Topic: Distinct quadratic factors

Question: Use partial fractions to evaluate the integral.

$$\int \frac{x^3 - 8x^2 - 1}{\left(x^2 + x - 6\right)\left(x^2 + 1\right)}\ dx$$

Answer choices:

A $\quad 2\ln|x+3| - \ln|x-2| - \tan^{-1}x + C$

B $\quad 2\ln|x+3| - \ln|x-2| + \tan^{-1}x + C$

C $\quad 2\ln|x+3| - \ln|x-2| + C$

D $\quad 2\ln|x+3| + \ln|x-2| + C$

Solution: A

First, factor the denominator.

$$\int \frac{x^3-8x^2-1}{\left(x^2+x-6\right)\left(x^2+1\right)}\ dx=\int \frac{x^3-8x^2-1}{(x+3)(x-2)\left(x^2+1\right)}\ dx$$

Using partial fractions decomposition containing a quadratic factor, we have

$$\frac{x^3-8x^2-1}{(x+3)(x-2)\left(x^2+1\right)}=\frac{A}{x+3}+\frac{B}{x-2}+\frac{Cx+D}{x^2+1}$$

Now we'll solve for constants.

$$\left[\frac{x^3-8x^2-1}{(x+3)(x-2)\left(x^2+1\right)}=\frac{A}{x+3}+\frac{B}{x-2}+\frac{Cx+D}{x^2+1}\right](x+3)(x-2)\left(x^2+1\right)$$

$$x^3-8x^2-1=A(x-2)\left(x^2+1\right)+B(x+3)\left(x^2+1\right)+(Cx+D)(x+3)(x-2)$$

$$x^3-8x^2-1=A\left(x^3-2x^2+x-2\right)+B\left(x^3+3x^2+x+3\right)+(Cx+D)\left(x^2+x-6\right)$$

$$x^3-8x^2-1=Ax^3-2Ax^2+Ax-2A+Bx^3+3Bx^2+Bx+3B$$

$$+Cx^3+Cx^2-6Cx+Dx^2+Dx-6D$$

$$x^3-8x^2-1=\left(Ax^3+Bx^3+Cx^3\right)+\left(-2Ax^2+3Bx^2+Cx^2+Dx^2\right)$$

$$+\left(Ax+Bx-6Cx+Dx\right)+\left(-2A+3B-6D\right)$$

$$x^3-8x^2-1=(A+B+C)x^3+(-2A+3B+C+D)x^2$$

$$+(A+B-6C+D)x+(-2A+3B-6D)$$

Equating coefficients on both sides, we get

[1] $A+B+C=1$

[2] $-2A+3B+C+D=-8$

[3] $A+B-6C+D=0$

[4] $-2A+3B-6D=-1$

Subtracting [4] from [2], we get

$$(-2A+3B+C+D=-8)-(-2A+3B-6D=-1)$$

$$-2A+2A+3B-3B+C+D+6D=-8+1$$

[5] $C+7D=-7$

Subtracting [1] from [3], we get

$$(A+B-6C+D=0)-(A+B+C=1)$$

$$A-A+B-B-6C-C+D=0-1$$

[6] $-7C+D=-1$

Multiplying [6] by 7, we get

$$7(-7C+D=-1)$$

[7] $-49C+7D=-7$

Subtracting [7] from [5], we get

$$(C + 7D = -7) - (-49C + 7D = -7)$$

$$C + 49C + 7D - 7D = -7 + 7$$

$$50C = 0$$

$$C = 0$$

Once we've solved for one constant it gets easier to solve for the others.

$$C + 7D = -7$$

$$0 + 7D = -7$$

$$D = -1$$

Plugging the values for C and D into [2] and [3], we get

$$-2A + 3B + 0 - 1 = -8$$

$$A + B - 6(0) - 1 = 0$$

and these simplify to

[8] $-2A + 3B = -7$

[9] $A + B = 1$

Multiplying [9] by 2, we get

$$A + B = 1$$

$$2(A + B = 1)$$

[10] $2A + 2B = 2$

Now we can add **[10]** to **[8]** and get

$$(-2A+3B=-7)+(2A+2B=2)$$

$$-2A+2A+3B+2B=-7+2$$

$$5B=-5$$

$$B=-1$$

Plugging $B=-1$ into $A+B=1$, we get

$$A+B=1$$

$$A-1=1$$

$$A=2$$

With values for all of the constants, we'll plug into the partial fractions decomposition.

$$\frac{x^3-8x^2-1}{(x+3)(x-2)\left(x^2+1\right)}=\frac{2}{x+3}+\frac{-1}{x-2}+\frac{0x-1}{x^2+1}$$

$$\frac{x^3-8x^2-1}{(x+3)(x-2)\left(x^2+1\right)}=\frac{2}{x+3}-\frac{1}{x-2}-\frac{1}{x^2+1}$$

Then we'll put the decomposition back into the integral in place of the original function.

$$\int\frac{x^3-8x^2-1}{\left(x^2+x-6\right)\left(x^2+1\right)}\ dx=\int\frac{2}{x+3}-\frac{1}{x-2}-\frac{1}{x^2+1}\ dx$$

$$2\int \frac{1}{x+3}\ dx - \int \frac{1}{x-2}\ dx - \int \frac{1}{x^2+1}\ dx$$

$$2\ln|x+3| - \ln|x-2| - \tan^{-1}x + C$$

Topic: Distinct quadratic factors

Question: Rewrite the integral using partial fractions. Do not solve it.

$$\int \frac{6x^3 + 2x^2 - x + 12}{(2x^2 - 1)\left(x^2 + 1\right)}\ dx$$

Answer choices:

A $\quad \frac{4}{3}\int \frac{x}{2x^2 - 1}\ dx + \frac{26}{3}\int \frac{1}{2x^2 - 1}\ dx + \frac{7}{3}\int \frac{x}{x^2 + 1}\ dx - \frac{10}{3}\int \frac{1}{x^2 + 1}\ dx$

B $\quad \frac{4}{3}\int \frac{x}{2x^2 - 1}\ dx + \frac{46}{3}\int \frac{1}{2x^2 - 1}\ dx - \frac{7}{3}\int \frac{x}{x^2 + 1}\ dx - \frac{10}{3}\int \frac{1}{x^2 + 1}\ dx$

C $\quad \frac{4}{3}\int \frac{x}{2x^2 + 1}\ dx + \frac{26}{3}\int \frac{1}{2x^2 + 1}\ dx + \frac{7}{3}\int \frac{x}{x^2 - 1}\ dx + \frac{10}{3}\int \frac{1}{x^2 - 1}\ dx$

D $\quad \frac{4}{3}\int \frac{x}{2x^2 - 1}\ dx + \frac{46}{3}\int \frac{1}{2x^2 - 1}\ dx + \frac{7}{3}\int \frac{x}{x^2 + 1}\ dx - \frac{10}{3}\int \frac{1}{x^2 + 1}\ dx$

Solution: A

The denominator is already factored as much as it can be, which means it's a product of irreducible factors.

$$\int \frac{6x^3+2x^2-x+12}{(2x^2-1)\left(x^2+1\right)}\ dx$$

Since the factors are quadratic, we know the numerators are going to be $Ax+B$, $Cx+D$, $Ex+F$, etc. For the partial fractions decomposition, we get

$$\frac{6x^3+2x^2-x+12}{(2x^2-1)\left(x^2+1\right)}=\frac{Ax+B}{2x^2-1}+\frac{Cx+D}{x^2+1}$$

Now we'll solve for constants.

$$\left[\frac{6x^3+2x^2-x+12}{\left(2x^2-1\right)\left(x^2+1\right)}=\frac{Ax+B}{2x^2-1}+\frac{Cx+D}{x^2+1}\right]\left(2x^2-1\right)\left(x^2+1\right)$$

$$6x^3+2x^2-x+12=(Ax+B)\left(x^2+1\right)+(Cx+D)\left(2x^2-1\right)$$

$$6x^3+2x^2-x+12=Ax^3+Ax+Bx^2+B+2Cx^3-Cx+2Dx^2-D$$

$$6x^3+2x^2-x+12=Ax^3+2Cx^3+Bx^2+2Dx^2+Ax-Cx+B-D$$

$$6x^3+2x^2-x+12=(A+2C)x^3+(B+2D)x^2+(A-C)x+(B-D)$$

Equating coefficients on both sides, we get

[1] $A+2C=6$

[2] $A-C=-1$

[3] $B+2D=2$

[4] $B-D=12$

We'll solve for A and C using [1] and [2], and for B and D using [3] and [4]. Subtracting [2] from [1], we get

$$(A+2C=6)-(A-C=-1)$$

$$A-A+2C+C=6+1$$

$$3C=7$$

$$C=\frac{7}{3}$$

Plugging the value for C back into [2] to solve for A, we get

$$A-\frac{7}{3}=-1$$

$$A=\frac{4}{3}$$

Subtracting [4] from [3], we get

$$(B+2D=2)-(B-D=12)$$

$$B-B+2D+D=2-12$$

$$3D=-10$$

$$D=-\frac{10}{3}$$

Plugging the value for D back into [4] to solve for B, we get

$$B-\left(-\frac{10}{3}\right)=12$$

$$B=\frac{26}{3}$$

Plugging the values for each of the four constants back into the partial fractions decomposition, and putting the decomposition back into the integral, we get

$$\int \frac{6x^3+2x^2-x+12}{(2x^2-1)(x^2+1)}\ dx=\int \frac{\frac{4}{3}x+\frac{26}{3}}{2x^2-1}+\frac{\frac{7}{3}x-\frac{10}{3}}{x^2+1}\ dx$$

$$\int \frac{\frac{4}{3}x}{2x^2-1}+\frac{\frac{26}{3}}{2x^2-1}+\frac{\frac{7}{3}x}{x^2+1}+\frac{-\frac{10}{3}}{x^2+1}\ dx$$

$$\frac{4}{3}\int \frac{x}{2x^2-1}\ dx+\frac{26}{3}\int \frac{1}{2x^2-1}\ dx+\frac{7}{3}\int \frac{x}{x^2+1}\ dx-\frac{10}{3}\int \frac{1}{x^2+1}\ dx$$

Topic: Distinct quadratic factors

Question: Rewrite the integral using partial fractions. Do not solve it.

$$\int \frac{x^3 - 2x^2 + 6x - 4}{\left(4x^2 - 2\right)\left(2x^2 + 6\right)}\ dx$$

Answer choices:

A $\quad -\frac{13}{14}\int \frac{x}{4x^2-2}\ dx - \frac{5}{7}\int \frac{1}{4x^2-2}\ dx - \frac{3}{14}\int \frac{x}{2x^2+6}\ dx - \frac{1}{7}\int \frac{1}{2x^2+6}\ dx$

B $\quad \frac{13}{14}\int \frac{x}{4x^2-2}\ dx - \frac{5}{7}\int \frac{1}{4x^2-2}\ dx + \frac{3}{14}\int \frac{x}{2x^2+6}\ dx + \frac{1}{7}\int \frac{1}{2x^2+6}\ dx$

C $\quad \frac{13}{14}\int \frac{x}{4x^2-2}\ dx + \frac{5}{7}\int \frac{1}{4x^2-2}\ dx - \frac{3}{14}\int \frac{x}{2x^2+6}\ dx - \frac{1}{7}\int \frac{1}{2x^2+6}\ dx$

D $\quad \frac{13}{14}\int \frac{x}{4x^2-2}\ dx - \frac{5}{7}\int \frac{1}{4x^2-2}\ dx - \frac{3}{14}\int \frac{x}{2x^2+6}\ dx - \frac{1}{7}\int \frac{1}{2x^2+6}\ dx$

Solution: D

The denominator is already factored as much as it can be, which means it's a product of irreducible factors.

$$\int \frac{x^3 - 2x^2 + 6x - 4}{\left(4x^2 - 2\right)\left(2x^2 + 6\right)}\ dx$$

Since the factors are quadratic, we know the numerators are going to be $Ax + B$, $Cx + D$, $Ex + F$, etc. For the partial fractions decomposition, we get

$$\frac{x^3 - 2x^2 + 6x - 4}{\left(4x^2 - 2\right)\left(2x^2 + 6\right)} = \frac{Ax + B}{4x^2 - 2} + \frac{Cx + D}{2x^2 + 6}$$

Now we'll solve for constants.

$$\left[\frac{x^3 - 2x^2 + 6x - 4}{\left(4x^2 - 2\right)\left(2x^2 + 6\right)} = \frac{Ax + B}{4x^2 - 2} + \frac{Cx + D}{2x^2 + 6}\right]\left(4x^2 - 2\right)\left(2x^2 + 6\right)$$

$$x^3 - 2x^2 + 6x - 4 = (Ax + B)\left(2x^2 + 6\right) + (Cx + D)\left(4x^2 - 2\right)$$

$$x^3 - 2x^2 + 6x - 4 = 2Ax^3 + 6Ax + 2Bx^2 + 6B + 4Cx^3 - 2Cx + 4Dx^2 - 2D$$

$$x^3 - 2x^2 + 6x - 4 = 2Ax^3 + 4Cx^3 + 2Bx^2 + 4Dx^2 + 6Ax - 2Cx + 6B - 2D$$

$$x^3 - 2x^2 + 6x - 4 = (2A + 4C)x^3 + (2B + 4D)x^2 + (6A - 2C)x + (6B - 2D)$$

Equating coefficients on both sides, we get

[1] $2A + 4C = 1$

[2] $6A - 2C = 6$

[3] $2B + 4D = -2$

[4] $6B - 2D = -4$

We'll solve for A and C using [1] and [2], and for B and D using [3] and [4]. Multiplying [1] by 3 so that both [1] and [2] contain $6A$, and then subtracting [2] from [1], we get

$$6A + 12C - (6A - 2C) = 3 - 6$$

$$6A + 12C - 6A + 2C = 3 - 6$$

$$14C = -3$$

$$C = -\frac{3}{14}$$

Plugging the value for C back into [2] to solve for A, we get

$$6A - 2\left(-\frac{3}{14}\right) = 6$$

$$A = \frac{13}{14}$$

Multiplying [3] by 3 so that both [3] and [4] contain $6B$, and then subtracting [4] from [3], we get

$$6B + 12D - (6B - 2D) = -6 - (-4)$$

$$6B + 12D - 6B + 2D = -6 - (-4)$$

$$14D = -2$$

$$D = -\frac{2}{14}$$

Plugging the value for D back into [4] to solve for B, we get

$$6B - 2\left(-\frac{1}{7}\right) = -4$$

$$B = -\frac{5}{7}$$

Plugging the values for each of the four constants back into the partial fractions decomposition, and putting the decomposition back into the integral, we get

$$\int \frac{x^3 - 2x^2 + 6x - 4}{(4x^2 - 2)(2x^2 + 6)}\ dx = \int \frac{\frac{13}{14}x - \frac{5}{7}}{4x^2 - 2} + \frac{-\frac{3}{14}x - \frac{1}{7}}{2x^2 + 6}\ dx$$

$$\int \frac{\frac{13}{14}x - \frac{5}{7}}{4x^2 - 2} - \frac{\frac{3}{14}x + \frac{1}{7}}{2x^2 + 6}\ dx$$

$$\int \frac{\frac{13}{14}x}{4x^2 - 2} + \frac{-\frac{5}{7}}{4x^2 - 2} - \frac{\frac{3}{14}x}{2x^2 + 6} - \frac{\frac{1}{7}}{2x^2 + 6}\ dx$$

$$\frac{13}{14}\int \frac{x}{4x^2 - 2}\ dx - \frac{5}{7}\int \frac{1}{4x^2 - 2}\ dx - \frac{3}{14}\int \frac{x}{2x^2 + 6}\ dx - \frac{1}{7}\int \frac{1}{2x^2 + 6}\ dx$$