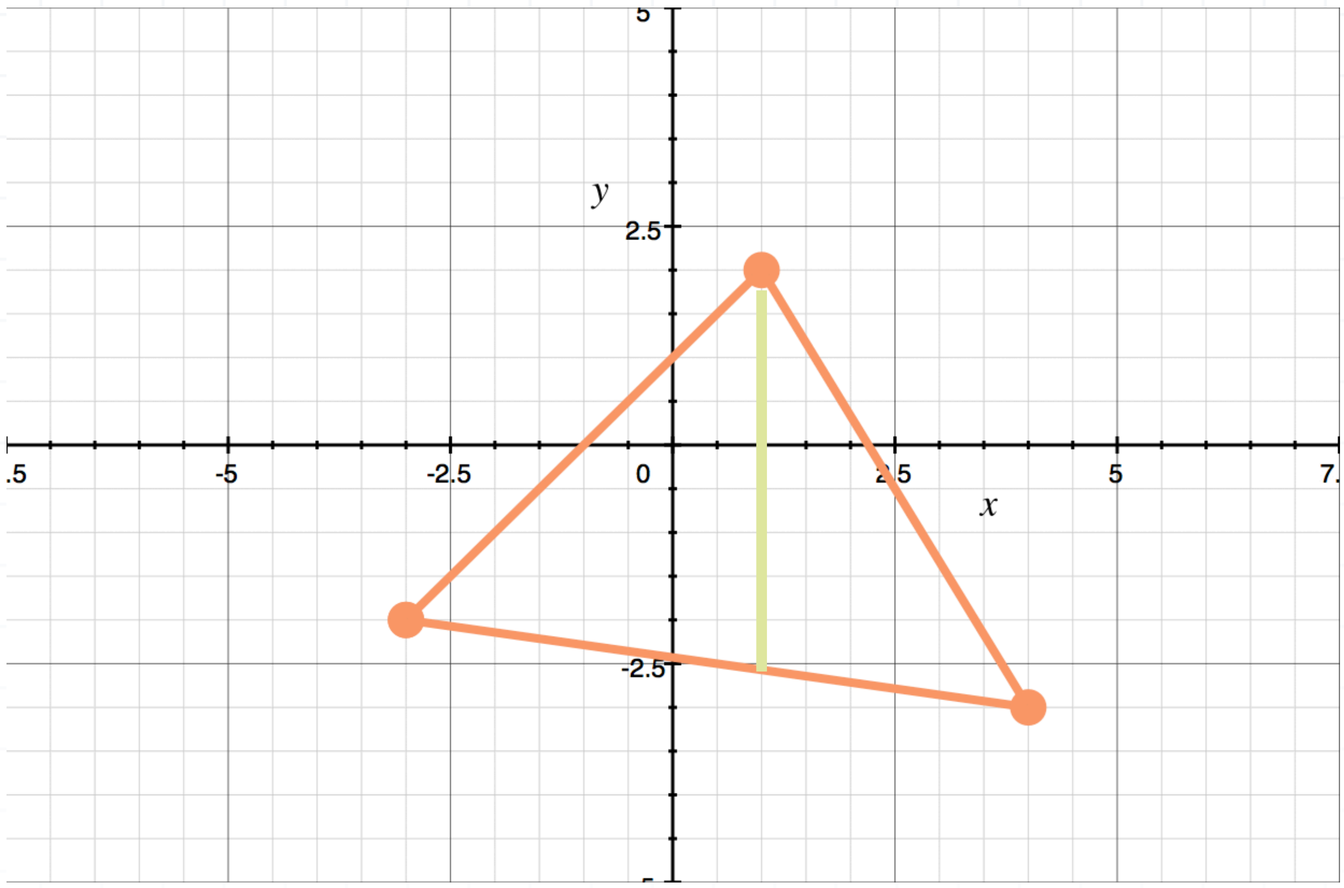
The area of the triangle is

$$A_L + A_R = \frac{32}{9} + \frac{112}{9} = \frac{144}{9} = 16$$

■ 3. Find the area of the triangle with vertices $G(-3,-2)$, $H(1,2)$, and $I(4,-3)$.

Solution:

A sketch of the region, separated by a vertical line from H is



The slope of the line connecting $G(-3,-2)$ and $H(1,2)$ is

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{2-(-2)}{1-(-3)} = \frac{4}{4} = 1$$

Then using $H(1,2)$ and the slope $m = 1$, the equation of that line is

$$y = 1(x-1) + 2$$

$$y = x - 1 + 2$$

$$y = x + 1$$

The slope of the line connecting $H(1,2)$ and $I(4,-3)$ is

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-3-2}{4-1} = -\frac{5}{3}$$

Then using $I(4,-3)$ and the slope $m = -5/3$, the equation of that line is

$$y = -\frac{5}{3}(x-4) - 3$$

$$y = -\frac{5}{3}x + \frac{20}{3} - 3$$

$$y = -\frac{5}{3}x + \frac{11}{3}$$

The slope of the line connecting $G(-3,-2)$ and $I(4,-3)$ is

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-3-(-2)}{4-(-3)} = -\frac{1}{7}$$

Then using $G(-3,-2)$ and the slope $m = -1/7$, the equation of that line is

$$y = -\frac{1}{7}(x+3) - 2$$

$$y = -\frac{1}{7}x - \frac{3}{7} - 2$$

$$y = -\frac{1}{7}x - \frac{17}{7}$$

Then the area to the left of the vertical line A_L is

$$A_L = \int_{-3}^{1} (x+1) - \left(-\frac{1}{7}x - \frac{17}{7}\right)\ dx$$

$$A_L = \int_{-3}^{1} \frac{7}{7}x + \frac{1}{7}x + \frac{7}{7} + \frac{17}{7}\ dx$$

$$A_L = \int_{-3}^{1} \frac{8}{7}x + \frac{24}{7}\ dx$$

Integrate, then evaluate over the interval.

$$A_L = \frac{4}{7}x^2 + \frac{24}{7}x\Big|_{-3}^{1}$$

$$A_L = \frac{4}{7}(1)^2 + \frac{24}{7}(1) - \left(\frac{4}{7}(-3)^2 + \frac{24}{7}(-3)\right)$$

$$A_L = \frac{4}{7} + \frac{24}{7} - \frac{36}{7} + \frac{72}{7}$$

$$A_L = \frac{64}{7}$$

The area to the right of the vertical line A_R is

$$A_R = \int_1^4 \left(-\frac{5}{3}x + \frac{11}{3}\right) - \left(-\frac{1}{7}x - \frac{17}{7}\right)\ dx$$

$$A_R = \int_1^4 -\frac{5}{3}x + \frac{11}{3} + \frac{1}{7}x + \frac{17}{7}\ dx$$

$$A_R = \int_1^4 \frac{3}{21}x - \frac{35}{21}x + \frac{77}{21} + \frac{51}{21}\ dx$$

$$A_R = \int_1^4 -\frac{32}{21}x + \frac{128}{21}\ dx$$

Integrate, then evaluate over the interval.

$$A_R = -\frac{16}{21}x^2 + \frac{128}{21}x\Big|_1^4$$

$$A_R = -\frac{16}{21}(4)^2 + \frac{128}{21}(4) - \left(-\frac{16}{21}(1)^2 + \frac{128}{21}(1)\right)$$

$$A_R = -\frac{256}{21} + \frac{512}{21} + \frac{16}{21} - \frac{128}{21}$$

$$A_R = \frac{144}{21}$$

$$A_R = \frac{48}{7}$$

The area of the triangle is

$$A_L + A_R = \frac{64}{7} + \frac{48}{7} = \frac{112}{7} = 16$$

SINGLE DEPOSIT, COMPOUNDED N TIMES, FUTURE VALUE

■ 1. Find the future value of $\$9,500$ after 7 years, at an annual interest rate of $2.25\ \%$, compounded quarterly.

Solution:

Using the future value formula, the future value is

$$FV = PV\left(1+\frac{r}{n}\right)^{nt}$$

$$FV = 9{,}500\left(1+\frac{0.0225}{4}\right)^{4\times 7}$$

$$FV = 9{,}500(1.005625)^{28}$$

$$FV = \$11{,}115.61$$

■ 2. Find the future value of $\$14,550$ after 3 years, at an annual interest rate of $1.95\ \%$, compounded monthly.

Solution:

Using the future value formula, the future value is

$$FV = PV\left(1 + \frac{r}{n}\right)^{nt}$$

$$FV = 14{,}550\left(1 + \frac{0.0195}{12}\right)^{12\times 3}$$

$$FV = 14{,}550(1.001625)^{36}$$

$$FV = \$15{,}425.83$$

■ 3. Find the future value of $\$7{,}595$ after 5 years, at an annual interest rate of $3.25\,\%$, compounded weekly.

Solution:

Using the future value formula, the future value is

$$FV = PV\left(1 + \frac{r}{n}\right)^{nt}$$

$$FV = 7{,}595\left(1 + \frac{0.0325}{52}\right)^{52\times 5}$$

$$FV = 7{,}595(1.000625)^{260}$$

$$FV = \$8{,}934.67$$

SINGLE DEPOSIT, COMPOUNDED N TIMES, PRESENT VALUE

■ 1. Find the present value of a deposit that, after 9 years, at an annual interest rate of 4.75%, compounded monthly, will have a value of $\$24{,}514.01$.

Solution:

Use the future value formula, then solve for present value.

$$FV = PV\left(1+\frac{r}{n}\right)^{nt}$$

$$24514.01 = PV\left(1+\frac{0.0475}{12}\right)^{12\times 9}$$

$$24514.01 = PV(1.003958333)^{108}$$

$$PV = \frac{24514.01}{1.003958333^{108}}$$

$$PV = \$16{,}000.00$$

■ 2. Find the present value of a deposit that, after 3 years, at an annual interest rate of 7.85%, compounded weekly, will have a value of $\$948.99$.

Solution:

Use the future value formula, then solve for present value.

$$FV = PV\left(1 + \frac{r}{n}\right)^{nt}$$

$$948.99 = PV\left(1 + \frac{0.0785}{52}\right)^{52\times 3}$$

$$948.99 = PV(1.001509615)^{156}$$

$$PV = \frac{948.99}{1.001509615^{156}}$$

$$PV = \$750.00$$

■ 3. Find the present value of a deposit that, after 6 years, at an annual interest rate of $3.95\,\%$, compounded quarterly, will have a value of $\$1,582,46$.

Solution:

Use the future value formula, then solve for present value.

$$FV = PV\left(1 + \frac{r}{n}\right)^{nt}$$

$$1,582.46 = PV\left(1 + \frac{0.0395}{4}\right)^{4\times 6}$$

$$1{,}582.46 = PV(1.009875)^{24}$$

$$PV = \frac{1{,}582.46}{1.009875^{24}}$$

$$PV = \$1{,}250.00$$

SINGLE DEPOSIT, COMPOUNDED CONTINUOUSLY, FUTURE VALUE

1. Find the future value of $\$2{,}850$, after 8 years, at an annual interest rate of $1.55\,\%$, compounded continuously.

Solution:

Use the future value formula for continuously compounded interest.

$$FV = PVe^{rt}$$

$$FV = 2{,}850e^{0.0155\times 8}$$

$$FV = 2{,}850e^{0.124}$$

$$FV = \$3{,}226.25$$

2. Find the future value of $\$9{,}875$, after 15 years, at an annual interest rate of $4.15\,\%$, compounded continuously.

Solution:

Use the future value formula for continuously compounded interest.

$$FV = PVe^{rt}$$

$$FV = 9{,}875e^{0.0415 \times 15}$$

$$FV = 9{,}875e^{0.6225}$$

$$FV = \$18{,}402.86$$

■ 3. Find the future value of $\$15{,}000$, after 18 years, at an annual interest rate of $8.5\,\%$, compounded continuously.

Solution:

Use the future value formula for continuously compounded interest.

$$FV = PVe^{rt}$$

$$FV = 15{,}000e^{0.085 \times 18}$$

$$FV = 15{,}000e^{1.53}$$

$$FV = \$69{,}272.65$$

SINGLE DEPOSIT, COMPOUNDED CONTINUOUSLY, PRESENT VALUE

■ 1. Find the present value of a deposit that, after 11 years, at an annual interest rate of $2.75\,\%$, compounded continuously, will have a value of $\$11{,}631.08$.

Solution:

Use the future value formula for continuously compounded interest, then solve for present value.

$$FV = PVe^{rt}$$

$$11{,}631.08 = PVe^{0.0275 \times 11}$$

$$11{,}631.08 = PVe^{0.3025}$$

$$PV = \frac{11{,}631.08}{e^{0.3025}}$$

$$PV = \$8{,}595.00$$

■ 2. Find the present value of a deposit that, after 7 years, at an annual interest rate of $6.17\,\%$, compounded continuously, will have a value of $\$3{,}850.45$.

Solution:

Use the future value formula for continuously compounded interest, then solve for present value.

$$FV = PVe^{rt}$$

$$3{,}850.45 = PVe^{0.0617\times7}$$

$$3{,}850.45 = PVe^{0.4319}$$

$$PV = \frac{3{,}850.45}{e^{0.4319}}$$

$$PV = \$2{,}500.00$$

■ 3. Find the present value of a deposit that, after 4 years, at an annual interest rate of $5.95\,\%$, compounded continuously, will have a value of $\$6{,}343.55$.

Solution:

Use the future value formula for continuously compounded interest, then solve for present value.

$$FV = PVe^{rt}$$

$$6{,}343.55 = PVe^{0.0595\times4}$$

$$6{,}343.55 = PVe^{0.238}$$

$$PV = \frac{6{,}343.55}{e^{0.238}}$$

$$PV = \$5{,}000.00$$

INCOME STREAM, COMPOUNDED CONTINUOUSLY, FUTURE VALUE

1. Money is invested at a rate of $\$10{,}000$ annually and the bank pays $8.85\,\%$ interest, compounded continuously. How many years will it take for the investment to grow to a balance of $\$300{,}000$?

Solution:

Use the future value formula for an income stream.

$$FV = \int_0^N S(t)e^{r(N-t)}\ dt$$

$$300{,}000 = \int_0^N 10{,}000e^{0.0885(N-t)}\ dt$$

$$300{,}000 = 10{,}000\int_0^N e^{0.0885(N-t)}\ dt$$

$$30 = \int_0^N e^{0.0885(N-t)}\ dt$$

$$30 = \int_0^N e^{0.0885N-0.0885t}\ dt$$

$$30 = \int_0^N e^{0.0885N}e^{-0.0885t}\ dt$$

$$30 = e^{0.0885N} \int_0^N e^{-0.0885t} \, dt$$

Integrate, then evaluate over the interval.

$$30 = e^{0.0885N} \left(\frac{1}{-0.0885} e^{-0.0885t} \right) \Bigg|_0^N$$

$$30 = e^{0.0885N} \left(\frac{1}{-0.0885} e^{-0.0885N} \right) - e^{0.0885N} \left(\frac{1}{-0.0885} e^{-0.0885(0)} \right)$$

$$30 = -e^{0.0885N} \left(\frac{e^{-0.0885N}}{0.0885} \right) + \frac{e^{0.0885N}}{0.0885}$$

$$30(0.0885) = -e^{0.0885N} e^{-0.0885N} + e^{0.0885N}$$

$$30(0.0885) = -1 + e^{0.0885N}$$

$$30(0.0885) + 1 = e^{0.0885N}$$

$$3.655 = e^{0.0885N}$$

Solve using the natural logarithm.

$$\ln 3.655 = \ln e^{0.0885N}$$

$$\ln 3.655 = 0.0885N \ln e$$

$$\ln 3.655 = 0.0885N$$

$$\frac{\ln 3.655}{0.0885} = N$$

$$N = 14.645$$

It will take the investment approximately 14.645 years to grow to a balance of $\$300{,}000$.

2. Money is invested at a rate of $\$5{,}000$ annually and the bank pays $6.75\,\%$ interest, compounded continuously. How many years will it take for the investment to grow to a balance of $\$100{,}000$?

Solution:

Use the future value formula for an income stream.

$$FV = \int_0^N S(t)e^{r(N-t)}\ dt$$

$$100{,}000 = \int_0^N 5{,}000e^{0.0675(N-t)}\ dt$$

$$100{,}000 = 5{,}000\int_0^N e^{0.0675(N-t)}\ dt$$

$$20 = \int_0^N e^{0.0675(N-t)}\ dt$$

$$20 = \int_0^N e^{0.0675N-0.0675t}\ dt$$

$$20 = \int_0^N e^{0.0675N} e^{-0.0675t} \ dt$$

$$20 = e^{0.0675N} \int_0^N e^{-0.0675t} \ dt$$

Integrate, then evaluate over the interval.

$$20 = e^{0.0675N} \left(\frac{1}{-0.0675} e^{-0.0675t} \right) \Bigg|_0^N$$

$$20 = e^{0.0675N} \left(\frac{1}{-0.0675} e^{-0.0675N} \right) - e^{0.0675N} \left(\frac{1}{-0.0675} e^{-0.0675(0)} \right)$$

$$20 = -e^{0.0675N} \left(\frac{e^{-0.0675N}}{0.0675} \right) + \frac{e^{0.0675N}}{0.0675}$$

$$20(0.0675) = -e^{0.0675N} e^{-0.0675N} + e^{0.0675N}$$

$$20(0.0675) = -1 + e^{0.0675N}$$

$$20(0.0675) + 1 = e^{0.0675N}$$

$$2.35 = e^{0.0675N}$$

Solve using the natural logarithm.

$$\ln 2.35 = \ln e^{0.0675N}$$

$$\ln 2.35 = 0.0675N \ln e$$

$$\ln 2.35 = 0.0675N$$

$$\frac{\ln 2.35}{0.0675} = N$$

$$N = 12.658$$

It will take the investment approximately 12.658 years to grow to a balance of $\$100{,}000$.

3. Money is invested at a rate of $\$2{,}500$ annually and the bank pays $5.25\,\%$ interest, compounded continuously. How many years will it take for the investment to grow to a balance of $\$25{,}000$?

Solution:

Use the future value formula for an income stream.

$$FV = \int_0^N S(t)e^{r(N-t)}\ dt$$

$$25{,}000 = \int_0^N 2{,}500e^{0.0525(N-t)}\ dt$$

$$25{,}000 = 2{,}500\int_0^N e^{0.0525(N-t)}\ dt$$

$$10 = \int_0^N e^{0.0525(N-t)}\ dt$$

$$10 = \int_0^N e^{0.0525N - 0.0525t}\ dt$$

$$10 = \int_0^N e^{0.0525N} e^{-0.0525t}\ dt$$

$$10 = e^{0.0525N} \int_0^N e^{-0.0525t}\ dt$$

Integrate, then evaluate over the interval.

$$10 = e^{0.0525N}\left(\frac{1}{-0.0525}e^{-0.0525t}\right) - e^{0.0525N}\left(\frac{1}{-0.0525}e^{-0.0525t}\right)\Bigg|_0^N$$

$$10 = e^{0.0525N}\left(\frac{1}{-0.0525}e^{-0.0525N}\right) - e^{0.0525N}\left(\frac{1}{-0.0525}e^{-0.0525(0)}\right)$$

$$10 = -e^{0.0525N}\left(\frac{e^{-0.0525N}}{0.0525}\right) + \frac{e^{0.0525N}}{0.0525}$$

$$10(0.0525) = -e^{0.0525N}e^{-0.0525N} + e^{0.0525N}$$

$$10(0.0525) = -1 + e^{0.0525N}$$

$$10(0.0525) + 1 = e^{0.0525N}$$

$$1.525 = e^{0.0525N}$$

Solve using the natural logarithm.

$$\ln 1.525 = \ln e^{0.0525N}$$

$$\ln 1.525 = 0.0525N \ln e$$

$$\ln 1.525 = 0.0525N$$

$$\frac{\ln 1.525}{0.0525} = N$$

$$N = 8.038$$

It will take the investment approximately 8.038 years to grow to a balance of $\$25{,}000$.

INCOME STREAM, COMPOUNDED CONTINUOUSLY, PRESENT VALUE

■ 1. Suppose that money is deposited steadily into an account at a constant rate of $\$15{,}000$ per year for 13 years. Find the present value of this income stream if the account pays $7.35\,\%$, compounded continuously.

Solution:

Use the present value formula for an income stream.

$$PV=\int_0^T S(t)e^{-rt}\ dt$$

$$PV=\int_0^{13} 15{,}000e^{-0.0735t}\ dt$$

Integrate, then evaluate over the interval.

$$PV=15{,}000\left(\frac{e^{-0.0735t}}{-0.0735}\right)\Bigg|_0^{13}$$

$$PV=15{,}000\left(\frac{e^{-0.0735(13)}}{-0.0735}-\frac{e^{-0.0735(0)}}{-0.0735}\right)$$

$$PV=15{,}000(8.372519911)$$

$$PV=\$125{,}587.80$$

2. Suppose that money is deposited steadily into a college fund at a constant rate of $\$3{,}000$ per year for 18 years. Find the present value of this income stream if the account pays $5.15\,\%$, compounded continuously.

Solution:

Use the present value formula for an income stream.

$$PV = \int_0^T S(t)e^{-rt}\ dt$$

$$PV = \int_0^{18} 3{,}000e^{-0.0515t}\ dt$$

Integrate, then evaluate over the interval.

$$PV = 3{,}000\left(\frac{e^{-0.0515t}}{-0.0515}\right)\Bigg|_0^{18}$$

$$PV = 3{,}000\left(\frac{e^{-0.0515(18)}}{-0.0515} - \frac{e^{-0.0515(0)}}{-0.0515}\right)$$

$$PV = 3{,}000(11.73322041)$$

$$PV = \$35{,}199.66$$

3. Suppose that money is deposited steadily into a new car account at a constant rate of $\$2{,}500$ per year for 8 years. Find the present value of this income stream if the account pays $7.5\,\%$, compounded continuously.

Solution:

Use the present value formula for an income stream.

$$PV = \int_0^T S(t)e^{-rt}\ dt$$

$$PV = \int_0^8 2{,}500e^{-0.075t}\ dt$$

Integrate, then evaluate over the interval.

$$PV = 2{,}500\left(\frac{e^{-0.075t}}{-0.075}\right)\Bigg|_0^8$$

$$PV = 2{,}500\left(\frac{e^{-0.075(8)}}{-0.075} - \frac{e^{-0.075(0)}}{-0.075}\right)$$

$$PV = 2{,}500(6.015844852)$$

$$PV = \$15{,}039.61$$

CONSUMER AND PRODUCER SURPLUS

1. Find the equilibrium quantity q_e and the equilibrium price p_e.

$$S(q) = 0.06q^2 + 5$$

$$D(q) = 0.1q + 17$$

Solution:

The equilibrium point is where the supply curve $S(q)$ and the demand curve $D(q)$ intersect. The equilibrium quantity is the x-value of the intersection point and the equilibrium price is the y-value of the intersection point. Set the supply equation equal to the demand equation and find their intersection point.

$$0.06q^2 + 5 = 0.1q + 17$$

$$0.06q^2 + 5 - 0.1q - 17 = 0$$

$$0.06q^2 - 0.1q - 12 = 0$$

$$6q^2 + 10q - 1{,}200 = 0$$

$$(6q + 80)(q - 15) = 0$$

Then the solutions for q are

$$6q + 80 = 0$$

$$6q = -80$$

$$q = -\frac{40}{3}$$

and

$$q - 15 = 0$$

$$q = 15$$

Since q is a quantity, the answer must be positive, so discard $q = -40/3$ as a possible solution, and accept the equilibrium quantity of $q = 15$. Use the equilibrium quantity to find the equilibrium price.

$$D(q) = 0.1q + 17$$

$$D(15) = 0.1(15) + 17$$

$$D(15) = 18.50$$

$$p = 18.50$$

Then the equilibrium quantity and equilibrium price are

$$q_e = 15$$

$$p_e = 18.50$$

■ 2. Find the consumer surplus.

$$S(q) = 0.05q^2 + 7$$

$$D(q) = -0.2q + 11.8$$

Solution:

Find equilibrium quantity by setting the curves equal to one another.

$$0.05q^2 + 7 = -0.2q + 11.8$$

$$0.05q^2 + 7 + 0.2q - 11.8 = 0$$

$$0.05q^2 + 0.2q - 4.8 = 0$$

$$5q^2 + 20q - 480 = 0$$

$$(5q - 40)(q + 12) = 0$$

Then the solutions for q are

$$5q - 40 = 0$$

$$5q = 40$$

$$q = 8$$

and

$$q + 12 = 0$$

$$q = -12$$

Since q is a quantity, the answer must be positive, so discard $q = -12$ as a possible solution, and accept the equilibrium quantity of $q = 8$. Use the equilibrium quantity to find the equilibrium price.

$$D(q) = -0.2q + 11.8$$

$$D(15) = -0.2(8) + 11.8$$

$$D(15) = 10.20$$

$$p = 10.20$$

Then the equilibrium quantity and equilibrium price are

$$q_e = 8$$

$$p_e = 10.20$$

Then the consumer surplus will be

$$CS = \int_0^{q_e} D(q)\ dq - p_e q_e$$

$$CS = \int_0^{8} -0.2q + 11.8\ dq - (10.20)(8)$$

$$CS = -0.1q^2 + 11.8q\Big|_0^8 - 81.6$$

$$CS = -0.1(8)^2 + 11.8(8) - (-0.1(0)^2 + 11.8(0)) - 81.6$$

$$CS = -6.4 + 94.4 - 81.6$$

$$CS = -6.4 + 94.4 - 81.6$$

$$CS = 6.4$$

■ 3. Find the equilibrium quantity q_e and the equilibrium price p_e.

$$S(q) = 0.09q^2 + 8$$

$$D(q) = 1.55q + 25.5$$

Solution:

The equilibrium point is where the supply curve $S(q)$ and the demand curve $D(q)$ intersect. The equilibrium quantity is the x-value of the intersection point and the equilibrium price is the y-value of the intersection point. Set the supply equation equal to the demand equation and find their intersection point.

$$0.09q^2 + 8 = 1.55q + 25.5$$

$$0.09q^2 + 8 - 1.55q - 25.5 = 0$$

$$0.09q^2 - 1.55q - 17.5 = 0$$

$$9q^2 - 155q - 1{,}750 = 0$$

$$(9q + 70)(q - 25) = 0$$

Then the solutions for q are

$$9q + 70 = 0$$

$$9q = -70$$

$$q = -\frac{70}{9}$$

and

$$q - 25 = 0$$

$$q = 25$$

Since q is a quantity, the answer must be positive, so discard $q = -70/9$ as a possible solution, and accept the equilibrium quantity of $q = 25$. Use the equilibrium quantity to find the equilibrium price.

$$D(q) = 1.55q + 25.5$$

$$D(15) = 1.55(25) + 25.5$$

$$D(15) = 64.25$$

$$p = 64.25$$

Then the equilibrium quantity and equilibrium price are

$$q_e = 25$$

$$p_e = 64.25$$

PROBABILITY DENSITY FUNCTIONS

■ 1. Given $f(x)$, find $P(0 \le x \le 2)$.

$$f(x) = \begin{cases} \frac{1}{32} & 0 \le x \le 32 \\ 0 & x < 0 \text{ or } x > 32 \end{cases}$$

Solution:

First ensure that the function meets the criteria to be a probability density function, in that $f(x) \ge 0$ on $-\infty \le x \le \infty$, and the integral of $f(x)$ on $-\infty \le x \le \infty$ equals 1.

The given function $f(x)$ is a piecewise constant function, and based on the function's definition, $f(x) \ge 0$ for all x.

The integral of $f(x)$ on $-\infty \le x \le \infty$ is

$$\int_{-\infty}^{\infty} f(x) \ dx$$

$$\int_{-\infty}^{0} f(x) \ dx + \int_{0}^{32} f(x) \ dx + \int_{32}^{\infty} f(x) \ dx$$

$$0 + \int_{0}^{32} \frac{1}{32} \ dx + 0$$

Integrate, then evaluate over the interval.

$$\frac{1}{32}x\Big|_0^{32}$$

$$\frac{1}{32}(32)-\frac{1}{32}(0)$$

$$1$$

Then $P(0 \le x \le 2)$ is

$$\int_0^2 f(x)\ dx=\int_0^2 \frac{1}{32}\ dx=\frac{1}{32}x\Big|_0^2=\frac{1}{32}(2)-\frac{1}{32}(0)=\frac{2}{32}=\frac{1}{16}$$

■ 2. Given $g(x)$, find $P(1 \le x \le 5)$.

$$g(x)=\begin{cases} e^{-x} & x \ge 0 \\ 0 & x < 0 \end{cases}$$

Solution:

First ensure that the function meets the criteria to be a probability density function, in that $g(x) \ge 0$ on $-\infty \le x \le \infty$, and the integral of $g(x)$ on $-\infty \le x \le \infty$ equals 1.

The given function $g(x)$ is a piecewise exponential function. So based on the function's definition, $g(x) \ge 0$ for all x.

The integral of $g(x)$ on $-\infty \le x \le \infty$ is

$$\int_{-\infty}^{\infty} g(x)\ dx$$

$$\int_{-\infty}^{0} g(x)\ dx + \int_{0}^{\infty} g(x)\ dx$$

$$\int_{-\infty}^{0} 0\ dx + \int_{0}^{\infty} e^{-x}\ dx$$

$$\lim_{a\to-\infty} \int_{a}^{0} 0\ dx + \lim_{b\to\infty} \int_{0}^{b} e^{-x}\ dx$$

Integrate, then evaluate over the interval.

$$\lim_{a\to-\infty} 0 + \lim_{b\to\infty} (-e^{-x})\Big|_{0}^{b}$$

$$\lim_{b\to\infty} -e^{-b} - (-e^{-0})$$

$$0 + 1$$

$$1$$

Then $P(1 \le x \le 5)$ is

$$\int_{1}^{5} e^{-x}\ dx = -e^{-x}\Big|_{1}^{5} = -e^{-5} - (-e^{-1}) = -\frac{1}{e^5} + \frac{1}{e} = \frac{1}{e} - \frac{1}{e^5}$$

■ 3. Given $h(x)$, find $P(-1 \le x \le 1)$.

$$h(x) = \begin{cases} \frac{1}{6} & -2 \leq x \leq 4 \\ 0 & x < -2 \text{ or } x > 4 \end{cases}$$

Solution:

First ensure that the function meets the criteria to be a probability density function, in that $h(x) \geq 0$ on $-\infty \leq x \leq \infty$, and the integral of $h(x)$ on $-\infty \leq x \leq \infty$ equals 1.

The given function $f(x)$ is a piecewise constant function, and based on the function's definition, $f(x) \geq 0$ for all x.

The integral of $h(x)$ on $-\infty \leq x \leq \infty$ is

$$\int_{-\infty}^{\infty} h(x)\ dx$$

$$\int_{-\infty}^{-2} h(x)\ dx + \int_{-2}^{4} h(x)\ dx + \int_{4}^{\infty} h(x)\ dx$$

$$\lim_{a \to -\infty} \int_{a}^{-2} h(x)\ dx + \int_{-2}^{4} h(x)\ dx + \lim_{b \to \infty} \int_{4}^{b} h(x)\ dx$$

$$\lim_{a \to -\infty} \int_{a}^{-2} 0\ dx + \int_{-2}^{4} \frac{1}{6}\ dx + \lim_{b \to \infty} \int_{4}^{b} 0\ dx$$

Integrate, then evaluate over the interval.

$$0 + \frac{1}{6}x \Big|_{-2}^{4} + 0$$

$$\frac{1}{6}(4)-\frac{1}{6}(-2)$$

$$\frac{4}{6}+\frac{2}{6}$$

$$\frac{6}{6}$$

$$1$$

Then $P(-1 \le x \le 1)$ is

$$\int_{-1}^{1} h(x)\ dx=\int_{-1}^{1} \frac{1}{6}\ dx$$

$$\frac{1}{6}x\Big|_{-1}^{1}$$

$$\frac{1}{6}(1)-\frac{1}{6}(-1)$$

$$\frac{1}{6}+\frac{1}{6}$$

$$\frac{2}{6}$$

$$\frac{1}{3}$$

CARDIAC OUTPUT

1. Find the cardiac output, in liters/second, if 8 mg of dye is injected into the heart and the amount of dye remaining in the heart t seconds after the injection is modeled by $C(t)=14te^{-0.6t}$. Assume $0\le t\le 20$.

Solution:

From the problem, we know $A=8$, $C(t)=14te^{-0.6t}$, and $T=20$. Substitute these values into the formula for blood flow.

$$F=\frac{A}{\int_0^T C(t)\ dt}$$

$$F=\frac{8}{\int_0^{20} 14te^{-0.6t}\ dt}$$

Work specifically on the integral.

$$\int_0^{20} te^{-0.6t}\ dt$$

Use integration by parts to solve it.

$$u=t$$

$$du=dt$$

$$dv = e^{-0.6t}\ dt$$

$$v = \frac{1}{-0.6}e^{-0.6t}$$

Find the antiderivative by integrating without the limits.

$$\int te^{-0.6t}\ dt$$

$$t\frac{e^{-0.6t}}{-0.6} - \int \frac{e^{-0.6t}}{-0.6}\ dt$$

$$-\frac{10t}{6}e^{-0.6t} + \frac{10}{6}\int e^{-0.6t}\ dt$$

$$-\frac{5t}{3}e^{-0.6t} + \frac{5}{3}\left(\frac{e^{-0.6t}}{-0.6}\right)$$

$$-\frac{5t}{3}e^{-0.6t} - \frac{25}{9}(e^{-0.6t})$$

Evaluating this over the interval gives

$$-\frac{5t}{3}e^{-0.6t} - \frac{25}{9}(e^{-0.6t})\Big|_0^{20}$$

$$\left[-\frac{5(20)}{3}e^{-0.6(20)} - \frac{25}{9}(e^{-0.6(20)})\right] - \left[-\frac{5(0)}{3}e^{-0.6(0)} - \frac{25}{9}(e^{-0.6(0)})\right]$$

$$-0.000221874 + \frac{25}{9}$$

$$2.777555$$

Then blood flow is

$$F = \frac{8}{14(2.777555)} = 0.2057307853 = 0.206 \text{ liters/second}$$

■ 2. Find the cardiac output, in liters/second, if 4 mg of dye is injected into the heart and the amount of dye remaining in the heart t seconds after the injection is modeled by $C(t) = 6te^{-0.2t}$. Assume $0 \le t \le 5$.

Solution:

From the problem, we know $A = 4$, $C(t) = 6te^{-0.2t}$, and $T = 5$. Substitute these values into the formula for blood flow.

$$F = \frac{A}{\int_0^T C(t)\ dt}$$

$$F = \frac{4}{\int_0^5 6te^{-0.2t}\ dt}$$

Work specifically on the integral.

$$\int_0^5 te^{-0.2t}\ dt$$

Use integration by parts to solve it.

$$u = t$$

$$du = dt$$

$$dv = e^{-0.2t}\ dt$$

$$v = \frac{1}{-0.2}e^{-0.2t}$$

Find the antiderivative by integrating without the limits.

$$\int te^{-0.2t}\ dt$$

$$t\frac{e^{-0.2t}}{-0.2} - \int \frac{e^{-0.2t}}{-0.2}\ dt$$

$$-5te^{-0.2t} + 5\int e^{-0.2t}\ dt$$

$$-5te^{-0.2t} + 5\left(\frac{e^{-0.2t}}{-0.2}\right)$$

$$-5te^{-0.2t} - 25(e^{-0.2t})$$

Evaluating this over the interval gives

$$-5te^{-0.2t} - 25(e^{-0.2t})\Big|_0^5$$

$$\left[-5(5)e^{-0.2(5)} - 25(e^{-0.2(5)})\right] - \left[-5(0)e^{-0.2(0)} - 25(e^{-0.2(0)})\right]$$

$$-18.39397206 + 25$$

$$6.6060279$$

Then blood flow is

$$F = \frac{4}{6(6.6060279)} = 0.1009179302 = 0.101 \text{ liters/second}$$

■ 3. Find the cardiac output, in liters/second, if 9 mg of dye is injected into the heart and the amount of dye remaining in the heart t seconds after the injection is modeled by $C(t) = 28te^{-0.85t}$. Assume $0 \le t \le 10$.

Solution:

From the problem, we know $A = 9$, $C(t) = 28te^{-0.85t}$, and $T = 10$. Substitute these values into the formula for blood flow.

$$F = \frac{A}{\int_0^T C(t)\ dt}$$

$$F = \frac{9}{\int_0^{10} 28te^{-0.85t}\ dt}$$

Work specifically on the integral.

$$\int_0^{10} te^{-0.85t}\ dt$$

Use integration by parts to solve it.

$$u = t$$

$$du = dt$$

$$dv = e^{-0.85t}\ dt$$

$$v = \frac{1}{-0.85}e^{-0.85t}$$

Find the antiderivative by integrating without the limits.

$$\int te^{-0.85t}\ dt$$

$$t\frac{e^{-0.85t}}{-0.85} - \int \frac{e^{-0.85t}}{-0.85}\ dt$$

$$-\frac{20t}{17}e^{-0.85t} + \frac{20}{17}\int e^{-0.85t}\ dt$$

$$-\frac{20t}{17}e^{-0.85t} + \frac{20}{17}\left(\frac{e^{-0.85t}}{-0.85}\right)$$

$$-\frac{20t}{17}e^{-0.85t} - \frac{400}{289}(e^{-0.85t})$$

Evaluating this over the interval gives

$$-\frac{20t}{17}e^{-0.85t} - \frac{400}{289}(e^{-0.85t})\Big|_0^{10}$$

$$\left[-\frac{20(10)}{17}e^{-0.85(10)} - \frac{400}{289}(e^{-0.85(10)})\right] - \left[-\frac{20(0)}{17}e^{-0.85(10)} - \frac{400}{289}(e^{-0.85(0)})\right]$$

$$-0.0026753626 + \frac{400}{289}$$

$$1.3814077$$

Then blood flow is

$$F = \frac{9}{28(1.3814077)} = 0.2326819023 = 0.233 \text{ liters/second}$$

POISEUILLE'S LAW

1. Use Poiseuille's law to find the flow of blood in the human artery in which $n = 0.031$, $R = 0.008$ cm, $L = 6$ cm, and $P = 3{,}900$ dynes/cm^2. Express the answer using scientific notation.

Solution:

The blood flow is

$$F = \frac{\pi P R^4}{8nL}$$

$$F = \frac{\pi(3{,}900)(0.008)^4}{8(0.031)(6)}$$

$$F = 3.37 \times 10^{-5} \text{ cm}^3\text{/sec}$$

2. Use Poiseuille's law to find the flow of blood in the human artery in which $n = 0.028$, $R = 0.007$ cm, $L = 3.5$ cm, and $P = 3{,}600$ dynes/cm^2. Express the answer using scientific notation.

Solution:

The blood flow is

$$F = \frac{\pi P R^4}{8nL}$$

$$F = \frac{\pi(3{,}600)(0.007)^4}{8(0.028)(3.5)}$$

$$F = 3.46 \times 10^{-5} \text{ cm}^3\text{/sec}$$

■ 3. Use Poiseuille's law to find the flow of blood in the human artery in which $n = 0.027$, $R = 0.006$ cm, $L = 2.5$ cm, and $P = 3{,}800$ dynes/cm^2. Express the answer using scientific notation.

Solution:

The blood flow is

$$F = \frac{\pi P R^4}{8nL}$$

$$F = \frac{\pi(3{,}800)(0.006)^4}{8(0.027)(2.5)}$$

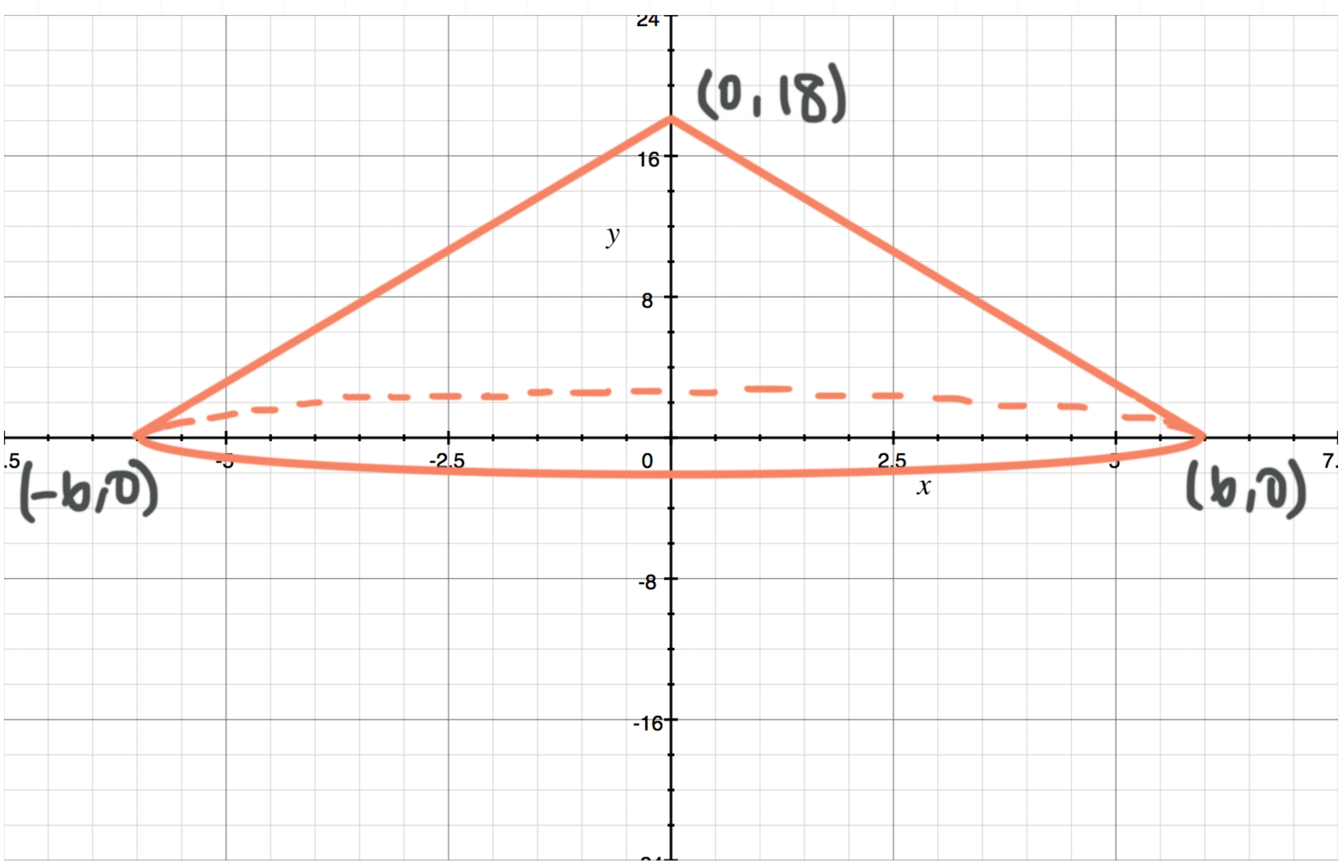
$$F = 2.87 \times 10^{-5} \text{ cm}^3\text{/sec}$$

THEOREM OF PAPPUS

■ 1. Use the Theorem of Pappus to find the exact volume of a right circular cone with radius 6 feet and height 18 feet.

Solution:

The right circular cone drawn with the center of the base at the origin is



The cross section that the Theorem of Pappus uses is the area of a triangle drawn from the vertex of the cone to the center of the base, and then to the edge of the cone. The area of this cross section is

$$A = \frac{1}{2}bh$$

$$A = \frac{1}{2}(6)(18)$$

$$A = 54$$

Two points on the cone are $(0,18)$ and $(6,0)$. Use these points to calculate the slope of the slant height.

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{0 - 18}{6 - 0} = \frac{-18}{6} = -3$$

Use $(0,18)$ and the slope $m = -3$ to write the equation.

$$y = mx + b$$

$$y = -3x + 18$$

$$f(x) = -3x + 18$$

Find the x-value of the centroid of the cross section, $\bar{x}$.

$$\bar{x} = \frac{1}{A}\int_a^b xf(x)\ dx$$

$$\bar{x} = \frac{1}{54}\int_0^6 x(-3x + 18)\ dx$$

$$\bar{x} = \frac{1}{18}\int_0^6 -x^2 + 6x\ dx$$

Integrate, then evaluate over the interval.

$$\bar{x} = \frac{1}{18}\left(-\frac{1}{3}x^3 + 3x^2\right)\Big|_0^6$$

$$\bar{x} = \frac{1}{18}\left(-\frac{1}{3}(6)^3 + 3(6)^2\right) - \frac{1}{18}\left(-\frac{1}{3}(0)^3 + 3(0)^2\right)$$

$$\bar{x} = \frac{1}{18}(-72 + 108)$$

$$\bar{x} = 2$$

Find the distance traveled by the x-value of the centroid.

$$d = 2\pi\bar{x}$$

$$d = 2\pi(2)$$

$$d = 4\pi$$

Then the volume is

$$V = Ad$$

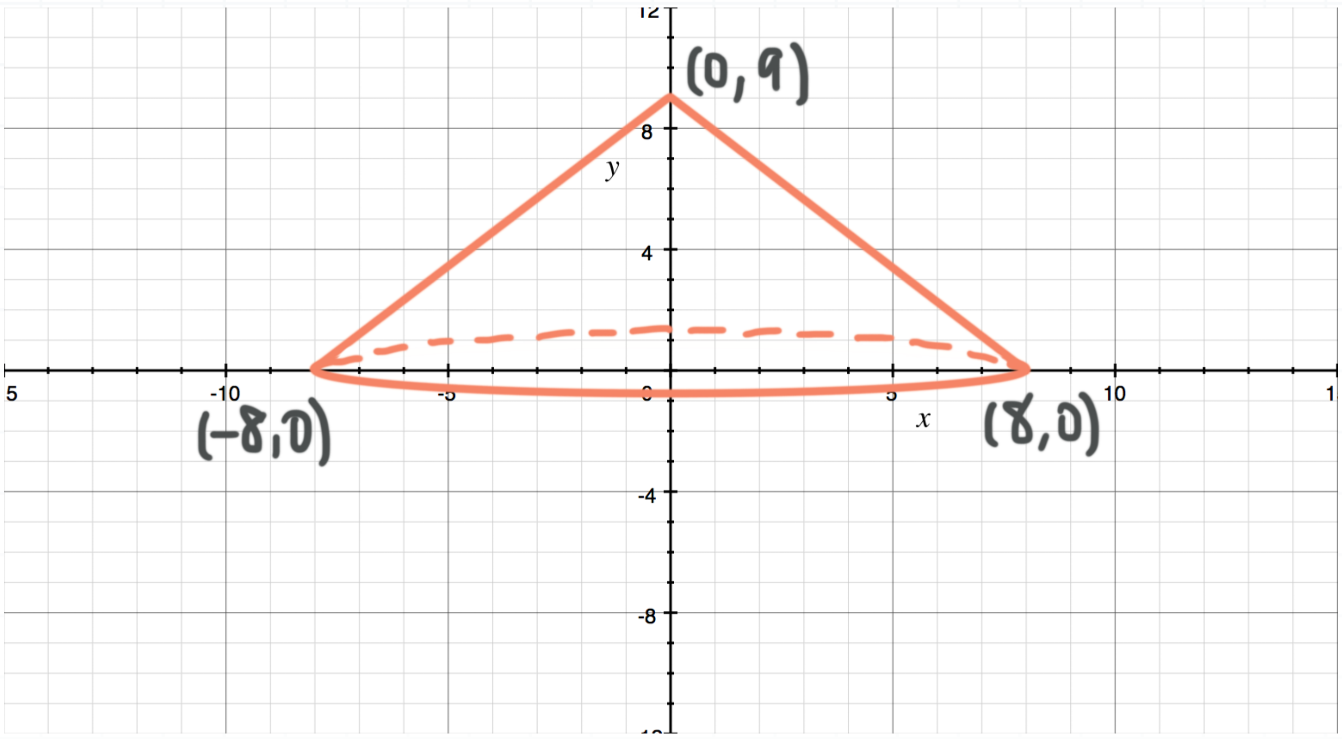
$$V = 54(4\pi)$$

$$V = 216\pi \text{ ft}^3$$

■ 2. Use the Theorem of Pappus to find the exact volume of a right circular cone with radius 8 inches and height 9 inches.

Solution:

The right circular cone drawn with the center of the base at the origin is



The cross section that the Theorem of Pappus uses is the area of a triangle drawn from the vertex of the cone to the center of the base, and then to the edge of the cone. The area of this cross section is

$$A = \frac{1}{2}bh$$

$$A = \frac{1}{2}(8)(9)$$

$$A = 36$$

Two points on the cone are $(0,9)$ and $(8,0)$. Use these points to calculate the slope of the slant height.

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{0-9}{8-0} = -\frac{9}{8}$$

Use $(0{,}9)$ and the slope $m = -9/8$ to write the equation.

$$y = mx + b$$

$$y = -\frac{9}{8}x + 9$$

$$f(x) = -\frac{9}{8}x + 9$$

Find the x-value of the centroid of the cross section, $\bar{x}$.

$$\bar{x} = \frac{1}{A}\int_a^b xf(x)\ dx$$

$$\bar{x} = \frac{1}{36}\int_0^8 x\left(-\frac{9}{8}x + 9\right)\ dx$$

$$\bar{x} = \frac{1}{4}\int_0^8 -\frac{1}{8}x^2 + x\ dx$$

Integrate, then evaluate over the interval.

$$\bar{x} = \frac{1}{4}\left(-\frac{1}{24}x^3 + \frac{1}{2}x^2\right)\Bigg|_0^8$$

$$\bar{x} = \frac{1}{4}\left(-\frac{1}{24}(8)^3 + \frac{1}{2}(8)^2\right) - \frac{1}{4}\left(-\frac{1}{24}(0)^3 + \frac{1}{2}(0)^2\right)$$

$$\bar{x} = \frac{1}{4}\left(-\frac{64}{3} + 32\right)$$

$$\bar{x} = -\frac{16}{3} + 8$$

$$\bar{x} = \frac{8}{3}$$

Find the distance traveled by the x-value of the centroid.

$$d = 2\pi\bar{x}$$

$$d = 2\pi\left(\frac{8}{3}\right)$$

$$d = \frac{16}{3}\pi$$

Then the volume is

$$V = Ad$$

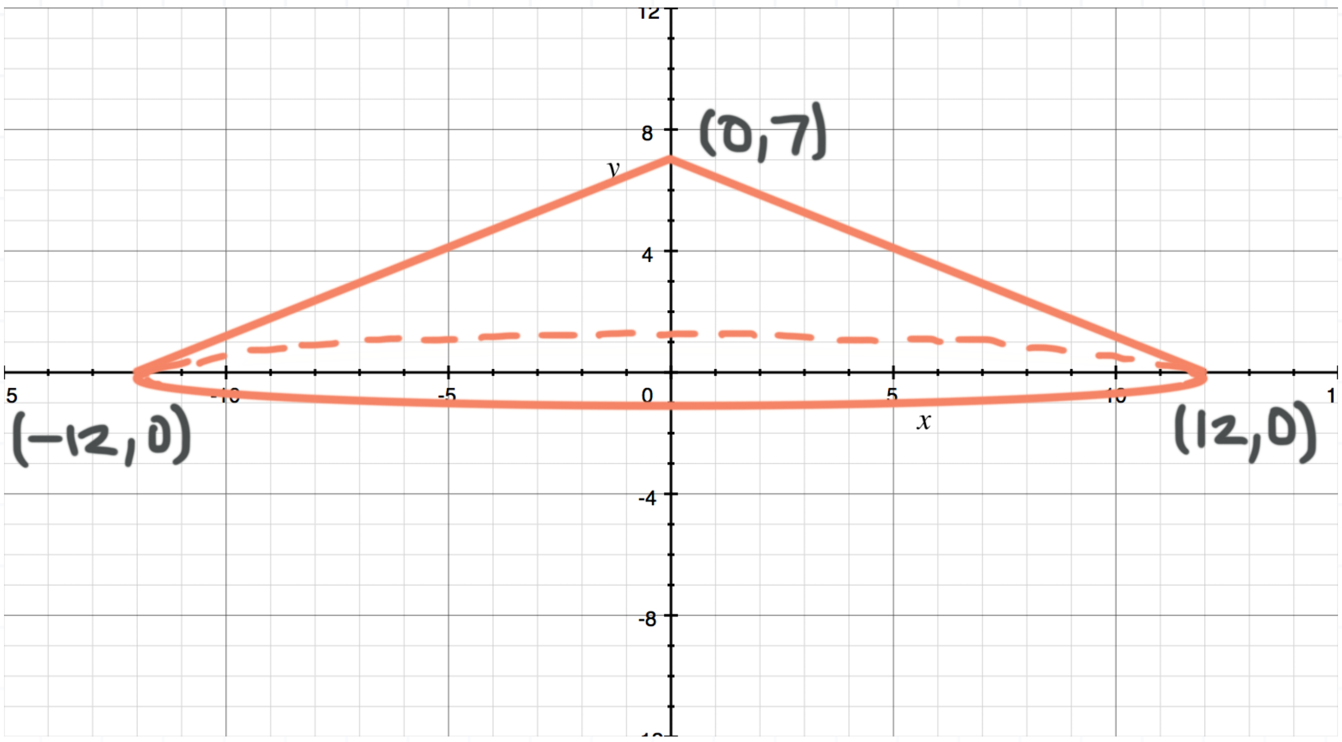
$$V = 36\left(\frac{16}{3}\pi\right)$$

$$V = 192\pi \text{ in}^3$$

3. Use the Theorem of Pappus to find the exact volume of a right circular cone with radius 12 centimeters and height 7 centimeters.

Solution:

The right circular cone drawn with the center of the base at the origin is



The cross section that the Theorem of Pappus uses is the area of a triangle drawn from the vertex of the cone to the center of the base, and then to the edge of the cone. The area of this cross section is

$$A = \frac{1}{2}bh$$

$$A = \frac{1}{2}(12)(7)$$

$$A = 42$$

Two points on the cone are $(0,7)$ and $(12,0)$. Use these points to calculate the slope of the slant height.

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{0 - 7}{12 - 0} = -\frac{7}{12}$$

Use $(0,7)$ and the slope $m = -7/12$ to write the equation.

$$y = mx + b$$

$$y = -\frac{7}{12}x + 7$$

$$f(x) = -\frac{7}{12}x + 7$$

Find the x-value of the centroid of the cross section, $\bar{x}$.

$$\bar{x} = \frac{1}{A}\int_a^b xf(x)\ dx$$

$$\bar{x} = \frac{1}{42}\int_0^{12} x\left(-\frac{7}{12}x + 7\right)\ dx$$

$$\bar{x} = \frac{1}{6}\int_0^{12} -\frac{1}{12}x^2 + x\ dx$$

Integrate, then evaluate over the interval.

$$\bar{x} = \frac{1}{6}\left(-\frac{1}{36}x^3 + \frac{1}{2}x^2\right)\Bigg|_0^{12}$$

$$\bar{x} = \frac{1}{6}\left(-\frac{1}{36}(12)^3 + \frac{1}{2}(12)^2\right) - \frac{1}{6}\left(-\frac{1}{36}(0)^3 + \frac{1}{2}(0)^2\right)$$

$$\bar{x} = \frac{1}{6}(-48 + 72)$$

$$\bar{x} = 4$$

Find the distance traveled by the x-value of the centroid.

$$d = 2\pi\bar{x}$$

$$d = 2\pi(4)$$

$$d = 8\pi$$

Then the volume is

$$V = Ad$$

$$V = 42(8\pi)$$

$$V = 336\pi \text{ cm}^3$$

ELIMINATING THE PARAMETER

1. Eliminate the parameter.

$$x = t^2 - 2$$

$$y = 8 - 3t$$

$$t \geq 0$$

Solution:

Solve $x = t^2 - 2$ for t and substitute the value of t into $y = 8 - 3t$.

$$x = t^2 - 2$$

$$x + 2 = t^2$$

$$t = \sqrt{x+2}$$

Then for $t \geq 0$,

$$y = 8 - 3t$$

$$y = 8 - 3\sqrt{x+2}$$

DERIVATIVES OF PARAMETRIC CURVES

■ 1. Find the derivative of the parametric curve.

$$x = 3 + \sqrt{t}$$

$$y = t^2 - 5t$$

Solution:

Find the derivatives of x and y with respect to t.

$$\frac{dy}{dt} = \frac{d}{dt}(t^2 - 5t) = 2t - 5$$

$$\frac{dx}{dt} = \frac{d}{dt}\left(3 + \sqrt{t}\right) = \frac{1}{2\sqrt{t}}$$

So the derivative of the parametric curve is

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{2t - 5}{\frac{1}{2\sqrt{t}}} = 2\sqrt{t}(2t - 5)$$

■ 2. Find the derivative of the parametric curve.

$$x = 4\cos t$$

$$y = t - 5\sin t$$

Solution:

Find the derivatives of x and y with respect to t.

$$\frac{dy}{dt} = \frac{d}{dt}(t - 5\sin t) = 1 - 5\cos t$$

$$\frac{dx}{dt} = \frac{d}{dt}(4\cos t) = -4\sin t$$

So the derivative of the parametric curve is

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{1 - 5\cos t}{-4\sin t} = \frac{5\cos t - 1}{4\sin t}$$

You could leave the answer this way, or rewrite it.

$$\frac{dy}{dx} = \frac{5\cos t}{4\sin t} - \frac{1}{4\sin t}$$

$$\frac{dy}{dx} = \frac{5}{4}\cot t - \frac{1}{4}\csc t$$

$$\frac{dy}{dx} = \frac{1}{4}(5\cot t - \csc t)$$

■ 3. Find the derivative of the parametric curve.

$$x = 7\cos t$$

$$y = 3t^2 - t$$

Solution:

Find the derivatives of x and y with respect to t.

$$\frac{dy}{dt} = \frac{d}{dt}(3t^2 - t) = 6t - 1$$

$$\frac{dx}{dt} = \frac{d}{dt}(7\cos t) = -7\sin t$$

So the derivative of the parametric curve is

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{6t - 1}{-7\sin t} = \frac{1 - 6t}{7\sin t}$$

■ 4. Find the derivative of the parametric curve.

$$x = e^t - 3t$$

$$y = e^{-t} + 2t$$

Solution:

Find the derivatives of x and y with respect to t.

$$\frac{dy}{dt} = \frac{d}{dt}(e^{-t} + 2t) = -e^{-t} + 2$$

$$\frac{dx}{dt} = \frac{d}{dt}(e^{t} - 3t) = e^{t} - 3$$

So the derivative of the parametric curve is

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{-e^{-t} + 2}{e^{t} - 3} = \frac{2 - e^{-t}}{e^{t} - 3}$$

■ 5. Find the derivative of the parametric curve.

$$x = 7t - 4$$

$$y = 5t^2 + 9t$$

Solution:

Find the derivatives of x and y with respect to t.

$$\frac{dy}{dt} = \frac{d}{dt}(5t^2 + 9t) = 10t + 9$$

$$\frac{dx}{dt} = \frac{d}{dt}(7t - 4) = 7$$

So the derivative of the parametric curve is

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{10t + 9}{7}$$

SECOND DERIVATIVES OF PARAMETRIC CURVES

■ 1. Find the second derivative of the parametric curve.

$$x = 1 - \cos^2 t$$

$$y = \sin t$$

Solution:

Find the derivatives of x and y with respect to t.

$$\frac{dy}{dt} = \frac{d}{dt}(\sin t) = \cos t$$

$$\frac{dx}{dt} = \frac{d}{dt}(1 - \cos^2 t) = -2\cos t \cdot (-\sin t) = 2\cos t \sin t$$

So the derivative of the parametric curve is

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\cos t}{2\cos t \cdot \sin t} = \frac{1}{2\sin t} = \frac{1}{2}\csc t$$

Take the derivative of dy/dx.

$$\frac{d}{dt}\left(\frac{dy}{dx}\right) = \frac{d}{dt}\left(\frac{1}{2}\csc t\right) = -\frac{1}{2}\csc t \cot t$$

Then the second derivative of the parametric curve is

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}} = \frac{-\frac{1}{2}\csc t \cot t}{2\cos t \sin t}$$

$$\frac{d^2y}{dx^2} = -\frac{\frac{1}{\sin t} \cdot \frac{\cos t}{\sin t}}{4\cos t \sin t}$$

$$\frac{d^2y}{dx^2} = -\frac{\cos t}{\sin^2 t} \cdot \frac{1}{4\cos t \sin t}$$

$$\frac{d^2y}{dx^2} = -\frac{1}{4\sin^3 t}$$

$$\frac{d^2y}{dx^2} = -\frac{1}{4}\csc^3 t$$

■ 2. Find the second derivative of the parametric curve.

$$x = e^{-3t}$$

$$y = e^{2t^2}$$

Solution:

Find the derivatives of x and y with respect to t.

$$\frac{dy}{dt} = \frac{d}{dt}(e^{2t^2}) = 4te^{2t^2}$$

$$\frac{dx}{dt} = \frac{d}{dt}(e^{-3t}) = -3e^{-3t}$$

So the derivative of the parametric curve is

$$\frac{dy}{dx}=\frac{\frac{dy}{dt}}{\frac{dx}{dt}}=\frac{4te^{2t^2}}{-3e^{-3t}}=-\frac{4te^{2t^2}}{3e^{-3t}}$$

Take the derivative of dy/dx.

$$\frac{d}{dt}\left(\frac{dy}{dx}\right)=\frac{d}{dt}\left(-\frac{4te^{2t^2}}{3e^{-3t}}\right)$$

$$\frac{d}{dt}\left(\frac{dy}{dx}\right)=-\frac{3e^{-3t}\left(16t^2e^{2t^2}+4e^{2t^2}\right)-4te^{2t^2}\cdot -9e^{-3t}}{9e^{-6t}}$$

$$\frac{d}{dt}\left(\frac{dy}{dx}\right)=-\frac{3e^{-3t}\left(16t^2e^{2t^2}+4e^{2t^2}\right)+36te^{2t^2}e^{-3t}}{9e^{-6t}}$$

$$\frac{d}{dt}\left(\frac{dy}{dx}\right)=-\frac{16t^2e^{2t^2}+4e^{2t^2}+12te^{2t^2}}{3e^{-3t}}$$

Then the second derivative of the parametric curve is

$$\frac{d^2y}{dx^2}=\frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}}=\frac{-\frac{16t^2e^{2t^2}+4e^{2t^2}+12te^{2t^2}}{3e^{-3t}}}{-3e^{-3t}}$$

$$\frac{d^2y}{dx^2}=-\frac{16t^2e^{2t^2}+4e^{2t^2}+12te^{2t^2}}{-9e^{-6t}}$$

$$\frac{d^2y}{dx^2}=-\frac{4e^{2t^2}\left(4t^2+3t+1\right)}{-9e^{-6t}}$$

■ 3. Find the second derivative of the parametric curve.

$$x = t^2 + 2t + 1$$

$$y = 3t + 4$$

Solution:

Find the derivatives of x and y with respect to t.

$$\frac{dy}{dt} = \frac{d}{dt}(3t + 4) = 3$$

$$\frac{dx}{dt} = \frac{d}{dt}(t^2 + 2t + 1) = 2t + 2$$

So the derivative of the parametric curve is

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{3}{2t + 2}$$

Take the derivative of dy/dx.

$$\frac{d}{dt}\left(\frac{dy}{dx}\right) = \frac{d}{dt}\left(\frac{3}{2t + 2}\right) = \frac{(2t + 2)(0) - 3(2)}{(2t + 2)^2} = -\frac{6}{(2t + 2)^2}$$

Then the second derivative of the parametric curve is

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}} = \frac{-\frac{6}{(2t+2)^2}}{2t+2} = -\frac{6}{(2t+2)^3}$$

SKETCHING PARAMETRIC CURVES BY PLOTTING POINTS

1. The graph of the parametric equation on the interval $0 \le t \le 2$ is a segment. What is the Cartesian equation in x and y? Find the left and right endpoints of the segment.

$$x = 2t + 3$$

$$y = 4t + 5$$

Solution:

To find the equation in cartesian coordinates, eliminate the parameter. First, solve one of the equations for t.

$$x = 2t + 3$$

$$x - 3 = 2t$$

$$t = \frac{x-3}{2}$$

Then,

$$y = 4t + 5$$

$$y = 4\left(\frac{x-3}{2}\right) + 5$$

$$y = 2(x-3) + 5$$

$$y = 2x - 6 + 5$$

$$y = 2x - 1$$

The equation in x and y is $y = 2x - 1$.

To find the endpoints, substitute the endpoints of the domain of t into the parametric equation. Plugging in $t = 0$ gives

$$x = 2(0) + 3 = 3$$

$$y = 4(0) + 5 = 5$$

Then the left endpoint is $(x, y) = (3,5)$. Plugging in $t = 2$ gives

$$x = 2(2) + 3 = 7$$

$$y = 4(2) + 5 = 13$$

Then the right endpoint is $(x, y) = (7,13)$.

2. What are the points on the curve for the parameter values $t = 1,\ 2,\ 3,$ and 4?

$$x = t^2 + t$$

$$y = t^2 - t$$

Solution:

To find the points, substitute the values of t into the parametric equation.

For $t = 1$:

$$x(1) = 1^2 + 1 = 2$$

$$y(1) = 1^2 - 1 = 0$$

$$(x, y) = (2,0)$$

For $t = 2$:

$$x(2) = 2^2 + 2 = 6$$

$$y(2) = 2^2 - 2 = 2$$

$$(x, y) = (6,2)$$

For $t = 3$:

$$x(3) = 3^2 + 3 = 12$$

$$y(3) = 3^2 - 3 = 6$$

$$(x, y) = (12,6)$$

For $t = 4$:

$$x(4) = 4^2 + 4 = 20$$

$$y(2) = 4^2 - 4 = 12$$

$$(x, y) = (20,12)$$

3. What are the points on the curve for the parameter values $t=0$, 1, 2, and 3?

$$x=3t^2-5$$

$$y=2t^3+1$$

Solution:

To find the points, substitute the values of t into the parametric equation.

For $t=0$:

$$x(0)=3(0)^2-5=-5$$

$$y(0)=2(0)^3+1=1$$

$$(x,y)=(-5,1)$$

For $t=1$:

$$x(1)=3(1)^2-5=-2$$

$$y(1)=2(1)^3+1=3$$

$$(x,y)=(-2,3)$$

For $t=2$:

$$x(2)=3(2)^2-5=7$$

$$y(2)=2(2)^3+1=17$$

$$(x, y) = (7,17)$$

For $t = 3$:

$$x(3) = 3(3)^2 - 5 = 22$$

$$y(3) = 2(3)^3 + 1 = 55$$

$$(x, y) = (22,55)$$

TANGENT LINES OF PARAMETRIC CURVES

1. Find the equation of the tangent line to the parametric curve at $t = 3$.

$$x = 3t + 5$$

$$y = 7t - 2$$

Solution:

Use the formula of the tangent line as $y - y_1 = m(x - x_1)$ and transform the equation into the form $y = mx + b$.

At $t = 3$, the slope of the parametric equation is

$$m = \frac{dy}{dx} = \frac{\frac{d}{dt}(7t - 2)}{\frac{d}{dt}(3t + 5)} = \frac{7}{3}$$

At $t = 3$, the parametric equation has the values

$$x(3) = 3(3) + 5 = 14$$

$$y(3) = 7(3) - 2 = 19$$

Putting these values together gives the coordinate point $(x, y) = (14,19)$. Plug everything into the equation of the tangent line.

$$y - y_1 = m(x - x_1)$$

$$y - 19 = \frac{7}{3}(x - 14)$$

$$y = \frac{7}{3}x - \frac{98}{3} + \frac{57}{3}$$

$$y = \frac{7}{3}x - \frac{41}{3}$$

■ 2. Find the equation of the tangent line to the parametric curve at $t = 4$.

$$x = 3t^2 - 12$$

$$y = 2t^3 + 6$$

Solution:

Use the formula of the tangent line as $y - y_1 = m(x - x_1)$ and transform the equation into the form $y = mx + b$.

At $t = 4$, the slope of the parametric equation is

$$m = \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\frac{d}{dt}(2t^3 + 6)}{\frac{d}{dt}(3t^2 - 12)} = \frac{6t^2}{6t} = t = 4$$

At $t = 4$, the parametric equation has the values

$$x(4) = 3(4)^2 - 12 = 36$$

$$y(4) = 2(4)^3 + 6 = 134$$

Putting these values together gives the coordinate point $(x, y) = (36,134)$. Plug everything into the equation of the tangent line.

$$y - y_1 = m(x - x_1)$$

$$y - 134 = 4(x - 36)$$

$$y = 4x - 144 + 134$$

$$y = 4x - 10$$

■ 3. Find the equation of the tangent line to the parametric curve at $t = \pi/3$.

$$x = \cos^2 t$$

$$y = \sin^2 t$$

Solution:

Use the formula of the tangent line as $y - y_1 = m(x - x_1)$ and transform the equation into the form $y = mx + b$.

At $t = 4$, the slope of the parametric equation is

$$m = \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\frac{d}{dt}(\sin^2 t)}{\frac{d}{dt}(\cos^2 t)} = \frac{2\sin t\cos t}{-2\cos t\sin t} = -1$$

At $t = 4$, the parametric equation has the values

$$x\left(\frac{\pi}{3}\right)=\cos^2\left(\frac{\pi}{3}\right)=\left(\frac{1}{2}\right)^2=\frac{1}{4}$$

$$y\left(\frac{\pi}{3}\right)=\sin^2\left(\frac{\pi}{3}\right)=\left(\frac{\sqrt{3}}{2}\right)^2=\frac{3}{4}$$

Putting these values together gives the coordinate point $(x,y)=(1/4,3/4)$. Plug everything into the equation of the tangent line.

$$y-y_1=m(x-x_1)$$

$$y-\frac{3}{4}=-1\left(x-\frac{1}{4}\right)$$

$$y=-x+\frac{1}{4}+\frac{3}{4}$$

$$y=-x+1$$

■ 4. Find the equation of the tangent line to the parametric curve at $t=4$.

$$x=t^2+t+3$$

$$y=t^2-3t+2$$

Solution:

Use the formula of the tangent line as $y-y_1=m(x-x_1)$ and transform the equation into the form $y=mx+b$.

At $t = 4$, the slope of the parametric equation is

$$m = \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\frac{d}{dt}(t^2 - 3t + 2)}{\frac{d}{dt}(t^2 + t + 3)} = \frac{2t - 3}{2t + 1} = \frac{2(4) - 3}{2(4) + 1} = \frac{5}{9}$$

At $t = 4$, the parametric equation has the values

$$x(4) = 4^2 + 4 + 3 = 23$$

$$y(4) = 4^2 - 3(4) + 2 = 6$$

Putting these values together gives the coordinate point $(x, y) = (23,6)$. Plug everything into the equation of the tangent line.

$$y - y_1 = m(x - x_1)$$

$$y - 6 = \frac{5}{9}(x - 23)$$

$$y = \frac{5}{9}x - \frac{115}{9} + \frac{54}{9}$$

$$y = \frac{5}{9}x - \frac{61}{9}$$

■ 5. Find the equation of the tangent line to the parametric curve at $t = 9$.

$$x = 3\sqrt{t}$$

$$y = 5t\sqrt{t}$$

Solution:

Use the formula of the tangent line as $y - y_1 = m(x - x_1)$ and transform the equation into the form $y = mx + b$.

At $t = 9$, the slope of the parametric equation is

$$m = \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\frac{d}{dt}\left(5t\sqrt{t}\right)}{\frac{d}{dt}\left(3\sqrt{t}\right)} = \frac{\frac{15}{2}\sqrt{t}}{\frac{3}{2\sqrt{t}}} = \frac{15}{2}\sqrt{t} \cdot \frac{2\sqrt{t}}{3} = 5t = 5(9) = 45$$

At $t = 9$, the parametric equation has the values

$$x(9) = 3\sqrt{9} = 3 \cdot 3 = 9$$

$$y(9) = 5 \cdot 9\sqrt{9} = 45 \cdot 3 = 135$$

Putting these values together gives the coordinate point $(x, y) = (9,135)$. Plug everything into the equation of the tangent line.

$$y - y_1 = m(x - x_1)$$

$$y - 135 = 45(x - 9)$$

$$y = 45x - 405 + 135$$

$$y = 45x - 270$$

AREA UNDER A PARAMETRIC CURVE

1. Find the area under the parametric curve.

$$x(t) = 3t^2$$

$$y(t) = t + 2$$

$$0 \le t \le 3$$

Solution:

Find the area under the curve on $a \le t \le b$ using the integral formula.

$$A = \int_a^b y(t)x'(t)\ dt$$

$$A = \int_0^3 (t+2)(6t)\ dt$$

$$A = \int_0^3 6t^2 + 12t\ dt$$

Integrate and evaluate over the interval.

$$A = \frac{6t^3}{3} + \frac{12t^2}{2}\Big|_0^3$$

$$A = 2t^3 + 6t^2 \Big|_0^3$$

$$A = \left(2(3)^3 + 6(3)^2\right) - \left(2(0)^3 + 6(0)^2\right)$$

$$A = 54 + 54$$

$$A = 108$$

2. Find the area under the parametric curve.

$$x(t) = 5t^2 - 3t + 4$$

$$y(t) = 6t - 1$$

$$0 \le t \le 5$$

Solution:

Find the area under the curve on $a \le t \le b$ using the integral formula.

$$A = \int_a^b y(t)x'(t)\ dt$$

$$A = \int_0^5 (6t - 1)(10t - 3)\ dt$$

$$A = \int_0^5 60t^2 - 28t + 3\ dt$$

Integrate and evaluate over the interval.

$$A = \frac{60t^3}{3} - \frac{28t^2}{2} + 3t\Big|_0^5$$

$$A = 20t^3 - 14t^2 + 3t\Big|_0^5$$

$$A = \left(20(5)^3 - 14(5)^2 + 3(5)\right) - \left(20(0)^3 - 14(0)^2 + 3(0)\right)$$

$$A = 2{,}500 - 350 + 15$$

$$A = 2{,}165$$

■ 3. Find the area under the parametric curve.

$$x(t) = t + \sin t$$

$$y(t) = 4 + \cos t$$

$$0 \le t \le 2\pi$$

Solution:

Find the area under the curve on $a \le t \le b$ using the integral formula.

$$A = \int_a^b y(t)x'(t)\ dt$$

$$A = \int_0^{2\pi} (4 + \cos t)(1 + \cos t)\ dt$$

$$A = \int_0^{2\pi} 4 + 5\cos t + \cos^2 t\ dt$$

Use the trig identity to make a substitution.

$$\cos^2 t = \frac{1}{2} + \frac{1}{2}\cos(2t)$$

Substitute.

$$A = \int_0^{2\pi} 4 + 5\cos t + \frac{1}{2} + \frac{1}{2}\cos(2t)\ dt$$

$$A = \int_0^{2\pi} \frac{9}{2} + 5\cos t + \frac{1}{2}\cos(2t)\ dt$$

Integrate and evaluate over the interval.

$$A = \frac{9}{2}t + 5\sin t + \frac{1}{4}\sin(2t)\Big|_0^{2\pi}$$

$$A = \frac{9}{2}(2\pi) + 5\sin(2\pi) + \frac{1}{4}\sin(2(2\pi)) - \left(\frac{9}{2}(0) + 5\sin(0) + \frac{1}{4}\sin(2(0))\right)$$

$$A = 9\pi + 5(0) + \frac{1}{4}(0)$$

$$A = 9\pi$$

4. Find the area under the parametric curve.

$$x(t) = t^2 + 5t - 8$$

$$y(t) = t^2 + 4t + 2$$

$$0 \le t \le 2$$

Solution:

Find the area under the curve on $a \le t \le b$ using the integral formula.

$$A = \int_a^b y(t)x'(t)\ dt$$

$$A = \int_0^2 (t^2 + 4t + 2)(2t + 5)\ dt$$

$$A = \int_0^2 2t^3 + 13t^2 + 24t + 10\ dt$$

Integrate and evaluate over the interval.

$$A = \frac{2t^4}{4} + \frac{13t^3}{3} + \frac{24t^2}{2} - 10t\Big|_0^2$$

$$A = \frac{t^4}{2} + \frac{13t^3}{3} + 12t^2 + 10t\Big|_0^2$$

$$A = \left(\frac{2^4}{2} + \frac{13(2)^3}{3} + 12(2)^2 + 10(2)\right) - \left(\frac{0^4}{2} + \frac{13(0)^3}{3} + 12(0)^2 + 10(0)\right)$$

$$A = 8 + \frac{104}{3} + 48 + 20$$

$$A = \frac{104}{3} + 76$$

$$A = \frac{104}{3} + \frac{228}{3}$$

$$A = \frac{332}{3}$$

AREA UNDER ONE ARC OR LOOP

1. Find the area in one loop of the parametric curve.

$$x(\theta) = 2\cos(2\theta)$$

$$y(\theta) = 4 + \sin(2\theta)$$

$$0 \le \theta \le \pi$$

Solution:

Plug the parametric equation and the given interval into the integral formula for the area under one arc or in one loop of the parametric curve.

$$A = \int_a^b y(\theta) \cdot x'(t)\ d\theta$$

$$A = \int_0^\pi (4 + \sin(2\theta))(-4\sin(2\theta))\ d\theta$$

$$A = \int_0^\pi -16\sin(2\theta) - 4\sin^2(2\theta)\ d\theta$$

Use the reduction formula

$$\sin^2(2\theta) = \frac{1}{2}(1 - \cos(4\theta)) = \frac{1}{2} - \frac{1}{2}\cos(4\theta)$$

Substitute that into the integral.

$$A=\int_0^{\pi}-16\sin(2\theta)-4\left(\frac{1}{2}-\frac{1}{2}\cos(4\theta)\right)\ d\theta$$

$$A=\int_0^{\pi}-16\sin(2\theta)-2+2\cos(4\theta)\ d\theta$$

Integrate, then evaluate over the interval.

$$A=-16\left(-\frac{\cos(2\theta)}{2}\right)-2\theta+2\left(\frac{\sin(4\theta)}{4}\right)\Bigg|_0^{\pi}$$

$$A=8\cos(2\theta)-2\theta+\frac{1}{2}\sin(4\theta)\Bigg|_0^{\pi}$$

$$A=8\cos(2\pi)-2\pi+\frac{1}{2}\sin(4\pi)-\left(8\cos(2(0))-2(0)+\frac{1}{2}\sin(4(0))\right)$$

$$A=8(1)-2\pi+\frac{1}{2}(0)-8(1)+0-\frac{1}{2}(0)$$

$$A=8-2\pi-8$$

$$A=-2\pi$$

$$A=\left|-2\pi\right|$$

$$A=2\pi$$

■ 2. Find the area in one loop of the parametric curve.

$$x(\theta)=2\sin\theta$$

$$y(\theta) = 5 + \cos\theta$$

$$0 \le \theta \le 2\pi$$

Solution:

Plug the parametric equation and the given interval into the integral formula for the area under one arc or in one loop of the parametric curve.

$$A = \int_a^b y(\theta) \cdot x'(t)\ d\theta$$

$$A = \int_0^{2\pi} (5 + \cos\theta)(2\cos\theta)\ d\theta$$

$$A = \int_0^{2\pi} 10\cos\theta + 2\cos^2\theta\ d\theta$$

Use the reduction formula

$$\cos^2\theta = \frac{1}{2}(1 + \cos(2\theta)) = \frac{1}{2} + \frac{1}{2}\cos(2\theta)$$

Substitute that into the integral.

$$A = \int_0^{2\pi} 10\cos\theta + 2\left(\frac{1}{2} + \frac{1}{2}\cos(2\theta)\right)\ d\theta$$

$$A = \int_0^{2\pi} 10\cos\theta + 1 + \cos(2\theta)\ d\theta$$

Integrate, then evaluate over the interval.

$$A = 10\sin\theta + \theta + \frac{\sin(2\theta)}{2}\Big|_0^{2\pi}$$

$$A = 10\sin(2\pi) + 2\pi + \frac{\sin(2(2\pi))}{2} - \left(10\sin(0) + 0 + \frac{\sin(2(0))}{2}\right)$$

$$A = 10(0) + 2\pi + \frac{0}{2} - 10(0) - 0 - \frac{0}{2}$$

$$A = 2\pi$$

3. Find the area in one loop of the parametric curve.

$$x(\theta) = 8 + 3\cos\theta$$

$$y(\theta) = 9 - 2\sin\theta$$

$$0 \le \theta \le 2\pi$$

Solution:

Plug the parametric equation and the given interval into the integral formula for the area under one arc or in one loop of the parametric curve.

$$A = \int_a^b y(\theta) \cdot x'(t)\ d\theta$$

$$A = \int_0^{2\pi} (9 - 2\sin\theta)(-3\sin\theta)\ d\theta$$

$$A = \int_0^{2\pi} -27\sin\theta + 6\sin^2\theta\ d\theta$$

Use the reduction formula

$$\sin^2(2\theta) = \frac{1}{2}(1 - \cos(4\theta)) = \frac{1}{2} - \frac{1}{2}\cos(4\theta)$$

Substitute that into the integral.

$$A = \int_0^{2\pi} -27\sin\theta + 6\left(\frac{1}{2} - \frac{1}{2}\cos(2\theta)\right)\ d\theta$$

$$A = \int_0^{2\pi} -27\sin\theta + 3 - 3\cos(2\theta)\ d\theta$$

Integrate, then evaluate over the interval.

$$A = 27\cos\theta + 3\theta - \frac{3}{2}\sin(2\theta)\Big|_0^{2\pi}$$

$$A = 27\cos(2\pi) + 3(2\pi) - \frac{3}{2}\sin(2(2\pi)) - \left(27\cos(0) + 3(0) - \frac{3}{2}\sin(2(0))\right)$$

$$A = 27(1) + 6\pi - \frac{3}{2}(0) - 27(1) - 0 + \frac{3}{2}(0)$$

$$A = 27 + 6\pi - 27$$

$$A = 6\pi$$

4. Find the area in one loop of the parametric curve.

$$x(\theta) = 12 + 6\sin\theta$$

$$y(\theta) = 12 - 6\cos\theta$$

$$0 \le \theta \le 2\pi$$

Solution:

Plug the parametric equation and the given interval into the integral formula for the area under one arc or in one loop of the parametric curve.

$$A = \int_a^b y(\theta) \cdot x'(t)\ d\theta$$

$$A = \int_0^{2\pi} (12 - 6\cos\theta)(6\cos\theta)\ d\theta$$

$$A = \int_0^{2\pi} 72\cos\theta - 36\cos^2\theta\ d\theta$$

Use the reduction formula

$$\cos^2\theta = \frac{1}{2}(1 + \cos(2\theta)) = \frac{1}{2} + \frac{1}{2}\cos(2\theta)$$

Substitute that into the integral.

$$A = \int_0^{2\pi} 72\cos\theta - 36\left(\frac{1}{2} + \frac{1}{2}\cos(2\theta)\right)\ d\theta$$

$$A = \int_0^{2\pi} 72\cos\theta - 18 - 18\cos(2\theta)\ d\theta$$

Integrate, then evaluate over the interval.

$$A = 72\sin\theta - 18\theta - 9\sin(2\theta)\Big|_0^{2\pi}$$

$$A = 72\sin(2\pi) - 18(2\pi) - 9\sin(2(2\pi)) - \big(72\sin(0) - 18(0) - 9\sin(2(0))\big)$$

$$A = 72(0) - 36\pi - 9(0) - 72(0) + 0 + 9(0)$$

$$A = -36\pi$$

$$A = \left|-36\pi\right|$$

$$A = 36\pi$$

■ 5. Find the area in one loop of the parametric curve.

$$x(\theta) = 15 - 5\cos\theta$$

$$y(\theta) = 5 + 15\sin\theta$$

$$0 \le \theta \le 2\pi$$

Solution:

Plug the parametric equation and the given interval into the integral formula for the area under one arc or in one loop of the parametric curve.

$$A = \int_a^b y(\theta) \cdot x'(t)\ d\theta$$

$$A = \int_0^{2\pi} (5 + 15 \sin\theta)(5 \sin\theta)\ d\theta$$

$$A = \int_0^{2\pi} 25 \sin\theta + 75 \sin^2\theta\ d\theta$$

Use the reduction formula

$$\sin^2(2\theta) = \frac{1}{2}(1 - \cos(4\theta)) = \frac{1}{2} - \frac{1}{2}\cos(4\theta)$$

Substitute that into the integral.

$$A = \int_0^{2\pi} 25 \sin\theta + 75\left(\frac{1}{2} - \frac{1}{2}\cos(2\theta)\right)\ d\theta$$

$$A = \int_0^{2\pi} 25 \sin\theta + \frac{75}{2} - \frac{75}{2}\cos(2\theta)\ d\theta$$

Integrate, then evaluate over the interval.

$$A = -25\cos\theta + \frac{75}{2}\theta - \frac{75}{4}\sin(2\theta)\Big|_0^{2\pi}$$

$$A = -25\cos(2\pi) + \frac{75}{2}(2\pi) - \frac{75}{4}\sin(2(2\pi)) - \left(-25\cos(0) + \frac{75}{2}(0) - \frac{75}{4}\sin(2(0))\right)$$

$$A = -25(1) + 75\pi - \frac{75}{4}(0) + 25(1) - 0 + \frac{75}{4}(0)$$

$$A = -25 + 75\pi + 25$$

$$A = 75\pi$$

ARC LENGTH OF PARAMETRIC CURVES

1. Find the length of the parametric curve on the given interval.

$$x(t) = 7 - 3t$$

$$y(t) = 5 + 8t$$

$$-1 \le t \le 4$$

Solution:

Plug the derivatives of $x(t)$ and $y(t)$ and the given interval into the integral formula for arc length.

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\ dt$$

$$L = \int_{-1}^{4} \sqrt{(-3)^2 + (8)^2}\ dt$$

$$L = \int_{-1}^{4} \sqrt{9 + 64}\ dt$$

$$L = \int_{-1}^{4} \sqrt{73}\ dt$$

Integrate, then evaluate over the interval.

$$L = \sqrt{73}t\Big|_{-1}^{4}$$

$$L = \sqrt{73}(4) - \sqrt{73}(-1)$$

$$L = 4\sqrt{73} + \sqrt{73}$$

$$L = 5\sqrt{73}$$

■ 2. Find the length of the parametric curve on the given interval.

$$x(t) = \cos^3 t$$

$$y(t) = \sin^3 t$$

$$0 \le t \le \frac{3\pi}{4}$$

Solution:

Plug the derivatives of $x(t)$ and $y(t)$ and the given interval into the integral formula for arc length.

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\ dt$$

$$L = \int_0^{\frac{3\pi}{4}} \sqrt{\left(3\cos^2 t(-\sin t)\right)^2 + \left(3\sin^2 t\cos t\right)^2}\ dt$$

$$L=\int_{0}^{\frac{3\pi}{4}}\sqrt{9\cos^4 t\sin^2 t+9\sin^4 t\cos^2 t}\ dt$$

$$L=\int_{0}^{\frac{3\pi}{4}}\sqrt{9\sin^2 t\cos^2 t(\cos^2 t+\sin^2 t)}\ dt$$

$$L=\int_{0}^{\frac{3\pi}{4}}\sqrt{9\sin^2 t\cos^2 t(1)}\ dt$$

$$L=\int_{0}^{\frac{3\pi}{4}}\sqrt{9\sin^2 t\cos^2 t}\ dt$$

$$L=\int_{0}^{\frac{3\pi}{4}}3\sin t\cos t\ dt$$

Use u-substitution.

$$u=\sin t$$

$$du=\cos t\ dt\text{, so }dt=\frac{du}{\cos t}$$

Substitute.

$$L=\int_{t=0}^{t=\frac{3\pi}{4}}3u\cos t\left(\frac{du}{\cos t}\right)$$

$$L=\int_{t=0}^{t=\frac{3\pi}{4}}3u\ du$$

Integrate, back-substitute, then evaluate over the interval.

$$L=\frac{3}{2}u^2\Big|_{t=0}^{t=\frac{3\pi}{4}}$$

$$L=\frac{3}{2}\sin^2 t\Big|_0^{\frac{3\pi}{4}}$$

$$L=\frac{3}{2}\sin^2\frac{3\pi}{4}-\frac{3}{2}\sin^2(0)$$

$$L=\frac{3}{2}\left(\frac{\sqrt{2}}{2}\right)^2-\frac{3}{2}(0)^2$$

$$L=\frac{3}{2}\left(\frac{2}{4}\right)$$

$$L=\frac{3}{4}$$

■ 3. Find the length of the parametric curve on the given interval.

$$x(t)=5t-5\sin t$$

$$y(t)=-5\cos t$$

$$0\le t\le 2\pi$$

Solution:

Plug the derivatives of $x(t)$ and $y(t)$ and the given interval into the integral formula for arc length.

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\ dt$$

$$L = \int_0^{2\pi} \sqrt{(5 - 5\cos t)^2 + (5\sin t)^2}\ dt$$

$$L = \int_0^{2\pi} \sqrt{25 - 50\cos t + 25\cos^2 t + 25\sin^2 t}\ dt$$

$$L = \int_0^{2\pi} \sqrt{25 - 50\cos t + 25(\cos^2 t + \sin^2 t)}\ dt$$

$$L = \int_0^{2\pi} \sqrt{25 - 50\cos t + 25(1)}\ dt$$

$$L = \int_0^{2\pi} \sqrt{50 - 50\cos t}\ dt$$

$$L = \int_0^{2\pi} \sqrt{50(1 - \cos t)}\ dt$$

$$L = \int_0^{2\pi} \sqrt{100 \cdot \frac{1}{2}(1 - \cos t)}\ dt$$

Use the reduction formula

$$\sin^2\theta = \frac{1}{2}(1 - \cos(2\theta)) = \frac{1}{2} - \frac{1}{2}\cos(2\theta)$$

Substitute.

$$L = \int_0^{2\pi} \sqrt{100 \sin^2 \frac{t}{2}} \ dt$$

$$L = \int_0^{2\pi} 10 \sin \frac{t}{2} \ dt$$

Integrate, then evaluate over the interval.

$$L = -20 \cos \frac{t}{2} \Big|_0^{2\pi}$$

$$L = -20 \cos \frac{2\pi}{2} - \left(-20 \cos \frac{0}{2} \right)$$

$$L = -20 \cos \pi - (-20 \cos 0)$$

$$L = -20(-1) + 20(1)$$

$$L = 20 + 20$$

$$L = 40$$

■ 4. Find the length of the parametric curve on the given interval.

$$x(t) = \cos t$$

$$y(t) = t + \sin t$$

$$0 \le t \le \pi$$

Solution:

Plug the derivatives of $x(t)$ and $y(t)$ and the given interval into the integral formula for arc length.

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\ dt$$

$$L = \int_0^\pi \sqrt{(-\sin t)^2 + (1 + \cos t)^2}\ dt$$

$$L = \int_0^\pi \sqrt{\sin^2 t + 1 + 2\cos t + \cos^2 t}\ dt$$

$$L = \int_0^\pi \sqrt{1 + 2\cos t + (\sin^2 t + \cos^2 t)}\ dt$$

$$L = \int_0^\pi \sqrt{1 + 2\cos t + (1)}\ dt$$

$$L = \int_0^\pi \sqrt{2 + 2\cos t}\ dt$$

$$L = \int_0^\pi \sqrt{2(1 + \cos t)}\ dt$$

$$L = \int_0^\pi \sqrt{4 \cdot \frac{1}{2}(1 + \cos t)}\ dt$$

Use the reduction formula

$$\cos^2\theta = \frac{1}{2}(1+\cos(2\theta)) = \frac{1}{2} + \frac{1}{2}\cos(2\theta)$$

Substitute.

$$L = \int_0^{\pi} \sqrt{4\cos^2\frac{t}{2}}\ dt$$

$$L = \int_0^{\pi} 2\cos\frac{t}{2}\ dt$$

Integrate, then evaluate over the interval.

$$L = 4\sin\frac{t}{2}\Big|_0^{\pi}$$

$$L = 4\sin\frac{\pi}{2} - 4\sin\frac{0}{2}$$

$$L = 4(1) - 4(0)$$

$$L = 4$$

SURFACE AREA OF REVOLUTION, HORIZONTAL AXIS

1. Find the surface area of revolution of the parametric curve on the interval $0 \le t \le 3$, rotated about the x-axis.

$$x = \frac{5}{3}t$$

$$y = 4t + 6$$

Solution:

Plug the derivatives and the interval into the integral formula formula for the surface area of revolution for a parametric curve about the x-axis.

$$S = \int_a^b 2\pi y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\ dt$$

$$S = \int_0^3 2\pi(4t+6)\sqrt{\left(\frac{5}{3}\right)^2 + (4)^2}\ dt$$

$$S = 4\pi \int_0^3 (2t+3)\sqrt{\frac{25}{9} + 16}\ dt$$

$$S = 4\pi \int_0^3 (2t+3)\sqrt{\frac{25}{9} + \frac{144}{9}}\ dt$$

$$S = 4\pi \int_0^3 (2t+3)\sqrt{\frac{169}{9}}\ dt$$

$$S = 4\pi \int_0^3 (2t+3)\frac{13}{3}\ dt$$

$$S = \frac{52\pi}{3}\int_0^3 2t+3\ dt$$

Integrate, then evaluate over the interval.

$$S = \frac{52\pi}{3}(t^2+3t)\Big|_0^3$$

$$S = \frac{52\pi}{3}(3^2+3(3)) - \frac{52\pi}{3}(0^2+3(0))$$

$$S = \frac{52\pi}{3}(18) - \frac{52\pi}{3}(0)$$

$$S = 312\pi$$

■ 2. Find the surface area of revolution of the parametric curve on the interval $0 \le t \le 2\pi$, rotated about the x-axis.

$$x = 3 + \cos t$$

$$y = 4 + \sin t$$

Solution:

Plug the derivatives and the interval into the integral formula formula for the surface area of revolution for a parametric curve about the x-axis.

$$S=\int_a^b 2\pi y\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\ dt$$

$$S=\int_0^{2\pi} 2\pi(4+\sin t)\sqrt{(-\sin t)^2+(\cos t)^2}\ dt$$

$$S=\int_0^{2\pi} 2\pi(4+\sin t)\sqrt{\sin^2 t+\cos^2 t}\ dt$$

$$S=2\pi\int_0^{2\pi} (4+\sin t)\sqrt{1}\ dt$$

$$S=2\pi\int_0^{2\pi} 4+\sin t\ dt$$

Integrate, then evaluate over the interval.

$$S=2\pi(4t-\cos t)\Big|_0^{2\pi}$$

$$S=2\pi(4(2\pi)-\cos(2\pi))-2\pi(4(0)-\cos(0))$$

$$S=2\pi(8\pi-1)-2\pi(0-1)$$

$$S=16\pi^2-2\pi+2\pi$$

$$S=16\pi^2$$

3. Find the surface area of revolution of the parametric curve on the interval $0 \le t \le 2\pi$, rotated about the x-axis.

$$x = 7 - 3\sin t$$

$$y = 6 + 3\cos t$$

Solution:

Plug the derivatives and the interval into the integral formula formula for the surface area of revolution for a parametric curve about the x-axis.

$$S = \int_a^b 2\pi y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\ dt$$

$$S = \int_0^{2\pi} 2\pi(6 + 3\cos t)\sqrt{(-3\cos t)^2 + (-3\sin t)^2}\ dt$$

$$S = 6\pi \int_0^{2\pi} (2 + \cos t)\sqrt{9\cos^2 t + 9\sin^2 t}\ dt$$

$$S = 6\pi \int_0^{2\pi} (2 + \cos t)\sqrt{9(\cos^2 t + \sin^2 t)}\ dt$$

$$S = 6\pi \int_0^{2\pi} (2 + \cos t)\sqrt{9(1)}\ dt$$

$$S = 6\pi \int_0^{2\pi} (2 + \cos t)(3)\ dt$$

$$S = 18\pi \int_0^{2\pi} 2 + \cos t\ dt$$

Integrate, then evaluate over the interval.

$$S = 18\pi(2t + \sin t)\Big|_0^{2\pi}$$

$$S = 18\pi(2(2\pi) + \sin(2\pi)) - 18\pi(2(0) + \sin(0))$$

$$S = 18\pi(4\pi + 0) - 18\pi(0 + 0)$$

$$S = 18\pi(4\pi)$$

$$S = 72\pi^2$$

■ 4. Find the surface area of revolution of the parametric curve on the interval $0 \le t \le \pi$, rotated about the x-axis.

$$x = 5 - \cos(2t)$$

$$y = 3 + \sin(2t)$$

Solution:

Plug the derivatives and the interval into the integral formula formula for the surface area of revolution for a parametric curve about the x-axis.

$$S = \int_a^b 2\pi y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \ dt$$

$$S = \int_0^\pi 2\pi(3 + \sin(2t))\sqrt{(2\sin(2t))^2 + (2\cos(2t))^2} \ dt$$

$$S = 2\pi \int_0^\pi (3 + \sin(2t))\sqrt{4\sin^2(2t) + 4\cos^2(2t)} \ dt$$

$$S = 2\pi \int_0^\pi (3 + \sin(2t))\sqrt{4(\sin^2(2t) + \cos^2(2t))} \ dt$$

$$S = 2\pi \int_0^\pi (3 + \sin(2t))\sqrt{4(1)} \ dt$$

$$S = 4\pi \int_0^\pi 3 + \sin(2t) \ dt$$

Integrate, then evaluate over the interval.

$$S = 4\pi\left(3t - \frac{1}{2}\cos(2t)\right)\Big|_0^\pi$$

$$S = 4\pi\left(3\pi - \frac{1}{2}\cos(2\pi)\right) - 4\pi\left(3(0) - \frac{1}{2}\cos(2(0))\right)$$

$$S = 4\pi\left(3\pi - \frac{1}{2}(1)\right) - 4\pi\left(0 - \frac{1}{2}(1)\right)$$

$$S = 12\pi^2 - 2\pi + 2\pi$$

$$S = 12\pi^2$$

SURFACE AREA OF REVOLUTION, VERTICAL AXIS

1. Find the surface area of revolution of the parametric curve on the interval $0 \le t \le \pi/3$, rotated about the y-axis.

$$x = 8 + \sin(6t)$$

$$y = 7 - \cos(6t)$$

Solution:

Plug the derivatives and the interval into the integral formula formula for the surface area of revolution for a parametric curve about the y-axis.

$$S = \int_a^b 2\pi x \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\ dt$$

$$S = \int_0^{\frac{\pi}{3}} 2\pi(8 + \sin(6t))\sqrt{(6\cos(6t))^2 + (6\sin(6t))^2}\ dt$$

$$S = 2\pi \int_0^{\frac{\pi}{3}} (8 + \sin(6t))\sqrt{36\cos^2(6t) + 36\sin^2(6t)}\ dt$$

$$S = 2\pi \int_0^{\frac{\pi}{3}} (8 + \sin(6t))\sqrt{36(\cos^2(6t) + \sin^2(6t))}\ dt$$

$$S = 2\pi \int_0^{\frac{\pi}{3}} (8 + \sin(6t))\sqrt{36(1)}\ dt$$

$$S = 12\pi \int_0^{\frac{\pi}{3}} 8 + \sin(6t)\ dt$$

Integrate, then evaluate over the interval.

$$S = 12\pi \left(8t - \frac{1}{6}\cos(6t) \right) \Big|_0^{\frac{\pi}{3}}$$

$$S = 12\pi \left(8 \cdot \frac{\pi}{3} - \frac{1}{6}\cos\left(6 \cdot \frac{\pi}{3}\right)\right) - 12\pi \left(8(0) - \frac{1}{6}\cos(6(0))\right)$$

$$S = 12\pi \left(\frac{8\pi}{3} - \frac{1}{6}\cos(2\pi)\right) - 12\pi \left(0 - \frac{1}{6}\cos(0)\right)$$

$$S = 12\pi \left(\frac{8\pi}{3} - \frac{1}{6}(1)\right) - 12\pi \left(0 - \frac{1}{6}(1)\right)$$

$$S = 32\pi^2 - 2\pi + 2\pi$$

$$S = 32\pi^2$$

■ 2. Find the surface area of revolution of the parametric curve on the interval $0 \le t \le 2\pi$, rotated about the y-axis.

$$x = 5 + 4\sin(t)$$

$$y = 5 + 4\cos(t)$$

Solution:

Plug the derivatives and the interval into the integral formula formula for the surface area of revolution for a parametric curve about the y-axis.

$$S=\int_a^b 2\pi x\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\ dt$$

$$S=\int_0^{2\pi} 2\pi(5+4\sin t)\sqrt{(4\cos t)^2+(-4\sin t)^2}\ dt$$

$$S=2\pi\int_0^{2\pi} (5+4\sin t)\sqrt{16\cos^2 t+16\sin^2 t}\ dt$$

$$S=2\pi\int_0^{2\pi} (5+4\sin t)\sqrt{16(\cos^2 t+16\sin^2 t)}\ dt$$

$$S=2\pi\int_0^{2\pi} (5+4\sin t)\sqrt{16(1)}\ dt$$

$$S=8\pi\int_0^{2\pi} 5+4\sin t\ dt$$

Integrate, then evaluate over the interval.

$$S=8\pi(5t-4\cos t)\Big|_0^{2\pi}$$

$$S=8\pi(5(2\pi)-4\cos(2\pi))-8\pi(5(0)-4\cos(0))$$

$$S=8\pi(10\pi-4(1))-8\pi(0-4(1))$$

$$S=80\pi^2-32\pi+32\pi$$

$$S = 80\pi^2$$

■ 3. Find the surface area of revolution of the parametric curve on the interval $0 \le t \le 2\pi$, rotated about the y-axis.

$$x = 12 - \sin t$$

$$y = 2 + \cos t$$

Solution:

Plug the derivatives and the interval into the integral formula formula for the surface area of revolution for a parametric curve about the y-axis.

$$S = \int_a^b 2\pi x \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\ dt$$

$$S = \int_0^{2\pi} 2\pi(12 - \sin t)\sqrt{(-\cos t)^2 + (-\sin t)^2}\ dt$$

$$S = \int_0^{2\pi} 2\pi(12 - \sin t)\sqrt{\cos^2 t + \sin^2 t}\ dt$$

$$S = 2\pi \int_0^{2\pi} (12 - \sin t)\sqrt{1}\ dt$$

$$S = 2\pi \int_0^{2\pi} 12 - \sin t\ dt$$

Integrate, then evaluate over the interval.

$$S = 2\pi(12t + \cos t)\Big|_0^{2\pi}$$

$$S = 2\pi(12(2\pi) + \cos(2\pi)) - 2\pi(12(0) + \cos(0))$$

$$S = 2\pi(24\pi + 1) - 2\pi(0 + 1)$$

$$S = 48\pi^2 + 2\pi - 2\pi$$

$$S = 48\pi^2$$

■ 4. Find the surface area of revolution of the parametric curve on the interval $0 \le t \le \pi$, rotated about the y-axis.

$$x = 4 - 3\sin(2t)$$

$$y = 4 - 3\cos(2t)$$

Solution:

Plug the derivatives and the interval into the integral formula formula for the surface area of revolution for a parametric curve about the y-axis.

$$S = \int_a^b 2\pi x\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\ dt$$

$$S = \int_0^{\pi} 2\pi(4 - 3\sin(2t))\sqrt{\left(-6\cos(2t)\right)^2 + \left(6\sin(2t)\right)^2}\ dt$$

$$S = 2\pi\int_0^{\pi} (4 - 3\sin(2t))\sqrt{36\cos^2(2t) + 36\sin^2(2t)}\ dt$$

$$S = 2\pi\int_0^{\pi} (4 - 3\sin(2t))\sqrt{36(\cos^2(2t) + \sin^2(2t))}\ dt$$

$$S = 2\pi\int_0^{\pi} (4 - 3\sin(2t))\sqrt{36(1)}\ dt$$

$$S = 12\pi\int_0^{\pi} 4 - 3\sin(2t)\ dt$$

Integrate, then evaluate over the interval.

$$S = 12\pi\left(4t + \frac{3}{2}\cos(2t)\right)\Big|_0^{\pi}$$

$$S = 12\pi\left(4\pi + \frac{3}{2}\cos(2\pi)\right) - 12\pi\left(4(0) + \frac{3}{2}\cos(2(0))\right)$$

$$S = 12\pi\left(4\pi + \frac{3}{2}(1)\right) - 12\pi\left(0 + \frac{3}{2}(1)\right)$$

$$S = 12\pi\left(4\pi + \frac{3}{2}\right) - 12\pi\left(\frac{3}{2}\right)$$

$$S = 48\pi^2 + 18\pi - 18\pi$$

$$S = 48\pi^2$$

5. Find the surface area of revolution of the parametric curve on the interval $0 \le t \le 4$, rotated about the y-axis.

$$x = 6t + 5$$

$$y = 8t + 7$$

Solution:

Plug the derivatives and the interval into the integral formula formula for the surface area of revolution for a parametric curve about the y-axis.

$$S = \int_a^b 2\pi x \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \ dt$$

$$S = \int_0^4 2\pi(6t + 5)\sqrt{(6)^2 + (8)^2} \ dt$$

$$S = 2\pi \int_0^4 (6t + 5)\sqrt{36 + 64} \ dt$$

$$S = 2\pi \int_0^4 (6t + 5)\sqrt{100} \ dt$$

$$S = 20\pi \int_0^4 6t + 5 \ dt$$

Integrate, then evaluate over the interval.

$$S = 20\pi(3t^2 + 5t)\Big|_0^4$$

$$S = 20\pi(3(4)^2 + 5(4)) - 20\pi(3(0)^2 + 5(0))$$

$$S = 20\pi(3(16) + 20) - 20\pi(0 + 0)$$

$$S = 20\pi(68)$$

$$S = 1{,}360\pi$$

VOLUME OF REVOLUTION, PARAMETRIC CURVES

■ 1. Find the volume of revolution of the parametric curve, rotated about the x-axis, over the interval $1 \le t \le 2$.

$$x(t) = 2t^2$$

$$y(t) = 4t^2$$

Solution:

Plug the interval and the parametric equation into the integral formula for volume of revolution for a parametric equation around the x-axis.

$$V_x = \int_a^b \pi y^2 \frac{dx}{dt}\ dt$$

$$V_x = \int_1^2 \pi(4t^2)^2(4t)\ dt$$

$$V_x = 64\pi \int_1^2 t^5\ dt$$

Integrate, then evaluate over the interval.

$$V_x = 64\pi \left(\frac{1}{6}t^6\right)\Big|_1^2$$

$$V_x = \frac{32\pi}{3} t^6 \Big|_1^2$$

$$V_x = \frac{32\pi}{3}(2)^6 - \frac{32\pi}{3}(1)^6$$

$$V_x = \frac{32\pi}{3}(64) - \frac{32\pi}{3}$$

$$V_x = \frac{32\pi}{3}(64 - 1)$$

$$V_x = \frac{32\pi}{3}(63)$$

$$V_x = 32\pi(21)$$

$$V_x = 672\pi$$

■ 2. Find the volume of revolution of the parametric curve, rotated about the y-axis, over the interval $1 \le t \le 3$.

$$x(t) = 3t$$

$$y(t) = 4t^2$$

Solution:

Plug the interval and the parametric equation into the integral formula for volume of revolution for a parametric equation around the y-axis.

$$V_y = \int_a^b \pi x^2 \frac{dy}{dt}\ dt$$

$$V_y = \int_1^3 \pi(3t)^2(8t)\ dt$$

$$V_y = 72\pi \int_1^3 t^3\ dt$$

Integrate, then evaluate over the interval.

$$V_y = 72\pi \left(\frac{1}{4}t^4\right)\Big|_1^3$$

$$V_y = \frac{72\pi}{4}t^4\Big|_1^3$$

$$V_y = \frac{72\pi}{4}(3)^4 - \frac{72\pi}{4}(1)^4$$

$$V_y = \frac{72\pi}{4}(81 - 1)$$

$$V_y = \frac{72\pi}{4}(80)$$

$$V_y = 72\pi(20)$$

$$V_y = 1{,}440\pi$$

3. Find the volume of revolution of the parametric curve, rotated about the x-axis, over the interval $1 \leq t \leq 3$.

$$x(t) = 2e^{2t} - 4t$$

$$y(t) = 6e^{\frac{5t}{2}}$$

Solution:

Plug the interval and the parametric equation into the integral formula for volume of revolution for a parametric equation around the x-axis.

$$V_x = \int_a^b \pi y^2 \frac{dx}{dt}\ dt$$

$$V_x = \int_1^3 \pi(6e^{\frac{5t}{2}})^2(4e^{2t} - 4)\ dt$$

$$V_x = 4\pi \int_1^3 (36e^{5t})(e^{2t} - 1)\ dt$$

$$V_x = 144\pi \int_1^3 e^{5t}e^{2t} - e^{5t}\ dt$$

$$V_x = 144\pi \int_1^3 e^{7t} - e^{5t}\ dt$$

Integrate, then evaluate over the interval.

$$V_x = 144\pi \left(\frac{1}{7}e^{7t} - \frac{1}{5}e^{5t} \right) \Big|_1^3$$

$$V_x = 144\pi \left(\frac{1}{7}e^{7(3)} - \frac{1}{5}e^{5(3)} \right) - 144\pi \left(\frac{1}{7}e^{7(1)} - \frac{1}{5}e^{5(1)} \right)$$

$$V_x = 144\pi \left(\frac{1}{7}e^{21} - \frac{1}{5}e^{15} \right) - 144\pi \left(\frac{1}{7}e^{7} - \frac{1}{5}e^{5} \right)$$

$$V_x = 144\pi \left(\frac{1}{7}e^{21} - \frac{1}{5}e^{15} - \frac{1}{7}e^{7} + \frac{1}{5}e^{5} \right)$$

$$V_x = 144\pi \left(\frac{e^{21} - e^{7}}{7} - \frac{e^{15} - e^{5}}{5} \right)$$

■ 4. Find the volume of revolution of the parametric curve, rotated about the y-axis, over the interval $0 \le t \le 1$.

$$x(t) = 3e^t$$

$$y(t) = e^t$$

Solution:

Plug the interval and the parametric equation into the integral formula for volume of revolution for a parametric equation around the y-axis.

$$V_y = \int_a^b \pi x^2 \frac{dy}{dt}\ dt$$

$$V_y = \int_0^1 \pi(3e^t)^2(e^t)\ dt$$

$$V_y = 9\pi \int_0^1 (e^{2t})(e^t)\ dt$$

$$V_y = 9\pi \int_0^1 e^{3t}\ dt$$

Integrate, then evaluate over the interval.

$$V_y = 9\pi \left(\frac{1}{3} e^{3t} \right) \Big|_0^1$$

$$V_y = 3\pi \left(e^{3t} \right) \Big|_0^1$$

$$V_y = 3\pi \left(e^{3(1)} \right) - 3\pi \left(e^{3(0)} \right)$$

$$V_y = 3\pi(e^3) - 3\pi(1)$$

$$V_y = 3\pi(e^3 - 1)$$

POLAR COORDINATES

■ 1. Convert the rectangular point $(2,-2)$ to a polar point.

Solution:

Use $x^2+y^2=r^2$ to find r.

$$2^2+(-2)^2=r^2$$

$$4+4=r^2$$

$$8=r^2$$

$$r=\sqrt{8}$$

$$r=2\sqrt{2}$$

Use $\theta=\tan^{-1}(y/x)$ to find θ.

$$\theta=\tan^{-1}\left(\frac{-2}{2}\right)$$

$$\theta=\tan^{-1}(-1)$$

$$\theta=\frac{3\pi}{4},\frac{7\pi}{4}$$

Since the point $(2,-2)$ is in quadrant IV, $\theta=7\pi/4$. Therefore, the polar point is

$$\left(2\sqrt{2}, \frac{7\pi}{4}\right)$$

■ 2. Convert the polar point $(3,\pi/4)$ to a rectangular point.

Solution:

Use $x = r\cos\theta$ and $y = r\sin\theta$ to find the rectangular point.

$$x = r\cos\theta$$

$$x = 3\cos\left(\frac{\pi}{4}\right)$$

$$x = 3\left(\frac{\sqrt{2}}{2}\right)$$

$$x = \frac{3\sqrt{2}}{2}$$

and

$$y = r\sin\theta$$

$$y = 3\sin\left(\frac{\pi}{4}\right)$$

$$y = 3\left(\frac{\sqrt{2}}{2}\right)$$

$$y = \frac{3\sqrt{2}}{2}$$

Therefore, the rectangular point is

$$\left(\frac{3\sqrt{2}}{2}, \frac{3\sqrt{2}}{2}\right)$$

3. Convert the rectangular point $\left(-5\sqrt{3}, 5\right)$ to a polar point.

Solution:

Use $x^2 + y^2 = r^2$ to find r.

$$\left(-5\sqrt{3}\right)^2 + (5)^2 = r^2$$

$$75 + 25 = r^2$$

$$100 = r^2$$

$$r = \sqrt{100}$$

$$r = 10$$

Use $\theta = \tan^{-1}(y/x)$ to find θ.

$$\theta = \tan^{-1}\left(\frac{5}{-5\sqrt{3}}\right)$$

$$\theta = \tan^{-1}\left(-\frac{1}{\sqrt{3}}\right)$$

$$\theta = \frac{5\pi}{6}, \frac{11\pi}{6}$$

Since the point $\left(-5\sqrt{3},5\right)$ is in quadrant II, $\theta = 5\pi/6$. Therefore, the polar point is

$$\left(10,\frac{5\pi}{6}\right)$$

4. Convert the polar point $(8,11\pi/6)$ to a rectangular point.

Solution:

Use $x = r\cos\theta$ and $y = r\sin\theta$ to find the rectangular point.

$$x = r\cos\theta$$

$$x = 8\cos\left(\frac{11\pi}{6}\right)$$

$$x = 8\left(\frac{\sqrt{3}}{2}\right)$$

$$x = 4\sqrt{3}$$

and

$$y = r\sin\theta$$

$$y = 8\sin\left(\frac{11\pi}{6}\right)$$

$$y = 8\left(-\frac{1}{2}\right)$$

$$y = -4$$

Therefore, the rectangular point is

$$\left(4\sqrt{3}, -4\right)$$

CONVERTING RECTANGULAR EQUATIONS

1. Convert the rectangular equation to an equivalent polar equation.

$$4x^2 + 4y^2 = 64$$

Solution:

When converting from rectangular to polar, use $x = r\cos\theta$ and $y = r\sin\theta$. Simplify the given equation and then substitute.

$$4x^2 + 4y^2 = 64$$

$$x^2 + y^2 = 16$$

$$(r\cos\theta)^2 + (r\sin\theta)^2 = 16$$

$$r^2\cos^2\theta + r^2\sin^2\theta = 16$$

$$r^2\left(\cos^2\theta + \sin^2\theta\right) = 16$$

$$r^2(1) = 16$$

$$r^2 = 16$$

$$r = 4$$

This is the equivalent polar equation.

2. Convert the rectangular equation to an equivalent polar equation.

$$\frac{x^2}{9}+\frac{y^2}{4}=1$$

Solution:

When converting from rectangular to polar, use $x=r\cos\theta$ and $y=r\sin\theta$. Eliminate the denominators and then substitute.

$$\frac{x^2}{9}+\frac{y^2}{4}=1$$

$$4x^2+9y^2=36$$

$$4(r\cos\theta)^2+9(r\sin\theta)^2=36$$

$$4r^2\cos^2\theta+9r^2\sin^2\theta=36$$

$$r^2\left(4\cos^2\theta+9\sin^2\theta\right)=36$$

$$r^2\left[4\cos^2\theta+4\sin^2\theta+5\sin^2\theta\right]=36$$

$$r^2\left[4\left(\cos^2\theta+\sin^2\theta\right)+5\sin^2\theta\right]=36$$

$$r^2(4(1)+5\sin^2\theta)=36$$

$$r^2(4+5\sin^2\theta)=36$$

$$r^2=\frac{36}{4+5\sin^2\theta}$$

$$r = \frac{6}{\sqrt{4 + 5\sin^2\theta}}$$

This is the equivalent polar equation.

3. Convert the rectangular equation to an equivalent polar equation.

$$(x-2)^2 + (y+2)^2 = 8$$

Solution:

When converting from rectangular to polar, use $x = r\cos\theta$ and $y = r\sin\theta$. Square the binomials and then substitute.

$$(x-2)^2 + (y+2)^2 = 8$$

$$x^2 - 4x + 4 + y^2 + 4y + 4 = 8$$

$$x^2 + y^2 - 4x + 4y = 0$$

$$r^2 - 4r\cos x + 4r\sin x = 0$$

$$r^2 = 4r\cos x - 4r\sin x$$

$$r = 4\cos x - 4\sin x$$

This is the equivalent polar equation.

4. Convert the rectangular equation to an equivalent polar equation.

$$\frac{x^2}{9} - \frac{y^2}{8} = 1$$

Solution:

When converting from rectangular to polar, use $x = r\cos\theta$ and $y = r\sin\theta$. Eliminate the denominators and then substitute.

$$\frac{x^2}{9} - \frac{y^2}{8} = 1$$

$$8x^2 - 9y^2 = 72$$

$$8(r\cos\theta)^2 - 9(r\sin\theta)^2 = 72$$

$$8r^2\cos^2\theta - 9r^2\sin^2\theta = 72$$

$$r^2\left(8\cos^2\theta - 9\sin^2\theta\right) = 72$$

$$r^2\left(8\cos^2\theta - 8\sin^2\theta - \sin^2\theta\right) = 72$$

$$8r^2\left(\cos^2\theta - \sin^2\theta\right) - r^2\sin^2\theta = 72$$

Use the identity $\cos(2\theta) = \cos^2\theta - \sin^2\theta$ to make a substitution.

$$8r^2\cos(2\theta) - r^2\sin^2\theta = 72$$

$$r^2\left(8\cos(2\theta) - \sin^2\theta\right) = 72$$

$$r^2 = \frac{72}{8\cos(2\theta) - \sin^2\theta}$$

$$r = \frac{6\sqrt{2}}{\sqrt{8\cos(2\theta) - \sin^2\theta}}$$

This is the equivalent polar equation.

CONVERTING POLAR EQUATIONS

1. Convert the polar equation to an equivalent rectangular equation.

$$r = 4\cos\theta + 4\sin\theta$$

Solution:

When converting from polar to rectangular, use $x = r\cos\theta$ and $y = r\sin\theta$. For this problem, rewrite those as

$$\cos\theta = \frac{x}{r}$$

$$\sin\theta = \frac{y}{r}$$

Substitute these values.

$$r = 4\cos\theta + 4\sin\theta$$

$$r = \frac{4x}{r} + \frac{4y}{r}$$

$$r^2 = 4x + 4y$$

Replace r^2 with $x^2 + y^2$.

$$x^2 + y^2 = 4x + 4y$$

$$x^2 - 4x + y^2 - 4y = 0$$

$$x^2 - 4x + 4 + y^2 - 4y + 4 = 4 + 4$$

$$\left(x^2 - 4x + 4\right) + \left(y^2 - 4y + 4\right) = 8$$

$$(x - 2)^2 + (y - 2)^2 = 8$$

This is the equivalent rectangular equation.

■ 2. Convert the polar equation to an equivalent rectangular equation.

$$r = 12\cos\theta - 12\sin\theta$$

Solution:

When converting from polar to rectangular, use $x = r\cos\theta$ and $y = r\sin\theta$. For this problem, rewrite those as

$$\cos\theta = \frac{x}{r}$$

$$\sin\theta = \frac{y}{r}$$

Substitute these values.

$$r = 12\cos\theta - 12\sin\theta$$

$$r = \frac{12x}{r} - \frac{12y}{r}$$

$$r^2 = 12x - 12y$$

$$x^2+y^2=12x-12y$$

$$x^2-12x+y^2+12y=0$$

$$x^2-12x+36+y^2+12y+36=36+36$$

$$\left(x^2-12x+36\right)+\left(y^2+12y+36\right)=72$$

$$(x-6)^2+(y+6)^2=72$$

This is the equivalent rectangular equation.

■ 3. Convert the polar equation to an equivalent rectangular equation.

$$r=3\sin\left(\theta+\frac{\pi}{4}\right)$$

Solution:

Use the identity $\sin(a+b)=\sin a\cos b+\cos a\sin b$ to rewrite the polar equation.

$$r=3\sin\left(\theta+\frac{\pi}{4}\right)$$

$$r=3\left(\sin\theta\cos\left(\frac{\pi}{4}\right)+\cos\theta\sin\left(\frac{\pi}{4}\right)\right)$$

$$r = 3\left(\frac{\sqrt{2}}{2}\sin\theta + \frac{\sqrt{2}}{2}\cos\theta\right)$$

$$r = \frac{3\sqrt{2}}{2}\sin\theta + \frac{3\sqrt{2}}{2}\cos\theta$$

When converting from polar to rectangular, use $x = r\cos\theta$ and $y = r\sin\theta$. For this problem, rewrite those as

$$\cos\theta = \frac{x}{r}$$

$$\sin\theta = \frac{y}{r}$$

Substitute these values.

$$r = \frac{3\sqrt{2}}{2}\frac{y}{r} + \frac{3\sqrt{2}}{2}\frac{x}{r}$$

$$r^2 = \frac{3\sqrt{2}}{2}y + \frac{3\sqrt{2}}{2}x$$

$$x^2 + y^2 = \frac{3\sqrt{2}}{2}y + \frac{3\sqrt{2}}{2}x$$

$$x^2 - \frac{3\sqrt{2}}{2}x + y^2 - \frac{3\sqrt{2}}{2}y = 0$$

Complete the square with respect to both variables.

$$x^2 - \frac{3\sqrt{2}}{2}x + \frac{9}{8} + y^2 - \frac{3\sqrt{2}}{2}y + \frac{9}{8} = \frac{9}{8} + \frac{9}{8}$$

$$\left(x^2-\frac{3\sqrt{2}}{2}x+\frac{9}{8}\right)+\left(y^2-\frac{3\sqrt{2}}{2}y+\frac{9}{8}\right)=\frac{9}{4}$$

$$\left(x-\frac{3\sqrt{2}}{4}\right)^2+\left(y-\frac{3\sqrt{2}}{4}\right)^2=\frac{9}{4}$$

This is the equivalent rectangular equation.

■ 4. Convert the polar equation to an equivalent rectangular equation.

$$r=6\cos\theta-10\sin\theta$$

Solution:

When converting from polar to rectangular, use $x=r\cos\theta$ and $y=r\sin\theta$. For this problem, rewrite those as

$$\cos\theta=\frac{x}{r}$$

$$\sin\theta=\frac{y}{r}$$

Substitute these values.

$$r=6\cos\theta-10\sin\theta$$

$$r=\frac{6x}{r}-\frac{10y}{r}$$

$$r^2 = 6x - 10y$$

$$x^2 + y^2 = 6x - 10y$$

$$x^2 - 6x + y^2 + 10y = 0$$

$$x^2 - 6x + 9 + y^2 + 10y + 25 = 9 + 25$$

$$\left(x^2 - 6x + 9\right) + \left(y^2 + 10y + 25\right) = 34$$

$$(x - 3)^2 + (y + 5)^2 = 34$$

This is the equivalent rectangular equation.

5. Convert the polar equation to an equivalent rectangular equation.

$$r = 12\sin\theta$$

Solution:

When converting from polar to rectangular, use $x = r\cos\theta$ and $y = r\sin\theta$. For this problem, rewrite those as

$$\cos\theta = \frac{x}{r}$$

$$\sin\theta = \frac{y}{r}$$

Substitute these values.

$$r = 12\sin\theta$$

$$r = \frac{12y}{r}$$

$$r^2 = 12y$$

$$x^2 + y^2 = 12y$$

$$x^2 + y^2 - 12y = 0$$

$$x^2 + y^2 - 12y + 36 = 36$$

$$x^2 + \left(y^2 - 12y + 36\right) = 36$$

$$x^2 + (y - 6)^2 = 36$$

This is the equivalent rectangular equation.

DISTANCE BETWEEN POLAR POINTS

1. Calculate the distance between the polar coordinate points.

$$\left(2,\frac{\pi}{3}\right) \text{ and } \left(2,\frac{11\pi}{6}\right)$$

Solution:

Find the distance between two polar coordinate points with the formula

$$D=\sqrt{{r_1}^2+{r_2}^2-2r_1r_2\cos\left(\theta_1-\theta_2\right)}$$

Plugging the points into this distance formula, we get

$$D=\sqrt{2^2+2^2-2(2)(2)\cos\left(\frac{11\pi}{6}-\frac{\pi}{3}\right)}$$

$$D=\sqrt{4+4-8\cos\left(\frac{3\pi}{2}\right)}$$

$$D=\sqrt{8-8(0)}$$

$$D=\sqrt{8}$$

$$D=2\sqrt{2}$$

This is the distance between the polar points.

2. Calculate the distance between the polar coordinate points.

$$\left(4,\frac{7\pi}{12}\right) \text{ and } \left(2,\frac{\pi}{12}\right)$$

Solution:

Find the distance between two polar coordinate points with the formula

$$D=\sqrt{{r_1}^2+{r_2}^2-2r_1r_2\cos\left(\theta_1-\theta_2\right)}$$

Plugging the points into this distance formula, we get

$$D=\sqrt{4^2+2^2-2(4)(2)\cos\left(\frac{7\pi}{12}-\frac{\pi}{12}\right)}$$

$$D=\sqrt{16+4-16\cos\left(\frac{\pi}{2}\right)}$$

$$D=\sqrt{20-16(0)}$$

$$D=\sqrt{20}$$

$$D=2\sqrt{5}$$

This is the distance between the polar points.

3. Calculate the distance between the polar coordinate points.

$$\left(4,\frac{\pi}{4}\right) \text{ and } \left(9,\frac{3\pi}{4}\right)$$

Solution:

Find the distance between two polar coordinate points with the formula

$$D = \sqrt{{r_1}^2 + {r_2}^2 - 2r_1r_2\cos\left(\theta_1 - \theta_2\right)}$$

Plugging the points into this distance formula, we get

$$D = \sqrt{4^2 + 9^2 - 2(4)(9)\cos\left(\frac{3\pi}{4} - \frac{\pi}{4}\right)}$$

$$D = \sqrt{16 + 81 - 72\cos\left(\frac{\pi}{2}\right)}$$

$$D = \sqrt{97 - 72(0)}$$

$$D = \sqrt{97}$$

This is the distance between the polar points.

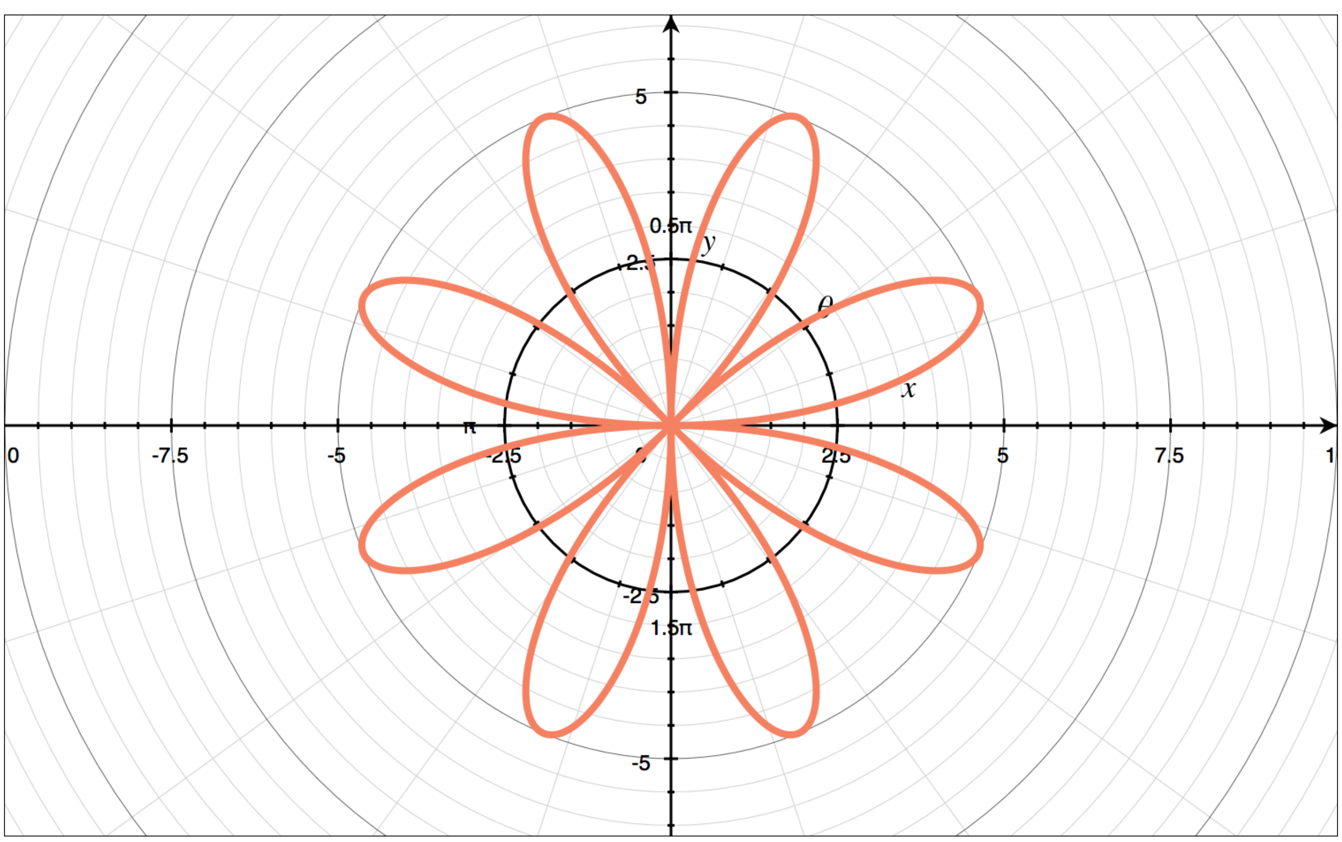
SKETCHING POLAR CURVES

■ 1. Graph the polar curve. How many petals does the curve have, and what is the length of each petal?

$$r = 5\sin(4\theta)$$

Solution:

The polar equation represents a rose. The length of the petals of a curve in the form $r = a\sin(b\theta)$ is a units. The number of petals depends on the value of b. If b is an odd number, then the graph has b petals. If b is an even number, then the graph has $2b$ petals. In this question, $a = 5$, $b = 4$. Therefore, the graph has 8 petals and the length of each petal is 5 units. The graph of the given polar equation is



TANGENT LINE TO THE POLAR CURVE

1. Find the tangent line to the polar curve at $\theta = 2\pi/3$.

$$r = 3\cos\theta$$

Solution:

The slope of the tangent line m is given by

$$m = \frac{dy}{dx} = \frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta}$$

Calculate $dr/d\theta$.

$$\frac{dr}{d\theta} = \frac{d}{d\theta}(3\cos\theta) = -3\sin\theta$$

Then m is

$$m = \frac{-3\sin\theta\sin\theta + 3\cos\theta\cos\theta}{-3\sin\theta\cos\theta - 3\cos\theta\sin\theta}$$

$$m = \frac{-\sin^2\theta + \cos^2\theta}{-2\sin\theta\cos\theta}$$

$$m = \frac{-\sin^2\left(\frac{2\pi}{3}\right) + \cos^2\left(\frac{2\pi}{3}\right)}{-2\sin\left(\frac{2\pi}{3}\right)\cos\left(\frac{2\pi}{3}\right)}$$

$$m = \frac{-\left(\frac{\sqrt{3}}{2}\right)^2 + \left(-\frac{1}{2}\right)^2}{-2\left(\frac{\sqrt{3}}{2}\right)\left(-\frac{1}{2}\right)}$$

$$m = \frac{-\frac{3}{4} + \frac{1}{4}}{\frac{\sqrt{3}}{2}} = \frac{-\frac{1}{2}}{\frac{\sqrt{3}}{2}} = -\frac{1}{\sqrt{3}} = -\frac{\sqrt{3}}{3}$$

Use $\theta = 2\pi/3$ and the conversion equations $x = r\cos\theta$ and $y = r\sin\theta$ to find a point on the tangent line.

$$x = r\cos\theta$$

$$x = 3\cos\theta\cos\theta$$

$$x_1 = 3\cos^2\left(\frac{3\pi}{2}\right)$$

$$x_1 = 3\left(-\frac{1}{2}\right)^2$$

$$x_1 = \frac{3}{4}$$

and

$$y = r\sin\theta$$

$$y = 3\cos\theta\sin\theta$$

$$y_1 = 3\cos\left(\frac{3\pi}{2}\right)\sin\left(\frac{3\pi}{2}\right)$$

$$y_1 = 3\left(-\frac{1}{2}\right)\left(\frac{\sqrt{3}}{2}\right)$$

$$y_1 = -\frac{3\sqrt{3}}{4}$$

Therefore, the equation of the tangent line is

$$y - y_1 = m(x - x_1)$$

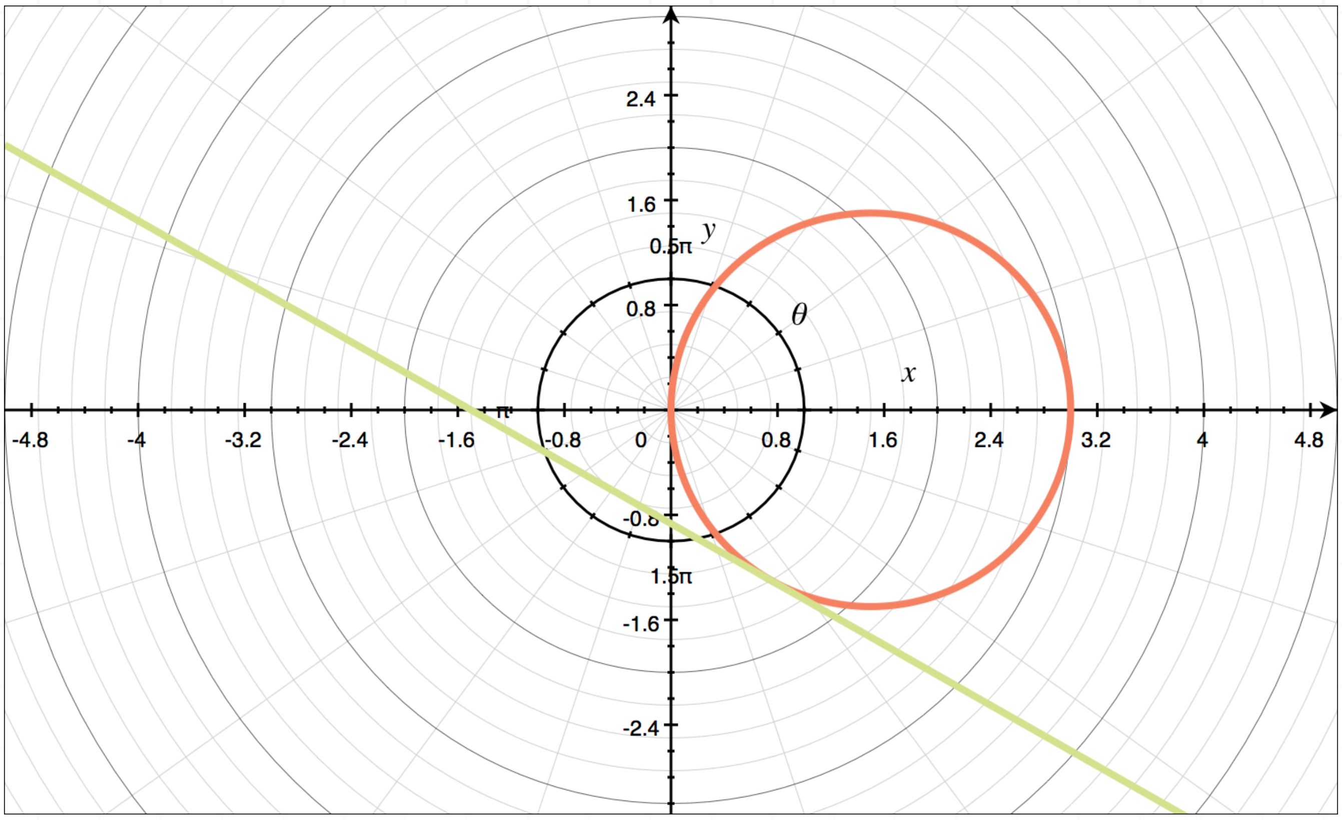
$$y + \frac{3\sqrt{3}}{4} = -\frac{\sqrt{3}}{3}\left(x - \frac{3}{4}\right)$$

$$y = -\frac{\sqrt{3}}{3}\left(x - \frac{3}{4}\right) - \frac{3\sqrt{3}}{4}$$

$$y = -\frac{\sqrt{3}}{3}x + \frac{\sqrt{3}}{4} - \frac{3\sqrt{3}}{4}$$

$$y = -\frac{\sqrt{3}}{3}x - \frac{\sqrt{3}}{2}$$

The graph shows the polar curve and the tangent line.



2. Find the tangent line to the polar curve at $\theta = \pi/3$.

$$r = 5\sin\theta$$

Solution:

The slope of the tangent line m is given by

$$m = \frac{dy}{dx} = \frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta}$$

Calculate $dr/d\theta$.

$$\frac{dr}{d\theta} = \frac{d}{d\theta}(5\sin\theta) = 5\cos\theta$$

Then m is

$$m=\frac{5\cos\theta\sin\theta+5\sin\theta\cos\theta}{5\cos\theta\cos\theta-5\sin\theta\sin\theta}$$

$$m=\frac{\cos\theta\sin\theta+\sin\theta\cos\theta}{\cos^2\theta-\sin^2\theta}$$

$$m=\frac{2\cos\theta\sin\theta}{\cos^2\theta-\sin^2\theta}$$

$$m=\frac{2\cos\left(\frac{\pi}{3}\right)\sin\left(\frac{\pi}{3}\right)}{\cos^2\left(\frac{\pi}{3}\right)-\sin^2\left(\frac{\pi}{3}\right)}$$

$$m=\frac{2\left(\frac{1}{2}\right)\left(\frac{\sqrt{3}}{2}\right)}{\left(\frac{1}{2}\right)^2-\left(\frac{\sqrt{3}}{2}\right)^2}$$

$$m=\frac{\frac{\sqrt{3}}{2}}{\frac{1}{4}-\frac{3}{4}}=-\frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}}=-\frac{\sqrt{3}}{1}=-\sqrt{3}$$

Use $\theta=\pi/3$ and the conversion equations $x=r\cos\theta$ and $y=r\sin\theta$ to find a point on the tangent line.

$$x=r\cos\theta$$

$$x=5\sin\theta\cos\theta$$

$$x_1=5\sin\left(\frac{\pi}{3}\right)\cos\left(\frac{\pi}{3}\right)$$

$$x_1 = 5\left(\frac{\sqrt{3}}{2}\right)\left(\frac{1}{2}\right)$$

$$x_1 = \frac{5\sqrt{3}}{4}$$

and

$$y = r\sin\theta$$

$$y = 5\sin\theta\sin\theta$$

$$y_1 = 5\sin^2\left(\frac{\pi}{3}\right)$$

$$y_1 = 5\left(\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{3}}{2}\right)$$

$$y_1 = \frac{15}{4}$$

Therefore, the equation of the tangent line is

$$y - y_1 = m(x - x_1)$$

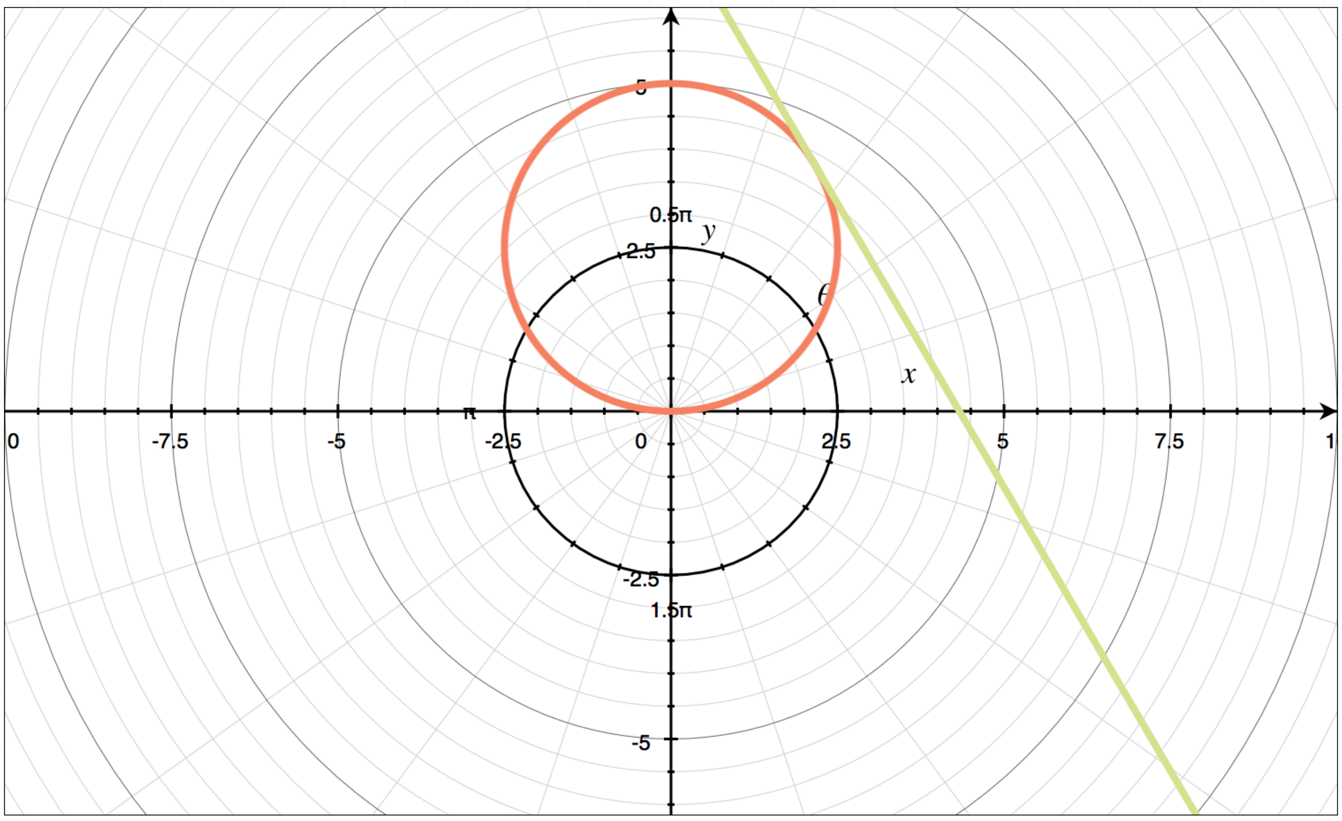
$$y - \frac{15}{4} = -\sqrt{3}\left(x - \frac{5\sqrt{3}}{4}\right)$$

$$y = -\sqrt{3}\left(x - \frac{5\sqrt{3}}{4}\right) + \frac{15}{4}$$

$$y = -\sqrt{3}x + \frac{15}{4} + \frac{15}{4}$$

$$y = -\sqrt{3}x + \frac{15}{2}$$

The graph shows the polar curve and the tangent line.



3. Find the tangent line to the polar curve at $\theta = \pi/4$.

$$r = 4 - 2\cos\theta$$

Solution:

The slope of the tangent line m is given by

$$m = \frac{dy}{dx} = \frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta}$$

Calculate $dr/d\theta$.

$$\frac{dr}{d\theta} = \frac{d}{d\theta}(4 - 2\cos\theta) = 2\sin\theta$$

Then m is

$$m = \frac{2\sin\theta\sin\theta + (4 - 2\cos\theta)\cos\theta}{2\sin\theta\cos\theta - (4 - 2\cos\theta)\sin\theta}$$

$$m = \frac{2\sin\theta\sin\theta + 4\cos\theta - 2\cos^2\theta}{2\sin\theta\cos\theta - 4\sin\theta + 2\cos\theta\sin\theta}$$

$$m = \frac{\sin\theta\sin\theta + 2\cos\theta - \cos^2\theta}{\sin\theta\cos\theta - 2\sin\theta + \cos\theta\sin\theta}$$

$$m = \frac{\sin\left(\frac{\pi}{4}\right)\sin\left(\frac{\pi}{4}\right) + 2\cos\left(\frac{\pi}{4}\right) - \cos^2\left(\frac{\pi}{4}\right)}{\sin\left(\frac{\pi}{4}\right)\cos\left(\frac{\pi}{4}\right) - 2\sin\left(\frac{\pi}{4}\right) + \cos\left(\frac{\pi}{4}\right)\sin\left(\frac{\pi}{4}\right)}$$

$$m = \frac{\left(\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{2}}{2}\right) + 2\left(\frac{\sqrt{2}}{2}\right) - \left(\frac{\sqrt{2}}{2}\right)^2}{\left(\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{2}}{2}\right) - 2\left(\frac{\sqrt{2}}{2}\right) + \left(\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{2}}{2}\right)}$$

$$m = \frac{\frac{1}{2} + \sqrt{2} - \frac{1}{2}}{\frac{1}{2} - \sqrt{2} + \frac{1}{2}} = \frac{\sqrt{2}}{1 - \sqrt{2}} = \frac{\sqrt{2}\left(1 + \sqrt{2}\right)}{\left(1 - \sqrt{2}\right)\left(1 + \sqrt{2}\right)} = \frac{2 + \sqrt{2}}{1 - 2} = -2 - \sqrt{2}$$

Use $\theta = \pi/4$ and the conversion equations $x = r\cos\theta$ and $y = r\sin\theta$ to find a point on the tangent line.

$$x = r\cos\theta$$

$$x = (4 - 2\cos\theta)\cos\theta$$

$$x_1 = 4\cos\left(\frac{\pi}{4}\right) - 2\cos^2\left(\frac{\pi}{4}\right)$$

$$x_1 = 4\left(\frac{\sqrt{2}}{2}\right) - 2\left(\frac{\sqrt{2}}{2}\right)^2$$

$$x_1 = 2\sqrt{2} - 1$$

and

$$y = r\sin\theta$$

$$y = (4 - 2\cos\theta)\sin\theta$$

$$y_1 = 4\sin\left(\frac{\pi}{4}\right) - 2\cos\left(\frac{\pi}{4}\right)\sin\left(\frac{\pi}{4}\right)$$

$$y_1 = 4\left(\frac{\sqrt{2}}{2}\right) - 2\left(\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{2}}{2}\right)$$

$$y_1 = 2\sqrt{2} - 1$$

Therefore, the equation of the tangent line is

$$y - y_1 = m(x - x_1)$$

$$y - \left(2\sqrt{2} - 1\right) = \left(-2 - \sqrt{2}\right)\left(x - \left(2\sqrt{2} - 1\right)\right)$$

$$y = \left(-2 - \sqrt{2}\right)\left(x - \left(2\sqrt{2} - 1\right)\right) + \left(2\sqrt{2} - 1\right)$$

$$y = -\left(2 + \sqrt{2}\right)\left(x - 2\sqrt{2} + 1\right) + 2\sqrt{2} - 1$$

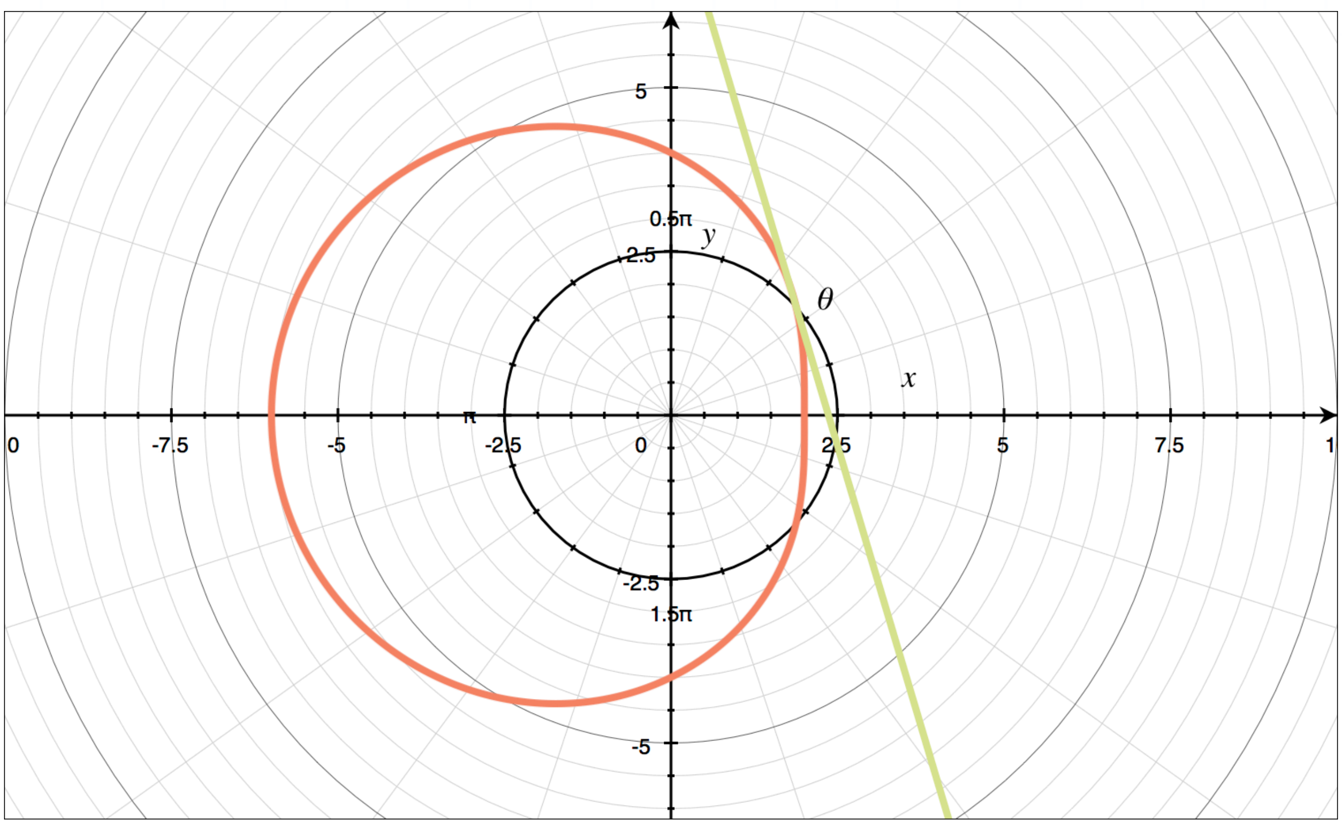
$$y = -\left(2x - 4\sqrt{2} + 2 + \sqrt{2}x - 4 + \sqrt{2}\right) + 2\sqrt{2} - 1$$

$$y = -2x + 4\sqrt{2} - 2 - \sqrt{2}x + 4 - \sqrt{2} + 2\sqrt{2} - 1$$

$$y = -2x - \sqrt{2}x + 5\sqrt{2} + 1$$

$$y = \left(-2 - \sqrt{2}\right)x + 5\sqrt{2} + 1$$

The graph shows the polar curve and the tangent line.



■ 4. Find the tangent line to the polar curve at $\theta = \pi$.

$$r = 8 - 5\sin\theta$$

Solution:

The slope of the tangent line m is given by

$$m = \frac{dy}{dx} = \frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta}$$

Calculate $dr/d\theta$.

$$\frac{dr}{d\theta} = \frac{d}{d\theta}(8 - 5\sin\theta) = -5\cos\theta$$

Then m is

$$m = \frac{-5\cos\theta\sin\theta + (8 - 5\sin\theta)\cos\theta}{-5\cos\theta\cos\theta - (8 - 5\sin\theta)\sin\theta}$$

$$m = \frac{-5\cos\theta\sin\theta + 8\cos\theta - 5\sin\theta\cos\theta}{-5\cos\theta\cos\theta - 8\sin\theta + 5\sin^2\theta}$$

$$m = \frac{-5\cos\pi\sin\pi + 8\cos\pi - 5\sin\pi\cos\pi}{-5\cos\pi\cos\pi - 8\sin\pi + 5\sin^2\pi}$$

$$m = \frac{-5(-1)(0) + 8(-1) - 5(0)(-1)}{-5(-1)(-1) - 8(0) + 5(0)^2}$$

$$m = \frac{0 - 8 + 0}{-5 - 0 + 0} = \frac{-8}{-5} = \frac{8}{5}$$

Use $\theta = \pi$ and the conversion equations $x = r\cos\theta$ and $y = r\sin\theta$ to find a point on the tangent line.

$$x = r\cos\theta$$

$$x = (8 - 5\sin\theta)\cos\theta$$

$$x_1 = (8 - 5\sin\pi)\cos\pi$$

$$x_1 = (8 - 5(0))(-1)$$

$$x_1 = 8(-1)$$

$$x_1 = -8$$

and

$$y = r\sin\theta$$

$$y = (8 - 5\sin\theta)\sin\theta$$

$$y_1 = (8 - 5\sin\pi)\sin\pi$$

$$y_1 = (8 - 5(0))(0)$$

$$y_1 = 0$$

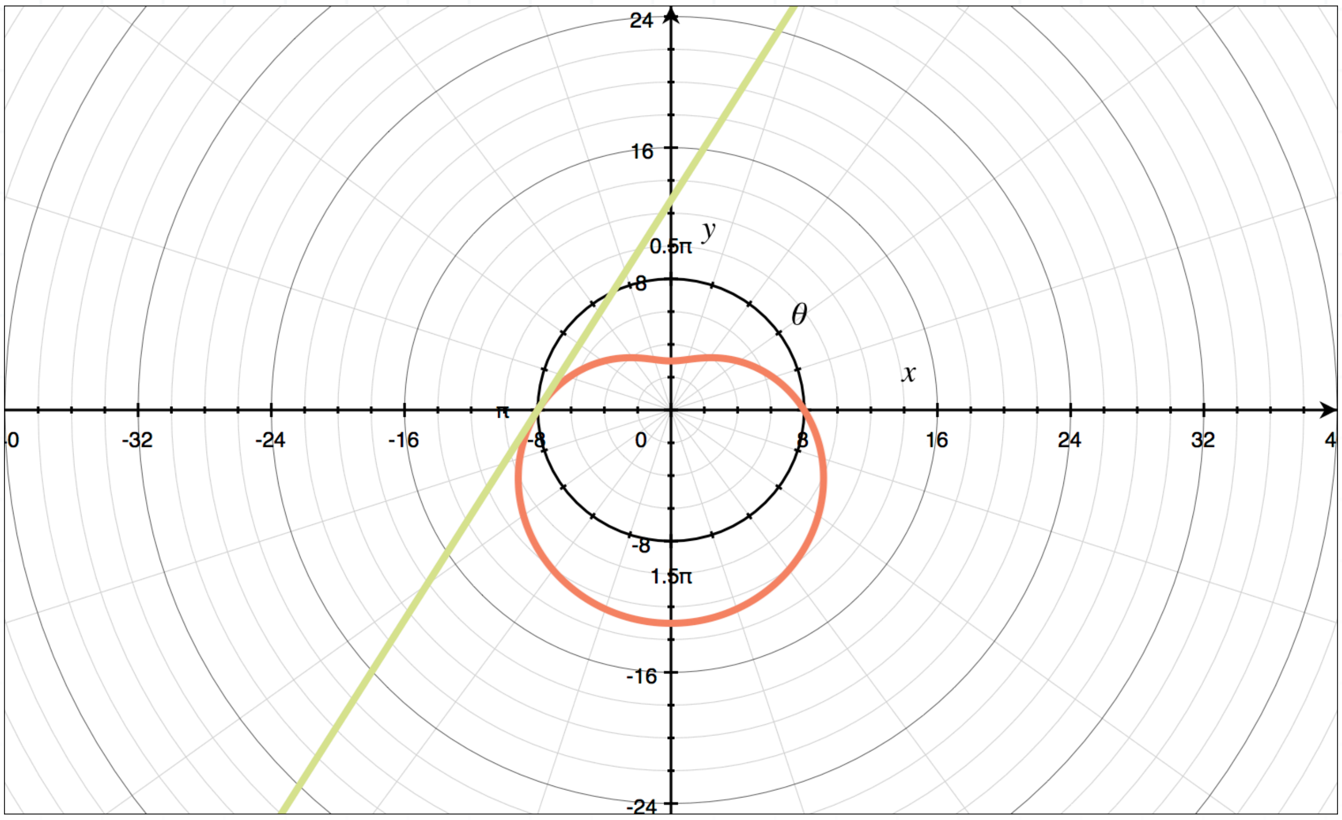
Therefore, the equation of the tangent line is

$$y - y_1 = m(x - x_1)$$

$$y - 0 = \frac{8}{5}(x + 8)$$

$$y = \frac{8}{5}x + \frac{64}{5}$$

The graph shows the polar curve and the tangent line.



■ 5. Find the tangent line to the polar curve at $\theta = \pi/2$.

$$r = 7 - 6\cos\theta$$

Solution:

The slope of the tangent line m is given by

$$m = \frac{dy}{dx} = \frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta}$$

Calculate $dr/d\theta$.

$$\frac{dr}{d\theta}=\frac{d}{d\theta}(7-6\cos\theta)=6\sin\theta$$

Then m is

$$m=\frac{6\sin\theta\sin\theta+\left(7-6\cos\theta\right)\cos\theta}{6\sin\theta\cos\theta-\left(7-6\cos\theta\right)\sin\theta}$$

$$m=\frac{6\sin\theta\sin\theta+7\cos\theta-6\cos^2\theta}{6\sin\theta\cos\theta-7\sin\theta+6\cos\theta\sin\theta}$$

$$m=\frac{6\sin\left(\frac{\pi}{2}\right)\sin\left(\frac{\pi}{2}\right)+7\cos\left(\frac{\pi}{2}\right)-6\cos^2\left(\frac{\pi}{2}\right)}{6\sin\left(\frac{\pi}{2}\right)\cos\left(\frac{\pi}{2}\right)-7\sin\left(\frac{\pi}{2}\right)+6\cos\left(\frac{\pi}{2}\right)\sin\left(\frac{\pi}{2}\right)}$$

$$m=\frac{6(1)(1)+7(0)-6(0)^2}{6(1)(0)-7(1)+6(0)(1)}=\frac{6+0-0}{0-7+0}=-\frac{6}{7}$$

Use $\theta=\pi/2$ and the conversion equations $x=r\cos\theta$ and $y=r\sin\theta$ to find a point on the tangent line.

$$x=r\cos\theta$$

$$x=(7-6\cos\theta)\cos\theta$$

$$x_1=\left(7-6\cos\left(\frac{\pi}{2}\right)\right)\cos\left(\frac{\pi}{2}\right)$$

$$x_1=\left(7-6(0)\right)(0)$$

$$x_1=0$$

and

$$y = r\sin\theta$$

$$y = (7 - 6\cos\theta)\sin\theta$$

$$y_1 = \left(7 - 6\cos\left(\frac{\pi}{2}\right)\right)\sin\left(\frac{\pi}{2}\right)$$

$$y_1 = (7 - 6(0))(1)$$

$$y_1 = 7$$

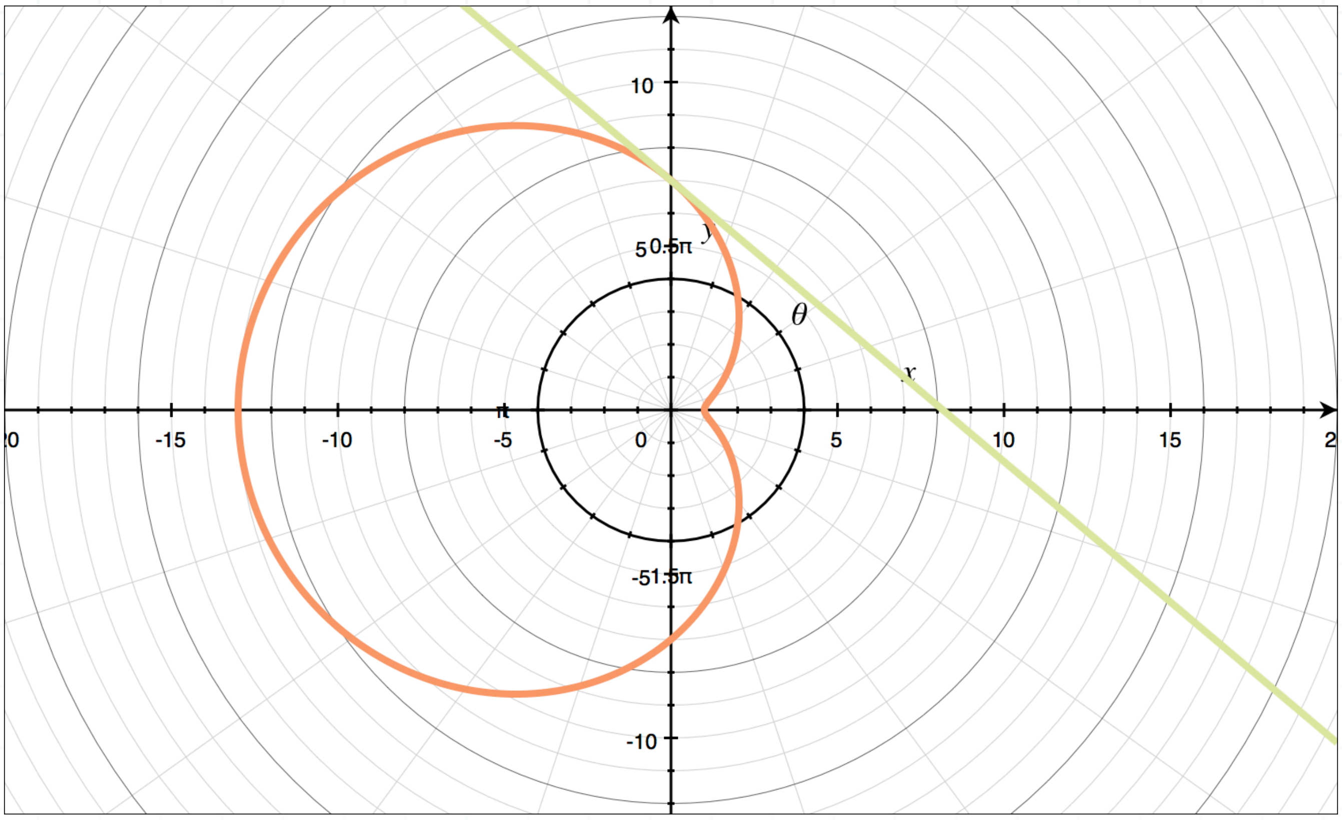
Therefore, the equation of the tangent line is

$$y - y_1 = m(x - x_1)$$

$$y - 7 = -\frac{6}{7}(x - 0)$$

$$y = -\frac{6}{7}x + 7$$

The graph shows the polar curve and the tangent line.


10
5
0.5π
y
θ
x
-15
-10
-5
π
0
5
10
15
-5
1.5π
-10

VERTICAL AND HORIZONTAL TANGENT LINES TO THE POLAR CURVE

1. At which points does the polar curve have horizontal tangent lines?

$$r = 4 - 4\sin\theta$$

Solution:

Use the conversion equations $x = r\cos\theta$ and $y = r\sin\theta$ and the polar equation into each of them.

$$x = r\cos\theta$$

$$x = (4 - 4\sin\theta)\cos\theta$$

$$x = 4\cos\theta - 4\sin\theta\cos\theta$$

and

$$y = r\sin\theta$$

$$y = (4 - 4\sin\theta)\sin\theta$$

$$y = 4\sin\theta - 4\sin^2\theta$$

Take the derivative of each equation.

$$\frac{dx}{d\theta} = \frac{d}{d\theta}(4\cos\theta - 4\sin\theta\cos\theta)$$

$$\frac{dx}{d\theta} = -4\sin\theta - 4(\cos\theta\cos\theta - \sin\theta\sin\theta)$$

$$\frac{dx}{d\theta} = 4\sin^2\theta - 4\cos^2\theta - 4\sin\theta$$

and

$$\frac{dy}{d\theta} = \frac{d}{d\theta}(4\sin\theta - 4\sin^2\theta)$$

$$\frac{dy}{d\theta} = 4\cos\theta - 8\sin\theta\cos\theta$$

Put these together to get dy/dx.

$$\frac{dy}{dx} = \frac{4\cos\theta - 8\sin\theta\cos\theta}{4\sin^2\theta - 4\cos^2\theta - 4\sin\theta}$$

$$\frac{dy}{dx} = \frac{\cos\theta - 2\sin\theta\cos\theta}{\sin^2\theta - \cos^2\theta - \sin\theta}$$

Horizontal tangent lines exist where the numerator is 0.

$$\cos\theta - 2\sin\theta\cos\theta = 0$$

$$\cos\theta(1 - 2\sin\theta) = 0$$

The solutions are

$\cos\theta = 0$ gives $\theta = \frac{\pi}{2}, \frac{3\pi}{2}$

$1 - 2\sin\theta = 0$ gives $\frac{\pi}{6}, \frac{5\pi}{6}$

Plug each of these angles into the equations for x and y. For $\theta = \pi/2$, we get

$$x = \left(4 - 4\sin\left(\frac{\pi}{2}\right)\right)\cos\left(\frac{\pi}{2}\right) \qquad y = \left(4 - 4\sin\left(\frac{\pi}{2}\right)\right)\sin\left(\frac{\pi}{2}\right)$$

$$x = (4 - 4(1))(0) \qquad y = (4 - 4(1))(1)$$

$$x = 0 \qquad y = 0$$

Plug $\theta = 3\pi/2$ into the equations for x and y.

$$x = \left(4 - 4\sin\left(\frac{3\pi}{2}\right)\right)\cos\left(\frac{3\pi}{2}\right) \qquad y = \left(4 - 4\sin\left(\frac{3\pi}{2}\right)\right)\sin\left(\frac{3\pi}{2}\right)$$

$$x = (4 - 4(-1))(0) \qquad y = (4 - 4(-1))(-1)$$

$$x = 0 \qquad y = -8$$

Plug $\theta = \pi/6$ into the equations for x and y.

$$x = \left(4 - 4\sin\left(\frac{\pi}{6}\right)\right)\cos\left(\frac{\pi}{6}\right) \qquad y = \left(4 - 4\sin\left(\frac{\pi}{6}\right)\right)\sin\left(\frac{\pi}{6}\right)$$

$$x = \left(4 - 4\left(\frac{1}{2}\right)\right)\left(\frac{\sqrt{3}}{2}\right) \qquad y = \left(4 - 4\left(\frac{1}{2}\right)\right)\left(\frac{1}{2}\right)$$

$$x = \sqrt{3} \qquad y = 1$$

Plug $\theta = 5\pi/6$ into the equations for x and y.

$$x = \left(4 - 4\sin\left(\frac{5\pi}{6}\right)\right)\cos\left(\frac{5\pi}{6}\right) \qquad y = \left(4 - 4\sin\left(\frac{5\pi}{6}\right)\right)\sin\left(\frac{5\pi}{6}\right)$$

$$x = \left(4 - 4\left(\frac{1}{2}\right)\right)\left(-\frac{\sqrt{3}}{2}\right) \qquad y = \left(4 - 4\left(\frac{1}{2}\right)\right)\left(\frac{1}{2}\right)$$

$$x = -\sqrt{3} \qquad y = 1$$

Therefore, the curve has horizontal tangent lines at

$$(0,0), (0,-8), (\sqrt{3},1), (-\sqrt{3},1)$$

■ 2. At which points does the polar curve have vertical tangent lines?

$$r = 6 - 6\cos\theta$$

Solution:

Use the conversion equations $x = r\cos\theta$ and $y = r\sin\theta$ and the polar equation into each of them.

$$x = r\cos\theta$$

$$x = (6 - 6\cos\theta)\cos\theta$$

$$x = 6\cos\theta - 6\cos^2\theta$$

and

$$y = r\sin\theta$$

$$y = (6 - 6\cos\theta)\sin\theta$$

$$y = 6\sin\theta - 6\cos\theta\sin\theta$$

Take the derivative of each equation.

$$\frac{dx}{d\theta} = \frac{d}{d\theta}(6\cos\theta - 6\cos^2\theta)$$

$$\frac{dx}{d\theta} = -6\sin\theta + 12\cos\theta\sin\theta$$

and

$$\frac{dy}{d\theta} = \frac{d}{d\theta}(6\sin\theta - 6\cos\theta\sin\theta)$$

$$\frac{dy}{d\theta} = 6\cos\theta - 6(\cos^2\theta - \sin^2\theta)$$

$$\frac{dy}{d\theta} = 6\cos\theta - 6\cos(2\theta)$$

Put these together to get dy/dx.

$$\frac{dy}{dx} = \frac{6\cos\theta - 6\cos(2\theta)}{-6\sin\theta + 12\cos\theta\sin\theta}$$

$$\frac{dy}{dx} = \frac{\cos\theta - \cos(2\theta)}{2\cos\theta\sin\theta - \sin\theta}$$

Vertical tangent lines exist where the denominator is 0.

$$2\cos\theta\sin\theta - \sin\theta = 0$$

$$\sin\theta(2\cos\theta - 1) = 0$$

The solutions are

$\sin\theta = 0$ gives $\theta = 0, \pi$

$2\cos\theta - 1 = 0$ gives $\theta = \frac{\pi}{3}, \frac{5\pi}{3}$

Plug $\theta = 0$ into the equations for x and y.

$$x = (6 - 6\cos(0))\cos(0) \qquad y = (6 - 6\sin(0))\sin(0)$$

$$x = (6 - 6(1))(1) \qquad y = (6 - 6(1))(0)$$

$$x = 0 \qquad y = 0$$

Plug $\theta = \pi$ into the equations for x and y.

$$x = (6 - 6\cos(\pi))\cos(\pi) \qquad y = (6 - 6\sin(\pi))\sin(\pi)$$

$$x = (6 - 6(-1))(-1) \qquad y = (6 - 6(0))(0)$$

$$x = -12 \qquad y = 0$$

Plug $\theta = \pi/3$ into the equations for x and y.

$$x = \left(6 - 6\cos\left(\frac{\pi}{3}\right)\right)\cos\left(\frac{\pi}{3}\right) \qquad y = \left(6 - 6\cos\left(\frac{\pi}{3}\right)\right)\sin\left(\frac{\pi}{3}\right)$$

$$x = \left(6 - 6\left(\frac{1}{2}\right)\right)\left(\frac{1}{2}\right) \qquad y = \left(6 - 6\left(\frac{1}{2}\right)\right)\left(\frac{\sqrt{3}}{2}\right)$$

$$x = \frac{3}{2} \qquad\qquad y = \frac{3\sqrt{3}}{2}$$

Plug $\theta = 5\pi/3$ into the equations for x and y.

$$x = \left(6 - 6\cos\left(\frac{5\pi}{3}\right)\right)\cos\left(\frac{5\pi}{3}\right) \qquad x = \left(6 - 6\cos\left(\frac{5\pi}{3}\right)\right)\sin\left(\frac{5\pi}{3}\right)$$

$$x = \left(6 - 6\left(\frac{1}{2}\right)\right)\left(\frac{1}{2}\right) \qquad y = \left(6 - 6\left(\frac{1}{2}\right)\right)\left(-\frac{\sqrt{3}}{2}\right)$$

$$x = \frac{3}{2} \qquad\qquad y = -\frac{3\sqrt{3}}{2}$$

Therefore, the curve has vertical tangent lines at

$$(0,0),\ (-12,0),\ \left(\frac{3}{2},\frac{3\sqrt{3}}{2}\right),\ \left(\frac{3}{2},-\frac{3\sqrt{3}}{2}\right)$$

■ 3. At which points does the polar curve have horizontal tangent lines?

$$r = 8 - 2\sin\theta$$

Solution:

Use the conversion equations $x = r\cos\theta$ and $y = r\sin\theta$ and the polar equation into each of them.

$$x = r\cos\theta$$

$$x = (8 - 2\sin\theta)\cos\theta$$

$$x = 8\cos\theta - 2\sin\theta\cos\theta$$

and

$$y = r\sin\theta$$

$$y = (8 - 2\sin\theta)\sin\theta$$

$$y = 8\sin\theta - 2\sin^2\theta$$

Take the derivative of each equation.

$$\frac{dx}{d\theta} = \frac{d}{d\theta}(8\cos\theta - 2\sin\theta\cos\theta)$$

$$\frac{dx}{d\theta} = -8\sin\theta - 2(\cos^2\theta - \sin^2\theta)$$

$$\frac{dx}{d\theta} = -8\sin\theta - 2\cos(2\theta)$$

and

$$\frac{dy}{d\theta} = \frac{d}{d\theta}(8\sin\theta - 2\sin^2\theta)$$

$$\frac{dy}{d\theta} = 8\cos\theta - 4\sin\theta\cos\theta$$

Put these together to get dy/dx.

$$\frac{dy}{dx} = \frac{8\cos\theta - 4\sin\theta\cos\theta}{-8\sin\theta - 2\cos(2\theta)}$$

$$\frac{dy}{dx} = \frac{4\cos\theta - 2\sin\theta\cos\theta}{-4\sin\theta - \cos(2\theta)}$$

Horizontal tangent lines exist where the numerator is 0.

$$4\cos\theta - 2\sin\theta\cos\theta = 0$$

$$2\cos\theta(2 - \sin\theta) = 0$$

$$\theta = \frac{\pi}{2}, \frac{3\pi}{2}$$

Plug $\theta = \pi/2$ into the equations for x and y.

$$x = \left(8 - 2\sin\left(\frac{\pi}{2}\right)\right)\cos\left(\frac{\pi}{2}\right) \qquad y = \left(8 - 2\sin\left(\frac{\pi}{2}\right)\right)\sin\left(\frac{\pi}{2}\right)$$

$$x = (8 - 2(1))(0) \qquad y = (8 - 2(1))(1)$$

$$x = 0 \qquad y = 6$$

Plug $\theta = 3\pi/2$ into the equations for x and y.

$$x = \left(8 - 2\sin\left(\frac{3\pi}{2}\right)\right)\cos\left(\frac{3\pi}{2}\right) \qquad y = \left(8 - 2\sin\left(\frac{3\pi}{2}\right)\right)\sin\left(\frac{3\pi}{2}\right)$$

$$x = (8 - 2(-1))(0) \qquad y = (8 - 2(-1))(-1)$$

$$x = 0 \qquad y = -10$$

Therefore, when $\theta = \pi/2$, the curve has a horizontal tangent line at $(0,6)$. And when $\theta = 3\pi/2$, the curve has a horizontal tangent at $(0, -10)$.

INTERSECTION OF THE POLAR CURVES

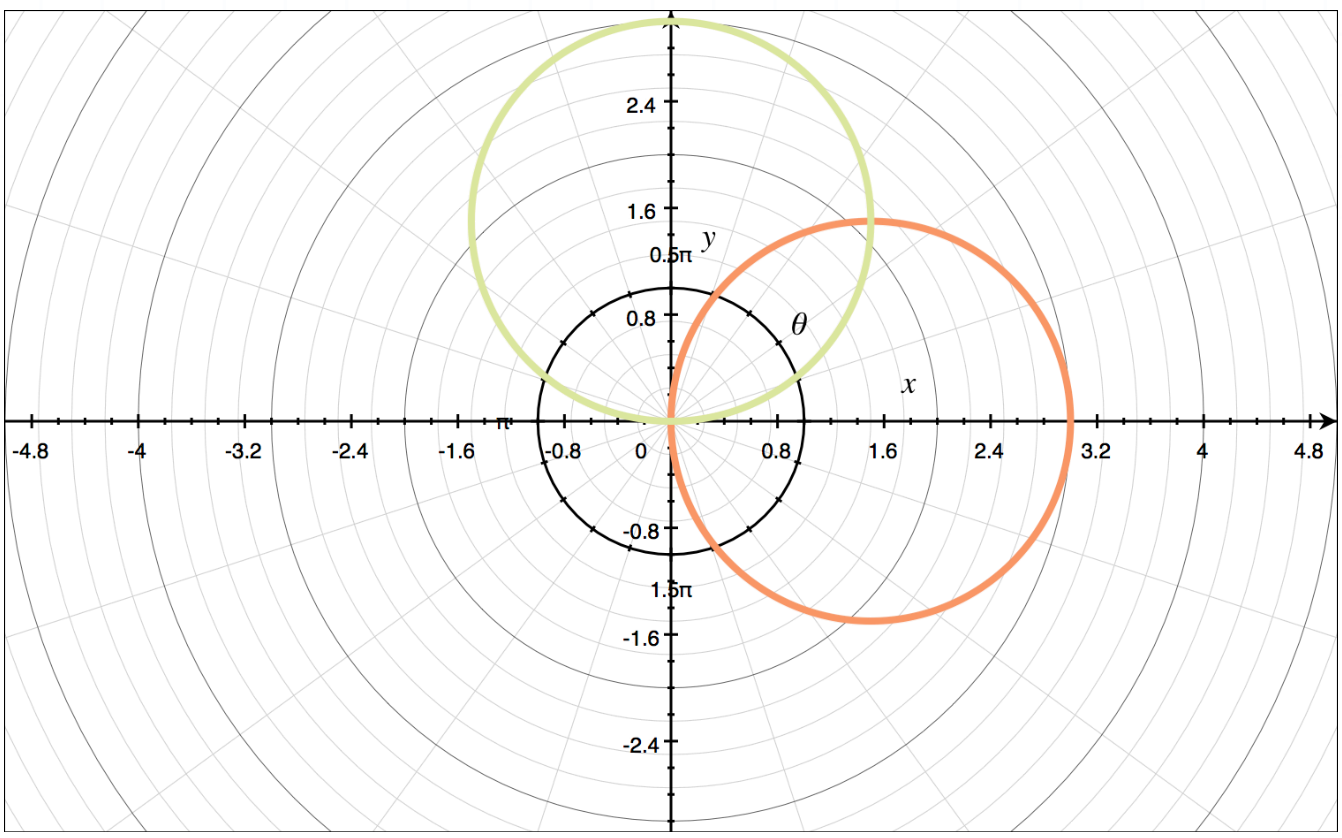
1. Find the rectangular points of intersection of the polar curves.

$$r = 3\cos\theta$$

$$r = 3\sin\theta$$

Solution:

A sketch of the polar curves is



Set the two equations equal to each other and solve for θ.

$$3\cos\theta = 3\sin\theta$$

$$\cos\theta = \sin\theta$$

$$\theta = \frac{\pi}{4}, \frac{5\pi}{4}$$

Plugging these values of θ back into $r = 3\cos\theta$ gives the polar points of intersection as

$$r = 3\cos\theta = 3\cos\frac{\pi}{4} = 3\left(\frac{\sqrt{2}}{2}\right) = \frac{3\sqrt{2}}{2}$$

$$r = 3\cos\theta = 3\cos\frac{5\pi}{4} = 3\left(-\frac{\sqrt{2}}{2}\right) = -\frac{3\sqrt{2}}{2}$$

The polar points of intersection are therefore

$$\left(\frac{3\sqrt{2}}{2}, \frac{\pi}{4}\right) \text{ and } \left(-\frac{3\sqrt{2}}{2}, \frac{5\pi}{4}\right)$$

If we convert these to rectangular points, we get

$$x = r\cos\theta \qquad y = r\sin\theta$$

$$x = \frac{3\sqrt{2}}{2}\cos\frac{\pi}{4} \qquad y = \frac{3\sqrt{2}}{2}\sin\frac{\pi}{4}$$

$$x = \frac{3\sqrt{2}}{2}\left(\frac{\sqrt{2}}{2}\right) = \frac{3(2)}{4} = \frac{3}{2} \qquad y = \frac{3\sqrt{2}}{2}\left(\frac{\sqrt{2}}{2}\right) = \frac{3(2)}{4} = \frac{3}{2}$$

and

$$x = r\cos\theta \qquad y = r\sin\theta$$

$$x = -\frac{3\sqrt{2}}{2}\cos\frac{5\pi}{4} \qquad y = -\frac{3\sqrt{2}}{2}\sin\frac{5\pi}{4}$$

$$x = -\frac{3\sqrt{2}}{2}\left(-\frac{\sqrt{2}}{2}\right) = \frac{3(2)}{4} = \frac{3}{2} \qquad y = -\frac{3\sqrt{2}}{2}\left(-\frac{\sqrt{2}}{2}\right) = \frac{3(2)}{4} = \frac{3}{2}$$

Both cases give the rectangular intersection point

$$\left(\frac{3}{2},\frac{3}{2}\right)$$

Notice though that from the sketch the curves also intersect at the pole $(0,0)$. Which means the rectangular points of intersection are

$$(0,0) \text{ and } \left(\frac{3}{2},\frac{3}{2}\right)$$

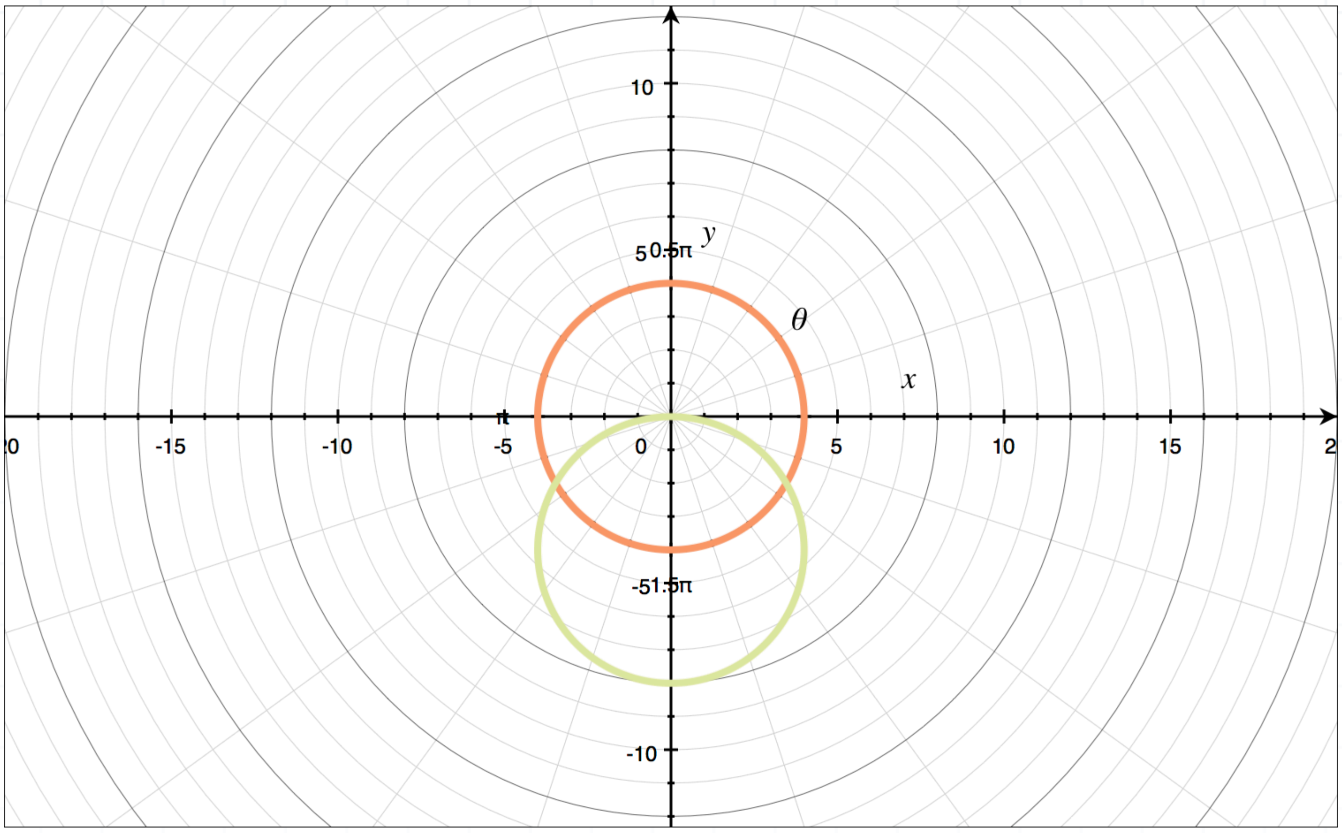
■ 2. Find the polar points of intersection of the polar curves.

$$r = 4$$

$$r = -8\sin\theta$$

Solution:

A sketch of the polar curves is



To find points of intersection, set the two equations equal to each other and solve for θ.

$$4 = -8\sin\theta$$

$$-\frac{1}{2} = \sin\theta$$

$$\theta = \sin^{-1}\left(-\frac{1}{2}\right)$$

$$\theta = \frac{7\pi}{6}, \frac{11\pi}{6}$$

Plugging these values of θ back into $r = 4$ gives the polar points of intersection as

$$\left(4,\frac{7\pi}{6}\right) \text{ and } \left(4,\frac{11\pi}{6}\right)$$

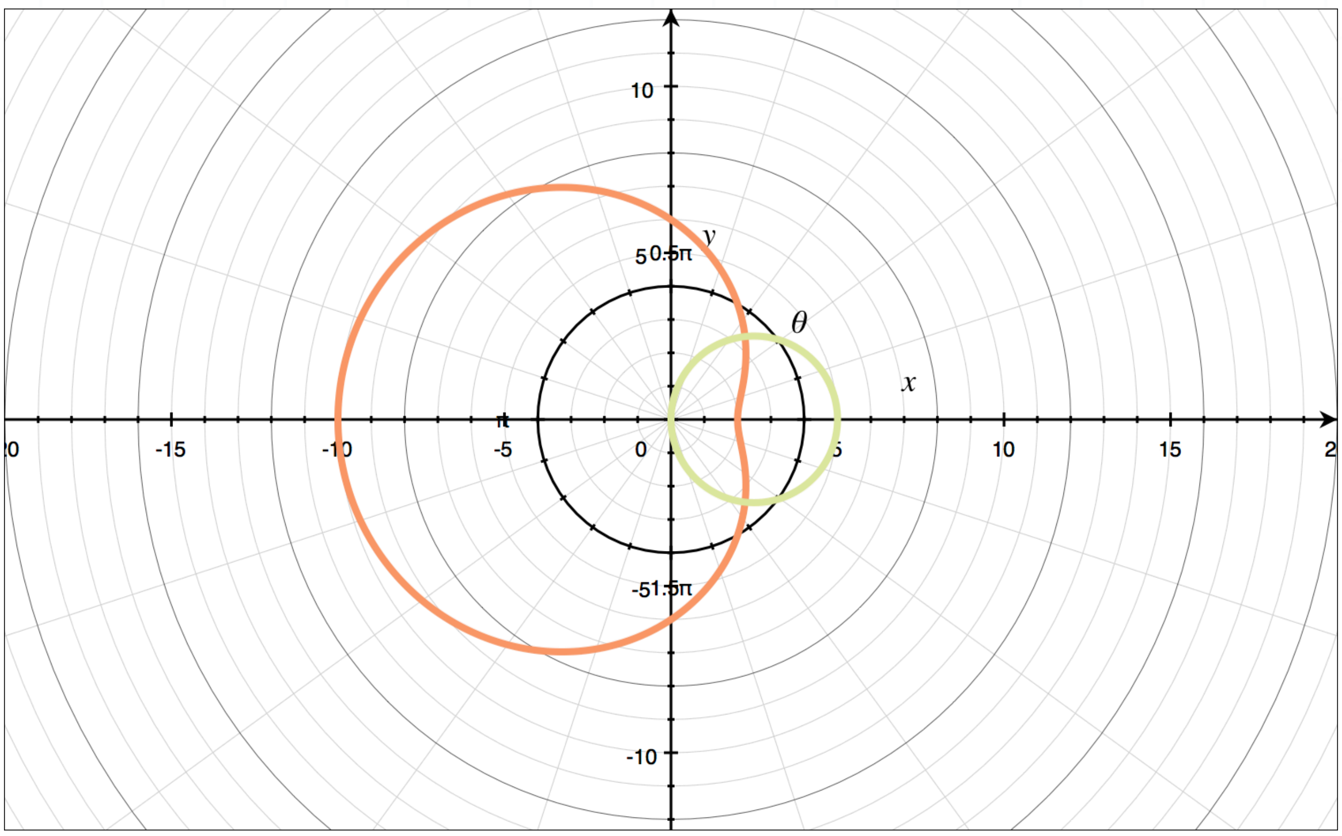
■ 3. Find the rectangular points of intersection of the polar curves.

$$r = 6 - 4\cos\theta$$

$$r = 5\cos\theta$$

Solution:

A sketch of the polar curves is



To find points of intersection, set the two equations equal to each other and solve for θ.

$$6 - 4\cos\theta = 5\cos\theta$$

$$6 = 9\cos\theta$$

$$\frac{6}{9} = \cos\theta$$

$$\theta = \cos^{-1}\left(\frac{2}{3}\right)$$

$$\theta = \pm\,0.841069$$

Plugging these values of θ back into $r = 5\cos\theta$ gives the polar points of intersection as

$$r = 5\cos(0.841069)$$

$$r = 5\left(\frac{2}{3}\right)$$

$$r = \frac{10}{3}$$

and

$$r = 5\cos(-0.841069)$$

$$r = 5\left(\frac{2}{3}\right)$$

$$r = \frac{10}{3}$$

So the polar points of intersection are

$$\left(\frac{10}{3}, 0.841069\right) \text{ and } \left(\frac{10}{3}, -0.841069\right)$$

Convert these to rectangular points. For x we get

$$x = r\cos\theta \qquad\qquad x = r\cos\theta$$

$$x_1 = \frac{10}{3}\cos(0.841069) \qquad\qquad x_2 = \frac{10}{3}\cos(-0.841069)$$

$$x_1 = \frac{10}{3}\left(\frac{2}{3}\right) \qquad\qquad x_2 = \frac{10}{3}\left(\frac{2}{3}\right)$$

$$x_1 = \frac{20}{9} \qquad\qquad x_2 = \frac{20}{9}$$

And for y we get

$$y = r\sin\theta \qquad\qquad y = r\sin\theta$$

$$y_1 = \frac{10}{3}\sin(0.841069) \qquad\qquad y_2 = \frac{10}{3}\sin(-0.841069)$$

$$y_1 = \frac{10}{3}(0.7453562121) \qquad\qquad y_2 = \frac{10}{3}(-0.7453562121)$$

$$y_1 = 2.48 \qquad\qquad y_2 = -2.48$$

So the rectangular points of intersection are

$$\left(\frac{20}{9}, 2.48\right) \text{ and } \left(\frac{20}{9}, -2.48\right)$$

$(2.22, 2.48)$ and $(2.22, -2.48)$

AREA INSIDE A POLAR CURVE

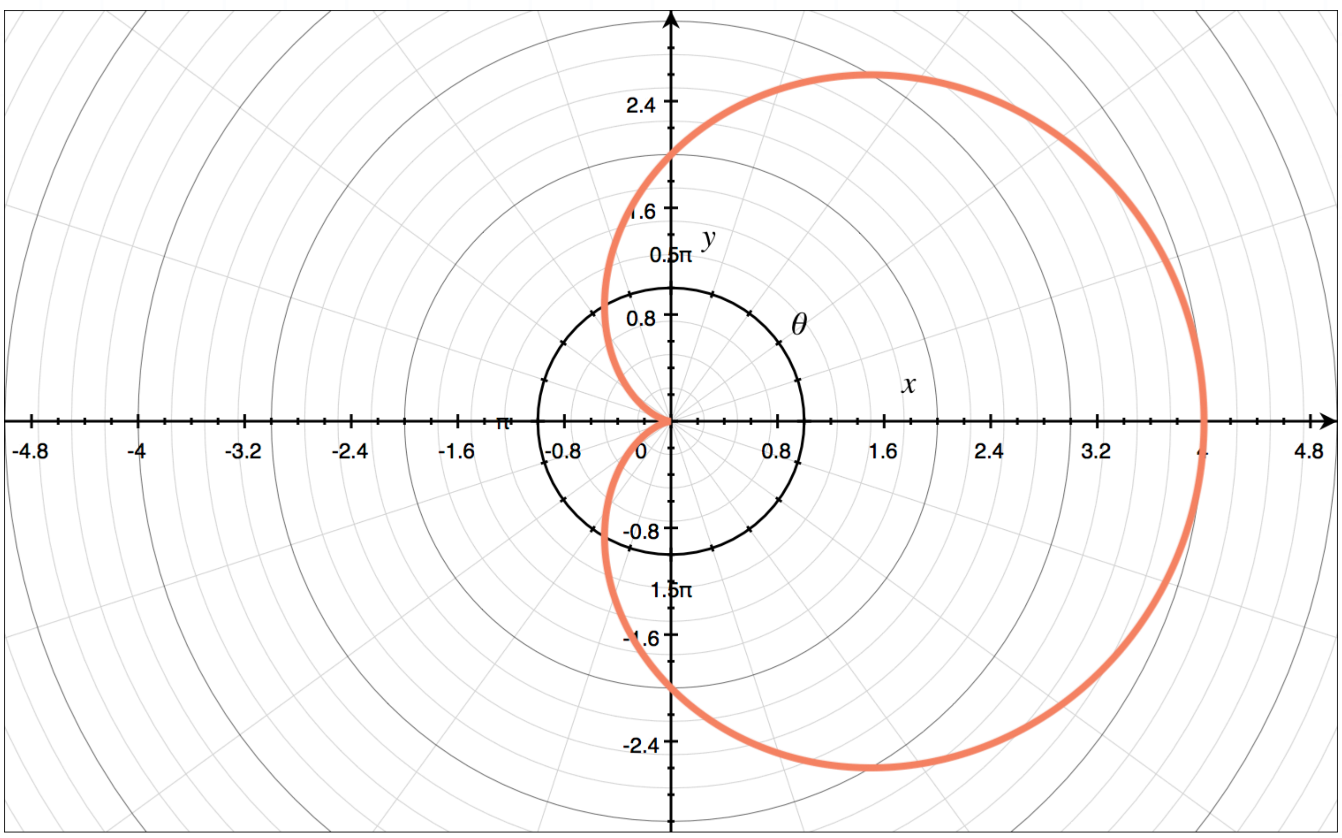
1. Find the area bounded by the polar curve over the interval.

$$r = 2 + 2\cos\theta$$

$$0 \le \theta \le 2\pi$$

Solution:

A sketch of the polar curve is



The area bounded by the curve over the interval is

$$A = \int_a^b \frac{1}{2}\left[f(\theta)\right]^2\ d\theta$$

$$A = \int_0^{2\pi} \frac{1}{2}(2 + 2\cos\theta)^2 \ d\theta$$

$$A = \frac{1}{2}\int_0^{2\pi} 4 + 8\cos\theta + 4\cos^2\theta \ d\theta$$

Use the trig identity

$$\cos^2\theta = \frac{1}{2}(1 + \cos(2\theta))$$

to substitute.

$$A = \frac{1}{2}\int_0^{2\pi} 4 + 8\cos\theta + 4 \cdot \frac{1}{2}(1 + \cos(2\theta)) \ d\theta$$

$$A = \frac{1}{2}\int_0^{2\pi} 4 + 8\cos\theta + 2 + 2\cos(2\theta) \ d\theta$$

$$A = \int_0^{2\pi} 3 + 4\cos\theta + \cos(2\theta) \ d\theta$$

Integrate, then evaluate over the interval.

$$A = 3\theta + 4\sin\theta + \frac{1}{2}\sin(2\theta)\Big|_0^{2\pi}$$

$$A = 3(2\pi) + 4\sin(2\pi) + \frac{1}{2}\sin(2 \cdot 2\pi) - \left(3(0) + 4\sin(0) + \frac{1}{2}\sin(2 \cdot 0)\right)$$

$$A = 6\pi + 4(0) + \frac{1}{2}(0)$$

$$A = 6\pi$$

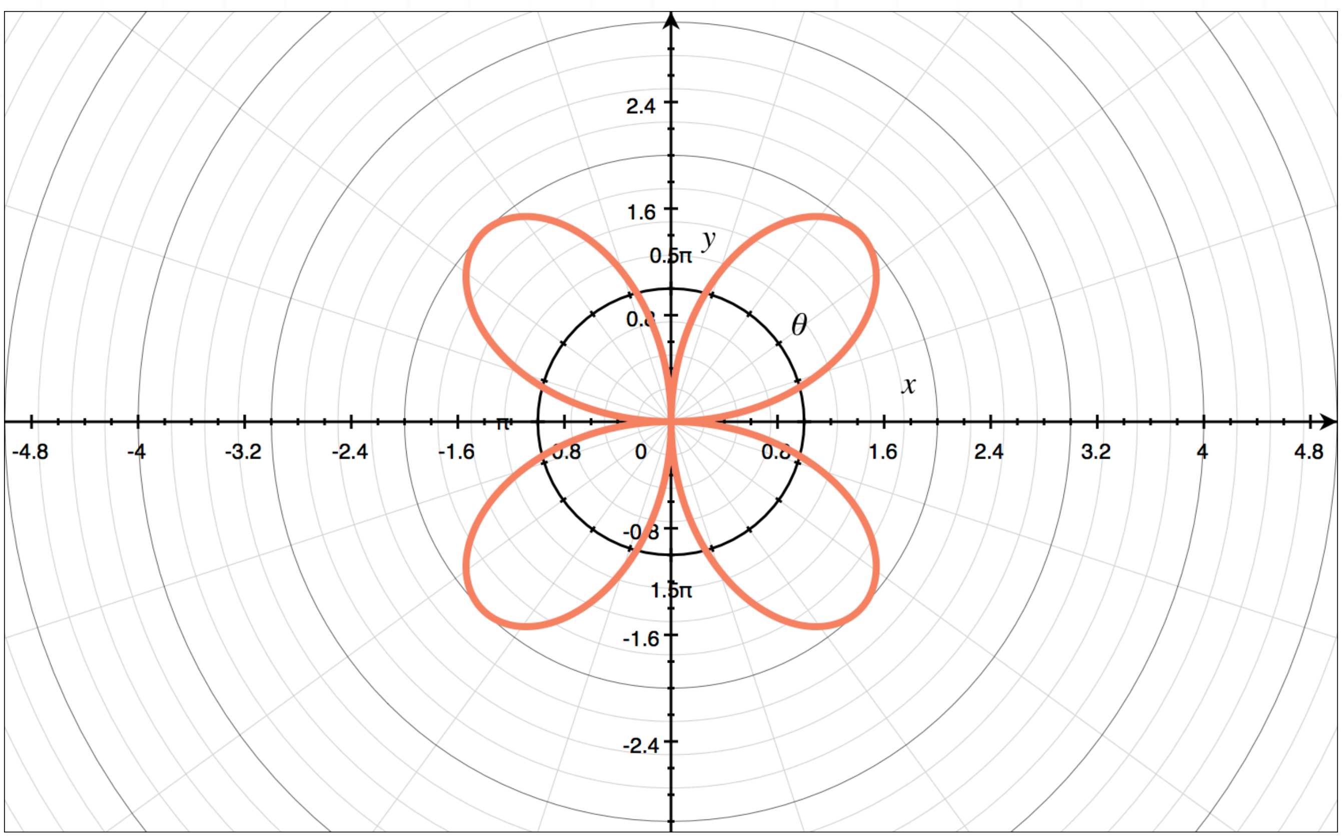
2. Find the area bounded by the polar curve over the interval.

$$r = 2\sin 2\theta$$

$$0 \le \theta \le 2\pi$$

Solution:

A sketch of the polar curve is



The area bounded by the curve over the interval is

$$A = \int_a^b \frac{1}{2}\left[f(\theta)\right]^2\ d\theta$$

$$A = \int_0^{2\pi} \frac{1}{2}(2\sin(2\theta))^2\ d\theta$$

$$A = \int_0^{2\pi} \frac{1}{2}(4\sin^2(2\theta))\ d\theta$$

$$A = 2\int_0^{2\pi} \sin^2(2\theta)\ d\theta$$

Use the trig identity

$$\sin^2\theta = \frac{1}{2}(1 - \cos(2\theta))$$

$$\sin^2(2\theta) = \frac{1}{2}(1 - \cos(4\theta))$$

to substitute.

$$A = 2\int_0^{2\pi} \frac{1}{2}(1 - \cos(4\theta))\ d\theta$$

$$A = \int_0^{2\pi} 1 - \cos(4\theta)\ d\theta$$

Integrate, then evaluate over the interval.

$$A = \theta - \frac{1}{4}\sin(4\theta)\Big|_0^{2\pi}$$

$$A = 2\pi - \frac{1}{4}\sin(4\cdot 2\pi) - \left(0 - \frac{1}{4}\sin(4\cdot 0)\right)$$

$$A=2\pi-\frac{1}{4}\sin(8\pi)$$

$$A=2\pi-\frac{1}{4}(0)$$

$$A=2\pi$$

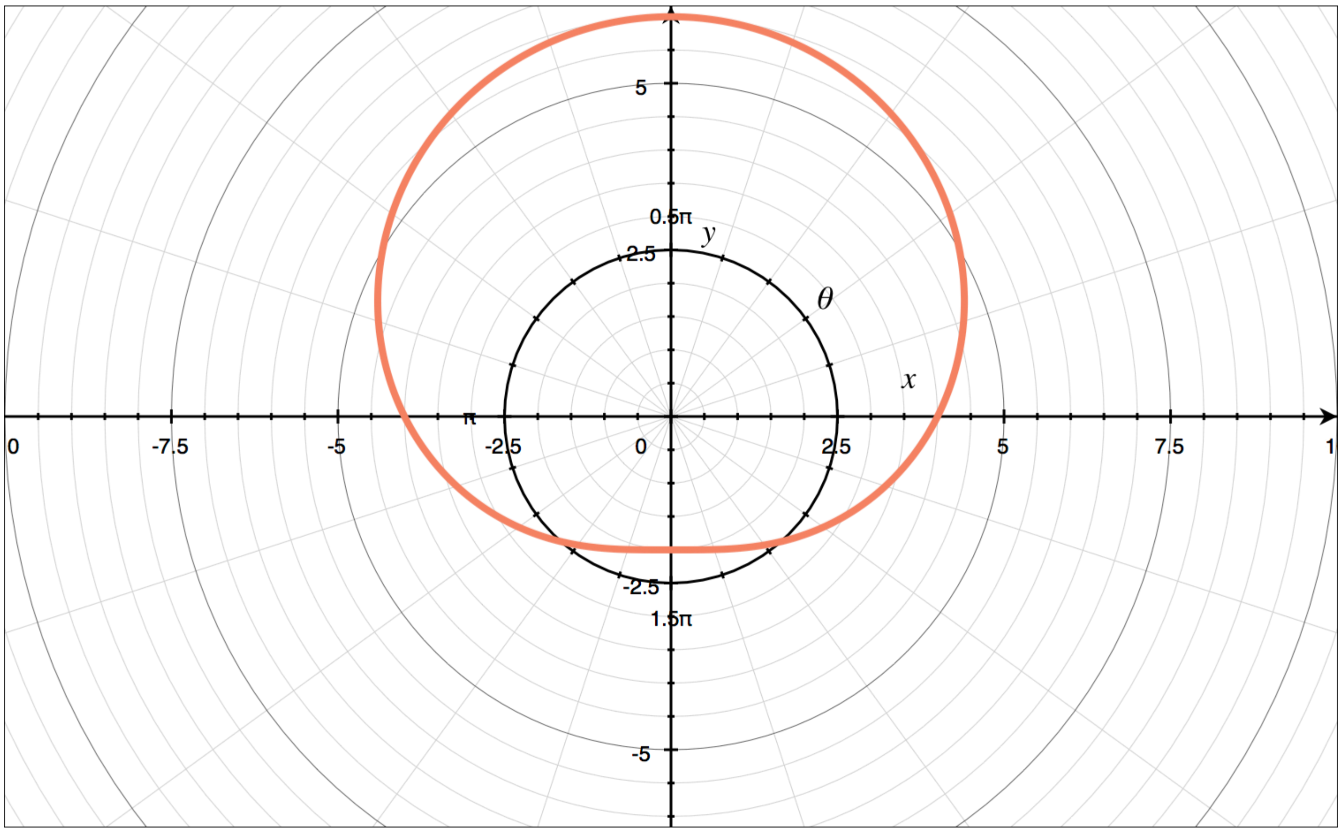
3. Find the area bounded by the polar curve over the interval.

$$r=4+2\sin\theta$$

$$0\leq\theta\leq2\pi$$

Solution:

A sketch of the polar curve is



The area bounded by the curve over the interval is

$$A = \int_a^b \frac{1}{2} \left[f(\theta)\right]^2 \ d\theta$$

$$A = \int_0^{2\pi} \frac{1}{2}(4 + 2\sin\theta)^2 \ d\theta$$

$$A = \frac{1}{2}\int_0^{2\pi} 16 + 16\sin\theta + 4\sin^2\theta \ d\theta$$

Use the trig identity

$$\sin^2\theta = \frac{1}{2}(1 - \cos(2\theta))$$

to substitute.

$$A = \frac{1}{2}\int_0^{2\pi} 16 + 16\sin\theta + 4 \cdot \frac{1}{2}(1 - \cos(2\theta)) \ d\theta$$

$$A = \frac{1}{2}\int_0^{2\pi} 16 + 16\sin\theta + 2 - 2\cos(2\theta) \ d\theta$$

$$A = \int_0^{2\pi} 9 + 8\sin\theta - \cos(2\theta) \ d\theta$$

Integrate, then evaluate over the interval.

$$A = 9\theta - 8\cos\theta - \frac{1}{2}\sin(2\theta)\Big|_0^{2\pi}$$

$$A = 9(2\pi) - 8\cos(2\pi) - \frac{1}{2}\sin(2 \cdot 2\pi) - \left(9(0) - 8\cos(0) - \frac{1}{2}\sin(2 \cdot 0)\right)$$

$$A = 18\pi - 8(1) - \frac{1}{2}(0) + 8(1)$$

$$A = 18\pi - 8 + 8$$

$$A = 18\pi$$

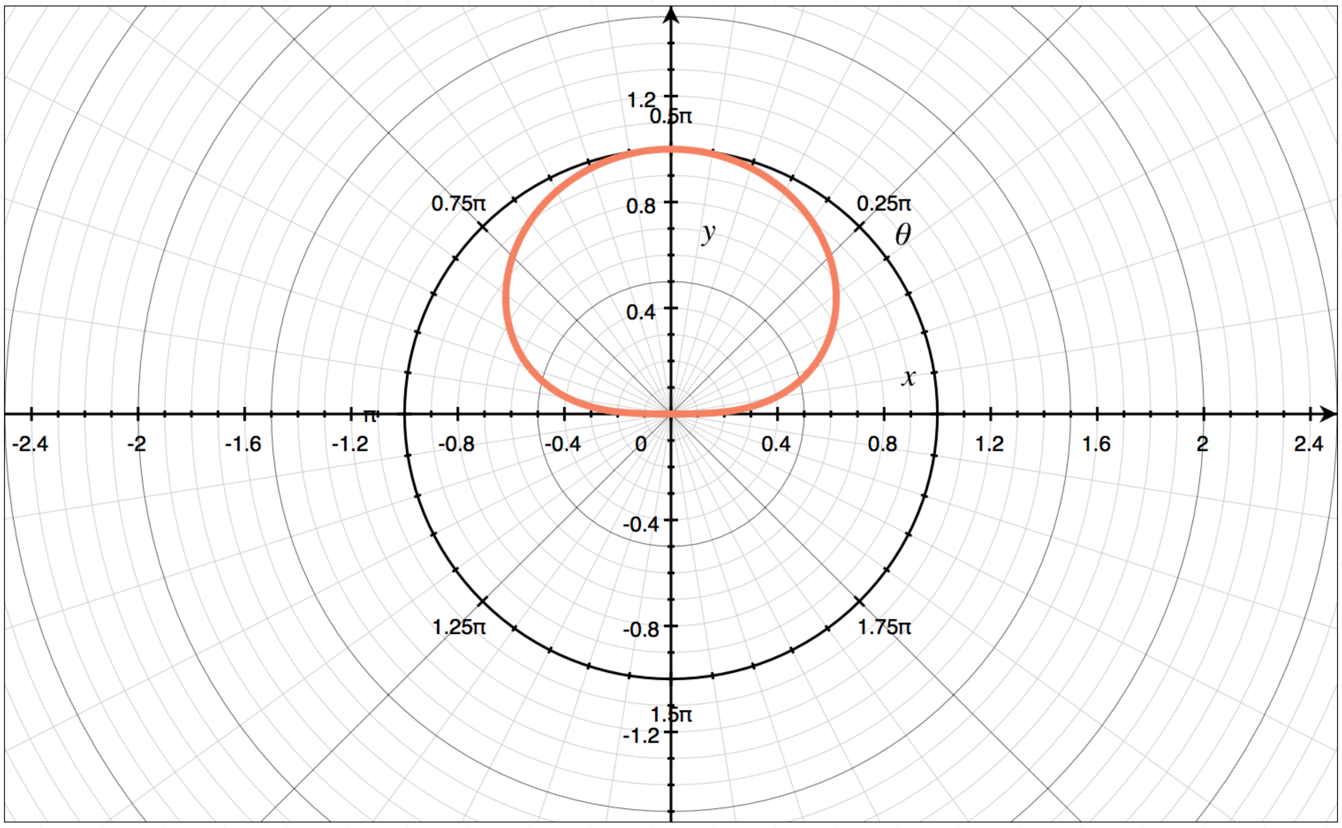
■ 4. Find the area bounded by the polar curve over the interval.

$$r^2 = \sin\theta$$

$$0 \le \theta \le \pi$$

Solution:

A sketch of the polar curve is



Since

$$r^2 = \sin\theta$$

$$r = \pm\sqrt{\sin\theta}$$

$r = \sqrt{\sin\theta}$ and $r = -\sqrt{\sin\theta}$

The area bounded by the curve over the interval is

$$A = \int_a^b \frac{1}{2}\left[f(\theta)\right]^2\ d\theta$$

$$A = \int_0^{\pi} \frac{1}{2}\left(\sqrt{\sin\theta}\right)^2\ d\theta + \int_0^{\pi} \frac{1}{2}\left(-\sqrt{\sin\theta}\right)^2\ d\theta$$

$$A = \frac{1}{2}\int_0^{\pi} \sin\theta\ d\theta + \frac{1}{2}\int_0^{\pi} \sin\theta\ d\theta$$

$$A = \int_0^{\pi} \sin\theta \ d\theta$$

Integrate, then evaluate over the interval.

$$A = -\cos\theta \Big|_0^{\pi}$$

$$A = -\cos\pi - (-\cos(0))$$

$$A = -(-1) + 1$$

$$A = 1 + 1$$

$$A = 2$$

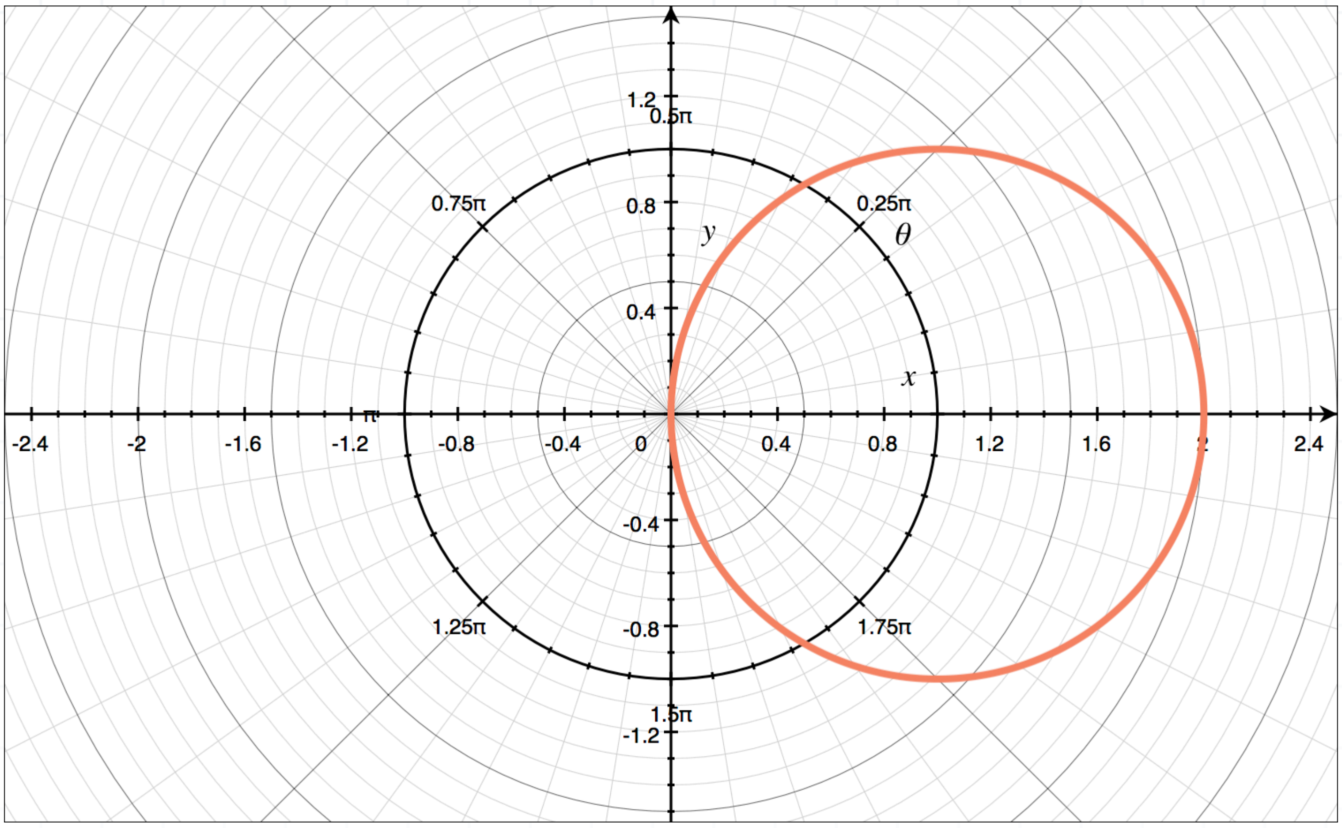
■ 5. Find the area bounded by the polar curve over the interval.

$$r = 2\cos\theta$$

$$-\frac{\pi}{4} \le \theta \le \frac{\pi}{4}$$

Solution:

The graph of the polar region is



The area bounded by the curve over the interval is

$$A=\int_a^b \frac{1}{2}\left[f(\theta)\right]^2\ d\theta$$

$$A=\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{1}{2}(2\cos\theta)^2\ d\theta$$

$$A=\frac{1}{2}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} 4\cos^2\theta\ d\theta$$

$$A=2\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \cos^2\theta\ d\theta$$

Use the trig identity

$$\cos^2\theta=\frac{1}{2}(1+\cos(2\theta))$$

to substitute.

$$A = 2\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{1}{2}(1+\cos(2\theta))\ d\theta$$

$$A = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} 1+\cos(2\theta)\ d\theta$$

Integrate, then evaluate over the interval.

$$A = \theta + \frac{1}{2}\sin(2\theta)\Big|_{-\frac{\pi}{4}}^{\frac{\pi}{4}}$$

$$A = \frac{\pi}{4} + \frac{1}{2}\sin\left(2\cdot\frac{\pi}{4}\right) - \left(-\frac{\pi}{4} + \frac{1}{2}\sin\left(2\cdot -\frac{\pi}{4}\right)\right)$$

$$A = \frac{\pi}{4} + \frac{1}{2}\sin\frac{\pi}{2} + \frac{\pi}{4} - \frac{1}{2}\sin\left(-\frac{\pi}{2}\right)$$

$$A = \frac{\pi}{2} + \frac{1}{2}\sin\frac{\pi}{2} - \frac{1}{2}\sin\left(-\frac{\pi}{2}\right)$$

$$A = \frac{\pi}{2} + \frac{1}{2}(1) - \frac{1}{2}(-1)$$

$$A = \frac{\pi}{2} + 1$$

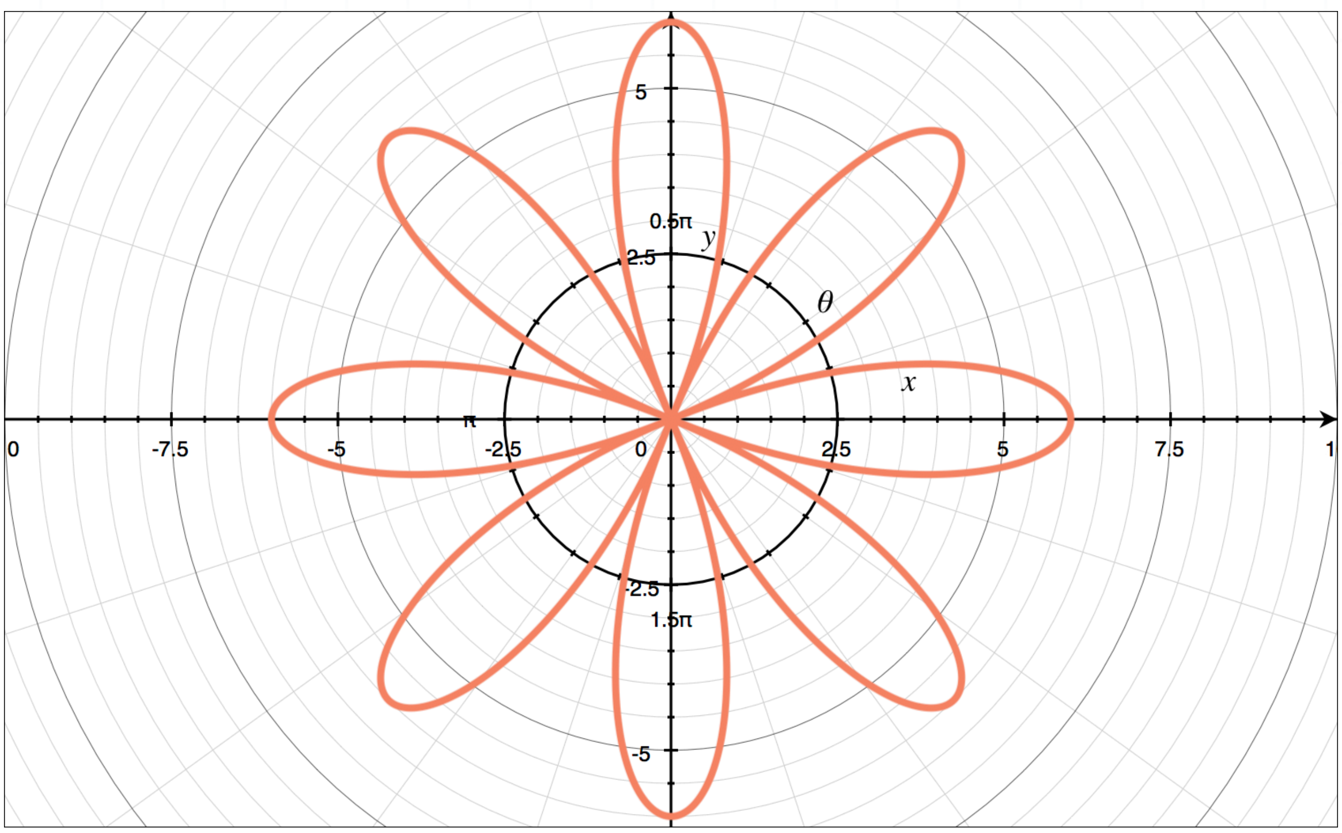
AREA BOUNDED BY ONE LOOP OF A POLAR CURVE

1. Find the area of one loop of the polar curve.

$$r = 6\cos(4\theta)$$

Solution:

A sketch of the polar curve is



Setting $4\theta = \pi/2$ gives $\theta = \pi/8$.

At $\theta = 0$, $r = 6\cos(4(0)) = 6$

At $\theta = \pi/8$, $r = 6\cos(4(\pi/8)) = 0$

So these angles define the top half of the loop that straddles the positive side of the horizontal axis. We'll use these limits of integration and then double the integral to get the area of the full loop.

$$A = 2\int_a^b \frac{1}{2}r^2\ d\theta$$

$$A = \int_a^b r^2\ d\theta$$

$$A = \int_0^{\frac{\pi}{8}} (6\cos(4\theta))^2\ d\theta$$

$$A = \int_0^{\frac{\pi}{8}} 36\cos^2(4\theta)\ d\theta$$

Use the trig identity

$$\cos^2\theta = \frac{1}{2}(1+\cos(2\theta))$$

$$\cos^2(4\theta) = \frac{1}{2}(1+\cos(8\theta))$$

to substitute.

$$A = \int_0^{\frac{\pi}{8}} 36\cdot\frac{1}{2}(1+\cos(8\theta))\ d\theta$$

$$A = 18\int_0^{\frac{\pi}{8}} 1+\cos(8\theta)\ d\theta$$

Integrate, then evaluate over the interval.

$$A = 18\left(\theta + \frac{1}{8}\sin(8\theta)\right)\Big|_0^{\frac{\pi}{8}}$$

$$A = 18\left(\frac{\pi}{8} + \frac{1}{8}\sin\left(8 \cdot \frac{\pi}{8}\right)\right) - 18\left(0 + \frac{1}{8}\sin(8 \cdot 0)\right)$$

$$A = 18\left(\frac{\pi}{8} + \frac{1}{8}\sin\pi\right)$$

$$A = \frac{9}{4}(\pi + 0)$$

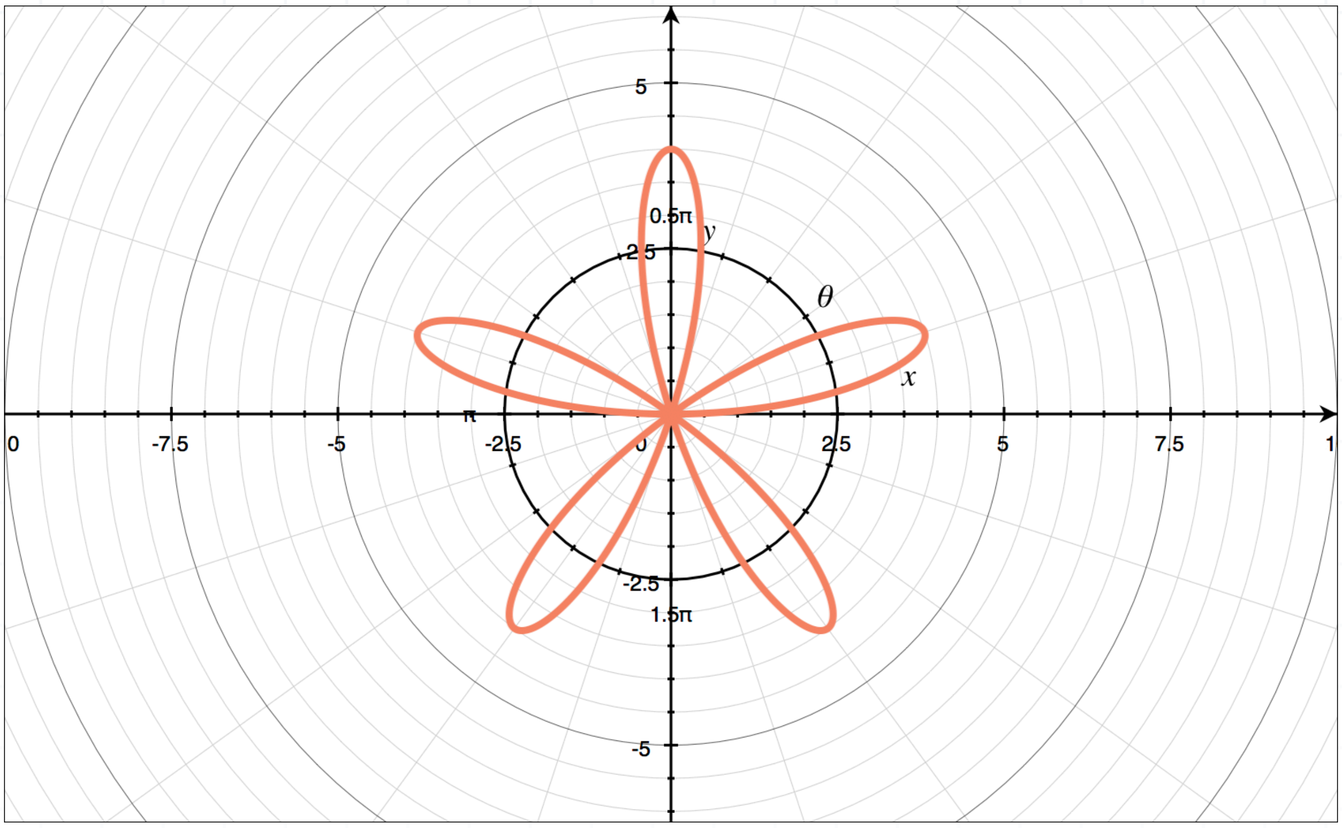
$$A = \frac{9\pi}{4}$$

■ 2. Find the area of one loop of the polar curve.

$$r = 4\sin(5\theta)$$

Solution:

A sketch of the polar curve is



Setting $5\theta = \pi/2$ gives $\theta = \pi/10$.

At $\theta = 0$, $r = 4\sin(5(0)) = 0$

At $\theta = \pi/10$, $r = 4\sin(5(\pi/10)) = 4$

At $\theta = \pi/5$, $r = 4\sin(5(\pi/5)) = 0$

So $\theta = 0$ and $\theta = \pi/5$ define the first loop in the first quadrant.

$$A = \int_a^b \frac{1}{2}r^2\ d\theta$$

$$A = \int_0^{\frac{\pi}{5}} \frac{1}{2}(4\sin(5\theta))^2\ d\theta$$

$$A = \int_0^{\frac{\pi}{5}} \frac{1}{2}(16\sin^2(5\theta))\ d\theta$$

$$A = 8\int_0^{\frac{\pi}{5}} \sin^2(5\theta)\ d\theta$$

Use the trig identity

$$\sin^2\theta = \frac{1}{2}(1 - \cos(2\theta))$$

$$\sin^2(5\theta) = \frac{1}{2}(1 - \cos(10\theta))$$

to substitute.

$$A = 8\int_0^{\frac{\pi}{5}} \frac{1}{2}(1 - \cos(10\theta))\ d\theta$$

$$A = 4\int_0^{\frac{\pi}{5}} 1 - \cos(10\theta)\ d\theta$$

Integrate, then evaluate over the interval.

$$A = 4\left(\theta - \frac{1}{10}\sin(10\theta)\right)\Bigg|_0^{\frac{\pi}{5}}$$

$$A = 4\left(\frac{\pi}{5} - \frac{1}{10}\sin\left(10 \cdot \frac{\pi}{5}\right)\right) - 4\left(0 - \frac{1}{10}\sin(10 \cdot 0)\right)$$

$$A = 4\left(\frac{\pi}{5} - \frac{1}{10}\sin(2\pi)\right)$$

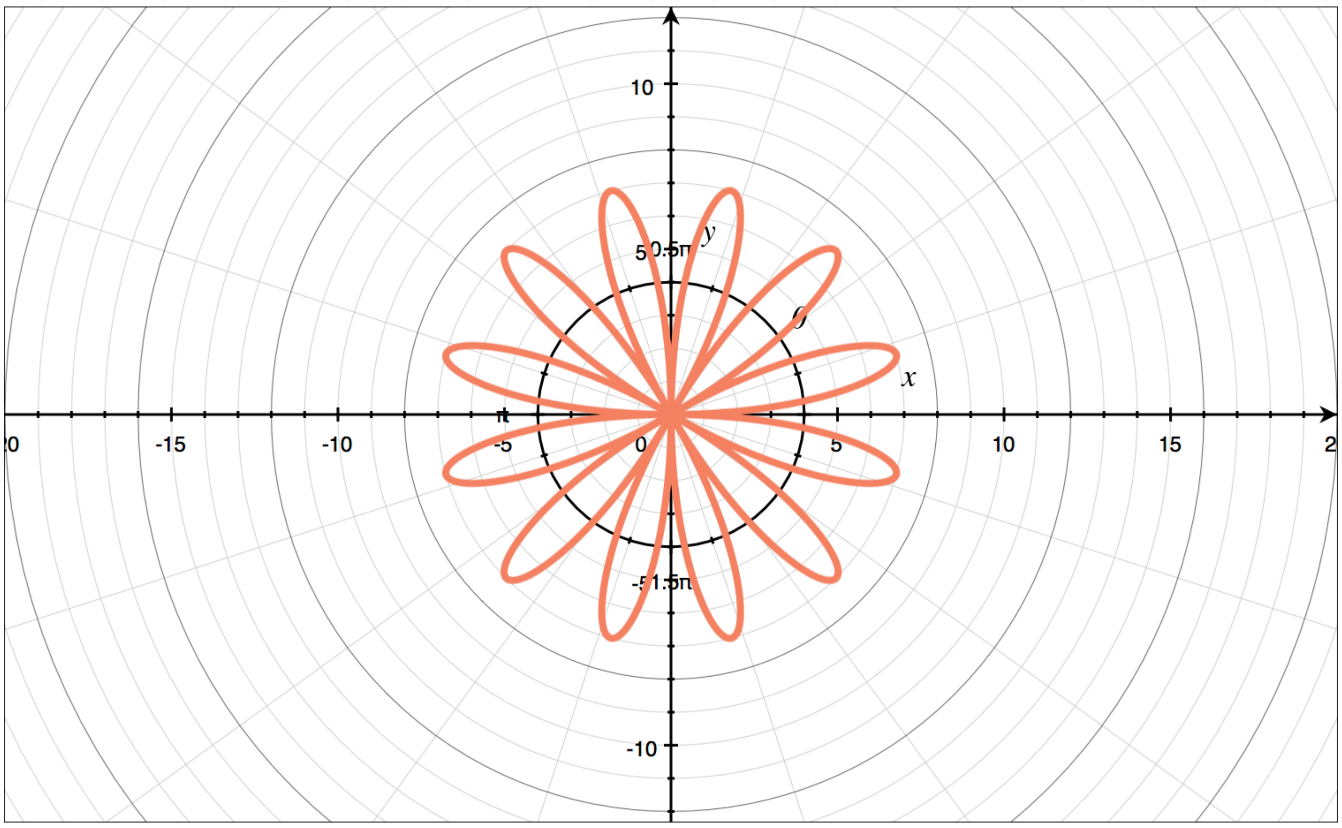
$$A = \frac{4\pi}{5}$$

3. Find the area of one loop of the polar curve.

$$r = 7\sin(6\theta)$$

Solution:

A sketch of the polar curve is



Setting $6\theta = \pi/2$ gives $\theta = \pi/12$.

At $\theta = 0$, $r = 7\sin(6(0)) = 0$

At $\theta = \pi/12$, $r = 7\sin(6(\pi/12)) = 7$

At $\theta = \pi/6$, $r = 7\sin(6(\pi/6)) = 0$

So $\theta = 0$ and $\theta = \pi/6$ define the first loop in the first quadrant.

$$A = \int_a^b \frac{1}{2} r^2 \ d\theta$$

$$A = \frac{1}{2} \int_0^{\frac{\pi}{6}} (7 \sin(6\theta))^2 \ d\theta$$

$$A = \frac{1}{2} \int_0^{\frac{\pi}{6}} 49 \sin^2(6\theta) \ d\theta$$

$$A = \frac{49}{2} \int_0^{\frac{\pi}{6}} \sin^2(6\theta) \ d\theta$$

Use the trig identity

$$\sin^2 \theta = \frac{1}{2}(1 - \cos(2\theta))$$

$$\sin^2(6\theta) = \frac{1}{2}(1 - \cos(12\theta))$$

to substitute.

$$A = \frac{49}{2} \int_0^{\frac{\pi}{6}} \frac{1}{2}(1 - \cos(12\theta)) \ d\theta$$

$$A = \frac{49}{4} \int_0^{\frac{\pi}{6}} 1 - \cos(12\theta) \ d\theta$$

Integrate, then evaluate over the interval.

$$A = \frac{49}{4} \left(\theta - \frac{1}{12} \sin(12\theta) \right) \Bigg|_0^{\frac{\pi}{6}}$$

$$A = \frac{49}{4}\left(\frac{\pi}{6} - \frac{1}{12}\sin\left(12 \cdot \frac{\pi}{6}\right)\right) - \frac{49}{4}\left(0 - \frac{1}{12}\sin(12 \cdot 0)\right)$$

$$A = \frac{49}{4}\left(\frac{\pi}{6} - \frac{1}{12}\sin(2\pi)\right)$$

$$A = \frac{49}{4}\left(\frac{\pi}{6}\right)$$

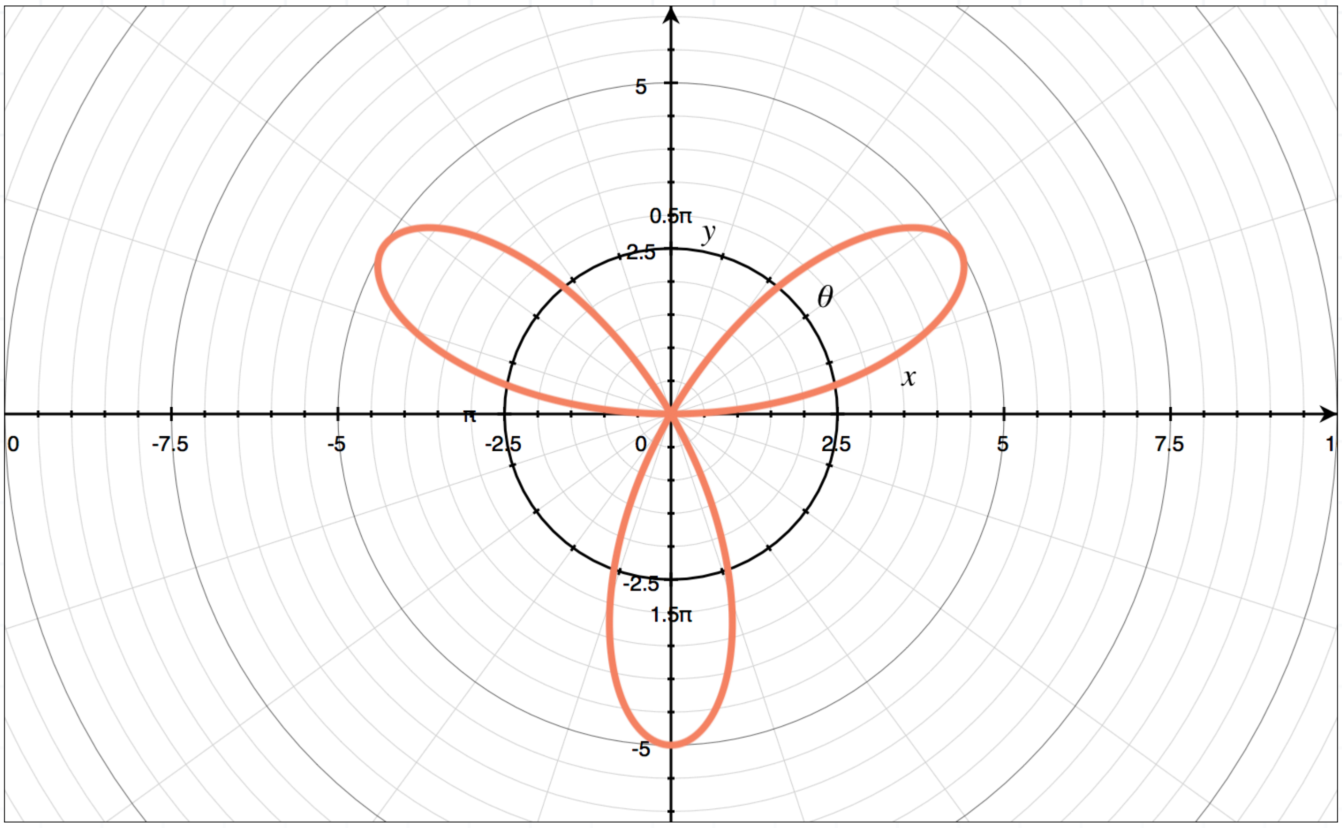
$$A = \frac{49\pi}{24}$$

■ 4. Find the area of one loop of the polar curve.

$$r = 5\sin(3\theta)$$

Solution:

A sketch of the polar curve is



Setting $3\theta = \pi/2$ gives $\theta = \pi/6$.

At $\theta = 0$, $r = 5\sin(3(0)) = 0$

At $\theta = \pi/6$, $r = 5\sin(3(\pi/6)) = 5$

At $\theta = \pi/3$, $r = 5\sin(3(\pi/3)) = 0$

So $\theta = 0$ and $\theta = \pi/3$ define the first loop in the first quadrant.

$$A = \int_a^b \frac{1}{2}r^2\ d\theta$$

$$A = \frac{1}{2}\int_0^{\frac{\pi}{3}} (5\sin(3\theta))^2\ d\theta$$

$$A = \frac{25}{2}\int_0^{\frac{\pi}{3}} \sin^2(3\theta)\ d\theta$$

Use the trig identity

$$\sin^2\theta = \frac{1}{2}(1 - \cos(2\theta))$$

$$\sin^2(3\theta) = \frac{1}{2}(1 - \cos(6\theta))$$

to substitute.

$$A = \frac{25}{2}\int_0^{\frac{\pi}{3}} \frac{1}{2}(1 - \cos(6\theta))\ d\theta$$

$$A = \frac{25}{4}\int_0^{\frac{\pi}{3}} 1 - \cos(6\theta)\ d\theta$$

Integrate, then evaluate over the interval.

$$A = \frac{25}{4}\left(\theta - \frac{1}{6}\sin(6\theta)\right)\Bigg|_0^{\frac{\pi}{3}}$$

$$A = \frac{25}{4}\left(\frac{\pi}{3} - \frac{1}{6}\sin\left(6\cdot\frac{\pi}{3}\right)\right) - \frac{25}{4}\left(0 - \frac{1}{6}\sin(6\cdot 0)\right)$$

$$A = \frac{25}{4}\left(\frac{\pi}{3} - \frac{1}{6}\sin(2\pi)\right)$$

$$A = \frac{25}{4}\left(\frac{\pi}{3}\right)$$

$$A = \frac{25\pi}{12}$$

AREA BETWEEN POLAR CURVES

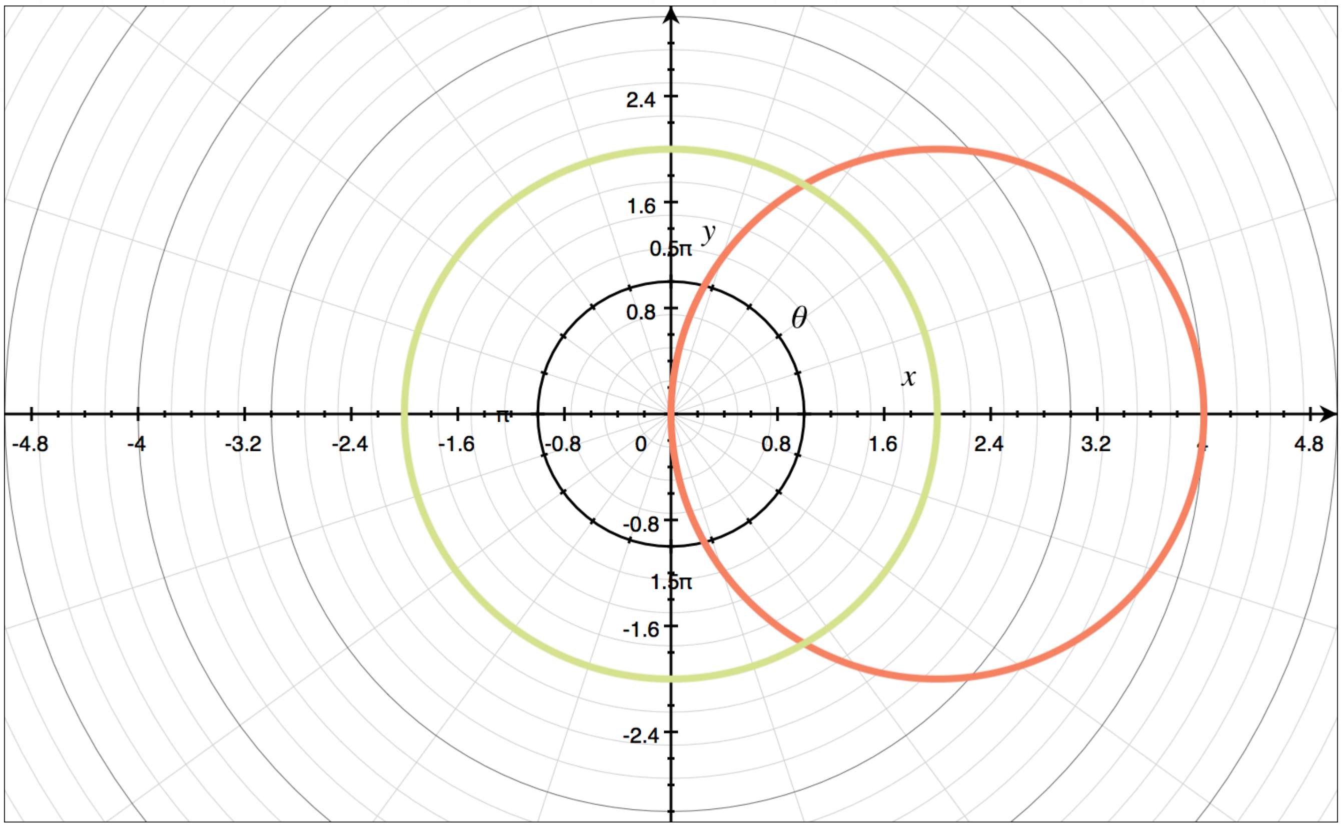
■ 1. Find the area of the region that inside both polar curves.

$$r = 4\cos\theta$$

$$r = 2$$

Solution:

A sketch of the curves is



Find the intersection points of the curves to get the bounds of integration.

$$2 = 4\cos\theta$$

$$\frac{1}{2} = \cos\theta$$

$$\theta = \cos^{-1}\left(\frac{1}{2}\right)$$

$$\theta = \frac{\pi}{3} \text{ and } \theta = -\frac{\pi}{3}$$

Looking at the sketch, we can see that $r = 2$ is outside of $r = 4\cos\theta$ on $[-\pi/3,\pi/3]$.

$$A = \frac{1}{2}\int_a^b r^2_{\text{outside}} - r^2_{\text{inside}}\ d\theta$$

$$A = \frac{1}{2}\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} (4\cos\theta)^2 - (2)^2\ d\theta$$

$$A = \frac{1}{2}\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} 16\cos^2\theta - 4\ d\theta$$

$$A = 2\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} 4\cos^2\theta - 1\ d\theta$$

Use the trig identity

$$\cos^2\theta = \frac{1}{2}(1 + \cos(2\theta))$$

to substitute.

$$A = 2\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} 4 \cdot \frac{1}{2}(1 + \cos(2\theta)) - 1\ d\theta$$

$$A=2\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}}2+2\cos(2\theta)-1\ d\theta$$

$$A=2\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}}1+2\cos(2\theta)\ d\theta$$

Integrate, then evaluate over the interval.

$$A=2(\theta+\sin(2\theta))\Big|_{-\frac{\pi}{3}}^{\frac{\pi}{3}}$$

$$A=2\theta+2\sin(2\theta)\Big|_{-\frac{\pi}{3}}^{\frac{\pi}{3}}$$

$$A=2\cdot\frac{\pi}{3}+2\sin\left(2\cdot\frac{\pi}{3}\right)-\left(2\left(-\frac{\pi}{3}\right)+2\sin\left(2\left(-\frac{\pi}{3}\right)\right)\right)$$

$$A=\frac{2\pi}{3}+2\sin\frac{2\pi}{3}-\left(-\frac{2\pi}{3}+2\sin\left(-\frac{2\pi}{3}\right)\right)$$

$$A=\frac{2\pi}{3}+2\cdot\frac{\sqrt{3}}{2}+\frac{2\pi}{3}-2\left(-\frac{\sqrt{3}}{2}\right)$$

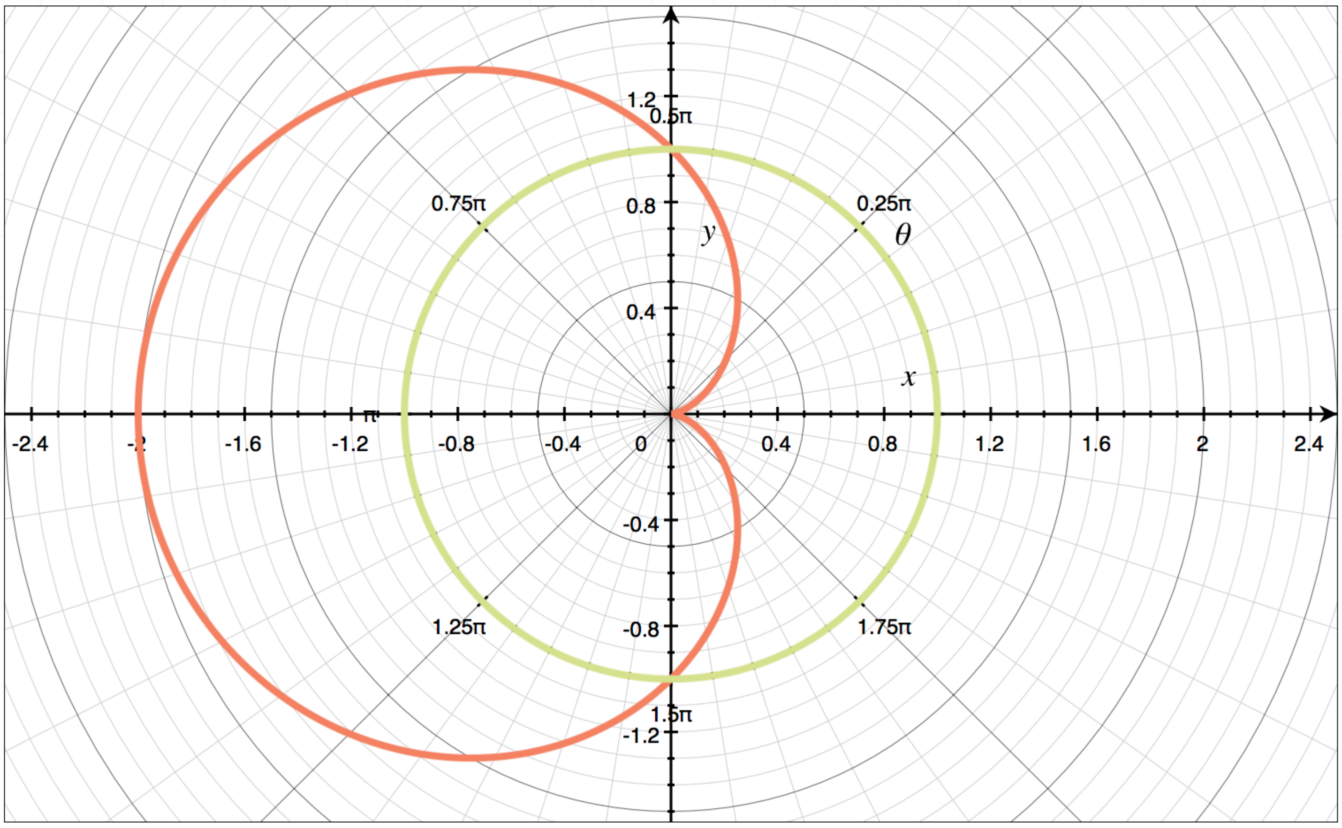
$$A=\frac{4\pi}{3}+\sqrt{3}+\sqrt{3}$$

$$A=\frac{4\pi}{3}+2\sqrt{3}$$

2. Find the area of the region inside $r=1-\cos\theta$ but outside $r=1$.

Solution:

A sketch of the curves is



Find the intersection points of the curves to get the bounds of integration.

$$1-\cos\theta=1$$

$$-\cos\theta=0$$

$$\cos\theta=0$$

$$\theta=\cos^{-1}(0)$$

$$\theta=\frac{\pi}{2} \text{ and } \theta=-\frac{\pi}{2}$$

Looking at the sketch, we can see that $r = 1 - \cos\theta$ is outside of $r = 1$ on $[\pi/2, 3\pi/2]$.

$$A = \frac{1}{2}\int_a^b r_{\text{outside}}^2 - r_{\text{inside}}^2\ d\theta$$

$$A = \frac{1}{2}\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} (1 - \cos\theta)^2 - (1)^2\ d\theta$$

$$A = \frac{1}{2}\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} 1 - 2\cos\theta + \cos^2\theta - 1\ d\theta$$

$$A = \frac{1}{2}\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \cos^2\theta - 2\cos\theta\ d\theta$$

Use the trig identity

$$\cos^2\theta = \frac{1}{2}(1 + \cos(2\theta))$$

to substitute.

$$A = \frac{1}{2}\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \frac{1}{2}(1 + \cos(2\theta)) - 2\cos\theta\ d\theta$$

$$A = \frac{1}{2}\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \frac{1}{2} + \frac{1}{2}\cos(2\theta) - 2\cos\theta\ d\theta$$

Integrate, then evaluate over the interval.

$$A=\frac{1}{2}\left(\frac{1}{2}\theta+\frac{1}{4}\sin(2\theta)-2\sin\theta\right)\Big|_{\frac{\pi}{2}}^{\frac{3\pi}{2}}$$

$$A=\frac{1}{2}\left(\frac{1}{2}\cdot\frac{3\pi}{2}+\frac{1}{4}\sin\left(2\cdot\frac{3\pi}{2}\right)-2\sin\frac{3\pi}{2}\right)-\frac{1}{2}\left(\frac{1}{2}\cdot\frac{\pi}{2}+\frac{1}{4}\sin\left(2\cdot\frac{\pi}{2}\right)-2\sin\frac{\pi}{2}\right)$$

$$A=\frac{1}{2}\left(\frac{3\pi}{4}+\frac{1}{4}\sin 3\pi-2\sin\frac{3\pi}{2}\right)-\frac{1}{2}\left(\frac{\pi}{4}+\frac{1}{4}\sin\pi-2\sin\frac{\pi}{2}\right)$$

$$A=\frac{1}{2}\left(\frac{3\pi}{4}+\frac{1}{4}(0)-2(-1)\right)-\frac{1}{2}\left(\frac{\pi}{4}+\frac{1}{4}(0)-2(1)\right)$$

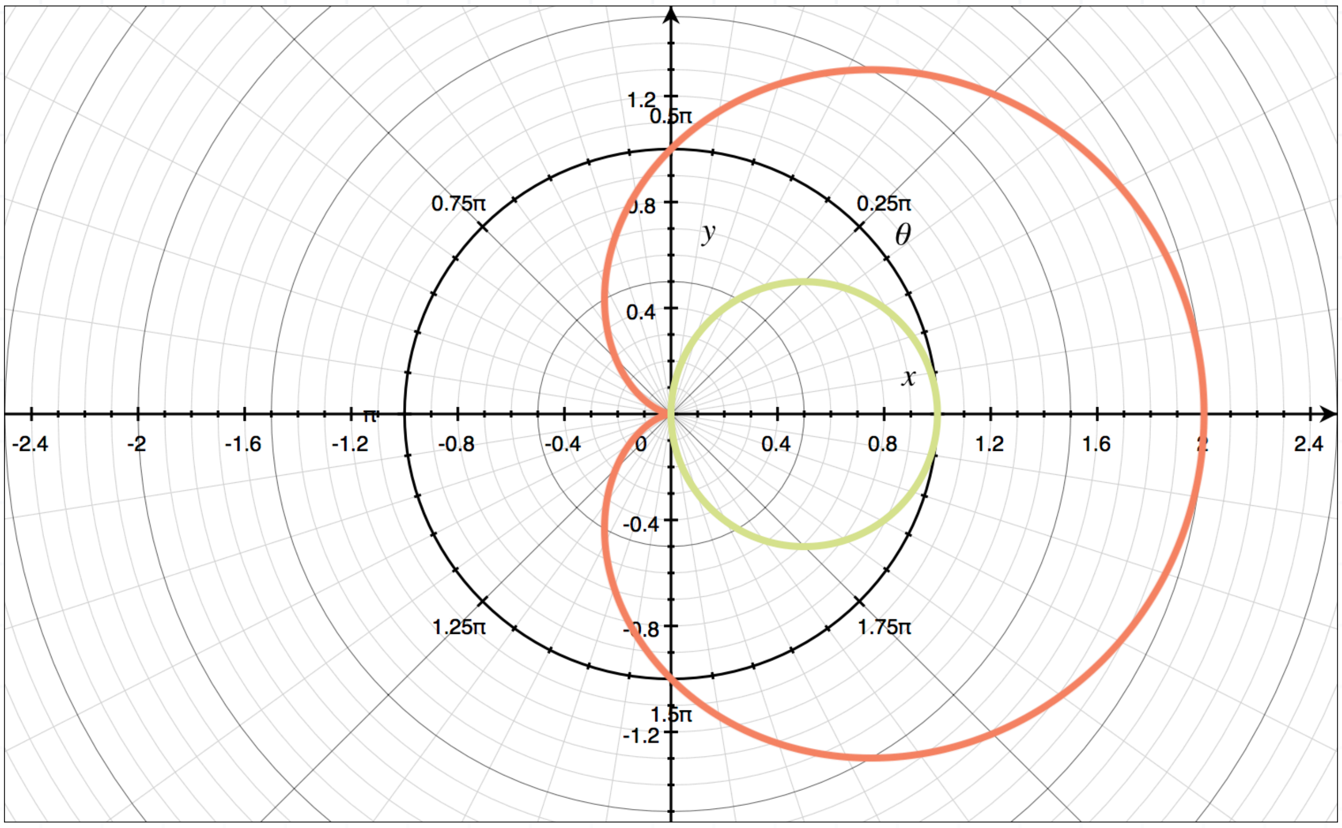
$$A=\frac{1}{2}\left(\frac{3\pi}{4}+2-\frac{\pi}{4}+2\right)$$

$$A=\frac{\pi}{4}+2$$

■ 3. Find the area of the region inside $r=1+\cos\theta$ but outside the circle $r=\cos\theta$.

Solution:

A sketch of the curves is



Since $r=1+\cos\theta$ is outside $r=\cos\theta$ everywhere, we can find the area between the curves by integrating the difference of the curves over $[0,2\pi]$.

$$A=\frac{1}{2}\int_0^{2\pi}(1+\cos\theta)^2-(\cos\theta)^2\ d\theta$$

$$A=\frac{1}{2}\int_0^{2\pi}(1+2\cos\theta+\cos^2\theta)-\cos^2\theta\ d\theta$$

$$A=\frac{1}{2}\int_0^{2\pi}1+2\cos\theta\ d\theta$$

Integrate, then evaluate over each interval.

$$A=\frac{1}{2}\left(\theta+2\sin\theta\right)\Big|_0^{2\pi}$$

$$A=\frac{1}{2}\theta+\sin\theta\Big|_0^{2\pi}$$

$$A = \frac{1}{2}(2\pi) + \sin(2\pi) - \left(\frac{1}{2}(0) + \sin(0)\right)$$

$$A = \pi + \sin(2\pi)$$

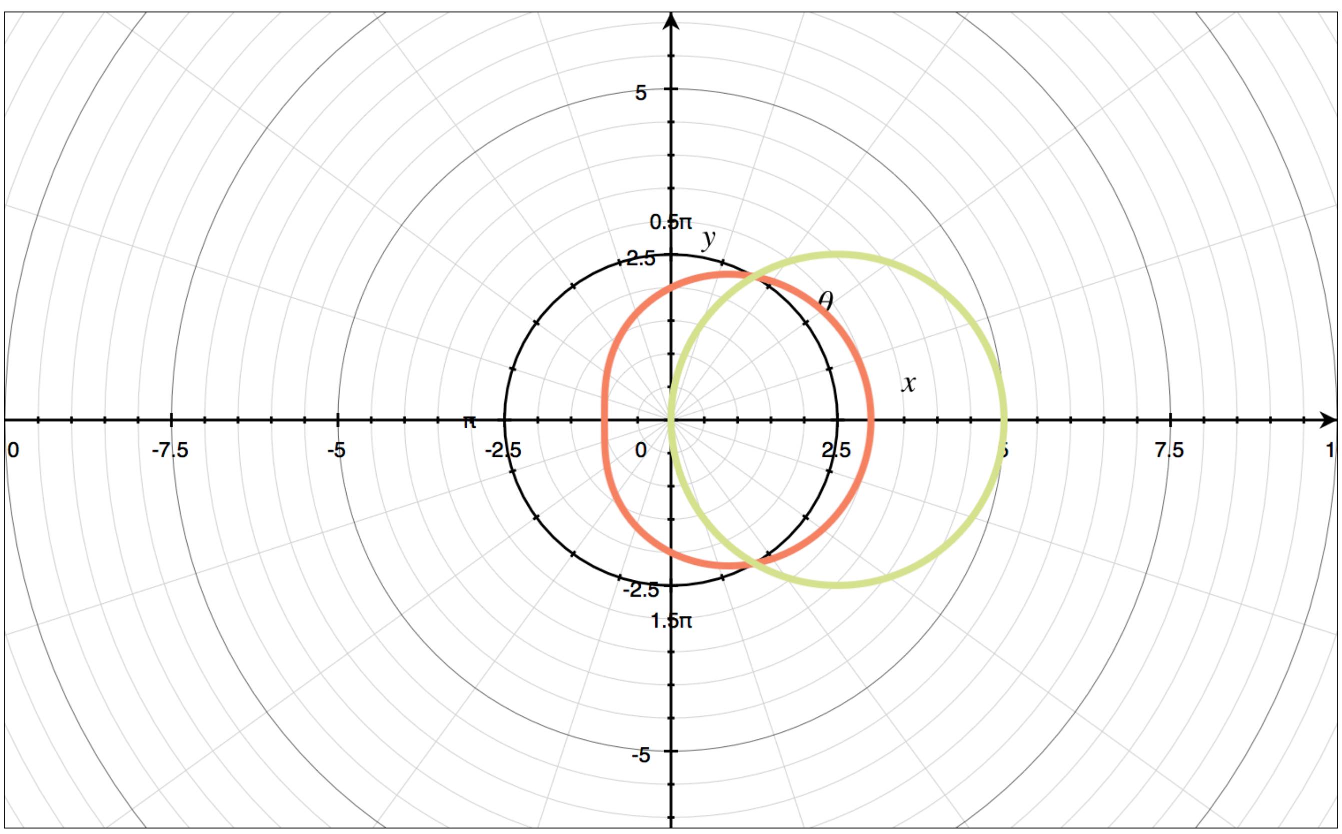
$$A = \pi + 0$$

$$A = \pi$$

■ 4. Find the area of the region inside $r = 2 + \cos\theta$ but outside the circle $r = 5\cos\theta$.

Solution:

A sketch of the curves is



Find the intersection points of the curves to get the bounds of integration.

$$5\cos\theta = 2 + \cos\theta$$

$$4\cos\theta = 2$$

$$\cos\theta = \frac{1}{2}$$

$$\theta = \cos^{-1}\left(\frac{1}{2}\right)$$

$$\theta = \frac{\pi}{3} \text{ and } \frac{5\pi}{3}$$

Looking at the sketch, we can see that $r = 2 + \cos\theta$ is outside of $r = 5\cos\theta$ on $[\pi/3, 5\pi/3]$.

$$A = \frac{1}{2}\int_a^b r_{\text{outside}}^2 - r_{\text{inside}}^2 \ d\theta$$

$$A = \frac{1}{2}\int_{\frac{\pi}{3}}^{\frac{5\pi}{3}} (2 + \cos\theta)^2 - (5\cos\theta)^2 \ d\theta$$

$$A = \frac{1}{2}\int_{\frac{\pi}{3}}^{\frac{5\pi}{3}} 4 + 4\cos\theta + \cos^2\theta - 25\cos^2\theta \ d\theta$$

$$A = \frac{1}{2}\int_{\frac{\pi}{3}}^{\frac{5\pi}{3}} 4 + 4\cos\theta - 24\cos^2\theta \ d\theta$$

$$A = 2\int_{\frac{\pi}{3}}^{\frac{5\pi}{3}} 1 + \cos\theta - 6\cos^2\theta \ d\theta$$

Use the trig identity

$$\cos^2\theta = \frac{1}{2}(1+\cos(2\theta))$$

to substitute.

$$A = 2\int_{\frac{\pi}{3}}^{\frac{5\pi}{3}} 1+\cos\theta - 6\cdot\frac{1}{2}(1+\cos(2\theta))\ d\theta$$

$$A = 2\int_{\frac{\pi}{3}}^{\frac{5\pi}{3}} 1+\cos\theta - 3 - 3\cos(2\theta)\ d\theta$$

$$A = 2\int_{\frac{\pi}{3}}^{\frac{5\pi}{3}} \cos\theta - 3\cos(2\theta) - 2\ d\theta$$

Integrate, then evaluate over the interval.

$$A = 2\left(\sin\theta - \frac{3}{2}\sin(2\theta) - 2\theta\right)\Bigg|_{\frac{\pi}{3}}^{\frac{5\pi}{3}}$$

$$A = 2\sin\theta - 3\sin(2\theta) - 4\theta\Bigg|_{\frac{\pi}{3}}^{\frac{5\pi}{3}}$$

$$A = 2\sin\frac{5\pi}{3} - 3\sin\left(2\cdot\frac{5\pi}{3}\right) - 4\cdot\frac{5\pi}{3} - \left(2\sin\frac{\pi}{3} - 3\sin\left(2\cdot\frac{\pi}{3}\right) - 4\cdot\frac{\pi}{3}\right)$$

$$A = -2\frac{\sqrt{3}}{2} - 3\sin\frac{10\pi}{3} - \frac{20\pi}{3} - 2\frac{\sqrt{3}}{2} + 3\sin\frac{2\pi}{3} + \frac{4\pi}{3}$$

$$A = -\sqrt{3} + 3\frac{\sqrt{3}}{2} - \frac{16\pi}{3} - \sqrt{3} + 3\frac{\sqrt{3}}{2}$$

$$A = -2\sqrt{3} + 3\sqrt{3} - \frac{16\pi}{3}$$

$$A = \sqrt{3} - \frac{16\pi}{3}$$

AREA INSIDE BOTH POLAR CURVES

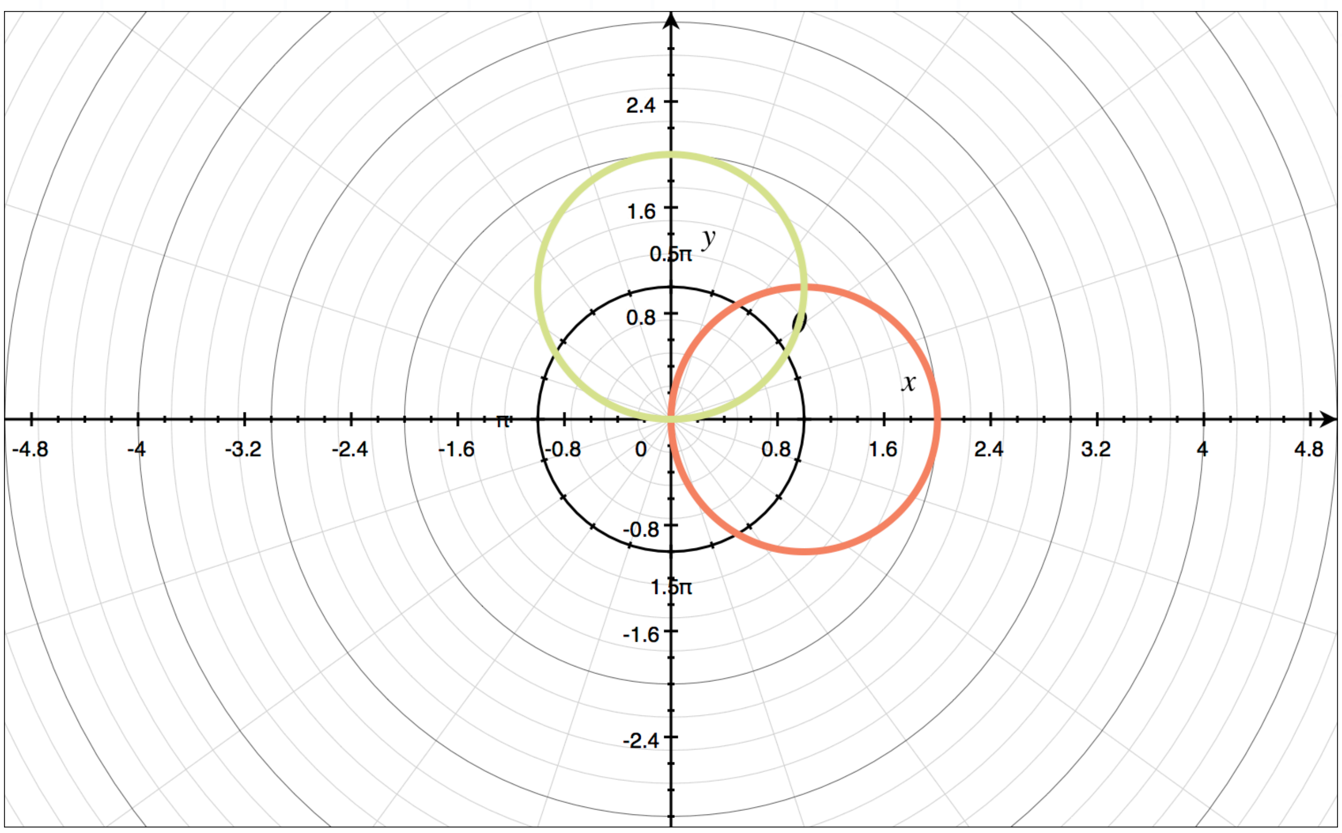
1. Find the area of the region that's inside both polar curves.

$$r = 2\cos\theta$$

$$r = 2\sin\theta$$

Solution:

A sketch of the curves is



Find points of intersection by setting the curves equal to each other.

$$2\cos\theta = 2\sin\theta$$

$$\cos\theta = \sin\theta$$

$$\theta = \frac{\pi}{4}$$

Integrating the $r = 2\sin\theta$ curve on $[0,\pi/4]$ will give the lower half of the area inside both curves, so we'll integrate the sine curve on that interval, and then double the result to get the total area inside both curves.

$$A = 2\left(\frac{1}{2}\int_0^{\frac{\pi}{4}} (2\sin\theta)^2\ d\theta\right)$$

$$A = \int_0^{\frac{\pi}{4}} 4\sin^2\theta\ d\theta$$

Use a double-angle identity to rewrite the integral.

$$A = \int_0^{\frac{\pi}{4}} 4\left(\frac{1}{2}(1-\cos(2\theta))\right)\ d\theta$$

$$A = \int_0^{\frac{\pi}{4}} 2 - 2\cos(2\theta)\ d\theta$$

Integrate, then evaluate over the interval.

$$A = 2\theta - \sin(2\theta)\Big|_0^{\frac{\pi}{4}}$$

$$A = 2\left(\frac{\pi}{4}\right) - \sin\left(2\cdot\frac{\pi}{4}\right) - (2(0) - \sin(2(0)))$$

$$A = \frac{\pi}{2} - 1$$

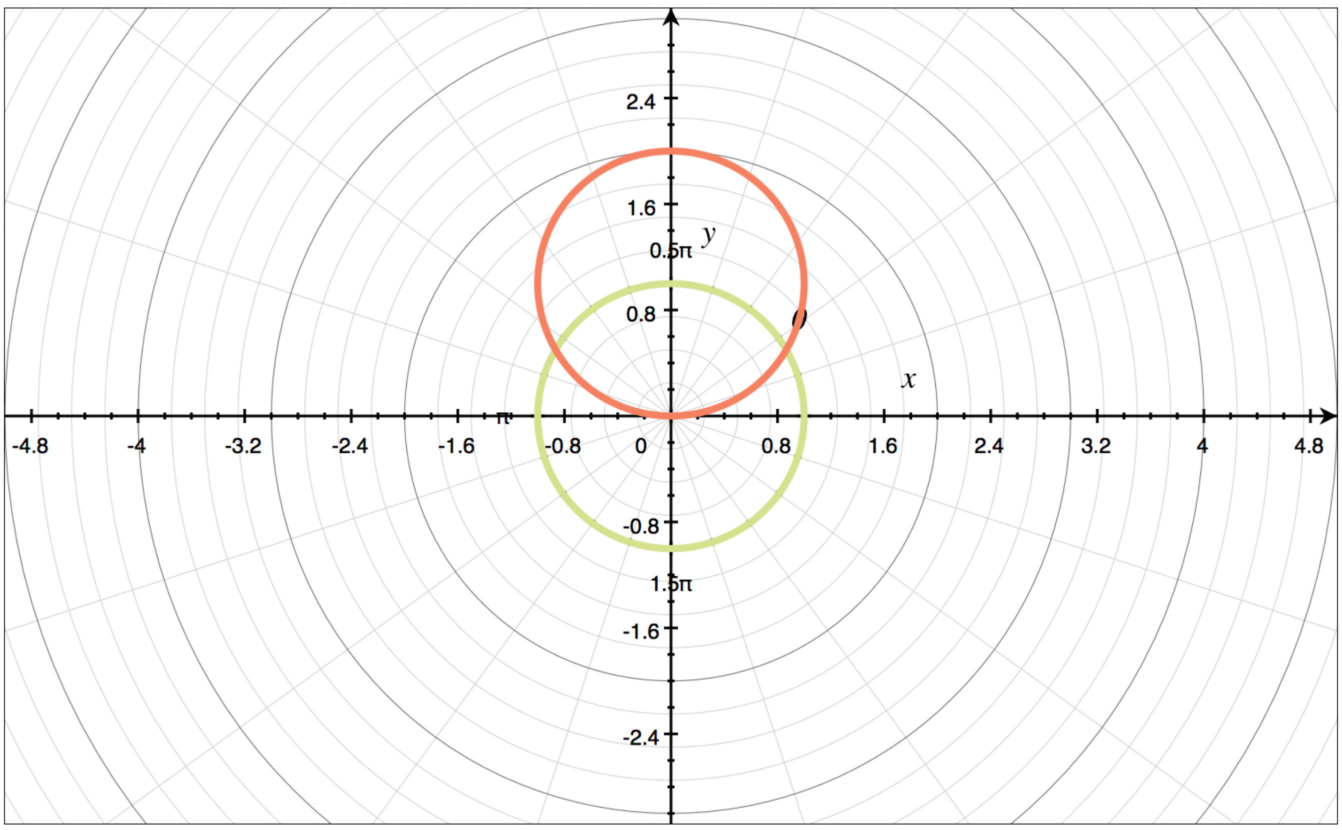
■ 2. Find the area of the region that's inside both polar curves.

$$r = 2\sin\theta$$

$$r = 1$$

Solution:

A sketch of the curves is



Find points of intersection by setting the curves equal to each other.

$$2\sin\theta = 1$$

$$\theta = \sin^{-1}\left(\frac{1}{2}\right)$$

$$\theta = \frac{\pi}{6}, \frac{5\pi}{6}$$

The overlapping area is given by the area inside $r=1$ between the points of intersection, plus the two slivers of area inside $r=2\sin\theta$ on $[0,\pi/6]$ and $[5\pi/6,\pi]$. Since the two slivers contain equal area, we can just double the area given on the interval $[0,\pi/6]$.

$$A=\frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}}(1)^2\ d\theta+2\left(\frac{1}{2}\int_0^{\frac{\pi}{6}}(2\sin\theta)^2\ d\theta\right)$$

$$A=\frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}}\ d\theta+\int_0^{\frac{\pi}{6}}4\sin^2\theta\ d\theta$$

Use a double-angle identity to rewrite the integral.

$$A=\frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}}\ d\theta+\int_0^{\frac{\pi}{6}}4\left(\frac{1}{2}(1-\cos(2\theta))\right)\ d\theta$$

$$A=\frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}}\ d\theta+2\int_0^{\frac{\pi}{6}}1-\cos(2\theta)\ d\theta$$

Integrate, then evaluate over the interval.

$$A=\frac{1}{2}\theta\Big|_{\frac{\pi}{6}}^{\frac{5\pi}{6}}+2\left(\theta-\frac{1}{2}\sin(2\theta)\right)\Big|_0^{\frac{\pi}{6}}$$

$$A=\frac{1}{2}\left(\frac{5\pi}{6}\right)-\frac{1}{2}\left(\frac{\pi}{6}\right)+2\left(\frac{\pi}{6}\right)-\sin\left(2\cdot\frac{\pi}{6}\right)-2(0)-\sin(2\cdot 0)$$

$$A=\frac{1}{2}\left(\frac{5\pi}{6}-\frac{\pi}{6}\right)+\frac{\pi}{3}-\sin\left(\frac{\pi}{3}\right)-\sin(0)$$

$$A = \frac{\pi}{3} + \frac{\pi}{3} - \frac{\sqrt{3}}{2}$$

$$A = \frac{2\pi}{3} - \frac{\sqrt{3}}{2}$$

$$A = \frac{4\pi - 3\sqrt{3}}{6}$$

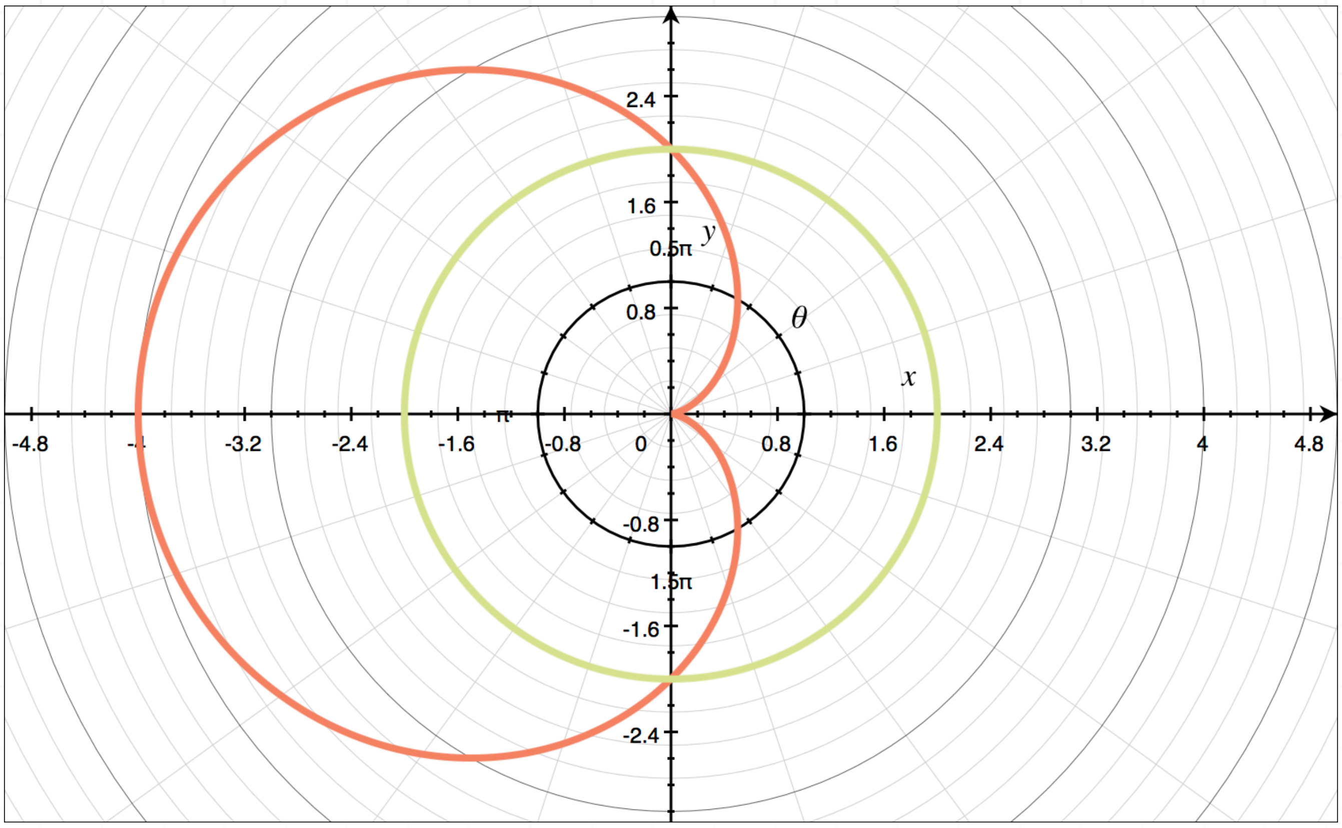
■ 3. Find the area of the region that's inside both polar curves.

$$r = 2(1 - \cos\theta)$$

$$r = 2$$

Solution:

A sketch of the curves is



Find points of intersection by setting the curves equal to each other.

$$2 - 2\cos\theta = 2$$

$$-2\cos\theta = 0$$

$$\cos\theta = 0$$

$$\theta = \frac{\pi}{2}, \frac{3\pi}{2}$$

The overlapping area is given by the area inside $r = 2$ between the points of intersection, plus the two slivers of area inside $r = 2(1 - \cos\theta)$ on $[0,\pi/2]$ and $[3\pi/2,2\pi]$. Since the two slivers contain equal area, we can just double the area given on the interval $[0,\pi/2]$.

$$A = \frac{1}{2}\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} (2)^2\ d\theta + 2\left(\frac{1}{2}\int_0^{\frac{\pi}{2}} (2 - 2\cos\theta)^2\ d\theta\right)$$

$$A = \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} 2\ d\theta + \int_{0}^{\frac{\pi}{2}} 4 - 8\cos\theta + 4\cos^2\theta\ d\theta$$

Use a double-angle identity to rewrite the integral.

$$A = \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} 2\ d\theta + \int_{0}^{\frac{\pi}{2}} 4 - 8\cos\theta + 4\left(\frac{1}{2}(1+\cos(2\theta))\right)\ d\theta$$

$$A = \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} 2\ d\theta + \int_{0}^{\frac{\pi}{2}} 4 - 8\cos\theta + 2 + 2\cos(2\theta)\ d\theta$$

$$A = \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} 2\ d\theta + \int_{0}^{\frac{\pi}{2}} 6 - 8\cos\theta + 2\cos(2\theta)\ d\theta$$

Integrate, then evaluate over the interval.

$$A = 2\theta\Big|_{\frac{\pi}{2}}^{\frac{3\pi}{2}} + 6\theta - 8\sin\theta + \sin(2\theta)\Big|_{0}^{\frac{\pi}{2}}$$

$$A = 2\left(\frac{3\pi}{2}\right) - 2\left(\frac{\pi}{2}\right) + 6\left(\frac{\pi}{2}\right) - 8\sin\left(\frac{\pi}{2}\right) + \sin\left(2\cdot\frac{\pi}{2}\right)$$

$$-(6(0) - 8\sin(0) + \sin(2(0)))$$

$$A = 3\pi - \pi + 3\pi - 8$$

$$A = 5\pi - 8$$

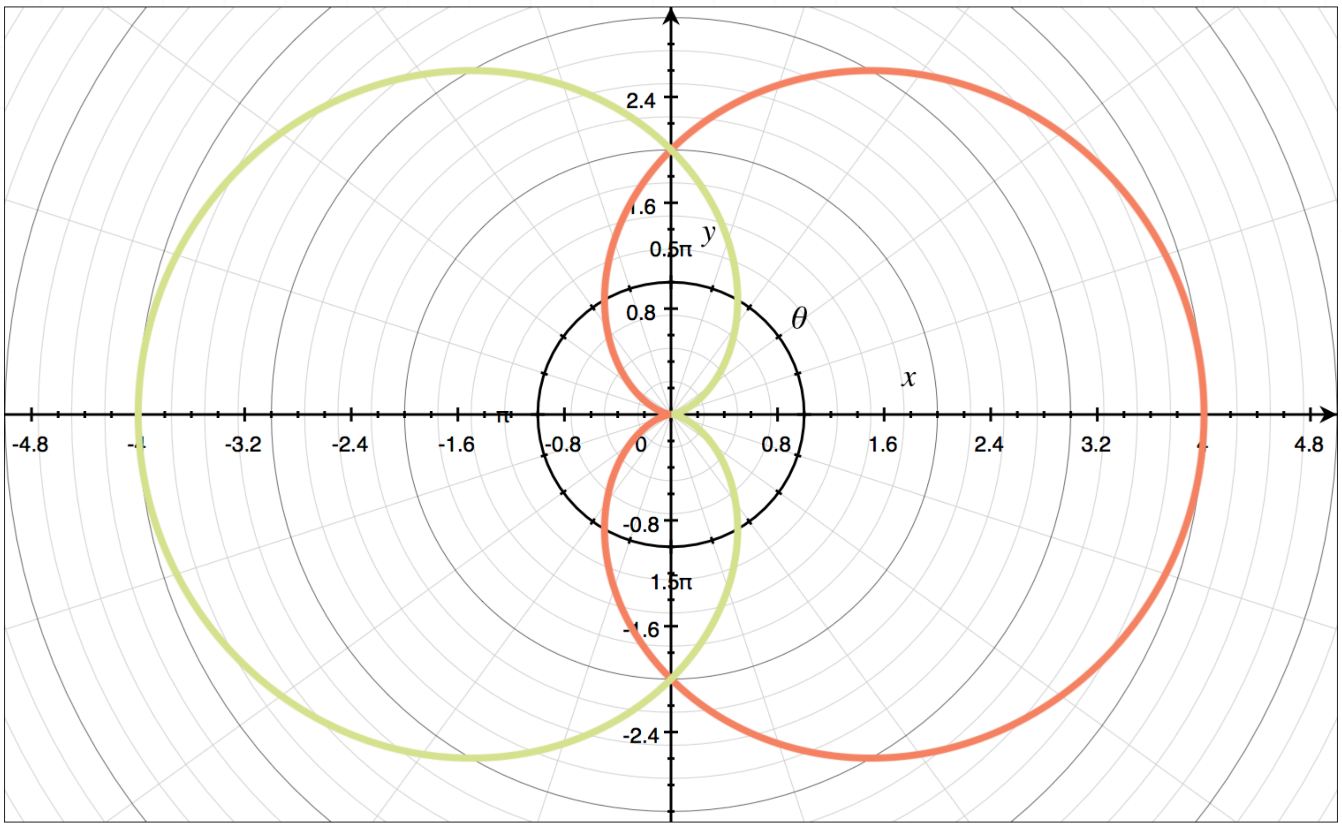
■ 4. Find the area of the region that's inside both polar curves.

$$r = 2(1 + \cos\theta)$$

$$r = 2(1 - \cos\theta)$$

Solution:

A sketch of the curves is



Find points of intersection by setting the curves equal to each other.

$$2(1 + \cos\theta) = 2(1 - \cos\theta)$$

$$2 + 2\cos\theta = 2 - 2\cos\theta$$

$$\cos\theta = -\cos\theta$$

$$\theta = \frac{\pi}{2}, \frac{3\pi}{2}$$

The overlapping area is given by the area inside $r=2(1-\cos\theta)$ on $[0,\pi/2]$, as long as we multiply that area by 4, which will give us the total area inside both curves.

$$A=4\left(\frac{1}{2}\int_0^{\frac{\pi}{2}}(2(1-\cos\theta))^2\ d\theta\right)$$

$$A=2\int_0^{\frac{\pi}{2}}4(1-\cos\theta)^2\ d\theta$$

$$A=8\int_0^{\frac{\pi}{2}}1-2\cos\theta+\cos^2\theta\ d\theta$$

Use a double-angle identity to rewrite the integral.

$$A=8\int_0^{\frac{\pi}{2}}1-2\cos\theta+\frac{1}{2}(1+\cos(2\theta))\ d\theta$$

$$A=8\int_0^{\frac{\pi}{2}}\frac{3}{2}-2\cos\theta+\frac{1}{2}\cos(2\theta)\ d\theta$$

Integrate, then evaluate over the interval.

$$A=8\left(\frac{3}{2}\theta-2\sin\theta+\frac{1}{4}\sin(2\theta)\right)\Bigg|_0^{\frac{\pi}{2}}$$

$$A=8\left(\frac{3}{2}\left(\frac{\pi}{2}\right)-2\sin\left(\frac{\pi}{2}\right)+\frac{1}{4}\sin\left(2\cdot\frac{\pi}{2}\right)\right)$$

$$-8\left(\frac{3}{2}(0)-2\sin(0)+\frac{1}{4}\sin(2(0))\right)$$

$$A = 8\left(\frac{3\pi}{4} - 2\right)$$

$$A = 6\pi - 16$$

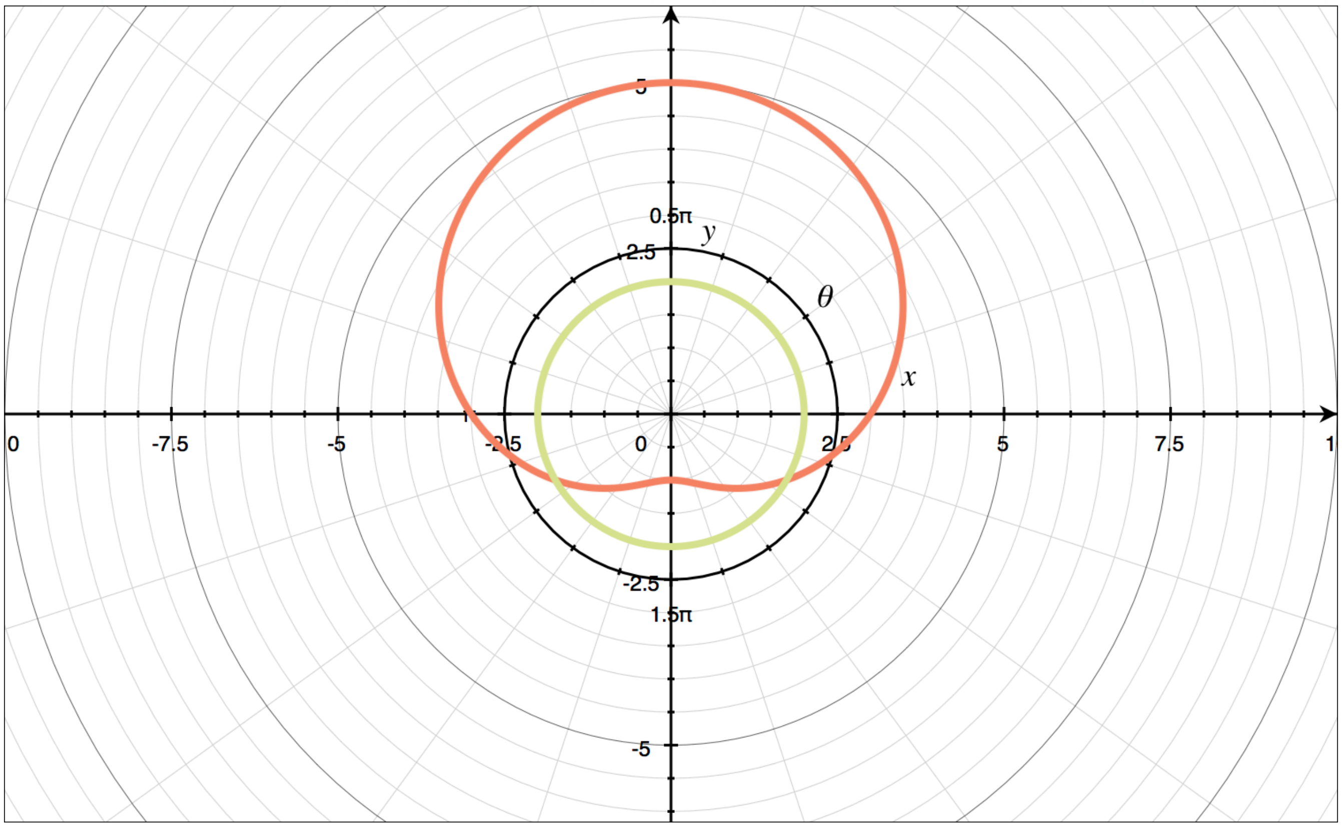
■ 5. Find the area of the region that's inside both polar curves.

$$r = 3 + 2\sin\theta$$

$$r = 2$$

Solution:

A sketch of the curves is



Find points of intersection by setting the curves equal to each other.

$$3 + 2\sin\theta = 2$$

$$2\sin\theta = -1$$

$$\theta = \frac{7\pi}{6}, \frac{11\pi}{6}$$

The overlapping area is given by the area inside $r = 2$ on $[0,2\pi]$ (the area of the full circle), minus the area between $r = 2$ and $r = 3 + 2\sin\theta$ on the interval $[7\pi/6,11\pi/6]$.

$$A = \frac{1}{2}\int_0^{2\pi} (2)^2\ d\theta - \frac{1}{2}\int_{\frac{7\pi}{6}}^{\frac{11\pi}{6}} 2^2 - (3 + 2\sin\theta)^2\ d\theta$$

$$A = 2\int_0^{2\pi}\ d\theta - \frac{1}{2}\int_{\frac{7\pi}{6}}^{\frac{11\pi}{6}} 4 - (9 + 12\sin\theta + 4\sin^2\theta)\ d\theta$$

$$A = 2\int_0^{2\pi}\ d\theta + \frac{1}{2}\int_{\frac{7\pi}{6}}^{\frac{11\pi}{6}} 5 + 12\sin\theta + 4\sin^2\theta\ d\theta$$

Use a double-angle identity to rewrite the integral.

$$A = 2\int_0^{2\pi}\ d\theta + \frac{1}{2}\int_{\frac{7\pi}{6}}^{\frac{11\pi}{6}} 5 + 12\sin\theta + 4\left(\frac{1}{2}(1 - \cos(2\theta))\right)\ d\theta$$

$$A = 2\int_0^{2\pi}\ d\theta + \frac{1}{2}\int_{\frac{7\pi}{6}}^{\frac{11\pi}{6}} 5 + 12\sin\theta + 2 - 2\cos(2\theta)\ d\theta$$

$$A = 2\int_0^{2\pi}\ d\theta + \frac{1}{2}\int_{\frac{7\pi}{6}}^{\frac{11\pi}{6}} 7 + 12\sin\theta - 2\cos(2\theta)\ d\theta$$

Integrate, then evaluate over the interval.

$$A = 2\theta \Big|_0^{2\pi} + \frac{1}{2}\left(7\theta - 12\cos\theta - \sin(2\theta)\right)\Big|_{\frac{7\pi}{6}}^{\frac{11\pi}{6}}$$

$$A = 2(2\pi) - 2(0) + \frac{1}{2}\left(7\left(\frac{11\pi}{6}\right) - 12\cos\left(\frac{11\pi}{6}\right) - \sin\left(2\cdot\frac{11\pi}{6}\right)\right)$$

$$-\frac{1}{2}\left(7\left(\frac{7\pi}{6}\right) - 12\cos\left(\frac{7\pi}{6}\right) - \sin\left(2\cdot\frac{7\pi}{6}\right)\right)$$

$$A = 4\pi + \frac{77\pi}{12} - 6\cos\left(\frac{11\pi}{6}\right) - \frac{1}{2}\sin\left(\frac{22\pi}{6}\right)$$

$$-\frac{49\pi}{12} + 6\cos\left(\frac{7\pi}{6}\right) + \frac{1}{2}\sin\left(\frac{7\pi}{3}\right)$$

$$A = 4\pi + \frac{77\pi}{12} - 6\left(\frac{\sqrt{3}}{2}\right) - \frac{1}{2}\left(-\frac{\sqrt{3}}{2}\right) - \frac{49\pi}{12} + 6\left(-\frac{\sqrt{3}}{2}\right) + \frac{1}{2}\left(\frac{\sqrt{3}}{2}\right)$$

$$A = \frac{19\pi}{3} - \frac{11\sqrt{3}}{2}$$

SURFACE AREA OF REVOLUTION OF A POLAR CURVE

1. Find the surface area generated by revolving the polar curve about the y-axis over the interval $0 \le \theta \le \pi$.

$$r = 2\cos\theta$$

Solution:

The derivative of $r = 2\cos\theta$ is

$$\frac{dr}{d\theta} = -2\sin\theta$$

So the surface area of revolution will be

$$S_y = \int_{\alpha}^{\beta} 2\pi x \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\ d\theta$$

$$S_y = \int_0^{\pi} 2\pi(2\cos\theta)\cos\theta\sqrt{(2\cos\theta)^2 + (-2\sin\theta)^2}\ d\theta$$

$$S_y = \int_0^{\pi} 4\pi\cos^2\theta\sqrt{4\cos^2\theta + 4\sin^2\theta}\ d\theta$$

$$S_y = 4\pi\int_0^{\pi} \cos^2\theta\sqrt{4(\cos^2\theta + \sin^2\theta)}\ d\theta$$

$$S_y = 4\pi \int_0^{\pi} \cos^2\theta \sqrt{4(1)}\ d\theta$$

$$S_y = 4\pi \int_0^{\pi} \cos^2\theta \sqrt{4}\ d\theta$$

$$S_y = 8\pi \int_0^{\pi} \cos^2\theta\ d\theta$$

Use the trig identity

$$\cos^2\theta = \frac{1}{2}(1 + \cos(2\theta))$$

to substitute.

$$S_y = 8\pi \int_0^{\pi} \frac{1}{2}(1 + \cos(2\theta))\ d\theta$$

$$S_y = 4\pi \int_0^{\pi} 1 + \cos(2\theta)\ d\theta$$

Integrate, then evaluate over the interval.

$$S_y = 4\pi \left(\theta + \frac{1}{2}\sin(2\theta) \right) \Big|_0^{\pi}$$

$$S_y = 4\pi\theta + 2\pi\sin(2\theta) \Big|_0^{\pi}$$

$$S_y = 4\pi^2 + 2\pi\sin(2\pi) - (4\pi(0) + 2\pi\sin(2(0)))$$

$$S_y = 4\pi^2 + 2\pi(0) - 4\pi(0) - 2\pi(0)$$

$$S_y = 4\pi^2$$

■ 2. Find the surface area generated by revolving the polar curve about the x-axis over the interval $0 \le \theta \le \pi/2$.

$$r = 4\cos\theta$$

Solution:

The derivative of $r = 4\cos\theta$ is

$$\frac{dr}{d\theta} = -4\sin\theta$$

So the surface area of revolution will be

$$S_x = \int_\alpha^\beta 2\pi y\sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\ d\theta$$

$$S_x = \int_0^{\frac{\pi}{2}} 2\pi(4\cos\theta)\sin\theta\sqrt{(4\cos\theta)^2 + (-4\sin\theta)^2}\ d\theta$$

$$S_x = 8\pi\int_0^{\frac{\pi}{2}} \cos\theta\sin\theta\sqrt{16\cos^2\theta + 16\sin^2\theta}\ d\theta$$

$$S_x = 8\pi\int_0^{\frac{\pi}{2}} \cos\theta\sin\theta\sqrt{16(\cos^2\theta + \sin^2\theta)}\ d\theta$$

$$S_x = 8\pi \int_0^{\frac{\pi}{2}} \cos\theta \sin\theta \sqrt{16(1)}\ d\theta$$

$$S_x = 8\pi \int_0^{\frac{\pi}{2}} \cos\theta \sin\theta \sqrt{16}\ d\theta$$

$$S_x = 32\pi \int_0^{\frac{\pi}{2}} \cos\theta \sin\theta\ d\theta$$

Integrate, then evaluate over the interval.

$$S_x = 32\pi\left(-\frac{\cos^2\theta}{2}\right)\Bigg|_0^{\frac{\pi}{2}}$$

$$S_x = -16\pi(\cos^2\theta)\Big|_0^{\frac{\pi}{2}}$$

$$S_x = -16\pi\left(\cos^2\frac{\pi}{2}\right) + 16\pi(\cos^2(0))$$

$$S_x = -16\pi(0^2) + 16\pi(1^2)$$

$$S_x = 16\pi$$

■ 3. Find the surface area generated by revolving the polar curve about the y-axis over the interval $0 \le \theta \le \pi/2$.

$$r = 8\sin\theta$$

Solution:

The derivative of $r = 8\sin\theta$ is

$$\frac{dr}{d\theta} = 8\cos\theta$$

So the surface area of revolution will be

$$S_y = \int_{\alpha}^{\beta} 2\pi x\sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\ d\theta$$

$$S_y = \int_0^{\frac{\pi}{2}} 2\pi(8\sin\theta)\cos\theta\sqrt{(8\sin\theta)^2 + (8\cos\theta)^2}\ d\theta$$

$$S_y = 16\pi\int_0^{\frac{\pi}{2}} \sin\theta\cos\theta\sqrt{64\sin^2\theta + 64\cos^2\theta}\ d\theta$$

$$S_y = 16\pi\int_0^{\frac{\pi}{2}} \sin\theta\cos\theta\sqrt{64(\sin^2\theta + \cos^2\theta)}\ d\theta$$

$$S_y = 16\pi\int_0^{\frac{\pi}{2}} \sin\theta\cos\theta\sqrt{64(1)}\ d\theta$$

$$S_y = 16\pi\int_0^{\frac{\pi}{2}} \sin\theta\cos\theta\sqrt{64}\ d\theta$$

$$S_y = 128\pi\int_0^{\frac{\pi}{2}} \sin\theta\cos\theta\ d\theta$$

Integrate, then evaluate over the interval.

$$S_y = 128\pi\left(\frac{\sin^2\theta}{2}\right)\Big|_0^{\frac{\pi}{2}}$$

$$S_y = 64\pi\sin^2\theta\Big|_0^{\frac{\pi}{2}}$$

$$S_y = 64\pi\sin^2\frac{\pi}{2} - 64\pi\sin^2(0)$$

$$S_y = 64\pi(1)^2 - 64\pi(0)^2$$

$$S_y = 64\pi(1)$$

$$S_y = 64\pi$$

■ 4. Find the surface area generated by revolving the polar curve about the x-axis over the interval $0 \le \theta \le \pi$.

$$r = 7\sin\theta$$

Solution:

The derivative of $r = 7\sin\theta$ is

$$\frac{dr}{d\theta} = 7\cos\theta$$

So the surface area of revolution will be

$$S_x = \int_{\alpha}^{\beta} 2\pi y \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\ d\theta$$

$$S_x = \int_0^{\pi} 2\pi(7\sin\theta)\sin\theta\sqrt{(7\sin\theta)^2 + (7\cos\theta)^2}\ d\theta$$

$$S_x = 14\pi\int_0^{\pi} \sin\theta\sin\theta\sqrt{49\sin^2\theta + 49\cos^2\theta}\ d\theta$$

$$S_x = 14\pi\int_0^{\pi} \sin\theta\sin\theta\sqrt{49(\sin^2\theta + \cos^2\theta)}\ d\theta$$

$$S_x = 14\pi\int_0^{\pi} \sin\theta\sin\theta\sqrt{49(1)}\ d\theta$$

$$S_x = 14\pi\int_0^{\pi} \sin\theta\sin\theta\sqrt{49}\ d\theta$$

$$S_x = 98\pi\int_0^{\pi} \sin^2\theta\ d\theta$$

Use the trig identities

$$\sin^2\theta = \frac{1}{2}(1 - \cos(2\theta))$$

to substitute.

$$S_x = 98\pi\int_0^{\pi} \frac{1}{2}(1 - \cos(2\theta))\ d\theta$$

$$S_x = 49\pi\int_0^{\pi} 1 - \cos(2\theta)\ d\theta$$

Integrate, then evaluate over the interval.

$$S_x = 49\pi \left(\theta - \frac{1}{2}\sin(2\theta) \right) \Big|_0^{\pi}$$

$$S_x = 49\pi^2 - \frac{49}{2}\pi\sin(2\pi) - \left(49\pi(0) - \frac{49}{2}\pi\sin(2(0)) \right)$$

$$S_x = 49\pi^2 - \frac{49}{2}\pi(0) - 49\pi(0) + \frac{49}{2}\pi(0)$$

$$S_x = 49\pi^2 - 49\pi(0)$$

$$S_x = 49\pi^2$$

SEQUENCES VS. SERIES

■ 1. Determine whether the expression is a sequence or a series.

$$5, 10, 15, 20\ , 25, 30$$

Solution:

A sequence is a list of terms. A series is the sum of a sequence of terms. The question provides a list, not a sum, so it's a sequence.

■ 2. Determine whether the expression is a sequence or a series.

$$\sum_{n=1}^{15} 5n - 2$$

Solution:

A sequence is a list of terms. A series is the sum of a sequence of terms. The question provides a sum of a sequence, so it's a series.

■ 3. Determine whether the expression is a sequence or a series.

$$3 + 6 + 9 + 12 + 15 + 18 + 21$$

Solution:

A sequence is a list of terms. A series is the sum of a sequence of terms. The question provides a sum of a sequence, so it's a series.

LISTING THE FIRST TERMS

■ 1. Write the first five terms of the sequence.

$$a_{n+1} = 3a_n + 4$$

$$a_1 = 4$$

Solution:

Since $a_1 = 4$, the first term of the sequence is 4. Use the rule for a_{n+1} to find the rest of the first five terms.

$$a_1 = 4$$

$$a_2 = 3a_1 + 4 = 3(4) + 4 = 16$$

$$a_3 = 3a_2 + 4 = 3(16) + 4 = 52$$

$$a_4 = 3a_3 + 4 = 3(52) + 4 = 160$$

$$a_5 = 3a_4 + 4 = 3(160) + 4 = 484$$

■ 2. Write the first five terms of the sequence.

$$a_{n+1} = 4a_n - 5$$

$$a_1 = 3$$

Solution:

Since $a_1 = 3$, the first term of the sequence is 3. Use the rule $a_{n+1} = 4a_n - 5$ to find the rest of the first five terms.

$$a_1 = 3$$

$$a_2 = 4a_1 - 5 = 4(3) - 5 = 7$$

$$a_3 = 4a_2 - 5 = 4(7) - 5 = 23$$

$$a_4 = 4a_3 - 5 = 4(23) - 5 = 87$$

$$a_5 = 4a_4 - 5 = 3(87) - 5 = 343$$

■ 3. Write the first five terms of the sequence.

$$a_{n+1} = a_n + 9$$

$$a_1 = 24$$

Solution:

Since $a_1 = 24$, the first term of the sequence is 24. Use the rule $a_{n+1} = a_n + 9$ to find the rest of the first five terms.

$$a_1 = 24$$

$$a_2 = a_1 + 9 = 24 + 9 = 33$$

$$a_3 = a_2 + 9 = 33 + 9 = 42$$

$$a_4 = a_3 + 9 = 42 + 9 = 51$$

$$a_5 = a_4 + 9 = 51 + 9 = 60$$

CALCULATING THE FIRST TERMS

1. Write the first five terms of the sequence and find the limit of the sequence a_n as $n \to \infty$.

$$a_n = \frac{5n^2 - 2}{n^2 + 3n - 2}$$

Solution:

To get the first five terms of the sequence, plug $n = 1, 2, 3, 4, 5$ into the formula for a_n.

n	$a_n = \frac{5n^2 - 2}{n^2 + 3n - 2}$	a_n
1	$a_1 = \frac{5(1)^2 - 2}{1^2 + 3(1) - 2}$	$a_1 = \frac{3}{2}$
2	$a_2 = \frac{5(2)^2 - 2}{2^2 + 3(2) - 2}$	$a_2 = \frac{9}{4}$
3	$a_3 = \frac{5(3)^2 - 2}{3^2 + 3(3) - 2}$	$a_3 = \frac{43}{16}$
4	$a_4 = \frac{5(4)^2 - 2}{4^2 + 3(4) - 2}$	$a_4 = 3$

$$5 \qquad a_5 = \frac{5(5)^2 - 2}{5^2 + 3(5) - 2} \qquad a_5 = \frac{123}{38}$$

Find the limit.

$$\lim_{n \to \infty} a_n$$

$$\lim_{n \to \infty} \frac{5n^2 - 2}{n^2 + 3n - 2}$$

$$\lim_{n \to \infty} \frac{5n^2 - 2}{n^2 + 3n - 2} \cdot \left(\frac{\frac{1}{n^2}}{\frac{1}{n^2}} \right)$$

$$\lim_{n \to \infty} \frac{\frac{5n^2}{n^2} - \frac{2}{n^2}}{\frac{n^2}{n^2} + \frac{3n}{n^2} - \frac{2}{n^2}}$$

$$\lim_{n \to \infty} \frac{5 - 0}{1 + 0 - 0}$$

$$\lim_{n \to \infty} \frac{5}{1}$$

$$5$$

■ 2. Write the first five terms of the sequence and find the limit of the sequence a_n as $n \to \infty$.

$$a_n = \frac{6n}{e^{2n}}$$

Solution:

To get the first five terms of the sequence, plug $n=1, 2, 3, 4, 5$ into the formula for a_n.

n	$a_n=\dfrac{6n}{e^{2n}}$	a_n
1	$a_1=\dfrac{6(1)}{e^{2(1)}}$	$a_1=\dfrac{6}{e^2}$
2	$a_2=\dfrac{6(2)}{e^{2(2)}}$	$a_2=\dfrac{12}{e^4}$
3	$a_3=\dfrac{6(3)}{e^{2(3)}}$	$a_3=\dfrac{18}{e^6}$
4	$a_4=\dfrac{6(4)}{e^{2(4)}}$	$a_4=\dfrac{24}{e^8}$
5	$a_4=\dfrac{6(5)}{e^{2(5)}}$	$a_4=\dfrac{30}{e^{10}}$

To find the limit, apply L'Hospital's Rule.

$$\lim_{n\to\infty} a_n$$

$$\lim_{n\to\infty} \frac{6n}{e^{2n}}$$

$$\lim_{n\to\infty} \frac{6}{2e^{2n}}$$

$$\lim_{n \to \infty} \frac{3}{e^{2n}}$$

$$\lim_{n \to \infty} \frac{0}{\infty}$$

$$\lim_{n \to \infty} 0$$

$$0$$

■ 3. Write the first five terms of the sequence and find the limit of the sequence a_n as $n \to \infty$.

$$a_n = \frac{n^2 + 1}{n^2 + 8n}$$

Solution:

To get the first five terms of the sequence, plug $n = 1, 2, 3, 4, 5$ into the formula for a_n.

n	$a_n = \dfrac{n^2 + 1}{n^2 + 8n}$	a_n
1	$a_1 = \dfrac{1^2 + 1}{1^2 + 8(1)}$	$a_1 = \dfrac{2}{9}$
2	$a_2 = \dfrac{2^2 + 1}{2^2 + 8(2)}$	$a_2 = \dfrac{1}{4}$

3	$a_3 = \dfrac{3^2+1}{3^2+8(3)}$	$a_3 = \dfrac{10}{33}$
4	$a_4 = \dfrac{4^2+1}{4^2+8(4)}$	$a_4 = \dfrac{17}{48}$
5	$a_5 = \dfrac{5^2+1}{5^2+8(5)}$	$a_5 = \dfrac{2}{5}$

Find the limit.

$$\lim_{n\to\infty} a_n$$

$$\lim_{n\to\infty} \frac{n^2+1}{n^2+8n}$$

$$\lim_{n\to\infty} \frac{n^2+1}{n^2+8n} \cdot \left(\frac{\frac{1}{n^2}}{\frac{1}{n^2}}\right)$$

$$\lim_{n\to\infty} \frac{\frac{n^2}{n^2}+\frac{1}{n^2}}{\frac{n^2}{n^2}+\frac{8n}{n^2}}$$

$$\lim_{n\to\infty} \frac{\frac{n^2}{n^2}+\frac{1}{n^2}}{\frac{n^2}{n^2}+\frac{8}{n}}$$

$$\lim_{n\to\infty} \frac{1+0}{1+0}$$

$$\lim_{n\to\infty} \frac{1}{1}$$

1

FORMULA FOR THE GENERAL TERM

■ 1. What is a formula for the general term of the sequence?

$$\frac{3}{4}, \frac{5}{8}, \frac{7}{12}, \frac{9}{16}, \frac{11}{20}$$

Solution:

In a sequence of fractions, consider the sequence of the numerators and the sequence of the denominators separately.

The sequence of the numerators is

n	1	2	3	4	5
a_n	3	5	7	9	11

$$a_n = 2n + 1$$

The sequence of the denominators is

n	1	2	3	4	5
a_n	4	8	12	16	20

$$a_n = 4n$$

So the rule for the sequence is

$$a_n = \frac{2n+1}{4n}$$

■ 2. What is a formula for the general term of the sequence?

$$5, 8, 13, 20, 29, 40$$

Solution:

The sequence of the terms is

n	1	2	3	4	5	6
a_n	5	8	13	20	29	40

We can build upon the chart, to see that the pattern is

n	1	2	3	4	5	6
n^2	1	4	9	16	25	36
Add 4	+4	+4	+4	+4	+4	+4
a_n	5	8	13	20	29	40

$$a_n = n^2 + 4$$

■ 3. What is a formula for the general term of the sequence?

$$-\frac{1}{6}, \frac{2}{7}, -\frac{3}{8}, \frac{4}{9}, -\frac{1}{2}, \frac{6}{11}$$

Solution:

In a sequence of fractions, consider the sequence of the numerators and the sequence of the denominators separately.

The sequence of the numerators is

n	1	2	3	4	5	6
a_n	-1	2	-3	4	-1	6

Notice that the 5th term seems out of sequence, but if the term is changed to match the pattern of the other terms, the sequence becomes

$$-\frac{1}{6}, \frac{2}{7}, -\frac{3}{8}, \frac{4}{9}, -\frac{5}{10}, \frac{6}{11}$$

So the new sequence of the numerators is

n	1	2	3	4	5	6
a_n	-1	2	-3	4	-5	6

The sign of the numerator is negative when n is odd and is positive when n is even. So the term value for the numerator is found using the formula $a_n = (-1)^n(n)$.

The sequence of the denominators is

n	1	2	3	4	5	6
a_n	6	7	8	9	10	11

The value of the denominator is consistently 5 more than the term number. So the formula for the denominator is $n+5$.

Therefore, the rule for the sequence is the rule for the numerator divided by the rule for the denominator.

$$a_n = \frac{(-1)^n(n)}{n+5}$$

CONVERGENCE OF A SEQUENCE

1. If the sequence converges, find its limit.

$$a_n = \frac{5n}{n^2 + 2n - 1}$$

Solution:

The sequence converges if the limit of the sequence as $n \to \infty$ exists and is finite. The sequence diverges if the limit does not exist or is infinite. Because

$$\lim_{n \to \infty} \frac{5n}{n^2 + 2n - 1} = \frac{\infty}{\infty}$$

is indeterminate, use L'Hospital's Rule to find the limit.

$$\lim_{n \to \infty} \frac{5n}{n^2 + 2n - 1} = \lim_{n \to \infty} \frac{5}{2n^2 + 2} = \frac{5}{\infty} = 0$$

2. If the sequence converges, find its limit.

$$a_n = \frac{9n^3 - 27n^2 + 5n}{3n^3 + 12n^2 - n}$$

Solution:

The sequence converges if the limit of the sequence as $n \to \infty$ exists and is finite. The sequence diverges if the limit does not exist or is infinite. Because

$$\lim_{n\to\infty} \frac{9n^3 - 27n^2 + 5n}{3n^3 + 12n^2 - n} = \frac{\infty}{\infty}$$

is indeterminate, use L'Hospital's Rule to find the limit.

$$\lim_{n\to\infty} \frac{9n^3 - 27n^2 + 5n}{3n^3 + 12n^2 - n} = \lim_{n\to\infty} \frac{27n^2 - 54n + 5}{9n^2 + 24n - 1} = \lim_{n\to\infty} \frac{54n - 54}{18n + 24} = \lim_{n\to\infty} \frac{54}{18} = 3$$

■ 3. If the sequence converges, find its limit.

$$a_n = \left(\frac{n^2 + 3}{n^3}\right)^2$$

Solution:

The sequence converges if the limit of the sequence as $n \to \infty$ exists and is finite. The sequence diverges if the limit does not exist or is infinite. Because

$$\lim_{n\to\infty} \left(\frac{n^2 + 3}{n^3}\right)^2 = \lim_{n\to\infty} \frac{(n^2 + 3)^2}{(n^3)^2} = \lim_{n\to\infty} \frac{n^4 + 6n^2 + 9}{n^6} = \frac{\infty}{\infty}$$

is indeterminate, use L'Hospital's Rule to find the limit.

$$\lim_{n\to\infty} \frac{n^4 + 6n^2 + 9}{n^6} = \lim_{n\to\infty} \frac{4n^3 + 12n}{6n^5} = \lim_{n\to\infty} \frac{12n^2 + 12}{30n^4}$$

$$= \lim_{n\to\infty} \frac{24n}{120n^3} = \lim_{n\to\infty} \frac{24}{120n^2} = \frac{24}{\infty} = 0$$

LIMIT OF A CONVERGENT SEQUENCE

■ 1. Find the limit of the convergent sequence.

$$a_n = \frac{3n^2 - 6}{9n^2 + 3n - 12}$$

Solution:

Since the sequence converges, the limit of the sequence as $n \to \infty$ exists and is finite. Find the limit. Because

$$\lim_{n\to\infty} \frac{3n^2 - 6}{9n^2 + 3n - 12} = \lim_{n\to\infty} \frac{n^2 - 2}{3n^2 + n - 4} = \frac{\infty}{\infty}$$

is indeterminate, use L'Hospital's Rule to find the limit.

$$\lim_{n\to\infty} \frac{2n}{6n + 1} = \lim_{n\to\infty} \frac{2}{6} = \frac{1}{3}$$

■ 2. Find the limit of the convergent sequence.

$$a_n = \frac{n^3}{3^n}$$

Solution:

Since the sequence converges, the limit of the sequence as $n \to \infty$ exists and is finite. Find the limit. Because

$$\lim_{n\to\infty} \frac{n^3}{3^n} = \frac{\infty}{\infty}$$

is indeterminate, use L'Hospital's Rule to find the limit.

$$\lim_{n\to\infty} \frac{n^3}{3^n} = \lim_{n\to\infty} \frac{3n^2}{3^n \cdot \ln 3} = \frac{1}{\ln 3} \lim_{n\to\infty} \frac{3n^2}{3^n} = \frac{1}{\ln 3} \lim_{n\to\infty} \frac{6n}{3^n \cdot \ln 3}$$

$$= \frac{1}{(\ln 3)^2} \lim_{n\to\infty} \frac{6n}{3^n} = \frac{1}{(\ln 3)^2} \lim_{n\to\infty} \frac{6}{3^n \cdot \ln 3} = \frac{1}{(\ln 3)^3} \lim_{n\to\infty} \frac{6}{3^n}$$

$$= \frac{1}{(\ln 3)^3} \cdot \frac{6}{\infty} = \frac{6}{\infty} = 0$$

■ 3. Find the limit of the convergent sequence.

$$a_n = n^5 e^{-2n}$$

Solution:

Since the sequence converges, the limit of the sequence as $n \to \infty$ exists and is finite. Find the limit. Because

$$\lim_{n\to\infty} n^5 e^{-2n} = \lim_{n\to\infty} \frac{n^5}{e^{2n}} = \frac{\infty}{\infty}$$

is indeterminate, use L'Hospital's Rule to find the limit.

$$\lim_{n\to\infty}\frac{n^5}{e^{2n}}=\lim_{n\to\infty}\frac{5n^4}{2e^{2n}}=\lim_{n\to\infty}\frac{20n^3}{4e^{2n}}=\lim_{n\to\infty}\frac{60n^2}{8e^{2n}}$$

$$=\lim_{n\to\infty}\frac{120n}{16e^{2n}}=\lim_{n\to\infty}\frac{120}{32e^{2n}}=\frac{120}{\infty}=0$$

INCREASING, DECREASING, AND NOT MONOTONIC

1. State whether the sequence is increasing, decreasing, and monotonic or not monotonic.

$$a_n = \frac{17}{4n^2 + 6n + 3}$$

Solution:

Calculate the value of the first few terms.

$$n = 1 \qquad a_1 = \frac{17}{4(1)^2 + 6(1) + 3} = \frac{17}{13}$$

$$n = 2 \qquad a_2 = \frac{17}{4(2)^2 + 6(2) + 3} = \frac{17}{31}$$

$$n = 3 \qquad a_3 = \frac{17}{4(3)^2 + 6(3) + 3} = \frac{17}{57}$$

$$n = 4 \qquad a_4 = \frac{17}{4(4)^2 + 6(4) + 3} = \frac{17}{91}$$

$$n = 5 \qquad a_5 = \frac{17}{4(5)^2 + 6(5) + 3} = \frac{17}{133}$$

Based on the first five terms, the value of the terms get consistently smaller as n gets larger, which means the sequence is decreasing, and also monotonic.

■ 2. State whether the sequence is increasing, decreasing, and monotonic or not monotonic.

$$a_n = \frac{3n^2 - 5}{4n + 2}$$

Solution:

Calculate the value of the first few terms.

$$n = 1 \qquad a_1 = \frac{3(1)^2 - 5}{4(1) + 2} = -\frac{1}{3}$$

$$n = 2 \qquad a_2 = \frac{3(2)^2 - 5}{4(2) + 2} = \frac{7}{10}$$

$$n = 3 \qquad a_3 = \frac{3(3)^2 - 5}{4(3) + 2} = \frac{11}{7}$$

$$n = 4 \qquad a_4 = \frac{3(4)^2 - 5}{4(4) + 2} = \frac{43}{18}$$

$$n = 5 \qquad a_5 = \frac{3(5)^2 - 5}{4(5) + 2} = \frac{35}{11}$$

Based on the first five terms, the value of the terms get consistently larger as n gets larger, which means the sequence is increasing, and also monotonic.

■ 3. State whether the sequence is increasing, decreasing, and monotonic or not monotonic.

$$a_n = n^5 + 1$$

Solution:

Calculate the value of the first few terms.

$$n = 1 \qquad a_1 = (1)^5 + 1 = 2$$

$$n = 2 \qquad a_2 = (2)^5 + 1 = 33$$

$$n = 3 \qquad a_3 = (3)^5 + 1 = 244$$

$$n = 4 \qquad a_4 = (4)^5 + 1 = 1{,}025$$

$$n = 5 \qquad a_5 = (5)^5 + 1 = 3{,}126$$

Based on the first five terms, the value of the terms get consistently larger as n gets larger, which means the sequence is increasing, and also monotonic.

BOUNDED SEQUENCES

1. Describe how the sequence is bounded by indicating the upper and lower bounds, or say whether there is no upper bound or now lower bound.

$$a_n = \frac{2n+5}{n^2}$$

Solution:

Find the first few terms of the sequence.

$$n = 1 \qquad a_1 = \frac{2(1)+5}{1^2} = 7$$

$$n = 2 \qquad a_2 = \frac{2(2)+5}{2^2} = \frac{9}{4}$$

$$n = 3 \qquad a_3 = \frac{2(3)+5}{3^2} = \frac{11}{9}$$

$$n = 4 \qquad a_4 = \frac{2(4)+5}{4^2} = \frac{13}{16}$$

$$n = 5 \qquad a_5 = \frac{2(5)+5}{5^2} = \frac{15}{25}$$

The sequence is bounded above at $a_1 = 7$. Now find the limit as $n \to \infty$. Because

$$\lim_{n \to \infty} \frac{2n+5}{n^2} = \frac{\infty}{\infty}$$

is indeterminate, use L'Hospital's Rule to find the limit.

$$\lim_{n \to \infty} \frac{2n+5}{n^2} = \lim_{n \to \infty} \frac{2}{2n} = \frac{2}{\infty} = 0$$

Therefore, the sequence is bounded below at 0, and above at $a_1 = 7$.

■ 2. Describe how the sequence is bounded by indicating the upper and lower bounds, or say whether there is no upper bound or now lower bound.

$$a_n = \frac{3n^3 + 2}{n^4}$$

Solution:

Find the first few terms of the sequence.

$$n = 1 \qquad a_1 = \frac{3(1)^3 + 2}{1^4} = 5$$

$$n = 2 \qquad a_2 = \frac{3(2)^3 + 2}{2^4} = \frac{13}{8}$$

$$n = 3 \qquad a_3 = \frac{3(3)^3 + 2}{3^4} = \frac{83}{81}$$

$$n = 4 \qquad a_4 = \frac{3(4)^3 + 2}{4^4} = \frac{97}{128}$$

$$n = 5 \qquad a_5 = \frac{3(5)^3 + 2}{5^4} = \frac{377}{625}$$

The sequence is bounded above at $a_1 = 5$. Now find the limit as $n \to \infty$. Because

$$\lim_{n \to \infty} \frac{3n^3 + 2}{n^4} = \frac{\infty}{\infty}$$

is indeterminate, use L'Hospital's Rule to find the limit.

$$\lim_{n \to \infty} \frac{3n^3 + 2}{n^4} = \lim_{n \to \infty} \frac{9n^2}{4n^3} = \lim_{n \to \infty} \frac{18n}{12n^2} = \lim_{n \to \infty} \frac{18}{24n} = \frac{18}{\infty} = 0$$

Therefore, the sequence is bounded below at 0, and above at $a_1 = 5$.

■ 3. Describe how the sequence is bounded by indicating the upper and lower bounds, or say whether there is no upper bound or now lower bound.

$$a_n = \frac{7n^3 + 15}{2n^3}$$

Solution:

Find the first few terms of the sequence.

$n = 1$ $\qquad a_1 = \frac{7(1)^3 + 15}{2(1)^3} = 11$

$n = 2$ $\qquad a_2 = \frac{7(2)^3 + 15}{2(2)^3} = \frac{71}{16}$

$n = 3$ $\qquad a_3 = \frac{7(3)^3 + 15}{2(3)^3} = \frac{34}{9}$

$n = 4$ $\qquad a_4 = \frac{7(4)^3 + 15}{2(4)^3} = \frac{463}{128}$

$n = 5$ $\qquad a_5 = \frac{7(5)^3 + 15}{2(5)^3} = \frac{89}{25}$

The sequence is bounded above at $a_1 = 11$. Now find the limit as $n \to \infty$. Because

$$\lim_{n \to \infty} \frac{7n^3 + 15}{2n^3} = \frac{\infty}{\infty}$$

is indeterminate, use L'Hospital's Rule to find the limit.

$$\lim_{n \to \infty} \frac{21n^2}{6n^2} = \lim_{n \to \infty} \frac{21}{6} = \frac{7}{2}$$

Therefore, the sequence is bounded below at $7/2$ and above at $a_1 = 11$.

4. Describe how the sequence is bounded by indicating the upper and lower bounds, or say whether there is no upper bound or now lower bound.

$$a_n = \frac{3n^4 + 9}{4n^3}$$

Solution:

Find the first few terms of the sequence.

$$n = 1 \qquad a_1 = \frac{3(1)^4 + 9}{4(1)^3} = 3$$

$$n = 2 \qquad a_2 = \frac{3(2)^4 + 9}{4(2)^3} = \frac{57}{32}$$

$$n = 3 \qquad a_3 = \frac{3(3)^4 + 9}{4(3)^3} = \frac{7}{3}$$

$$n = 4 \qquad a_4 = \frac{3(4)^4 + 9}{4(4)^3} = \frac{777}{256}$$

$$n = 5 \qquad a_5 = \frac{3(5)^4 + 9}{4(5)^3} = \frac{471}{125}$$

The sequence is bounded below at $a_2 = 57/32$. Because

$$\lim_{n \to \infty} \frac{3n^4 + 9}{4n^3} = \frac{\infty}{\infty}$$

is indeterminate, use L'Hospital's Rule to find the limit.

$$\lim_{n\to\infty}\frac{12n^3}{12n^2}=\lim_{n\to\infty}\frac{n}{1}=\infty$$

Therefore, the sequence is bounded below at $a_2 = 57/32$ and has no upper bound.

CALCULATING THE FIRST TERMS OF A SERIES OF PARTIAL SUMS

1. Approximate the first four terms of the series of partial sums.

$$\sum_{n=1}^{\infty} \frac{7n}{3n^2+2}$$

Solution:

Make a table and calculate the value of each term.

$$n=1 \qquad a_1 = \frac{7(1)}{3(1)^2+2} = \frac{7}{5} \approx 1.40 \qquad s_1 = 1.40$$

$$n=2 \qquad a_2 = \frac{7(2)}{3(2)^2+2} = \frac{14}{14} = 1 = 1.00 \qquad s_2 = 1.40 + 1.00 = 2.40$$

$$n=3 \qquad a_3 = \frac{7(3)}{3(3)^2+2} = \frac{21}{29} \approx 0.72 \qquad s_3 = 2.40 + 0.72 = 3.12$$

$$n=4 \qquad a_4 = \frac{7(4)}{3(4)^2+2} = \frac{28}{50} = \frac{14}{25} \approx 0.56 \qquad s_4 = 3.12 + 0.56 = 3.68$$

2. Approximate the first four terms of the series of partial sums.

$$\sum_{n=1}^{\infty} \frac{5n^2}{7n+4}$$

Solution:

Make a table and calculate the value of each term.

$$n=1 \qquad a_1=\frac{5(1)^2}{7(1)+4}=\frac{5}{11}\approx 0.45 \qquad s_1=0.45$$

$$n=2 \qquad a_2=\frac{5(2)^2}{7(2)+4}=\frac{20}{18}=\frac{10}{9}\approx 1.11 \qquad s_2=0.45+1.11=1.56$$

$$n=3 \qquad a_3=\frac{5(3)^2}{7(3)+4}=\frac{45}{25}=\frac{9}{5}\approx 1.80 \qquad s_3=1.56+1.80=3.36$$

$$n=4 \qquad a_4=\frac{5(4)^2}{7(4)+4}=\frac{80}{32}=\frac{5}{2}\approx 2.50 \qquad s_4=3.36+2.50=5.86$$

■ 3. Approximate the first four terms of the series of partial sums.

$$\sum_{n=1}^{\infty}\frac{9n^3}{8n^2+13}$$

Solution:

Make a table and calculate the value of each term.

$$n=1 \qquad a_1=\frac{9(1)^3}{8(1)^2+13}=\frac{9}{21}=\frac{3}{7}\approx 0.43 \qquad s_1=0.43$$

$$n = 2 \qquad a_2 = \frac{9(2)^3}{8(2)^2 + 13} = \frac{72}{45} = \frac{8}{5} = 1.60 \qquad s_2 = 0.43 + 1.60 = 2.03$$

$$n = 3 \qquad a_3 = \frac{9(3)^3}{8(3)^2 + 13} = \frac{243}{85} \approx 2.86 \qquad s_3 = 2.03 + 2.86 = 4.89$$

$$n = 4 \qquad a_4 = \frac{9(4)^3}{8(4)^2 + 13} = \frac{576}{141} \approx 4.09 \qquad s_4 = 4.89 + 4.09 = 8.98$$

SUM OF THE SERIES OF PARTIAL SUMS

■ 1. Use the partial sums equation to find the sum of the series.

$$s_n = 12 + \frac{9}{n}$$

Solution:

The sum of the series with terms a_n is given by

$$\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty}\left(12 + \frac{9}{n}\right)$$

$$\sum_{n=1}^{\infty} a_n = 12 + \frac{9}{\infty}$$

$$\sum_{n=1}^{\infty} a_n = 12 + 0$$

$$\sum_{n=1}^{\infty} a_n = 12$$

■ 2. Use the partial sums equation to find the sum of the series.

$$s_n = \frac{7n^2 + 9n}{n^2 - 6}$$

Solution:

The sum of the series with terms a_n is given by

$$\sum_{n=1}^{\infty} a_n = \lim_{n \to \infty} \left(\frac{7n^2 + 9n}{n^2 - 6} \right)$$

Because evaluating the limit gives an indeterminate form, use L'Hospital's Rule to evaluate the limit.

$$\lim_{n \to \infty} \left(\frac{7n^2 + 9n}{n^2 - 6} \right) = \lim_{n \to \infty} \left(\frac{14n + 9}{2n} \right) = \lim_{n \to \infty} \left(\frac{14}{2} \right) = 7$$

So the sum of the series a_n is 7.

■ 3. Use the partial sums equation to find the sum of the series.

$$s_n = \frac{9n^3 + 7n + 9}{8n^3 + 2n^2 + 5}$$

Solution:

The sum of the series with terms a_n is given by

$$\sum_{n=1}^{\infty} a_n = \lim_{n \to \infty} \left(\frac{9n^3 + 7n + 9}{8n^3 + 2n^2 + 5} \right)$$

Because evaluating the limit gives an indeterminate form, use L'Hospital's Rule to evaluate the limit.

$$\lim_{n\to\infty}\left(\frac{9n^3+7n+9}{8n^3+2n^2+5}\right)=\lim_{n\to\infty}\left(\frac{27n^2+7}{24n^2+4n}\right)=\lim_{n\to\infty}\left(\frac{54n}{48n+4}\right)=\frac{54}{48}=\frac{9}{8}$$

So the sum of the series a_n is $9/8$.

■ 4. Use the partial sums equation to find the sum of the series.

$$s_n=\frac{13}{15n^3}+\frac{12}{n}+5$$

Solution:

The sum of the series with terms a_n is given by

$$\sum_{n=1}^{\infty}a_n=\lim_{n\to\infty}\left(\frac{13}{15n^3}+\frac{12}{n}+5\right)$$

$$\sum_{n=1}^{\infty}a_n=\frac{13}{\infty}+\frac{12}{\infty}+5$$

$$\sum_{n=1}^{\infty}a_n=0+0+5$$

$$\sum_{n=1}^{\infty}a_n=5$$

5. Use the partial sums equation to find the sum of the series.

$$s_n = \frac{14n^2}{15n^3} - \frac{n}{16n^2} - \frac{1}{4n} + \frac{1}{3}$$

Solution:

The sum of the series with terms a_n is given by

$$\sum_{n=1}^{\infty} a_n = \lim_{n \to \infty} \left(\frac{14n^2}{15n^3} - \frac{n}{16n^2} - \frac{1}{4n} + \frac{1}{3} \right)$$

$$\sum_{n=1}^{\infty} a_n = \lim_{n \to \infty} \left(\frac{14}{15n} - \frac{1}{16n} - \frac{1}{4n} + \frac{1}{3} \right)$$

$$\sum_{n=1}^{\infty} a_n = \frac{14}{\infty} - \frac{1}{\infty} - \frac{1}{\infty} + \frac{1}{3}$$

$$\sum_{n=1}^{\infty} a_n = 0 + 0 + 0 + \frac{1}{3}$$

$$\sum_{n=1}^{\infty} a_n = \frac{1}{3}$$

GEOMETRIC SERIES TEST

■ 1. Use the geometric series test to say whether the geometric series converges or diverges, then give the value of the common ratio r.

$$\sum_{n=1}^{\infty} 6\left(\frac{2}{3}\right)^{n-1}$$

Solution:

In the series, $a = 6$ and $r = 2/3$. Since $|r| < 1$, the series converges.

■ 2. Use the geometric series test to say whether the geometric series converges or diverges, then give the value of the common ratio r.

$$\sum_{n=1}^{\infty} \left(\frac{3}{7}\right)^{n-1}$$

Solution:

In the series, $a = 1$ and $r = 3/7$. Since $|r| < 1$, the series converges.

■ 3. Use the geometric series test to say whether the geometric series converges or diverges, then give the value of the common ratio r.

$$\frac{\pi}{2} + \frac{\pi^2}{6} + \frac{\pi^3}{18} + \frac{\pi^4}{54} + \cdots$$

Solution:

In the series, $a = \pi/2$ and $r = \pi/3$. Since $|r| > 1$, the series diverges.

■ 4. Use the geometric series test to say whether the geometric series converges or diverges, then give the value of the common ratio r.

$$1 - \frac{1}{3} + \frac{1}{9} - \frac{1}{27} + \cdots + \left(-\frac{1}{3}\right)^{n-1} + \cdots$$

Solution:

In the series, $a = 1$ and $r = -1/3$. Since $|r| < 1$, the series converges.

■ 5. Use the geometric series test to say whether the geometric series converges or diverges, then give the value of the common ratio r.

$$\sum_{n=1}^{\infty} \left(\frac{e}{\pi}\right)^n$$

Solution:

In the series, $a = e/\pi$ and $r = e/\pi$. Since $|r| < 1$, the series converges.

SUM OF THE GEOMETRIC SERIES

1. Find the sum of the geometric series.

$$\sum_{n=1}^{\infty} 7\left(\frac{3}{8}\right)^{n-1}$$

Solution:

In the series, $a = 7$ and $r = 3/8$, so $|r| < 1$. Then the series converges to the sum

$$S = \frac{a}{1-r} = \frac{7}{1-\frac{3}{8}} = \frac{\frac{7}{1}}{\frac{5}{8}} = \frac{7}{1} \cdot \frac{8}{5} = \frac{56}{5}$$

2. Find the sum of the geometric series.

$$\sum_{n=1}^{\infty} 9\left(\frac{5}{14}\right)^{n-1}$$

Solution:

In the series, $a = 9$ and $r = 5/14$, so $|r| < 1$. Then the series converges to the sum

$$S=\frac{a}{1-r}=\frac{9}{1-\frac{5}{14}}=\frac{\frac{9}{1}}{\frac{9}{14}}=\frac{9}{1}\cdot\frac{14}{9}=14$$

3. Find the sum of the geometric series.

$$\frac{1}{3}-\frac{2}{9}+\frac{4}{27}-\frac{8}{81}+\cdots$$

Solution:

In the series, $a=1/3$ and $r=-2/3$, so $|r|<1$. Then the series converges to the sum

$$S=\frac{a}{1-r}=\frac{\frac{1}{3}}{1-\left(-\frac{2}{3}\right)}=\frac{\frac{1}{3}}{\frac{5}{3}}=\frac{1}{3}\cdot\frac{3}{5}=\frac{1}{5}$$

4. Find the sum of the geometric series.

$$\sum_{n=1}^{\infty}\left(\frac{e}{\pi}\right)^{n}$$

Solution:

In the series, $a = e/\pi$ and $r = e/\pi$, so $|r| < 1$. Then the series converges to the sum

$$S = \frac{a}{1-r} = \frac{\frac{e}{\pi}}{1-\frac{e}{\pi}} = \frac{\frac{e}{\pi}}{\frac{\pi}{\pi}-\frac{e}{\pi}} = \frac{\frac{e}{\pi}}{\frac{\pi-e}{\pi}} = \frac{e}{\pi} \cdot \frac{\pi}{\pi-e} = \frac{e}{\pi-e}$$

VALUES FOR WHICH THE SERIES CONVERGES

1. Find the values of x for which the geometric series converges.

$$\sum_{n=1}^{\infty} \frac{17}{3} x^{n-1}$$

Solution:

Expand the series.

$$\sum_{n=1}^{\infty} \frac{17}{3} x^{n-1} = \frac{17}{3} + \frac{17}{3}x + \frac{17}{3}x^2 + \frac{17}{3}x^3 + \frac{17}{3}x^4 + \cdots$$

The common ratio between each term is x. So we'll set up the inequality $|r| < 1$ to solve for the values where the series converges.

$$|x| < 1$$

$$-1 < x < 1$$

2. Find the values of x for which the geometric series converges.

$$\sum_{n=1}^{\infty} 5 \left(\frac{x-2}{3} \right)^{n-1}$$

Solution:

Expand the series.

$$\sum_{n=1}^{\infty} 5\left(\frac{x-2}{3}\right)^{n-1} = 5 + 5\left(\frac{x-2}{3}\right) + 5\left(\frac{x-2}{3}\right)^2 + 5\left(\frac{x-2}{3}\right)^3$$

$$+5\left(\frac{x-2}{3}\right)^4 + 5\left(\frac{x-2}{3}\right)^5 + \cdots$$

The common ratio between each term is $(x-2)/3$. So we'll set up the inequality $|r| < 1$ to solve for the values where the series converges.

$$\left|\frac{x-2}{3}\right| < 1$$

$$-1 < \frac{x-2}{3} < 1$$

$$-3 < x-2 < 3$$

$$-1 < x < 5$$

3. Find the values of x for which the geometric series converges.

$$\sum_{n=0}^{\infty} 4^n x^n$$

Solution:

Expand the series.

$$\sum_{n=0}^{\infty} 4^n x^n = 1 + 4x + 16x^2 + 64x^3 + 256x^4 + \cdots$$

The common ratio between each term is $4x$. So we'll set up the inequality $|r| < 1$ to solve for the values where the series converges.

$$|4x| < 1$$

$$-1 < 4x < 1$$

$$-\frac{1}{4} < x < \frac{1}{4}$$

GEOMETRIC SERIES FOR REPEATING DECIMALS

■ 1. Express the repeating decimal $0.\overline{17}$ as a geometric series.

Solution:

The repeating decimal can be re-written as

$$0.\overline{17}$$

$$0.17171717171717...$$

$$0.17 + 0.0017 + 0.000017 + 0.00000017 + \dots$$

$$\frac{17}{100} + \frac{17}{10{,}000} + \frac{17}{1{,}000{,}000} + \frac{17}{100{,}000{,}000} + \dots$$

$$\frac{17}{100}\left(1 + \frac{1}{100} + \frac{1}{10{,}000} + \frac{1}{1{,}000{,}000} + \dots\right)$$

Now that the repeating decimal is written as a series, we can identify $a = 17/100$ and $r = 1/100$. So the series is

$$\sum_{n=1}^{\infty} a_1 r^{n-1}$$

$$\sum_{n=1}^{\infty} \frac{17}{100}\left(\frac{1}{100}\right)^{n-1}$$

■ 2. Express the repeating decimal $23.\overline{23}$ as a geometric series.

Solution:

The repeating decimal can be re-written as

$$23.\overline{23}$$

$$23.23232323...$$

$$23 + 0.23 + 0.0023 + 0.000023 + 0.00000023 + \dots$$

$$23 + \frac{23}{100} + \frac{23}{10,000} + \frac{23}{1,000,000} + \frac{23}{100,000,000} + \dots$$

$$23 + \frac{23}{100}\left(1 + \frac{1}{100} + \frac{1}{10,000} + \frac{1}{1,000,000} + \dots\right)$$

Now that the repeating decimal is written as a series, we can identify $a = 23/100$ and $r = 1/100$. So the series is

$$\sum_{n=1}^{\infty} a_1 r^{n-1}$$

$$23 + \sum_{n=1}^{\infty} \frac{23}{100}\left(\frac{1}{100}\right)^{n-1}$$

3. Express the repeating decimal $6.7\overline{2}$ as a geometric series.

Solution:

The repeating decimal can be re-written as

$$6.7\overline{2}$$

$$6.722222222...$$

$$6.7 + 0.02 + 0.002 + 0.0002 + 0.00002 + \ldots$$

$$6.7 + \frac{2}{100} + \frac{2}{1{,}000} + \frac{2}{10{,}000} + \frac{2}{100{,}000} + \ldots$$

$$6.7 + \frac{1}{50} + \frac{1}{500} + \frac{1}{5{,}000} + \frac{1}{50{,}000} + \ldots$$

$$6.7 + \frac{1}{50}\left(1 + \frac{1}{10} + \frac{1}{100} + \frac{1}{1{,}000} + \ldots\right)$$

Now that the repeating decimal is written as a series, we can identify $a = 1/50$ and $r = 1/10$. So the series is

$$\sum_{n=1}^{\infty} a_1 r^{n-1}$$

$$6.7 + \sum_{n=1}^{\infty} \frac{1}{50}\left(\frac{1}{10}\right)^{n-1}$$

4. Express the repeating decimal $9.15\overline{65}$ as a geometric series.

Solution:

The repeating decimal can be re-written as

$$9.15\overline{65}$$

$$9.1565656565...$$

$$9.15 + 0.0065 + 0.000065 + 0.00000065 + 0.0000000065 + \dots$$

$$9.15 + \frac{65}{10,000} + \frac{65}{1,000,000} + \frac{65}{100,000,000} + \frac{65}{10,000,000,000} + \dots$$

$$9.15 + \frac{13}{2,000} + \frac{13}{200,000} + \frac{13}{20,000,000} + \frac{13}{2,000,000,000} + \dots$$

$$9.15 + \frac{13}{2,000}\left(1 + \frac{1}{100} + \frac{1}{10,000} + \frac{1}{1,000,000} + \dots\right)$$

Now that the repeating decimal is written as a series, we can identify $a = 13/2,000$ and $r = 1/100$. So the series is

$$\sum_{n=1}^{\infty} a_1 r^{n-1}$$

$$9.15 + \sum_{n=0}^{\infty} \frac{13}{2,000}\left(\frac{1}{100}\right)^{n-1}$$

CONVERGENCE OF A TELESCOPING SERIES

1. Say whether the telescoping series converges or diverges.

$$\sum_{n=1}^{\infty} \left(5^n - 5^{n-1}\right)$$

Solution:

Rewrite the series.

$$\sum_{n=1}^{\infty} \left(5^n - 5^{n-1}\right)$$

$$\sum_{n=1}^{\infty} \left(5 \cdot 5^{n-1} - 5^{n-1}\right)$$

$$\sum_{n=1}^{\infty} (5 - 1)5^{n-1}$$

$$\sum_{n=1}^{\infty} (4)5^{n-1}$$

Matching this to

$$\sum_{n=1}^{\infty} a_1 \cdot r^{n-1}$$

tells us that $r=5$. Because the series only converges when $|r|<1$, this series diverges.

■ 2. Say whether the telescoping series converges or diverges.

$$\sum_{n=1}^{\infty} \frac{1}{n(n+2)}$$

Solution:

Using a partial fractions decomposition on the series,

$$\sum_{n=1}^{\infty} \frac{1}{n(n+2)} = \sum_{n=1}^{\infty} \frac{1}{2n} - \frac{1}{2(n+2)}$$

Find the first few terms of this rewritten series.

$$n=1 \qquad \frac{1}{2(1)} - \frac{1}{2(1+2)} = \frac{1}{2} - \frac{1}{6}$$

$$n=2 \qquad \frac{1}{2(2)} - \frac{1}{2(2+2)} = \frac{1}{4} - \frac{1}{8}$$

$$n=3 \qquad \frac{1}{2(3)} - \frac{1}{2(3+2)} = \frac{1}{6} - \frac{1}{10}$$

$$n=4 \qquad \frac{1}{2(4)} - \frac{1}{2(4+2)} = \frac{1}{8} - \frac{1}{12}$$

$$n = 5 \qquad \frac{1}{2(5)} - \frac{1}{2(5+2)} = \frac{1}{10} - \frac{1}{14}$$

If we use these terms to write out the expanded series, we get

$$\left(\frac{1}{2} - \frac{1}{6}\right) + \left(\frac{1}{4} - \frac{1}{8}\right) + \left(\frac{1}{6} - \frac{1}{10}\right)$$

$$+\left(\frac{1}{8} - \frac{1}{12}\right) + \left(\frac{1}{10} - \frac{1}{14}\right) + \dots + \left(\frac{1}{2n} - \frac{1}{2(n+2)}\right) + \dots$$

$$\frac{1}{2} - \frac{1}{6} + \frac{1}{4} - \frac{1}{8} + \frac{1}{6} - \frac{1}{10}$$

$$+\frac{1}{8} - \frac{1}{12} + \frac{1}{10} - \frac{1}{14} + \dots + \frac{1}{2n} - \frac{1}{2(n+2)} + \dots$$

$$\frac{1}{2} - \frac{1}{6} + \frac{1}{6} + \frac{1}{4} - \frac{1}{8} + \frac{1}{8}$$

$$-\frac{1}{10} + \frac{1}{10} - \frac{1}{12} - \frac{1}{14} + \dots + \frac{1}{2n} - \frac{1}{2(n+2)} + \dots$$

$$\frac{1}{2} + \frac{1}{4} - \frac{1}{12} - \frac{1}{14} + \dots + \frac{1}{2n} - \frac{1}{2(n+2)} + \dots$$

But if we were to continue with the pattern, those middle terms cancel, and we're left with only

$$\frac{1}{2} + \frac{1}{4} - \frac{1}{2((n-1)+2)} - \frac{1}{2(n+2)}$$

$$\frac{1}{2} + \frac{1}{4} - \frac{1}{2(n+1)} - \frac{1}{2(n+2)}$$

$$\frac{3}{4}-\frac{1}{2(n+1)}-\frac{1}{2(n+2)}$$

Take the limit as $n \to \infty$.

$$\lim_{n\to\infty}\frac{3}{4}-\frac{1}{2(n+1)}-\frac{1}{2(n+2)}$$

$$\frac{3}{4}-\frac{1}{2(\infty+1)}-\frac{1}{2(\infty+2)}$$

$$\frac{3}{4}-0-0$$

$$\frac{3}{4}$$

The series converges.

■ 3. Say whether the telescoping series converges or diverges.

$$\sum_{n=1}^{\infty}\frac{2}{n^2+n}$$

Solution:

Using a partial fractions decomposition on the series,

$$\sum_{n=1}^{\infty}\frac{2}{n^2+n}=\sum_{n=1}^{\infty}\frac{2}{n}-\frac{2}{n+1}$$

Find the first few terms of this rewritten series.

$n = 1$ $\qquad \frac{2}{n} - \frac{2}{n+1} = \frac{2}{1} - \frac{2}{1+1} = 2 - 1 = 1$

$n = 2$ $\qquad \frac{2}{n} - \frac{2}{n+1} = \frac{2}{2} - \frac{2}{2+1} = 1 - \frac{2}{3} = \frac{1}{3}$

$n = 3$ $\qquad \frac{2}{n} - \frac{2}{n+1} = \frac{2}{3} - \frac{2}{3+1} = \frac{2}{3} - \frac{1}{2} = \frac{1}{6}$

$n = 4$ $\qquad \frac{2}{n} - \frac{2}{n+1} = \frac{2}{4} - \frac{2}{4+1} = \frac{1}{2} - \frac{2}{5} = \frac{1}{10}$

$n = 5$ $\qquad \frac{2}{n} - \frac{2}{n+1} = \frac{2}{5} - \frac{2}{5+1} = \frac{2}{5} - \frac{1}{3} = \frac{1}{15}$

If we use these terms to write out the expanded series, we get

$$(2-1) + \left(1 - \frac{2}{3}\right) + \left(\frac{2}{3} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{2}{5}\right) + \left(\frac{2}{5} - \frac{1}{3}\right) + \ldots + \left(\frac{2}{n} - \frac{2}{n+1}\right) + \ldots$$

$$2 - 1 + 1 - \frac{2}{3} + \frac{2}{3} - \frac{1}{2} + \frac{1}{2} - \frac{2}{5} + \frac{2}{5} - \frac{1}{3} + \ldots + \frac{2}{n} - \frac{2}{n+1} + \ldots$$

$$2 - \frac{1}{3} + \ldots + \frac{2}{n} - \frac{2}{n+1} + \ldots$$

But if we were to continue with the pattern, those middle terms cancel, and we're left with only

$$2 - \frac{2}{n+1}$$

Take the limit as $n \to \infty$.

$$\lim_{n\to\infty} 2-\frac{2}{n+1}=2-\frac{2}{\infty+1}=2-0=2$$

The series converges.

4. Say whether the telescoping series converges or diverges.

$$\sum_{n=1}^{\infty}\frac{4}{n^2+3n+2}$$

Solution:

Using a partial fractions decomposition on the series,

$$\sum_{n=1}^{\infty}\frac{4}{n^2+3n+2}=\sum_{n=1}^{\infty}\frac{4}{n+1}-\frac{4}{n+2}$$

Find the first few terms of this rewritten series.

$$n=1 \qquad \frac{4}{n+1}-\frac{4}{n+2}=\frac{4}{1+1}-\frac{4}{1+2}=2-\frac{4}{3}$$

$$n=2 \qquad \frac{4}{n+1}-\frac{4}{n+2}=\frac{4}{2+1}-\frac{4}{2+2}=\frac{4}{3}-1$$

$$n=3 \qquad \frac{4}{n+1}-\frac{4}{n+2}=\frac{4}{3+1}-\frac{4}{3+2}=1-\frac{4}{5}$$

$$n=4 \qquad \frac{4}{n+1}-\frac{4}{n+2}=\frac{4}{4+1}-\frac{4}{4+2}=\frac{4}{5}-\frac{2}{3}$$

$n = 5$ $$\frac{4}{n+1} - \frac{4}{n+2} = \frac{4}{5+1} - \frac{4}{5+2} = \frac{2}{3} - \frac{4}{7}$$

If we use these terms to write out the expanded series, we get

$$\left(2 - \frac{4}{3}\right) + \left(\frac{4}{3} - 1\right) + \left(1 - \frac{4}{5}\right)$$

$$+ \left(\frac{4}{5} - \frac{2}{3}\right) + \left(\frac{2}{3} - \frac{4}{7}\right) + \ldots + \left(\frac{4}{n+1} - \frac{4}{n+2}\right) + \ldots$$

$$2 - \frac{4}{3} + \frac{4}{3} - 1 + 1 - \frac{4}{5} + \frac{4}{5} - \frac{2}{3} + \frac{2}{3} - \frac{4}{7} + \ldots + \frac{4}{n+1} - \frac{4}{n+2} + \ldots$$

$$2 - \frac{4}{7} + \ldots + \frac{4}{n+1} - \frac{4}{n+2} + \ldots$$

But if we were to continue with the pattern, those middle terms cancel, and we're left with only

$$2 - \frac{4}{n+2}$$

Take the limit as $n \to \infty$.

$$\lim_{n \to \infty} 2 - \frac{4}{n+2} = 2 - \frac{4}{\infty + 2} = 2 - 0 = 2$$

The series converges.

■ 5. Say whether the telescoping series converges or diverges.

$$\sum_{n=1}^{\infty} \frac{5}{n+1} - \frac{5}{n+2}$$

Solution:

Find the first few terms of this rewritten series.

$n = 1$ $\quad \frac{5}{n+1} - \frac{5}{n+2} = \frac{5}{1+1} - \frac{5}{1+2} = \frac{5}{2} - \frac{5}{3} = \frac{5}{6}$

$n = 2$ $\quad \frac{5}{n+1} - \frac{5}{n+2} = \frac{5}{2+1} - \frac{5}{2+2} = \frac{5}{3} - \frac{5}{4} = \frac{5}{12}$

$n = 3$ $\quad \frac{5}{n+1} - \frac{5}{n+2} = \frac{5}{3+1} - \frac{5}{3+2} = \frac{5}{4} - 1 = \frac{1}{4}$

$n = 4$ $\quad \frac{5}{n+1} - \frac{5}{n+2} = \frac{5}{4+1} - \frac{5}{4+2} = 1 - \frac{5}{6} = \frac{1}{6}$

$n = 5$ $\quad \frac{5}{n+1} - \frac{5}{n+2} = \frac{5}{5+1} - \frac{5}{5+2} = \frac{5}{6} - \frac{5}{7} = \frac{5}{42}$

If we use these terms to write out the expanded series, we get

$$\left(\frac{5}{2} - \frac{5}{3}\right) + \left(\frac{5}{3} - \frac{5}{4}\right) + \left(\frac{5}{4} - 1\right)$$

$$+\left(1 - \frac{5}{6}\right) + \left(\frac{5}{6} - \frac{5}{7}\right) + \dots + \left(\frac{5}{n+1} - \frac{5}{n+2}\right) + \dots$$

$$\frac{5}{2} - \frac{5}{3} + \frac{5}{3} - \frac{5}{4} + \frac{5}{4} - 1 + 1 - \frac{5}{6} + \frac{5}{6} - \frac{5}{7} + \dots + \frac{5}{n+1} - \frac{5}{n+2} + \dots$$

$$\frac{5}{2} - \frac{5}{7} + \ldots + \frac{5}{n+1} - \frac{5}{n+2} + \ldots$$

But if we were to continue with the pattern, those middle terms cancel, and we're left with only

$$\frac{5}{2} - \frac{5}{n+2}$$

Take the limit as $n \to \infty$.

$$\lim_{n\to\infty} \frac{5}{2} - \frac{5}{n+2} = \frac{5}{2} - \frac{5}{\infty + 1} = \frac{5}{2} - 0 = \frac{5}{2}$$

The series converges.

SUM OF A TELESCOPING SERIES

1. Calculate the sum of the telescoping series.

$$\sum_{n=1}^{\infty} \frac{3}{n^2+n}$$

Solution:

Use a partial fractions decomposition to rewrite the series.

$$\sum_{n=1}^{\infty} \frac{3}{n^2+n} = \sum_{n=1}^{\infty} \frac{3}{n} - \frac{3}{n+1}$$

Then the first few terms of the series are

$$n = 1 \qquad \frac{3}{n} - \frac{3}{n+1} = \frac{3}{1} - \frac{3}{1+1} = 3 - \frac{3}{2}$$

$$n = 2 \qquad \frac{3}{n} - \frac{3}{n+1} = \frac{3}{2} - \frac{3}{2+1} = \frac{3}{2} - 1$$

$$n = 3 \qquad \frac{3}{n} - \frac{3}{n+1} = \frac{3}{3} - \frac{3}{3+1} = 1 - \frac{3}{4}$$

$$n = 4 \qquad \frac{3}{n} - \frac{3}{n+1} = \frac{3}{4} - \frac{3}{4+1} = \frac{3}{4} - \frac{3}{5}$$

$$n = 5 \qquad \frac{3}{n} - \frac{3}{n+1} = \frac{3}{5} - \frac{3}{5+1} = \frac{3}{5} - \frac{1}{2}$$

If we use these terms to write out the expanded series, we get

$$\left(3-\frac{3}{2}\right)+\left(\frac{3}{2}-1\right)+\left(1-\frac{3}{4}\right)$$

$$+\left(\frac{3}{4}-\frac{3}{5}\right)+\left(\frac{3}{5}-\frac{1}{2}\right)+\ldots+\left(\frac{3}{n}-\frac{3}{n+1}\right)+\ldots$$

$$3-\frac{3}{2}+\frac{3}{2}-1+1-\frac{3}{4}+\frac{3}{4}-\frac{3}{5}+\frac{3}{5}-\frac{1}{2}+\ldots+\left(\frac{3}{n}-\frac{3}{n+1}\right)+\ldots$$

$$3-\frac{1}{2}+\ldots+\left(\frac{3}{n}-\frac{3}{n+1}\right)+\ldots$$

But if we were to continue with the pattern, those middle terms cancel, and we're left with only

$$3-\frac{3}{n+1}$$

Take the limit as $n \to \infty$.

$$\lim_{n\to\infty} 3-\frac{3}{n+1}=3-\frac{3}{\infty+1}=3-0=3$$

The sum of the series is 3.

2. Calculate the sum of the telescoping series.

$$\sum_{n=1}^{\infty}\frac{4}{n^2+3n+2}$$

Solution:

Use a partial fractions decomposition to rewrite the series.

$$\sum_{n=1}^{\infty} \frac{4}{n^2+3n+2} = \sum_{n=1}^{\infty} \frac{4}{n+1} - \frac{4}{n+2}$$

Then the first few terms of the series are

$$n=1 \qquad \frac{4}{n+1} - \frac{4}{n+2} = \frac{4}{1+1} - \frac{4}{1+2} = 2 - \frac{4}{3} = \frac{2}{3}$$

$$n=2 \qquad \frac{4}{n+1} - \frac{4}{n+2} = \frac{4}{2+1} - \frac{4}{2+2} = \frac{4}{3} - 1 = \frac{1}{3}$$

$$n=3 \qquad \frac{4}{n+1} - \frac{4}{n+2} = \frac{4}{3+1} - \frac{4}{3+2} = 1 - \frac{4}{5} = \frac{1}{5}$$

$$n=4 \qquad \frac{4}{n+1} - \frac{4}{n+2} = \frac{4}{4+1} - \frac{4}{4+2} = \frac{4}{5} - \frac{2}{3} = \frac{2}{15}$$

$$n=5 \qquad \frac{4}{n+1} - \frac{4}{n+2} = \frac{4}{5+1} - \frac{4}{5+2} = \frac{2}{3} - \frac{4}{7} = \frac{2}{21}$$

If we use these terms to write out the expanded series, we get

$$\left(2-\frac{4}{3}\right)+\left(\frac{4}{3}-1\right)+\left(1-\frac{4}{5}\right)$$

$$+\left(\frac{4}{5}-\frac{2}{3}\right)+\left(\frac{2}{3}-\frac{4}{7}\right)+\dots+\left(\frac{4}{n+1}-\frac{4}{n+2}\right)+\dots$$

$$2-\frac{4}{3}+\frac{4}{3}-1+1-\frac{4}{5}+\frac{4}{5}-\frac{2}{3}+\frac{2}{3}-\frac{4}{7}+\ldots+\frac{4}{n+1}-\frac{4}{n+2}+\ldots$$

$$2-\frac{4}{7}+\ldots+\frac{4}{n+1}-\frac{4}{n+2}+\ldots$$

But if we were to continue with the pattern, those middle terms cancel, and we're left with only

$$2-\frac{4}{n+2}$$

Take the limit as $n \to \infty$.

$$\lim_{n\to\infty} 2-\frac{4}{n+2}=2-\frac{4}{\infty+2}=2-0=2$$

The sum of the series is 2.

■ 3. Calculate the sum of the telescoping series.

$$\sum_{n=1}^{\infty}\frac{6}{n+2}-\frac{6}{n+3}$$

Solution:

Then the first few terms of the series are

$$n=1 \qquad \frac{6}{n+2}-\frac{6}{n+3}=\frac{6}{1+2}-\frac{6}{1+3}=2-\frac{3}{2}$$

$$n=2 \qquad \frac{6}{n+2}-\frac{6}{n+3}=\frac{6}{2+2}-\frac{6}{2+3}=\frac{3}{2}-\frac{6}{5}$$

$$n=3 \qquad \frac{6}{n+2}-\frac{6}{n+3}=\frac{6}{3+2}-\frac{6}{3+3}=\frac{6}{5}-1$$

$$n=4 \qquad \frac{6}{n+2}-\frac{6}{n+3}=\frac{6}{4+2}-\frac{6}{4+3}=1-\frac{6}{7}$$

$$n=5 \qquad \frac{6}{n+2}-\frac{6}{n+3}=\frac{6}{5+2}-\frac{6}{5+3}=\frac{6}{7}-\frac{3}{4}$$

If we use these terms to write out the expanded series, we get

$$\left(2-\frac{3}{2}\right)+\left(\frac{3}{2}-\frac{6}{5}\right)+\left(\frac{6}{5}-1\right)$$

$$+\left(1-\frac{6}{7}\right)+\left(\frac{6}{7}-\frac{3}{4}\right)+\ldots+\left(\frac{6}{n+2}-\frac{6}{n+3}\right)+\ldots$$

$$2-\frac{3}{2}+\frac{3}{2}-\frac{6}{5}+\frac{6}{5}-1+1-\frac{6}{7}+\frac{6}{7}-\frac{3}{4}+\ldots+\frac{6}{n+2}-\frac{6}{n+3}+\ldots$$

$$2-\frac{3}{4}+\ldots.+\frac{6}{n+2}-\frac{6}{n+3}+\ldots$$

But if we were to continue with the pattern, those middle terms cancel, and we're left with only

$$2-\frac{6}{n+3}$$

Take the limit as $n \to \infty$.

$$\lim_{n\to\infty} 2 - \frac{6}{n+3} = 2 - \frac{6}{\infty + 3} = 2 - 0 = 2$$

The sum of the series is 2.

LIMIT VS. SUM OF THE SERIES

1. Find the limit of the series, and if it converges, find its sum.

$$\sum_{n=1}^{\infty} 3e^{-n} + 2^{-n}$$

Solution:

The limit of the series is given by

$$\lim_{n\to\infty} 3e^{-n} + \frac{1}{2^n} = \lim_{n\to\infty} \frac{3}{e^n} + \lim_{n\to\infty} \frac{1}{2^n}$$

Notice that the denominator of both expressions gets bigger and bigger, but the numerator is a constant. So the value of each fraction approaches 0.

$$\lim_{n\to\infty} \frac{3}{e^n} + \lim_{n\to\infty} \frac{1}{2^n} = \frac{3}{\infty} + \frac{1}{\infty} = 0 + 0 = 0$$

To find the sum of the series, rewrite it as

$$\sum_{n=1}^{\infty} 3e^{-n} + 2^{-n}$$

$$\sum_{n=1}^{\infty} \frac{3}{e^n} + \frac{1}{2^n}$$

The first few terms of the series are

$$a_1 = \frac{3}{e^1} + \frac{1}{2^1} = \frac{3}{e} + \frac{1}{2}$$

$$a_2 = \frac{3}{e^2} + \frac{1}{2^2} = \frac{3}{e^2} + \frac{1}{4}$$

$$a_3 = \frac{3}{e^3} + \frac{1}{2^3} = \frac{3}{e^3} + \frac{1}{8}$$

$$a_4 = \frac{3}{e^4} + \frac{1}{2^4} = \frac{3}{e^4} + \frac{1}{16}$$

Then the sum of the series is

$$S = \frac{3}{e} + \frac{1}{2} + \frac{3}{e^2} + \frac{1}{4} + \frac{3}{e^3} + \frac{1}{8} + \frac{3}{e^4} + \frac{1}{16} + \ldots + \frac{3}{e^n} + \frac{1}{2^n} + \ldots$$

Split this into two separate sums.

$$S_1 = \frac{3}{e} + \frac{3}{e^2} + \frac{3}{e^3} + \frac{3}{e^4} + \ldots + \frac{3}{e^n} + \ldots$$

$$S_2 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ldots + \frac{1}{2^n} + \ldots$$

S_1 is a geometric series with $a = 3/e$ and $r = 1/e$. So S_1 is

$$S_1 = \frac{a}{1-r} = \frac{\frac{3}{e}}{1-\frac{1}{e}} = \frac{\frac{3}{e}}{\frac{e}{e}-\frac{1}{e}} = \frac{\frac{3}{e}}{\frac{e-1}{e}} = \frac{3}{e-1}$$

S_2 is a geometric series with $a = 1/2$ and $r = 1/2$. So S_2 is

$$S_2 = \frac{a}{1-r} = \frac{\frac{1}{2}}{1-\frac{1}{2}} = \frac{\frac{1}{2}}{\frac{1}{2}} = 1$$

So the sum of the series and limit of the series are

$$S_1 + S_2 = \frac{3}{e-1} + 1$$

$$\lim_{n\to\infty} \frac{3}{e^n} + \frac{1}{2^n} = 0$$

2. Find the limit of the series, and if it converges, find its sum.

$$\sum_{n=1}^{\infty} \frac{3^n + 2^n}{6^n}$$

Solution:

The limit of the series is given by

$$\lim_{n\to\infty} \frac{3^n + 2^n}{6^n}$$

$$\lim_{n\to\infty} \frac{3^n}{6^n} + \lim_{n\to\infty} \frac{2^n}{6^n}$$

$$\lim_{n\to\infty} \left(\frac{3}{6}\right)^n + \lim_{n\to\infty} \left(\frac{2}{6}\right)^n$$

$$\lim_{n\to\infty}\left(\frac{1}{2}\right)^n+\lim_{n\to\infty}\left(\frac{1}{3}\right)^n$$

$$\lim_{n\to\infty}\frac{1}{2^n}+\lim_{n\to\infty}\frac{1}{3^n}$$

Notice that the denominator of both expressions gets bigger and bigger, but the numerator is a constant. So the value of each fraction approaches 0.

$$\lim_{n\to\infty}\frac{1}{2^n}+\lim_{n\to\infty}\frac{1}{3^n}=\frac{1}{\infty}+\frac{1}{\infty}=0+0=0$$

To find the sum of the series, rewrite it as

$$\sum_{n=1}^{\infty}\frac{3^n+2^n}{6^n}$$

$$\sum_{n=1}^{\infty}\frac{1}{2^n}+\frac{1}{3^n}$$

The first few terms of the series are

$$a_1=\frac{1}{2^1}+\frac{1}{3^1}=\frac{1}{2}+\frac{1}{3}$$

$$a_2=\frac{1}{2^2}+\frac{1}{3^2}=\frac{1}{4}+\frac{1}{9}$$

$$a_3=\frac{1}{2^3}+\frac{1}{3^3}=\frac{1}{8}+\frac{1}{27}$$

$$a_4=\frac{1}{2^4}+\frac{1}{3^4}=\frac{1}{16}+\frac{1}{81}$$

Then the sum of the series is

$$S = \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{9} + \frac{1}{8} + \frac{1}{27} + \frac{1}{16} + \frac{1}{81} + \ldots + \frac{1}{2^n} + \frac{1}{3^n} + \ldots$$

Split this into two separate sums.

$$S_1 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ldots + \frac{1}{2^n} + \ldots$$

$$S_2 = \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \frac{1}{81} + \ldots + \frac{1}{3^n} + \ldots$$

S_1 is a geometric series with $a = 1/2$ and $r = 1/2$. So S_1 is

$$S_1 = \frac{a}{1-r} = \frac{\frac{1}{2}}{1-\frac{1}{2}} = \frac{\frac{1}{2}}{\frac{1}{2}} = 1$$

S_2 is a geometric series with $a = 1/3$ and $r = 1/3$. So S_2 is

$$S_2 = \frac{a}{1-r} = \frac{\frac{1}{3}}{1-\frac{1}{3}} = \frac{\frac{1}{3}}{\frac{2}{3}} = \frac{1}{2}$$

So the sum of the series and limit of the series are

$$S_1 + S_2 = 1 + \frac{1}{2} = \frac{3}{2}$$

$$\lim_{n\to\infty} \frac{1}{2^n} + \frac{1}{3^n} = 0$$

■ 3. Find the limit of the series, and if it converges, find its sum.

$$\sum_{n=1}^{\infty} \frac{3}{5^n} + \frac{2}{n}$$

Solution:

The limit of the series is given by

$$\lim_{n\to\infty} \frac{3}{5^n} + \frac{2}{n} = \lim_{n\to\infty} \frac{3}{5^n} + \lim_{n\to\infty} \frac{2}{n}$$

Notice that the denominator of both expressions gets bigger and bigger, but the numerator is a constant. So the value of each fraction approaches 0.

$$\lim_{n\to\infty} \frac{1}{2^n} + \lim_{n\to\infty} \frac{1}{3^n} = \frac{1}{\infty} + \frac{1}{\infty} = 0 + 0 = 0$$

To find the sum of the series, rewrite it as

$$\sum_{n=1}^{\infty} \frac{3}{5^n} + \frac{2}{n}$$

$$\sum_{n=1}^{\infty} \frac{3}{5^n} + \sum_{n=1}^{\infty} \frac{2}{n}$$

The first few terms of the first series are

$$a_1 = \frac{3}{5^1} = \frac{3}{5}$$

$$a_2 = \frac{3}{5^2} = \frac{3}{25}$$

$$a_3 = \frac{3}{5^3} = \frac{3}{125}$$

$$a_4 = \frac{3}{5^4} = \frac{3}{625}$$

Then the sum of the series is

$$S_1 = \frac{3}{5} + \frac{3}{25} + \frac{3}{125} + \frac{3}{625} + \dots + \frac{3}{5^n} + \dots$$

S_1 is a geometric series with $a = 3/5$ and $r = 1/5$. So S_1 is

$$S_1 = \frac{a}{1-r} = \frac{\frac{3}{5}}{1-\frac{1}{5}} = \frac{\frac{3}{5}}{\frac{4}{5}} = \frac{3}{4}$$

The first few terms of the second series are

$$a_1 = \frac{2}{1}$$

$$a_2 = \frac{2}{2}$$

$$a_3 = \frac{2}{3}$$

$$a_4 = \frac{2}{4}$$

Notice that this series is a p-series with $p = 1$, which means the series diverges and has no sum. Since part of the given series has no sum, the whole series has no sum.

INTEGRAL TEST

1. Use the integral test to say whether the series converges or diverges. If it converges, give the value to which it converges.

$$\sum_{n=1}^{\infty} \frac{7}{n^{\frac{3}{2}}}$$

Solution:

Every term of the series is positive, every term is less than the preceding term, and the series is defined for every term because $n \geq 1$, so the integral test will apply.

So express the series as a function.

$$f(x) = \frac{7}{x^{\frac{3}{2}}} = 7x^{-\frac{3}{2}}$$

Then set up the integral.

$$\int_1^{\infty} f(x)\ dx = \lim_{b \to \infty} \int_1^b 7x^{-\frac{3}{2}}\ dx = 7 \lim_{b \to \infty} \int_1^b x^{-\frac{3}{2}}\ dx$$

Integrate, then evaluate over the interval.

$$7 \lim_{b \to \infty} -2x^{-\frac{1}{2}} \Big|_1^b$$

$$7\lim_{b\to\infty} -\frac{2}{\sqrt{x}}\Big|_1^b$$

$$7\lim_{b\to\infty} -\frac{2}{\sqrt{b}} - \left(-\frac{2}{\sqrt{1}}\right)$$

$$7\lim_{b\to\infty} -\frac{2}{\sqrt{b}} + 2$$

$$7(0+2)$$

$$14$$

The integral converges to a real number value, which means the series also converges. To find the value to which the series converges, we'll take the limit of the series as $n \to \infty$.

$$\lim_{n\to\infty} \frac{7}{n^{\frac{3}{2}}}$$

$$\lim_{n\to\infty} \frac{7}{\sqrt{n^3}}$$

$$0$$

Therefore, the series is convergent, and converges to 0.

■ 2. Use the integral test to say whether the series converges or diverges. If it converges, give the value to which it converges.

$$\sum_{n=1}^{\infty} \frac{9}{n+1}$$

Solution:

Every term of the series is positive, every term is less than the preceding term, and the series is defined for every term because $n \geq 1$, so the integral test will apply.

So express the series as a function.

$$f(x) = \frac{9}{x+1}$$

Then set up the integral.

$$\int_1^{\infty} f(x)\ dx = \lim_{b \to \infty} \int_1^b \frac{9}{x+1}\ dx = 9 \lim_{b \to \infty} \int_1^b \frac{1}{x+1}\ dx$$

Integrate, then evaluate over the interval.

$$9 \lim_{b \to \infty} \ln|x+1| \Big|_1^b$$

$$9 \lim_{b \to \infty} \ln|b+1| - \ln|1+1|$$

$$9(\infty - \ln 2)$$

$$\infty$$

The integral diverges, which means the series also diverges.

3. Use the integral test to say whether the series converges or diverges. If it converges, give the value to which it converges.

$$\sum_{n=1}^{\infty} \frac{9}{7n-2}$$

Solution:

Every term of the series is positive, every term is less than the preceding term, and the series is defined for every term because $n \geq 1$, so the integral test will apply.

So express the series as a function.

$$f(x) = \frac{9}{7x-2}$$

Then set up the integral.

$$\int_1^{\infty} f(x)\ dx = \lim_{b\to\infty} \int_1^b \frac{9}{7x-2}\ dx = 9 \lim_{b\to\infty} \int_1^b \frac{1}{7x-2}\ dx$$

Integrate, then evaluate over the interval.

$$\frac{9}{7} \lim_{b\to\infty} \ln|7x-2| \Big|_1^b$$

$$\frac{9}{7} \lim_{b\to\infty} \ln|7b-2| - \ln|7(1)-2|$$

$$\frac{9}{7}(\infty - \ln 5)$$

$$\infty$$

The integral diverges, which means the series also diverges.

P-SERIES TEST

1. Use the p-series test to say whether the series converges of diverges.

$$\sum_{n=1}^{\infty} \frac{23}{4\sqrt[3]{n}}$$

Solution:

Rewrite the series as

$$\sum_{n=1}^{\infty} \frac{23}{4\sqrt[3]{n}} = \frac{23}{4} \sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n}} = \frac{23}{4} \sum_{n=1}^{\infty} \frac{1}{n^{\frac{1}{3}}}$$

Now that the series is in standard form for a p-series, and $p = 1/3 \leq 1$, we know the series diverges.

2. Use the p-series test to say whether the series converges of diverges.

$$\sum_{n=1}^{\infty} \frac{7}{5n^3}$$

Solution:

Rewrite the series as

$$\sum_{n=1}^{\infty}\frac{7}{5n^3}=\frac{7}{5}\sum_{n=1}^{\infty}\frac{1}{n^3}$$

Now that the series is in standard form for a p-series, and $p=3>1$, we know the series converges.

■ 3. Use the p-series test to say whether the series converges of diverges.

$$\sum_{n=1}^{\infty}\frac{6n^2+2n}{9n^4}$$

Solution:

Rewrite the series as

$$\sum_{n=1}^{\infty}\frac{6n^2+2n}{9n^4}=\sum_{n=1}^{\infty}\frac{6n^2}{9n^4}+\frac{2n}{9n^4}=\sum_{n=1}^{\infty}\frac{2}{3n^2}+\sum_{n=1}^{\infty}\frac{2}{9n^3}=\frac{2}{3}\sum_{n=1}^{\infty}\frac{1}{n^2}+\frac{2}{9}\sum_{n=1}^{\infty}\frac{1}{n^3}$$

In both series, $1/n^2$ and $1/n^3$, $p>1$ (because p is 2 in the first series and 3 in the second series), which means both series converge, which means the series in general converges.

NTH TERM TEST

1. Use the nth term test to say whether the series diverges, or whether the nth term test is inconclusive.

$$\sum_{n=1}^{\infty} \frac{1}{2n-1}$$

Solution:

First find the limit of the series.

$$\lim_{n\to\infty} \frac{1}{2n-1} = \frac{1}{2 \cdot \infty - 1} = \frac{1}{\infty} = 0$$

Because the limit is 0, the nth term test is inconclusive.

2. Use the nth term test to say whether the series diverges, or whether the nth term test is inconclusive.

$$\sum_{n=1}^{\infty} a_n = 8 + 2 + \frac{1}{2} + \frac{1}{8} + \frac{1}{32} + \frac{1}{128} + \dots$$

Solution:

First find the limit of the series.

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} 8 \cdot \left(\frac{1}{4}\right)^{n-1} = \lim_{n\to\infty} \frac{8}{4^{n-1}} = \frac{8}{4^{\infty-1}} = \frac{8}{4^{\infty}} = \frac{8}{\infty} = 0$$

Because the limit is 0, the nth term test is inconclusive.

■ 3. Use the nth term test to say whether the series diverges, or whether the nth term test is inconclusive.

$$\sum_{n=1}^{\infty} \frac{11^n}{10^n}$$

Solution:

First find the limit of the series.

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{11^n}{10^n} = \lim_{n\to\infty} \left(\frac{11}{10}\right)^n = \infty$$

Because the limit is not 0, the nth term test tells us the series will diverge.

■ 4. Use the nth term test to say whether the series diverges, or whether the nth term test is inconclusive.

$$\sum_{n=1}^{\infty} \frac{n}{n+1}$$

Solution:

First find the limit of the series.

$$\lim_{n\to\infty} \frac{n}{n+1} = \frac{\infty}{\infty+1} = \frac{\infty}{\infty} = 1$$

Because the limit is not 0, the nth term test tells us the series will diverge.

COMPARISON TEST

1. Use the comparison test to say whether or not the series converges.

$$\sum_{n=0}^{\infty} \frac{4}{3^n + n}$$

Solution:

Identify the comparison series.

$$a_n = \frac{4}{3^n + n} \qquad b_n = \frac{4}{3^n}$$

For all $n \geq 0$,

$$\frac{4}{3^n + n} \leq \frac{4}{3^n}$$

Rewrite the comparison series.

$$\sum_{n=0}^{\infty} \frac{4}{3^n}$$

$$\sum_{n=1}^{\infty} 4 \left(\frac{1}{3} \right)^{n-1}$$

The comparison series is a geometric series with $a = 4$ and $r = 1/3$. The sum of the geometric comparison series is

$$\sum_{n=0}^{\infty} \frac{4}{3^n} = \frac{a}{1-r} = \frac{4}{1-\frac{1}{3}} = \frac{4}{\frac{2}{3}} = \frac{4}{1} \cdot \frac{3}{2} = 6$$

Because we know the original series is always less than or equal to the comparison series, we can also say

$$\sum_{n=0}^{\infty} \frac{4}{3^n + n} \leq \sum_{n=0}^{\infty} \frac{4}{3^n}$$

$$\sum_{n=0}^{\infty} \frac{4}{3^n + n} \leq 6$$

Therefore, the series converges.

■ 2. Use the comparison test to say whether or not the series converges.

$$\sum_{n=1}^{\infty} \frac{n}{n^4 + 7}$$

Solution:

Identify the comparison series.

$$a_n = \frac{n}{n^4 + 7} \qquad\qquad b_n = \frac{n}{n^4} = \frac{1}{n^3}$$

The original series a_n will be less than the comparison series b_n,

$$\frac{n}{n^4+7} \le \frac{1}{n^3}$$

$$n \le \frac{n^4+7}{n^3}$$

$$n \le \frac{n^4}{n^3} + \frac{7}{n^3}$$

$$n \le n + \frac{7}{n^3}$$

$$0 \le \frac{7}{n^3}$$

for any $n > 0$. The comparison series is a p-series with $p = 3 > 1$, which means the comparison series converges, and because we know the original series is always less than or equal to the comparison series, we can say that the original series also converges.

■ 3. Use the comparison test to say whether or not the series converges.

$$\sum_{n=2}^{\infty} \frac{5}{\ln n}$$

Solution:

Identify the comparison series.

$$a_n = \frac{5}{\ln n} \qquad\qquad b_n = \frac{5}{n}$$

For all $n \geq 2$,

$$\frac{5}{\ln n} \geq \frac{5}{n}$$

The comparison series is

$$\sum_{n=2}^{\infty} \frac{5}{n}$$

The comparison series is a p-series with $p = 1 \leq 1$, which means the comparison series diverges.

Therefore, because b_n diverges, a_n also diverges.

LIMIT COMPARISON TEST

1. Use the limit comparison test to say whether or not the series converges.

$$\sum_{n=1}^{\infty} \frac{3n+2}{(2n-1)^4}$$

Solution:

Let the comparison series be

$$\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n^3}$$

Use the limit comparison test.

$$\lim_{b\to\infty} \frac{a_n}{b_n} = \lim_{b\to\infty} \frac{3n+2}{(2n-1)^4} \cdot \frac{n^3}{1} = \lim_{b\to\infty} \frac{3n^4+2n^3}{(2n-1)^4} = \lim_{b\to\infty} \frac{\frac{3n^4}{n^4}+\frac{2n^3}{n^4}}{\frac{(2n-1)^4}{n^4}}$$

$$= \lim_{b\to\infty} \frac{3+\frac{2}{n}}{\left(2-\frac{1}{n}\right)^4} = \frac{3+\frac{2}{\infty}}{\left(2-\frac{1}{\infty}\right)^4} = \frac{3+0}{(2-0)^4} = \frac{3}{16}$$

So the value of L is $L = 3/16 > 0$. We know also that the comparison series converges by the p-series test since for that series $p = 3 > 1$. Therefore, because b_n converges, a_n also converges.

2. Use the limit comparison test to say whether or not the series converges.

$$\sum_{n=1}^{\infty} \frac{12n^2+5}{n^3-7}$$

Solution:

Let the comparison series be

$$\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{n^2}{n^3} = \sum_{n=1}^{\infty} \frac{1}{n}$$

Use the limit comparison test.

$$\lim_{b\to\infty} \frac{a_n}{b_n} = \lim_{b\to\infty} \frac{12n^2+5}{n^3-7} \cdot \frac{n}{1} = \lim_{b\to\infty} \frac{12n^3+5n}{n^3-7} = \lim_{b\to\infty} \frac{\frac{12n^3}{n^3}+\frac{5n}{n^3}}{\frac{n^3}{n^3}-\frac{7}{n^3}}$$

$$= \lim_{b\to\infty} \frac{12+\frac{5}{n^2}}{1-\frac{7}{n^3}} = \frac{12+\frac{5}{\infty}}{1-\frac{7}{\infty}} = \frac{12+0}{1-0} = 12$$

So the value of L is $L = 12 > 0$. We know also that the comparison series diverges by the p-series test since for that series $p = 1 \leq 1$. Therefore, because b_n diverges, a_n also diverges.

3. Use the limit comparison test to say whether or not the series converges.

$$\sum_{n=1}^{\infty} \frac{n^4 + 3n^2}{7n^6 + 3n^4}$$

Solution:

Let the comparison series be

$$\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{n^4}{n^6} = \sum_{n=1}^{\infty} \frac{1}{n^2}$$

Use the limit comparison test.

$$\lim_{b \to \infty} \frac{a_n}{b_n} = \lim_{b \to \infty} \frac{n^4 + 3n^2}{7n^6 + 3n^4} \cdot \frac{n^2}{1} = \lim_{b \to \infty} \frac{n^6 + 3n^4}{7n^6 + 3n^4} = \lim_{b \to \infty} \frac{\frac{n^6}{n^6} + \frac{3n^4}{n^6}}{\frac{7n^6}{n^6} + \frac{3n^4}{n^6}}$$

$$= \lim_{b \to \infty} \frac{1 + \frac{3}{n^2}}{7 + \frac{3}{n^2}} = \frac{1 + \frac{3}{\infty}}{7 + \frac{3}{\infty}} = \frac{1 + 0}{7 + 0} = \frac{1}{7}$$

So the value of L is $L = 1/7 > 0$. We know also that the comparison series converges by the p-series test since for that series $p = 2 > 1$. Therefore, because b_n converges, a_n also converges.

ERROR OR REMAINDER OF A SERIES

1. Estimate the remainder of the series using the first three terms.

$$\sum_{n=1}^{\infty} \frac{3}{7n^3 + 2n^2 + 3}$$

Solution:

To find the remainder, estimate the total sum by calculating a partial sum for the series, determine whether the series converges or diverges using the comparison test, and use the integral test to solve for the remainder.

$$n = 1 \qquad a_1 = \frac{3}{7(1)^3 + 2(1)^2 + 3} = \frac{3}{12} = \frac{1}{4}$$

$$s_1 = a_1 = \frac{1}{4} = 0.25$$

$$n = 2 \qquad a_2 = \frac{3}{7(2)^3 + 2(2)^2 + 3} = \frac{3}{67}$$

$$s_2 = a_1 + a_2 = \frac{1}{4} + \frac{3}{67} = \frac{79}{268} \approx 0.295$$

$$n = 3 \qquad a_3 = \frac{3}{7(3)^3 + 2(3)^2 + 3} = \frac{3}{210} = \frac{1}{70}$$

$$s_3 = a_1 + a_2 + a_3 = \frac{1}{4} + \frac{3}{67} + \frac{1}{70} = \frac{2{,}899}{9{,}380} \approx 0.309$$

Use the comparison test to determine convergence or divergence, using the comparison series

$$\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{3}{n^3}$$

Apply the comparison test.

$$\lim_{b \to \infty} \frac{a_n}{b_n} = \lim_{b \to \infty} \frac{3}{7n^3 + 2n^2 + 3} \cdot \frac{n^3}{3} = \lim_{b \to \infty} \frac{3n^3}{21n^3 + 6n^2 + 9} = \lim_{b \to \infty} \frac{\frac{3n^3}{n^3}}{\frac{21n^3}{n^3} + \frac{6n^2}{n^3} + \frac{9}{n^3}}$$

$$= \lim_{b \to \infty} \frac{3}{21 + \frac{6}{n} + \frac{9}{n^3}} = \frac{3}{21 + \frac{6}{\infty} + \frac{9}{\infty}} = \frac{3}{21 + 0 + 0} = \frac{1}{7}$$

This value for L is $L = 1/7 > 0$. Now check convergence or divergence of the comparison series.

$$\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{3}{n^3} = 3 \sum_{n=1}^{\infty} \frac{1}{n^3}$$

This is a p-series with $p = 3 > 1$, which means the comparison series converges. Which means the original series a_n also converges.

Use the integral test to find the remainder of a_n after the first three terms.

$$R_3 \le T_3 \le \int_3^{\infty} b_n \, dx = \int_3^{\infty} f(x) \, dx = \int_3^{\infty} \frac{3}{x^3} \, dx = \int_3^{\infty} 3x^{-3} \, dx$$

$$R_3 \le \int_3^{\infty} 3x^{-3} \, dx$$

Integrate, then evaluate over the interval.

$$R_3 \le \lim_{b\to\infty} \frac{3x^{-2}}{-2}\Bigg|_3^b$$

$$R_3 \le \lim_{b\to\infty} -\frac{3}{2x^2}\Bigg|_3^b$$

$$R_3 \le \lim_{b\to\infty} -\frac{3}{2b^2} - \left(-\frac{3}{2(3)^2}\right)$$

$$R_3 \le \lim_{b\to\infty} -\frac{3}{2b^2} + \frac{1}{6}$$

$$R_3 \le -0 + \frac{1}{6}$$

$$R_3 \le \frac{1}{6}$$

$$R_3 \le 0.167$$

The third partial sum of the series a_n is $s_3 \approx 0.309$, with error $R_3 \le 0.167$.

■ 2. Estimate the remainder of the series using the first three terms.

$$\sum_{n=1}^{\infty} \frac{5}{\sqrt{n^4+6}}$$

Solution:

To find the remainder, estimate the total sum by calculating a partial sum for the series, determine whether the series converges or diverges using the comparison test, and use the integral test to solve for the remainder.

$$n = 1 \qquad a_1 = \frac{5}{\sqrt{1^4 + 6}} = \frac{5}{\sqrt{7}}$$

$$s_1 = a_1 = \frac{5}{\sqrt{7}} \approx 1.890$$

$$n = 2 \qquad a_2 = \frac{5}{\sqrt{2^4 + 6}} = \frac{5}{\sqrt{22}} \approx 1.066$$

$$s_2 = a_1 + a_2 = 1.890 + 1.066 \approx 2.956$$

$$n = 3 \qquad a_3 = \frac{5}{\sqrt{3^4 + 6}} = \frac{5}{\sqrt{87}} \approx 0.836$$

$$s_3 = a_1 + a_2 + a_3 = 1.890 + 1.066 + 0.536 \approx 3.492$$

Use the comparison test to determine convergence or divergence, using the comparison series

$$\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{5}{\sqrt{n^4}} = \sum_{n=1}^{\infty} \frac{5}{n^2}$$

Apply the comparison test.

$$\lim_{b \to \infty} \frac{a_n}{b_n} = \lim_{b \to \infty} \frac{5}{\sqrt{n^4 + 6}} \cdot \frac{n^2}{5} = \lim_{b \to \infty} \frac{5n^2}{5\sqrt{n^4 + 6}} = \lim_{b \to \infty} \frac{\frac{5n^2}{n^2}}{5\frac{\sqrt{n^4+6}}{n^2}}$$

$$= \lim_{b\to\infty} \frac{\frac{5n^2}{n^2}}{5\sqrt{\frac{n^4}{n^4}+\frac{6}{n^4}}} = \lim_{b\to\infty} \frac{5}{5\sqrt{1+\frac{6}{n^4}}} = \frac{5}{5\sqrt{1+\frac{6}{\infty}}}$$

$$= \frac{5}{5\sqrt{1+0}} = \frac{5}{5} = 1$$

This value for L is $L = 1 > 0$. Now check convergence or divergence of the comparison series.

$$\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{5}{n^2} = 5\sum_{n=1}^{\infty} \frac{1}{n^2}$$

This is a p-series with $p = 2 > 1$, which means the comparison series converges. Which means the original series a_n also converges.

Use the integral test to find the remainder of a_n after the first three terms.

$$R_3 \le T_3 \le \int_3^{\infty} b_n\ dx = \int_3^{\infty} f(x)\ dx = \int_3^{\infty} \frac{5}{x^2}\ dx = \int_3^{\infty} 5x^{-2}\ dx$$

$$R_3 \le \int_3^{\infty} 5x^{-2}\ dx$$

Integrate, then evaluate over the interval.

$$R_3 \le \lim_{b\to\infty} \frac{5x^{-1}}{-1}\Bigg|_3^b$$

$$R_3 \le \lim_{b\to\infty} -\frac{5}{x}\Bigg|_3^b$$

$$R_3 \le \lim_{b \to \infty} -\frac{5}{b} - \left(-\frac{5}{3}\right)$$

$$R_3 \le \lim_{b \to \infty} -\frac{5}{b} + \frac{5}{3}$$

$$R_3 \le -0 + \frac{5}{3}$$

$$R_3 \le \frac{5}{3}$$

$$R_3 \le 0.167$$

The third partial sum of the series a_n is $s_3 \approx 3.492$, with error $R_3 \le 0.167$.

■ 3. Estimate the remainder of the series using the first three terms.

$$\sum_{n=1}^{\infty} \frac{4n^2}{n^5 + n^2 + 3}$$

Solution:

To find the remainder, estimate the total sum by calculating a partial sum for the series, determine whether the series converges or diverges using the comparison test, and use the integral test to solve for the remainder.

$$n = 1 \qquad a_1 = \frac{4(1)^2}{1^5 + 1^2 + 3} = \frac{4}{5}$$

$$s_1 = a_1 = \frac{4}{5} = 0.8$$

$n = 2$ $$a_2 = \frac{4(2)^2}{2^5 + 2^2 + 3} = \frac{16}{39} \approx 0.410$$

$$s_2 = a_1 + a_2 = 0.8 + 0.410 \approx 1.210$$

$n = 3$ $$a_3 = \frac{4(3)^2}{3^5 + 3^2 + 3} = \frac{36}{255} = \frac{12}{85} \approx 0.141$$

$$s_3 = a_1 + a_2 + a_3 = 0.8 + 0.410 + 0.141 \approx 1.351$$

Use the comparison test to determine convergence or divergence, using the comparison series

$$\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{4n^2}{n^5} = \sum_{n=1}^{\infty} \frac{4}{n^3}$$

Apply the comparison test.

$$\lim_{b\to\infty} \frac{a_n}{b_n} = \lim_{b\to\infty} \frac{4n^2}{n^5 + n^2 + 3} \cdot \frac{n^3}{4} = \lim_{b\to\infty} \frac{n^5}{n^5 + n^2 + 3} = \lim_{b\to\infty} \frac{\frac{n^5}{n^5}}{\frac{n^5}{n^5} + \frac{n^2}{n^5} + \frac{3}{n^5}}$$

$$= \lim_{b\to\infty} \frac{1}{1 + \frac{1}{n^3} + \frac{3}{n^5}} = \frac{1}{1 + \frac{1}{\infty} + \frac{3}{\infty}} = \frac{1}{1 + 0 + 0} = 1$$

This value for L is $L = 1 > 0$. Now check convergence or divergence of the comparison series.

$$\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{4}{n^3} = 4\sum_{n=1}^{\infty} \frac{1}{n^3}$$

This is a p-series with $p = 3 > 1$, which means the comparison series converges. Which means the original series a_n also converges.

Use the integral test to find the remainder of a_n after the first three terms.

$$R_3 \le T_3 \le \int_3^\infty b_n \, dx = \int_3^\infty f(x) \, dx = \int_3^\infty \frac{4}{x^3} \, dx = \int_3^\infty 4x^{-3} \, dx$$

$$R_3 \le \int_3^\infty 4x^{-3} \, dx$$

Integrate, then evaluate over the interval.

$$R_3 \le \lim_{b \to \infty} \frac{4x^{-2}}{-2} \bigg|_3^b$$

$$R_3 \le \lim_{b \to \infty} -\frac{2}{x^2} \bigg|_3^b$$

$$R_3 \le \lim_{b \to \infty} -\frac{2}{b^2} - \left(-\frac{2}{3^2}\right)$$

$$R_3 \le \lim_{b \to \infty} -\frac{2}{b^2} + \frac{2}{9}$$

$$R_3 \le -0 + \frac{2}{9}$$

$$R_3 \le \frac{2}{9}$$

$$R_3 \le 0.222$$

The third partial sum of the series a_n is $s_3 \approx 1.351$, with error $R_3 \le 0.222$.

RATIO TEST

1. Use the ratio test to determine the convergence of the series.

$$\sum_{n=1}^{\infty} \frac{7^n}{n^3}$$

Solution:

Apply the ratio test.

$$L = \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right|$$

$$L = \lim_{n \to \infty} \left| \frac{\frac{7^{n+1}}{(n+1)^3}}{\frac{7^n}{n^3}} \right| = \lim_{n \to \infty} \left| \frac{7^{n+1}}{(n+1)^3} \cdot \frac{n^3}{7^n} \right|$$

$$L = \lim_{n \to \infty} \left| \frac{7}{(n+1)^3} \cdot \frac{n^3}{1} \right| = \lim_{n \to \infty} \left| \frac{7n^3}{(n+1)^3} \right|$$

$$L = 7 \lim_{n \to \infty} \left| \frac{n^3}{(n+1)^3} \right| = 7 \cdot 1$$

$$L = 7 > 1$$

The series converges if $L < 1$ and diverges if $L > 1$, which means this series diverges.

■ 2. Use the ratio test to determine the convergence of the series.

$$\sum_{n=1}^{\infty} \frac{9(n+3)}{n^2}$$

Solution:

Apply the ratio test.

$$L = \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right|$$

$$L = \lim_{n \to \infty} \left| \frac{\frac{9(n+4)}{(n+1)^2}}{\frac{9(n+3)}{n^2}} \right| = \lim_{n \to \infty} \left| \frac{9(n+4)}{(n+1)^2} \cdot \frac{n^2}{9(n+3)} \right|$$

$$L = \lim_{n \to \infty} \left| \frac{(n+4)}{(n+1)^2} \cdot \frac{n^2}{(n+3)} \right| = \lim_{n \to \infty} \left| \frac{n^3 + 4n^2}{n^3 + 5n^2 + 7n + 3} \right|$$

$$L = 1$$

The ratio test is inconclusive when $L = 1$, so we can't use it to determine convergence for this particular series.

3. Use the ratio test to determine the convergence of the series.

$$\sum_{n=1}^{\infty} \frac{10^n}{5^{3n+1}(n+2)}$$

Solution:

Apply the ratio test.

$$L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$$

$$L = \lim_{n\to\infty} \left| \frac{\frac{10^{n+1}}{5^{3n+4}(n+3)}}{\frac{10^n}{5^{3n+1}(n+2)}} \right| = \lim_{n\to\infty} \left| \frac{10^{n+1}}{5^{3n+4}(n+3)} \cdot \frac{5^{3n+1}(n+2)}{10^n} \right|$$

$$L = \lim_{n\to\infty} \left| \frac{10}{5^3(n+3)} \cdot \frac{(n+2)}{1} \right| = \lim_{n\to\infty} \left| \frac{10(n+2)}{125(n+3)} \right|$$

$$L = \lim_{n\to\infty} \left| \frac{2(n+2)}{25(n+3)} \right| = \frac{2}{25} \lim_{n\to\infty} \left| \frac{n+2}{n+3} \right|$$

$$L = \frac{2}{25} \cdot 1$$

$$L = \frac{2}{25}$$

The series converges if $L < 1$ and diverges if $L > 1$, which means this series converges.

4. Use the ratio test to determine the convergence of the series.

$$\sum_{n=1}^{\infty} \frac{6n+17}{3^{2n+1}}$$

Solution:

Apply the ratio test.

$$L = \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right|$$

$$L = \lim_{n \to \infty} \left| \frac{\frac{6n+23}{3^{2n+3}}}{\frac{6n+17}{3^{2n+1}}} \right| = \lim_{n \to \infty} \left| \frac{6n+23}{3^{2n+3}} \cdot \frac{3^{2n+1}}{6n+17} \right|$$

$$L = \lim_{n \to \infty} \left| \frac{6n+23}{9} \cdot \frac{1}{6n+17} \right| = \lim_{n \to \infty} \left| \frac{6n+23}{9(6n+17)} \right|$$

$$L = \frac{1}{9} \lim_{n \to \infty} \left| \frac{6n+23}{6n+17} \right|$$

$$L = \frac{1}{9} \cdot 1$$

$$L = \frac{1}{9}$$

The series converges if $L < 1$ and diverges if $L > 1$, which means this series converges.

■ 5. Use the ratio test to determine the convergence of the series.

$$\sum_{n=1}^{\infty} \frac{(-1)^n \cdot 5^{n+3}}{6^{n+1}}$$

Solution:

Apply the ratio test.

$$L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$$

$$L = \lim_{n\to\infty} \left| \frac{\frac{(-1)^{n+1} \cdot 5^{n+4}}{6^{n+2}}}{\frac{(-1)^n \cdot 5^{n+3}}{6^{n+1}}} \right| = \lim_{n\to\infty} \left| \frac{\frac{5^{n+4}}{6^{n+2}}}{\frac{5^{n+3}}{6^{n+1}}} \right|$$

$$L = \lim_{n\to\infty} \left| \frac{5^{n+4}}{6^{n+2}} \cdot \frac{6^{n+1}}{5^{n+3}} \right| = \lim_{n\to\infty} \left| \frac{5}{6} \right|$$

$$L = \frac{5}{6}$$

The series converges if $L < 1$ and diverges if $L > 1$, which means this series converges.

RATIO TEST WITH FACTORIALS

1. Use the ratio test to determine the convergence of the series.

$$\sum_{n=1}^{\infty} \frac{n^3}{(2n-1)!}$$

Solution:

Apply the ratio test.

$$L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$$

$$L = \lim_{n\to\infty} \left| \frac{\frac{(n+1)^3}{(2n+1)!}}{\frac{n^3}{(2n-1)!}} \right| = \lim_{n\to\infty} \left| \frac{(n+1)^3}{(2n+1)!} \cdot \frac{(2n-1)!}{n^3} \right|$$

$$L = \lim_{n\to\infty} \left| \frac{(n+1)^3}{(2n+1)(2n)(2n-1)!} \cdot \frac{(2n-1)!}{n^3} \right| = \lim_{n\to\infty} \left| \frac{(n+1)^3}{(2n+1)(2n)} \cdot \frac{1}{n^3} \right|$$

$$L = \lim_{n\to\infty} \left| \frac{(n+1)^3}{n^3(2n+1)(2n)} \right| = \lim_{n\to\infty} \left| \frac{n^3+3n^2+3n+1}{4n^5+2n^4} \right|$$

$$L = 0$$

The series converges if $L < 1$ and diverges if $L > 1$, which means this series converges.

2. Use the ratio test to determine the convergence of the series.

$$\sum_{n=1}^{\infty} \frac{8^n}{2^{n+1} \cdot n!}$$

Solution:

Apply the ratio test.

$$L = \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right|$$

$$L = \lim_{n \to \infty} \left| \frac{\frac{8^{n+1}}{2^{n+2} \cdot (n+1)!}}{\frac{8^n}{2^{n+1} \cdot n!}} \right| = \lim_{n \to \infty} \left| \frac{8^{n+1}}{2^{n+2} \cdot (n+1)!} \cdot \frac{2^{n+1} \cdot n!}{8^n} \right|$$

$$L = \lim_{n \to \infty} \left| \frac{8}{2 \cdot (n+1) \cdot n!} \cdot \frac{1 \cdot n!}{1} \right| = \lim_{n \to \infty} \left| \frac{4}{n+1} \right|$$

$$L = 4 \lim_{n \to \infty} \left| \frac{1}{n+1} \right| = 4 \cdot 0$$

$$L = 0$$

The series converges if $L < 1$ and diverges if $L > 1$, which means this series converges.

3. Use the ratio test to determine the convergence of the series.

$$\sum_{n=1}^{\infty} \frac{(-1)^n n!}{n^3+1}$$

Solution:

Apply the ratio test.

$$L = \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right|$$

$$L = \lim_{n \to \infty} \left| \frac{\frac{(-1)^{n+1}(n+1)!}{(n+1)^3+1}}{\frac{(-1)^n n!}{n^3+1}} \right| = \lim_{n \to \infty} \left| \frac{\frac{(n+1)!}{(n+1)^3+1}}{\frac{n!}{n^3+1}} \right|$$

$$L = \lim_{n \to \infty} \left| \frac{(n+1)!}{(n+1)^3+1} \cdot \frac{n^3+1}{n!} \right|$$

$$L = \lim_{n \to \infty} \left| \frac{(n+1)n!}{(n+1)^3+1} \cdot \frac{n^3+1}{n!} \right| = \lim_{n \to \infty} \left| \frac{(n+1)}{(n+1)^3+1} \cdot \frac{n^3+1}{1} \right|$$

$$L = \lim_{n\to\infty}\left|\frac{(n+1)(n^3+1)}{(n+1)^3+1}\right| = \lim_{n\to\infty}\left|\frac{n^4+n^3+n+1}{n^3+3n^2+3n+2}\right|$$

$$L = \infty$$

The series converges if $L < 1$ and diverges if $L > 1$, which means this series diverges.

4. Use the ratio test to determine the convergence of the series.

$$\sum_{n=0}^{\infty}\frac{(n+2)!}{(3n)^2+7}$$

Solution:

Apply the ratio test.

$$L = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|$$

$$L = \lim_{n\to\infty}\left|\frac{\frac{(n+3)!}{(3n+3)^2+7}}{\frac{(n+2)!}{(3n)^2+7}}\right| = \lim_{n\to\infty}\left|\frac{(n+3)!}{(3n+3)^2+7}\cdot\frac{(3n)^2+7}{(n+2)!}\right|$$

$$L = \lim_{n\to\infty}\left|\frac{(n+3)(n+2)!}{(3n+3)^2+7}\cdot\frac{9n^2+7}{(n+2)!}\right| = \lim_{n\to\infty}\left|\frac{(n+3)}{(3n+3)^2+7}\cdot\frac{9n^2+7}{1}\right|$$

$$L = \lim_{n \to \infty} \left| \frac{(n+3)(9n^2+7)}{(3n+3)^2+7} \right| = \lim_{n \to \infty} \left| \frac{9n^3+27n^2+7n+21}{9n^2+36n+16} \right|$$

$$L = \infty$$

The series converges if $L < 1$ and diverges if $L > 1$, which means this series diverges.

■ 5. Use the ratio test to determine the convergence of the series.

$$\sum_{n=0}^{\infty} \frac{4^n(n+1)}{n!}$$

Solution:

Apply the ratio test.

$$L = \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right|$$

$$L = \lim_{n \to \infty} \left| \frac{\frac{4^{n+1}(n+2)}{(n+1)!}}{\frac{4^n(n+1)}{n!}} \right| = \lim_{n \to \infty} \left| \frac{4^{n+1}(n+2)}{(n+1)!} \cdot \frac{n!}{4^n(n+1)} \right|$$

$$L = \lim_{n \to \infty} \left| \frac{4(n+2)}{(n+1)n!} \cdot \frac{n!}{(n+1)} \right| = \lim_{n \to \infty} \left| \frac{4(n+2)}{(n+1)} \cdot \frac{1}{(n+1)} \right|$$

$$L = \lim_{n \to \infty} \left| \frac{4(n+2)}{(n+1)(n+1)} \right| = \lim_{n \to \infty} \left| \frac{4n+2}{n^2+2n+1} \right|$$

$$L = 0$$

The series converges if $L < 1$ and diverges if $L > 1$, which means this series converges.

ROOT TEST

1. Use the root test to determine the convergence of the series.

$$\sum_{n=3}^{\infty}\left(\frac{5n^3+3n^2-6}{\sqrt{6n^6+7n^4-8}}\right)^n$$

Solution:

Apply the root test.

$$L=\lim_{n\to\infty}\sqrt[n]{|a_n|}$$

$$L=\lim_{n\to\infty}|a_n|^{\frac{1}{n}}$$

$$L=\lim_{n\to\infty}\left|\left(\frac{5n^3+3n^2-6}{\sqrt{6n^6+7n^4-8}}\right)^n\right|^{\frac{1}{n}}$$

$$L=\lim_{n\to\infty}\left|\frac{5n^3+3n^2-6}{\sqrt{6n^6+7n^4-8}}\right|$$

Divide through by the highest-degree term.

$$L = \lim_{n \to \infty} \left| \frac{\frac{5n^3 + 3n^2 - 6}{n^3}}{\frac{\sqrt{6n^6 + 7n^4 - 8}}{n^3}} \right|$$

$$L = \lim_{n \to \infty} \left| \frac{\frac{5n^3}{n^3} + \frac{3n^2}{n^3} - \frac{6}{n^3}}{\sqrt{\frac{6n^6}{n^6} + \frac{7n^4}{n^6} - \frac{8}{n^6}}} \right|$$

$$L = \lim_{n \to \infty} \left| \frac{5 + \frac{3}{n} \frac{-6}{n^3}}{\sqrt{6 + \frac{7}{n^2} - \frac{8}{n^6}}} \right|$$

$$L = \lim_{n \to \infty} \left| \frac{5 + 0 - 0}{\sqrt{6 + 0 - 0}} \right|$$

$$L = \frac{5}{\sqrt{6}}$$

The series converges absolutely if $L < 1$ but diverges if $L > 1$, so the series diverges.

■ 2. Use the root test to determine the convergence of the series.

$$\sum_{n=1}^{\infty} \frac{7n^3}{e^{2n^2}}$$

Solution:

Apply the root test.

$$L = \lim_{n\to\infty} \sqrt[n]{|a_n|}$$

$$L = \lim_{n\to\infty} |a_n|^{\frac{1}{n}}$$

$$L = \lim_{n\to\infty} \left|\frac{7n^3}{e^{2n^2}}\right|^{\frac{1}{n}}$$

$$L = \lim_{n\to\infty} \left|\frac{\sqrt[n]{7n^3}}{\sqrt[n]{e^{2n^2}}}\right|$$

$$L = \lim_{n\to\infty} \left|\frac{\sqrt[n]{7} \cdot n^{\frac{3}{n}}}{e^{\frac{2n^2}{n}}}\right|$$

$$L = \left|\frac{1 \cdot 1}{\infty}\right|$$

$$L = 0$$

The series converges absolutely if $L < 1$ but diverges if $L > 1$, so the series converges absolutely.

■ 3. Use the root test to determine the convergence of the series.

$$\sum_{n=0}^{\infty} \left(\frac{7n - 6n^4}{9n^4 + 3} \right)^n$$

Solution:

Apply the root test.

$$L = \lim_{n \to \infty} \sqrt[n]{|a_n|}$$

$$L = \lim_{n \to \infty} |a_n|^{\frac{1}{n}}$$

$$L = \lim_{n \to \infty} \left| \left(\frac{7n - 6n^4}{9n^4 + 3} \right)^n \right|^{\frac{1}{n}}$$

$$L = \lim_{n \to \infty} \left| \frac{7n - 6n^4}{9n^4 + 3} \right|$$

Divide through by the highest-degree term.

$$L = \lim_{n \to \infty} \left| \frac{\frac{7n}{n^4} - \frac{6n^4}{n^4}}{\frac{9n^4}{n^4} + \frac{3}{n^4}} \right|$$

$$L = \lim_{n \to \infty} \left| \frac{\frac{7}{n^3} - 6}{9 + \frac{3}{n^4}} \right|$$

$$L=\left|\frac{0-6}{9+0}\right|$$

$$L=\frac{2}{3}$$

The series converges absolutely if $L<1$ but diverges if $L>1$, so the series converges absolutely.

ABSOLUTE AND CONDITIONAL CONVERGENCE

■ 1. Use the root test to determine the absolute or conditional convergence of the series.

$$\sum_{n=1}^{\infty}\left(\frac{6n}{8n+5}\right)^n$$

Solution:

Both the ratio and root tests can determine absolute or conditional convergence. The series converges absolutely if $a_n = |a_n|$ and converges conditionally if $a_n \neq |a_n|$.

By the root test,

$$L = \lim_{n\to\infty}\left|\left(\frac{6n}{8n+5}\right)^n\right|^{\frac{1}{n}}$$

$$L = \lim_{n\to\infty}\left|\frac{6n}{8n+5}\right|$$

Divide through by the highest-degree term.

$$L = \lim_{n\to\infty}\left|\frac{\frac{6n}{n}}{\frac{8n}{n}+\frac{5}{n}}\right|$$

$$L = \lim_{n\to\infty} \left| \frac{6}{8 + \frac{5}{n}} \right|$$

$$L = \left| \frac{6}{8+0} \right|$$

$$L = \frac{3}{4} < 1$$

The series converges, so check for absolute vs. conditional convergence by comparing

$$\lim_{n\to\infty} \left| \frac{\frac{6n}{n}}{\frac{8n}{n} + \frac{5}{n}} \right| \text{ and } \lim_{n\to\infty} \frac{\frac{6n}{n}}{\frac{8n}{n} + \frac{5}{n}}$$

We've already found the first value, but the second value is

$$L = \lim_{n\to\infty} \frac{\frac{6n}{n}}{\frac{8n}{n} + \frac{5}{n}} = \frac{6}{8+0} = \frac{3}{4}$$

Since the values are equal, the series converges absolutely.

■ 2. Use the ratio test to determine the absolute or conditional convergence of the series, or say if the series diverges of if the ratio test is inconclusive.

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{25n}$$

Solution:

Both the ratio and root tests can determine absolute or conditional convergence. The series converges absolutely if $a_n = \left| a_n \right|$ and converges conditionally if $a_n \neq \left| a_n \right|$.

By the ratio test,

$$L = \lim_{n \to \infty} \left| \frac{\frac{(-1)^{n+1}}{25n+25}}{\frac{(-1)^n}{25n}} \right|$$

$$L = \lim_{n \to \infty} \left| \frac{1}{25n+25} \cdot \frac{25n}{1} \right|$$

$$L = \lim_{n \to \infty} \left| \frac{25n}{25n+25} \right| = \lim_{n \to \infty} \left| \frac{n}{n+1} \right|$$

$$L = \lim_{n \to \infty} \left| \frac{n}{n+1} \right|$$

$$L = 1$$

The series converges if $L < 1$ and diverges if $L > 1$, but the ratio test is inconclusive when $L = 1$. So the ratio test is inconclusive, and we can't determine absolute or conditional convergence.

3. Use the root test to determine the absolute or conditional convergence of the series.

$$\sum_{n=1}^{\infty} \left(\frac{8n - 9n^5}{14n^5 + 7} \right)^n$$

Solution:

Both the ratio and root tests can determine absolute or conditional convergence. The series converges absolutely if $a_n = |a_n|$ and converges conditionally if $a_n \neq |a_n|$.

By the root test,

$$L = \lim_{n \to \infty} \left| \left(\frac{8n - 9n^5}{14n^5 + 7} \right)^n \right|^{\frac{1}{n}}$$

$$L = \lim_{n \to \infty} \left| \frac{8n - 9n^5}{14n^5 + 7} \right|$$

Divide through by the highest-degree term.

$$L = \lim_{n \to \infty} \left| \frac{\frac{8n}{n^5} - \frac{9n^5}{n^5}}{\frac{14n^5}{n^5} + \frac{7}{n^5}} \right|$$

$$L = \lim_{n\to\infty}\left|\frac{\frac{8}{n^4}-9}{14+\frac{7}{n^5}}\right|$$

$$R = \left|\frac{0-9}{14+0}\right|$$

$$R = \frac{9}{14} < 1$$

The series converges, so check for absolute vs. conditional convergence by comparing

$$\lim_{n\to\infty}\left|\frac{\frac{8}{n^4}-9}{14+\frac{7}{n^5}}\right| \text{ and } \lim_{n\to\infty}\frac{\frac{8}{n^4}-9}{14+\frac{7}{n^5}}$$

We've already found the first value, but the second value is

$$L = \lim_{n\to\infty}\frac{\frac{8}{n^4}-9}{14+\frac{7}{n^5}} = \frac{0-9}{14+0} = -\frac{9}{14}$$

Since the values are unequal, the series conditionally converges.

■ 4. Use the ratio test to determine the absolute or conditional convergence of the series, or say if the series diverges of if the ratio test is inconclusive.

$$\sum_{n=1}^{\infty}\frac{n!}{9^n}$$

Solution:

Both the ratio and root tests can determine absolute or conditional convergence. The series converges absolutely if $a_n = \left|a_n\right|$ and converges conditionally if $a_n \neq \left|a_n\right|$.

By the ratio test,

$$L = \lim_{n\to\infty} \left| \frac{\frac{(n+1)!}{9^{n+1}}}{\frac{n!}{9^n}} \right|$$

$$L = \lim_{n\to\infty} \left| \frac{(n+1)!}{9^{n+1}} \cdot \frac{9^n}{n!} \right|$$

$$L = \lim_{n\to\infty} \left| \frac{(n+1)n!}{9 \cdot 9^n} \cdot \frac{9^n}{n!} \right| = \lim_{n\to\infty} \left| \frac{n+1}{9} \right|$$

$$L = 9 \lim_{n\to\infty} \left| n+1 \right|$$

$$L = 9 \cdot \infty$$

$$L = \infty$$

The series converges if $L < 1$ and diverges if $L > 1$, so the series diverges.

ALTERNATING SERIES TEST

■ 1. Use the alternating series test to say whether the series converges or diverges.

$$\sum_{n=1}^{\infty} (-1)^{n-1} \left(\frac{3}{5n+6} \right)$$

Solution:

An alternating series will only converge if the series is decreasing. Check the first few terms of the series.

$$a_n = \frac{3}{5n+6}$$

$$a_1 = \frac{3}{5(1)+6} = \frac{3}{5+6} = \frac{3}{11}$$

$$a_2 = \frac{3}{5(2)+6} = \frac{3}{10+6} = \frac{3}{17}$$

$$a_3 = \frac{3}{5(3)+6} = \frac{3}{15+6} = \frac{3}{21}$$

$$a_4 = \frac{3}{5(4)+6} = \frac{3}{20+6} = \frac{3}{26}$$

We can see that the series is decreasing. The limit as $n \to \infty$ must also be 0 if the alternating series is going to converge.

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{3}{5n+6} = \frac{3}{5(\infty)+6} = \frac{3}{\infty} = 0$$

Because these two conditions are met, the alternating series converges.

■ 2. Use the alternating series test to say whether the series converges or diverges.

$$\sum_{n=1}^{\infty} (-1)^{n-1} n \left(\frac{2}{7}\right)^n$$

Solution:

An alternating series will only converge if the series is decreasing. Check the first few terms of the series.

$$a_n = n\left(\frac{2}{7}\right)^n$$

$$a_1 = 1\left(\frac{2}{7}\right)^1 = \frac{2}{7} \approx 0.2857$$

$$a_2 = 2\left(\frac{2}{7}\right)^2 = 2 \cdot \frac{4}{49} = \frac{8}{49} \approx 0.1633$$

$$a_3 = 3\left(\frac{2}{7}\right)^3 = 3 \cdot \frac{8}{343} = \frac{24}{343} \approx 0.0988$$

$$a_4 = 4\left(\frac{2}{7}\right)^4 = 4 \cdot \frac{16}{2401} = \frac{64}{2401} \approx 0.0267$$

$$a_5 = 5\left(\frac{2}{7}\right)^5 = 5 \cdot \frac{32}{16{,}807} = \frac{160}{16807} \approx 0.0095$$

$$a_6 = 6\left(\frac{2}{7}\right)^6 = 6 \cdot \frac{64}{117{,}649} = \frac{384}{117{,}649} \approx 0.0033$$

$$a_7 = 7\left(\frac{2}{7}\right)^7 = 7 \cdot \frac{128}{823{,}453} = \frac{896}{823{,}453} \approx 0.0011$$

$$a_8 = 8\left(\frac{2}{7}\right)^8 = 8 \cdot \frac{256}{5{,}764{,}801} = \frac{800}{5{,}764{,}801} \approx 0.00039$$

We can see that the series is decreasing. The limit as $n \to \infty$ must also be 0 if the alternating series is going to converge.

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} n\left(\frac{2}{7}\right)^n = \lim_{n\to\infty} \frac{n}{\left(\frac{7}{2}\right)^n}$$

Use L'Hospital's Rule to evaluate the limit.

$$\lim_{n\to\infty} \frac{n}{\left(\frac{7}{2}\right)^n} = \lim_{n\to\infty} \frac{1}{\left(\frac{7}{2}\right)^n \ln\left(\frac{7}{2}\right)} = 0$$

Because these two conditions are met, the alternating series converges.

3. Use the alternating series test to say whether the series converges or diverges.

$$\sum_{n=3}^{\infty} (-1)^{n+1} \frac{n^3}{n!}$$

Solution:

An alternating series will only converge if the series is decreasing. Check the first few terms of the series.

$$a_n = \frac{n^3}{n!}$$

$$a_3 = \frac{3^3}{3!} = \frac{27}{6} \approx 4.5$$

$$a_4 = \frac{4^3}{4!} = \frac{81}{24} \approx 3.375$$

$$a_5 = \frac{5^3}{5!} = \frac{125}{120} \approx 1.042$$

$$a_6 = \frac{6^3}{6!} = \frac{216}{720} \approx 0.3$$

$$a_7 = \frac{7^3}{7!} = \frac{343}{5040} \approx 0.681$$

We can see that the series is decreasing. The limit as $n \to \infty$ must also be 0 if the alternating series is going to converge.

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} (-1)^{n-1}\frac{n^3}{n!}$$

Use the ratio test to find the limit.

$$\lim_{n\to\infty} (-1)^{n-1}\frac{n^3}{n!}$$

$$\lim_{n\to\infty} \left| \frac{(-1)^{n+1-1}\frac{(n+1)^3}{(n+1)!}}{(-1)^{n-1}\frac{n^3}{n!}} \right|$$

$$\lim_{n\to\infty} \left| \frac{\frac{(n+1)^3}{(n+1)!}}{\frac{n^3}{n!}} \right|$$

$$\lim_{n\to\infty} \left| \frac{(n+1)^3}{(n+1)!} \cdot \frac{n!}{n^3} \right|$$

$$\lim_{n\to\infty} \left| \frac{(n+1)^3}{(n+1)n!} \cdot \frac{n!}{n^3} \right|$$

$$\lim_{n\to\infty} \left| \frac{(n+1)^3}{(n+1)} \cdot \frac{1}{n^3} \right|$$

$$\lim_{n\to\infty} \left| \frac{(n+1)^3}{n^3(n+1)} \right|$$

$$\lim_{n \to \infty} \left| \frac{n + 3n^2 + 3n + 1}{n^4 + n^3} \right|$$

$$\lim_{n \to \infty} \left| \frac{\frac{n}{n^4} + \frac{3n^2}{n^4} + \frac{3n}{n^4} + \frac{1}{n^4}}{\frac{n^4}{n^4} + \frac{n^3}{n^4}} \right|$$

$$\lim_{n \to \infty} \left| \frac{\frac{1}{n^3} + \frac{3}{n^2} + \frac{3}{n^3} + \frac{1}{n^4}}{1 + \frac{1}{n}} \right|$$

$$\left| \frac{0 + 0 + 0 + 0}{1 + 0} \right|$$

$$0$$

Because these two conditions are met, the alternating series converges.

ALTERNATING SERIES ESTIMATION THEOREM

1. Approximate the sum of the alternating series to three decimal places, using the first 5 terms. Then find the remainder of the approximation, to the nearest six decimal places.

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}n^3}{12^n}$$

Solution:

The first several terms of the series are

$$a_n = \frac{(-1)^{n-1}n^3}{12^n}$$

$$a_1 = \frac{(-1)^{1-1}(1)^3}{12^1} = \frac{1 \cdot 1}{12} = \frac{1}{12} \approx 0.083333$$

$$a_2 = \frac{(-1)^{2-1}(2)^3}{12^2} = \frac{-1 \cdot 8}{144} = \frac{-8}{144} = -\frac{1}{18} \approx -0.055556$$

$$a_3 = \frac{(-1)^{3-1}(3)^3}{12^3} = \frac{1 \cdot 27}{1728} = \frac{27}{1728} = \frac{1}{64} \approx 0.015625$$

$$a_4 = \frac{(-1)^{4-1}(4)^3}{12^4} = \frac{-1 \cdot 81}{20736} = -\frac{81}{20736} = -\frac{1}{256} \approx -0.003086$$

$$a_5 = \frac{(-1)^{5-1}(5)^3}{12^5} = \frac{1 \cdot 125}{248,832} = \frac{125}{248,832} \approx 0.0005023$$

$$a_6 = \frac{(-1)^{6-1}(6)^3}{12^6} = \frac{-1 \cdot 216}{2,985,984} = -\frac{216}{2,985,984} = -\frac{27}{373,248} \approx -0.0000723$$

Then the first five partial sums are

$$s_1 = a_1 = 0.08333$$

$$s_2 = a_1 + a_2 = 0.08333 - 0.05556 = 0.02777$$

$$s_3 = a_1 + a_2 + a_3 = 0.08333 - 0.05556 + 0.015625 = 0.043395$$

$$s_4 = a_1 + a_2 + a_3 + a_4$$

$$= 0.08333 - 0.05556 + 0.015625 - 0.003086 = 0.04031$$

$$s_5 = a_1 + a_2 + a_3 + a_4 + a_5$$

$$= 0.08333 - 0.05556 + 0.015625 - 0.003086 + 0.000502 = 0.040811$$

$$s_6 = a_1 + a_2 + a_3 + a_4 + a_5 + a_6$$

$$= 0.08333 - 0.05556 + 0.015625 - 0.003086 + 0.000502 - 0.0000723$$

$$= 0.040739$$

The approximation to three decimal places is $s = 0.041$. Verify that the series is decreasing, $b_{n+1} \leq b_n$.

$$b_n = \frac{n^3}{12^n}$$

$$b_1 = \frac{(1)^3}{12^1} = \frac{1}{12} \approx 0.08333$$

$$b_2 = \frac{(2)^3}{12^2} = \frac{8}{144} = \frac{8}{144} = \frac{1}{18} \approx 0.05556$$

$$b_3 = \frac{(3)^3}{12^3} = \frac{27}{1728} = \frac{1}{64} \approx 0.015625$$

$$b_4 = \frac{(4)^3}{12^4} = \frac{64}{20{,}736} = \frac{1}{324} \approx 0.003086$$

$$b_5 = \frac{(5)^3}{12^5} = \frac{125}{248{,}832} \approx 0.000502$$

$$b_6 = \frac{(6)^3}{12^6} = \frac{216}{2{,}985{,}984} \approx 0.0000723$$

Verify that the limit as $n \to \infty$ is 0. Use L'Hospital's rule.

$$\lim_{n\to\infty} \frac{n^3}{12^n} = \lim_{n\to\infty} \frac{3n^2}{12^n \ln 12} = \lim_{n\to\infty} \frac{6n}{12^n(\ln 12)^2} = \lim_{n\to\infty} \frac{6}{12^n(\ln 12)^3} = \frac{6}{\infty} = 0$$

Find the remainder.

$$\left|R_n\right| = \left|S - S_n\right| \le b_{n+1}$$

$$\left|R_5\right| = \left|S - S_5\right| \le b_{5+1}$$

$$\left|R_5\right| \le b_6$$

$$\left|R_5\right| \le 0.0000723$$

So the approximation of the sum of the alternating series is $S_5 \approx 0.041$, with an error of $\left|R_5\right| \le 0.0000723$.

2. Approximate the sum of the alternating series to three decimal places, using the first 12 terms. Then find the remainder of the approximation, to the nearest six decimal places.

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{3n^3}$$

Solution:

The first several terms of the series are

$$a_n = \frac{(-1)^{n+1}}{3n^3}$$

$$a_1 = \frac{(-1)^{1+1}}{3(1)^3} = \frac{1}{3} \approx 0.333333$$

$$a_2 = \frac{(-1)^{2+1}}{3(2)^3} = \frac{-1}{24} \approx -0.041667$$

$$a_3 = \frac{(-1)^{3+1}}{3(3)^3} = \frac{1}{81} \approx 0.012346$$

$$a_4 = \frac{(-1)^{4+1}}{3(4)^3} = \frac{-1}{192} \approx -0.005208$$

$$a_5 = \frac{(-1)^{5+1}}{3(5)^3} = \frac{1}{729} \approx 0.002667$$

$$a_6 = \frac{(-1)^{6+1}}{3(6)^3} = \frac{-1}{648} \approx -0.001543$$

$$a_7 = \frac{(-1)^{7+1}}{3(7)^3} = \frac{1}{1029} \approx 0.000972$$

$$a_8 = \frac{(-1)^{8+1}}{3(8)^3} = \frac{-1}{1536} \approx -0.000651$$

$$a_9 = \frac{(-1)^{9+1}}{3(9)^3} = \frac{1}{2187} \approx 0.000457$$

$$a_{10} = \frac{(-1)^{10+1}}{3(10)^3} = \frac{-1}{3000} \approx -0.000333$$

$$a_{11} = \frac{(-1)^{11+1}}{3(11)^3} = \frac{1}{3993} \approx 0.000250$$

$$a_{12} = \frac{(-1)^{12+1}}{3(12)^3} = \frac{-1}{5184} \approx -0.000193$$

$$a_{13} = \frac{(-1)^{13+1}}{3(13)^3} = \frac{1}{6591} \approx 0.000152$$

Then the first five partial sums are

$$s_1 = a_1 = 0.333333$$

$$s_2 = a_1 + a_2 = 0.333333 - 0.041667 = 0.291666$$

$$s_3 = a_1 + a_2 + a_3 = 0.333333 - 0.041667 + 0.012346 = 0.304012$$

$$s_4 = a_1 + a_2 + a_3 + a_4$$

$$= 0.333333 - 0.041667 + 0.012346 - 0.005208 = 0.298804$$

$$s_5 = a_1 + a_2 + a_3 + a_4 + a_5$$

$$= 0.333333 - 0.041667 + 0.012346 - 0.005208 + 0.002667 = 0.301471$$

$$s_6 = a_1 + a_2 + a_3 + a_4 + a_5 + a_6$$

$$= 0.333333 - 0.041667 + 0.012346 - 0.005208 + 0.002667 - 0.001543$$

$$= 0.299928$$

$$s_7 = a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + a_7$$

$$= 0.333333 - 0.041667 + 0.012346 - 0.005208$$

$$+0.002667 - 0.001543 + 0.000972$$

$$= 0.300900$$

$$s_8 = a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + a_7 + a_8$$

$$= 0.333333 - 0.041667 + 0.012346 - 0.005208$$

$$+0.002667 - 0.001543 + 0.000972 - 0.000651$$

$$= 0.300249$$

$$s_9 = a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + a_7 + a_8 + a_9$$

$$= 0.333333 - 0.041667 + 0.012346 - 0.005208 + 0.002667$$

$$-0.001543 + 0.000972 - 0.000651 + 0.000457$$

$$= 0.300706$$

$$s_{10} = a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + a_7 + a_8 + a_9 + a_{10}$$

$$= 0.333333 - 0.041667 + 0.012346 - 0.005208 + 0.002667$$

$$-0.001543 + 0.000972 - 0.000651 + 0.000457 - 0.000333$$

$$= 0.300373$$

$$s_{11} = a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + a_7 + a_8 + a_9 + a_{10} + a_{11}$$

$$= 0.333333 - 0.041667 + 0.012346 - 0.005208 + 0.002667 - 0.001543$$

$$+0.000972 - 0.000651 + 0.000457 - 0.000333 + 0.000250$$

$$= 0.300623$$

$$s_{12} = a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + a_7 + a_8 + a_9 + a_{10} + a_{11} + a_{12}$$

$$= 0.333333 - 0.041667 + 0.012346 - 0.005208$$

$$+0.002667 - 0.001543 + 0.000972 - 0.000651$$

$$+0.000457 - 0.000333 + 0.000250 - 0.000193$$

$$= 0.300430$$

$$s_{13} = a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + a_7 + a_8 + a_9 + a_{10} + a_{11} + a_{12} + a_{13}$$

$$= 0.333333 - 0.041667 + 0.012346 - 0.005208$$

$$+0.002667 - 0.001543 + 0.000972 - 0.000651$$

$$+0.000457 - 0.000333 + 0.000250 - 0.000193 + 0.000152$$

$$= 0.300195$$

The approximation to three decimal places is $s = 0.300$. Verify that the series is decreasing, $b_{n+1} \le b_n$.

$$b_n = \frac{1}{3n^3}$$

$$b_1 = \frac{1}{3(1)^3} = \frac{1}{3} \approx 0.333333$$

$$b_2 = \frac{1}{3(2)^3} = \frac{1}{24} \approx 0.041667$$

$$b_3 = \frac{1}{3(3)^3} = \frac{1}{81} \approx 0.012346$$

$$b_4 = \frac{1}{3(4)^3} = \frac{1}{192} \approx 0.005208$$

$$b_5 = \frac{1}{3(5)^3} = \frac{1}{729} \approx 0.002667$$

$$b_6 = \frac{1}{3(6)^3} = \frac{1}{648} \approx 0.001543$$

$$b_7 = \frac{1}{3(7)^3} = \frac{1}{1029} \approx 0.000972$$

$$b_8 = \frac{1}{3(8)^3} = \frac{1}{1536} \approx 0.000651$$

$$b_9 = \frac{1}{3(9)^3} = \frac{1}{2187} \approx 0.000457$$

$$b_{10} = \frac{1}{3(10)^3} = \frac{1}{3000} \approx 0.000333$$

$$b_{11} = \frac{1}{3(11)^3} = \frac{1}{3993} \approx 0.000250$$

$$b_{12} = \frac{1}{3(12)^3} = \frac{1}{5184} \approx 0.000193$$

$$b_{13} = \frac{1}{3(13)^3} = \frac{1}{6591} \approx 0.000152$$

Verify that the limit as $n \to \infty$ is 0.

$$\lim_{n\to\infty} \frac{1}{3n^3} = \frac{1}{3} \lim_{n\to\infty} \frac{1}{n^3} = \frac{1}{3} \cdot 0 = 0$$

Find the remainder.

$$\left|R_n\right| = \left|S - S_n\right| \le b_{n+1}$$

$$\left|R_{12}\right| = \left|S - S_{10}\right| \le b_{12+1}$$

$$\left|R_{12}\right| \le b_{13}$$

$$\left|R_{12}\right| \le 0.000152$$

So the approximation of the sum of the alternating series is $S_{10} \approx 0.300$, with an error of $\left|R_{10}\right| \leq 0.000152$.

3. Approximate the sum of the alternating series to three decimal places, using the first 10 terms. Then find the remainder of the approximation.

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1} \cdot 3}{12n^3 + 4n^2}$$

Solution:

The first several terms of the series are

$$a_n = \frac{(-1)^{n-1} \cdot 3}{12n^3 + 4n^2}$$

$$a_1 = \frac{(-1)^{1-1} \cdot 3}{12(1)^3 + 4(1)^2} = \frac{1 \cdot 3}{12 + 4} = \frac{3}{16} = 0.187500$$

$$a_2 = \frac{(-1)^{2-1} \cdot 3}{12(2)^3 + 4(2)^2} = \frac{-1 \cdot 3}{96 + 16} = -\frac{3}{112} \approx -0.026786$$

$$a_3 = \frac{(-1)^{3-1} \cdot 3}{12(3)^3 + 4(3)^2} = \frac{1 \cdot 3}{324 + 36} = \frac{3}{360} \approx 0.008333$$

$$a_4 = \frac{(-1)^{4-1} \cdot 3}{12(4)^3 + 4(4)^2} = \frac{-1 \cdot 3}{768 + 64} = -\frac{3}{832} \approx -0.003606$$

$$a_5 = \frac{(-1)^{5-1} \cdot 3}{12(5)^3 + 4(5)^2} = \frac{1 \cdot 3}{1500 + 100} = \frac{3}{1600} \approx 0.001875$$

$$a_6 = \frac{(-1)^{6-1} \cdot 3}{12(6)^3 + 4(6)^2} = \frac{-1 \cdot 3}{2592 + 144} = -\frac{3}{2736} \approx -0.001096$$

$$a_7 = \frac{(-1)^{7-1} \cdot 3}{12(7)^3 + 4(7)^2} = \frac{1 \cdot 3}{4116 + 196} = \frac{3}{4312} \approx 0.000696$$

$$a_8 = \frac{(-1)^{8-1} \cdot 3}{12(8)^3 + 4(8)^2} = \frac{-1 \cdot 3}{6144 + 256} = -\frac{3}{6400} \approx -0.000469$$

$$a_9 = \frac{(-1)^{9-1} \cdot 3}{12(9)^3 + 4(9)^2} = \frac{1 \cdot 3}{8748 + 324} = \frac{3}{9072} \approx 0.000331$$

$$a_{10} = \frac{(-1)^{10-1} \cdot 3}{12(10)^3 + 4(10)^2} = \frac{-1 \cdot 3}{12000 + 400} = -\frac{3}{12400} \approx -0.000242$$

$$a_{11} = \frac{(-1)^{11-1} \cdot 3}{12(11)^3 + 4(11)^2} = \frac{1 \cdot 3}{15972 + 484} = \frac{3}{16456} \approx 0.000182$$

Then the first five partial sums are

$$s_1 = a_1 = 0.187500$$

$$s_2 = a_1 + a_2 = 0.1875 - 0.026786 = 0.160714$$

$$s_3 = a_1 + a_2 + a_3 = 0.1875 - 0.026786 + 0.008333 = 0.169047$$

$$s_4 = a_1 + a_2 + a_3 + a_4$$

$$= 0.1875 - 0.026786 + 0.008333 - 0.003606 = 0.165441$$

$$s_5 = a_1 + a_2 + a_3 + a_4 + a_5$$

$$= 0.1875 - 0.026786 + 0.008333 - 0.003606 + 0.001875 = 0.167316$$

$$s_6 = a_1 + a_2 + a_3 + a_4 + a_5 + a_6$$

$$= 0.1875 - 0.026786 + 0.008333 - 0.003606 + 0.001875 - 0.001096$$

$$= 0.166220$$

$$s_7 = a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + a_7$$

$$= 0.1875 - 0.026786 + 0.008333 - 0.003606$$

$$+0.001875 - 0.001096 + 0.000696$$

$$= 0.166916$$

$$s_8 = a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + a_7 + a_8$$

$$= 0.1875 - 0.026786 + 0.008333 - 0.003606$$

$$+0.001875 - 0.001096 + 0.000696 - 0.000469$$

$$= 0.166447$$

$$s_9 = a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + a_7 + a_8 + a_9$$

$$= 0.1875 - 0.026786 + 0.008333 - 0.003606 + 0.001875$$

$$-0.001096 + 0.000696 - 0.000469 + 0.000331$$

$$= 0.166778$$

$$s_{10} = a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + a_7 + a_8 + a_9 + a_{10}$$

$$= 0.1875 - 0.026786 + 0.008333 - 0.003606 + 0.001875$$

$$-0.001096 + 0.000696 - 0.000469 + 0.000331 - 0.000242$$

$$= 0.166536'$$

$$s_{11} = a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + a_7 + a_8 + a_9 + a_{10} + a_{11}$$

$$= 0.1875 - 0.026786 + 0.008333 - 0.003606$$

$$+0.001875 - 0.001096 + 0.000696 - 0.000469$$

$$+0.000331 - 0.000242 + 0.000182$$

$$= 0.166718$$

The approximation to three decimal places is $s = 0.167$. Verify that the series is decreasing, $b_{n+1} \leq b_n$.

$$b_n = \frac{3}{12n^3 + 4n^2}$$

$$b_1 = \frac{3}{12(1)^3 + 4(1)^2} = \frac{3}{12 + 4} = \frac{3}{16} = 0.187500$$

$$b_2 = \frac{3}{12(2)^3 + 4(2)^2} = \frac{3}{96 + 16} = \frac{3}{112} \approx 0.026786$$

$$b_3 = \frac{3}{12(3)^3 + 4(3)^2} = \frac{3}{324 + 36} = \frac{3}{360} \approx 0.008333$$

$$b_4 = \frac{3}{12(4)^3 + 4(4)^2} = \frac{3}{768 + 64} = \frac{3}{832} \approx 0.003606$$

$$b_5 = \frac{3}{12(5)^3 + 4(5)^2} = \frac{3}{1500 + 100} = \frac{3}{1600} \approx 0.001875$$

$$b_6 = \frac{3}{12(6)^3 + 4(6)^2} = \frac{3}{2592 + 144} = \frac{3}{2736} \approx 0.001096$$

$$b_7 = \frac{3}{12(7)^3 + 4(7)^2} = \frac{3}{4116 + 196} = \frac{3}{4312} \approx 0.000696$$

$$b_8 = \frac{3}{12(8)^3 + 4(8)^2} = \frac{3}{6144 + 256} = \frac{3}{6400} \approx 0.000469$$

$$b_9 = \frac{3}{12(9)^3 + 4(9)^2} = \frac{3}{8748 + 324} = \frac{3}{9072} \approx 0.000331$$

$$b_{10} = \frac{3}{12(10)^3 + 4(10)^2} = \frac{3}{12000 + 400} = \frac{3}{12400} \approx 0.000242$$

$$b_{11} = \frac{3}{12(11)^3 + 4(11)^2} = \frac{3}{15972 + 484} = \frac{3}{16456} \approx 0.000182$$

Verify that the limit as $n \to \infty$ is 0. Use L'Hospital's rule.

$$\lim_{n \to \infty} \frac{3}{12n^3 + 4n^2} = \lim_{n \to \infty} \frac{0}{6n^2 + 8n} = 0$$

Find the remainder.

$$\left| R_n \right| = \left| S - S_n \right| \leq b_{n+1}$$

$$\left|R_{10}\right| = \left|S - S_{10}\right| \le b_{10+1}$$

$$\left|R_{10}\right| \le b_{11}$$

$$\left|R_{10}\right| \le 0.000182$$

So the approximation of the sum of the alternating series is $S_{10} \approx 0.167$, with an error of $\left|R_{10}\right| \le 0.000182$.

POWER SERIES REPRESENTATION

■ 1. Find the power series representation of the function.

$$f(x)=\frac{3x}{7+x^2}$$

Solution:

The standard form of a power series is

$$\frac{1}{1-x}=\sum_{n=0}^{\infty}x^n$$

Manipulate the function until it's in the form of the standard power series.

$$\frac{1}{1-x}=\frac{3x}{7+x^2}$$

$$\frac{1}{1-x}=(3x)\frac{1}{7+x^2}$$

$$\frac{1}{1-x}=(3x)\frac{1}{7\left(1+\frac{x^2}{7}\right)}$$

$$\frac{1}{1-x}=\left(\frac{3x}{7}\right)\frac{1}{\left(1+\frac{x^2}{7}\right)}$$

$$\frac{1}{1-x}=\left(\frac{3x}{7}\right)\frac{1}{1-\left(-\frac{x^2}{7}\right)}$$

Then the power series representation of the function is

$$\frac{3x}{7}\sum_{n=0}^{\infty}\left(-\frac{x^2}{7}\right)^n$$

$$\sum_{n=0}^{\infty}\frac{3^1x^1}{7^1}\left(\frac{(-1)x^2}{7}\right)^n$$

$$\sum_{n=0}^{\infty}\frac{3^1x^1(-1)^n}{7^1}\left(\frac{x^{2n}}{7^n}\right)$$

$$\sum_{n=0}^{\infty}\frac{(-1)^n3x^{2n+1}}{7^{n+1}}$$

■ 2. Find the power series representation of the function.

$$f(x)=\frac{5}{4-6x}$$

Solution:

The standard form of a power series is

$$\frac{1}{1-x}=\sum_{n=0}^{\infty}x^n$$

Manipulate the function until it's in the form of the standard power series.

$$\frac{1}{1-x}=\frac{5}{4-6x}$$

$$\frac{1}{1-x}=(5)\frac{1}{4-6x}$$

$$\frac{1}{1-x}=(5)\frac{1}{4\left(1-\frac{6x}{4}\right)}$$

$$\frac{1}{1-x}=(5)\frac{1}{4\left(1-\frac{3x}{2}\right)}$$

$$\frac{1}{1-x}=\left(\frac{5}{4}\right)\frac{1}{\left(1-\frac{3x}{2}\right)}$$

$$\frac{1}{1-x}=\left(\frac{5}{4}\right)\frac{1}{1-\frac{3x}{2}}$$

Then the power series representation of the function is

$$\frac{5}{4}\sum_{n=0}^{\infty}\left(\frac{3x}{2}\right)^n$$

$$\sum_{n=0}^{\infty}\frac{5^1}{4^1}\left(\frac{3x}{2}\right)^n$$

$$\sum_{n=0}^{\infty} \frac{5^1}{2^2} \left(\frac{3x}{2} \right)^n$$

$$\sum_{n=0}^{\infty} \frac{5(3x)^n}{2^{n+2}}$$

■ 3. Find the power series representation of the function.

$$f(x) = \frac{4}{x^2 - x^3}$$

Solution:

The standard form of a power series is

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

Manipulate the function until it's in the form of the standard power series.

$$\frac{1}{1-x} = \frac{4}{x^2 - x^3}$$

$$\frac{1}{1-x} = (4)\frac{1}{x^2 - x^3}$$

$$\frac{1}{1-x} = (4)\frac{1}{x^2(1-x)}$$

$$\frac{1}{1-x} = \left(\frac{4}{x^2}\right)\frac{1}{1-x}$$

Then the power series representation of the function is

$$\frac{4}{x^2}\sum_{n=0}^{\infty} x^n$$

$$\sum_{n=0}^{\infty} \frac{4^1}{x^2}(x)^n$$

$$\sum_{n=0}^{\infty} \frac{4^1 x^n}{x^2}$$

$$\sum_{n=0}^{\infty} 4x^{n-2}$$

■ 4. Find the power series representation of the function.

$$f(x) = \frac{5x^2}{1+x^3}$$

Solution:

The standard form of a power series is

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

Manipulate the function until it's in the form of the standard power series.

$$\frac{1}{1-x}=\frac{5x^2}{1+x^3}$$

$$\frac{1}{1-x}=(5x^2)\frac{1}{1+x^3}$$

$$\frac{1}{1-x}=(5x^2)\frac{1}{1-(-x^3)}$$

Then the power series representation of the function is

$$5x^2\sum_{n=0}^{\infty}(-x^3)^n$$

$$\sum_{n=0}^{\infty}5^1x^2(-x^3)^n$$

$$\sum_{n=0}^{\infty}5^1x^2(-1)^n(x^3)^n$$

$$\sum_{n=0}^{\infty}5x^2(-1)^nx^{3n}$$

$$\sum_{n=0}^{\infty}(-1)^n5x^{3n+2}$$

■ 5. Find the power series representation of the function.

$$f(x) = \frac{x}{8-x}$$

Solution:

The standard form of a power series is

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

Manipulate the function until it's in the form of the standard power series.

$$\frac{1}{1-x} = \frac{x}{8-x}$$

$$\frac{1}{1-x} = (x)\frac{1}{8-x}$$

$$\frac{1}{1-x} = (x)\frac{1}{8\left(1-\frac{x}{8}\right)}$$

$$\frac{1}{1-x} = \left(\frac{x}{8}\right)\frac{1}{1-\frac{x}{8}}$$

Then the power series representation of the function is

$$\frac{x}{8}\sum_{n=0}^{\infty}\left(\frac{x}{8}\right)^n$$

$$\sum_{n=0}^{\infty}\frac{x^1}{8^1}\left(\frac{x}{8}\right)^n$$

$$\sum_{n=0}^{\infty} \frac{x^1}{8^1} \cdot \frac{x^n}{8^n}$$

$$\sum_{n=0}^{\infty} \frac{x^{n+1}}{8^{n+1}}$$

POWER SERIES MULTIPLICATION

■ 1. Use power series multiplication to find the first four non-zero terms of the Maclaurin series.

$$y = \cos(3x)e^{3x}$$

Solution:

The given series is the product of two other series.

$$y = \cos(3x)$$

$$y = e^{3x}$$

For the first one, start with the common series

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots$$

Substitute $3x$ for x.

$$\cos(3x) = 1 - \frac{(3x)^2}{2!} + \frac{(3x)^4}{4!} - \frac{(3x)^6}{6!} + \frac{(3x)^8}{8!} - \cdots$$

$$\cos(3x) = 1 - \frac{9x^2}{2} + \frac{81x^4}{24} - \frac{729x^6}{720} + \frac{6{,}561x^8}{40{,}320} - \cdots$$

$$\cos(3x) = 1 - \frac{9x^2}{2} + \frac{27x^4}{8} - \frac{81x^6}{80} + \frac{729x^8}{4{,}480} - \cdots$$

For the second, start with the common series

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots$$

Substitute $3x$ for x.

$$e^{3x} = 1 + \frac{3x}{1!} + \frac{(3x)^2}{2!} + \frac{(3x)^3}{3!} + \frac{(3x)^4}{4!} + \frac{(3x)^5}{5!} + \cdots$$

$$e^{3x} = 1 + 3x + \frac{9x^2}{2} + \frac{9x^3}{2} + \frac{27x^4}{8} + \frac{81x^5}{40} + \cdots$$

Multiply the series together.

$$\cos(3x)e^{3x} = \left(1 - \frac{9x^2}{2} + \frac{27x^4}{8} - \frac{81x^6}{80} + \frac{729x^8}{4{,}480} - \cdots\right)$$

$$\left(1 + 3x + \frac{9x^2}{2} + \frac{9x^3}{2} + \frac{27x^4}{8} + \frac{81x^5}{40} + \cdots\right)$$

Multiply every term in the first series by every term in the second series.

$$\cos(3x)e^{3x} = 1\left(1 + 3x + \frac{9x^2}{2} + \frac{9x^3}{2} + \frac{27x^4}{8} + \frac{81x^5}{40} + \cdots\right)$$

$$-\frac{9x^2}{2}\left(1 + 3x + \frac{9x^2}{2} + \frac{9x^3}{2} + \frac{27x^4}{8} + \frac{81x^5}{40} + \cdots\right)$$

$$+\frac{27x^4}{8}\left(1 + 3x + \frac{9x^2}{2} + \frac{9x^3}{2} + \frac{27x^4}{8} + \frac{81x^5}{40} + \cdots\right)$$

$$-\frac{81x^6}{80}\left(1+3x+\frac{9x^2}{2}+\frac{9x^3}{2}+\frac{27x^4}{8}+\frac{81x^5}{40}+\cdots\right)$$

$$+\cdots$$

$$\cos(3x)e^{3x}=1+3x+\frac{9x^2}{2}+\frac{9x^3}{2}+\frac{27x^4}{8}+\frac{81x^5}{40}+\cdots$$

$$-\frac{9x^2}{2}-\frac{27x^3}{2}-\frac{81x^4}{4}-\frac{81x^5}{4}-\frac{243x^6}{16}-\frac{729x^7}{80}-\cdots$$

$$+\frac{27x^4}{8}+\frac{81x^5}{8}+\frac{243x^6}{16}+\frac{243x^7}{16}+\frac{729x^8}{64}+\frac{2{,}187x^9}{320}+\cdots$$

$$-\frac{81x^6}{80}-\frac{243x^7}{80}-\frac{729x^8}{160}-\frac{729x^9}{160}-\frac{2{,}187x^{10}}{60}-\frac{6{,}561x^{11}}{3{,}200}-\cdots$$

To get the first four non-zero terms, we only need terms through x^4.

$$\cos(3x)e^{3x}=1+3x+\frac{9x^2}{2}-\frac{9x^2}{2}+\frac{9x^3}{2}-\frac{27x^3}{2}+\frac{27x^4}{8}-\frac{81x^4}{4}+\frac{27x^4}{8}+\ldots$$

$$\cos(3x)e^{3x}=1+3x-9x^3-\frac{27x^4}{4}+\ldots$$

2. Use power series multiplication to find the first four non-zero terms of the Maclaurin series.

$$y=\arctan(2x)\sin x$$

Solution:

The given series is the product of two other series.

$$y = \arctan(2x)$$

$$y = \sin x$$

For the first one, start with the common series

$$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \cdots$$

Substitute $2x$ for x.

$$\arctan(2x) = 2x - \frac{(2x)^3}{3} + \frac{(2x)^5}{5} - \frac{(2x)^7}{7} + \frac{(2x)^9}{9} - \cdots$$

$$\arctan(2x) = 2x - \frac{8x^3}{3} + \frac{32x^5}{5} - \frac{128x^7}{7} + \frac{512x^9}{9} - \cdots$$

For the second, use the common series

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \cdots$$

$$\sin x = x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{5{,}040} + \frac{x^9}{362{,}880} - \cdots$$

Multiply the series together.

$$\arctan(2x)\sin x = \left(2x - \frac{8x^3}{3} + \frac{32x^5}{5} - \frac{128x^7}{7} + \frac{512x^9}{9} - \cdots\right)$$

$$\left(x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{5,040} + \frac{x^9}{362,880} - \cdots\right)$$

Multiply every term in the first series by every term in the second series.

$$\arctan(2x)\sin x = x\left(2x - \frac{8x^3}{3} + \frac{32x^5}{5} - \frac{128x^7}{7} + \frac{512x^9}{9} - \cdots\right)$$

$$-\frac{x^3}{6}\left(2x - \frac{8x^3}{3} + \frac{32x^5}{5} - \frac{128x^7}{7} + \frac{512x^9}{9} - \cdots\right)$$

$$+\frac{x^5}{120}\left(2x - \frac{8x^3}{3} + \frac{32x^5}{5} - \frac{128x^7}{7} + \frac{512x^9}{9} - \cdots\right)$$

$$-\frac{x^7}{5,040}\left(2x - \frac{8x^3}{3} + \frac{32x^5}{5} - \frac{128x^7}{7} + \frac{512x^9}{9} - \cdots\right)$$

$$+\frac{x^9}{362,880}\left(2x - \frac{8x^3}{3} + \frac{32x^5}{5} - \frac{128x^7}{7} + \frac{512x^9}{9} - \cdots\right)$$

$$-\cdots$$

$$\arctan(2x)\sin x = 2x^2 - \frac{8x^4}{3} + \frac{32x^6}{5} - \frac{128x^8}{7} + \frac{512x^{10}}{9} + \cdots$$

$$-\frac{x^4}{3} + \frac{4x^6}{9} - \frac{16x^8}{15} + \frac{128x^{10}}{42} - \cdots$$

$$+\frac{x^6}{60} - \frac{x^8}{45} + \frac{4x^{10}}{75} - \cdots$$

$$-\frac{x^8}{2{,}520}+\frac{8x^{10}}{15{,}120}-\cdots$$

$$+\frac{x^{10}}{181{,}440}-\cdots$$

To get the first four non-zero terms, we only need terms through x^8.

$$\arctan(2x)\sin x=2x^2-\frac{8x^4}{3}-\frac{x^4}{3}+\frac{32x^6}{5}+\frac{4x^6}{9}+\frac{x^6}{60}$$

$$-\frac{128x^8}{7}-\frac{16x^8}{15}-\frac{x^8}{45}-\frac{x^8}{2{,}520}+\cdots$$

$$\arctan(2x)\sin x=2x^2-3x^4+\frac{247x^6}{36}-\frac{155x^8}{8}+\cdots$$

3. Use power series multiplication to find the first four non-zero terms of the Maclaurin series.

$$y=e^{-2x}\cos(2x)$$

Solution:

The given series is the product of two other series.

$$y=e^{-2x}$$

$$y=\cos(2x)$$

For the first one, start with the common series

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots$$

Substitute $-2x$ for x.

$$e^{-2x} = 1 + \frac{-2x}{1!} + \frac{(-2x)^2}{2!} + \frac{(-2x)^3}{3!} + \frac{(-2x)^4}{4!} + \frac{(-2x)^5}{5!} + \cdots$$

$$e^{-2x} = 1 - 2x + 2x^2 - \frac{4x^3}{3} + \frac{2x^4}{3} - \frac{4x^5}{15} + \cdots$$

For the second, use the common series

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots$$

$$\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} + \frac{x^8}{40{,}320} - \cdots$$

Substitute $2x$ for x.

$$\cos(2x) = 1 - \frac{(2x)^2}{2} + \frac{(2x)^4}{24} - \frac{(2x)^6}{720} + \frac{(2x)^8}{40{,}320} - \cdots$$

$$\cos(2x) = 1 - \frac{4x^2}{2} + \frac{16x^4}{24} - \frac{64x^6}{720} + \frac{256x^8}{40{,}320} - \cdots$$

$$\cos(2x) = 1 - 2x^2 + \frac{2x^4}{3} - \frac{4x^6}{45} + \frac{2x^8}{315} - \cdots$$

Multiply the series together.

$$e^{-2x}\cos(2x) = \left(1 - 2x + 2x^2 - \frac{4x^3}{3} + \frac{2x^4}{3} - \frac{4x^5}{15} + \cdots\right)$$

$$\left(1 - 2x^2 + \frac{2x^4}{3} - \frac{4x^6}{45} + \frac{2x^8}{315} - \cdots\right)$$

Multiply every term in the first series by every term in the second series.

$$e^{-2x}\cos(2x) = 1\left(1 - 2x^2 + \frac{2x^4}{3} - \frac{4x^6}{45} + \frac{2x^8}{315} - \cdots\right)$$

$$-2x\left(1 - 2x^2 + \frac{2x^4}{3} - \frac{4x^6}{45} + \frac{2x^8}{315} - \cdots\right)$$

$$+2x^2\left(1 - 2x^2 + \frac{2x^4}{3} - \frac{4x^6}{45} + \frac{2x^8}{315} - \cdots\right)$$

$$-\frac{4x^3}{3}\left(1 - 2x^2 + \frac{2x^4}{3} - \frac{4x^6}{45} + \frac{2x^8}{315} - \cdots\right)$$

$$+\frac{2x^4}{3}\left(1 - 2x^2 + \frac{2x^4}{3} - \frac{4x^6}{45} + \frac{2x^8}{315} - \cdots\right)$$

$$-\frac{4x^5}{15}\left(1 - 2x^2 + \frac{2x^4}{3} - \frac{4x^6}{45} + \frac{2x^8}{315} - \cdots\right)$$

$$+\cdots$$

$$e^{-2x}\cos(2x) = 1 - 2x^2 + \frac{2x^4}{3} - \frac{4x^6}{45} + \frac{2x^8}{315} - \cdots$$

$$-2x + 4x^3 - \frac{4x^5}{3} + \frac{8x^7}{45} - \frac{4x^9}{315} + \cdots$$

$$+2x^2 - 4x^4 + \frac{4x^6}{3} - \frac{8x^8}{45} + \frac{4x^{10}}{315} - \cdots$$

$$-\frac{4x^3}{3} + \frac{8x^5}{3} - \frac{8x^7}{9} + \frac{16x^9}{135} - \frac{8x^{11}}{945} + \cdots$$

$$+\frac{2x^4}{3} - \frac{4x^6}{3} + \frac{4x^8}{9} - \frac{8x^{10}}{135} + \frac{4x^{12}}{945} - \cdots$$

$$-\frac{4x^5}{15} + \frac{8x^7}{15} - \frac{8x^9}{45} + \frac{16x^{11}}{675} - \frac{8x^{13}}{4{,}725} + \cdots$$

To get the first four non-zero terms, we only need terms through x^4.

$$e^{-2x}\cos(2x) = 1 - 2x - 2x^2 + 2x^2 + 4x^3 - \frac{4x^3}{3} + \frac{2x^4}{3} - 4x^4 + \frac{2x^4}{3}$$

$$e^{-2x}\cos(2x) = 1 - 2x + \frac{8x^3}{3} - \frac{8x^4}{3}$$

■ 4. Use power series multiplication to find the first four non-zero terms of the Maclaurin series.

$$y = e^{5x}\ln(1 + 3x)$$

Solution:

The given series is the product of two other series.

$$y = e^{5x}$$

$$y = \ln(1 + 3x)$$

For the first one, start with the common series

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots$$

Substitute $5x$ for x.

$$e^{(5x)} = 1 + \frac{5x}{1!} + \frac{(5x)^2}{2!} + \frac{(5x)^3}{3!} + \frac{(5x)^4}{4!} + \frac{(5x)^5}{5!} + \cdots$$

$$e^{(5x)} = 1 + 5x + \frac{25x^2}{2} + \frac{125x^3}{6} + \frac{625x^4}{24} + \frac{3{,}125x^5}{120} + \cdots$$

For the second, use the common series

$$\ln(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \frac{x^6}{6} + \frac{x^7}{7} - \cdots$$

Substitute $3x$ for x.

$$\ln(1 + 3x) = 3x - \frac{(3x)^2}{2} + \frac{(3x)^3}{3} - \frac{(3x)^4}{4} + \frac{(3x)^5}{5} - \cdots$$

$$\ln(1 + 3x) = 3x - \frac{9x^2}{2} + \frac{27x^3}{3} - \frac{81x^4}{4} + \frac{273x^5}{5} - \cdots$$

Multiply the series together.

$$e^{5x}\ln(1+3x)=\left(1+5x+\frac{25x^2}{2}+\frac{125x^3}{6}+\frac{625x^4}{24}+\frac{3{,}125x^5}{120}+\cdots\right)$$

$$\left(3x-\frac{9x^2}{2}+\frac{27x^3}{3}-\frac{81x^4}{4}+\frac{273x^5}{5}-\cdots\right)$$

Multiply every term in the first series by every term in the second series.

$$e^{5x}\ln(1+3x)=1\left(3x-\frac{9x^2}{2}+\frac{27x^3}{3}-\frac{81x^4}{4}+\frac{273x^5}{5}-\cdots\right)$$

$$+5x\left(3x-\frac{9x^2}{2}+\frac{27x^3}{3}-\frac{81x^4}{4}+\frac{273x^5}{5}-\cdots\right)$$

$$+\frac{25x^2}{2}\left(3x-\frac{9x^2}{2}+\frac{27x^3}{3}-\frac{81x^4}{4}+\frac{273x^5}{5}-\cdots\right)$$

$$+\frac{125x^3}{6}\left(3x-\frac{9x^2}{2}+\frac{27x^3}{3}-\frac{81x^4}{4}+\frac{273x^5}{5}-\cdots\right)$$

$$+\frac{625x^4}{24}\left(3x-\frac{9x^2}{2}+\frac{27x^3}{3}-\frac{81x^4}{4}+\frac{273x^5}{5}-\cdots\right)$$

$$+\frac{3{,}125x^5}{120}\left(3x-\frac{9x^2}{2}+\frac{27x^3}{3}-\frac{81x^4}{4}+\frac{273x^5}{5}-\cdots\right)$$

$$e^{5x}\ln(1+3x)=3x-\frac{9x^2}{2}+\frac{27x^3}{3}-\frac{81x^4}{4}+\frac{273x^5}{5}-\cdots$$

$$+15x^2 - \frac{45x^3}{2} + \frac{135x^4}{3} - \frac{405x^5}{4} + \frac{1{,}365x^6}{5} - \cdots$$

$$+\frac{75x^3}{2} - \frac{225x^4}{4} + \frac{675x^5}{6} - \frac{2{,}025x^6}{8} + \frac{6{,}825x^7}{10} - \cdots$$

$$+\frac{125x^4}{2} - \frac{375x^5}{4} + \frac{375x^6}{2} - \frac{3{,}375x^7}{8} + \frac{11{,}375x^8}{10} - \cdots$$

$$+\frac{625x^5}{8} - \frac{1{,}875x^6}{16} + \frac{1{,}875x^7}{8} - \frac{16{,}875x^8}{32} + \frac{56{,}875x^9}{40} - \cdots$$

$$+\frac{3{,}125x^6}{40} - \frac{9{,}375x^7}{80} + \frac{9{,}375x^8}{40} - \frac{84{,}375x^9}{160} + \frac{284{,}375x^{10}}{200} - \cdots$$

To get the first four non-zero terms, we only need terms through x^4.

$$e^{5x}\ln(1+3x) = 3x - \frac{9x^2}{2} + 15x^2 + \frac{27x^3}{3} - \frac{45x^3}{2} + \frac{75x^3}{2} - \frac{81x^4}{4} + \frac{135x^4}{3} - \frac{225x^4}{4} + \frac{125x^4}{2}$$

$$e^{5x}\ln(1+3x) = 3x + \frac{21x^2}{2} + 24x^3 + 31x^4$$

■ 5. Use power series multiplication to find the first four non-zero terms of the Maclaurin series.

$$y = e^{3x} \cdot \frac{3}{1-x}$$

Solution:

The given series is the product of two other series.

$$y = e^{3x}$$

$$y = \frac{3}{1-x}$$

For the first one, start with the common series

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots$$

Substitute $3x$ for x.

$$e^{3x} = 1 + \frac{3x}{1!} + \frac{(3x)^2}{2!} + \frac{(3x)^3}{3!} + \frac{(3x)^4}{4!} + \frac{(3x)^5}{5!} + \cdots$$

$$e^{3x} = 1 + 3x + \frac{9x^2}{2} + \frac{9x^3}{2} + \frac{27x^4}{8} + \frac{81x^5}{40} + \cdots$$

For the second, use the common series

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + x^5 + x^6 + \cdots$$

Multiply through by 3.

$$\frac{3}{1-x} = 3 + 3x + 3x^2 + 3x^3 + 3x^4 + 3x^5 + 3x^6 + \cdots$$

Multiply the series together.

$$e^{3x} \cdot \frac{3}{1-x} = \left(1 + 3x + \frac{9x^2}{2} + \frac{9x^3}{2} + \frac{27x^4}{8} + \frac{81x^5}{40} + \cdots \right)$$

$$(3 + 3x + 3x^2 + 3x^3 + 3x^4 + 3x^5 + 3x^6 + \cdots)$$

Multiply every term in the first series by every term in the second series.

$$e^{3x} \cdot \frac{3}{1-x} = 1(3 + 3x + 3x^2 + 3x^3 + 3x^4 + 3x^5 + 3x^6 + \cdots)$$

$$+3x(3 + 3x + 3x^2 + 3x^3 + 3x^4 + 3x^5 + 3x^6 + \cdots)$$

$$+\frac{9x^2}{2}(3 + 3x + 3x^2 + 3x^3 + 3x^4 + 3x^5 + 3x^6 + \cdots)$$

$$+\frac{9x^3}{2}(3 + 3x + 3x^2 + 3x^3 + 3x^4 + 3x^5 + 3x^6 + \cdots)$$

$$+\frac{27x^4}{8}(3 + 3x + 3x^2 + 3x^3 + 3x^4 + 3x^5 + 3x^6 + \cdots)$$

$$+\frac{81x^5}{40}(3 + 3x + 3x^2 + 3x^3 + 3x^4 + 3x^5 + 3x^6 + \cdots)$$

$$e^{3x} \cdot \frac{3}{1-x} = 3 + 3x + 3x^2 + 3x^3 + 3x^4 + 3x^5 + 3x^6 + \cdots$$

$$+9x + 9x^2 + 9x^3 + 9x^4 + 9x^5 + 9x^6 + 9x^7 + \cdots$$

$$+\frac{27x^2}{2} + \frac{27x^3}{2} + \frac{27x^4}{2} + \frac{27x^5}{2} + \frac{27x^6}{2} + \frac{27x^7}{2} + \frac{27x^8}{2} + \cdots$$

$$+\frac{27x^3}{2} + \frac{27x^4}{2} + \frac{27x^5}{2} + \frac{27x^6}{2} + \frac{27x^7}{2} + \frac{27x^8}{2} + \frac{27x^9}{2} + \cdots$$

$$+\frac{81x^4}{8} + \frac{81x^5}{8} + \frac{81x^6}{8} + \frac{81x^7}{8} + \frac{81x^8}{8} + \frac{81x^9}{8} + \frac{81x^{10}}{8} + \cdots$$

$$+\frac{243x^5}{40}+\frac{243x^6}{40}+\frac{243x^7}{40}+\frac{243x^8}{40}+\frac{243x^9}{40}+\frac{243x^{10}}{40}+\frac{243x^{11}}{40}+\cdots$$

To get the first four non-zero terms, we only need terms through x^3.

$$e^{3x}\cdot\frac{3}{1-x}=3+3x+9x+3x^2+9x^2+\frac{27x^2}{2}+3x^3+9x^3+\frac{27x^3}{2}+\frac{27x^3}{2}$$

$$e^{3x}\cdot\frac{3}{1-x}=3+12x+\frac{51x^2}{2}+39x^3$$

POWER SERIES DIVISION

■ 1. Use power series division to find the first four non-zero terms of the Maclaurin series.

$$y = \frac{e^{3x}}{x^2}$$

Solution:

Start with the common series

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

Substitute $3x$ for x.

$$e^{3x} = 1 + \frac{3x}{1!} + \frac{(3x)^2}{2!} + \frac{(3x)^3}{3!} + \dots$$

$$e^{3x} = 1 + \frac{3x}{1!} + \frac{9x^2}{2!} + \frac{27x^3}{3!} + \dots$$

Divide through by x^2.

$$\frac{e^{3x}}{x^2} = \frac{1}{x^2} + \frac{3x}{1!x^2} + \frac{9x^2}{2!x^2} + \frac{27x^3}{3!x^2} + \dots$$

$$\frac{e^{3x}}{x^2} = \frac{1}{x^2} + \frac{3}{x} + \frac{9}{2} + \frac{9x}{2} + \dots$$

2. Use power series division to find the first four non-zero terms of the Maclaurin series.

$$y = \frac{6x}{\ln(1+6x)}$$

Solution:

Start with the common series

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \frac{x^6}{6} + \frac{x^7}{7} - \cdots$$

Substitute $6x$ for x.

$$\ln(1+6x) = 6x - \frac{(6x)^2}{2} + \frac{(6x)^3}{3} - \frac{(6x)^4}{4} + \frac{(6x)^5}{5} - \frac{(6x)^6}{6} + \frac{(6x)^7}{7} - \cdots$$

$$\ln(1+6x) = 6x - \frac{36x^2}{2} + \frac{216x^3}{3} - \frac{1{,}296x^4}{4} + \frac{7{,}776x^5}{5} - \frac{46{,}656x^6}{6} + \cdots$$

$$\ln(1+6x) = 6x - 18x^2 + 72x^3 - 324x^4 + 1{,}552.2x^5 - 7{,}776x^6 + \cdots$$

Divide through by $6x$.

$$\frac{6x}{\ln(1+6x)} = \frac{6x}{6x} - \frac{6x}{18x^2} + \frac{6x}{72x^3} - \frac{6x}{324x^4} + \frac{60x}{15{,}522x^5} - \frac{6x}{7{,}776x^6} + \cdots$$

$$\frac{6x}{\ln(1+6x)} = 1 - \frac{1}{3x} + \frac{1}{12x^2} - \frac{1}{54x^3} + \frac{10}{2587x^4} - \frac{1}{1{,}296x^5} + \cdots$$

To get the first four non-zero terms, we only need

$$\frac{6x}{\ln(1+6x)} = 1 - \frac{1}{3x} + \frac{1}{12x^2} - \frac{1}{54x^3}$$

■ 3. Use power series division to find the first four non-zero terms of the Maclaurin series.

$$y = \frac{\cos(2x)}{2x^3}$$

Solution:

Start with the common series

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots$$

Substitute $2x$ for x.

$$\cos(2x) = 1 - \frac{(2x)^2}{2!} + \frac{(2x)^4}{4!} - \frac{(2x)^6}{6!} + \frac{(2x)^8}{8!} - \cdots$$

$$\cos(2x) = 1 - \frac{4x^2}{2!} + \frac{16x^4}{4!} - \frac{64x^6}{6!} + \frac{256x^8}{8!} - \cdots$$

Divide through by $2x^3$.

$$\frac{\cos(2x)}{2x^3} = \frac{1}{2x^3} - \frac{4x^2}{2!2x^3} + \frac{16x^4}{4!2x^3} - \frac{64x^6}{6!2x^3} + \frac{256x^8}{8!2x^3} - \ldots$$

$$\frac{\cos(2x)}{2x^3} = \frac{1}{2x^3} - \frac{4x^2}{4x^3} + \frac{16x^4}{48x^3} - \frac{64x^6}{1{,}440x^3} + \frac{256x^8}{80{,}640x^3} - \dots$$

$$\frac{\cos(2x)}{2x^3} = \frac{1}{2x^3} - \frac{1}{x} + \frac{x}{3} - \frac{2x^3}{45} + \frac{x^5}{315} - \dots$$

To get the first four non-zero terms, we only need

$$\frac{\cos(2x)}{2x^3} = \frac{1}{2x^3} - \frac{1}{x} + \frac{x}{3} - \frac{2x^3}{45}$$

4. Use power series division to find the first four non-zero terms of the Maclaurin series.

$$y = \frac{\sin(3x)}{3x^2}$$

Solution:

Start with the common series

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \cdots$$

Substitute $3x$ for x.

$$\sin(3x) = 3x - \frac{(3x)^3}{3!} + \frac{(3x)^5}{5!} - \frac{(3x)^7}{7!} + \frac{(3x)^9}{9!} - \cdots$$

$$\sin(3x) = 3x - \frac{27x^3}{6} + \frac{243x^5}{120} - \frac{2{,}187x^7}{5{,}040} + \frac{19{,}683x^9}{362{,}880} - \cdots$$

$$\sin(3x) = 3x - \frac{9x^3}{2} + \frac{81x^5}{40} - \frac{243x^7}{560} + \frac{243x^9}{4,480} - \cdots$$

Divide this series by $3x^2$.

$$\frac{\sin(3x)}{3x^2} = \frac{3x}{3x^2} - \frac{9x^3}{2(3x^2)} + \frac{81x^5}{40(3x^2)} - \frac{243x^7}{560(3x^2)} + \frac{243x^9}{4,480(3x^2)} - \cdots$$

$$\frac{\sin(3x)}{3x^2} = x^{-1} - \frac{3}{2}x + \frac{27}{40}x^3 - \frac{81}{560}x^5 + \frac{81}{4,480}x^7 - \cdots$$

To get the first four non-zero terms, we only need

$$\frac{\sin(3x)}{3x^2} = x^{-1} - \frac{3}{2}x + \frac{27}{40}x^3 - \frac{81}{560}x^5 + \ldots$$

■ 5. Use power series division to find the first four non-zero terms of the Maclaurin series.

$$y = \frac{\arctan(4x)}{4x^2}$$

Solution:

Start with the common series

$$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \cdots$$

Substitute $4x$ for x.

$$\arctan(4x) = 4x - \frac{(4x)^3}{3} + \frac{(4x)^5}{5} - \frac{(4x)^7}{7} + \frac{(4x)^9}{9} - \cdots$$

$$\arctan(4x) = 4x - \frac{64x^3}{3} + \frac{1{,}024x^5}{5} - \frac{16{,}384x^7}{7} + \frac{262{,}144x^9}{9} - \cdots$$

Divide through by $4x^2$.

$$\frac{\arctan(4x)}{4x^2} = \frac{4x}{4x^2} - \frac{64x^3}{3(4x^2)} + \frac{1{,}024x^5}{5(4x^2)} - \frac{16{,}384x^7}{7(4x^2)} + \frac{262{,}144x^9}{9(4x^2)} - \cdots$$

$$\frac{\arctan(4x)}{4x^2} = \frac{1}{x} - \frac{16x}{3} + \frac{256x^3}{5} - \frac{4{,}096x^5}{7} + \frac{65{,}536x^7}{9} - \cdots$$

To get the first four non-zero terms, we only need

$$\frac{\arctan(4x)}{4x^2} = \frac{1}{x} - \frac{16x}{3} + \frac{256x^3}{5} - \frac{4{,}096x^5}{7} + \cdots$$

POWER SERIES DIFFERENTIATION

1. Differentiate to find the power series representation of the function.

$$f(x) = \frac{5}{(3-x)^2}$$

Solution:

Integrate the given function using u-substitution.

$$\int \frac{5}{(3-x)^2}\ dx$$

$$u = 3 - x$$

$$du = -\ dx$$

$$dx = -\ du$$

$$\int \frac{5}{(3-x)^2}\ dx = \int \frac{5}{u^2}(-du) = \int -5u^{-2}\ du$$

$$\frac{-5u^{-1}}{-1} + C = \frac{5}{u} + C = \frac{5}{3-x} + C$$

Starting with the standard form of a power series,

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + x^5 + \ \cdots = \sum_{n=0}^{\infty} x^n$$

find the series that represents the integrated function.

$$\frac{1}{1-x}=\frac{5}{3-x}$$

$$\frac{1}{1-x}=(5)\frac{1}{3-x}$$

$$\frac{1}{1-x}=(5)\frac{1}{3\left(1-\frac{x}{3}\right)}$$

$$\frac{1}{1-x}=\left(\frac{5}{3}\right)\frac{1}{1-\frac{x}{3}}$$

The power series representation of this is

$$\frac{5}{3}\sum_{n=0}^{\infty}\left(\frac{x}{3}\right)^n=\sum_{n=0}^{\infty}\frac{5}{3}\left(\frac{x}{3}\right)^n=\sum_{n=0}^{\infty}\frac{5^1\cdot x^n}{3^1\cdot 3^n}=\sum_{n=0}^{\infty}\frac{5x^n}{3^{n+1}}$$

So the integrated function can be written as

$$\frac{5}{3-x}=\frac{5}{3}+\frac{5}{3}\left(\frac{x}{3}\right)+\frac{5}{3}\left(\frac{x}{3}\right)^2+\frac{5}{3}\left(\frac{x}{3}\right)^3+\frac{5}{3}\left(\frac{x}{3}\right)^4+\frac{5}{3}\left(\frac{x}{3}\right)^5+\cdots=\sum_{n=0}^{\infty}\frac{5x^n}{3^{n+1}}$$

$$\frac{5}{3-x}=\frac{5}{3}+\frac{5x}{9}+\frac{5x^2}{27}+\frac{5x^3}{81}+\frac{5x^4}{243}+\frac{5x^5}{729}+\cdots=\sum_{n=0}^{\infty}\frac{5x^n}{3^{n+1}}$$

Now differentiate this entire equation, include the function on the left, the terms in the middle, and the series on the right.

$$\frac{(3-x)(0)-5(-1)}{(3-x)^2}=\frac{5}{9}+\frac{10x}{27}+\frac{15x^2}{81}+\frac{20x^3}{243}+\frac{25x^4}{729}+\cdots=\sum_{n=0}^{\infty}\frac{5nx^{n-1}}{3^{n+1}}$$

$$\frac{5}{(3-x)^2}=\frac{5}{9}+\frac{10x}{27}+\frac{15x^2}{81}+\frac{20x^3}{243}+\frac{25x^4}{729}+\cdots=\sum_{n=0}^{\infty}\frac{5nx^{n-1}}{3^{n+1}}$$

So the power series representation of the original function is

$$\frac{5}{(3-x)^2}=\sum_{n=0}^{\infty}\frac{5nx^{n-1}}{3^{n+1}}$$

When $n=0$,

$$\frac{5(0)x^{0-1}}{3^{0+1}}=\frac{0}{3x}=0$$

so we can start the index at $n=1$ instead of $n=0$, without changing the value of the series.

$$\frac{5}{(3-x)^2}=\sum_{n=1}^{\infty}\frac{5nx^{n-1}}{3^{n+1}}$$

■ 2. Differentiate to find the power series representation of the function.

$$f(x)=\frac{3}{(4+x)^2}$$

Solution:

Integrate the given function using u-substitution.

$$\int \frac{3}{(4+x)^2}\ dx$$

$$u = 4 + x$$

$$du = dx$$

$$dx = du$$

$$\int \frac{3}{(4+x)^2}\ dx = \int \frac{3}{u^2}\ du = \int 3u^{-2}\ du$$

$$\frac{3u^{-1}}{-1} + C = -\frac{3}{u} + C = -\frac{3}{4+x} + C$$

Starting with the standard form of a power series,

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + x^5 + \ \cdots = \sum_{n=0}^{\infty} x^n$$

find the series that represents the integrated function.

$$\frac{1}{1-x} = -\frac{3}{4+x}$$

$$\frac{1}{1-x} = -\frac{3}{4-(-x)}$$

$$\frac{1}{1-x} = (-3)\frac{1}{4\left(1-\left(-\frac{x}{4}\right)\right)}$$

$$\frac{1}{1-x} = (-3)\frac{1}{4\left(1-\left(-\frac{x}{4}\right)\right)}$$

$$\frac{1}{1-x}=\left(-\frac{3}{4}\right)\frac{1}{1-\left(-\frac{x}{4}\right)}$$

The power series representation of this is

$$-\frac{3}{4}\sum_{n=0}^{\infty}\left(-\frac{x}{4}\right)^n=\sum_{n=0}^{\infty}-\frac{3}{4}\left(-\frac{x}{4}\right)^n=\sum_{n=0}^{\infty}\frac{-3^1\cdot(-1)^n x^n}{4^1\cdot 4^n}=\sum_{n=0}^{\infty}\frac{(-1)^{n+1}3x^n}{4^{n+1}}$$

So the original function can be written as

$$-\frac{3}{4+x}=-\frac{3}{4}\left(1+\left(-\frac{x}{4}\right)+\left(-\frac{x}{4}\right)^2+\left(-\frac{x}{4}\right)^3+\left(-\frac{x}{4}\right)^4+\left(-\frac{x}{4}\right)^5+\cdots\right)$$

$$=\sum_{n=0}^{\infty}\frac{(-1)^{n+1}3x^n}{4^{n+1}}$$

$$-\frac{3}{4+x}=-\frac{3}{4}\left(1-\frac{x}{4}+\frac{x^2}{16}-\frac{x^3}{64}+\frac{x^4}{256}-\frac{x^5}{1{,}024}+\cdots\right)$$

$$=\sum_{n=0}^{\infty}\frac{(-1)^{n+1}3x^n}{4^{n+1}}$$

$$-\frac{3}{4+x}=\left(-\frac{3}{4}\right)-\frac{x}{4}\left(-\frac{3}{4}\right)+\frac{x^2}{16}\left(-\frac{3}{4}\right)-\frac{x^3}{64}\left(-\frac{3}{4}\right)+\frac{x^4}{256}\left(-\frac{3}{4}\right)-\frac{x^5}{1{,}024}\left(-\frac{3}{4}\right)+\cdots$$

$$=\sum_{n=0}^{\infty}\frac{(-1)^{n+1}3x^n}{4^{n+1}}$$

$$-\frac{3}{4+x}=-\frac{3}{4}+\frac{3x}{16}-\frac{3x^2}{64}+\frac{3x^3}{256}-\frac{3x^4}{1{,}024}+\frac{3x^5}{4{,}096}-\cdots$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^{n+1}3x^n}{4^{n+1}}$$

Now differentiate this entire equation, include the function on the left, the terms in the middle, and the series on the right.

$$-\frac{(4+x)(0)-3(1)}{(4+x)^2} = \frac{3}{16} - \frac{6x}{64} + \frac{9x^2}{256} - \frac{12x^3}{1,024} + \frac{15x^4}{4,096} - \cdots = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}3nx^{n-1}}{4^{n+1}}$$

$$\frac{3}{(4+x)^2} = \frac{3}{16} - \frac{6x}{64} + \frac{9x^2}{256} - \frac{12x^3}{1,024} + \frac{15x^4}{4,096} - \cdots = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}3nx^{n-1}}{4^{n+1}}$$

So the power series representation of the original function is

$$\frac{3}{(4+x)^2} = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}3nx^{n-1}}{4^{n+1}}$$

$$\frac{3}{(4+x)^2} = \sum_{n+1=0}^{\infty} \frac{(-1)^{n+1+1}3(n+1)x^{n+1-1}}{4^{n+1+1}}$$

$$\frac{3}{(4+x)^2} = \sum_{n=-1}^{\infty} \frac{(-1)^{n+2}3(n+1)x^n}{4^{n+2}}$$

If we plug in $n = -1$, the start of the new index, we get a zero value because of the $n+1$ factor in the numerator. Which means we can start the index at $n = 0$ instead.

$$\frac{3}{(4+x)^2} = \sum_{n=0}^{\infty} \frac{(-1)^{n+2}3(n+1)x^n}{4^{n+2}}$$

3. Differentiate to find the power series representation of the function.

$$f(x) = \frac{1}{(-5-x)^2}$$

Solution:

Integrate the given function using u-substitution.

$$\int \frac{1}{(-5-x)^2}\ dx$$

$$u = -5 - x$$

$$du = -dx$$

$$dx = -du$$

$$\int \frac{1}{(-5-x)^2}\ dx = \int \frac{1}{u^2}(-du) = -\int u^{-2}\ du$$

$$-\frac{u^{-1}}{-1} + C = \frac{1}{u} + C = \frac{1}{-5-x} + C$$

Starting with the standard form of a power series,

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + x^5 + \ \cdots = \sum_{n=0}^{\infty} x^n$$

find the series that represents the integrated function.

$$\frac{1}{1-x}=\frac{1}{-5-x}$$

$$\frac{1}{1-x}=-\frac{3}{1-(6+x)}$$

The power series representation of this is

$$\sum_{n=0}^{\infty}(x+6)^n$$

So the original function can be written as

$$\frac{1}{1-(x+6)}=1+(x+6)+(x+6)^2+(x+6)^3+(x+6)^4+(x+6)^5+\ \cdots=\sum_{n=0}^{\infty}(x+6)^n$$

$$\frac{1}{-5-x}=1+(x+6)+(x+6)^2+(x+6)^3+(x+6)^4+(x+6)^5+\ \cdots=\sum_{n=0}^{\infty}(x+6)^n$$

Now differentiate this entire equation, include the function on the left, the terms in the middle, and the series on the right.

$$\frac{(-5-x)(0)-1(-1)}{(-5-x)^2}=1+2(x+6)+3(x+6)^2+4(x+6)^3+5(x+6)^4+\ \cdots=\sum_{n=0}^{\infty}n(x+6)^{n-1}$$

$$\frac{1}{(-5-x)^2}=1+2(x+6)+3(x+6)^2+4(x+6)^3+5(x+6)^4+\ \cdots=\sum_{n=0}^{\infty}n(x+6)^{n-1}$$

So the power series representation of the original function is

$$\frac{1}{(-5-x)^2}=\sum_{n=0}^{\infty}n(x+6)^{n-1}$$

$$\frac{1}{(-5-x)^2}=\sum_{n+1=0}^{\infty}(n+1)(x+6)^{n+1-1}$$

$$\frac{1}{(-5-x)^2}=\sum_{n=-1}^{\infty}(n+1)(x+6)^{n}$$

■ 4. Differentiate to find the power series representation of the function.

$$f(x)=\frac{3}{(6-3x)^2}$$

Solution:

Integrate the given function using u-substitution.

$$\int \frac{3}{(6-3x)^2}\ dx$$

$$u=6-3x$$

$$du=-3dx$$

$$dx=-\frac{du}{3}$$

$$\int \frac{3}{(6-3x)^2}\ dx=\int \frac{3}{u^2}\left(-\frac{du}{3}\right)=-\int u^{-2}\ du$$

$$-\frac{u^{-1}}{-1}+C=\frac{1}{u}+C=\frac{1}{6-3x}+C$$

Starting with the standard form of a power series,

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + x^5 + \ \cdots = \sum_{n=0}^{\infty} x^n$$

find the series that represents the integrated function.

$$\frac{1}{1-x} = \frac{1}{6-3x}$$

$$\frac{1}{1-x} = \frac{1}{6\left(1-\frac{x}{2}\right)}$$

$$\frac{1}{1-x} = \frac{1}{6} \cdot \frac{1}{1-\frac{x}{2}}$$

The power series representation of this is

$$\frac{1}{6}\sum_{n=0}^{\infty}\left(\frac{x}{2}\right)^n = \sum_{n=0}^{\infty}\frac{1}{6}\left(\frac{x}{2}\right)^n$$

So the original function can be written as

$$\frac{1}{6-3x} = \frac{1}{6}\left(1 + \frac{x}{2} + \left(\frac{x}{2}\right)^2 + \left(\frac{x}{2}\right)^3 + \left(\frac{x}{2}\right)^4 + \left(\frac{x}{2}\right)^5 + \ldots\right) = \sum_{n=0}^{\infty}\frac{1}{6}\left(\frac{x}{2}\right)^n$$

$$\frac{1}{6-3x} = \frac{1}{6}\left(1 + \frac{x}{2} + \frac{x^2}{4} + \frac{x^3}{8} + \frac{x^4}{16} + \frac{x^5}{32} + \ldots\right) = \sum_{n=0}^{\infty}\frac{1}{6}\left(\frac{x}{2}\right)^n$$

$$\frac{1}{6-3x} = \frac{1}{6} + \frac{x}{12} + \frac{x^2}{24} + \frac{x^3}{48} + \frac{x^4}{96} + \frac{x^5}{192} + \ldots = \sum_{n=0}^{\infty}\frac{1}{6}\left(\frac{x}{2}\right)^n$$

Now differentiate this entire equation, including the function on the left, the terms in the middle, and the series on the right.

$$\frac{(0)(6-3x)-(1)(-3)}{(6-3x)^2}=\frac{1}{12}+\frac{x}{12}+\frac{x^2}{16}+\frac{x^3}{24}+\frac{5x^4}{192}+\ldots=\sum_{n=0}^{\infty}\frac{1}{6}n\left(\frac{x}{2}\right)^{n-1}\cdot\frac{1}{2}$$

$$\frac{3}{(6-3x)^2}=\frac{1}{12}+\frac{x}{12}+\frac{x^2}{16}+\frac{x^3}{24}+\frac{5x^4}{192}+\ldots=\sum_{n=0}^{\infty}\frac{n}{12}\left(\frac{x}{2}\right)^{n-1}$$

So the power series representation of the original function is

$$\frac{3}{(6-3x)^2}=\sum_{n=0}^{\infty}\frac{n}{12}\left(\frac{x}{2}\right)^{n-1}$$

5. Differentiate to find the power series representation of the function.

$$f(x)=\frac{2}{(1-2x)^2}$$

Solution:

Integrate the given function using u-substitution.

$$\int\frac{2}{(1-2x)^2}\ dx$$

$$u=1-2x$$

$$du = -2dx$$

$$dx = -\frac{du}{2}$$

$$\int \frac{2}{(1-2x)^2}\ dx = \int \frac{2}{u^2}\left(-\frac{du}{2}\right) = -\int u^{-2}\ du$$

$$-\frac{u^{-1}}{-1} + C = \frac{1}{u} + C = \frac{1}{1-2x} + C$$

Starting with the standard form of a power series,

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + x^5 + \ \cdots = \sum_{n=0}^{\infty} x^n$$

find the series that represents the integrated function.

$$\frac{1}{1-x} = \frac{1}{1-2x}$$

The power series representation of this is

$$\sum_{n=0}^{\infty} (2x)^n$$

So the original function can be written as

$$\frac{1}{1-2x} = 1 + (2x) + (2x)^2 + (2x)^3 + (2x)^4 + (2x)^5 + \ \cdots = \sum_{n=0}^{\infty} (2x)^n$$

$$\frac{1}{1-2x} = 1 + 2x + 4x^2 + 8x^3 + 16x^4 + 32x^5 + \ \cdots = \sum_{n=0}^{\infty} (2x)^n$$

Now differentiate this entire equation, include the function on the left, the terms in the middle, and the series on the right.

$$\frac{(1-2x)(0)-1(-2)}{(1-2x)^2} = 2 + 8x + 24x^2 + 64x^3 + 160x^4 + \cdots = \sum_{n=0}^{\infty} n(2x)^{n-1} \cdot 2$$

$$\frac{1}{(1-2x)^2} = 2 + 8x + 24x^2 + 64x^3 + 160x^4 + \cdots = \sum_{n=0}^{\infty} 2n(2x)^{n-1}$$

So the power series representation of the original function is

$$\frac{2}{(1-2x)^2} = \sum_{n=0}^{\infty} 4n(2x)^{n-1}$$

$$\frac{2}{(1-2x)^2} = \sum_{n+1=0}^{\infty} 4(n+1)(2x)^{n+1-1}$$

$$\frac{2}{(1-2x)^2} = \sum_{n=-1}^{\infty} 4(n+1)(2x)^{n}$$

RADIUS OF CONVERGENCE

1. Find the radius of convergence of the series.

$$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{4 \cdot 2^{2n}}$$

Solution:

Apply the ratio test.

$$L = \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right|$$

$$L = \lim_{n \to \infty} \left| \frac{\frac{(-1)^{n+1} x^{2(n+1)}}{4 \cdot 2^{2(n+1)}}}{\frac{(-1)^n x^{2n}}{4 \cdot 2^{2n}}} \right|$$

$$L = \lim_{n \to \infty} \left| \frac{\frac{x^{2(n+1)}}{4 \cdot 2^{2(n+1)}}}{\frac{x^{2n}}{4 \cdot 2^{2n}}} \right|$$

$$L = \lim_{n \to \infty} \left| \frac{x^{2n+2}}{4 \cdot 2^{2n+2}} \cdot \frac{4 \cdot 2^{2n}}{x^{2n}} \right|$$

$$L = \lim_{n \to \infty} \left| \frac{x^2}{2^2} \cdot \frac{1}{1} \right|$$

$$L = \lim_{n \to \infty} \left| \frac{x^2}{2^2} \right|$$

$$L = \frac{x^2}{4}$$

Then the interval of convergence is given by the inequality

$$\frac{x^2}{4} < 1$$

$$x^2 < 4$$

$$-2 < x < 2$$

The interval of convergence spans -2 to 2, which is 4 units wide. The radius of convergence will be half that, so the radius of convergence is 2.

■ 2. Find the radius of convergence of the series.

$$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

Solution:

Apply the ratio test.

$$L = \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right|$$

$$L = \lim_{n\to\infty} \left| \frac{\frac{(-1)^{n+1}x^{2(n+1)+1}}{(2(n+1)+1)!}}{\frac{(-1)^n x^{2n+1}}{(2n+1)!}} \right|$$

$$L = \lim_{n\to\infty} \left| \frac{\frac{x^{2n+3}}{(2n+3)!}}{\frac{x^{2n+1}}{(2n+1)!}} \right|$$

$$L = \lim_{n\to\infty} \left| \frac{x^{2n+3}}{(2n+3)!} \cdot \frac{(2n+1)!}{x^{2n+1}} \right|$$

$$L = \lim_{n\to\infty} \left| \frac{x^2}{(2n+3)(2n+2)(2n+1)!} \cdot \frac{(2n+1)!}{1} \right|$$

$$L = \lim_{n\to\infty} \left| \frac{x^2}{(2n+3)(2n+2)} \cdot \frac{1}{1} \right|$$

$$L = x^2 \lim_{n\to\infty} \left| \frac{1}{(2n+3)(2n+2)} \right|$$

$$L = x^2 \cdot 0$$

$$L = 0$$

The series converges if $L < 1$ and diverges if $L > 1$, which means the series converges everywhere, so the interval of convergence is ∞, and the radius of convergence is, too.

■ 3. Find the radius of convergence of the series.

$$\sum_{n=0}^{\infty} \frac{x^n}{n+4}$$

Solution:

Apply the ratio test.

$$L = \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right|$$

$$L = \lim_{n \to \infty} \left| \frac{\frac{x^{n+1}}{n+1+4}}{\frac{x^n}{n+4}} \right|$$

$$L = \lim_{n \to \infty} \left| \frac{\frac{x^{n+1}}{n+5}}{\frac{x^n}{n+4}} \right|$$

$$L = \lim_{n \to \infty} \left| \frac{x^{n+1}}{n+5} \cdot \frac{n+4}{x^n} \right|$$

$$L = \lim_{n \to \infty} \left| \frac{x}{n+5} \cdot \frac{n+4}{1} \right|$$

$$L = |x| \lim_{n \to \infty} \left| \frac{n+4}{n+5} \right|$$

$$L = |x| \cdot 1$$

$$L = |x|$$

Then the interval of convergence is given by the inequality

$$|x| < 1$$

$$-1 < x < 1$$

The interval of convergence spans -1 to 1, which is 2 units wide. The radius of convergence will be half that, so the radius of convergence is 1.

■ 4. Find the radius of convergence of the series.

$$\sum_{n=0}^{\infty} \frac{3^n(x+2)^n}{n!}$$

Solution:

Apply the ratio test.

$$L = \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right|$$

$$L = \lim_{n \to \infty} \left| \frac{\frac{3^{n+1}(x+2)^{n+1}}{(n+1)!}}{\frac{3^n(x+2)^n}{n!}} \right|$$

$$L = \lim_{n \to \infty} \left| \frac{3^{n+1}(x+2)^{n+1}}{(n+1)!} \cdot \frac{n!}{3^n(x+2)^n} \right|$$

$$L = \lim_{n \to \infty} \left| \frac{3(x+2)}{(n+1)n!} \cdot \frac{n!}{1} \right|$$

$$L = \lim_{n \to \infty} \left| \frac{3(x+2)}{(n+1)} \cdot \frac{1}{1} \right|$$

$$L = \left| 3(x+2) \right| \lim_{n \to \infty} \left| \frac{1}{n+1} \right|$$

$$L = \left| 3(x+2) \right| \cdot 0$$

$$L = 0$$

The series converges if $L < 1$ and diverges if $L > 1$, which means the series converges everywhere, so the interval of convergence is ∞, and the radius of convergence is, too.

5. Find the radius of convergence of the series.

$$\sum_{n=0}^{\infty} \frac{3^n(x+2)^n}{n+1}$$

Solution:

Apply the ratio test.

$$L = \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right|$$

$$L = \lim_{n \to \infty} \left| \frac{\frac{3^{n+1}(x+2)^{n+1}}{n+1+1}}{\frac{3^n(x+2)^n}{n+1}} \right|$$

$$L = \lim_{n \to \infty} \left| \frac{3^{n+1}(x+2)^{n+1}}{n+2} \cdot \frac{n+1}{3^n(x+2)^n} \right|$$

$$L = \lim_{n \to \infty} \left| \frac{3(x+2)}{n+2} \cdot \frac{n+1}{1} \right|$$

$$L = \left| 3(x+2) \right| \lim_{n \to \infty} \left| \frac{n+1}{n+2} \right|$$

$$L = \left| 3(x+2) \right| \cdot 1$$

$$L = \left| 3(x+2) \right|$$

Then the interval of convergence is given by the inequality

$$\left| 3(x+2) \right| < 1$$

$$-1 < 3(x+2) < 1$$

$$-\frac{1}{3} < x+2 < \frac{1}{3}$$

$$-\frac{7}{3} < x + 2 < -\frac{5}{3}$$

The interval of convergence spans $-7/3$ to $-5/3$, which is $2/3$ units wide. The radius of convergence will be half that, so the radius of convergence is $1/3$.

INTERVAL OF CONVERGENCE

1. Find the interval of convergence of the series.

$$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$$

Solution:

Apply the ratio test.

$$L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$$

$$L = \lim_{n\to\infty} \left| \frac{\frac{(-1)^{n+1} x^{2(n+1)+1}}{2(n+1)+1}}{\frac{(-1)^n x^{2n+1}}{2n+1}} \right|$$

$$L = \lim_{n\to\infty} \left| \frac{\frac{x^{2n+3}}{2n+3}}{\frac{x^{2n+1}}{2n+1}} \right|$$

$$L = \lim_{n\to\infty} \left| \frac{x^{2n+3}}{2n+3} \cdot \frac{2n+1}{x^{2n+1}} \right|$$

$$L = \lim_{n\to\infty} \left| \frac{x^2}{2n+3} \cdot \frac{2n+1}{1} \right|$$

$$L = x^2 \lim_{n\to\infty} \left| \frac{2n+1}{2n+3} \right|$$

$$L = x^2 \cdot 1$$

$$L = x^2$$

Then the interval of convergence is given by the inequality

$$x^2 < 1$$

$$-1 < x < 1$$

Check the endpoints of the interval.

At $x = -1$,

$$\lim_{n\to\infty} \frac{(-1)^n(-1)^{2n+1}}{2n+1} = \lim_{n\to\infty} \frac{(-1)^{3n+1}}{2n+1}$$

This converges by the alternating series test.

At $x = 1$,

$$\lim_{n\to\infty} \frac{(-1)^n(1)^{2n+1}}{2n+1} = \lim_{n\to\infty} \frac{(-1)^n}{2n+1}$$

This converges by the alternating series test.

So the interval of convergence is

$$-1 \le x \le 1$$

2. Find the interval of convergence of the series.

$$\sum_{n=0}^{\infty} \frac{(-1)^n(x-3)^n}{n+1}$$

Solution:

Apply the ratio test.

$$L = \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right|$$

$$L = \lim_{n \to \infty} \left| \frac{\frac{(-1)^{n+1}(x-3)^{n+1}}{n+1+1}}{\frac{(-1)^n(x-3)^n}{n+1}} \right|$$

$$L = \lim_{n \to \infty} \left| \frac{\frac{(x-3)^{n+1}}{n+2}}{\frac{(x-3)^n}{n+1}} \right|$$

$$L = \lim_{n \to \infty} \left| \frac{(x-3)^{n+1}}{n+2} \cdot \frac{n+1}{(x-3)^n} \right|$$

$$L = \lim_{n \to \infty} \left| \frac{x-3}{n+2} \cdot \frac{n+1}{1} \right|$$

$$L = |x-3| \lim_{n \to \infty} \left| \frac{n+1}{n+2} \right|$$

$$L = |x-3| \cdot 1$$

$$L = |x-3|$$

Then the interval of convergence is given by the inequality

$$|x-3| < 1$$

$$-1 < x-3 < 1$$

$$2 < x < 4$$

Check the endpoints of the interval.

At $x = 2$,

$$\lim_{n\to\infty} \frac{(-1)^n(2-3)^n}{n+1} = \lim_{n\to\infty} \frac{(-1)^n(-1)^n}{n+1} = \lim_{n\to\infty} \frac{(-1)^{2n}}{n+1} = \lim_{n\to\infty} \frac{1}{n+1}$$

This is a harmonic series that diverges.

At $x = 4$,

$$\lim_{n\to\infty} \frac{(-1)^n(4-3)^n}{n+1} = \lim_{n\to\infty} \frac{(-1)^n(1)^n}{n+1} = \lim_{n\to\infty} \frac{(-1)^n}{n+1}$$

This converges by the alternating series test.

So the interval of convergence is

$$2 < x \le 4$$

ESTIMATING DEFINITE INTEGRALS

1. Evaluate the definite integral as a power series, using the first four terms.

$$\int_0^2 \frac{24}{x^2+4}\ dx$$

Solution:

Rewrite the integral.

$$\int_0^2 \frac{24}{x^2+4}\ dx = 24\int_0^2 \frac{1}{x^2+4}\ dx = 24\int_0^2 \frac{1}{x^2+2^2}\ dx = 24\int_0^2 \frac{\frac{1}{4}}{\left(\frac{x}{2}\right)^2+1}\ dx$$

Integrate, then evaluate over the interval.

$$\frac{24}{2}\arctan\left(\frac{x}{2}\right)\Big|_0^2 = 12\arctan\left(\frac{2}{2}\right) = 12\arctan 1 = 12\left(\frac{\pi}{4}\right) = 3\pi$$

Write the original function in the same format as the common series.

$$\frac{1}{1+x} = \frac{24}{x^2+4}$$

$$\frac{1}{1+x} = (24)\frac{1}{x^2+4}$$

$$\frac{1}{1+x}=(24)\frac{1}{4\left(\left(\frac{x}{2}\right)^2+1\right)}$$

$$\frac{1}{1+x}=(6)\frac{1}{\left(\frac{x}{2}\right)^2+1}$$

Substitute into the common series.

$$\frac{1}{1+x}=\sum_{n=0}^{\infty}(-1)^n x^n$$

$$\frac{24}{x^2+4}=6\sum_{n=0}^{\infty}(-1)^n\left(\frac{x}{2}\right)^{2n}=\sum_{n=0}^{\infty}6(-1)^n\frac{x^{2n}}{2^{2n}}$$

Now integrate this series, then evaluate over the interval.

$$\int_0^2 6(-1)^n\frac{x^{2n}}{2^{2n}}\ dx$$

$$\frac{6(-1)^n}{2^{2n}}\int_0^2 x^{2n}\ dx$$

$$\frac{6(-1)^n}{2^{2n}}\cdot\frac{x^{2n+1}}{2n+1}\Bigg|_0^2$$

$$\frac{6(-1)^n}{2^{2n}}\cdot\frac{2^{2n+1}}{2n+1}$$

$$\frac{6(-1)^n}{2^{2n}}\cdot\frac{0^{2n+1}}{2n+1}$$

$$\frac{6(-1)^n}{2^{2n}} \cdot \frac{2^{2n+1}}{2n+1}$$

$$\frac{6(-1)^n}{1} \cdot \frac{2}{2n+1}$$

$$\frac{12(-1)^n}{2n+1}$$

Then we can set up an equation with the integral and the new series.

$$\int_0^2 \frac{24}{x^2+4}\ dx = \sum_{n=0}^{\infty} \frac{12(-1)^n}{2n+1}$$

$$\int_0^2 \frac{24}{x^2+4}\ dx = \frac{12(-1)^0}{2(0)+1} + \frac{12(-1)^1}{2(1)+1} + \frac{12(-1)^2}{2(2)+1} + \frac{12(-1)^3}{2(3)+1} + \dots$$

$$\int_0^2 \frac{24}{x^2+4}\ dx = 12 - 4 + 2.4 - 1.714 + 1.333$$

Using the first four terms of the series,

$$\int_0^2 \frac{24}{x^2+4}\ dx \approx 12 - 4 + 2.4 - 1.714 \approx 8.686$$

■ 2. Evaluate the definite integral as a power series, using the first four terms.

$$\int_0^1 3x\cos(x^3)\ dx$$

Solution:

Rework the common series so that it matches the integrand. First substitute x^3 for x.

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

$$\cos(x^3) = \sum_{n=0}^{\infty} \frac{(-1)^n (x^3)^{2n}}{(2n)!}$$

$$\cos(x^3) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{6n}}{(2n)!}$$

Multiply by $3x$.

$$3x\cos(x^3) = \sum_{n=0}^{\infty} \frac{3x(-1)^n (x^3)^{2n}}{(2n)!}$$

$$3x\cos(x^3) = \sum_{n=0}^{\infty} \frac{3(-1)^n x^{6n+1}}{(2n)!}$$

Now integrate this series, then evaluate over the interval.

$$\int_0^1 \frac{3(-1)^n x^{6n+1}}{(2n)!} \ dx$$

$$\frac{3(-1)^n}{(2n)!} \int_0^1 x^{6n+1} \ dx$$

$$\frac{3(-1)^n}{(2n)!} \cdot \frac{x^{6n+2}}{6n+2}\Bigg|_0^1$$

$$\frac{3(-1)^n}{(2n)!} \cdot \frac{1^{6n+2}}{6n+2} - \frac{3(-1)^n}{(2n)!} \cdot \frac{0^{6n+2}}{6n+2}$$

$$\frac{3(-1)^n}{(2n)!(6n+2)}$$

Then we can set up an equation with the integral and the new series.

$$\int_0^1 3x\cos(x^3)\ dx = \sum_{n=0}^{\infty} \frac{3(-1)^n}{(2n)!(6n+2)}$$

$$\int_0^1 3x\cos(x^3)\ dx = \frac{3(-1)^0}{(2\cdot 0)!(6(0)+2)} + \frac{3(-1)^1}{(2\cdot 1)!(6(1)+2)}$$

$$+\frac{3(-1)^2}{(2\cdot 2)!(6(2)+2)} + \frac{3(-1)^3}{(2\cdot 3)!(6(3)+2)} + \dots$$

$$\int_0^1 3x\cos(x^3)\ dx \approx 1.5 - 0.1875 + 0.00893 - 0.000208$$

Using the first four terms of the series,

$$\int_0^1 3x\cos(x^3)\ dx \approx 1.5 - 0.1875 + 0.00893 - 0.000208 \approx 1.321222$$

■ 3. Evaluate the definite integral as a power series, using the first four terms.

$$\int_0^1 4e^{x^2}\ dx$$

Solution:

Rework the common series so that it matches the integrand. First substitute x^2 for x.

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$e^{x^2} = \sum_{n=0}^{\infty} \frac{(x^2)^n}{n!}$$

$$e^{x^2} = \sum_{n=0}^{\infty} \frac{x^{2n}}{n!}$$

Multiply by 4.

$$4e^{x^2} = 4\sum_{n=0}^{\infty} \frac{(x^2)^n}{n!} = \sum_{n=0}^{\infty} \frac{4x^{2n}}{n!}$$

Now integrate this series, then evaluate over the interval.

$$\int \frac{4x^{2n}}{n!}\ dx = \frac{4}{n!}\int x^{2n}\ dx$$

$$\frac{4}{n!} \cdot \frac{x^{2n+1}}{2n+1}\Bigg|_0^1$$

$$\frac{4}{n!} \cdot \frac{1^{2n+1}}{2n+1} - \frac{4}{n!} \cdot \frac{0^{2n+1}}{2n+1}$$

$$\frac{4}{n!(2n+1)}$$

Then we can set up an equation with the integral and the new series.

$$\int_0^1 4e^{x^2}\,dx = \sum_{n=0}^{\infty} \frac{4}{n!(2n+1)}$$

$$\int_0^1 4e^{x^2}\,dx = \frac{4}{0!(2(0)+1)} + \frac{4}{1!(2(1)+1)} + \frac{4}{2!(2(2)+1)}$$

$$+\frac{4}{3!(2(3)+1)} + \frac{4}{4!(2(4)+1)} + \dots$$

$$\int_0^1 4e^{x^2}\,dx \approx 4 + 1.3333 + 0.4 + 0.0953 + 0.01852 + \dots$$

Using the first four terms of the series,

$$\int_0^1 4e^{x^2}\,dx \approx 4 + 1.3333 + 0.4 + 0.0953 + 0.01852 \approx 5.84712$$

ESTIMATING INDEFINITE INTEGRALS

1. Evaluate the indefinite integral as a power series.

$$\int x^2 \sin(x^2)\ dx$$

Solution:

Start with a common series, substitute x^2 for x,

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

$$\sin x^2 = \sum_{n=0}^{\infty} \frac{(-1)^n (x^2)^{2n+1}}{(2n+1)!}$$

$$\sin x^2 = \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+2}}{(2n+1)!}$$

and then multiply by x^2.

$$x^2 \sin x^2 = x^2 \sum_{n=0}^{\infty} \frac{(-1)^n (x^2)^{2n+1}}{(2n+1)!}$$

$$x^2 \sin x^2 = \sum_{n=0}^{\infty} \frac{(-1)^n x^2 \cdot x^{4n+2}}{(2n+1)!}$$

$$x^2 \sin x^2 = \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+4}}{(2n+1)!}$$

Integrate this series.

$$\int \frac{(-1)^n x^{4n+4}}{(2n+1)!} \ dx$$

$$\frac{(-1)^n}{(2n+1)!} \int x^{4n+4} \ dx$$

$$\frac{(-1)^n}{(2n+1)!} \cdot \frac{x^{4n+5}}{4n+5} + C$$

$$\frac{(-1)^n x^{4n+5}}{(4n+5)(2n+1)!} + C$$

$$\frac{(-1)^n x^{4n+5}}{(4n+5)(2n+1)!}$$

So this is the value of the integral, integrated as a power series.

$$\int x^2 \sin(x^2) \ dx = \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+5}}{(4n+5)(2n+1)!}$$

■ 2. Evaluate the indefinite integral as a power series.

$$\int \ln(1+2x) \ dx$$

Solution:

Start with a common series, substitute $2x$ for x.

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}x^n}{n}$$

$$\ln(1+2x) = 2x - \frac{(2x)^2}{2} + \frac{(2x)^3}{3} - \frac{(2x)^4}{4} + \frac{(2x)^5}{5} - \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(2x)^n}{n}$$

$$\ln(1+2x) = 2x - \frac{4x^2}{2} + \frac{8x^3}{3} - \frac{16x^4}{4} + \frac{32x^5}{5} - \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}2^n x^n}{n}$$

Integrate this series.

$$\int \frac{(-1)^{n-1}2^n x^n}{n} \ dx$$

$$\frac{(-1)^{n-1}2^n}{n} \int x^n \ dx$$

$$\frac{(-1)^{n-1}2^n}{n} \left(\frac{x^{n+1}}{n+1} \right) + C$$

$$\frac{(-1)^{n-1}2^n x^{n+1}}{n(n+1)} + C$$

So this is the value of the integral, integrated as a power series.

$$\int \ln(1+2x) \ dx = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}2^n x^{n+1}}{n(n+1)}$$

■ 3. Evaluate the indefinite integral as a power series.

$$\int x^2 \cos(x^3)\ dx$$

Solution:

Start with a common series, substitute x^3 for x,

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

$$\cos x^3 = \sum_{n=0}^{\infty} \frac{(-1)^n (x^3)^{2n}}{(2n)!}$$

$$\cos x^3 = \sum_{n=0}^{\infty} \frac{(-1)^n x^{6n}}{(2n)!}$$

and then multiply by x^2.

$$x^2 \cos x^3 = x^2 \sum_{n=0}^{\infty} \frac{(-1)^n x^{6n}}{(2n)!}$$

$$x^2 \cos x^3 = \sum_{n=0}^{\infty} \frac{(-1)^n x^{6n+2}}{(2n)!}$$

Integrate this series.

$$\int \frac{(-1)^n x^{6n+2}}{(2n)!}\ dx$$

$$\frac{(-1)^n}{(2n)!}\int x^{6n+2}\ dx$$

$$\frac{(-1)^n}{(2n)!}\cdot\frac{x^{6n+3}}{6n+3}+C$$

$$\frac{(-1)^n x^{6n+3}}{(2n)!(6n+3)}+C$$

$$\frac{(-1)^n x^{6n+3}}{(2n)!(6n+3)}$$

So this is the value of the integral, integrated as a power series.

$$\int x^2\cos(x^3)\ dx=\sum_{n=0}^{\infty}\frac{(-1)^n x^{6n+3}}{(2n)!(6n+3)}$$

BINOMIAL SERIES

1. Use a binomial series to expand the function as a power series.

$$f(x) = (3+x)^5$$

Solution:

Begin with the binomial series.

$$(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n = 1 + kx + \frac{k(k-1)}{2!}x^2$$

$$+\frac{k(k-1)(k-2)}{3!}x^3 + \frac{k(k-1)(k-2)(k-3)}{4!}x^4 + \ldots$$

Replace x with $x+2$ and k with 5.

$$(1+x+2)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n = 1 + \frac{k(x+2)}{1!} + \frac{k(k-1)}{2!}(x+2)^2$$

$$+\frac{k(k-1)(k-2)}{3!}(x+2)^3 + \frac{k(k-1)(k-2)(k-3)}{4!}(x+2)^4 + \ldots$$

$$(3+x)^5 = \sum_{n=0}^{\infty} \binom{5}{n} x^n = \frac{1}{0!}(x+2)^0 + \frac{5}{1!}(x+2)^1 + \frac{5(5-1)}{2!}(x+2)^2$$

$$+\frac{5(5-1)(5-2)}{3!}(x+2)^3 + \frac{5(5-1)(5-2)(5-3)}{4!}(x+2)^4 + \ldots$$

Simplify the right side.

$$(3+x)^5 = \sum_{n=0}^{\infty} \binom{5}{n} x^n = \frac{1}{0!}(x+2)^0 + \frac{5}{1!}(x+2)^1 + \frac{5(4)}{2!}(x+2)^2$$

$$+\frac{5(4)(3)}{3!}(x+2)^3 + \frac{5(4)(3)(2)}{4!}(x+2)^4 + \dots$$

Match the terms to their corresponding n-values.

$n=0$ $\qquad$ $\frac{1}{0!}(x+2)^0$

$n=1$ $\qquad$ $\frac{5}{1!}(x+2)^1$

$n=2$ $\qquad$ $\frac{5(4)}{2!}(x+2)^2$

$n=3$ $\qquad$ $\frac{5(4)(3)}{3!}(x+2)^3$

$n=4$ $\qquad$ $\frac{5(4)(3)(2)}{4!}(x+2)^4$

Then the power series is

$$(3+x)^5 = 1 + \sum_{n=1}^{\infty} \frac{5 \cdot 4 \cdot 3 \cdot 2 \cdot \dots \cdot (6-n)}{n!}(x+2)^n$$

■ 2. Use a binomial series to expand the function as a power series.

$$f(x) = (6-x)^4$$

Solution:

Begin with the binomial series.

$$(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n = 1 + kx + \frac{k(k-1)}{2!}x^2$$

$$+\frac{k(k-1)(k-2)}{3!}x^3 + \frac{k(k-1)(k-2)(k-3)}{4!}x^4 + \dots$$

Replace x with $-x-5$ and k with 4.

$$(1+(-x-5))^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n = 1 + \frac{k}{1!}(-x-5) + \frac{k(k-1)}{2!}(-x-5)^2$$

$$+\frac{k(k-1)(k-2)}{3!}(-x-5)^3 + \frac{k(k-1)(k-2)(k-3)}{4!}(-x-5)^4 + \dots$$

$$(6-x)^4 = \sum_{n=0}^{\infty} \binom{4}{n} x^n = \frac{1}{0!}(-x-5)^0 + \frac{4}{1!}(-x-5) + \frac{4(4-1)}{2!}(-x-5)^2$$

$$+\frac{4(4-1)(4-2)}{3!}(-x-5)^3 + \frac{4(4-1)(4-2)(4-3)}{4!}(-x-5)^4 + \dots$$

Simplify the right side.

$$(6-x)^4 = \sum_{n=0}^{\infty} \binom{4}{n} x^n = \frac{1}{0!}(-x-5)^0 + \frac{4}{1!}(-x-5)^1 + \frac{4(3)}{2!}(-x-5)^2$$

$$+\frac{4(3)(2)}{3!}(-x-5)^3 + \frac{4(3)(2)(1)}{4!}(-x-5)^4 + \dots$$

Match the terms to their corresponding n-values.

$n=0$ $\qquad \frac{1}{0!}(-x-5)^0$

$n=1$ $\qquad \frac{4}{1!}(-x-5)^1$

$n=2$ $\qquad \frac{4(3)}{2!}(-x-5)^2$

$n=3$ $\qquad \frac{4(3)(2)}{3!}(-x-5)^3$

$n=4$ $\qquad \frac{4(3)(2)(1)}{4!}(-x-5)^4$

Then the power series is

$$(6-x)^4 = 1 + \sum_{n=1}^{\infty} \frac{4 \cdot 3 \cdot 2 \cdot \ldots \cdot (5-n)}{n!}(-x-5)^n$$

■ 3. Use a binomial series to expand the function as a power series.

$$f(x) = (-4+x)^5$$

Solution:

Begin with the binomial series.

$$(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n = 1 + kx + \frac{k(k-1)}{2!}x^2$$

$$+\frac{k(k-1)(k-2)}{3!}x^3 + \frac{k(k-1)(k-2)(k-3)}{4!}x^4 + \dots$$

Replace x with $x-5$ and k with 5.

$$(1+x-5)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n = 1 + \frac{k(x+2)}{1!} + \frac{k(k-1)}{2!}(x-5)^2$$

$$+\frac{k(k-1)(k-2)}{3!}(x-5)^3 + \frac{k(k-1)(k-2)(k-3}{4!}(x-5)^4 + \dots$$

$$(-4+x)^5 = \sum_{n=0}^{\infty} \binom{5}{n} x^n = \frac{1}{0!}(x-5)^0 + \frac{5}{1!}(x-5)^1 + \frac{5(5-1)}{2!}(x-5)^2$$

$$+\frac{5(5-1)(5-2)}{3!}(x-5)^3 + \frac{5(5-1)(5-2)(5-3)}{4!}(x-5)^4 + \dots$$

Simplify the right side.

$$(-4+x)^5 = \sum_{n=0}^{\infty} \binom{5}{n} x^n = \frac{1}{0!}(x-5)^0 + \frac{5}{1!}(x-5)^1$$

$$+\frac{5(4)}{2!}(x-5)^2 + \frac{5(4)(3)}{3!}(x-5)^3 + \frac{5(4)(3)(2)}{4!}(x-5)^4 + \dots$$

Match the terms to their corresponding n-values.

$n=0$ $\qquad\qquad$ $\frac{1}{0!}(x-5)^0$

$$n = 1 \qquad \frac{5}{1!}(x-5)^1$$

$$n = 2 \qquad \frac{5(4)}{2!}(x-5)^2$$

$$n = 3 \qquad \frac{5(4)(3)}{3!}(x-5)^3$$

$$n = 4 \qquad \frac{5(4)(3)(2)}{4!}(x-5)^4$$

Then the power series is

$$(-4+x)^5 = 1 + \sum_{n=1}^{\infty} \frac{5 \cdot 4 \cdot 3 \cdot 2 \cdot \ldots \cdot (6-n)}{n!}(x-5)^n$$

■ 4. Use a binomial series to expand the function as a power series.

$$f(x) = (7-x)^6$$

Solution:

Begin with the binomial series.

$$(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n = 1 + kx + \frac{k(k-1)}{2!}x^2$$

$$+\frac{k(k-1)(k-2)}{3!}x^3 + \frac{k(k-1)(k-2)(k-3)}{4!}x^4 + \ldots$$

Replace x with $-x+6$ and k with 6.

$$(1+(-x+6))^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n = 1 + \frac{k}{1!}(-x+6) + \frac{k(k-1)}{2!}(-x+6)^2$$

$$+\frac{k(k-1)(k-2)}{3!}(-x+6)^3 + \frac{k(k-1)(k-2)(k-3)}{4!}(-x+6)^4 + \dots$$

$$(7-x)^6 = \sum_{n=0}^{\infty} \binom{6}{n} x^n = \frac{1}{0!}(-x+6)^0 + \frac{6}{1!}(-x+6) + \frac{6(6-1)}{2!}(-x+6)^2$$

$$+\frac{6(6-1)(6-2)}{3!}(-x+6)^3 + \frac{6(6-1)(6-2)(6-3)}{4!}(-x+6)^4 + \dots$$

Simplify the right side.

$$(7-x)^6 = \sum_{n=0}^{\infty} \binom{6}{n} x^n = \frac{1}{0!}(-x+6)^0 + \frac{6}{1!}(-x+6) + \frac{6(5)}{2!}(-x+6)^2$$

$$+\frac{6(5)(4)}{3!}(-x+6)^3 + \frac{6(5)(4)(3)}{4!}(-x+6)^4 + \dots$$

Match the terms to their corresponding n-values.

$n=0$ $\qquad \frac{1}{0!}(-x+6)^0$

$n=1$ $\qquad \frac{6}{1!}(-x+6)^1$

$n=2$ $\qquad \frac{6(5)}{2!}(-x+6)^2$

$n=3$ $\qquad \frac{6(5)(4)}{3!}(-x+6)^3$

$$n = 4 \qquad \frac{6(5)(4)(3)}{4!}(-x+6)^4$$

Then the power series is

$$(7-x)^6 = 1 + \sum_{n=1}^{\infty} \frac{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot \ldots \cdot (7-n)}{n!}(-x+6)^n$$

5. Use a binomial series to expand the function as a power series.

$$f(x) = (8+x)^7$$

Solution:

Begin with the binomial series.

$$(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n = 1 + kx + \frac{k(k-1)}{2!}x^2$$

$$+\frac{k(k-1)(k-2)}{3!}x^3 + \frac{k(k-1)(k-2)(k-3)}{4!}x^4 + \ldots$$

Replace x with $x+7$ and k with 7.

$$(1+x+7)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n = 1 + \frac{k}{1!}(x+7) + \frac{k(k-1)}{2!}(x+7)^2$$

$$+\frac{k(k-1)(k-2)}{3!}(x+7)^3 + \frac{k(k-1)(k-2)(k-3)}{4!}(x+7)^4 + \ldots$$

$$(8+x)^7 = \sum_{n=0}^{\infty} \binom{7}{n} x^n = \frac{1}{0!}(x+7)^0 + \frac{7}{1!}(x+7)^1 + \frac{7(7-1)}{2!}(x+7)^2$$

$$+\frac{7(7-1)(7-2)}{3!}(x+7)^3 + \frac{7(7-1)(7-2)(7-3)}{4!}(x+7)^4 + \ldots$$

Simplify the right side.

$$(8+x)^7 = \sum_{n=0}^{\infty} \binom{7}{n} x^n = \frac{1}{0!}(x+7)^0 + \frac{7}{1!}(x+7)^1$$

$$+\frac{7(6)}{2!}(x+7)^2 + \frac{7(6)(5)}{3!}(x+7)^3 + \frac{7(6)(5)(4)}{4!}(x+7)^4 + \ldots$$

Match the terms to their corresponding n-values.

$n = 0$ $\qquad \dfrac{1}{0!}(x+7)^0$

$n = 1$ $\qquad \dfrac{7}{1!}(x+7)^1$

$n = 2$ $\qquad \dfrac{7(6)}{2!}(x+7)^2$

$n = 3$ $\qquad \dfrac{7(6)(5)}{3!}(x+7)^3$

$n = 4$ $\qquad \dfrac{7(6)(5)(4)}{4!}(x+7)^4$

Then the power series is

$$(8+x)^7 = 1 + \sum_{n=1}^{\infty} \frac{7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot \ldots \cdot (8-n)}{n!}(x+7)^n$$

TAYLOR SERIES

■ 1. Find the third-degree Taylor polynomial and use it to approximate $f(5)$.

$$f(x) = 3\sqrt{x+1}$$

$n = 3$ and $a = 3$

Solution:

The formula for the Taylor polynomial of $f(x)$ at a is

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots + \frac{f^n(a)}{n!}(x-a)^n$$

Use the original function and the first three derivatives.

$$f(x) = 3(x+1)^{\frac{1}{2}} \qquad f(3) = 3(3+1)^{\frac{1}{2}} \qquad f(3) = 6$$

$$f'(x) = \frac{3}{2(x+1)^{\frac{1}{2}}} \qquad f'(3) = \frac{3}{2(3+1)^{\frac{1}{2}}} \qquad f'(3) = \frac{3}{4}$$

$$f''(x) = -\frac{3}{4(x+1)^{\frac{3}{2}}} \qquad f''(3) = -\frac{3}{4(3+1)^{\frac{3}{2}}} \qquad f''(3) = -\frac{3}{32}$$

$$f'''(x) = \frac{9}{8(x+1)^{\frac{5}{2}}} \qquad f'''(3) = \frac{9}{8(3+1)^{\frac{5}{2}}} \qquad f'''(3) = \frac{9}{256}$$

So the third-degree Taylor polynomial is

$$f^{(3)}(x) = 6 + \frac{3}{4}(x-3) - \frac{3}{32 \cdot 2!}(x-3)^2 + \frac{9}{256 \cdot 3!}(x-3)^3$$

$$f^{(3)}(x) = 6 + \frac{3}{4}(x-3) - \frac{3}{64}(x-3)^2 + \frac{3}{512}(x-3)^3$$

Use the polynomial to estimate $f(5)$.

$$f^{(3)}(5) \approx 6 + \frac{3}{4}(5-3) - \frac{3}{64}(5-3)^2 + \frac{3}{512}(5-3)^3$$

$$f^{(3)}(5) \approx 6 + \frac{3}{4}(2) - \frac{3}{64}(4) + \frac{3}{512}(8)$$

$$f^{(3)}(5) \approx 6 + \frac{3}{2} - \frac{3}{16} + \frac{3}{64}$$

$$f^{(3)}(5) \approx 7.359$$

2. Find the third-degree Taylor polynomial and use it to approximate $f(4)$.

$$f(x) = e^{2x} + 9$$

$n = 3$ and $a = 2$

Solution:

The formula for the Taylor polynomial of $f(x)$ at a is

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots + \frac{f^n(a)}{n!}(x-a)^n$$

Use the original function and the first three derivatives.

$$f(x) = e^{2x} + 9 \qquad f(2) = e^{2(2)} + 9 \qquad f(2) = e^4 + 9$$

$$f'(x) = 2e^{2x} \qquad f'(2) = 2e^{2(2)} \qquad f'(2) = 2e^4$$

$$f''(x) = 4e^{2x} \qquad f''(2) = 4e^{2(2)} \qquad f''(2) = 4e^4$$

$$f'''(x) = 8e^{2x} \qquad f'''(2) = 8e^{2(2)} \qquad f'''(2) = 8e^4$$

So the third-degree Taylor polynomial is

$$f^{(3)}(x) = e^4 + 9 + 2e^4(x-2) + \frac{4e^4}{2!}(x-2)^2 + \frac{8e^4}{3!}(x-2)^3$$

$$f^{(3)}(x) = e^4 + 9 + 2e^4(x-2) + \frac{4e^4}{2}(x-2)^2 + \frac{8e^4}{6}(x-2)^3$$

$$f^{(3)}(x) = e^4 + 9 + 2e^4(x-2) + 2e^4(x-2)^2 + \frac{4e^4}{3}(x-2)^3$$

Use the polynomial to estimate $f(4)$.

$$f^{(3)}(4) = e^4 + 9 + 2e^4(4-2) + 2e^4(4-2)^2 + \frac{4e^4}{3}(4-2)^3$$

$$f^{(3)}(4) = e^4 + 9 + 2e^4(2) + 2e^4(2)^2 + \frac{4}{3}e^4(2)^3$$

$$f^{(3)}(4) = e^4 + 9 + 4e^4 + 8e^4 + \frac{32}{3}e^4$$

$$f^{(3)}(4) \approx 1{,}301.156$$

3. Find the fourth-degree Taylor polynomial and use it to approximate $f(\pi/24)$.

$$f(x) = \sin(6x) + 5$$

$n = 4$ and $a = \dfrac{\pi}{12}$

Solution:

The formula for the Taylor polynomial of $f(x)$ at a is

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots + \frac{f^n(a)}{n!}(x-a)^n$$

Use the original function and the first four derivatives.

$$f(x) = \sin 6x + 5 \qquad f\left(\frac{\pi}{12}\right) = \sin\left(6 \cdot \frac{\pi}{12}\right) + 5 \qquad f\left(\frac{\pi}{12}\right) = 6$$

$$f'(x) = 6\cos 6x \qquad f'\left(\frac{\pi}{12}\right) = 6\cos\left(6 \cdot \frac{\pi}{12}\right) \qquad f'\left(\frac{\pi}{12}\right) = 0$$

$$f''(x) = -36\sin 6x \qquad f''\left(\frac{\pi}{12}\right) = -36\sin\left(6 \cdot \frac{\pi}{12}\right) \qquad f''\left(\frac{\pi}{12}\right) = -36$$

$$f'''(x) = -216\cos 6x \qquad f'''\left(\frac{\pi}{12}\right) = -216\cos\left(6 \cdot \frac{\pi}{12}\right) \qquad f'''\left(\frac{\pi}{12}\right) = 0$$

$$f''''(x) = 1{,}296\sin 6x \qquad f''''\left(\frac{\pi}{12}\right) = 1{,}296\sin\left(6 \cdot \frac{\pi}{12}\right) \qquad f''''\left(\frac{\pi}{12}\right) = 1{,}296$$

So the fourth-degree Taylor polynomial is

$$f^{(4)}(x)=6+0\left(x-\frac{\pi}{12}\right)-\frac{36}{2!}\left(x-\frac{\pi}{12}\right)^2+\frac{0}{3!}\left(x-\frac{\pi}{12}\right)^3+\frac{1{,}296}{4!}\left(x-\frac{\pi}{12}\right)^4$$

$$f^{(4)}(x)=6+0-\frac{36}{2}\left(x-\frac{\pi}{12}\right)^2+0+\frac{1{,}296}{24}\left(x-\frac{\pi}{12}\right)^4$$

$$f^{(4)}(x)=6+0-18\left(x-\frac{\pi}{12}\right)^2+0+54\left(x-\frac{\pi}{12}\right)^4$$

$$f^{(4)}(x)=6-18\left(x-\frac{\pi}{12}\right)^2+54\left(x-\frac{\pi}{12}\right)^4$$

Use the polynomial to estimate $f(\pi/24)$.

$$f^{(4)}\left(\frac{\pi}{24}\right)=6-18\left(\frac{\pi}{24}-\frac{\pi}{12}\right)^2+54\left(\frac{\pi}{24}-\frac{\pi}{12}\right)^4$$

$$f^{(4)}\left(\frac{\pi}{24}\right)=6-18\left(-\frac{\pi}{24}\right)^2+54\left(-\frac{\pi}{24}\right)^4$$

$$f^{(4)}\left(\frac{\pi}{24}\right)=6-18\left(\frac{\pi^2}{576}\right)+54\left(\frac{\pi^4}{331{,}776}\right)$$

$$f^{(4)}\left(\frac{\pi}{24}\right)\approx 6-0.30843+0.01585$$

$$f^{(4)}\left(\frac{\pi}{24}\right)\approx 5.707$$

RADIUS AND INTERVAL OF CONVERGENCE OF A TAYLOR SERIES

1. Find the radius of convergence of the Taylor polynomial.

$$P_{(3)}(x) = 1 + 2(x-3) + 4(x-3)^2 + 8(x-3)^3$$

Solution:

Rewrite the Taylor polynomial

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots + \frac{f^n(a)}{n!}(x-a)^n$$

as a power series in the form

$$f(x) = \sum_{n=1}^{\infty} \frac{f^n(a)}{n!}(x-a)^n = \sum_{n=0}^{\infty} \frac{f^{n-1}(a)}{(n-1)!}(x-a)^{n-1}$$

Rewrite the Taylor polynomial.

$$P_{(3)}(x) = 1 + 2(x-3) + 4(x-3)^2 + 8(x-3)^3$$

$$P_{(3)}(x) = 1(x-3)^0 + 2(x-3)^1 + 4(x-3)^2 + 8(x-3)^3$$

$$P_{(3)}(x) = 2^0(x-3)^0 + 2^1(x-3)^1 + 2^2(x-3)^2 + 2^3(x-3)^3$$

Then its power series representation is

$$P(x) = 2^0(x-3)^0 + 2^1(x-3)^1 + 2^2(x-3)^2 + 2^3(x-3)^3 + \cdots = \sum_{n=0}^{\infty} 2^n(x-3)^n$$

Apply the ratio test.

$$L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$$

$$L = \lim_{n\to\infty} \left| \frac{2^{n+1}(x-3)^{n+1}}{2^n(x-3)^n} \right|$$

$$L = \lim_{n\to\infty} \left| 2^{n+1-n}(x-3)^{n+1-n} \right|$$

$$L = \lim_{n\to\infty} \left| 2(x-3) \right|$$

$$L = \left| 2(x-3) \right|$$

Then set up an inequality.

$$\left| 2(x-3) \right| < 1$$

$$-1 < 2(x-3) < 1$$

$$-\frac{1}{2} < x-3 < \frac{1}{2}$$

$$\frac{5}{2} < x < \frac{7}{2}$$

The interval of convergence spans from $5/2$ to $7/2$, which is a distance of 1 unit. The radius of convergence is will be half that distance, so the radius of convergence is $1/2$.

■ 2. Find the radius of convergence of the Taylor polynomial.

$$P_{(3)}(x) = 4 - 4(x-5) + 16(x-5)^2 - 64(x-5)^3$$

Solution:

Rewrite the Taylor polynomial

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots + \frac{f^n(a)}{n!}(x-a)^n$$

as a power series in the form

$$f(x) = \sum_{n=1}^{\infty} \frac{f^n(a)}{n!}(x-a)^n = \sum_{n=0}^{\infty} \frac{f^{n-1}(a)}{(n-1)!}(x-a)^{n-1}$$

Rewrite the Taylor polynomial.

$$P_{(3)}(x) = 4 - 4(x-5) + 16(x-5)^2 - 64(x-5)^3$$

$$P_{(3)}(x) = 4^1(x-5)^0 - 4^1(x-5)^1 + 4(4)(x-5)^2 - 4(4)^2(x-5)^3$$

Then its power series representation is

$$P(x) = 4 - 4^1(x-5)^1 + 4(4)(x-5)^2 - 4(4)^2(x-5)^3 + \cdots = 4 + \sum_{n=1}^{\infty} (-4)^n(x-5)^n$$

Apply the ratio test.

$$L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$$

$$L = \lim_{n\to\infty} \left| \frac{(-4)^{n+1}(x-5)^{n+1}}{(-4)^n(x-5)^n} \right|$$

$$L = \lim_{n\to\infty} \left| (-4)^{n+1-n}(x-5)^{n+1-n} \right|$$

$$L = \lim_{n\to\infty} \left| (4)(x-5) \right|$$

$$L = \left| 4(x-5) \right|$$

Then set up an inequality.

$$\left| 4(x-5) \right| < 1$$

$$\left| 4x-20 \right| < 1$$

$$-1 < 4x - 20 < 1$$

$$19 < 4x < 21$$

$$\frac{19}{4} < x < \frac{21}{4}$$

The interval of convergence spans from $19/4$ to $21/4$, which is a distance of $2/4 = 1/2$ unit. The radius of convergence is will be half that distance, so the radius of convergence is $1/4$.

■ 3. Find the radius of convergence of the Taylor polynomial.

$$P_{(3)}(x) = \frac{1}{4} - \frac{1}{4}(x-4) + \frac{1}{8}(x-4)^2 - \frac{1}{24}(x-4)^3$$

Solution:

Rewrite the Taylor polynomial

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots + \frac{f^n(a)}{n!}(x-a)^n$$

as a power series in the form

$$f(x) = \sum_{n=1}^{\infty} \frac{f^n(a)}{n!}(x-a)^n = \sum_{n=0}^{\infty} \frac{f^{n-1}(a)}{(n-1)!}(x-a)^{n-1}$$

Rewrite the Taylor polynomial.

$$P_{(3)}(x) = \frac{1}{4} - \frac{1}{4}(x-4) + \frac{1}{8}(x-4)^2 - \frac{1}{24}(x-4)^3$$

$$P_{(3)}(x) = \frac{1}{4 \cdot 1}(x-4)^0 - \frac{1}{4 \cdot 1}(x-4)^1 + \frac{1}{4 \cdot 2}(x-4)^2 - \frac{1}{4 \cdot 6}(x-4)^3$$

$$P_{(3)}(x) = \frac{(-1)^0}{4 \cdot 0!}(x-4)^0 + \frac{(-1)^1}{4 \cdot 1!}(x-4)^1 + \frac{(-1)^2}{4 \cdot 2!}(x-4)^2 + \frac{(-1)^3}{4 \cdot 3!}(x-4)^3$$

Then its power series representation is

$$P(x) = \frac{(-1)^0}{4 \cdot 0!}(x-4)^0 + \frac{(-1)^1}{4 \cdot 1!}(x-4)^1 + \frac{(-1)^2}{4 \cdot 2!}(x-4)^2 + \frac{(-1)^3}{4 \cdot 3!}(x-4)^3 + \cdots$$

$$P(x) = \sum_{n=0}^{\infty} \frac{(-1)^n(x-4)^n}{4 \cdot n!}$$

Apply the ratio test.

$$L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$$

$$L = \lim_{n\to\infty} \left| \frac{\frac{(-1)^{n+1}(x-4)^{n+1}}{4 \cdot (n+1)!}}{\frac{(-1)^n(x-4)^n}{4 \cdot n!}} \right|$$

$$L = \lim_{n\to\infty} \left| \frac{\frac{(x-4)^{n+1}}{4 \cdot (n+1)!}}{\frac{(x-4)^n}{4 \cdot n!}} \right|$$

$$L = \lim_{n\to\infty} \left| \frac{(x-4)^{n+1}}{4 \cdot (n+1)!} \cdot \frac{4 \cdot n!}{(x-4)^n} \right|$$

$$L = \lim_{n\to\infty} \left| \frac{x-4}{(n+1)n!} \cdot \frac{n!}{1} \right|$$

$$L = \lim_{n\to\infty} \left| \frac{x-4}{(n+1)} \cdot \frac{1}{1} \right|$$

$$L = \left| x-4 \right| \lim_{n\to\infty} \left| \frac{1}{(n+1)} \right|$$

$$L = \left| x-4 \right| \cdot 0$$

$$L = 0$$

Then set up an inequality.

$$0 < 1$$

This inequality is true everywhere, so the interval of convergence is infinite, which means the radius of convergence is, too. The radius of convergence is ∞.

TAYLOR'S INEQUALITY

■ 1. Find Taylor's inequality for the function.

$$f(x) = 5\cos x$$

Solution:

Modify Taylor's inequality to state that

If $\left|f^{n+1}(x)\right| \le M$ for $|x| \le d$, then $\left|R_n(x)\right| \le \frac{M}{(n+1)!}|x|^{n+1}$ for $|x| \le d$.

The function $y = \cos x$ is represented by

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

So $f(x) = 5\cos x$ is represented by

$$5\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n 5x^{2n}}{(2n)!}$$

Take the first few derivatives of $f(x) = 5\cos x$ in order to find a value for $f^{(n+1)}(x)$.

$$n = 0 \qquad f^{n+1}(x) = f^{0+1}(x) = f^1(x) = f'(x) = -5\sin x$$

$$n = 1 \qquad f^{n+1}(x) = f^{1+1}(x) = f^2(x) = f''(x) = -5\cos x$$

$$n = 2 \qquad f^{n+1}(x) = f^{2+1}(x) = f^3(x) = f'''(x) = 5\sin x$$

$$n = 3 \qquad f^{n+1}(x) = f^{3+1}(x) = f^4(x) = f''''(x) = 5\cos x$$

$$n = 4 \qquad f^{n+1}(x) = f^{4+1}(x) = f^5(x) = f'''''(x) = -5\sin x$$

Then

$\left| f^{(n+1)}(x) \right| \le 5\cos x$ or $\left| f^{(n+1)}(x) \right| \le 5\sin x$

Since both $\cos x$ and $\sin x$ exist only between -1 and 1, both $5\cos x$ and $5\sin x$ exist only between -5 and 5. So

$$-5 \le \left| f^{(n+1)}(x) \right| \le 5$$

But the absolute value in the inequality requires only positive values, so

$$0 \le \left| f^{(n+1)}(x) \right| \le 5$$

■ 2. Find Taylor's inequality for the function.

$$f(x) = 3\sin x$$

Solution:

Modify Taylor's inequality to state that

If $\left| f^{n+1}(x) \right| \le M$ for $|x| \le d$, then $\left| R_n(x) \right| \le \dfrac{M}{(n+1)!}|x|^{n+1}$ for $|x| \le d$.

The function $y = \sin x$ is represented by

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

So $f(x) = 3\sin x$ is represented by

$$3\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n 3x^{2n+1}}{(2n+1)!}$$

Take the first few derivatives of $f(x) = 5\cos x$ in order to find a value for $f^{(n+1)}(x)$.

$$n = 0 \qquad f^{n+1}(x) = f^{0+1}(x) = f^{1}(x) = f'(x) = 3\cos x$$

$$n = 1 \qquad f^{n+1}(x) = f^{1+1}(x) = f^{2}(x) = f''(x) = -3\sin x$$

$$n = 2 \qquad f^{n+1}(x) = f^{2+1}(x) = f^{3}(x) = f'''(x) = -3\cos x$$

$$n = 3 \qquad f^{n+1}(x) = f^{3+1}(x) = f^{4}(x) = f''''(x) = 3\sin x$$

$$n = 4 \qquad f^{n+1}(x) = f^{4+1}(x) = f^{5}(x) = f'''''(x) = 3\cos x$$

Then

$$\left| f^{(n+1)}(x) \right| \le 3\cos x \text{ or } \left| f^{(n+1)}(x) \right| \le 3\sin x$$

Since both $\cos x$ and $\sin x$ exist only between -1 and 1, both $3\cos x$ and $3\sin x$ exist only between -3 and 3.

$$-3 \le \left| f^{(n+1)}(x) \right| \le 3$$

But the absolute value in the inequality requires only positive values, so

$$0 \le \left| f^{(n+1)}(x) \right| \le 3$$

■ 3. Find Taylor's inequality for the function.

$$f(x) = 7\sin x + 5$$

Solution:

Modify Taylor's inequality to state that

If $\left| f^{n+1}(x) \right| \le M$ for $|x| \le d$, then $\left| R_n(x) \right| \le \dfrac{M}{(n+1)!}|x|^{n+1}$ for $|x| \le d$.

The function $y = \sin x$ is represented by

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

So $f(x) = 7\sin x + 5$ is represented by

$$7\sin x + 5 = 5 + \sum_{n=0}^{\infty} \frac{(-1)^n 7x^{2n+1}}{(2n+1)!}$$

Take the first few derivatives of $f(x) = 7\sin x + 5$ in order to find a value for $f^{(n+1)}(x)$.

$$n = 0 \qquad f^{n+1}(x) = f^{0+1}(x) = f^{1}(x) = f'(x) = 7\cos x$$

$$n = 1 \qquad f^{n+1}(x) = f^{1+1}(x) = f^2(x) = f''(x) = -7\sin x$$

$$n = 2 \qquad f^{n+1}(x) = f^{2+1}(x) = f^3(x) = f'''(x) = -7\cos x$$

$$n = 3 \qquad f^{n+1}(x) = f^{3+1}(x) = f^4(x) = f''''(x) = 7\sin x$$

$$n = 4 \qquad f^{n+1}(x) = f^{4+1}(x) = f^5(x) = f'''''(x) = 7\cos x$$

Then

$$\left| f^{(n+1)}(x) \right| \le 7\cos x \text{ or } \left| f^{(n+1)}(x) \right| \le 7\sin x$$

Since both $\cos x$ and $\sin x$ exist only between -1 and 1, both $7\cos x$ and $7\sin x$ exist only between -7 and 7.

$$-7 \le \left| f^{(n+1)}(x) \right| \le 7$$

But the absolute value in the inequality requires only positive values, so

$$0 \le \left| f^{(n+1)}(x) \right| \le 7$$

■ 4. Find Taylor's inequality for the function.

$$f(x) = \pi \cos x$$

Solution:

Modify Taylor's inequality to state that

If $\left|f^{n+1}(x)\right| \leq M$ for $|x| \leq d$, then $\left|R_n(x)\right| \leq \frac{M}{(n+1)!}|x|^{n+1}$ for $|x| \leq d$.

The function $y = \cos x$ is represented by

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

So $f(x) = \pi \cos x$ is represented by

$$\pi \cos x = \sum_{n=0}^{\infty} \frac{(-1)^n \pi x^{2n}}{(2n)!}$$

Take the first few derivatives of $f(x) = \pi \cos x$ in order to find a value for $f^{(n+1)}(x)$.

$$n = 0 \qquad f^{n+1}(x) = f^{0+1}(x) = f^{1}(x) = f'(x) = -\pi \sin x$$

$$n = 1 \qquad f^{n+1}(x) = f^{1+1}(x) = f^{2}(x) = f''(x) = -\pi \cos x$$

$$n = 2 \qquad f^{n+1}(x) = f^{2+1}(x) = f^{3}(x) = f'''(x) = \pi \sin x$$

$$n = 3 \qquad f^{n+1}(x) = f^{3+1}(x) = f^{4}(x) = f''''(x) = \pi \cos x$$

$$n = 4 \qquad f^{n+1}(x) = f^{4+1}(x) = f^{5}(x) = f'''''(x) = -\pi \sin x$$

Then

$\left|f^{(n+1)}(x)\right| \leq \pi \cos x$ or $\left|f^{(n+1)}(x)\right| \leq \pi \sin x$

Since both $\cos x$ and $\sin x$ exist only between -1 and 1, both $\pi \cos x$ and $\pi \sin x$ exist only between $-\pi$ and π.

$$-\pi \le \left| f^{(n+1)}(x) \right| \le \pi$$

But the absolute value in the inequality requires only positive values, so

$$0 \le \left| f^{(n+1)}(x) \right| \le \pi$$

■ 5. Find Taylor's inequality for the function.

$$f(x) = e \sin x$$

Solution:

Modify Taylor's inequality to state that

If $\left| f^{n+1}(x) \right| \le M$ for $|x| \le d$, then $\left| R_n(x) \right| \le \dfrac{M}{(n+1)!} |x|^{n+1}$ for $|x| \le d$.

The function $y = \sin x$ is represented by

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

So $f(x) = e \sin x$ is represented by

$$e \sin x = \sum_{n=0}^{\infty} \frac{(-1)^n e x^{2n+1}}{(2n+1)!}$$

Take the first few derivatives of $f(x) = e \sin x$ in order to find a value for $f^{(n+1)}(x)$.

$$n = 0 \qquad f^{n+1}(x) = f^{0+1}(x) = f^1(x) = f'(x) = e\cos x$$

$$n = 1 \qquad f^{n+1}(x) = f^{1+1}(x) = f^2(x) = f''(x) = -e\sin x$$

$$n = 2 \qquad f^{n+1}(x) = f^{2+1}(x) = f^3(x) = f'''(x) = -e\cos x$$

$$n = 3 \qquad f^{n+1}(x) = f^{3+1}(x) = f^4(x) = f''''(x) = e\sin x$$

$$n = 4 \qquad f^{n+1}(x) = f^{4+1}(x) = f^5(x) = f'''''(x) = e\cos x$$

Then

$$\left|f^{(n+1)}(x)\right| \le e\cos x \text{ or } \left|f^{(n+1)}(x)\right| \le e\sin x$$

Since both $\cos x$ and $\sin x$ exist only between -1 and 1, both $e\cos x$ and $e\sin x$ exist only between $-e$ and e.

$$-e \le \left|f^{(n+1)}(x)\right| \le e$$

But the absolute value in the inequality requires only positive values, so

$$0 \le \left|f^{(n+1)}(x)\right| \le e$$

MACLAURIN SERIES

■ 1. Write the first four non-zero terms of the Maclaurin series and use it to estimate $f(\pi/9)$.

$$f(x) = \cos(3x)$$

Solution:

The common Maclaurin series for $f(x) = \cos x$ is

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots + (-1)^n \frac{x^{2n}}{(2n)!} + \cdots$$

So the Maclaurin series for $f(x) = \cos(3x)$ is

$$\cos(3x) = 1 - \frac{(3x)^2}{2!} + \frac{(3x)^4}{4!} - \frac{(3x)^6}{6!} + \cdots + (-1)^n \frac{(3x)^{2n}}{(2n)!} + \cdots$$

$$\cos(3x) = 1 - \frac{9x^2}{2} + \frac{81x^4}{24} - \frac{729x^6}{720} + \cdots + (-1)^n \frac{(3x)^{2n}}{(2n)!} + \cdots$$

$$\cos(3x) = 1 - \frac{9x^2}{2} + \frac{27x^4}{8} - \frac{81x^6}{80} + \cdots + (-1)^n \frac{(3x)^{2n}}{(2n)!} + \cdots$$

Then $f(\pi/9)$ is

$$\cos\left(3 \cdot \frac{\pi}{9}\right) = 1 - \frac{9\left(\frac{\pi}{9}\right)^2}{2} + \frac{27\left(\frac{\pi}{9}\right)^4}{8} - \frac{81\left(\frac{\pi}{9}\right)^6}{80}$$

$$\cos\left(\frac{\pi}{3}\right) \approx 1 - 0.548311 + 0.050108 - 0.001832$$

$$\cos\left(\frac{\pi}{3}\right) \approx 0.499965$$

$$\cos\left(\frac{\pi}{3}\right) \approx 0.500$$

■ 2. Write the first three non-zero terms of the Maclaurin series and use it to estimate $f(2\pi/3)$.

$$f(x) = \cos^2 x$$

Solution:

Find the first few terms of the series.

$n = 0$	$f(x) = \cos^2 x$	$f(0) = 1$
$n = 1$	$f'(x) = -2\cos x \sin x$	$f'(0) = 0$
$n = 2$	$f''(x) = 2 - 4\cos^2 x$	$f''(0) = -2$
$n = 3$	$f'''(x) = 8\sin x \cos x$	$f'''(0) = 0$
$n = 4$	$f^{(4)}(x) = 16\cos^2 x - 8$	$f^{(4)}(0) = 8$

So the Maclaurin series for $f(x) = \cos^2 x$ is

$$f(x) = \frac{f(0)x^0}{0!} + \frac{f'(0)x^1}{1!} + \frac{f''(0)x^2}{2!} + \frac{f'''(0)x^3}{3!} + \cdots + \frac{f^{(n)}(0)x^n}{n!}$$

$$f_{(4)}(x) = \frac{f(0)x^0}{0!} + \frac{f'(0)x^1}{1!} + \frac{f''(0)x^2}{2!} + \frac{f'''(0)x^3}{3!} + \frac{f^{(4)}(0)x^4}{4!}$$

$$f_{(4)}(x) = \frac{1x^0}{0!} + \frac{0 \cdot x^1}{1!} + \frac{-2 \cdot x^2}{2!} + \frac{0 \cdot x^3}{3!} + \frac{8 \cdot x^4}{4!}$$

$$f_{(4)}(x) = 1 + 0x - x^2 + 0x^3 + \frac{x^4}{3}$$

$$f_{(4)}(x) = 1 - x^2 + \frac{x^4}{3}$$

Then $f(2\pi/3)$ is

$$f\left(\frac{2\pi}{3}\right) \approx 1 - \left(\frac{2\pi}{3}\right)^2 + \frac{\left(\frac{2\pi}{3}\right)^4}{3}$$

$$f\left(\frac{2\pi}{3}\right) \approx 1 - 4.386491 + 6.413767$$

$$f\left(\frac{2\pi}{3}\right) \approx 3.027276$$

■ 3. Write the first four non-zero terms of the Maclaurin series and use it to estimate $f(2)$.

$$f(x) = (x+4)^{\frac{3}{2}}$$

Solution:

Find the first few terms of the series.

$n=0$ $\quad f(x)=(x+4)^{\frac{3}{2}}$ $\quad f(0)=(0+4)^{\frac{3}{2}}=8$

$n=1$ $\quad f'(x)=\dfrac{3(x+4)^{\frac{1}{2}}}{2}$ $\quad f'(0)=\dfrac{3(0+4)^{\frac{1}{2}}}{2}=3$

$n=2$ $\quad f''(x)=\dfrac{3}{4(x+4)^{\frac{1}{2}}}$ $\quad f''(0)=\dfrac{3}{4(0+4)^{\frac{1}{2}}}=\dfrac{3}{8}$

$n=3$ $\quad f'''(x)=\dfrac{-3}{8(x+4)^{\frac{3}{2}}}$ $\quad f'''(0)=\dfrac{-3}{8(0+4)^{\frac{3}{2}}}=-\dfrac{3}{64}$

$n=4$ $\quad f^{(4)}(x)=\dfrac{9}{16(x+4)^{\frac{5}{2}}}$ $\quad f^{(4)}(0)=\dfrac{9}{16(0+4)^{\frac{5}{2}}}=\dfrac{9}{512}$

So the Maclaurin series for $f(x)=(x+4)^{\frac{3}{2}}$ is

$$f(x)=\frac{f(0)x^0}{0!}+\frac{f'(0)x^1}{1!}+\frac{f''(0)x^2}{2!}+\frac{f'''(0)x^3}{3!}+\cdots+\frac{f^{(n)}(0)x^n}{n!}$$

$$f_{(4)}(x)=\frac{f(0)x^0}{0!}+\frac{f'(0)x^1}{1!}+\frac{f''(0)x^2}{2!}+\frac{f'''(0)x^3}{3!}+\frac{f^{(4)}(0)x^4}{4!}$$

$$f_{(4)}(x)=\frac{8x^0}{0!}+\frac{3\cdot x^1}{1!}+\frac{3\cdot x^2}{8\cdot 2!}+\frac{-3\cdot x^3}{64\cdot 3!}+\frac{9\cdot x^4}{512\cdot 4!}$$

$$f_{(4)}(x)=8+3x+\frac{3x^2}{16}-\frac{x^3}{128}+\frac{3x^4}{4{,}096}$$

$$f_{(3)}(x) = 8 + 3x + \frac{3x^2}{16} - \frac{x^3}{128}$$

Then $f(2)$ is

$$f(2) \approx 8 + 3x + \frac{3x^2}{16} - \frac{x^3}{128}$$

$$f(2) \approx 8 + 3(2) + \frac{3(2)^2}{16} - \frac{(2)^3}{128}$$

$$f(2) \approx 8 + 6 + \frac{12}{16} - \frac{8}{128}$$

$$f(2) \approx 14.6875$$

SUM OF THE MACLAURIN SERIES

1. Find the sum of the Maclaurin series.

$$\sum_{n=0}^{\infty} \frac{7(x+4)^n}{n!}$$

Solution:

Begin with the common series

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

and then start manipulating it until it matches the given series.

$$e^{x+4} = \sum_{n=0}^{\infty} \frac{(x+4)^n}{n!}$$

$$7e^{x+4} = \sum_{n=0}^{\infty} \frac{7(x+4)^n}{n!}$$

So the sum of the series is $7e^{x+4}$.

2. Find the sum of the Maclaurin series.

$$\sum_{n=0}^{\infty} \frac{6(-1)^n (x-\pi)^{2n+1}}{7(2n+1)!}$$

Solution:

Begin with the common series

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n (x)^{2n+1}}{(2n+1)!}$$

and then start manipulating it until it matches the given series.

$$\sin(x-\pi) = \sum_{n=0}^{\infty} \frac{(-1)^n (x-\pi)^{2n+1}}{(2n+1)!}$$

$$\frac{6}{7}\sin(x-\pi) = \sum_{n=0}^{\infty} \frac{6(-1)^n (x-\pi)^{2n+1}}{7(2n+1)!}$$

So the sum of the series is $(6/7)\sin(x-\pi)$.

■ 3. Find the sum of the Maclaurin series.

$$4 + \sum_{n=0}^{\infty} \frac{e(-1)^n (x+\pi)^{2n}}{3(2n)!}$$

Solution:

Begin with the common series

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$$

and then start manipulating it until it matches the given series.

$$\cos(x+\pi) = \sum_{n=0}^{\infty} \frac{(-1)^n (x+\pi)^{2n}}{(2n)!}$$

$$\frac{e}{3}\cos(x+\pi) = \sum_{n=0}^{\infty} \frac{e(-1)^n (x+\pi)^{2n}}{3(2n)!}$$

$$4 + \frac{e}{3}\cos(x+\pi) = 4 + \sum_{n=0}^{\infty} \frac{e(-1)^n (x+\pi)^{2n}}{3(2n)!}$$

So the sum of the series is

$$4 + \frac{e}{3}\cos(x+\pi)$$

RADIUS AND INTERVAL OF CONVERGENCE OF A MACLAURIN SERIES

1. Find the radius of convergence of the Maclaurin series.

$$f(x) = \frac{5}{1-x^3}$$

Solution:

Start with the common series.

$$\frac{1}{1-x} = \sum_{n=1}^{\infty} x^n$$

and then start manipulating it until it matches the given series.

$$\frac{5}{1-x} = 5\sum_{n=1}^{\infty} x^n = \sum_{n=1}^{\infty} 5x^n$$

$$\frac{5}{1-x^3} = \sum_{n=1}^{\infty} 5(x^3)^n = \sum_{n=1}^{\infty} 5x^{3n}$$

Use the ratio test.

$$L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$$

$$L = \lim_{n\to\infty} \left| \frac{5x^{3(n+1)}}{5x^{3n}} \right|$$

$$L = \lim_{n\to\infty} \left| \frac{5x^{3n+3}}{5x^{3n}} \right|$$

$$L = \lim_{n\to\infty} \left| x^3 \right| < 1$$

So the interval of convergence is

$$-1 < x^3 < 1$$

$$\sqrt[3]{-1} < x < \sqrt[3]{1}$$

$$-1 < x < 1$$

The interval of convergence spans from -1 to 1, which is 2 units wide. The radius of convergence is half that, which means the radius of convergence is 1.

■ 2. Find the radius of convergence of the Maclaurin series.

$$f(x) = 4\cos(x^2)$$

Solution:

Start with the common series.

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

and then start manipulating it until it matches the given series.

$$4\cos x = \sum_{n=0}^{\infty} \frac{4(-1)^n x^{2n}}{(2n)!}$$

$$4\cos(x^2) = \sum_{n=0}^{\infty} \frac{4(-1)^n (x^2)^{2n}}{(2n)!}$$

$$4\cos(x^2) = \sum_{n=0}^{\infty} \frac{4(-1)^n x^{4n}}{(2n)!}$$

Use the ratio test.

$$L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$$

$$L = \lim_{n\to\infty} \left| \frac{\frac{4(-1)^{n+1}x^{4(n+1)}}{(2(n+1))!}}{\frac{4(-1)^n x^{4n}}{(2n)!}} \right|$$

$$L = \lim_{n\to\infty} \left| \frac{\frac{4x^{4(n+1)}}{(2(n+1))!}}{\frac{4x^{4n}}{(2n)!}} \right|$$

$$L = \lim_{n\to\infty} \left| \frac{\frac{x^{4(n+1)}}{(2(n+1))!}}{\frac{x^{4n}}{(2n)!}} \right|$$

$$L = \lim_{n\to\infty} \left| \frac{x^{4n+4}}{(2n+2)!} \cdot \frac{(2n)!}{x^{4n}} \right|$$

$$L = \lim_{n\to\infty} \left| \frac{x^4}{(2n+2)(2n+1)(2n)!} \cdot \frac{(2n)!}{1} \right|$$

$$L = \lim_{n\to\infty} \left| \frac{x^4}{(2n+2)(2n+1)} \cdot \frac{1}{1} \right|$$

$$L = \lim_{n\to\infty} \left| \frac{x^4}{(2n+2)(2n+1)} \right|$$

$$L = x^4 \lim_{n\to\infty} \left| \frac{1}{(2n+2)(2n+1)} \right|$$

$$L = x^4 \cdot 0$$

$$L = 0$$

The series converges if $L < 1$ and diverges if $L > 1$, so this series converges everywhere, which means the interval of convergence is ∞, and therefore the radius of conversion is ∞, too.

■ 3. Find the radius of convergence of the Maclaurin series.

$$\sum_{n=1}^{\infty} \frac{x^n \cdot 3^n}{n}$$

Solution:

Use the ratio test.

$$L = \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right|$$

$$L = \lim_{n \to \infty} \left| \frac{\frac{x^{n+1} \cdot 3^{n+1}}{n+1}}{\frac{x^n \cdot 3^n}{n}} \right|$$

$$L = \lim_{n \to \infty} \left| \frac{x^{n+1} \cdot 3^{n+1}}{n+1} \cdot \frac{n}{x^n \cdot 3^n} \right|$$

$$L = \lim_{n \to \infty} \left| \frac{x \cdot 3}{n+1} \cdot \frac{n}{1} \right|$$

$$L = \left| 3x \right| \lim_{n \to \infty} \left| \frac{1}{n+1} \cdot \frac{n}{1} \right|$$

$$L = \left| 3x \right| \lim_{n \to \infty} \left| \frac{n}{n+1} \right|$$

$$L = \left| 3x \right| \cdot 1$$

$$L = \left| 3x \right|$$

So the interval of convergence is

$$|3x| < 1$$

$$-1 < 3x < 1$$

$$-\frac{1}{3} < x < \frac{1}{3}$$

The interval of convergence spans from $-1/3$ to $1/3$, which is a width of $2/3$. The radius of convergence is half that, so the radius of convergence is $1/3$.

INDEFINITE INTEGRAL AS AN INFINITE SERIES

■ 1. Use an infinite series to evaluate the indefinite integral.

$$\int x^2 \cos(x^3)\ dx$$

Solution:

Start with the known Maclaurin series

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

and then start manipulating it until it matches the given series, first by substituting x^3 for x,

$$\cos x^3 = \sum_{n=0}^{\infty} \frac{(-1)^n (x^3)^{2n}}{(2n)!}$$

$$\cos x^3 = \sum_{n=0}^{\infty} \frac{(-1)^n x^{6n}}{(2n)!}$$

and then multiplying by x^2.

$$x^2 \cos x^3 = x^2 \sum_{n=0}^{\infty} \frac{(-1)^n x^{6n}}{(2n)!}$$

$$x^2 \cos x^3 = \sum_{n=0}^{\infty} \frac{(-1)^n x^2 \cdot x^{6n}}{(2n)!}$$

$$x^2 \cos x^3 = \sum_{n=0}^{\infty} \frac{(-1)^n x^{6n+2}}{(2n)!}$$

Integrate the series.

$$\int x^2 \cos(x^3)\ dx = \int \sum_{n=0}^{\infty} \frac{(-1)^n x^{6n+2}}{(2n)!}\ dx$$

$$\int x^2 \cos(x^3)\ dx = \frac{(-1)^n}{(2n)!} \int \sum_{n=0}^{\infty} x^{6n+2}\ dx$$

$$\int x^2 \cos(x^3)\ dx = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \cdot \frac{x^{6n+3}}{6n+3} + C$$

$$\int x^2 \cos(x^3)\ dx = C + \sum_{n=0}^{\infty} \frac{(-1)^n x^{6n+3}}{(6n+3)(2n)!}$$

■ 2. Use an infinite series to evaluate the indefinite integral.

$$\int 4x^3 \sin(x^4)\ dx$$

Solution:

Start with the known Maclaurin series

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

and then start manipulating it until it matches the given series, first by substituting x^4 for x,

$$\sin(x^4) = \sum_{n=0}^{\infty} \frac{(-1)^n (x^4)^{2n+1}}{(2n+1)!}$$

$$\sin(x^4) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{8n+4}}{(2n+1)!}$$

and then multiplying by $4x^3$.

$$4x^3 \sin(x^4) = \sum_{n=0}^{\infty} \frac{4x^3(-1)^n x^{8n+4}}{(2n+1)!}$$

$$4x^3 \sin(x^4) = \sum_{n=0}^{\infty} \frac{4(-1)^n x^{8n+7}}{(2n+1)!}$$

Integrate the series.

$$\int 4x^3 \sin(x^4)\ dx = \int \sum_{n=0}^{\infty} \frac{4(-1)^n x^{8n+7}}{(2n+1)!}\ dx$$

$$\int 4x^3 \sin(x^4)\ dx = \frac{4(-1)^n}{(2n+1)!} \int \sum_{n=0}^{\infty} x^{8n+7}\ dx$$

$$\int 4x^3 \sin(x^4)\ dx = \sum_{n=0}^{\infty} \frac{4(-1)^n}{(2n+1)!} \cdot \frac{x^{8n+8}}{8n+8} + C$$

$$\int 4x^3 \sin(x^4)\ dx = C + \sum_{n=0}^{\infty} \frac{4(-1)^n x^{8n+8}}{(8n+8)(2n+1)!}$$

■ 3. Use an infinite series to evaluate the indefinite integral.

$$\int 2x \ln(1+x^2)\ dx$$

Solution:

Start with the known Maclaurin series

$$\ln(1+x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n}$$

and then start manipulating it until it matches the given series, first by substituting x^2 for x,

$$\ln(1+x^2) = \sum_{n=0}^{\infty} \frac{(-1)^n (x^2)^n}{n}$$

$$\ln(1+x^2) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n}$$

and then multiplying by $2x$.

$$2x \ln(1+x^2) = 2x \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n}$$

$$2x\ln(1+x^2)=\sum_{n=0}^{\infty}\frac{2x^1(-1)^n x^{2n}}{n}$$

$$2x\ln(1+x^2)=\sum_{n=0}^{\infty}\frac{2(-1)^n x^{2n+1}}{n}$$

Integrate the series.

$$\int 2x\ln(1+x^2)\ dx=\int\sum_{n=0}^{\infty}\frac{2(-1)^n x^{2n+1}}{n}\ dx$$

$$\int 2x\ln(1+x^2)\ dx=\frac{2(-1)^n}{n}\int\sum_{n=0}^{\infty}x^{2n+1}\ dx$$

$$\int 2x\ln(1+x^2)\ dx=\sum_{n=0}^{\infty}\frac{2(-1)^n}{n}\cdot\frac{x^{2n+2}}{2n+2}+C$$

$$\int 2x\ln(1+x^2)\ dx=C+\sum_{n=0}^{\infty}\frac{2(-1)^n x^{2n+2}}{2n^2+2n}$$

MACLAURIN SERIES TO ESTIMATE AN INDEFINITE INTEGRAL

1. Use a Maclaurin series to estimate the indefinite integral.

$$\int \frac{\sin(2x)}{4x}\ dx$$

Solution:

Begin with the common series

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n (x)^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \cdots$$

and then start manipulating it until it matches the given series, first by substituting $2x$ for x,

$$\sin(2x) = \sum_{n=0}^{\infty} \frac{(-1)^n (2x)^{2n+1}}{(2n+1)!} = 2x - \frac{(2x)^3}{3!} + \frac{(2x)^5}{5!} - \frac{(2x)^7}{7!} + \frac{(2x)^9}{9!} - \cdots$$

$$\sin(2x) = \sum_{n=0}^{\infty} \frac{(-1)^n (2x)^{2n+1}}{(2n+1)!} = 2x - \frac{8x^3}{6} + \frac{32x^5}{120} - \frac{128x^7}{5{,}040} + \frac{512x^9}{362{,}880} - \cdots$$

$$\sin(2x) = \sum_{n=0}^{\infty} \frac{(-1)^n (2x)^{2n+1}}{(2n+1)!} = 2x - \frac{4x^3}{3} + \frac{4x^5}{15} - \frac{8x^7}{315} + \frac{4x^9}{2{,}835} - \cdots$$

and then dividing by $4x$.

$$\frac{\sin(2x)}{4x} = \sum_{n=0}^{\infty} \frac{(-1)^n(2x)^{2n+1}}{4x(2n+1)!} = \frac{2x}{4x} - \frac{4x^3}{3 \cdot 4x} + \frac{4x^5}{15 \cdot 4x} - \frac{8x^7}{315 \cdot 4x} + \frac{4x^9}{2,835 \cdot 4x} - \cdots$$

$$\frac{\sin(2x)}{4x} = \sum_{n=0}^{\infty} \frac{(-1)^n(2x)^{2n+1}}{4x(2n+1)!} = \frac{1}{2} - \frac{x^2}{3} + \frac{x^4}{15} - \frac{2x^6}{315} + \frac{x^8}{2,835} - \cdots$$

$$\frac{\sin(2x)}{4x} = \sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n+1} x^{2n+1}}{2^2 x^1 (2n+1)!} = \frac{1}{2} - \frac{x^2}{3} + \frac{x^4}{15} - \frac{2x^6}{315} + \frac{x^8}{2,835} - \cdots$$

$$\frac{\sin(2x)}{4x} = \sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n-1} x^{2n}}{(2n+1)!} = \frac{1}{2} - \frac{x^2}{3} + \frac{x^4}{15} - \frac{2x^6}{315} + \frac{x^8}{2,835} - \cdots$$

Integrate the series term by term.

$$\int \frac{\sin(2x)}{4x} \, dx = \int \frac{1}{2} - \frac{x^2}{3} + \frac{x^4}{15} - \frac{2x^6}{315} + \frac{x^8}{2,835} - \ldots \, dx$$

$$\int \frac{\sin(2x)}{4x} \, dx = C + \frac{x}{2} - \frac{x^{2+1}}{3 \cdot 3} + \frac{x^{4+1}}{15 \cdot 5} - \frac{2x^{6+1}}{315 \cdot 7} + \frac{x^{8+1}}{2,835 \cdot 9} - \ldots$$

$$\int \frac{\sin(2x)}{4x} \, dx = C + \frac{x}{2} - \frac{x^3}{9} + \frac{x^5}{75} - \frac{2x^7}{2,205} + \frac{x^9}{25,515} - \ldots$$

Then the indefinite integral can be expressed as

$$\int \frac{\sin(2x)}{4x} \, dx = \sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n-1} x^{2n+1}}{(2n+1)(2n+1)!} + C$$

■ 2. Use a Maclaurin series to estimate the indefinite integral.

$$\int \frac{\cos x}{x^2}\, dx$$

Solution:

Begin with the common series

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \dots$$

and then start manipulating it until it matches the given series, by dividing through by x^2.

$$\frac{\cos x}{x^2} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{x^2 (2n)!} = \frac{1}{x^2} - \frac{x^2}{2!x^2} + \frac{x^4}{4!x^2} - \frac{x^6}{6!x^2} + \frac{x^8}{8!x^2} - \cdots$$

$$\frac{\cos x}{x^2} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n-2}}{(2n)!} = \frac{1}{x^2} - \frac{1}{2} + \frac{x^2}{24} - \frac{x^4}{720} + \frac{x^6}{40{,}320} - \cdots$$

Integrate the series term by term.

$$\int \frac{\cos x}{x^2}\, dx = \int \frac{1}{x^2} - \frac{1}{2} + \frac{x^2}{24} - \frac{x^4}{720} + \frac{x^6}{40{,}320} - \dots\, dx$$

$$\int \frac{\cos x}{x^2}\, dx = \int x^{-2} - \frac{1}{2} + \frac{x^2}{24} - \frac{x^4}{720} + \frac{x^6}{40{,}320} - \dots\, dx$$

$$\int \frac{\cos x}{x^2}\, dx = C - \frac{1}{x} - \frac{x}{2} + \frac{x^3}{24 \cdot 3} - \frac{x^5}{720 \cdot 5} + \frac{x^7}{40{,}320 \cdot 7} - \dots$$

$$\int \frac{\cos x}{x^2}\,dx = C - \frac{1}{x} - \frac{x}{2} + \frac{x^3}{72} - \frac{x^5}{3{,}600} + \frac{x^7}{282{,}240} - \dots$$

Then the indefinite integral can be expressed as this series:

$$\int \frac{\cos x}{x^2}\,dx = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n-1}}{(2n-1)(2n)!} + C$$

■ 3. Use a Maclaurin series to estimate the indefinite integral.

$$\int \frac{\arctan x}{x^2}\,dx$$

Solution:

Begin with the common series

$$\arctan x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \dots$$

and then start manipulating it until it matches the given series, by dividing through by x^2.

$$\frac{\arctan x}{x^2} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)x^2} = \frac{x}{x^2} - \frac{x^3}{3x^2} + \frac{x^5}{5x^2} - \frac{x^7}{7x^2} + \frac{x^9}{9x^2} - \cdots$$

$$\frac{\arctan x}{x^2} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n-1}}{(2n+1)} = \frac{1}{x} - \frac{x}{3} + \frac{x^3}{5} - \frac{x^5}{7} + \frac{x^7}{9} - \cdots$$

Integrate the series term by term.

$$\int \frac{\arctan x}{x^2}\ dx = \int \frac{1}{x} - \frac{x}{3} + \frac{x^3}{5} - \frac{x^5}{7} + \frac{x^7}{9} - \dots\ dx$$

$$\int \frac{\arctan x}{x^2}\ dx = C + \ln|x| - \frac{x^2}{3 \cdot 2} + \frac{x^4}{5 \cdot 4} - \frac{x^6}{7 \cdot 6} + \frac{x^8}{9 \cdot 8} - \dots$$

$$\int \frac{\arctan x}{x^2}\ dx = C + \ln|x| - \frac{x^2}{6} + \frac{x^4}{20} - \frac{x^6}{42} + \frac{x^8}{72} - \dots$$

Then the indefinite integral can be expressed as this series:

$$\int \frac{\arctan x}{x^2} dx = C + \ln|x| + \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n}}{2n(2n+1)}$$

MACLAURIN SERIES TO ESTIMATE A DEFINITE INTEGRAL

1. Use a Maclaurin series to estimate the value of the definite integral.

$$\int_0^3 3xe^{\frac{1}{2}x^2}\ dx$$

Solution:

Start with the common series

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

and then start manipulating it until it matches the given series, first by substituting $(1/2)x^2$ for x,

$$e^{\frac{1}{2}x^2} = \sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}x^2\right)^n}{n!} = \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n \frac{x^{2n}}{n!} = \sum_{n=0}^{\infty} \frac{x^{2n}}{2^n n!}$$

and then multiplying by $3x$.

$$3xe^{\frac{1}{2}x^2} = 3x\sum_{n=0}^{\infty} \frac{x^{2n}}{2^n n!} = \sum_{n=0}^{\infty} \frac{3x^1 \cdot x^{2n}}{2^n n!} = \sum_{n=0}^{\infty} \frac{3x^{2n+1}}{2^n n!}$$

Integrate the power series.

$$\int_0^3 3xe^{\frac{1}{2}x^2}\ dx = \int_0^3 \sum_{n=0}^{\infty} \frac{3x^{2n+1}}{2^n n!}\ dx$$

$$\int_0^3 3xe^{\frac{1}{2}x^2}\ dx = \frac{3}{2^n n!} \int_0^3 \sum_{n=0}^{\infty} x^{2n+1}\ dx$$

$$\int_0^3 3xe^{\frac{1}{2}x^2}\ dx = \sum_{n=0}^{\infty} \frac{3}{2^n n!} \cdot \frac{x^{2n+2}}{2n+2} \Bigg|_0^3$$

Expand the power series through its first eight terms.

$$\int_0^3 3xe^{\frac{1}{2}x^2}\ dx = \frac{3x^2}{2} + \frac{3x^4}{8} + \frac{3x^6}{48} + \frac{3x^8}{384} + \frac{3x^{10}}{3{,}840}$$

$$+\frac{3x^{12}}{46{,}080} + \frac{3x^{14}}{645{,}120} + \frac{3x^{16}}{10{,}321{,}920} \Bigg|_0^3$$

$$\int_0^3 3xe^{\frac{1}{2}x^2}\ dx = \frac{3(3)^2}{2} + \frac{3(3)^4}{8} + \frac{3(3)^6}{48} + \frac{3(3)^8}{384} + \frac{3(3)^{10}}{3{,}840}$$

$$+\frac{3(3)^{12}}{46{,}080} + \frac{3(3)^{14}}{645{,}120} + \frac{3(3)^{16}}{10{,}321{,}920}$$

$$-\left(\frac{3(0)^2}{2} + \frac{3(0)^4}{8} + \frac{3(0)^6}{48} + \frac{3(0)^8}{384} + \frac{3(0)^{10}}{3{,}840}\right.$$

$$\left.+\frac{3(0)^{12}}{46{,}080} + \frac{3(0)^{14}}{645{,}120} \frac{3(0)^{16}}{10{,}321{,}920}\right)$$

$$\int_0^3 3xe^{\frac{1}{2}x^2}\ dx \approx 13.5 + 30.375 + 45.5625 + 51.257813 + 46.132031$$

$$+34.599023 + 22.242229 + 12.511254$$

$$\int_0^3 3xe^{\frac{1}{2}x^2}\ dx \approx 256.180$$

■ 2. Use a Maclaurin series to estimate the value of the definite integral.

$$\int_0^{\sqrt{\pi/2}} 12\cos(x^2)\ dx$$

Solution:

Start with the common series

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

and then start manipulating it until it matches the given series, first by substituting x^2 for x,

$$\cos(x^2) = \sum_{n=0}^{\infty} \frac{(-1)^n (x^2)^{2n}}{(2n)!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n}}{(2n)!}$$

and then multiplying by 12.

$$12\cos(x^2)=12\sum_{n=0}^{\infty}\frac{(-1)^n x^{4n}}{(2n)!}=\sum_{n=0}^{\infty}\frac{12\cdot(-1)^n x^{4n}}{(2n)!}$$

Integrate the power series.

$$\int_0^{\sqrt{\pi/2}} 12\cos(x^2)\ dx=\int_0^{\sqrt{\pi/2}}\sum_{n=0}^{\infty}\frac{12\cdot(-1)^n x^{4n}}{(2n)!}\ dx$$

$$\int_0^{\sqrt{\pi/2}} 12\cos(x^2)\ dx=\frac{12\cdot(-1)^n}{(2n)!}\int_0^{\sqrt{\pi/2}}\sum_{n=0}^{\infty}x^{4n}\ dx$$

$$\int_0^{\sqrt{\pi/2}} 12\cos(x^2)\ dx=\frac{12\cdot(-1)^n}{(2n)!}\cdot\frac{x^{4n+1}}{4n+1}\Bigg|_0^{\sqrt{\pi/2}}$$

Expand the power series through its first six terms.

$$\int_0^{\sqrt{\pi/2}} 12\cos(x^2)\ dx=\frac{12(-1)^0}{(2\cdot 0)!}\cdot\frac{x^{4(0)+1}}{4(0)+1}+\frac{12(-1)^1}{(2\cdot 1)!}\cdot\frac{x^{4(1)+1}}{4(1)+1}$$

$$+\frac{12(-1)^2}{(2\cdot 2)!}\cdot\frac{x^{4(2)+1}}{4(2)+1}+\frac{12(-1)^3}{(2\cdot 3)!}\cdot\frac{x^{4(3)+1}}{4(3)+1}+\frac{12(-1)^4}{(2\cdot 4)!}\cdot\frac{x^{4(4)+1}}{4(4)+1}$$

$$+\frac{12(-1)^5}{(2\cdot 5)!}\cdot\frac{x^{4(5)+1}}{4(5)+1}\Bigg|_0^{\sqrt{\pi/2}}$$

$$\int_0^{\sqrt{\pi/2}} 12\cos(x^2)\ dx=\frac{12x^1}{1}-\frac{12x^5}{10}+\frac{12x^9}{216}-\frac{12x^{13}}{9{,}360}+\frac{x^{17}}{57{,}120}-\frac{12x^{21}}{76{,}204{,}800}\Bigg|_0^{\sqrt{\pi/2}}$$

$$\int_0^{\sqrt{\pi/2}} 12\cos(x^2)\ dx=12x-\frac{6x^5}{5}+\frac{x^9}{18}-\frac{x^{13}}{780}+\frac{x^{17}}{57{,}120}-\frac{x^{21}}{6{,}350{,}400}\Bigg|_0^{\sqrt{\pi/2}}$$

$$\int_0^{\sqrt{\pi/2}} 12\cos(x^2)\ dx$$

$$= 12\sqrt{\pi/2} - \frac{6\sqrt{\pi/2}^5}{5} + \frac{\sqrt{\pi/2}^9}{18} - \frac{\sqrt{\pi/2}^{13}}{780} + \frac{\sqrt{\pi/2}^{17}}{57{,}120} - \frac{\sqrt{\pi/2}^{21}}{6{,}350{,}400}$$

$$-\left(12(0) - \frac{6(0)^5}{5} + \frac{0^9}{18} - \frac{0^{13}}{780} + \frac{0^{17}}{57{,}120} - \frac{0^{21}}{6{,}350{,}400}\right)$$

$$\int_0^{\sqrt{\pi/2}} 12\cos(x^2)\ dx \approx 15.039770 - 3.710914 + 0.423903$$

$$-0.024137 + 0.000813 - 0.000018$$

$$\int_0^{\sqrt{\pi/2}} 12\cos(x^2)\ dx \approx 11.729417$$

3. Use a Maclaurin series to estimate the value of the definite integral.

$$\int_0^{\sqrt[3]{\pi}} 15\sin(x^3)\ dx$$

Solution:

Start with the common series

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

and then start manipulating it until it matches the given series, first by substituting x^3 for x,

$$\sin(x^3) = \sum_{n=0}^{\infty} \frac{(-1)^n (x^3)^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{6n+3}}{(2n+1)!}$$

and then multiplying by 15.

$$15\sin(x^3) = 15\sum_{n=0}^{\infty} \frac{(-1)^n x^{6n+3}}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{15(-1)^n x^{6n+3}}{(2n+1)!}$$

Integrate the power series.

$$\int_0^{\sqrt[3]{\pi}} 15\sin(x^3)\ dx = \int_0^{\sqrt[3]{\pi}} \sum_{n=0}^{\infty} \frac{15(-1)^n x^{6n+3}}{(2n+1)!}\ dx$$

$$\int_0^{\sqrt[3]{\pi}} 15\sin(x^3)\ dx = \frac{15(-1)^n}{(2n+1)!} \int_0^{\sqrt[3]{\pi}} \sum_{n=0}^{\infty} x^{6n+3}\ dx$$

$$\int_0^{\sqrt[3]{\pi}} 15\sin(x^3)\ dx = \frac{15(-1)^n}{(2n+1)!} \cdot \frac{x^{6n+4}}{6n+4}\Bigg|_0^{\sqrt[3]{\pi}}$$

Expand the power series through its first six terms.

$$\int_0^{\sqrt[3]{\pi}} 15\sin(x^3)\ dx = \frac{15(-1)^0}{(2(0)+1)!} \cdot \frac{x^{6(0)+4}}{6(0)+4} + \frac{15(-1)^1}{(2(1)+1)!} \cdot \frac{x^{6(1)+4}}{6(1)+4}$$

$$+\frac{15(-1)^2}{(2(2)+1)!}\cdot\frac{x^{6(2)+4}}{6(2)+4}+\frac{15(-1)^3}{(2(3)+1)!}\cdot\frac{x^{6(3)+4}}{6(3)+4}$$

$$+\frac{15(-1)^4}{(2(4)+1)!}\cdot\frac{x^{6(4)+4}}{6(4)+4}+\frac{15(-1)^5}{(2(5)+1)!}\cdot\frac{x^{6(5)+4}}{6(5)+4}\Bigg|_0^{\sqrt[3]{\pi}}$$

$$\int_0^{\sqrt[3]{\pi}} 15\sin(x^3)\ dx = \frac{15x^4}{4}-\frac{x^{10}}{4}+\frac{x^{16}}{128}-\frac{x^{22}}{7{,}392}+\frac{x^{28}}{677{,}376}-\frac{x^{34}}{90{,}478{,}080}\Bigg|_0^{\sqrt[3]{\pi}}$$

$$\int_0^{\sqrt[3]{\pi}} 15\sin(x^3)\ dx = \frac{15\sqrt[3]{\pi}^4}{4}-\frac{\sqrt[3]{\pi}^{10}}{4}+\frac{\sqrt[3]{\pi}^{16}}{128}-\frac{\sqrt[3]{\pi}^{22}}{7{,}392}+\frac{\sqrt[3]{\pi}^{28}}{677{,}376}-\frac{\sqrt[3]{\pi}^{34}}{90{,}478{,}080}$$

$$-\left(\frac{15(0)^4}{4}-\frac{0^{10}}{4}+\frac{0^{16}}{128}-\frac{0^{22}}{7{,}392}+\frac{0^{28}}{677{,}376}-\frac{0^{34}}{90{,}478{,}080}\right)$$

$$\int_0^{\sqrt[3]{\pi}} 15\sin(x^3)\ dx \approx 17.254317-11.352885+3.501515$$

$$-0.598417+0.064452-0.004762$$

$$\int_0^{\sqrt[3]{\pi}} 15\sin(x^3)\ dx \approx 8.86422$$

MACLAURIN SERIES TO EVALUATE A LIMIT

1. Use a Maclaurin series to evaluate the limit.

$$\lim_{x \to 0} \frac{e^{2x} - 1 - 2x}{x^2}$$

Solution:

The Maclaurin series expansion of e^x is

$$e^x = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \cdots$$

So the Maclaurin series expansion of e^{2x} is

$$e^{2x} = 1 + 2x + \frac{1}{2}(2x)^2 + \frac{1}{6}(2x)^3 + \frac{1}{24}(2x)^4 + \cdots$$

$$e^{2x} = 1 + 2x + \frac{1}{2}(4x^2) + \frac{1}{6}(8x^3) + \frac{1}{24}(16x^4) + \cdots$$

$$e^{2x} = 1 + 2x + 2x^2 + \frac{4}{3}x^3 + \frac{2}{3}x^4 + \cdots$$

Subtract 1 and subtract $2x$.

$$e^{2x} - 1 - 2x = 1 + 2x + 2x^2 + \frac{4}{3}x^3 + \frac{2}{3}x^4 + \cdots - 1 - 2x$$

$$e^{2x} - 1 - 2x = 2x^2 + \frac{4}{3}x^3 + \frac{2}{3}x^4 + \cdots$$

Divide by x^2.

$$\frac{e^{2x}-1-2x}{x^2}=\frac{2x^2}{x^2}+\frac{\frac{4}{3}x^3}{x^2}+\frac{\frac{2}{3}x^4}{x^2}+\cdots$$

$$\frac{e^{2x}-1-2x}{x^2}=2+\frac{4}{3}x+\frac{2}{3}x^2+\cdots$$

Substitute into the given limit and then evaluate.

$$\lim_{x\to 0}\frac{e^{2x}-1-2x}{x^2}=\lim_{x\to 0}\left(2+\frac{4}{3}x+\frac{2}{3}x^2+\cdots\right)$$

$$\lim_{x\to 0}\frac{e^{2x}-1-2x}{x^2}=2+\frac{4}{3}(0)+\frac{2}{3}(0)^2+\cdots$$

$$\lim_{x\to 0}\frac{e^{2x}-1-2x}{x^2}=2$$

■ 2. Use a Maclaurin series to evaluate the limit.

$$\lim_{x\to 0}\frac{\arctan x-x}{x^3}$$

Solution:

The Maclaurin series for the expansion of $\arctan x$ is

$$\arctan x=x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+\frac{x^9}{9}-\cdots$$

Subtract x.

$$\arctan x - x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \cdots - x$$

$$\arctan x - x = -\frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \cdots$$

Divide by x^3.

$$\frac{\arctan x - x}{x^3} = -\frac{x^3}{x^3 \cdot 3} + \frac{x^5}{x^3 \cdot 5} - \frac{x^7}{x^3 \cdot 7} + \frac{x^9}{x^3 \cdot 9} - \cdots$$

$$\frac{\arctan x - x}{x^3} = -\frac{1}{3} + \frac{x^2}{5} - \frac{x^4}{7} + \frac{x^6}{9} - \cdots$$

Substitute into the given limit and then evaluate.

$$\lim_{x \to 0} \frac{\arctan x - x}{x^3} = \lim_{x \to 0} \left(-\frac{1}{3} + \frac{x^2}{5} - \frac{x^4}{7} + \frac{x^6}{9} - \cdots \right)$$

$$\lim_{x \to 0} \frac{\arctan x - x}{x^3} = -\frac{1}{3} + \frac{0^2}{5} - \frac{0^4}{7} + \frac{0^6}{9} - \cdots$$

$$\lim_{x \to 0} \frac{\arctan x - x}{x^3} = -\frac{1}{3}$$

3. Use a Maclaurin series to evaluate the limit.

$$\lim_{x \to 0} \frac{\cos(3x) + \frac{9}{2}x^2 - 1}{x^4}$$

Solution:

The Maclaurin series for the expansion of $\cos x$ is

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots$$

So the Maclaurin series expansion of $\cos(3x)$ is

$$\cos(3x) = 1 - \frac{(3x)^2}{2!} + \frac{(3x)^4}{4!} - \frac{(3x)^6}{6!} + \frac{(3x)^8}{8!} - \cdots$$

$$\cos(3x) = 1 - \frac{9x^2}{2!} + \frac{81x^4}{4!} - \frac{729x^6}{6!} + \frac{6561x^8}{8!} - \cdots$$

Add $(9/2)x^2$ and subtract 1.

$$\cos(3x) + \frac{9}{2}x^2 - 1 = 1 - \frac{9x^2}{2!} + \frac{81x^4}{4!} - \frac{729x^6}{6!} + \frac{6561x^8}{8!} - \cdots + \frac{9}{2}x^2 - 1$$

$$\cos(3x) + \frac{9}{2}x^2 - 1 = \frac{81x^4}{4!} - \frac{729x^6}{6!} + \frac{6561x^8}{8!} - \cdots$$

Divide by x^4.

$$\frac{\cos(3x) + \frac{9}{2}x^2 - 1}{x^4} = \frac{81x^4}{x^4 \cdot 4!} - \frac{729x^6}{x^4 \cdot 6!} + \frac{6561x^8}{x^4 \cdot 8!} - \cdots$$

$$\frac{\cos(3x) + \frac{9}{2}x^2 - 1}{x^4} = \frac{81}{4!} - \frac{729x^2}{6!} + \frac{6561x^4}{8!} - \cdots$$

Substitute the terms into the given limit and then evaluate.

$$\lim_{x\to 0}\frac{\cos(3x)+\frac{9}{2}x^2-1}{x^4}=\lim_{x\to 0}\left(\frac{81}{4!}-\frac{729x^2}{6!}+\frac{6561x^4}{8!}-\cdots\right)$$

$$\lim_{x\to 0}\frac{\cos(3x)+\frac{9}{2}x^2-1}{x^4}=\frac{81}{4!}-\frac{729(0)^2}{6!}+\frac{6561(0)^4}{8!}-\cdots$$

$$\lim_{x\to 0}\frac{\cos(3x)+\frac{9}{2}x^2-1}{x^4}=\frac{81}{4!}$$

$$\lim_{x\to 0}\frac{\cos(3x)+\frac{9}{2}x^2-1}{x^4}=\frac{27}{8}$$

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