Topic: Area under a parametric curve

Question: Find the area under the parametric curve.

$$x = t - \sin t$$

$$y = 6(1 - \cos t)$$

$$0 \le t \le 2\pi$$

Answer choices:

A 18

B 231π

C 18π

D $18\pi - 13$

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Solution: C

We'll find the area under the curve using the integral formula

$$A = \int_{\alpha}^{\beta} y(t)x'(t)\ dt$$

Since the question defines an interval for our parameter, we'll be able to find a real-number answer for the area.

We've already been given $y(t)$, but we need to find $x'(t)$ before we can plug into the area formula.

$$x(t) = t - \sin t$$

$$x'(t) = 1 - \cos t$$

Plugging into our area formula, we get

$$A = \int_{0}^{2\pi} 6(1 - \cos t)(1 - \cos t)\ dt$$

$$A = 6\int_{0}^{2\pi} 1 - 2\cos t + \cos^2 t\ dt$$

Using the power reduction formula from trigonometry,

$$\cos^2 t = \frac{1}{2} + \frac{1}{2}\cos 2t$$

we get

$$A = 6\int_{0}^{2\pi} 1 - 2\cos t + \frac{1}{2} + \frac{1}{2}\cos 2t\ dt$$

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$$A = 6\int_0^{2\pi} \frac{3}{2} - 2\cos t + \frac{1}{2}\cos 2t \ dt$$

$$A = 6\left(\frac{3}{2}t - 2\sin t + \frac{1}{4}\sin 2t\right)\Big|_0^{2\pi}$$

$$A = 6\left[\left(\frac{3}{2}(2\pi) - 2\sin(2\pi) + \frac{1}{4}\sin 2(2\pi)\right) - \left(\frac{3}{2}(0) - 2\sin(0) + \frac{1}{4}\sin 2(0)\right)\right]$$

$$A = 6\left[\left(3\pi - 2(0) + \frac{1}{4}\sin 4\pi\right) - \left(0 - 2(0) + \frac{1}{4}(0)\right)\right]$$

$$A = 6\left(3\pi + \frac{1}{4}(0)\right)$$

$$A = 6(3\pi)$$

$$A = 18\pi$$

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Topic: Area under a parametric curve

Question: Find the area under the parametric curve.

$$f(t) = 4t^2$$

$$g(t) = t - 4$$

$$0 \le t \le 4$$

Answer choices:

A 256

B $\dfrac{256}{3}$

C -256

D $-\dfrac{256}{3}$

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Solution: D

We'll find the area under the curve using the integral formula

$$A = \int_{\alpha}^{\beta} y(t)x'(t)\ dt$$

Since the question defines an interval for our parameter, we'll be able to find a real-number answer for the area.

We've already been given $y(t)$ as $g(t) = t - 4$, but we need to find $x'(t)$ as $f'(t)$ before we can plug into the area formula.

$$f(t) = 4t^2$$

$$f'(t) = 8t$$

Plugging into our area formula, we get

$$A = \int_{0}^{4} (t - 4)(8t)\ dt$$

$$A = \int_{0}^{4} 8t^2 - 32t\ dt$$

$$A = \frac{8}{3}t^3 - 16t^2 \Big|_0^4$$

$$A = \frac{8}{3}(4)^3 - 16(4)^2 - \left[\frac{8}{3}(0)^3 - 16(0)^2\right]$$

$$A = \frac{512}{3} - \frac{768}{3}$$

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$$A = -\frac{256}{3}$$

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Topic: Area under a parametric curve

Question: Find the area under the parametric curve.

$$f(t) = e^t$$

$$g(t) = 1 - t^2$$

$$0 \le t \le 4$$

Answer choices:

A $9e^4 - 1$

B $12e^4$

C $1 - 9e^4$

D $9e^4$

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Solution: C

We'll find the area under the curve using the integral formula

$$A=\int_{\alpha}^{\beta} y(t)x'(t)\ dt$$

Since the question defines an interval for our parameter, we'll be able to find a real-number answer for the area.

We've already been given $y(t)$ as $g(t)=1-t^2$, but we need to find $x'(t)$ as $f'(t)$ before we can plug into the area formula.

$$f(t)=e^t$$

$$f'(t)=e^t$$

Plugging into our area formula, we get

$$A=\int_0^4 \left(1-t^2\right)\left(e^t\right)\ dt$$

$$A=\int_0^4 e^t-t^2e^t\ dt$$

$$A=\int_0^4 e^t\ dt-\int_0^4 t^2e^t\ dt$$

For the second integral, we'll use integration by parts, setting

$$u=t^2$$

$$du=2t\ dt$$

and

$$dv = e^t \ dt$$

$$v = e^t$$

Replacing the second integral using the integration by parts formula

$$\int u \ dv = uv - \int v \ du$$

we get

$$A = \int_0^4 e^t \ dt - \left[t^2 e^t \Big|_0^4 - \int_0^4 2te^t \ dt \right]$$

We'll need to use integration by parts again on this new integral. Setting

$$u = 2t$$

$$du = 2 \ dt$$

and

$$dv = e^t \ dt$$

$$v = e^t$$

Using the integration by parts formula to replace the integral again, we get

$$A = \int_0^4 e^t \ dt - \left[t^2 e^t \Big|_0^4 - \left[2te^t \Big|_0^4 - \int_0^4 2e^t \ dt \right] \right]$$

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$$A = \int_0^4 e^t \ dt - \left[t^2e^t \Big|_0^4 - 2te^t \Big|_0^4 + \int_0^4 2e^t \ dt \right]$$

$$A = \int_0^4 e^t \ dt - t^2e^t \Big|_0^4 + 2te^t \Big|_0^4 - \int_0^4 2e^t \ dt$$

$$A = e^t \Big|_0^4 - t^2e^t \Big|_0^4 + 2te^t \Big|_0^4 - 2e^t \Big|_0^4$$

$$A = \left(e^t - t^2e^t + 2te^t - 2e^t \right) \Big|_0^4$$

$$A = \left[e^4 - 4^2e^4 + 2(4)e^4 - 2e^4 \right] - \left[e^0 - 0^2e^0 + 2(0)e^0 - 2e^0 \right]$$

$$A = e^4 - 16e^4 + 8e^4 - 2e^4 - \left(1 - 0 + 0 - 2(1) \right)$$

$$A = -9e^4 + 1$$

$$A = 1 - 9e^4$$

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