Limit vs. sum of the series

Sometimes it's easy to forget that there's a difference between the *limit* of an infinite series and the *sum* of an infinite series.

> The **limit** of a series is the value the series' terms are approaching as $n \to \infty$.
>
> The **sum** of a series is the value of all the series' terms added together.

They're two very different things, and we use a different calculation to find each one. Let's find both the limit and the sum of the same series so that we can see the difference.

Example

Find the limit and the sum of the series.

$$\sum_{n=1}^{\infty} \frac{2^{3n}}{64^{\frac{n}{2}}}$$

To find the limit of the series, we'll identify the series as a_n, and then take the limit of a_n as $n \to \infty$.

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{2^{3n}}{64^{\frac{n}{2}}}$$

$$\lim_{n\to\infty} \frac{\left(2^3\right)^n}{\left(64^{\frac{1}{2}}\right)^n}$$

$$\lim_{n\to\infty} \frac{8^n}{\left(\sqrt{64}\right)^n}$$

$$\lim_{n\to\infty} \frac{8^n}{8^n}$$

$$\lim_{n\to\infty} 1$$

$$1$$

The limit of the series is 1.

To find the sum of the series, we'll expand the series.

$$\sum_{n=1}^{\infty} \frac{2^{3n}}{64^{\frac{n}{2}}} = \frac{2^{3(1)}}{64^{\frac{1}{2}}} + \frac{2^{3(2)}}{64^{\frac{2}{2}}} + \frac{2^{3(3)}}{64^{\frac{3}{2}}} + \dots$$

$$\sum_{n=1}^{\infty} \frac{2^{3n}}{64^{\frac{n}{2}}} = \frac{2^3}{\sqrt{64}} + \frac{2^6}{\left(\sqrt{64}\right)^2} + \frac{2^9}{\left(\sqrt{64}\right)^3} + \dots$$

$$\sum_{n=1}^{\infty} \frac{2^{3n}}{64^{\frac{n}{2}}} = \frac{8}{8} + \frac{64}{8^2} + \frac{512}{8^3} + \dots$$

$$\sum_{n=1}^{\infty} \frac{2^{3n}}{64^{\frac{n}{2}}} = \frac{8}{8} + \frac{64}{64} + \frac{512}{512} + \dots$$

$$\sum_{n=1}^{\infty} \frac{2^{3n}}{64^{\frac{n}{2}}} = 1 + 1 + 1 + \dots$$

$$\sum_{n=1}^{\infty} \frac{2^{3n}}{64^{\frac{n}{2}}} = \infty$$

Every term in our series will be equal to 1. Since we have an infinite number of terms in our series, we can say that the sum is infinite.

We can see that the limit of the series is 1, but the sum of the same series is ∞.