Centroids of plane regions

The centroid of a plane region is the region's exact center point. If we imagine the plane region as a flat sheet of paper, and we attached a string to its centroid, the paper would hang perfectly flat from the string. In other words, the centroid of a plane region is like the region's balancing point.

To find the centroid of a region over the interval $[a, b]$, we have to start by calculating the area of the region. If the region is defined above by $f(x)$ and below by $g(x)$, over the interval $[a, b]$, then the area of the region is given by

$$A = \int_a^b f(x) - g(x)\ dx$$

Keep in mind that, if only one curve is given, then it's likely implied that $g(x) = 0$. Once we've found the area of the plane region, we can find the coordinates coordinates of the centroid $(\bar{x}, \bar{y})$ as

$$\bar{x} = \frac{1}{A}\int_a^b x(f(x) - g(x))\ dx$$

$$\bar{y} = \frac{1}{A}\int_a^b \frac{1}{2}[(f(x))^2 - (g(x))^2]\ dx$$

Let's work through an example where we find the centroid of a rectangular region.

Example

Find the centroid of the region bounded by the curves.

$x = 1$ and $x = 6$

$y = 0$ and $y = 4$

We know $[a, b] = [1,6]$, and because $y = 4$ is above $y = 0$, we'll say $f(x) = 4$ and $g(x) = 0$. Then the area of the plane region will be

$$A = \int_1^6 4 - 0 \ dx$$

$$A = 4 \int_1^6 \ dx$$

$$A = 4x \Big|_1^6$$

$$A = 4(6) - 4(1)$$

$$A = 20$$

Then the coordinates of the centroid will be

$$\bar{x} = \frac{1}{A} \int_a^b x(f(x) - g(x)) \ dx$$

$$\bar{x} = \frac{1}{20} \int_1^6 x(4 - 0) \ dx$$

$$\bar{x} = \frac{1}{5} \int_1^6 x \ dx$$

$$\bar{x}=\frac{1}{5}\left(\frac{x^2}{2}\right)\Big|_1^6$$

$$\bar{x}=\frac{x^2}{10}\Big|_1^6$$

$$\bar{x}=\frac{6^2}{10}-\frac{1^2}{10}$$

$$\bar{x}=\frac{35}{10}$$

$$\bar{x}=\frac{7}{2}$$

and

$$\bar{y}=\frac{1}{A}\int_a^b\frac{1}{2}[(f(x))^2-(g(x))^2]\ dx$$

$$\bar{y}=\frac{1}{20}\int_1^6\frac{1}{2}(4^2-0^2)\ dx$$

$$\bar{y}=\frac{2}{5}\int_1^6 dx$$

$$\bar{y}=\frac{2x}{5}\Big|_1^6$$

$$\bar{y}=\frac{2(6)}{5}-\frac{2(1)}{5}$$

$$\bar{y}=2$$

So the centroid of the region is at $(7/2,2)$, which we can confirm visually by graphing the region and the centroid that we found.