Topic: Taylor series

Question: Find the Taylor polynomial and use it to approximate the given value.

$$e^x$$

when $n = 3$ and $a = 2$

Find $e^{1.23}$

Answer choices:

A $\quad (x-1) + \frac{1}{2}(x-2)^2 + \frac{1}{6}(x-2)^3$ and $e^{1.23} = 3.427744$

B $\quad e\left[(x-1) + \frac{1}{2}(x-2)^2 + \frac{1}{6}(x-2)^3\right]$ and $e^{1.23} = 3.327744$

C $\quad e^2\left[(x-1) + (x-2)^2 + (x-2)^3\right]$ and $e^{1.23} = 4.427744$

D $\quad e^2\left[(x-1) + \frac{1}{2}(x-2)^2 + \frac{1}{6}(x-2)^3\right]$ and $e^{1.23} = 3.327744$

Solution: D

The formula for the taylor polynomial of f at a is

$$f^{(n)}(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \ldots + \frac{f^n(a)}{n!}(x-a)^n$$

To find the third-degree taylor polynomial, we need the original function, plus its first three derivatives.

$f(x) = e^x$ and $f(2) = e^2$

$f'(x) = e^x$ and $f'(2) = e^2$

$f''(x) = e^x$ and $f''(2) = e^2$

$f'''(x) = e^x$ and $f'''(2) = e^2$

Therefore, the third-degree taylor polynomial is

$$f^{(3)}(x) = e^2 + e^2(x-2) + \frac{e^2}{2!}(x-2)^2 + \frac{e^2}{3!}(x-2)^3$$

$$f^{(3)}(x) = e^2\left[1 + (x-2) + \frac{1}{2}(x-2)^2 + \frac{1}{6}(x-2)^3\right]$$

$$f^{(3)}(x) = e^2\left[x - 1 + \frac{1}{2}(x-2)^2 + \frac{1}{6}(x-2)^3\right]$$

Using the series to estimate $e^{1.23}$, we get

$$e^{1.23} \approx f^{(3)}(1.23) \approx e^2\left[1.23 - 1 + \frac{1}{2}(1.23-2)^2 + \frac{1}{6}(1.23-2)^3\right]$$

$$e^{1.23} \approx f^{(3)}(1.23) \approx 3.327744$$

Topic: Taylor series

Question: Find the Taylor polynomial and use it to approximate the given value.

$\tan x$

when $n = 3$ and $a = \frac{\pi}{4}$

Find $\tan \frac{\pi}{8}$

Answer choices:

A $\quad 1 + 2\left(x - \frac{\pi}{4}\right) + 2\left(x - \frac{\pi}{4}\right)^2 + \frac{8}{3}\left(x - \frac{\pi}{4}\right)^3$ and $\tan \frac{\pi}{8} = 0.361536$

B $\quad 1 + 2\left(x - \frac{\pi}{4}\right) + \frac{2}{3}\left(x - \frac{\pi}{4}\right)^2 + \frac{8}{3}\left(x - \frac{\pi}{4}\right)^3$ and $\tan \frac{\pi}{8} = 0.414214$

C $\quad 1 + 2\left(x - \frac{\pi}{4}\right)^2 + \frac{8}{3}\left(x - \frac{\pi}{4}\right)^3$ and $\tan \frac{\pi}{8} = 0.365136$

D $\quad 2 + 2\left(x - \frac{\pi}{4}\right) + 2\left(x - \frac{\pi}{4}\right)^2 + \frac{8}{3}\left(x - \frac{\pi}{4}\right)^3$ and $\tan \frac{\pi}{8} = 0.414214$

Solution: A

The formula for the taylor polynomial of f at a is

$$f(x)=f(a)+f'(a)(x-a)+\frac{f''(a)}{2!}(x-a)^2+\ldots+\frac{f^n(a)}{n!}(x-a)^n$$

To find the third-degree taylor polynomial, we need the original function, plus its first three derivatives.

$$f(x)=\tan x \quad \text{and} \quad f\left(\frac{\pi}{4}\right)=1$$

$$f'(x)=\sec^2 x \quad \text{and} \quad f'\left(\frac{\pi}{4}\right)=2$$

$$f''(x)=2\sec^2 x\tan x \quad \text{and} \quad f''\left(\frac{\pi}{4}\right)=4$$

$$f'''(x)=2\sec^2 x\left(\sec^2 x+2\tan^2 x\right) \quad \text{and} \quad f'''\left(\frac{\pi}{4}\right)=16$$

Therefore, the third-degree taylor polynomial is

$$f^{(3)}(x)=1+2\left(x-\frac{\pi}{4}\right)+\frac{4}{2!}\left(x-\frac{\pi}{4}\right)^2+\frac{16}{3!}\left(x-\frac{\pi}{4}\right)^3$$

$$f^{(3)}(x)=1+2\left(x-\frac{\pi}{4}\right)+2\left(x-\frac{\pi}{4}\right)^2+\frac{8}{3}\left(x-\frac{\pi}{4}\right)^3$$

Using the series to estimate $\tan \pi/8$, we get

$$\tan\left(\frac{\pi}{8}\right)\approx f^{(3)}\left(\frac{\pi}{8}\right)\approx 1+2\left(\frac{\pi}{8}-\frac{\pi}{4}\right)+2\left(\frac{\pi}{8}-\frac{\pi}{4}\right)^2+\frac{8}{3}\left(\frac{\pi}{8}-\frac{\pi}{4}\right)^3$$

$$\tan\left(\frac{\pi}{8}\right) \approx f^{(3)}\left(\frac{\pi}{8}\right) \approx 1 + 2\left(-\frac{\pi}{8}\right) + 2\left(-\frac{\pi}{8}\right)^2 + \frac{8}{3}\left(-\frac{\pi}{8}\right)^3$$

$$\tan\left(\frac{\pi}{8}\right) \approx f^{(3)}\left(\frac{\pi}{8}\right) \approx 1 - \frac{\pi}{4} + \frac{\pi^2}{32} - \frac{\pi^3}{192}$$

$$\tan\left(\frac{\pi}{8}\right) \approx f^{(3)}\left(\frac{\pi}{8}\right) \approx 0.361536$$

Topic: Taylor series

Question: Find the Taylor polynomial.

$$f(x) = 2x^2 - x + 4$$

when $n = 2$ and $a = 1$

Answer choices:

A $\quad 5 + 4(x + 1) + 3(x + 1)^2$

B $\quad 5 + 3(x - 1) + 2(x - 1)^2$

C $\quad 5 + 4(x - 1) + 3(x - 1)^2$

D $\quad 5 + 3(x + 1) + 2(x + 1)^2$

Solution: B

The formula for the taylor polynomial of f at a is

$$f(x)=f(a)+f'(a)(x-a)+\frac{f''(a)}{2!}(x-a)^2+\dots+\frac{f^n(a)}{n!}(x-a)^n$$

To find the second-degree taylor polynomial, we need the original function, plus its first two derivatives.

$f(x)=2x^2-x+4$ and $f(1)=2(1)^2-(1)+4=5$

$f'(x)=4x-1$ and $f'(1)=4(1)-1=3$

$f''(x)=4$ and $f''(1)=4$

Therefore, the second-degree taylor polynomial is

$$f^{(2)}(x)=5+3(x-1)+\frac{4}{2!}(x-1)^2$$

$$f^{(2)}(x)=5+3(x-1)+2(x-1)^2$$