Hydrostatic pressure and force

The word “hydrostatic” refers to a liquid at rest. So “hydrostatic force” refers to the force exerted on a solid object by a liquid at rest, and “hydrostatic pressure” refers to the pressure exerted on a solid object by a liquid at rest.

Calculus problems involving hydrostatic pressure and force usually involve calculating the hydrostatic pressure and force that a liquid exerts on the container it’s being held in. In these types of problems, your first step will be to solve for hydrostatic pressure using the formula

$$P = \rho g d$$

where P is hydrostatic pressure, ρ is density of the liquid, g is the gravitational constant 9.8m/s^2, and d is the depth of the liquid (not the depth of the container). The units for pressure are Pascals Pa, or equivalently, kg/ms^2.

With a value for pressure, you’ll then solve for hydrostatic force exerted by the liquid on the bottom of the container, using the formula

$$F = PA$$

where F is hydrostatic force, P is hydrostatic pressure you found earlier, and A is the square area of the bottom of the container. The units for force are Newtons N, or equivalently, kg m/s^2.

If you need to solve for hydrostatic force on the *end* of your container, instead of on the *bottom*, or on an upright plate inserted into your container, you'll use the modified force equation

$$F = WAd$$

where F is hydrostatic force, W is weight (density $\times$ gravity), A is the area of the vertical surface, and d is the depth of the liquid (not the depth of the container).

Example

A tank is 6 m wide, 12 m long and 4 m deep. It's filled with water of density $1{,}000$ kg/m^3 to a depth of 3 m.

a) Find the hydrostatic pressure at the bottom of the tank.

b) Find the hydrostatic force on the bottom of the tank.

c) Find the hydrostatic force on one end of the tank.

Hydrostatic pressure

To find hydrostatic pressure at the bottom of the tank, we'll use $P = \rho g d$. We're told that $\rho = 1{,}000$ kg/m^3 and that $d = 3$ m. Remember that depth refers to the depth of the liquid, not the depth of the tank (although the two are equal when the tank is completely full). Finally, we know that $g = 9.8$ m/s^2.

$$P=\left(\frac{1{,}000\text{kg}}{\text{m}^3}\right)\left(\frac{9.8\text{m}}{\text{s}^2}\right)(3\text{m})$$

$$P=\frac{29{,}400\text{kg}}{\text{ms}^2}$$

$$P=2.94\times10^4\ \text{kg/ms}^2$$

or

$$P=2.94\times10^4\ \text{Pa}$$

The hydrostatic pressure at the bottom of the tank is $P=2.94\times10^4$ Pa.

Hydrostatic force on the bottom

To find hydrostatic force on the bottom of the tank, we'll use $F=PA$. From the previous part, we know that $P=29{,}400$ Pa. To calculate area, we'll use the fact that the tank is 6 m wide and 12 m long. Which means the area of the bottom of the tank is

$$A=(6\ \text{m})(12\ \text{m})$$

$$A=72\ \text{m}^2$$

Next, we can solve for the hydrostatic force at the bottom of the pool.

$$F=\left(\frac{29{,}400\ \text{kg}}{\text{ms}^2}\right)\left(72\ \text{m}^2\right)$$

$$F=\frac{2{,}116{,}800\ \text{kg}\cdot\text{m}}{\text{s}^2}$$

$$F = 2.12 \times 10^6 \text{ kg} \cdot \text{m/s}^2$$

or

$$F = 2.12 \times 10^6 \text{ N}$$

The hydrostatic force at the bottom of the tank is $F = 2.12 \times 10^6$ N.

Hydrostatic force on the end

To find hydrostatic force on one end of the tank, we'll use the modified force equation $F = WAd$.

Since weight is density $\times$ gravity, weight is

$$W = \left(\frac{1{,}000 \text{ kg}}{\text{m}^3}\right)\left(\frac{9.8 \text{ m}}{\text{s}^2}\right)$$

$$W = \left(\frac{9{,}800 \text{ kg}}{\text{m}^2\text{s}^2}\right)$$

Since we're looking for force against a vertical surface and force at deeper depths is greater than force at shallower depths, we can't use the area of the entire surface in our force equation. Instead, we have to divide the surface into small horizontal strips so that we can assume that the force against each strip is roughly the same throughout the strip.

If we divide the end of the tank into tiny slices of equal depth, then each strip is 6 m wide and Δx tall, and sitting at a depth of x_i. The area of one strip is $A_i = 6 \cdot \Delta x$. The force against one strip is

$$F = WAd$$

$$F_i = (9{,}800)(6\Delta x)(x_i)$$

In order to solve for the force against the end of the tank, instead of against a small strip of it, we need to sum together the force against all of the slices, and take the limit as the number of slices approaches infinity, $n \to \infty$. Let's put this all together and see how it looks.

$$F = \lim_{n\to\infty} \sum_{i=1}^{n} (9{,}800)(6\Delta x)(x_i)$$

We need to remember that taking the limit as $n \to \infty$ of the sum of the force against all of the slices is the same as taking the integral of our force equation over the interval of the depth, $[0,3]$. Remember, when we move this into an integral, x_i becomes x, and Δx becomes dx. Let's put this all together and see how it looks.

$$F = \int_0^3 (9{,}800)(6\ dx)x$$

$$F = 58{,}800 \int_0^3 x\ dx$$

$$F = 58{,}800 \left(\frac{x^2}{2}\right)\Bigg|_0^3$$

$$F = 29{,}400x^2\Big|_0^3$$

$$F = 29{,}400(3)^2 - 29{,}400(0)^2$$

$$F = 264{,}600$$

$$F = 2.65 \times 10^5$$

The hydrostatic force on the end is $F = 2.65 \times 10^5$ N.