Calculus 2 Workbook Solutions

Work

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WORK DONE TO LIFE A WEIGHT OR MASS

1. Find the work required to lift a 50-pound load from ground level up into a tree house that's 60 feet above the ground, if the chain being used to lift the weight itself weighs 1 pound per foot.

Solution:

The work required to lift the load is

$$50 \text{ lbs} \cdot 60 \text{ ft} = 3{,}000 \text{ ft-lbs}$$

The work required to lift the chain is

$$\int_0^{60} x\ dx = \frac{x^2}{2}\Bigg|_0^{60} = \frac{60^2}{2} - \frac{0^2}{2} = 1{,}800 \text{ ft-lbs}$$

So the total work required to lift the load is

$$3{,}000 + 1{,}800 = 4{,}800 \text{ ft-lbs}$$

2. Find the work required to lift a 40-pound box of roofing nails from ground level up onto a roof that's 35 feet above the ground, if the rope being used to lift the weight itself weighs 2 ounces per foot.

Solution:

The work required to lift the box is

$$40 \text{ lbs} \cdot 35 \text{ ft} = 1{,}400 \text{ ft-lbs}$$

2 ounces is equivalent to $1/8$ pounds, which means the work required to lift the rope is

$$\int_0^{35} \frac{1}{8}x\ dx = \frac{x^2}{16}\Big|_0^{35} = \frac{35^2}{16} - \frac{0^2}{16} = 76.5625 \text{ ft-lbs}$$

So the total work required to lift the load is

$$1{,}400 + 76.5625 = 1{,}476.5625 \approx 1{,}477 \text{ ft-lbs}$$

■ 3. Find the work required to lift a $5{,}500$-pound load of concrete from ground level up onto a construction platform that's 75 feet above the ground, if the cable being used to lift the weight itself weighs 8 pounds per foot.

Solution:

The work required to lift the box is

$$5{,}500 \text{ lbs} \cdot 75 \text{ ft} = 412{,}500 \text{ ft-lbs}$$

The work required to lift the cable is

$$\int_0^{75} 8x \ dx = 4x^2 \Big|_0^{75} = 4(75)^2 - 4(0)^2 = 22{,}500 \text{ ft-lbs}$$

So the total work required to lift the load is

$$412{,}500 + 22{,}500 = 435{,}000 \text{ ft-lbs}$$

■ 4. Find the work required to lift a 5-gallon bucket of water, with each gallon of water weighing 6.75 pounds and the bucket weighing 2 pounds, from ground level up onto a scaffold that's 14 feet above the ground, if the rope being used to lift the weight itself weighs 8 ounces per foot.

Solution:

The weight of the bucket of water is

$$2 + 5 \cdot 6.75 = 35.75 \text{ lbs}$$

The work required to lift the box is

$$35.75 \text{ lbs} \cdot 14 \text{ ft} = 500.5 \text{ ft-lbs}$$

8 ounces is $8/16$, or $1/2$ pounds, which means the work required to lift the rope is

$$\int_0^{14} \frac{1}{2}x \ dx = \frac{x^2}{4}\Big|_0^{14} = \frac{14^2}{4} - \frac{0^2}{4} = 49 \text{ ft-lbs}$$

So the total work required to lift the load is

$500.5 + 49 = 549.5$ ft-lbs

5. Find the work required to lift a $7{,}200$-pound load of rocks from ground level up into a dump truck that's 13 feet above the ground, if the chain being used to lift the weight itself weighs 12 pounds per foot.

Solution:

The work required to lift the load of concrete is

$7{,}200$ lbs $\cdot$ 13 ft $= 93{,}600$ ft-lbs

The work required to lift the chain is

$$\int_0^{13} 12x\ dx = 6x^2 \Big|_0^{13} = 6(13)^2 - 6(0)^2 = 1{,}014 \text{ ft-lbs}$$

So the total work required to lift the load is

$93{,}600 + 1{,}014 = 94{,}614$ ft-lbs

WORK DONE ON ELASTIC SPRINGS

■ 1. Find the work required to stretch a spring 3 feet beyond its normal length, if a force of $5s$ lbs is required to stretch the spring s feet beyond its normal length.

Solution:

The work needed to stretch a spring a feet is

$$W = \int_0^a F(s)\ ds$$

Using Hooke's Law, $F(s) = ks$ where k is a constant, and s is the distance, we have $ks = 5s$, so $k = 5$ and $F(s) = 5s$. So the work to stretch the spring 3 feet beyond its normal length is

$$W = \int_0^3 5s\ ds = \frac{5s^2}{2}\Big|_0^3 = \frac{5(3)^2}{2} - \frac{5(0)^2}{2} = \frac{45}{2} = 22.5 \text{ ft-lbs}$$

■ 2. Find the work required to stretch a spring 7 inches beyond its normal length, if a force of $9s$ lbs is required to stretch the spring s inches beyond its normal length.

Solution:

The work needed to stretch a spring a inches is

$$W = \int_0^a F(s)\ ds$$

Using Hooke's Law, $F(s) = ks$ where k is a constant, and s is the distance, we have $ks = 9s$, so $k = 9$ and $F(s) = 9s$. So the work to stretch the spring 7 inches beyond its normal length is

$$W = \int_0^7 9s\ ds = \frac{9s^2}{2}\Big|_0^7 = \frac{9(7)^2}{2} - \frac{9(0)^2}{2} = \frac{441}{2} = 220.5 \text{ in-lbs}$$

■ 3. Find the work required to stretch a spring 6 feet beyond its normal length, if a force of $15s$ lbs is required to stretch the spring s feet beyond its normal length.

Solution:

The work needed to stretch a spring a feet is

$$W = \int_0^a F(s)\ ds$$

Using Hooke's Law, $F(s) = ks$ where k is a constant, and s is the distance, we have $ks = 15s$, so $k = 15$ and $F(s) = 15s$. So the work to stretch the spring 6 feet beyond its normal length is

$$W = \int_0^6 15s\ ds = \frac{15s^2}{2}\Big|_0^6 = \frac{15(6)^2}{2} - \frac{15(0)^2}{2} = \frac{540}{2} = 270 \text{ ft-lbs}$$

■ 4. Find the work required to stretch a spring 1 foot beyond its normal length, if a force of $3.5s$ lbs is required to stretch the spring s feet beyond its normal length.

Solution:

The work needed to stretch a spring a feet is

$$W = \int_0^a F(s)\ ds$$

Using Hooke's Law, $F(s) = ks$ where k is a constant, and s is the distance, we have $ks = 0.5s$, so $k = 3.5$ and $F(s) = 3.5s$. So the work to stretch the spring 1 foot beyond its normal length is

$$W = \int_0^1 3.5s\ ds = \frac{3.5s^2}{2}\Big|_0^1 = \frac{3.5(1)^2}{2} - \frac{3.5(0)^2}{2} = \frac{3.5}{2} = 1.75 \text{ ft-lbs}$$

■ 5. Find the work required, in foot pounds, to stretch a spring 58 inches beyond its normal length, if a force of $4s$ lbs is required to stretch the spring s feet beyond its normal length.

Solution:

The work needed to stretch a spring a feet is

$$W=\int_0^a F(s)\ ds$$

Using Hooke's Law, $F(s)=ks$ where k is a constant, and s is the distance, we have $ks=4s$, so $k=4$ and $F(s)=4s$. But 58 inches is equivalent to $58/12$, or $29/6$ feet. So the work to stretch the spring 58 inches beyond its normal length is

$$W=\int_0^{\frac{29}{6}} 4s\ ds=\frac{4s^2}{2}\Big|_0^{\frac{29}{6}}=2s^2\Big|_0^{\frac{29}{6}}=2\left(\frac{29}{6}\right)^2-2(0)^2=\frac{841}{18}\text{ ft-lbs}$$

WORK DONE TO EMPTY A TANK

1. Find the work required to empty a tank that is 6 feet wide, 8 feet tall, 12 feet long, and completely full. The tank will be emptied by pumping the liquid in the tank through a hose to a height of 2 feet above the top of the tank. The liquid in the tank has a density of 58.9 lbs/ft^3.

Solution:

The volume of a slice of the liquid is

$$6 \cdot 12 \cdot dy \text{ ft}^3$$

$$72\ dy \text{ ft}^3$$

The force needed to pump a slice of the liquid, which is weight times volume, is

$$58.9 \cdot 72\ dy \text{ ft}^3$$

The distance the liquid will be pumped is $10 - y$ feet. The liquid will be pumped from an original height of 0 to 8 feet. So the work required is

$$W = \int_0^8 (10 - y)(58.9 \cdot 72)\ dy$$

$$W = 4{,}240.8 \int_0^8 10 - y\ dy$$

$$W = 4{,}240.8\left(10y - \frac{y^2}{2}\right)\Bigg|_0^8$$

$$W = 4{,}240.8\left[\left(10(8) - \frac{8^2}{2}\right) - \left(10(0) - \frac{0^2}{2}\right)\right]$$

$$W = 4{,}240.8(48 - 0)$$

$$W = 203{,}558.4 \text{ ft-lbs}$$

2. Find the work required to empty an in-ground swimming pool that is 20 feet wide, 4 feet deep, 18 feet long, and completely full. The pool will be emptied by pumping the water in the pool through a hose over the top of the pool. The water in the pool has a density of 62.43 lbs/ft^3.

Solution:

The volume of a slice of the water is

$$20 \cdot 18 \cdot dy \text{ ft}^3$$

$$360\ dy \text{ ft}^3$$

The force needed to pump a slice of the water, which is weight times volume, is

$$62.43 \cdot 360\ dy \text{ ft}^3$$

The distance the water will be pumped is $4-y$ feet. The water will be pumped from an original height of 0 to 4 feet. So the work required is

$$W=\int_0^4 (4-y)(62.43\cdot 360)\ dy$$

$$W=22{,}474.8\int_0^4 4-y\ dy$$

$$W=22{,}474.8\left(4y-\frac{y^2}{2}\right)\Bigg|_0^4$$

$$W=22{,}474.8\left[\left(4(4)-\frac{4^2}{2}\right)-\left(4(0)-\frac{0^2}{2}\right)\right]$$

$$W=22{,}474.8(8-0)$$

$$W=179{,}798.4 \text{ ft-lbs}$$

■ 3. Find the work required to empty a cylindrical tank that is 12 feet tall, has a radius of 6 feet, and is half full of diesel fuel. The tank will be emptied by pumping the fuel in the tank through a hose to a height of 6 feet above the top of the tank. The diesel fuel in the tank has a density of 53.5 lbs/ft^3.

Solution:

The volume of a slice of the fuel is

$$\pi \cdot 6^2 \cdot dy \text{ ft}^3$$

$$36\pi\ dy \text{ ft}^3$$

The force needed to pump a slice of the fuel, which is weight times volume, is

$$53.5 \cdot 36\pi\ dy \text{ ft}^3$$

The distance the fuel will be pumped is $18 - y$ feet. The fuel will be pumped from an original height of 0 to 6 feet. So the work required is

$$W = \int_0^6 (18 - y)(53.5 \cdot 36\pi)\ dy$$

$$W = 1{,}926\pi \int_0^6 18 - y\ dy$$

$$W = 1{,}926\pi \left(18y - \frac{y^2}{2} \right) \Big|_0^6$$

$$W = 1{,}926\pi \left[\left(18(6) - \frac{6^2}{2} \right) - \left(18(0) - \frac{0^2}{2} \right) \right]$$

$$W = 1{,}926\pi(90 - 0)$$

$$W = 173{,}340\pi \text{ ft-lbs}$$

■ 4. Find the work required to empty an above-ground child's pool that is 2 feet tall, has a diameter of 8 feet, and is three-fourths full. The pool will be emptied by pumping the water in the pool through a hose over the top of the pool. The water in the pool has a density of 62.4 lbs/ft^3.

Solution:

The volume of a slice of the water is

$$\pi \cdot 4^2 \cdot dy \text{ ft}^3$$

$$16\pi \ dy \text{ ft}^3$$

The force needed to pump a slice of the water, which is weight times volume, is

$$62.4 \cdot 16\pi \ dy \text{ ft}^3$$

The distance the water will be pumped is $2 - y$ feet. The water will be pumped from an original height of 0 to 1.5 feet. So the work required is

$$W = \int_0^{1.5} (2 - y)(62.4 \cdot 16\pi) \ dy$$

$$W = 998.4\pi \int_0^{1.5} 2 - y \ dy$$

$$W = 998.4\pi \left(2y - \frac{y^2}{2} \right) \Big|_0^{1.5}$$

$$W = 998.4\pi \left[\left(2(1.5) - \frac{1.5^2}{2} \right) - \left(2(0) - \frac{0^2}{2} \right) \right]$$

$$W = 998.4\pi(1.875 - 0)$$

$$W = 5{,}881.061448 \text{ ft-lbs}$$

■ 5. Find the work required to empty a cylindrical tank that is 8 feet tall, has a radius of 9 feet, and is three-fourths full of gasoline. The tank will be emptied by pumping the gas in the tank through a hose into a truck that's 8 feet above the top of the tank. The gasoline in the tank has a density of 54.5 lbs/ft^3.

Solution:

The volume of a slice of the water is

$$\pi \cdot 9^2 \cdot dy \text{ ft}^3$$

$$81\pi\ dy \text{ ft}^3$$

The force needed to pump a slice of the gasoline, which is weight times volume, is

$$54.5 \cdot 81\pi\ dy \text{ ft}^3$$

The distance the gas will be pumped is $16 - y$ feet. The gas will be pumped from an original height of 0 to 6 feet. So the work required is

$$W = \int_0^6 (16 - y)(54.5 \cdot 81\pi)\ dy$$

$$W = 4{,}414.5\pi \int_0^6 16 - y\ dy$$

$$W = 4{,}414.5\pi \left(16y - \frac{y^2}{2} \right) \Big|_0^6$$

$$W = 4{,}414.5\pi \left[\left(16(6) - \frac{6^2}{2} \right) - \left(16(0) - \frac{0^2}{2} \right) \right]$$

$$W = 4{,}414.5\pi(78 - 0)$$

$W = 344{,}331\pi$ ft-lbs

WORK DONE BY A VARIABLE FORCE

■ 1. Calculate the variable force on the interval $[0,2]$.

$$F(x) = 3x^2 + 2x$$

Solution:

Plugging the force equation and the interval into the integral formula for work done by a variable force, we get

$$W = \int_a^b F(x)\ dx$$

$$W = \int_0^2 3x^2 + 2x\ dx$$

$$W = x^3 + x^2 \Big|_0^2$$

$$W = 2^3 + 2^2 - (0^3 + 0^2)$$

$$W = 8 + 4$$

$$W = 12$$

■ 2. Calculate the variable force on the interval $[0,\pi/2]$.

$$F(x) = 3\sin(2x) + x$$

Solution:

Plugging the force equation and the interval into the integral formula for work done by a variable force, we get

$$W = \int_a^b F(x)\ dx$$

$$W = \int_0^{\frac{\pi}{2}} 3\sin(2x) + x\ dx$$

$$W = -\frac{3\cos(2x)}{2} + \frac{x^2}{2}\Bigg|_0^{\frac{\pi}{2}}$$

$$W = -\frac{3\cos\left(2\cdot\frac{\pi}{2}\right)}{2} + \frac{\left(\frac{\pi}{2}\right)^2}{2} - \left(-\frac{3\cos(2(0))}{2} + \frac{(0)^2}{2}\right)$$

$$W = -\frac{3\cos\pi}{2} + \frac{\frac{\pi^2}{4}}{2} + \frac{3\cos 0}{2}$$

$$W = -\frac{3(-1)}{2} + \frac{\pi^2}{8} + \frac{3(1)}{2}$$

$$W = 3 + \frac{\pi^2}{8}$$

3. Calculate the variable force on the interval $[1,6]$.

$$F(x) = x^2 + x + 1$$

Solution:

Plugging the force equation and the interval into the integral formula for work done by a variable force, we get

$$W = \int_a^b F(x)\ dx$$

$$W = \int_1^6 x^2 + x + 1\ dx$$

$$W = \frac{x^3}{3} + \frac{x^2}{2} + x \Big|_1^6$$

$$W = \frac{6^3}{3} + \frac{6^2}{2} + 6 - \left(\frac{1^3}{3} + \frac{1^2}{2} + 1\right)$$

$$W = \frac{216}{3} + \frac{36}{2} + 6 - \frac{1}{3} - \frac{1}{2} - 1$$

$$W = \frac{432}{6} + \frac{108}{6} + \frac{36}{6} - \frac{2}{6} - \frac{3}{6} - \frac{6}{6}$$

$$W = \frac{565}{6}$$

4. Calculate the variable force on the interval $[0,\pi/3]$.

$$F(x)=2\tan^2 x$$

Solution:

Plugging the force equation and the interval into the integral formula for work done by a variable force, we get

$$W=\int_a^b F(x)\ dx$$

$$W=\int_0^{\frac{\pi}{3}} 2\tan^2 x\ dx$$

$$W=2\int_0^{\frac{\pi}{3}} \sec^2 x-1\ dx$$

$$W=2(\tan x-x)\Big|_0^{\frac{\pi}{3}}$$

$$W=2\left(\tan\frac{\pi}{3}-\frac{\pi}{3}\right)-2(\tan 0-0)$$

$$W=2\left(\sqrt{3}-\frac{\pi}{3}\right)-2(0-0)$$

$$W=2\sqrt{3}-\frac{2\pi}{3}$$

5. Calculate the variable force on the interval $[1.2,3.5]$.

$$F(x) = 4(x-2)^3 - 2(x-2) + 1$$

Solution:

Plugging the force equation and the interval into the integral formula for work done by a variable force, we get

$$W = \int_a^b F(x)\ dx$$

$$W = \int_{1.2}^{3.5} 4(x-2)^3 - 2(x-2) + 1\ dx$$

$$W = (x-2)^4 - (x-2)^2 + x \Big|_{1.2}^{3.5}$$

$$W = (3.5-2)^4 - (3.5-2)^2 + 3.5 - \left((1.2-2)^4 - (1.2-2)^2 + 1.2\right)$$

$$W = 1.5^4 - 1.5^2 + 3.5 - 0.8^4 + 0.8^2 - 1.2$$

$$W = 5.0625 - 2.25 + 3.5 - 0.4096 + 0.64 - 1.2$$

$$W = 5.0625 - 2.25 + 3.5 - 0.4096 + 0.64 - 1.2$$

$$W = 5.3429$$

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