Calculus 2 Workbook Solutions

Improper integrals

krista king MATH

IMPROPER INTEGRALS, CASE 1

1. Evaluate the improper integral.

$$\int_1^\infty \frac{5}{x^3}\ dx$$

Solution:

Replace the upper limit, rewriting the integral as

$$\lim_{b\to\infty}\int_1^b \frac{5}{x^3}\ dx$$

$$\lim_{b\to\infty}\int_1^b 5x^{-3}\ dx$$

Integrate, then evaluate over the interval.

$$\lim_{b\to\infty}\frac{5x^{-2}}{-2}\Big|_1^b$$

$$-\frac{5}{2}\lim_{b\to\infty}\frac{1}{x^2}\Big|_1^b$$

$$-\frac{5}{2}\lim_{b\to\infty}\frac{1}{b^2}-\frac{1}{1^2}$$

$$-\frac{5}{2}\left(\frac{1}{\infty^2}-1\right)$$

$$-\frac{5}{2}(0-1)=\frac{5}{2}$$

■ 2. Evaluate the improper integral.

$$\int_3^\infty \frac{7}{(x-2)^2}\ dx$$

Solution:

Replace the upper limit, rewriting the integral as

$$\lim_{b\to\infty}\int_3^b \frac{7}{(x-2)^2}\ dx$$

$$\lim_{b\to\infty}\int_3^b 7(x-2)^{-2}\ dx$$

Integrate, then evaluate over the interval.

$$\lim_{b\to\infty}\frac{7(x-2)^{-1}}{-1}\Bigg|_3^b$$

$$-7\lim_{b\to\infty}\frac{1}{x-2}\Bigg|_3^b$$

$$-7 \lim_{b\to\infty} \frac{1}{b-2} - \frac{1}{3-2}$$

$$-7\left(\frac{1}{\infty-2} - \frac{1}{1}\right)$$

$$-7(0-1)$$

$$7$$

■ 3. Evaluate the improper integral.

$$\int_0^{\infty} 2e^{-2x}\ dx$$

Solution:

Replace the upper limit, rewriting the integral as

$$\lim_{b\to\infty} \int_0^b 2e^{-2x}\ dx$$

Integrate, then evaluate over the interval.

$$\lim_{b\to\infty} -e^{-2x}\Big|_0^b$$

$$-\lim_{b\to\infty} \frac{1}{e^{2x}}\Big|_0^b$$

$$-\lim_{b\to\infty}\frac{1}{e^{2b}}-\frac{1}{e^{2(0)}}$$

$$-\left(\frac{1}{e^{2(\infty)}}-\frac{1}{1}\right)$$

$$-(0-1)$$

$$1$$

4. Evaluate the improper integral.

$$\int_0^\infty \frac{3x}{2+2x^2}\ dx$$

Solution:

Simplify the integrand.

$$\int_0^\infty \frac{3x}{2+2x^2}\ dx=\int_0^\infty \frac{3x}{2(1+x^2)}\ dx=\frac{3}{2}\int_0^\infty \frac{x}{1+x^2}\ dx$$

Replace the upper limit, rewriting the integral as

$$\lim_{b\to\infty}\frac{3}{2}\int_0^b \frac{x}{1+x^2}\ dx$$

Let

$$u=1+x^2$$

$$du = 2x\ dx, \text{ so } dx = \frac{du}{2x}$$

Substitute, then integrate.

$$\lim_{b\to\infty} \frac{3}{2} \int_{x=0}^{x=b} \frac{x}{u} \left(\frac{du}{2x}\right)$$

$$\lim_{b\to\infty} \frac{3}{4} \int_{x=0}^{x=b} \frac{1}{u}\ du$$

$$\lim_{b\to\infty} \frac{3}{4} \ln|u| \Big|_{x=0}^{x=b}$$

Back-substitute.

$$\lim_{b\to\infty} \frac{3}{4} \ln|1+x^2| \Big|_{x=0}^{x=b}$$

$$\lim_{b\to\infty} \frac{3}{4} \ln(1+x^2) \Big|_{0}^{b}$$

Evaluate over the interval.

$$\lim_{b\to\infty} \frac{3}{4} \ln(1+b^2) - \frac{3}{4} \ln(1+0^2)$$

$$\lim_{b\to\infty} \frac{3}{4} \ln(1+b^2) - \frac{3}{4} \ln(1)$$

$$\lim_{b\to\infty} \frac{3}{4} \ln(1+b^2) - \frac{3}{4}(0)$$

$$\lim_{b \to \infty} \frac{3}{4} \ln(1 + b^2)$$

$$\frac{3}{4} \ln(1 + \infty^2)$$

$$\frac{3}{4} \ln(\infty)$$

$$\frac{3}{4}(\infty)$$

$$\infty$$

IMPROPER INTEGRALS, CASE 2

1. Evaluate the improper integral.

$$\int_{-\infty}^{0} e^{3x}\ dx$$

Solution:

Replace the lower limit, rewriting the integral as

$$\lim_{a\to-\infty}\int_{a}^{0} e^{3x}\ dx$$

Integrate, then evaluate over the interval.

$$\lim_{a\to-\infty}\frac{e^{3x}}{3}\Big|_{a}^{0}$$

$$\lim_{a\to-\infty}\frac{e^{3(0)}}{3}-\frac{e^{3a}}{3}$$

$$\lim_{a\to-\infty}\frac{1}{3}-\frac{e^{3a}}{3}$$

$$\frac{1}{3}-\frac{e^{3(-\infty)}}{3}$$

$$\frac{1}{3}-\frac{1}{3e^{\infty}}$$

$$\frac{1}{3} - 0$$

$$\frac{1}{3}$$

■ 2. Evaluate the improper integral.

$$\int_{-\infty}^{1} xe^{x^2}\ dx$$

Solution:

Replace the lower limit, rewriting the integral as

$$\lim_{a \to -\infty} \int_{a}^{1} xe^{x^2}\ dx$$

Use u-substitution.

$$u = x^2$$

$du = 2x\ dx$, so $dx = \dfrac{du}{2x}$

Substitute.

$$\lim_{a \to -\infty} \int_{x=a}^{x=1} xe^{u} \left(\frac{du}{2x} \right)$$

$$\frac{1}{2}\lim_{a\to-\infty}\int_{x=a}^{x=1} e^u \ du$$

Integrate, back-substitute, then evaluate over the interval.

$$\frac{1}{2}\lim_{a\to-\infty} e^u\Big|_{x=a}^{x=1}$$

$$\frac{1}{2}\lim_{a\to-\infty} e^{x^2}\Big|_{a}^{1}$$

$$\frac{1}{2}\lim_{a\to-\infty} e^{1^2}-e^{a^2}$$

$$\frac{1}{2}\lim_{a\to-\infty} e-e^{a^2}$$

$$\frac{1}{2}\left(e-e^{(-\infty)^2}\right)$$

$$\frac{1}{2}(e-\infty)$$

$$\frac{1}{2}(-\infty)$$

$$-\infty$$

■ 3. Evaluate the improper integral.

$$\int_{-\infty}^{-2}\frac{2}{x-1}-\frac{2}{x+1}\ dx$$

Solution:

Replace the lower limit, rewriting the integral as

$$\lim_{a\to-\infty}\int_a^{-2}\frac{2}{x-1}-\frac{2}{x+1}\ dx$$

Integrate, then evaluate over the interval.

$$\lim_{a\to-\infty}2\ln|x-1|-2\ln|x+1|\Big|_a^{-2}$$

$$2\lim_{a\to-\infty}\ln|x-1|-\ln|x+1|\Big|_a^{-2}$$

$$2\lim_{a\to-\infty}\ln|-2-1|-\ln|-2+1|-\left(\ln|a-1|-\ln|a+1|\right)$$

$$2\lim_{a\to-\infty}\ln|-3|-\ln|-1|-\ln|a-1|+\ln|a+1|$$

$$2\lim_{a\to-\infty}\ln 3-\ln 1-\ln|a-1|+\ln|a+1|$$

$$2\lim_{a\to-\infty}\ln 3-\ln|a-1|+\ln|a+1|$$

$$2\lim_{a\to-\infty}\ln 3+\ln\frac{|a+1|}{|a-1|}$$

$$2\ln 3+2\ln\left(\lim_{a\to-\infty}\frac{|a+1|}{|a-1|}\right)$$

$$2\ln 3 + 2\ln\left(\lim_{a\to-\infty}\frac{|1+\frac{1}{a}|}{|1-\frac{1}{a}|}\right)$$

$$2\ln 3 + 2\ln\left(\frac{1}{1}\right)$$

$$2\ln 3 + 2(0)$$

$$2\ln 3$$

$$\ln 9$$

■ 4. Evaluate the improper integral.

$$\int_{-\infty}^{3}\frac{3}{x^2+9}\ dx$$

Solution:

Replace the lower limit, rewriting the integral as

$$\lim_{a\to-\infty}\int_{a}^{3}\frac{3}{x^2+9}\ dx$$

Integrate, then evaluate over the interval.

$$\lim_{a\to-\infty}\arctan\frac{x}{3}\Big|_{a}^{3}$$

$$\lim_{a \to -\infty} \arctan \frac{3}{3} - \arctan \frac{a}{3}$$

$$\lim_{a \to -\infty} \arctan 1 - \arctan \frac{a}{3}$$

$$\lim_{a \to -\infty} \frac{\pi}{4} - \arctan \frac{a}{3}$$

$$\frac{\pi}{4} - \arctan \frac{-\infty}{3}$$

$$\frac{\pi}{4} - \arctan(-\infty)$$

$$\frac{\pi}{4} - \left(-\frac{\pi}{2}\right)$$

$$\frac{\pi}{4} + \frac{\pi}{2}$$

$$\frac{3\pi}{4}$$

■ 5. Evaluate the improper integral.

$$\int_{-\infty}^{0} \frac{2\ dx}{e^x}$$

Solution:

Replace the lower limit, rewriting the integral as

$$\lim_{a \to -\infty} \int_a^0 \frac{2\ dx}{e^x}$$

$$2 \lim_{a \to -\infty} \int_a^0 \frac{1}{e^x}\ dx$$

$$2 \lim_{a \to -\infty} \int_a^0 e^{-x}\ dx$$

Integrate, then evaluate over the interval.

$$2 \lim_{a \to -\infty} -e^{-x} \Big|_a^0$$

$$2 \lim_{a \to -\infty} -e^{-0} - (-e^{-a})$$

$$2 \lim_{a \to -\infty} -e^{-0} + e^{-a}$$

$$2 \lim_{a \to -\infty} -1 + e^{-a}$$

$$2 \lim_{a \to -\infty} e^{-a} - 1$$

$$2(e^{-(-\infty)} - 1)$$

$$2(e^{\infty} - 1)$$

$$2(\infty - 1)$$

$$2(\infty)$$

$$\infty$$

6. Evaluate the improper integral.

$$\int_{-\infty}^{0} 4e^{-4x}\ dx$$

Solution:

Replace the lower limit, rewriting the integral as

$$\lim_{a \to -\infty} \int_{a}^{0} 4e^{-4x}\ dx$$

Integrate, then evaluate over the interval.

$$\lim_{a \to -\infty} \frac{4}{-4} e^{-4x} \Big|_{a}^{0}$$

$$\lim_{a \to -\infty} -e^{-4x} \Big|_{a}^{0}$$

$$\lim_{a \to -\infty} -e^{-4(0)} - (-e^{-4(a)})$$

$$\lim_{a \to -\infty} -1 + e^{-4a}$$

$$\lim_{a \to -\infty} e^{-4a} - 1$$

$$e^{-4(-\infty)} - 1$$

$$e^{4\infty} - 1$$

$$e^{\infty} - 1$$

$$\infty - 1$$

$$\infty$$

IMPROPER INTEGRALS, CASE 3

1. Evaluate the improper integral.

$$\int_{-\infty}^{\infty} 2xe^{-x^2}\ dx$$

Solution:

Separate the integral in two, splitting the interval at 0.

$$\int_{-\infty}^{0} 2xe^{-x^2}\ dx + \int_{0}^{\infty} 2xe^{-x^2}\ dx$$

Replace the infinite limits, rewriting the integral as

$$\lim_{a \to -\infty} \int_{a}^{0} 2xe^{-x^2}\ dx + \lim_{b \to \infty} \int_{0}^{b} 2xe^{-x^2}\ dx$$

Use u-substitution.

$$u = -x^2$$

$du = -2x\ dx$, so $dx = \dfrac{du}{-2x}$

Substitute.

$$\lim_{a \to -\infty} \int_{x=a}^{x=0} 2xe^{u} \left(\frac{du}{-2x} \right) + \lim_{b \to \infty} \int_{x=0}^{x=b} 2xe^{u} \left(\frac{du}{-2x} \right)$$

$$\lim_{a \to -\infty} \int_{x=a}^{x=0} -e^u \ du + \lim_{b \to \infty} \int_{x=0}^{x=b} -e^u \ du$$

Integrate, then evaluate over the interval.

$$\lim_{a \to -\infty} -e^u \Big|_{x=a}^{x=0} + \lim_{b \to \infty} -e^u \Big|_{x=0}^{x=b}$$

$$\lim_{a \to -\infty} -e^{-x^2} \Big|_{a}^{0} + \lim_{b \to \infty} -e^{-x^2} \Big|_{0}^{b}$$

$$\lim_{a \to -\infty} -e^{-0^2} - (-e^{-a^2}) + \lim_{b \to \infty} -e^{-b^2} - (-e^{-0^2})$$

$$\lim_{a \to -\infty} -e^0 + e^{-a^2} + \lim_{b \to \infty} -e^{-b^2} + e^0$$

$$\lim_{a \to -\infty} e^{-a^2} - 1 + \lim_{b \to \infty} 1 - e^{-b^2}$$

$$e^{-(-\infty)^2} - 1 + 1 - e^{-(\infty)^2}$$

$$e^{-\infty} - e^{-\infty}$$

$$0$$

■ 2. Evaluate the improper integral.

$$\int_{-\infty}^{\infty} \frac{3 \ dx}{x^2 + 1}$$

Solution:

Separate the integral in two, splitting the interval at 0.

$$\int_{-\infty}^{0} \frac{3\ dx}{x^2+1} + \int_{0}^{\infty} \frac{3\ dx}{x^2+1}$$

Replace the infinite limits, rewriting the integral as

$$\lim_{a\to-\infty} \int_{a}^{0} \frac{3\ dx}{x^2+1} + \lim_{b\to\infty} \int_{0}^{b} \frac{3\ dx}{x^2+1}$$

$$3 \lim_{a\to-\infty} \int_{a}^{0} \frac{1}{x^2+1}\ dx + 3 \lim_{b\to\infty} \int_{0}^{b} \frac{1}{x^2+1}\ dx$$

Integrate, then evaluate over the interval.

$$3 \lim_{a\to-\infty} \arctan x \Big|_{a}^{0} + 3 \lim_{b\to\infty} \arctan x \Big|_{0}^{b}$$

$$3 \lim_{a\to-\infty} \arctan 0 - \arctan a + 3 \lim_{b\to\infty} \arctan b - \arctan 0$$

$$3(\arctan 0 - \arctan(-\infty)) + 3(\arctan(\infty) - \arctan 0)$$

$$3\left(0 - \left(-\frac{\pi}{2}\right)\right) + 3\left(\frac{\pi}{2} - 0\right)$$

$$\frac{3\pi}{2} + \frac{3\pi}{2}$$

$$3\pi$$

3. Evaluate the improper integral.

$$\int_{-\infty}^{\infty} x^2 + 7x + 1 \ dx$$

Solution:

Separate the integral in two, splitting the interval at 0.

$$\int_{-\infty}^{0} x^2 + 7x + 1 \ dx + \int_{0}^{\infty} x^2 + 7x + 1 \ dx$$

Replace the infinite limits, rewriting the integral as

$$\lim_{a \to -\infty} \int_{a}^{0} x^2 + 7x + 1 \ dx + \lim_{b \to \infty} \int_{0}^{b} x^2 + 7x + 1 \ dx$$

Integrate, then evaluate over the interval.

$$\lim_{a \to -\infty} \frac{1}{3}x^3 + \frac{7}{2}x^2 + x \Big|_{a}^{0} + \lim_{b \to \infty} \frac{1}{3}x^3 + \frac{7}{2}x^2 + x \Big|_{0}^{b}$$

$$\lim_{a \to -\infty} \frac{1}{3}(0)^3 + \frac{7}{2}(0)^2 + 0 - \left(\frac{1}{3}(a)^3 + \frac{7}{2}(a)^2 + a\right) + \lim_{b \to \infty} \frac{1}{3}(b)^3 + \frac{7}{2}(b)^2 + b - \left(\frac{1}{3}(0)^3 + \frac{7}{2}(0)^2 + 0\right)$$

$$\lim_{a \to -\infty} -\frac{1}{3}a^3 - \frac{7}{2}a^2 - a + \lim_{b \to \infty} \frac{1}{3}b^3 + \frac{7}{2}b^2 + b$$

$$-\frac{1}{3}(-\infty)^3 - \frac{7}{2}(-\infty)^2 - (-\infty) + \frac{1}{3}(\infty)^3 + \frac{7}{2}(\infty)^2 + (\infty)$$

$$\frac{1}{3}\infty - \frac{7}{2}\infty + \infty + \frac{1}{3}\infty + \frac{7}{2}\infty + \infty$$

$$\frac{2}{3}\infty + \infty$$

The value diverges to ∞.

4. Evaluate the improper integral.

$$\int_{-\infty}^{\infty} 3x^2e^{-x^3}\ dx$$

Solution:

Separate the integral in two, splitting the interval at 0.

$$\int_{-\infty}^{0} 3x^2e^{-x^3}\ dx + \int_{0}^{\infty} 3x^2e^{-x^3}\ dx$$

Replace the infinite limits, rewriting the integral as

$$\lim_{a\to-\infty}\int_{a}^{0} 3x^2e^{-x^3}\ dx + \lim_{b\to\infty}\int_{0}^{b} 3x^2e^{-x^3}\ dx$$

Use u-substitution.

$$u = -x^3$$

$$du=-3x^2\ dx\text{, so } dx=\frac{du}{-3x^2}$$

Substitute.

$$\lim_{a\to-\infty}\int_{x=a}^{x=0}3x^2e^u\left(\frac{du}{-3x^2}\right)+\lim_{b\to\infty}\int_{x=0}^{x=b}3x^2e^u\left(\frac{du}{-3x^2}\right)$$

$$\lim_{a\to-\infty}\int_{x=a}^{x=0}-e^u\ du+\lim_{b\to\infty}\int_{x=0}^{x=b}-e^u\ du$$

Integrate, then evaluate over the interval.

$$\lim_{a\to-\infty}-e^u\Big|_{x=a}^{x=0}+\lim_{b\to\infty}-e^u\Big|_{x=0}^{x=b}$$

$$\lim_{a\to-\infty}-e^{-x^3}\Big|_{a}^{0}+\lim_{b\to\infty}-e^{-x^3}\Big|_{0}^{b}$$

$$\lim_{a\to-\infty}-e^{-0^3}-(-e^{-a^3})+\lim_{b\to\infty}-e^{-b^3}-(-e^{-0^3})$$

$$\lim_{a\to-\infty}-e^{-0^3}+e^{-a^3}+\lim_{b\to\infty}-e^{-b^3}+e^{-0^3}$$

$$\lim_{a\to-\infty}e^{-a^3}-1+\lim_{b\to\infty}1-e^{-b^3}$$

$$e^{-(-\infty)^3}-1+1-e^{-(\infty)^3}$$

$$e^{-(-\infty)^3}-e^{-(\infty)^3}$$

$$e^{\infty}-e^{-\infty}$$

$$\infty-0$$

∞

IMPROPER INTEGRALS, CASE 4

1. Evaluate the improper integral.

$$\int_{-\frac{\pi}{2}}^{0} \frac{3\cos x}{2\sin x}\ dx$$

Solution:

The integrand is undefined at the upper bound, $x = 0$. Therefore, rewrite the integral as

$$\lim_{b\to 0^-} \int_{-\frac{\pi}{2}}^{b} \frac{3\cos x}{2\sin x}\ dx$$

$$\frac{3}{2} \lim_{b\to 0^-} \int_{-\frac{\pi}{2}}^{b} \frac{\cos x}{\sin x}\ dx$$

Use u-substitution.

$$u = \sin x$$

$du = \cos x\ dx$, so $dx = \dfrac{du}{\cos x}$

Substitute.

$$\frac{3}{2} \lim_{b\to 0^-} \int_{x=-\frac{\pi}{2}}^{x=b} \frac{\cos x}{u} \left(\frac{du}{\cos x} \right)$$

$$\frac{3}{2}\lim_{b\to 0^-}\int_{x=-\frac{\pi}{2}}^{x=b}\frac{1}{u}\ du$$

Integrate, then evaluate over the interval.

$$\frac{3}{2}\lim_{b\to 0^-}\ln|u|\Big|_{x=-\frac{\pi}{2}}^{x=b}$$

$$\frac{3}{2}\lim_{b\to 0^-}\ln|\sin x|\Big|_{-\frac{\pi}{2}}^{b}$$

$$\frac{3}{2}\lim_{b\to 0^-}\ln|\sin b|-\ln\left|\sin\left(-\frac{\pi}{2}\right)\right|$$

$$\frac{3}{2}\lim_{b\to 0^-}\ln|\sin b|-\ln|-1|$$

$$\frac{3}{2}\lim_{b\to 0^-}\ln|\sin b|-\ln 1$$

$$\frac{3}{2}\lim_{b\to 0^-}\ln|\sin b|$$

$$\frac{3}{2}(-\infty)$$

$$-\infty$$

■ 2. Evaluate the improper integral.

$$\int_{-8}^{0} \frac{e^x \ dx}{e^x - 1}$$

Solution:

The integrand is undefined at the upper bound, $x = 0$. Therefore, rewrite the integral as

$$\lim_{b \to 0^-} \int_{-8}^{b} \frac{e^x \ dx}{e^x - 1}$$

Use u-substitution.

$$u = e^x - 1$$

$du = e^x \ dx$, so $dx = \dfrac{du}{e^x}$

Substitute.

$$\lim_{b \to 0^-} \int_{x=-8}^{x=b} \frac{e^x}{u} \left(\frac{du}{e^x} \right)$$

$$\lim_{b \to 0^-} \int_{x=-8}^{x=b} \frac{1}{u} \ du$$

Integrate, then evaluate over the interval.

$$\lim_{b \to 0^-} \ln|u| \Big|_{x=-8}^{x=b}$$

$$\lim_{b \to 0^-} \ln|e^x - 1| \Big|_{-8}^{b}$$

$$\lim_{b \to 0^-} \ln|e^b - 1| - \ln|e^{-8} - 1|$$

$$\ln|1 - 1| - \ln|e^{-8} - 1|$$

$$\ln|0| - \ln|e^{-8} - 1|$$

$$-\infty - \ln|e^{-8} - 1|$$

$$-\infty$$

3. Evaluate the improper integral.

$$\int_{-9}^{0} \frac{e^{\sqrt{-x}}\ dx}{\sqrt{-x}}$$

Solution:

The integrand is undefined at the upper bound, $x = 0$. Therefore, rewrite the integral as

$$\lim_{b \to 0^-} \int_{-9}^{b} \frac{e^{\sqrt{-x}}\ dx}{\sqrt{-x}}$$

Use u-substitution.

$$u = \sqrt{-x}$$

$$du = -\frac{1}{2\sqrt{-x}}\ dx, \text{ so } dx = -2\sqrt{-x}\ du$$

Substitute.

$$\lim_{b \to 0^-} \int_{x=-9}^{x=b} \frac{e^u}{u} \left(-2\sqrt{-x}\ du\right)$$

$$-2 \lim_{b \to 0^-} \int_{x=-9}^{x=b} \frac{e^u}{u} \left(u\ du\right)$$

$$-2 \lim_{b \to 0^-} \int_{x=-9}^{x=b} e^u\ du$$

Integrate, then evaluate over the interval.

$$-2 \lim_{b \to 0^-} e^u \Big|_{x=-9}^{x=b}$$

$$-2 \lim_{b \to 0^-} e^{\sqrt{-x}} \Big|_{-9}^{b}$$

$$-2 \lim_{b \to 0^-} e^{\sqrt{-b}} - e^{\sqrt{-(-9)}}$$

$$-2 \lim_{b \to 0^-} e^{\sqrt{-b}} - e^{\sqrt{9}}$$

$$-2(e^{\sqrt{-0}} - e^3)$$

$$-2(1 - e^3)$$

$$-2 + 2e^3$$

$$2e^3 - 2$$

4. Evaluate the improper integral.

$$\int_1^3 \frac{2x - 3}{\sqrt{3x - x^2}}\ dx$$

Solution:

The integrand is undefined at the upper bound, $x = 3$. Therefore, rewrite the integral as

$$\lim_{b \to 3^-} \int_1^b \frac{2x - 3}{\sqrt{3x - x^2}}\ dx$$

Use u-substitution.

$$u = 3x - x^2$$

$du = (3 - 2x)\ dx$, so $dx = \dfrac{du}{3 - 2x}$

Substitute.

$$\lim_{b \to 3^-} \int_{x=1}^{x=b} \frac{2x - 3}{\sqrt{u}} \left(\frac{du}{3 - 2x} \right)$$

$$-\lim_{b\to 3^-}\int_{x=1}^{x=b}\frac{2x-3}{\sqrt{u}}\left(\frac{du}{-(3-2x)}\right)$$

$$-\lim_{b\to 3^-}\int_{x=1}^{x=b}\frac{2x-3}{\sqrt{u}}\left(\frac{du}{2x-3}\right)$$

$$-\lim_{b\to 3^-}\int_{x=1}^{x=b}\frac{1}{\sqrt{u}}\ du$$

$$-\lim_{b\to 3^-}\int_{x=1}^{x=b}u^{-\frac{1}{2}}\ du$$

Integrate, then evaluate over the interval.

$$-\lim_{b\to 3^-}2u^{\frac{1}{2}}\Big|_{x=1}^{x=b}$$

$$-\lim_{b\to 3^-}2\sqrt{u}\Big|_{x=1}^{x=b}$$

$$-\lim_{b\to 3^-}2\sqrt{3x-x^2}\Big|_{1}^{b}$$

$$-\lim_{b\to 3^-}2\sqrt{3b-b^2}-2\sqrt{3(1)-1^2}$$

$$-\lim_{b\to 3^-}2\sqrt{3b-b^2}-2\sqrt{2}$$

$$\lim_{b\to 3^-}2\sqrt{2}-2\sqrt{3b-b^2}$$

$$2\sqrt{2}-2\sqrt{3(3)-3^2}$$

$$2\sqrt{2}-2\sqrt{9-9}$$

$$2\sqrt{2}-2(0)$$

$$2\sqrt{2}$$

5. Evaluate the improper integral.

$$\int_0^{2\sqrt{2}} \frac{x}{\sqrt{8-x^2}}\ dx$$

Solution:

The integrand is undefined at the upper bound, $x=2\sqrt{2}$. Therefore, rewrite the integral as

$$\lim_{b\to 2\sqrt{2}^-}\int_0^b \frac{x}{\sqrt{8-x^2}}\ dx$$

Use u-substitution.

$$u=8-x^2$$

$du=-2x\ dx$, so $dx=\dfrac{du}{-2x}$

Substitute.

$$\lim_{b\to 2\sqrt{2}^-}\int_{x=0}^{x=b}\frac{x}{\sqrt{u}}\left(\frac{du}{-2x}\right)$$

$$-\frac{1}{2}\lim_{b\to 2\sqrt{2}^-}\int_{x=0}^{x=b}\frac{1}{\sqrt{u}}\ du$$

$$-\frac{1}{2}\lim_{b\to 2\sqrt{2}^-}\int_{x=0}^{x=b}u^{-\frac{1}{2}}\ du$$

Integrate, then evaluate over the interval.

$$-\frac{1}{2}\lim_{b\to 2\sqrt{2}^-}2u^{\frac{1}{2}}\Big|_{x=0}^{x=b}$$

$$-\lim_{b\to 2\sqrt{2}^-}\sqrt{u}\Big|_{x=0}^{x=b}$$

$$-\lim_{b\to 2\sqrt{2}^-}\sqrt{8-x^2}\Big|_0^b$$

$$-\lim_{b\to 2\sqrt{2}^-}\sqrt{8-b^2}-\sqrt{8-0^2}$$

$$-\left(\sqrt{8-(2\sqrt{2})^2}-\sqrt{8}\right)$$

$$-\left(\sqrt{8-4(2)}-\sqrt{8}\right)$$

$$-\sqrt{8-8}+\sqrt{8}$$

$$\sqrt{8}-\sqrt{0}$$

$$2\sqrt{2}$$

6. Evaluate the improper integral.

$$\int_1^3 \frac{x-1}{x^2-4x+3}\ dx$$

Solution:

Simplify the integrand by factoring the denominator.

$$\int_1^3 \frac{x-1}{(x-1)(x-3)}\ dx$$

$$\int_1^3 \frac{1}{x-3}\ dx$$

The integrand is undefined at the upper bound, $x = 3$. Therefore, rewrite the integral as

$$\lim_{b \to 3^-} \int_1^b \frac{1}{x-3}\ dx$$

Integrate, then evaluate over the interval.

$$\lim_{b \to 3^-} \ln|x-3| \Big|_1^b$$

$$\lim_{b \to 3^-} \ln|b-3| - \ln|1-3|$$

$$\lim_{b \to 3^-} \ln|b-3| - \ln|-2|$$

$$\lim_{b \to 3^-} \ln|b-3| - \ln 2$$

$$\ln|3-3| - \ln 2$$

$$\ln 0 - \ln 2$$

$$-\infty - \ln 2$$

$$-\infty$$

IMPROPER INTEGRALS, CASE 5

1. Evaluate the improper integral.

$$\int_0^2 \frac{3}{\sqrt[3]{x}}\ dx$$

Solution:

The integrand is undefined at the lower bound, $x = 0$. Therefore, rewrite the integral as

$$\lim_{a \to 0^+} \int_a^2 \frac{3}{\sqrt[3]{x}}\ dx$$

$$\lim_{a \to 0^+} \int_a^2 3x^{-\frac{1}{3}}\ dx$$

Integrate, then evaluate over the interval.

$$\lim_{a \to 0^+} \frac{9}{2}x^{\frac{2}{3}} \Big|_a^2$$

$$\lim_{a \to 0^+} \frac{9}{2}(2)^{\frac{2}{3}} - \frac{9}{2}(a)^{\frac{2}{3}}$$

$$\frac{9}{2}(2)^{\frac{2}{3}} - \frac{9}{2}(0)^{\frac{2}{3}}$$

$$\frac{9}{2}(2)^{\frac{2}{3}}$$

$$\frac{9}{2}\sqrt[3]{4}$$

■ 2. Evaluate the improper integral.

$$\int_{-1}^{5} \frac{3}{\sqrt{x+1}}\ dx$$

Solution:

The integrand is undefined at the lower bound, $x = -1$. Therefore, rewrite the integral as

$$\lim_{a \to -1^+} \int_{a}^{5} \frac{3}{\sqrt{x+1}}\ dx$$

$$3 \lim_{a \to -1^+} \int_{a}^{5} (x+1)^{-\frac{1}{2}}\ dx$$

Integrate, then evaluate over the interval.

$$3 \lim_{a \to -1^+} 2(x+1)^{\frac{1}{2}} \Big|_{a}^{5}$$

$$6 \lim_{a \to -1^+} \sqrt{x+1} \Big|_{a}^{5}$$

$$6 \lim_{a \to -1^+} \sqrt{5+1} - \sqrt{a+1}$$

$$6\left(\sqrt{6} - \sqrt{-1+1}\right)$$

$$6\sqrt{6}$$

3. Evaluate the improper integral.

$$\int_3^7 \frac{5}{x-3} \ dx$$

Solution:

The integrand is undefined at the lower bound, $x = 3$. Therefore, rewrite the integral as

$$\lim_{a \to 3^+} \int_a^7 \frac{5}{x-3} \ dx$$

Integrate, then evaluate over the interval.

$$\lim_{a \to 3^+} 5 \ln|x-3| \Big|_a^7$$

$$\lim_{a \to 3^+} 5 \ln|7-3| - 5 \ln|a-3|$$

$$5 \ln 4 - 5 \ln|3-3|$$

$$5 \ln 4 - 5 \ln 0$$

$$5 \ln 4 - (-\infty)$$

$$5 \ln 4 + \infty$$

$$\infty$$

4. Evaluate the improper integral.

$$\int_0^6 \frac{9}{5\sqrt[4]{x^3}} \, dx$$

Solution:

The integrand is undefined at the lower bound, $x = 0$. Therefore, rewrite the integral as

$$\lim_{a \to 0^+} \int_a^6 \frac{9}{5\sqrt[4]{x^3}} \, dx$$

$$\frac{9}{5} \lim_{a \to 0^+} \int_a^6 x^{-\frac{3}{4}} \, dx$$

Integrate, then evaluate over the interval.

$$\frac{9}{5} \lim_{a \to 0^+} 4x^{\frac{1}{4}} \Big|_a^6$$

$$\frac{36}{5}\lim_{a\to 0^+} x^{\frac{1}{4}}\Big|_a^6$$

$$\frac{36}{5}\lim_{a\to 0^+} 6^{\frac{1}{4}} - a^{\frac{1}{4}}$$

$$\frac{36}{5}\left(6^{\frac{1}{4}} - 0^{\frac{1}{4}}\right)$$

$$\frac{36}{5}\sqrt[4]{6}$$

■ 5. Evaluate the improper integral.

$$\int_{-1}^{7} \frac{x^2}{x^3+1}\ dx$$

Solution:

The integrand is undefined at the lower bound, $x = -1$. Therefore, rewrite the integral as

$$\lim_{a\to -1^+} \int_{a}^{7} \frac{x^2}{x^3+1}\ dx$$

Use u-substitution.

$$u = x^3 + 1$$

$$du = 3x^2\ dx\text{, so } dx = \frac{du}{3x^2}$$

Substitute.

$$\lim_{a \to -1^+} \int_{x=a}^{x=7} \frac{x^2}{u} \left(\frac{du}{3x^2} \right)$$

$$\frac{1}{3} \lim_{a \to -1^+} \int_{x=a}^{x=7} \frac{1}{u}\ du$$

Integrate, then evaluate over the interval.

$$\frac{1}{3} \lim_{a \to -1^+} \ln|u| \Big|_{x=a}^{x=7}$$

$$\frac{1}{3} \lim_{a \to -1^+} \ln|x^3 + 1| \Big|_{a}^{7}$$

$$\frac{1}{3} \lim_{a \to -1^+} \ln|7^3 + 1| - \ln|a^3 + 1|$$

$$\frac{1}{3} \lim_{a \to -1^+} \ln 344 - \ln|a^3 + 1|$$

$$\frac{1}{3}(\ln 344 - \ln 0)$$

$$\frac{1}{3}(\ln 344 - (-\infty))$$

$$\frac{1}{3}(\ln 344 + \infty)$$

$$\frac{1}{3}(\infty)$$

$$\infty$$

6. Evaluate the improper integral.

$$\int_{-4}^{4} \frac{x+4}{x^2+8x+16}\ dx$$

Solution:

Simplify the integrand by factoring the denominator.

$$\int_{-4}^{4} \frac{x+4}{(x+4)(x+4)}\ dx$$

$$\int_{-4}^{4} \frac{1}{x+4}\ dx$$

The integrand is undefined at the lower bound, $x = -4$. Therefore, rewrite the integral as

$$\lim_{a \to -4^+} \int_{a}^{4} \frac{1}{x+4}\ dx$$

Integrate, then evaluate over the interval.

$$\lim_{a \to -4^+} \ln|x+4| \Big|_a^4$$

$$\lim_{a \to -4^+} \ln|4+4| - \ln|a+4|$$

$$\lim_{a \to -4^+} \ln 8 - \ln|a+4|$$

$$\ln 8 - \ln|-4+4|$$

$$\ln 8 - \ln 0$$

$$\ln 8 - (-\infty)$$

$$\ln 8 + \infty$$

$$\infty$$

IMPROPER INTEGRALS, CASE 6

1. Evaluate the improper integral.

$$\int_{-2}^{2} \frac{3}{2\sqrt[5]{x^3}}\ dx$$

Solution:

The integrand is undefined between the lower and upper bounds, at $x = 0$. So we'll split the integral in two at $x = 0$.

$$\int_{-2}^{0} \frac{3}{2\sqrt[5]{x^3}}\ dx + \int_{0}^{2} \frac{3}{2\sqrt[5]{x^3}}\ dx$$

The first integral is undefined at the upper bound, and the second integral is undefined at the lower bound, so rewrite the expression as

$$\lim_{b \to 0^-} \int_{-2}^{b} \frac{3}{2\sqrt[5]{x^3}}\ dx + \lim_{a \to 0^+} \int_{a}^{2} \frac{3}{2\sqrt[5]{x^3}}\ dx$$

$$\frac{3}{2} \lim_{b \to 0^-} \int_{-2}^{b} \frac{1}{\sqrt[5]{x^3}}\ dx + \frac{3}{2} \lim_{a \to 0^+} \int_{a}^{2} \frac{1}{\sqrt[5]{x^3}}\ dx$$

$$\frac{3}{2} \lim_{b \to 0^-} \int_{-2}^{b} x^{-\frac{3}{5}}\ dx + \frac{3}{2} \lim_{a \to 0^+} \int_{a}^{2} x^{-\frac{3}{5}}\ dx$$

Integrate, then evaluate over the interval.

$$\frac{3}{2}\lim_{b\to 0^-}\frac{5}{2}x^{\frac{2}{5}}\Big|_{-2}^{b}+\frac{3}{2}\lim_{a\to 0^+}\frac{5}{2}x^{\frac{2}{5}}\Big|_{a}^{2}$$

$$\frac{3}{2}\lim_{b\to 0^-}\frac{5}{2}b^{\frac{2}{5}}-\frac{5}{2}(-2)^{\frac{2}{5}}+\frac{3}{2}\lim_{a\to 0^+}\frac{5}{2}(2)^{\frac{2}{5}}-\frac{5}{2}a^{\frac{2}{5}}$$

$$\frac{3}{2}\left(\frac{5}{2}(0)^{\frac{2}{5}}-\frac{5}{2}(-2)^{\frac{2}{5}}\right)+\frac{3}{2}\left(\frac{5}{2}(2)^{\frac{2}{5}}-\frac{5}{2}(0)^{\frac{2}{5}}\right)$$

$$\frac{3}{2}\left(-\frac{5}{2}(-2)^{\frac{2}{5}}\right)+\frac{3}{2}\left(\frac{5}{2}(2)^{\frac{2}{5}}\right)$$

$$-\frac{15}{4}(-2)^{\frac{2}{5}}+\frac{15}{4}(2)^{\frac{2}{5}}$$

$$-\frac{15}{4}\sqrt[5]{4}+\frac{15}{4}\sqrt[5]{4}$$

$$\frac{15}{4}\sqrt[5]{4}-\frac{15}{4}\sqrt[5]{4}$$

$$0$$

2. Evaluate the improper integral.

$$\int_0^4 \frac{7\ dx}{2(x-2)^2}$$

Solution:

The integrand is undefined between the lower and upper bounds, at $x = 2$. So we'll split the integral in two at $x = 2$.

$$\int_0^2 \frac{7\ dx}{2(x-2)^2} + \int_2^4 \frac{7\ dx}{2(x-2)^2}$$

The first integral is undefined at the upper bound, and the second integral is undefined at the lower bound, so rewrite the expression as

$$\lim_{a \to 2^-} \int_0^a \frac{7\ dx}{2(x-2)^2} + \lim_{a \to 2^+} \int_a^4 \frac{7\ dx}{2(x-2)^2}$$

$$\frac{7}{2} \lim_{a \to 2^-} \int_0^a \frac{1}{(x-2)^2}\ dx + \frac{7}{2} \lim_{a \to 2^+} \int_a^4 \frac{1}{(x-2)^2}\ dx$$

$$\frac{7}{2} \lim_{a \to 2^-} \int_0^a (x-2)^{-2}\ dx + \frac{7}{2} \lim_{a \to 2^+} \int_a^4 (x-2)^{-2}\ dx$$

Integrate, then evaluate over the interval.

$$\frac{7}{2} \lim_{a \to 2^-} -(x-2)^{-1} \Big|_0^a + \frac{7}{2} \lim_{a \to 2^+} -(x-2)^{-1} \Big|_a^4$$

$$\frac{7}{2} \lim_{a \to 2^-} -\frac{1}{(x-2)} \Big|_0^a + \frac{7}{2} \lim_{a \to 2^+} -\frac{1}{(x-2)} \Big|_a^4$$

$$\frac{7}{2} \lim_{a \to 2^-} -\frac{1}{(a-2)} - \left(-\frac{1}{(0-2)}\right) + \frac{7}{2} \lim_{a \to 2^+} -\frac{1}{(4-2)} - \left(-\frac{1}{(a-2)}\right)$$

$$\frac{7}{2} \lim_{a \to 2^-} -\frac{1}{a-2} + \frac{1}{0-2} + \frac{7}{2} \lim_{a \to 2^+} -\frac{1}{4-2} + \frac{1}{a-2}$$

$$\frac{7}{2}\lim_{a\to 2^-} -\frac{1}{a-2}-\frac{1}{2}+\frac{7}{2}\lim_{a\to 2^+}\frac{1}{a-2}-\frac{1}{2}$$

$$\frac{7}{2}(\infty)+\frac{7}{2}(\infty)$$

$$\infty$$

■ 3. Evaluate the improper integral.

$$\int_{-27}^{8}\frac{3\ dx}{\sqrt[3]{x}}$$

Solution:

The integrand is undefined between the lower and upper bounds, at $x=0$. So we'll split the integral in two at $x=0$.

$$\int_{-27}^{0}\frac{3\ dx}{\sqrt[3]{x}}+\int_{0}^{8}\frac{3\ dx}{\sqrt[3]{x}}$$

The first integral is undefined at the upper bound, and the second integral is undefined at the lower bound, so rewrite the expression as

$$\lim_{b\to 0^-}\int_{-27}^{b}\frac{3\ dx}{\sqrt[3]{x}}+\lim_{a\to 0^+}\int_{a}^{8}\frac{3\ dx}{\sqrt[3]{x}}$$

$$3\lim_{b\to 0^-}\int_{-27}^{b}x^{-\frac{1}{3}}\ dx+3\lim_{a\to 0^+}\int_{a}^{8}x^{-\frac{1}{3}}\ dx$$

Integrate, then evaluate over the interval.

$$3\left(\frac{3}{2}\right)\lim_{b\to 0^-} x^{\frac{2}{3}}\Big|_{-27}^{b}+3\left(\frac{3}{2}\right)\lim_{a\to 0^+} x^{\frac{2}{3}}\Big|_{a}^{8}$$

$$\frac{9}{2}\lim_{b\to 0^-} x^{\frac{2}{3}}\Big|_{-27}^{b}+\frac{9}{2}\lim_{a\to 0^+} x^{\frac{2}{3}}\Big|_{a}^{8}$$

$$\frac{9}{2}\lim_{b\to 0^-} b^{\frac{2}{3}}-(-27)^{\frac{2}{3}}+\frac{9}{2}\lim_{a\to 0^+} 8^{\frac{2}{3}}-a^{\frac{2}{3}}$$

$$\frac{9}{2}\lim_{b\to 0^-} b^{\frac{2}{3}}-(-3)^2+\frac{9}{2}\lim_{a\to 0^+} 2^2-a^{\frac{2}{3}}$$

$$\frac{9}{2}\lim_{b\to 0^-} b^{\frac{2}{3}}-(-3)^2+\frac{9}{2}\lim_{a\to 0^+} 2^2-a^{\frac{2}{3}}$$

$$\frac{9}{2}\lim_{b\to 0^-} b^{\frac{2}{3}}-9+\frac{9}{2}\lim_{a\to 0^+} 4-a^{\frac{2}{3}}$$

$$\frac{9}{2}(0^{\frac{2}{3}}-9)+\frac{9}{2}(4-0^{\frac{2}{3}})$$

$$\frac{9}{2}(-9)+\frac{9}{2}(4)$$

$$-\frac{81}{2}+\frac{36}{2}$$

$$-\frac{45}{2}$$

■ 4. Evaluate the improper integral.

$$\int_{-3}^{3} \frac{x+2}{x^2-4}\ dx$$

Solution:

The integrand is undefined between the lower and upper bounds, at $x = 2$. So we'll split the integral in two at $x = 2$.

$$\int_{-3}^{2} \frac{x+2}{x^2-4}\ dx + \int_{2}^{3} \frac{x+2}{x^2-4}\ dx$$

The first integral is undefined at the upper bound, and the second integral is undefined at the lower bound, so rewrite the expression as

$$\lim_{b\to 2^-} \int_{-3}^{b} \frac{x+2}{x^2-4}\ dx + \lim_{a\to 2^+} \int_{a}^{3} \frac{x+2}{x^2-4}\ dx$$

$$\lim_{b\to 2^-} \int_{-3}^{b} \frac{x+2}{(x+2)(x-2)}\ dx + \lim_{a\to 2^+} \int_{a}^{3} \frac{x+2}{(x+2)(x-2)}\ dx$$

$$\lim_{b\to 2^-} \int_{-3}^{b} \frac{1}{x-2}\ dx + \lim_{a\to 2^+} \int_{a}^{3} \frac{1}{x-2}\ dx$$

Integrate, then evaluate over the interval.

$$\lim_{b\to 2^-} \ln|x-2| \Big|_{-3}^{b} + \lim_{a\to 2^+} \ln|x-2| \Big|_{a}^{3}$$

$$\lim_{b\to 2^-} \ln|b-2| - \ln|-3-2| + \lim_{a\to 2^+} \ln|3-2| - \ln|a-2|$$

$$\lim_{b\to 2^-} \ln|b-2| - \ln 5 + \lim_{a\to 2^+} \ln 1 - \ln|a-2|$$

$$\lim_{b\to 2^-} \ln|b-2| - \ln 5 + \lim_{a\to 2^+} -\ln|a-2|$$

$$\ln|2-2| - \ln 5 + (-\ln|2-2|)$$

$$\ln 0 - \ln 5 - \ln 0$$

$$-\infty - \ln 5 - (-\infty)$$

$$-\infty - \ln 5 + \infty$$

This value is indeterminate, which means that the integral does not converge.

■ 5. Evaluate the improper integral.

$$\int_0^6 \frac{4}{x-3} - \frac{4}{x+3}\ dx$$

Solution:

The integrand is undefined between the lower and upper bounds, at $x = 3$. So we'll split the integral in two at $x = 3$.

$$\int_0^3 \frac{4}{x-3} - \frac{4}{x+3}\ dx + \int_3^6 \frac{4}{x-3} - \frac{4}{x+3}\ dx$$

The first integral is undefined at the upper bound, and the second integral is undefined at the lower bound, so rewrite the expression as

$$\lim_{b\to 3^-}\int_0^b \frac{4}{x-3}-\frac{4}{x+3}\ dx+\lim_{a\to 3^+}\int_a^6 \frac{4}{x-3}-\frac{4}{x+3}\ dx$$

Integrate, then evaluate over the interval.

$$\lim_{b\to 3^-} 4\ln|x-3|-4\ln|x+3|\Big|_0^b+\lim_{a\to 3^+} 4\ln|x-3|-4\ln|x+3|\Big|_a^6$$

$$\lim_{b\to 3^-} 4\ln|b-3|-4\ln|b+3|-(4\ln|0-3|-4\ln|0+3|)$$

$$+\lim_{a\to 3^+} 4\ln|6-3|-4\ln|6+3|-(4\ln|a-3|-4\ln|a+3|)$$

$$\lim_{b\to 3^-} 4\ln|b-3|-4\ln|b+3|-4\ln 3+4\ln 3$$

$$+\lim_{a\to 3^+} 4\ln 3-4\ln 9-4\ln|a-3|+4\ln|a+3|$$

$$\lim_{b\to 3^-} 4(\ln|b-3|-\ln|b+3|)+\lim_{a\to 3^+} 4\ln 3-4\ln 9+4(\ln|a+3|-\ln|a-3|)$$

$$4\ln 3+4\ln 6-4\ln 6-4\ln 9+\lim_{b\to 3^-} 4\ln|b-3|-4\lim_{a\to 3^+}\ln|a-3|$$

$$4\ln\frac{1}{3}+\lim_{b\to 3^-} 4\ln|b-3|-4\lim_{a\to 3^+}\ln|a-3|$$

$$4\ln\frac{1}{3}+\infty-\infty$$

The integral does not converge.

COMPARISON THEOREM

1. Use the Comparison Theorem to say whether the integral converges or diverges.

$$\int_1^\infty \frac{1}{2+2x^2}\ dx$$

Solution:

Let

$$f(x)=\frac{1}{x^2}$$

$$g(x)=\frac{1}{2+2x^2}$$

Compare the two functions using limits and L'Hospital's Rule.

$$\lim_{x\to\infty}\frac{g(x)}{f(x)}=\lim_{x\to\infty}\frac{\frac{1}{2+2x^2}}{\frac{1}{x^2}}=\lim_{x\to\infty}\frac{x^2}{2+2x^2}=\lim_{x\to\infty}\frac{2x}{4x}=\lim_{x\to\infty}\frac{2}{4}=\frac{1}{2}$$

So $g(x)\le f(x)$ on $[1,\infty)$. Now compare the integral of both functions on $[1,\infty)$.

$$\int_1^\infty \frac{1}{x^2}\ dx=\lim_{b\to\infty}\int_1^b \frac{1}{x^2}\ dx=\lim_{b\to\infty}\int_1^b x^{-2}\ dx$$

Integrate and evaluate over the interval.

$$\lim_{b\to\infty} \frac{x^{-1}}{-1}\Bigg|_1^b$$

$$\lim_{b\to\infty} -\frac{1}{x}\Bigg|_1^b$$

$$\lim_{b\to\infty} -\frac{1}{b} - \left(-\frac{1}{1}\right)$$

$$\lim_{b\to\infty} 1 - \frac{1}{b}$$

$$1 - \frac{1}{\infty}$$

$$1 - 0$$

$$1$$

So $f(x)$ converges on $[1,\infty)$. Since $g(x) \le f(x)$ on $[1,\infty)$, and $f(x)$ converges on $[1,\infty)$, $g(x)$ also converges on $[1,\infty)$.

■ 2. Use the Comparison Theorem to say whether the integral converges or diverges.

$$\int_1^\infty \frac{1}{5x + e^x}\ dx$$

Solution:

Let

$$f(x) = \frac{1}{e^x}$$

$$g(x) = \frac{1}{5x + e^x}$$

Compare the two functions.

$$\frac{g(x)}{f(x)} = \frac{\frac{1}{5x + e^x}}{\frac{1}{e^x}} = \frac{e^x}{5x + e^x}$$

So

$$0 < \frac{e^x}{5x + e^x} < 1$$

on $[1,\infty)$. Now compare the integral of both functions on $[1,\infty)$.

$$\int_1^{\infty} \frac{1}{e^x} \ dx = \lim_{b \to \infty} \int_1^b \frac{1}{e^x} \ dx = \lim_{b \to \infty} \int_1^b e^{-x} \ dx$$

Integrate and evaluate over the interval.

$$\lim_{b \to \infty} -e^{-x} \Big|_1^b$$

$$\lim_{b \to \infty} -\frac{1}{e^x} \Big|_1^b$$

$$\lim_{b\to\infty} -\frac{1}{e^b} - \left(-\frac{1}{e}\right)$$

$$\lim_{b\to\infty} \frac{1}{e} - \frac{1}{e^b}$$

$$\frac{1}{e} - \frac{1}{e^\infty}$$

$$\frac{1}{e}$$

So $f(x)$ converges on $[1,\infty)$. Since $g(x) \le f(x)$ on $[1,\infty)$, and $f(x)$ converges on $[1,\infty)$, $g(x)$ also converges on $[1,\infty)$.

■ 3. Can we use the harmonic series $1/x$ as a comparison series to say whether or not the integral converges?

$$\int_1^\infty \frac{x}{x^2+1}\ dx$$

Solution:

We'll call the original function

$$g(x) = \frac{x}{x^2+1}$$

If we take just the leading terms from the numerator and denominator, we get

$$f(x) = \frac{x}{x^2} = \frac{1}{x}$$

This comparison series is the harmonic series, which we know diverges. Therefore, as long as $g(x) \geq f(x)$, we know the original series will diverge as well.

$$\frac{x}{x^2 + 1} \geq \frac{1}{x}$$

$$x \geq \frac{x^2 + 1}{x}$$

$$x^2 \geq x^2 + 1$$

This inequality will always be false, which means the harmonic series is useless to us as a comparison series, and we therefore can't use the harmonic series to as a comparison series to say whether or not the integral converges.

W W W . K R I S T A K I N G M A T H . C O M