Topic: Arc length of y=f(x)

Question: Find the arc length of the curve over the given interval.

$$8y = x^4 + 2x^{-2}$$

on the interval $[1,2]$

Answer choices:

A $\frac{1}{16}$

B $\frac{33}{16}$

C $\frac{27}{16}$

D $\frac{33}{8}$

Solution: B

The formula for arc length for a curve defined as $y=f(x)$ and with limits of integration given as $x=a$ and $x=b$ is

$$L=\int_a^b \sqrt{1+\left[f'(x)\right]^2}\ dx$$

We already know that $a=1$ and $b=2$. The only other thing we need for our formula is $f'(x)$, which we'll find by taking the derivative of our original function and solving for y'.

$$8y'=4x^3-4x^{-3}$$

$$y'=\frac{1}{2}x^3-\frac{1}{2}x^{-3}$$

Plugging all of these values into the formula, we get

$$L=\int_1^2 \sqrt{1+\left(\frac{1}{2}x^3-\frac{1}{2}x^{-3}\right)^2}\ dx$$

$$L=\int_1^2 \sqrt{1+\left(\frac{x^3}{2}-\frac{1}{2x^3}\right)^2}\ dx$$

Find a common denominator and combine fractions.

$$L=\int_1^2 \sqrt{1+\left(\frac{x^6}{2x^3}-\frac{1}{2x^3}\right)^2}\ dx$$

$$L=\int_1^2\sqrt{1+\left(\frac{x^6-1}{2x^3}\right)^2}\ dx$$

$$L=\int_1^2\sqrt{1+\frac{\left(x^6-1\right)^2}{4x^6}}\ dx$$

Find a common denominator and combine fractions.

$$L=\int_1^2\sqrt{\frac{4x^6}{4x^6}+\frac{\left(x^6-1\right)^2}{4x^6}}\ dx$$

$$L=\int_1^2\sqrt{\frac{4x^6+\left(x^6-1\right)^2}{4x^6}}\ dx$$

Take the square root of the numerator and denominator separately.

$$L=\frac{1}{2}\int_1^2\frac{1}{x^3}\sqrt{4x^6+x^{12}-2x^6+1}\ dx$$

$$L=\frac{1}{2}\int_1^2\frac{1}{x^3}\sqrt{x^{12}+2x^6+1}\ dx$$

$$L=\frac{1}{2}\int_1^2\frac{1}{x^3}\sqrt{\left(x^6+1\right)^2}\ dx$$

$$L=\frac{1}{2}\int_1^2\frac{1}{x^3}\left(x^6+1\right)\ dx$$

$$L=\frac{1}{2}\int_1^2\frac{x^6}{x^3}+\frac{1}{x^3}\ dx$$

$$L = \frac{1}{2}\int_1^2 x^3 + x^{-3}\ dx$$

Take the integral, then evaluate over the given interval.

$$L = \frac{1}{2}\left(\frac{1}{4}x^4 + \frac{1}{-2}x^{-2}\right)\Bigg|_1^2$$

$$L = \frac{1}{2}\left(\frac{x^4}{4} - \frac{1}{2x^2}\right)\Bigg|_1^2$$

$$L = \frac{1}{4}\left(\frac{x^4}{2} - \frac{1}{x^2}\right)\Bigg|_1^2$$

$$L = \frac{1}{4}\left[\left(\frac{(2)^4}{2} - \frac{1}{(2)^2}\right) - \left(\frac{(1)^4}{2} - \frac{1}{(1)^2}\right)\right]$$

$$L = \frac{1}{4}\left[\left(8 - \frac{1}{4}\right) - \left(\frac{1}{2} - 1\right)\right]$$

$$L = \frac{1}{4}\left(8 - \frac{1}{4} - \frac{1}{2} + 1\right)$$

$$L = \frac{1}{4}\left(\frac{72}{8} - \frac{2}{8} - \frac{4}{8}\right)$$

$$L = \frac{66}{32}$$

$$L = \frac{33}{16}$$

Topic: Arc length of y=f(x)

Question: Find the arc length of the curve over the given interval.

$$y = \frac{1}{3}\left(x^2 + 2\right)^{\frac{3}{2}}$$

on the interval $[0,3]$

Answer choices:

A 12

B 16

C 18

D 3

Solution: A

The formula for the arc length for a curve defined as $y=f(x)$ and with limits of integration give as $x=a$ and $x=b$ is

$$L=\int_a^b \sqrt{1+\left[f'(x)\right]^2}\ dx$$

We already know that $a=0$ and $b=3$. The only other thing we need for our formula is $f'(x)$, which we'll find by taking the derivative of our original function.

$$y'=x\left(x^2+2\right)^{\frac{1}{2}}$$

Plugging all of these values into the formula, we get

$$L=\int_0^3 \sqrt{1+\left[x\left(x^2+2\right)^{\frac{1}{2}}\right]^2}\ dx$$

$$L=\int_0^3 \sqrt{1+x^2\left(x^2+2\right)}\ dx$$

$$L=\int_0^3 \sqrt{1+x^4+2x^2}\ dx$$

$$L=\int_0^3 \sqrt{x^4+2x^2+1}\ dx$$

$$L=\int_0^3 \sqrt{\left(x^2+1\right)^2}\ dx$$

$$L=\int_0^3 x^2+1\ dx$$

Take the integral, then evaluate over the given interval.

$$L = \frac{1}{3}x^3 + x \Big|_0^3$$

$$L = \left[\frac{1}{3}(3)^3 + 3\right] - \left[\frac{1}{3}(0)^3 + 0\right]$$

$$L = 9 + 3$$

$$L = 12$$

Topic: Arc length of y=f(x)

Question: Find the arc length of the curve over the given interval.

$$y = 2x^{\frac{3}{2}}$$

on the interval $\left[\frac{1}{3},7\right]$

Answer choices:

A $\frac{1,008}{25}$

B $\frac{112}{3}$

C $\frac{1,008}{3}$

D $\frac{112}{9}$

Solution: B

Since our curve is defined as $y = f(x)$ and the limits of integration are $x = a$ and $x = b$, we know that the applicable arc length formula is

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\ dx$$

We already know that our limits of integration are $a = 1/3$ and $b = 7$. The derivative of our original function is $y' = 3x^{\frac{1}{2}}$, so the arc length of the curve on the given interval is

$$L = \int_{\frac{1}{3}}^{7} \sqrt{1 + \left(3x^{\frac{1}{2}}\right)^2}\ dx$$

$$L = \int_{\frac{1}{3}}^{7} \sqrt{1 + 9x}\ dx$$

Use u-substitution and let

$$u = 1 + 9x$$

$$du = 9\ dx$$

$$dx = \frac{du}{9}$$

Plugging the substitution into the integral, we get

$$L = \int_{x=\frac{1}{3}}^{x=7} \sqrt{u}\ \frac{du}{9}$$

$$L = \frac{1}{9} \int_{x=\frac{1}{3}}^{x=7} \sqrt{u} \ du$$

Take the integral, then evaluate over the given interval.

$$L = \frac{1}{9} \left(\frac{2}{3} u^{\frac{3}{2}} \right) \Bigg|_{x=\frac{1}{3}}^{x=7}$$

Back substitute to get the problem back in terms of x.

$$L = \frac{1}{9} \left[\frac{2}{3} (1 + 9x)^{\frac{3}{2}} \right] \Bigg|_{\frac{1}{3}}^{7}$$

$$L = \frac{2}{27} (1 + 9x)^{\frac{3}{2}} \Bigg|_{\frac{1}{3}}^{7}$$

$$L = \frac{2}{27} \left[1 + 9(7) \right]^{\frac{3}{2}} - \frac{2}{27} \left[1 + 9 \left(\frac{1}{3} \right) \right]^{\frac{3}{2}}$$

$$L = \frac{2}{27} (64)^{\frac{3}{2}} - \frac{2}{27} (4)^{\frac{3}{2}}$$

$$L = \frac{2}{27} \left(64^{\frac{1}{2}} \right)^3 - \frac{2}{27} \left(4^{\frac{1}{2}} \right)^3$$

$$L = \frac{2}{27} \left[\left(64^{\frac{1}{2}} \right)^3 - \left(4^{\frac{1}{2}} \right)^3 \right]$$

$$L = \frac{2}{27} (512 - 8)$$

$$L = \frac{2}{27} (504)$$

$$L=\frac{2}{3}(56)$$

$$L=\frac{112}{3}$$