Radius and interval of convergence of a Taylor series

Sometimes we'll be asked for the radius and interval of convergence of a Taylor series. In order to find these things, we'll first have to find a power series representation for the Taylor series.

Once we have the Taylor series represented as a power series, we'll identify a_n and a_{n+1} and plug them into the limit formula from the ratio test in order to say where the series is convergent.

Example

Using the chart below, find the third-degree Taylor series about $a=3$ for $f(x)=\ln(2x)$. Then find the power series representation of the Taylor series, and the radius and interval of convergence.

n	$n!$	$f^{(n)}(x)$	$f^{(n)}(a)$	$\frac{f^{(n)}(a)}{n!}$
0	1	$\ln(2x)$	$\ln 6$	$\ln 6$
1	1	$\frac{1}{x}$	$\frac{1}{3}$	$\frac{1}{3}$
2	2	$-\frac{1}{x^2}$	$-\frac{1}{9}$	$-\frac{1}{18}$
3	6	$\frac{2}{x^3}$	$\frac{2}{27}$	$\frac{1}{81}$

Taylor series

Since we already have the chart done, the value in the far right column becomes the coefficient on each term in the Taylor polynomial, in the form

$$\frac{f^{(n)}(a)}{n!}(x-a)^n$$

With the whole chart filled in, we can build each term of the Taylor polynomial.

$$n=0 \qquad \frac{f^{(n)}(a)}{n!}(x-a)^n = \ln(6)(x-3)^0 \qquad \ln 6$$

$$n=1 \qquad \frac{f^{(n)}(a)}{n!}(x-a)^n = \frac{1}{3}(x-3)^1 \qquad \frac{1}{3}(x-3)$$

$$n=2 \qquad \frac{f^{(n)}(a)}{n!}(x-a)^n = -\frac{1}{18}(x-3)^2 \qquad -\frac{1}{18}(x-3)^2$$

$$n=3 \qquad \frac{f^{(n)}(a)}{n!}(x-a)^n = \frac{1}{81}(x-3)^3 \qquad \frac{1}{81}(x-3)^3$$

Putting all of the terms together, we get the third-degree Taylor polynomial.

$$\ln 6 + \frac{1}{3}(x-3) - \frac{1}{18}(x-3)^2 + \frac{1}{81}(x-3)^3$$

Power series representation

We want to find a power series representation for the Taylor series above. The first thing we can see is that the exponent of each $(x-3)$ is equal to the n value of that term, which means that

$$(x-3)^n$$

will be part of the power series representation. The fractional coefficient in front of the $(x-3)$ terms can be represented by

$$\frac{1}{n3^n}$$

Finally, we need to deal with the negative sign in front of the $n=2$ term. If we multiply our terms by

$$(-1)^{n+1}$$

the $n=2$ term will be negative and the $n=1$ and $n=3$ terms will be positive. Remember, none of these generalizations apply to our $n=0$ term, so we'll leave this term outside of the power series representation.

$$\ln 6+\sum_{n=1}^{\infty}\frac{(-1)^{n+1}(x-3)^n}{n3^n}$$

Notice the sum starts at $n=1$, since the $n=0$ term is not included in the sum.

Radius and interval of convergence

To find the radius of convergence, we'll identify a_n and a_{n+1} using the power series representation we just found.

$$a_n = \frac{(-1)^{n+1}(x-3)^n}{n3^n}$$

$$a_{n+1} = \frac{(-1)^{n+2}(x-3)^{n+1}}{3^{n+1}(n+1)}$$

We can plug a_n and a_{n+1} into the limit formula from the ratio test.

$$L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$$

$$L = \lim_{n\to\infty} \left| \frac{\frac{(-1)^{n+2}(x-3)^{n+1}}{(n+1)3^{n+1}}}{\frac{(-1)^{n+1}(x-3)^n}{n3^n}} \right|$$

$$L = \lim_{n\to\infty} \left| \frac{(-1)^{n+2}(x-3)^{n+1}}{(n+1)3^{n+1}} \cdot \frac{n3^n}{(-1)^{n+1}(x-3)^n} \right|$$

$$L = \lim_{n\to\infty} \left| \frac{(-1)^{n+2}}{(-1)^{n+1}} \cdot \frac{(x-3)^{n+1}}{(x-3)^n} \cdot \frac{n}{n+1} \cdot \frac{3^n}{3^{n+1}} \right|$$

$$L = \lim_{n\to\infty} \left| (-1)^{n+2-(n+1)} \cdot (x-3)^{n+1-n} \cdot \frac{n}{n+1} \cdot 3^{n-(n+1)} \right|$$

$$L = \lim_{n\to\infty} \left| (-1)^{n+2-n-1} \cdot (x-3)^{n+1-n} \cdot 3^{n-n-1} \cdot \frac{n}{n+1} \right|$$

$$L=\lim_{n\to\infty}\left|(-1)^1\cdot(x-3)^1\cdot 3^{-1}\cdot\frac{n}{n+1}\right|$$

$$L=\lim_{n\to\infty}\left|-\frac{1}{3}(x-3)\frac{n}{n+1}\right|$$

Since we're dealing with absolute value, the -1 can be removed.

$$L=\lim_{n\to\infty}\left|\frac{n(x-3)}{3(n+1)}\right|$$

The limit only effects n, so we can remove the $(x-3)$.

$$L=|x-3|\lim_{n\to\infty}\left|\frac{n}{3(n+1)}\right|$$

$$L=|x-3|\lim_{n\to\infty}\left|\frac{n}{3n+3}\right|$$

Since we'll get the indeterminate form ∞/∞ if we try to evaluate the limit, we'll divide the numerator and denominator by the highest-degree variable in order to reduce the fraction.

$$L=|x-3|\lim_{n\to\infty}\left|\frac{n}{3n+3}\left(\frac{\frac{1}{n}}{\frac{1}{n}}\right)\right|$$

$$L=|x-3|\lim_{n\to\infty}\left|\frac{\frac{n}{n}}{\frac{3n}{n}+\frac{3}{n}}\right|$$

$$L = |x-3| \lim_{n \to \infty} \left| \frac{1}{3 + \frac{3}{n}} \right|$$

$$L = |x-3| \left| \frac{1}{3 + \frac{3}{\infty}} \right|$$

$$L = |x-3| \left| \frac{1}{3+0} \right|$$

$$L = |x-3| \left| \frac{1}{3} \right|$$

$$L = \frac{1}{3}|x-3|$$

Since the ratio test tells us that the series will converge when $L < 1$, so we'll set up the inequality.

$$\frac{1}{3}|x-3| < 1$$

$$|x-3| < 3$$

Since the inequality is in the form $|x-a| < R$, we can say that the radius of convergence is $R = 3$.

To find the interval of convergence, we'll take the inequality we used to find the radius of convergence, and solve it for x.

$$|x-3| < 3$$

$$-3 < x - 3 < 3$$

$$-3 + 3 < x - 3 + 3 < 3 + 3$$

$$0 < x < 6$$

We need to test the endpoints of the inequality by plugging them into the power series representation. We'll start with $x = 0$.

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(0-3)^n}{n3^n}$$

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(-3)^n}{n3^n}$$

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(-1)^n(3)^n}{n3^n}$$

$$\sum_{n=1}^{\infty} \frac{(-1)^{2n+1}}{n}$$

The exponent on the -1 will always be odd, so the sum is going to simplify to

$$\sum_{n=1}^{\infty} -\frac{1}{n}$$

This is a divergent p-series, so the series diverges at the endpoint $x = 0$. Now we'll test $x = 6$.

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(6-3)^n}{n3^n}$$

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}3^n}{n3^n}$$

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$$

This is an alternating series where

$$a_n = \frac{1}{n}$$

The alternating series test for convergence says that a series converges if $\lim_{n\to\infty} a_n = 0$.

$$\lim_{n\to\infty} \frac{1}{n}$$

$$\frac{1}{\infty}$$

$$0$$

The series converges at the endpoint $x = 6$.

We've shown that the series diverges at $x = 0$ and converges at $x = 6$, which means the interval of convergence is

$$0 < x \le 6$$

We'll summarize our findings.

3rd-degree Taylor polynomial $\quad \ln 6 + \frac{1}{3}(x-3) - \frac{1}{18}(x-3)^2 + \frac{1}{81}(x-3)^3$

Power series representation	$\ln 6 + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(x-3)^n}{n3^n}$
Radius of convergence	$R = 3$
Interval of convergence	$0 < x \le 6$