Limit comparison test

The limit comparison test for convergence lets us determine the convergence or divergence of the given series a_n by *comparing* it to a similar, but simpler comparison series b_n.

We're usually trying to find a comparison series that's a geometric or p-series, since it's very easy to determine the convergence of a geometric or p-series.

We can use the limit comparison test to show that

the original series a_n is **diverging** if

$a_n \geq 0$ and $b_n > 0$,

$L = \lim_{n\to\infty} \frac{a_n}{b_n}$ and $0 < L < \infty$, and

the comparison series b_n is diverging

the original series a_n is **converging** if

$a_n \geq 0$ and $b_n > 0$,

$L = \lim_{n\to\infty} \frac{a_n}{b_n}$ and $0 < L < \infty$, and

the comparison series b_n is converging

Example

Use the limit comparison test to say whether or not the series is converging.

$$\sum_{n=1}^{\infty} \frac{6n}{2n^3+3}$$

We need to find a series that's similar to the original series, but simpler. The original series is

$$a_n = \frac{6n}{2n^3+3}$$

For the comparison series, we'll use the same numerator as the original series, since it's already pretty simple, but we'll drop the 6 since it has little effect on the series as $n \to \infty$. Looking at the denominator, we can see that the first term $2n^3$ carries more weight and will affect our series more than the second term 3, so we'll just use the first term from the original denominator for the denominator of our comparison series, but drop the 2, and the comparison series is

$$b_n = \frac{n}{n^3}$$

$$b_n = \frac{1}{n^2}$$

We can see that this simplified version of b_n is just a p-series, where $p = 2$. We'll use the p-series test for convergence to say whether or not b_n converges. Remember, the p-series test says that the series will

converge when $p > 1$

diverge when $p \le 1$

Since $p = 2$ in b_n, we know that b_n converges.

That means we need to show that $a_n > 0$ and $b_n > 0$ and that

$$\lim_{n\to\infty} \frac{a_n}{b_n} = L > 0$$

in order to prove that the original series a_n is also converging.

Let's try to verify that $a_n > 0$ and $b_n > 0$ by checking a few points for both a_n and b_n, like $n = 1$, $n = 2$ and $n = 3$.

		a_n		b_n
$n = 1$	$\frac{6(1)}{2(1)^3 + 3}$	$\frac{6}{5}$	$\frac{1}{(1)^2}$	1
$n = 2$	$\frac{6(2)}{2(2)^3 + 3}$	$\frac{12}{19}$	$\frac{1}{(2)^2}$	$\frac{1}{4}$
$n = 3$	$\frac{6(3)}{2(3)^3 + 3}$	$\frac{18}{57}$	$\frac{1}{(3)^2}$	$\frac{1}{9}$

Looking at these three terms, we can see that $a_n > 0$ and $b_n > 0$. There's no positive value of n that will make a term in a_n or b_n negative.

The last thing we need to verify is

$$\lim_{n\to\infty} \frac{a_n}{b_n} = L > 0$$

Plugging a_n and b_n into the limit formula gives

$$L = \lim_{n \to \infty} \frac{\frac{6n}{2n^3 + 3}}{\frac{1}{n^2}}$$

$$L = \lim_{n \to \infty} \frac{6n}{2n^3 + 3} \left(\frac{n^2}{1} \right)$$

$$L = \lim_{n \to \infty} \frac{6n^3}{2n^3 + 3}$$

$$L = \lim_{n \to \infty} \frac{6n^3}{2n^3 + 3} \left(\frac{\frac{1}{n^3}}{\frac{1}{n^3}} \right)$$

$$L = \lim_{n \to \infty} \frac{\frac{6n^3}{n^3}}{\frac{2n^3}{n^3} + \frac{3}{n^3}}$$

$$L = \lim_{n \to \infty} \frac{6}{2 + \frac{3}{n^3}}$$

$$L = \frac{6}{2 + \frac{3}{\infty}}$$

$$L = \frac{6}{2 + 0}$$

$$L = 3$$

Since

$L = 3 > 0,$

$a_n > 0$ and $b_n > 0$, and

the comparison series b_n is converging,

we can say the the original series a_n is also converging.