Calculus 2 Workbook Solutions

Ratio and root tests

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MATH

RATIO TEST

1. Use the ratio test to determine the convergence of the series.

$$\sum_{n=1}^{\infty} \frac{7^n}{n^3}$$

Solution:

Apply the ratio test.

$$L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$$

$$L = \lim_{n\to\infty} \left| \frac{\frac{7^{n+1}}{(n+1)^3}}{\frac{7^n}{n^3}} \right| = \lim_{n\to\infty} \left| \frac{7^{n+1}}{(n+1)^3} \cdot \frac{n^3}{7^n} \right|$$

$$L = \lim_{n\to\infty} \left| \frac{7}{(n+1)^3} \cdot \frac{n^3}{1} \right| = \lim_{n\to\infty} \left| \frac{7n^3}{(n+1)^3} \right|$$

$$L = 7 \lim_{n\to\infty} \left| \frac{n^3}{(n+1)^3} \right| = 7 \cdot 1$$

$$L = 7 > 1$$

The series converges if $L < 1$ and diverges if $L > 1$, which means this series diverges.

2. Use the ratio test to determine the convergence of the series.

$$\sum_{n=1}^{\infty} \frac{9(n+3)}{n^2}$$

Solution:

Apply the ratio test.

$$L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$$

$$L = \lim_{n\to\infty} \left| \frac{\frac{9(n+4)}{(n+1)^2}}{\frac{9(n+3)}{n^2}} \right| = \lim_{n\to\infty} \left| \frac{9(n+4)}{(n+1)^2} \cdot \frac{n^2}{9(n+3)} \right|$$

$$L = \lim_{n\to\infty} \left| \frac{(n+4)}{(n+1)^2} \cdot \frac{n^2}{(n+3)} \right| = \lim_{n\to\infty} \left| \frac{n^3 + 4n^2}{n^3 + 5n^2 + 7n + 3} \right|$$

$$L = 1$$

The ratio test is inconclusive when $L = 1$, so we can't use it to determine convergence for this particular series.

3. Use the ratio test to determine the convergence of the series.

$$\sum_{n=1}^{\infty} \frac{10^n}{5^{3n+1}(n+2)}$$

Solution:

Apply the ratio test.

$$L = \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right|$$

$$L = \lim_{n \to \infty} \left| \frac{\frac{10^{n+1}}{5^{3n+4}(n+3)}}{\frac{10^n}{5^{3n+1}(n+2)}} \right| = \lim_{n \to \infty} \left| \frac{10^{n+1}}{5^{3n+4}(n+3)} \cdot \frac{5^{3n+1}(n+2)}{10^n} \right|$$

$$L = \lim_{n \to \infty} \left| \frac{10}{5^3(n+3)} \cdot \frac{(n+2)}{1} \right| = \lim_{n \to \infty} \left| \frac{10(n+2)}{125(n+3)} \right|$$

$$L = \lim_{n \to \infty} \left| \frac{2(n+2)}{25(n+3)} \right| = \frac{2}{25} \lim_{n \to \infty} \left| \frac{n+2}{n+3} \right|$$

$$L = \frac{2}{25} \cdot 1$$

$$L = \frac{2}{25}$$

The series converges if $L < 1$ and diverges if $L > 1$, which means this series converges.

4. Use the ratio test to determine the convergence of the series.

$$\sum_{n=1}^{\infty} \frac{6n+17}{3^{2n+1}}$$

Solution:

Apply the ratio test.

$$L = \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right|$$

$$L = \lim_{n \to \infty} \left| \frac{\frac{6n+23}{3^{2n+3}}}{\frac{6n+17}{3^{2n+1}}} \right| = \lim_{n \to \infty} \left| \frac{6n+23}{3^{2n+3}} \cdot \frac{3^{2n+1}}{6n+17} \right|$$

$$L = \lim_{n \to \infty} \left| \frac{6n+23}{9} \cdot \frac{1}{6n+17} \right| = \lim_{n \to \infty} \left| \frac{6n+23}{9(6n+17)} \right|$$

$$L = \frac{1}{9} \lim_{n \to \infty} \left| \frac{6n+23}{6n+17} \right|$$

$$L = \frac{1}{9} \cdot 1$$

$$L = \frac{1}{9}$$

The series converges if $L < 1$ and diverges if $L > 1$, which means this series converges.

5. Use the ratio test to determine the convergence of the series.

$$\sum_{n=1}^{\infty} \frac{(-1)^n \cdot 5^{n+3}}{6^{n+1}}$$

Solution:

Apply the ratio test.

$$L = \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right|$$

$$L = \lim_{n \to \infty} \left| \frac{\frac{(-1)^{n+1} \cdot 5^{n+4}}{6^{n+2}}}{\frac{(-1)^n \cdot 5^{n+3}}{6^{n+1}}} \right| = \lim_{n \to \infty} \left| \frac{\frac{5^{n+4}}{6^{n+2}}}{\frac{5^{n+3}}{6^{n+1}}} \right|$$

$$L = \lim_{n \to \infty} \left| \frac{5^{n+4}}{6^{n+2}} \cdot \frac{6^{n+1}}{5^{n+3}} \right| = \lim_{n \to \infty} \left| \frac{5}{6} \right|$$

$$L = \frac{5}{6}$$

The series converges if $L < 1$ and diverges if $L > 1$, which means this series converges.

RATIO TEST WITH FACTORIALS

■ 1. Use the ratio test to determine the convergence of the series.

$$\sum_{n=1}^{\infty} \frac{n^3}{(2n-1)!}$$

Solution:

Apply the ratio test.

$$L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$$

$$L = \lim_{n\to\infty} \left| \frac{\frac{(n+1)^3}{(2n+1)!}}{\frac{n^3}{(2n-1)!}} \right| = \lim_{n\to\infty} \left| \frac{(n+1)^3}{(2n+1)!} \cdot \frac{(2n-1)!}{n^3} \right|$$

$$L = \lim_{n\to\infty} \left| \frac{(n+1)^3}{(2n+1)(2n)(2n-1)!} \cdot \frac{(2n-1)!}{n^3} \right| = \lim_{n\to\infty} \left| \frac{(n+1)^3}{(2n+1)(2n)} \cdot \frac{1}{n^3} \right|$$

$$L = \lim_{n\to\infty} \left| \frac{(n+1)^3}{n^3(2n+1)(2n)} \right| = \lim_{n\to\infty} \left| \frac{n^3+3n^2+3n+1}{4n^5+2n^4} \right|$$

$$L = 0$$

The series converges if $L < 1$ and diverges if $L > 1$, which means this series converges.

2. Use the ratio test to determine the convergence of the series.

$$\sum_{n=1}^{\infty} \frac{8^n}{2^{n+1} \cdot n!}$$

Solution:

Apply the ratio test.

$$L = \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right|$$

$$L = \lim_{n \to \infty} \left| \frac{\frac{8^{n+1}}{2^{n+2} \cdot (n+1)!}}{\frac{8^n}{2^{n+1} \cdot n!}} \right| = \lim_{n \to \infty} \left| \frac{8^{n+1}}{2^{n+2} \cdot (n+1)!} \cdot \frac{2^{n+1} \cdot n!}{8^n} \right|$$

$$L = \lim_{n \to \infty} \left| \frac{8}{2 \cdot (n+1) \cdot n!} \cdot \frac{1 \cdot n!}{1} \right| = \lim_{n \to \infty} \left| \frac{4}{n+1} \right|$$

$$L = 4 \lim_{n \to \infty} \left| \frac{1}{n+1} \right| = 4 \cdot 0$$

$$L = 0$$

The series converges if $L<1$ and diverges if $L>1$, which means this series converges.

3. Use the ratio test to determine the convergence of the series.

$$\sum_{n=1}^{\infty}\frac{(-1)^n n!}{n^3+1}$$

Solution:

Apply the ratio test.

$$L=\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|$$

$$L=\lim_{n\to\infty}\left|\frac{\frac{(-1)^{n+1}(n+1)!}{(n+1)^3+1}}{\frac{(-1)^n n!}{n^3+1}}\right|=\lim_{n\to\infty}\left|\frac{\frac{(n+1)!}{(n+1)^3+1}}{\frac{n!}{n^3+1}}\right|$$

$$L=\lim_{n\to\infty}\left|\frac{(n+1)!}{(n+1)^3+1}\cdot\frac{n^3+1}{n!}\right|$$

$$L=\lim_{n\to\infty}\left|\frac{(n+1)n!}{(n+1)^3+1}\cdot\frac{n^3+1}{n!}\right|=\lim_{n\to\infty}\left|\frac{(n+1)}{(n+1)^3+1}\cdot\frac{n^3+1}{1}\right|$$

$$L = \lim_{n \to \infty} \left| \frac{(n+1)(n^3+1)}{(n+1)^3+1} \right| = \lim_{n \to \infty} \left| \frac{n^4+n^3+n+1}{n^3+3n^2+3n+2} \right|$$

$$L = \infty$$

The series converges if $L < 1$ and diverges if $L > 1$, which means this series diverges.

■ 4. Use the ratio test to determine the convergence of the series.

$$\sum_{n=0}^{\infty} \frac{(n+2)!}{(3n)^2+7}$$

Solution:

Apply the ratio test.

$$L = \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right|$$

$$L = \lim_{n \to \infty} \left| \frac{\frac{(n+3)!}{(3n+3)^2+7}}{\frac{(n+2)!}{(3n)^2+7}} \right| = \lim_{n \to \infty} \left| \frac{(n+3)!}{(3n+3)^2+7} \cdot \frac{(3n)^2+7}{(n+2)!} \right|$$

$$L = \lim_{n \to \infty} \left| \frac{(n+3)(n+2)!}{(3n+3)^2+7} \cdot \frac{9n^2+7}{(n+2)!} \right| = \lim_{n \to \infty} \left| \frac{(n+3)}{(3n+3)^2+7} \cdot \frac{9n^2+7}{1} \right|$$

$$L = \lim_{n \to \infty} \left| \frac{(n+3)(9n^2+7)}{(3n+3)^2+7} \right| = \lim_{n \to \infty} \left| \frac{9n^3+27n^2+7n+21}{9n^2+36n+16} \right|$$

$$L = \infty$$

The series converges if $L < 1$ and diverges if $L > 1$, which means this series diverges.

5. Use the ratio test to determine the convergence of the series.

$$\sum_{n=0}^{\infty} \frac{4^n(n+1)}{n!}$$

Solution:

Apply the ratio test.

$$L = \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right|$$

$$L = \lim_{n \to \infty} \left| \frac{\frac{4^{n+1}(n+2)}{(n+1)!}}{\frac{4^n(n+1)}{n!}} \right| = \lim_{n \to \infty} \left| \frac{4^{n+1}(n+2)}{(n+1)!} \cdot \frac{n!}{4^n(n+1)} \right|$$

$$L = \lim_{n \to \infty} \left| \frac{4(n+2)}{(n+1)n!} \cdot \frac{n!}{(n+1)} \right| = \lim_{n \to \infty} \left| \frac{4(n+2)}{(n+1)} \cdot \frac{1}{(n+1)} \right|$$

$$L = \lim_{n\to\infty} \left| \frac{4(n+2)}{(n+1)(n+1)} \right| = \lim_{n\to\infty} \left| \frac{4n+2}{n^2+2n+1} \right|$$

$$L = 0$$

The series converges if $L < 1$ and diverges if $L > 1$, which means this series converges.

ROOT TEST

1. Use the root test to determine the convergence of the series.

$$\sum_{n=3}^{\infty}\left(\frac{5n^3+3n^2-6}{\sqrt{6n^6+7n^4-8}}\right)^n$$

Solution:

Apply the root test.

$$L=\lim_{n\to\infty}\sqrt[n]{\left|a_n\right|}$$

$$L=\lim_{n\to\infty}\left|a_n\right|^{\frac{1}{n}}$$

$$L=\lim_{n\to\infty}\left|\left(\frac{5n^3+3n^2-6}{\sqrt{6n^6+7n^4-8}}\right)^n\right|^{\frac{1}{n}}$$

$$L=\lim_{n\to\infty}\left|\frac{5n^3+3n^2-6}{\sqrt{6n^6+7n^4-8}}\right|$$

Divide through by the highest-degree term.

$$L = \lim_{n \to \infty} \left| \frac{\frac{5n^3 + 3n^2 - 6}{n^3}}{\frac{\sqrt{6n^6 + 7n^4 - 8}}{n^3}} \right|$$

$$L = \lim_{n \to \infty} \left| \frac{\frac{5n^3}{n^3} + \frac{3n^2}{n^3} - \frac{6}{n^3}}{\sqrt{\frac{6n^6}{n^6} + \frac{7n^4}{n^6} - \frac{8}{n^6}}} \right|$$

$$L = \lim_{n \to \infty} \left| \frac{5 + \frac{3}{n} \frac{-6}{n^3}}{\sqrt{6 + \frac{7}{n^2} - \frac{8}{n^6}}} \right|$$

$$L = \lim_{n \to \infty} \left| \frac{5 + 0 - 0}{\sqrt{6 + 0 - 0}} \right|$$

$$L = \frac{5}{\sqrt{6}}$$

The series converges absolutely if $L < 1$ but diverges if $L > 1$, so the series diverges.

■ 2. Use the root test to determine the convergence of the series.

$$\sum_{n=1}^{\infty} \frac{7n^3}{e^{2n^2}}$$

Solution:

Apply the root test.

$$L=\lim_{n\to\infty}\sqrt[n]{\left|a_n\right|}$$

$$L=\lim_{n\to\infty}\left|a_n\right|^{\frac{1}{n}}$$

$$L=\lim_{n\to\infty}\left|\frac{7n^3}{e^{2n^2}}\right|^{\frac{1}{n}}$$

$$L=\lim_{n\to\infty}\left|\frac{\sqrt[n]{7n^3}}{\sqrt[n]{e^{2n^2}}}\right|$$

$$L=\lim_{n\to\infty}\left|\frac{\sqrt[n]{7}\cdot n^{\frac{3}{n}}}{e^{\frac{2n^2}{n}}}\right|$$

$$L=\left|\frac{1\cdot 1}{\infty}\right|$$

$$L=0$$

The series converges absolutely if $L<1$ but diverges if $L>1$, so the series converges absolutely.

■ 3. Use the root test to determine the convergence of the series.

$$\sum_{n=0}^{\infty} \left(\frac{7n - 6n^4}{9n^4 + 3} \right)^n$$

Solution:

Apply the root test.

$$L = \lim_{n \to \infty} \sqrt[n]{|a_n|}$$

$$L = \lim_{n \to \infty} |a_n|^{\frac{1}{n}}$$

$$L = \lim_{n \to \infty} \left| \left(\frac{7n - 6n^4}{9n^4 + 3} \right)^n \right|^{\frac{1}{n}}$$

$$L = \lim_{n \to \infty} \left| \frac{7n - 6n^4}{9n^4 + 3} \right|$$

Divide through by the highest-degree term.

$$L = \lim_{n \to \infty} \left| \frac{\frac{7n}{n^4} - \frac{6n^4}{n^4}}{\frac{9n^4}{n^4} + \frac{3}{n^4}} \right|$$

$$L = \lim_{n \to \infty} \left| \frac{\frac{7}{n^3} - 6}{9 + \frac{3}{n^4}} \right|$$

$$L=\left|\frac{0-6}{9+0}\right|$$

$$L=\frac{2}{3}$$

The series converges absolutely if $L<1$ but diverges if $L>1$, so the series converges absolutely.

ABSOLUTE AND CONDITIONAL CONVERGENCE

1. Use the root test to determine the absolute or conditional convergence of the series.

$$\sum_{n=1}^{\infty} \left(\frac{6n}{8n+5} \right)^n$$

Solution:

Both the ratio and root tests can determine absolute or conditional convergence. The series converges absolutely if $a_n = \left| a_n \right|$ and converges conditionally if $a_n \neq \left| a_n \right|$.

By the root test,

$$L = \lim_{n \to \infty} \left| \left(\frac{6n}{8n+5} \right)^n \right|^{\frac{1}{n}}$$

$$L = \lim_{n \to \infty} \left| \frac{6n}{8n+5} \right|$$

Divide through by the highest-degree term.

$$L = \lim_{n \to \infty} \left| \frac{\frac{6n}{n}}{\frac{8n}{n} + \frac{5}{n}} \right|$$

$$L = \lim_{n\to\infty}\left|\frac{6}{8+\frac{5}{n}}\right|$$

$$L = \left|\frac{6}{8+0}\right|$$

$$L = \frac{3}{4} < 1$$

The series converges, so check for absolute vs. conditional convergence by comparing

$$\lim_{n\to\infty}\left|\frac{\frac{6n}{n}}{\frac{8n}{n}+\frac{5}{n}}\right| \text{ and } \lim_{n\to\infty}\frac{\frac{6n}{n}}{\frac{8n}{n}+\frac{5}{n}}$$

We've already found the first value, but the second value is

$$L = \lim_{n\to\infty}\frac{\frac{6n}{n}}{\frac{8n}{n}+\frac{5}{n}} = \frac{6}{8+0} = \frac{3}{4}$$

Since the values are equal, the series converges absolutely.

■ 2. Use the ratio test to determine the absolute or conditional convergence of the series, or say if the series diverges of if the ratio test is inconclusive.

$$\sum_{n=1}^{\infty}\frac{(-1)^n}{25n}$$

Solution:

Both the ratio and root tests can determine absolute or conditional convergence. The series converges absolutely if $a_n = \left| a_n \right|$ and converges conditionally if $a_n \neq \left| a_n \right|$.

By the ratio test,

$$L = \lim_{n \to \infty} \left| \frac{\frac{(-1)^{n+1}}{25n+25}}{\frac{(-1)^n}{25n}} \right|$$

$$L = \lim_{n \to \infty} \left| \frac{1}{25n+25} \cdot \frac{25n}{1} \right|$$

$$L = \lim_{n \to \infty} \left| \frac{25n}{25n+25} \right| = \lim_{n \to \infty} \left| \frac{n}{n+1} \right|$$

$$L = \lim_{n \to \infty} \left| \frac{n}{n+1} \right|$$

$$L = 1$$

The series converges if $L < 1$ and diverges if $L > 1$, but the ratio test is inconclusive when $L = 1$. So the ratio test is inconclusive, and we can't determine absolute or conditional convergence.

3. Use the root test to determine the absolute or conditional convergence of the series.

$$\sum_{n=1}^{\infty} \left(\frac{8n - 9n^5}{14n^5 + 7} \right)^n$$

Solution:

Both the ratio and root tests can determine absolute or conditional convergence. The series converges absolutely if $a_n = |a_n|$ and converges conditionally if $a_n \neq |a_n|$.

By the root test,

$$L = \lim_{n \to \infty} \left| \left(\frac{8n - 9n^5}{14n^5 + 7} \right)^n \right|^{\frac{1}{n}}$$

$$L = \lim_{n \to \infty} \left| \frac{8n - 9n^5}{14n^5 + 7} \right|$$

Divide through by the highest-degree term.

$$L = \lim_{n \to \infty} \left| \frac{\frac{8n}{n^5} - \frac{9n^5}{n^5}}{\frac{14n^5}{n^5} + \frac{7}{n^5}} \right|$$

$$L = \lim_{n\to\infty} \left| \frac{\frac{8}{n^4} - 9}{14 + \frac{7}{n^5}} \right|$$

$$R = \left| \frac{0 - 9}{14 + 0} \right|$$

$$R = \frac{9}{14} < 1$$

The series converges, so check for absolute vs. conditional convergence by comparing

$$\lim_{n\to\infty} \left| \frac{\frac{8}{n^4} - 9}{14 + \frac{7}{n^5}} \right| \text{ and } \lim_{n\to\infty} \frac{\frac{8}{n^4} - 9}{14 + \frac{7}{n^5}}$$

We've already found the first value, but the second value is

$$L = \lim_{n\to\infty} \frac{\frac{8}{n^4} - 9}{14 + \frac{7}{n^5}} = \frac{0 - 9}{14 + 0} = -\frac{9}{14}$$

Since the values are unequal, the series conditionally converges.

■ 4. Use the ratio test to determine the absolute or conditional convergence of the series, or say if the series diverges of if the ratio test is inconclusive.

$$\sum_{n=1}^{\infty} \frac{n!}{9^n}$$

Solution:

Both the ratio and root tests can determine absolute or conditional convergence. The series converges absolutely if $a_n = \left|a_n\right|$ and converges conditionally if $a_n \neq \left|a_n\right|$.

By the ratio test,

$$L = \lim_{n \to \infty} \left| \frac{\frac{(n+1)!}{9^{n+1}}}{\frac{n!}{9^n}} \right|$$

$$L = \lim_{n \to \infty} \left| \frac{(n+1)!}{9^{n+1}} \cdot \frac{9^n}{n!} \right|$$

$$L = \lim_{n \to \infty} \left| \frac{(n+1)n!}{9 \cdot 9^n} \cdot \frac{9^n}{n!} \right| = \lim_{n \to \infty} \left| \frac{n+1}{9} \right|$$

$$L = 9 \lim_{n \to \infty} \left| n+1 \right|$$

$$L = 9 \cdot \infty$$

$$L = \infty$$

The series converges if $L < 1$ and diverges if $L > 1$, so the series diverges.

W W W . K R I S T A K I N G M A T H . C O M