Topic: Net change theorem

Question: Water is being pumped into a tank at a rate (in gallons per minute) given by $w(t) = 60 - 10\sqrt{t}$, with $0 \le t \le 120$ where t is the time in minutes since the pumping began. The tank had $1{,}200$ gallons of water in it when pumping began. Use the Net Change Theorem to determine how much water will be in the tank after 49 minutes of pumping.

Answer choices:

A $653\frac{1}{3}$ gallons

B $1{,}853\frac{1}{3}$ gallons

C $6{,}426\frac{1}{3}$ gallons

D $4{,}140$ gallons

Solution: B

The question asks us to determine the amount of water in a tank that has an initial volume of $1{,}200$ gallons, with water being pumped into the tank at a rate (in gallons per minute) of $w(t) = 60 - 10\sqrt{t}$ for 49 minutes. We'll use the Net Change Theorem to answer this question.

The Net Change Theorem states that if we integrate a rate of change expression of a function, we will find the net amount of the change of the function during the period of integration. Thus, if $f'(x)$ is the derivative of $f(x)$, or in other words, if $f'(x)$ is the rate of change of $f(x)$, then

$$\int_a^b f'(x)\ dx = f(b) - f(a)$$

This means that the integral of the rate of change is the net change. This theorem uses the concepts of the Fundamental Theorem of Calculus for integration.

Let's begin by writing the integral for this problem. Since the tank initially has $1{,}200$ gallons in it, we'll add $1{,}200$ to the integral. Also, since the question asks us to find the volume of water in the tank after 49 minutes of pumping, the integration of the pumping rate will be performed on the interval $[0,49]$.

$$1{,}200 + \int_0^{49} 60 - 10\sqrt{t}\ dt$$

To integrate, we'll find the anti-derivative of the integrand, so to make the integration a little easier, we will write the integrand using exponents.

$$1{,}200+\int_0^{49}60-10t^{\frac{1}{2}}\ dt$$

Now, integrate using the power rule for integration; adding 1 to the exponent and dividing by the new exponent in each term.

$$1{,}200+\int_0^{49}60-10t^{\frac{1}{2}}\ dt=1{,}200+\left[60t-10\left(\frac{t^{\frac{3}{2}}}{\frac{3}{2}}\right)\right]\Bigg|_0^{49}$$

We can change the division by a fraction to multiplication by the inverse of the fraction. Then we'll evaluate the anti-derivative at the upper and lower limits.

$$1{,}200+\left[60t-10\left(\frac{2}{3}t^{\frac{3}{2}}\right)\right]\Bigg|_0^{49}$$

$$1{,}200+\left[60(49)-\frac{20}{3}(49)^{\frac{3}{2}}\right]-\left[60(0)-\frac{20}{3}(0)^{\frac{3}{2}}\right]$$

$$1{,}200+\left[60(49)-\frac{20}{3}(49)^{\frac{3}{2}}\right]$$

$$1{,}200+2{,}940-\frac{20}{3}(343)$$

$$\frac{3{,}600}{3}+\frac{8{,}820}{3}-\frac{6{,}860}{3}$$

$$\frac{5{,}560}{3}$$

$1,853.33$

After 49 minutes of pumping, this is the number of gallons in the tank.

Topic: Net change theorem

Question: Beginning at $15{,}000$ feet, a commercial jetliner begins climbing at a rate (in feet per minute) given by $f(t)=e^{0.4t}+8$, with $0 \le t \le 20$, where t is the time in minutes since the aircraft began climbing. Use the Net Change Theorem to determine the aircraft's elevation after climbing for 15 minutes. Round the answer to the nearest foot.

Answer choices:

A $\quad 1{,}126$ feet

B $\quad 15{,}526$ feet

C $\quad 16{,}126$ feet

D $\quad 13{,}874$ feet

Solution: C

The question asks us to determine the elevation of an aircraft that, from an initial elevation of $15{,}000$ feet, climbs at a rate (in feet per minute) of $f(t) = e^{0.4t} + 8$ for 15 minutes. We will use the Net Change Theorem to answer this question.

The Net Change Theorem states that if we integrate a rate of change expression of a function, we will find the net amount of the change of the function during the period of integration. Thus, if $f'(x)$ is the derivative of $f(x)$, or in other words, if $f'(x)$ is the rate of change of $f(x)$, then

$$\int_a^b f'(x)\ dx = f(b) - f(a)$$

This means that the integral of the rate of change is the net change. This theorem uses the concepts of the Fundamental Theorem of Calculus for integration.

Let's begin by writing the integral for this problem. Since the aircraft is initially at an altitude of $15{,}000$ feet, we'll add $15{,}000$ to the integral. Also, since the question asks us to find the aircraft's elevation after 15 minutes of climbing, the integration of the climbing rate will be performed on the interval $[0,15]$.

$$15{,}000 + \int_0^{15} e^{0.4t} + 8\ dt$$

To integrate, we'll find the anti-derivative of the integrand, so we'll use the chain rule to integrate $e^{0.4t}$. The integration rule of an expression e^u is e^u/du. In this case, $u = 0.4t$ so $du = 0.4$.

Now, integrate using the chain rule and the power rule for integration; adding 1 to the exponent and dividing by the new exponent.

$$15{,}000+\int_0^{15} e^{0.4t}+8\ dt=15{,}000+\left(\frac{e^{0.4t}}{0.4}+8t\right)\Big|_0^{15}$$

Now, we will evaluate the result of the integration at the upper and lower limits.

$$15{,}000+\left(\frac{e^{0.4t}}{0.4}+8t\right)_0^{15}=15{,}000+\left[\frac{e^{0.4(15)}}{0.4}+8(15)\right]-\left[\frac{e^{0.4(0)}}{0.4}+8(0)\right]$$

$$15{,}000+\left[\frac{e^{0.4(15)}}{0.4}+8(15)\right]-\left[\frac{e^{0.4(0)}}{0.4}+8(0)\right]$$

$$15{,}000+1008.571984+120-2.5$$

$$16{,}126.071984$$

Thus, after climbing for 15 minutes from an elevation of $15{,}000$ feet, the aircraft is flying at an elevation of $16{,}126$ feet.

Topic: Net change theorem

Question: A tank with $1,000$ gallons of solvent is leaking at a rate (in gallons per hour) given by the the function $l(t)$, where t is the time in hours since the tank began leaking. Use the Net Change Theorem to determine the amount of solvent in the tank after 2 days. Round the answer to the nearest gallon.

$$l(t) = \frac{1}{4}\sqrt{t} + 3$$

with $0 \leq t \leq 240$

Answer choices:

A 801 gallons

B $1,199$ gallons

C 199 gallons

D 994 gallons

Solution: A

The question asks us to determine the amount of solvent in a tank that initially contains $1{,}000$ gallons, and leaks at a rate (in gallons per hour) of

$$l(t) = \frac{1}{4}\sqrt{t} + 3$$

for 2 days. We'll use the Net Change Theorem to answer this question.

The Net Change Theorem states that if we integrate a rate of change expression of a function, we will find the net amount of the change of the function during the period of integration. Thus, if $f'(x)$ is the derivative of $f(x)$, or in other words, if $f'(x)$ is the rate of change of $f(x)$, then

$$\int_a^b f'(x)\ dx = f(b) - f(a)$$

This means that the integral of the rate of change is the net change. This theorem uses the concepts of the Fundamental Theorem of Calculus for integration.

Let's begin by writing the integral for this problem. Since the tank initially has $1{,}000$ gallons of solvent in it, and it is leaking, we will subtract the integral from $1{,}000$ to find the amount of solvent remaining in the tank after two days. Furthermore, the question states the rate of the leak is in gallons per hour, and we are asked to find the amount in the tank after two days of leaking, we know that the tank leaked for 48 hours. Thus, our integration will be performed on the interval $[0,48]$.

$$1{,}000-\int_0^{48}\frac{1}{4}\sqrt{t}+3\ dt$$

To integrate, we will find the anti-derivative of the integrand, so we will use the exponent rule to integrate the term $(1/4)\sqrt{x}$. The integration rule of an expression $\sqrt{x}$ is

$$\frac{x^{\frac{3}{2}}}{\frac{3}{2}}$$

To integrate, we'll find the anti-derivative of the integrand, so to make the integration a little easier, we'll write the integrand using exponents.

$$1{,}000-\int_0^{48}\frac{1}{4}t^{\frac{1}{2}}+3\ dt$$

Now, integrate using the power rule for integration; adding 1 to the exponent and dividing by the new exponent.

$$1{,}000-\left[\left(\frac{1}{4}\right)\frac{t^{\frac{3}{2}}}{\frac{3}{2}}+3t\right]\Bigg|_0^{48}$$

Since we're dividing by a fraction, we can change to multiplying by the reciprocal of the denominator. Then we will evaluate the anti-derivative at the upper and lower limits.

$$1{,}000-\left[\left(\frac{1}{4}\right)\left(\frac{2}{3}\right)t^{\frac{3}{2}}+3t\right]\Bigg|_0^{48}$$

$$1{,}000 - \left(\frac{1}{6}t^{\frac{3}{2}} + 3t\right)\Bigg|_0^{48}$$

$$1{,}000 - \frac{1}{6}(48)^{\frac{3}{2}} - 3(48)$$

$$1{,}000 - 55.425358 - 144$$

$$800.574642$$

The question asked us to round the answer to the nearest whole gallon, so the tank will have 801 gallons of solvent after leaking for 2 days.