Radius of the balloon

Previously, we learned how to use implicit differentiation to take the derivative of an equation in which the x and y were mixed together on the same side of the equation. For instance,

$$x^2y^2 = 2x + 4y$$

is an equation with x and y variables mixed together on both sides of the equation, and no matter how much we try to manipulate it, we can't rewrite the equation in a way that completely separates the variables from each other, putting y and one side and x on the other.

Therefore, to differentiate that equation, we'd have to use implicit differentiation.

Related rates

At this point, we want to turn our attention toward related rates problems, which we can only solve using implicit differentiation.

Related rates problems are usually easy to spot, because they ask us to find how quickly one variable is changing when we know how quickly another variable is changing. For example, these are all related rates questions:

- How fast is the radius of a tire increasing if air is being pumped into the tire at a particular rate?

- How fast is the water level in a swimming pool increasing if water is being fed into the pool at a particular rate?

- How fast is the distance between you and your friend decreasing if you're walking toward him at a particular rate?

These sound tricky, but we'll actually follow a fairly consistent set of steps in order to solve every related rates problem. In general, we want to follow these steps:

1. Build an equation containing all the relevant variables, solving for some of them using other information, if necessary.

2. Implicitly differentiate the equation with respect to time t, before plugging in any of the values we know.

3. Plug in all the values we know, leaving only the one we're trying to solving for.

4. Solve for the unknown variable.

Related rates is one of those concepts where practicing lots and lots of problems really does make a difference, because the more problems you do, the better feel you get for this pattern of problem solving.

Radius of the balloon

There are an infinite number of different kinds of related rates problems we can do, but there are also some related rates problems in particular that are really common. In this section we're going to focus each lesson on one of these most common types, starting here with the inflating/deflating balloon problem.

The scenario is that we have either an inflating balloon, or a deflating balloon. We're usually told to treat the balloon as a perfect sphere, and then we're given information about the volume of the balloon, and/or the radius of the balloon.

Remember from geometry that that the formula for the volume of a sphere and the surface area of a sphere are

$$V = \frac{4}{3}\pi r^3$$

$$S = 4\pi r^2$$

where V is the volume of the sphere and r is the radius of the sphere. We'll usually use this formula in these kinds of related rates problems.

Let's work through an example where we're given the rate at which volume is increasing, and asked to find the rate at which the length of the radius is increasing.

Example

How fast is the radius of a spherical balloon increasing when the radius is 100 cm, if air is being pumped into it at 400 cm^3/s?

In this example, we're asked to find the rate of change of the radius, given the rate of change of the volume. So right away we know we're looking for a formula that relates the volume of a sphere to the radius of a sphere, and we can get that from the formula for volume of a sphere:

$$V = \frac{4}{3}\pi r^3$$

Before doing anything else, we use implicit differentiation to differentiate both sides with respect to time t. When we used implicit differentiation in the past, we'd treat x like a normal variable, but we'd treat y as a function of x. That meant that, every time we took the derivative of y, we had to multiply by y'.

For these related rates problems, we treat t like a normal variable, and we treat every other variable as a function of t. Which means that, every time we take the derivative of one of these other variables, we have to multiply by the derivative of that variable.

So when we use implicit differentiation to differentiate both sides of the equation for the volume of a sphere, we'll multiply by V' when we differentiate V, and multiply by r' when we differentiate r.

$$1(V') = \frac{4}{3}\pi(3r^2)(r')$$

$$V' = 4\pi r^2 r'$$

Notice how we took the derivative of V and r like normal, but then we multiplied by V' and r'. Remember that π is a constant, so the $(4/3)\pi$ just acts like a coefficient in front of the r^3, and it stays right where it is.

Because V and r are both functions with respect to time t, we can replace their derivatives V' and r' with dV/dt and dr/dt. Keep in mind that dV/dt

represents the rate at which the volume is changing, dr/dt is the rate at which the radius is changing, and r is the length of the radius at a specific moment in time.

$$\frac{dV}{dt}=4\pi r^2\frac{dr}{dt}$$

Now that we're done with the differentiation, we can substitute for any values we already know. The problem tells us that air is being pumped into the balloon, which means the volume of the balloon will be changing at that rate, so the rate of change of the volume is 400 cm^3/s.

$$400=4\pi r^2\frac{dr}{dt}$$

We've also been told that the length of the radius at the specific moment we're interested in is 100.

$$400=4\pi(100)^2\frac{dr}{dt}$$

$$400=4\pi(10{,}000)\frac{dr}{dt}$$

$$400=40{,}000\pi\frac{dr}{dt}$$

We've been asked to figure out how fast the radius of the balloon is changing. The rate of change of the radius is dr/dt, so we need to solve this equation for dr/dt.

$$\frac{dr}{dt}=\frac{400}{40{,}000\pi}$$

$$\frac{dr}{dt} = \frac{1}{100\pi}$$

This equation says that the rate of change of the radius is $1/100\pi$ cm per second, so we can say that, if air is being pumped into the balloon at 400 cm^3/s, then the length of the radius is increasing at $1/100\pi$ cm/s at the exact moment in time when the length of the radius is 100 cm.