Topic: Maclaurin series

Question: Write the first four terms of the maclaurin series and use it to estimate the given value.

$$f(x) = \sqrt{1+x}$$

Find $\sqrt{10}$

Answer choices:

A $\quad 1 + \frac{1}{4}x - \frac{1}{8}x^2 + \frac{1}{6}x^3 \quad$ and $\quad \sqrt{10} = 3.17$

B $\quad 1 + \frac{1}{2}x^2 - \frac{1}{8}x^3 + \frac{1}{16}x^4 \quad$ and $\quad \sqrt{10} = 3.15$

C $\quad 1 + \frac{1}{2}x - \frac{1}{4}x^2 + \frac{1}{8}x^3 \quad$ and $\quad \sqrt{10} = 3.20$

D $\quad 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 \quad$ and $\quad \sqrt{10} = 3.16$

Solution: D

The Maclaurin formula is

$$f(x)=\frac{f(0)x^0}{0!}+\frac{f'(0)x^1}{1!}+\frac{f''(0)x^2}{2!}+\frac{f'''(0)x^3}{3!}+\dots+\frac{f^n(0)x^n}{n!}$$

To find the first four terms, we need the original function, plus the first three derivatives.

$$f(x)=(1+x)^{\frac{1}{2}} \quad \text{and} \quad f(0)=(1+0)^{\frac{1}{2}}=1$$

$$f'(x)=\frac{1}{2}(1+x)^{-\frac{1}{2}} \quad \text{and} \quad f'(0)=\frac{1}{2}(1+0)^{-\frac{1}{2}}=\frac{1}{2}$$

$$f''(x)=-\frac{1}{4}(1+x)^{-\frac{3}{2}} \quad \text{and} \quad f''(0)=-\frac{1}{4}(1+0)^{-\frac{3}{2}}=-\frac{1}{4}$$

$$f'''(x)=\frac{3}{8}(1+x)^{-\frac{5}{2}} \quad \text{and} \quad f'''(0)=\frac{3}{8}(1+0)^{-\frac{5}{2}}=\frac{3}{8}$$

Therefore, the first four terms of the maclaurin series are

$$f^{(3)}(x)=\frac{(1)x^0}{0!}+\frac{\left(\frac{1}{2}\right)x^1}{1!}+\frac{\left(-\frac{1}{4}\right)x^2}{2!}+\frac{\left(\frac{3}{8}\right)x^3}{3!}$$

$$f^{(3)}(x)=1+\frac{1}{2}x-\frac{1}{8}x^2+\frac{1}{16}x^3$$

Using the series to estimate $\sqrt{10}$, we get

$$\sqrt{10}=\sqrt{1+9}$$

$$\sqrt{10} = \sqrt{9\left(\frac{1}{9} + 1\right)}$$

$$\sqrt{10} = 3\sqrt{1 + \frac{1}{9}}$$

Now that we have a format similar to the original function, we can say that $x = 1/9$. Because we have a 3 in front of the square root, we need to remember to multiply our series by 3.

$$\sqrt{10} \approx 3\left[f^{(3)}\left(\frac{1}{9}\right)\right] \approx 3\left[1 + \frac{1}{2}\left(\frac{1}{9}\right) - \frac{1}{8}\left(\frac{1}{9}\right)^2 + \frac{1}{16}\left(\frac{1}{9}\right)^3\right]$$

$$\sqrt{10} \approx 3\left[f^{(3)}\left(\frac{1}{9}\right)\right] \approx 3\left(1 + \frac{1}{18} - \frac{1}{648} + \frac{1}{11{,}664}\right)$$

$$\sqrt{10} \approx 3\left[f^{(3)}\left(\frac{1}{9}\right)\right] \approx 3.16$$

Topic: Maclaurin series

Question: Find the maclaurin series.

Find the maclaurin series of $f(x) = \sin x$ that includes x^7. Then use the ratio test to show that the series converges in its domain.

Answer choices:

A $\quad \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}$

B $\quad x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}$

C $\quad x^2 - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}$

D $\quad x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!}$

Solution: B

The Maclaurin formula is

$$f(x)=\frac{f(0)x^0}{0!}+\frac{f'(0)x^1}{1!}+\frac{f''(0)x^2}{2!}+\frac{f'''(0)x^3}{3!}+\ldots+\frac{f^n(0)x^n}{n!}$$

To find the maclaurin series that includes x^7, we need the original function, plus the first seven derivatives.

$f(x)=\sin x$ and $f(0)=\sin 0=0$

$f'(x)=\cos x$ and $f'(0)=\cos 0=1$

$f''(x)=-\sin x$ and $f''(0)=-\sin 0=0$

$f'''(x)=-\cos x$ and $f'''(0)=-\cos 0=-1$

$f^{(4)}(x)=\sin x$ and $f^{(4)}(0)=\sin 0=0$

$f^{(5)}(x)=\cos x$ and $f^{(5)}(0)=\cos 0=1$

$f^{(6)}(x)=-\sin x$ and $f^{(6)}(0)=-\sin 0=0$

$f^{(7)}(x)=-\cos x$ and $f^{(7)}(0)=-\cos 0=-1$

Therefore, the maclaurin series containing x^7 is

$$f^{(7)}(x)=\sin x=\frac{(0)x^0}{0!}+\frac{(1)x^1}{1!}+\frac{(0)x^2}{2!}+\frac{(-1)x^3}{3!}+\frac{(0)x^4}{4!}+\frac{(1)x^5}{5!}+\frac{(0)x^6}{6!}+\frac{(-1)x^7}{7!}$$

$$f^{(7)}(x)=\sin x=0+\frac{x}{1}+0-\frac{x^3}{3!}+0+\frac{x^5}{5!}+0-\frac{x^7}{7!}$$

$$f^{(7)}(x) = \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}$$

The powers and coefficients on each term are in arithmetic series. We can use the formula for the general term of an arithmetic series $T_n = a + (n-1)d$ for the powers and coefficients to generate a_n. The signs also alternate $+, -, +, -, \ldots$ and to achieve this alternation, the general term must be multiplied by $(-1)^{n-1}$.

So for the powers and coefficients $(1,3,5,\ldots)$, the arithmetic series with $a = 1$ and $d = 2$ may be used and

$$T_n = 1 + (n-1)2$$

$$T_n = 1 + 2n - 2$$

$$T_n = 2n - 1$$

So the general term of $\sin x$ is

$$a_n = (-1)^{n-1}\frac{x^{2n-1}}{(2n-1)!}$$

Using the ratio test to show that the series is convergent, we get

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(-1)^{(n+1)-1}\frac{x^{2(n+1)-1}}{(2(n+1)-1)!}}{(-1)^{n-1}\frac{x^{2n-1}}{(2n-1)!}}\right|$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{(-1)^n x^{2n}}{(2n+1)!}}{\frac{(-1)^{n-1}x^{2n-1}}{(2n-1)!}}\right|$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\lim_{n\to\infty}\left|\frac{(-1)^n x^{2n}}{(2n+1)!}\cdot\frac{(2n-1)!}{(-1)^{n-1}x^{2n-1}}\right|$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\lim_{n\to\infty}\left|(-1)^{n-(n-1)}x^{2n-(2n-1)}\cdot\frac{(2n-1)!}{(2n+1)!}\right|$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\lim_{n\to\infty}\left|(-1)^{n-n+1}x^{2n-2n+1}\cdot\frac{(2n-1)(2n-1-1)(2n-1-2)(2n-1-3)\dots}{(2n+1)(2n+1-1)(2n+1-2)(2n+1-3)\dots}\right|$$

Since the limit only effects n, we can pull x outside of the limit.

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=-x\lim_{n\to\infty}\left|\frac{(2n-1)(2n-2)(2n-3)(2n-4)\dots}{(2n+1)(2n)(2n-1)(2n-2)\dots}\right|$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=-x\lim_{n\to\infty}\left|\frac{1}{(2n+1)(2n)}\right|$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=-x\lim_{n\to\infty}\left|\frac{1}{4n^2+2n}\right|$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=-x(0)$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=0$$

Therefore, the series is convergent for all $x\in R$.

Topic: Maclaurin series

Question: Find the first four non-zero terms of the function's Maclaurin series.

$$f(x) = \frac{1}{1-2x}$$

Answer choices:

A $1 + 2x + 4x^2 + 8x^3$

B $1 + x + x^2 + x^3$

C $2 + 2x + 4x^2 + 8x^3$

D $1 + 2x + 4x^2 + 6x^3$

Solution: A

The Maclaurin series is a specific Taylor series where $a = 0$.

The Maclaurin series for

$$f(x) = \frac{1}{1-x}$$

is

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} 1 + x + x^2 + x^3 + \dots$$

Since this is extremely close to the given series, we can manipulate it until they match. We'll just replace each x with $2x$, and then simplify.

$$\frac{1}{1-2x} = \sum_{n=0}^{\infty} 1 + 2x + (2x)^2 + (2x)^3 + \dots$$

$$\frac{1}{1-2x} = \sum_{n=0}^{\infty} 1 + 2x + 4x^2 + 8x^3 + \dots$$

Since we've been asked for the first four terms of the series, we can say that the first four terms are

$$1 + 2x + 4x^2 + 8x^3$$