Arc length of a polar curve

The arc length of a polar curve is simply the length of a section of a polar parametric curve between two points a and b. We use the formula

$$L = \int_a^b \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} \, d\theta$$

where L is the arc length

r is the equation of the polar curve

$\frac{dr}{d\theta}$ is the derivative of the polar curve

a and b are the endpoints of the section

Example

Find the arc length of the polar curve over the given interval.

$$r = \cos^2 \frac{\theta}{2}$$

$$0 \le \theta \le \frac{\pi}{2}$$

Before we can plug into the arc length formula, we need to find $dr/d\theta$.

$$\frac{dr}{d\theta} = 2\cos\frac{\theta}{2}\left[-\sin\frac{\theta}{2}\right]\left(\frac{1}{2}\right)$$

$$\frac{dr}{d\theta} = -\cos\frac{\theta}{2}\sin\frac{\theta}{2}$$

Now we can go ahead and solve for the arc length

$$L = \int_0^{\frac{\pi}{2}} \sqrt{\left(\cos^2\frac{\theta}{2}\right)^2 + \left(-\cos\frac{\theta}{2}\sin\frac{\theta}{2}\right)^2}\ d\theta$$

$$L = \int_0^{\frac{\pi}{2}} \sqrt{\cos^4\frac{\theta}{2} + \cos^2\frac{\theta}{2}\sin^2\frac{\theta}{2}}\ d\theta$$

$$L = \int_0^{\frac{\pi}{2}} \sqrt{\cos^2\frac{\theta}{2}\left(\cos^2\frac{\theta}{2} + \sin^2\frac{\theta}{2}\right)}\ d\theta$$

Since $\cos^2 x + \sin^2 x = 1$, we get

$$L = \int_0^{\frac{\pi}{2}} \sqrt{\left(\cos^2\frac{\theta}{2}\right)(1)}\ d\theta$$

$$L = \int_0^{\frac{\pi}{2}} \cos\frac{\theta}{2}\ d\theta$$

$$L = 2\sin\frac{\theta}{2}\Big|_0^{\frac{\pi}{2}}$$

$$L = 2\sin\left(\frac{\frac{\pi}{2}}{2}\right) - 2\sin\left(\frac{0}{2}\right)$$

$$L = 2\sin\frac{\pi}{4} - 2\sin 0$$

$$L = 2 \cdot \frac{\sqrt{2}}{2} - 2(0)$$

$$L = \sqrt{2}$$

Let's do another example.

Example

Find the arc length of the polar curve over the given interval.

$$r = e^{2\theta}$$

$$0 \le \theta \le \pi$$

Before we can plug into the arc length formula, we need to find $dr/d\theta$.

$$\frac{dr}{d\theta} = 2e^{2\theta}$$

Plugging everything into the formula, we get

$$s = \int_0^{\pi} \sqrt{\left(e^{2\theta}\right)^2 + \left(2e^{2\theta}\right)^2}\ d\theta$$

$$s = \int_0^{\pi} \sqrt{e^{4\theta} + 4e^{4\theta}}\ d\theta$$

$$s=\int_0^\pi \sqrt{5e^{4\theta}}\ d\theta$$

$$s=\sqrt{5}\int_0^\pi e^{2\theta}\ d\theta$$

$$s=\frac{\sqrt{5}}{2}e^{2\theta}\Big|_0^\pi$$

$$s=\frac{\sqrt{5}}{2}e^{2(\pi)}-\frac{\sqrt{5}}{2}e^{2(0)}$$

$$s=\frac{\sqrt{5}}{2}e^{2\pi}-\frac{\sqrt{5}}{2}$$

$$s=\frac{\sqrt{5}\left(e^{2\pi}-1\right)}{2}$$