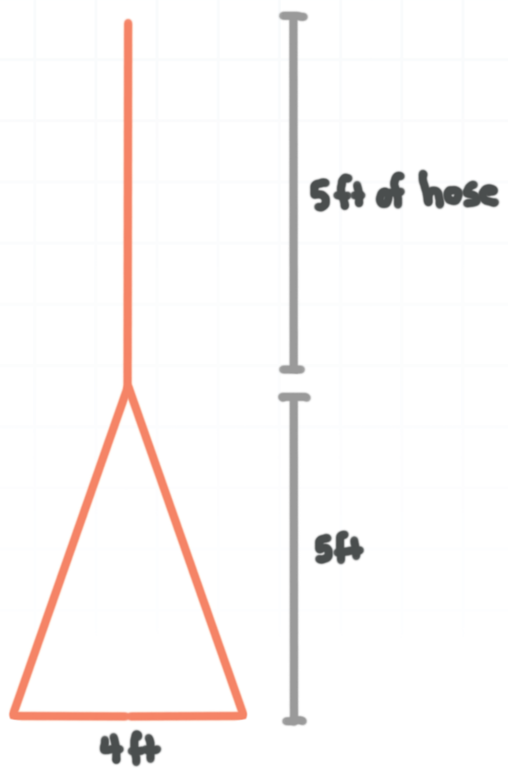
Topic: Work done to empty a tank

Question: A triangular water tank that's 4 feet wide, 5 feet tall, and 10 feet long is completely full of water and needs to be emptied by pumping the water through a hose to a height of 5 feet above the top of the tank. Find the work required to empty the tank.



Answer choices:

A $52,000$ ft-lbs

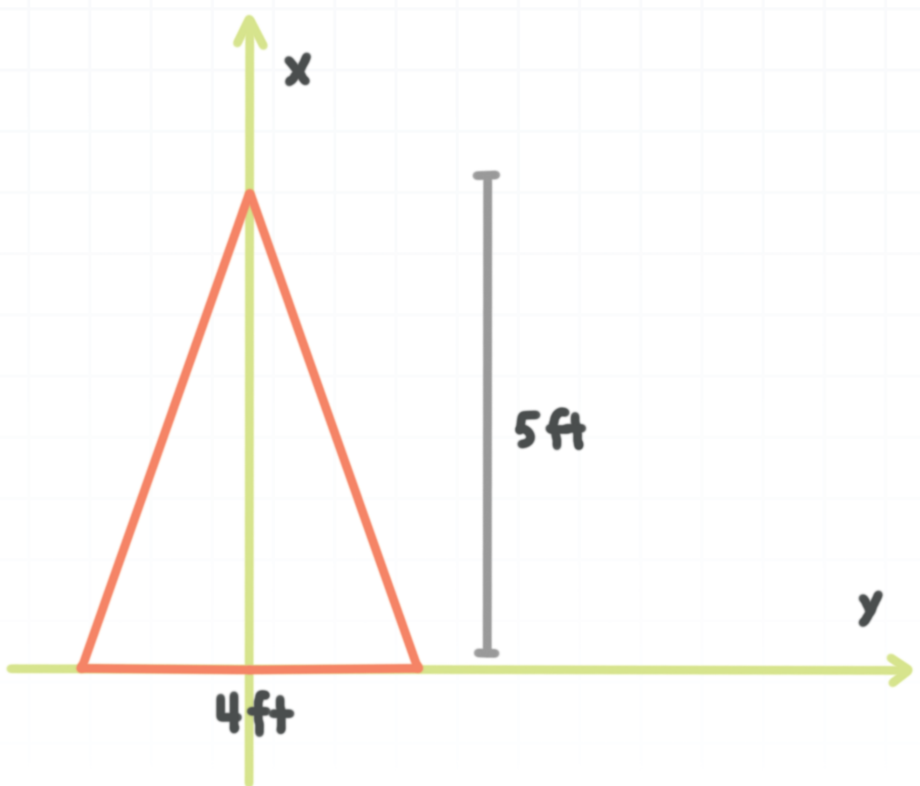
B $68,000$ ft-lbs

C $81,000$ ft-lbs

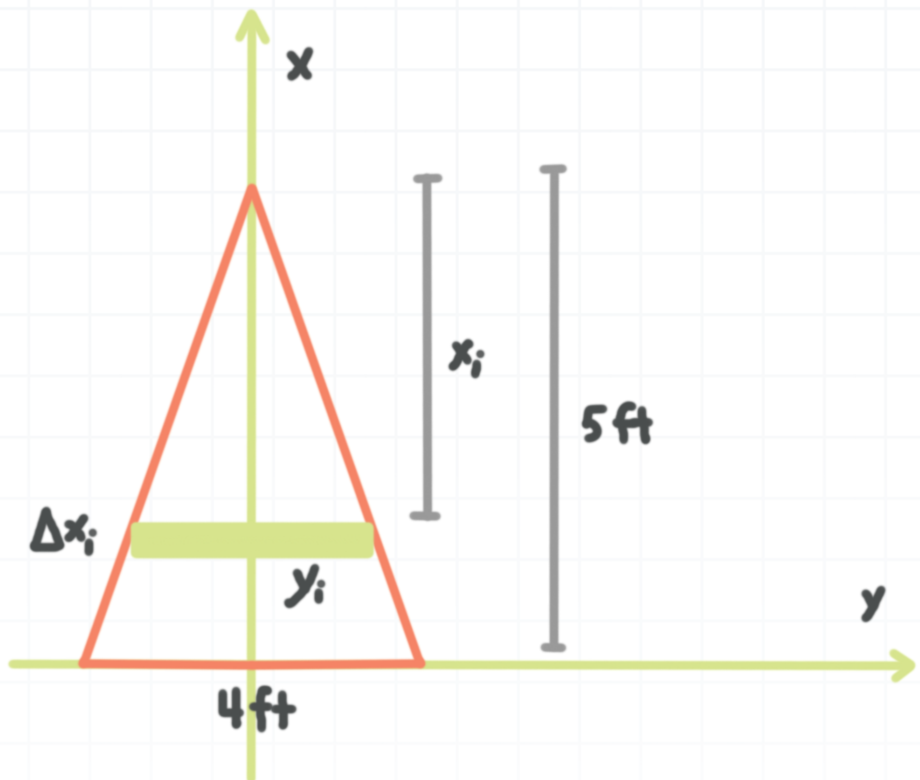
D $96,000$ ft-lbs

Solution: A

We can place the tank in a coordinate system, with the center of the base at the origin, and the top of the tank extending out along the x-axis. Notice that we've flipped the axes to accomplish this. The reason we sketch it this way is so that we can keep the whole problem in terms of x, instead of y.



If we divide the water in the tank into cross sections and we sketch in one cross section (which we'll call the ith cross section, where "ith" is just representing the 1st, 2nd, 3rd, 4th, cross section, etc.) then we can label the height of the cross section along the x-axis as Δx_i, the width of the cross section as y_i, and the distance between the cross section and the top of the tank as x_i.



We know that the formula for work is $W = Fd$, where W is the work done to pump out the water, F is the force required to lift the water, and d is distance traveled by the water to get out of the tank.

Since the measurements of the tank were given in feet, we'll be solving for work in terms of foot-pounds (ft-lbs), which means we need to look at the weight of the water in terms of pounds. "Weight" already accounts for gravity, so we don't need to factor gravity into our calculations.

Starting from the equation for work $W = Fd$, we can say that force F is the same as weight, and we know that weight is the product of density and volume. The weight of water in terms of feet and pounds is

$$\rho = 62.4\ \frac{\text{lb}}{\text{ft}^3}$$

We said that weight would be the product of density and volume, and we just gave density, so now we need to find the volume of the ith cross section. Since the cross section is a rectangular slice and the tank is 10 feet long, we can say

$$V = lwh$$

$$V = 10 \cdot y_i \cdot \Delta x_i$$

We want to do this whole problem in terms of x, which means we'll need to solve for a value of y_i that's in terms of x_i, and then make a substitution into our volume equation. To do that, we'll use similar triangles and relate the dimensions of the triangular tank to the dimensions of the triangle whose base is the cross section of water.

$$\frac{\text{base of the big triangle}}{\text{height of the big triangle}} = \frac{\text{base of the little triangle}}{\text{height of the little triangle}}$$

$$\frac{4}{5} = \frac{y_i}{x_i}$$

$$y_i = \frac{4}{5}x_i$$

Now we can substitute into the volume equation for y_i.

$$V = 10 \cdot y_i \cdot \Delta x_i$$

$$V = 10\left(\frac{4}{5}x_i\right)\Delta x_i$$

$$V = 8x_i\Delta x_i$$

Now that we have volume, and we know that weight is the product of volume and density, we can say that the weight of the ith cross section is

$$\text{weight}_i = 62.4(8x_i\Delta x_i)$$

$$\text{weight}_i = 499.2x_i\Delta x_i$$

Because weight already accounts for gravity, we can make a direct substitution into the work equation for force F.

$$W = Fd$$

$$W_i = (499.2x_i\Delta x_i)d_i$$

The ith cross section must move a distance of x_i to get to the top of the tank. In addition, we know that all of the water has to be pumped 5 feet above the top of the tank, which means that the ith cross section has to travel a distance of $d_i = x_i + 5$.

Therefore,

$$W_i = (499.2x_i\Delta x_i)(x_i + 5)$$

$$W_i = (499.2x_i)(x_i + 5)\Delta x_i$$

$$W_i = 499.2(x_i)(x_i + 5)\Delta x_i$$

$$W_i = 499.2(x_i^2 + 5x_i)\Delta x_i$$

This is the equation that represents the work required to pump the water in the ith cross section out of the tank. But of course we're not just interested in this *one* cross section. We want an equation that will give us the work required to pump *all* of the water out of the tank.

The way that we do this is by pretending that we have an infinite number of cross sections, each of which is infinitely thin. We'll use a Riemann sum to add up the work required to pump out an infinite number n of infinitely thin cross sections, and we'll get

$$W=\lim_{n\to\infty}\sum_{i=1}^{n}499.2(x_i^2+5x_i)\Delta x_i$$

Now we'll translate this Riemann sum into integral notation.

$$W=\int 499.2(x^2+5x)\ dx$$

$$W=499.2\int x^2+5x\ dx$$

We're integrating with respect to x, and we've set up our diagram so that x represents the distance from the top of the tank to any particular slice of water. Therefore, the limits of integration need to be $[0,5]$, since the water at the top of the tank has a distance of 0 ft from the top of the tank, while the water at the bottom of the tank has a distance of 5 ft from the top of the tank.

$$W=499.2\int_0^5 x^2+5x\ dx$$

$$W=499.2\left(\frac{1}{3}x^3+\frac{5}{2}x^2\right)\Bigg|_0^5$$

$$W=499.2\left[\left(\frac{1}{3}(5)^3+\frac{5}{2}(5)^2\right)-\left(\frac{1}{3}(0)^3+\frac{5}{2}(0)^2\right)\right]$$

$$W = 499.2\left(\frac{125}{3} + \frac{125}{2}\right)$$

$$W = 499.2\left(\frac{250}{6} + \frac{375}{6}\right)$$

$$W = 499.2\left(\frac{625}{6}\right)$$

$W = 52{,}000$ ft-lbs

Topic: Work done to empty a tank

Question: Find the work required to empty the tank.

A cylindrical tank standing on end, with base radius 5 feet and height 10 feet, is full of oil and must be emptied by pumping the oil over the edge of the tank. Assume oil density is

$$\rho = 50\ \frac{\text{lb}}{\text{ft}^3}$$

Answer choices:

A $\quad 62{,}400\pi$ ft-lbs

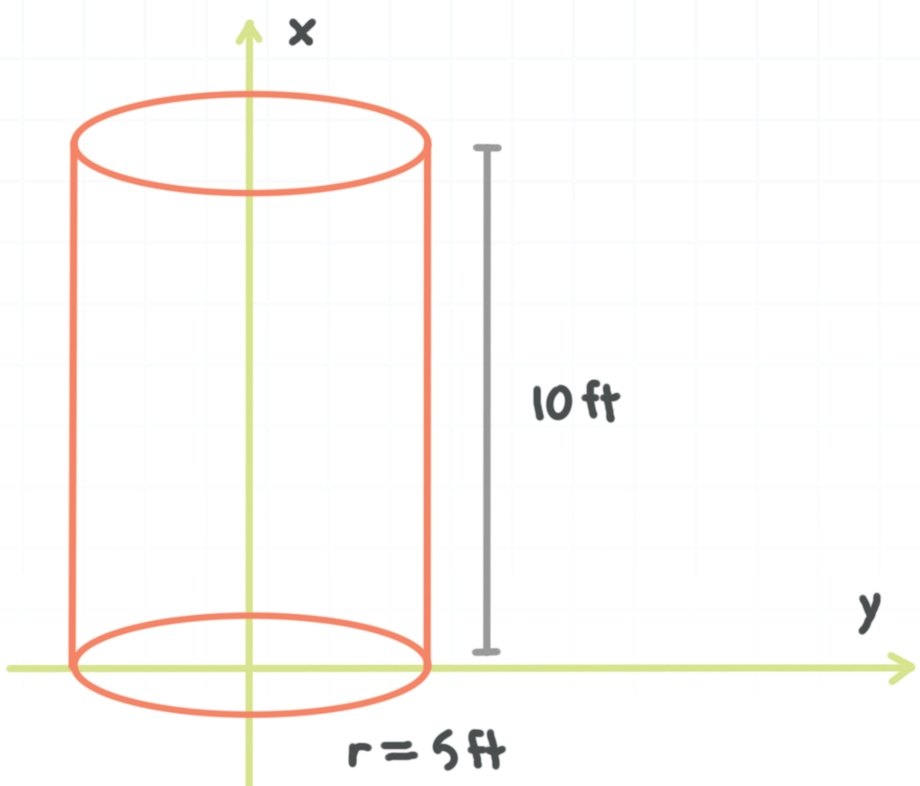
B $\quad 62{,}500\pi$ ft-lbs

C $\quad 60{,}400$ ft-lbs

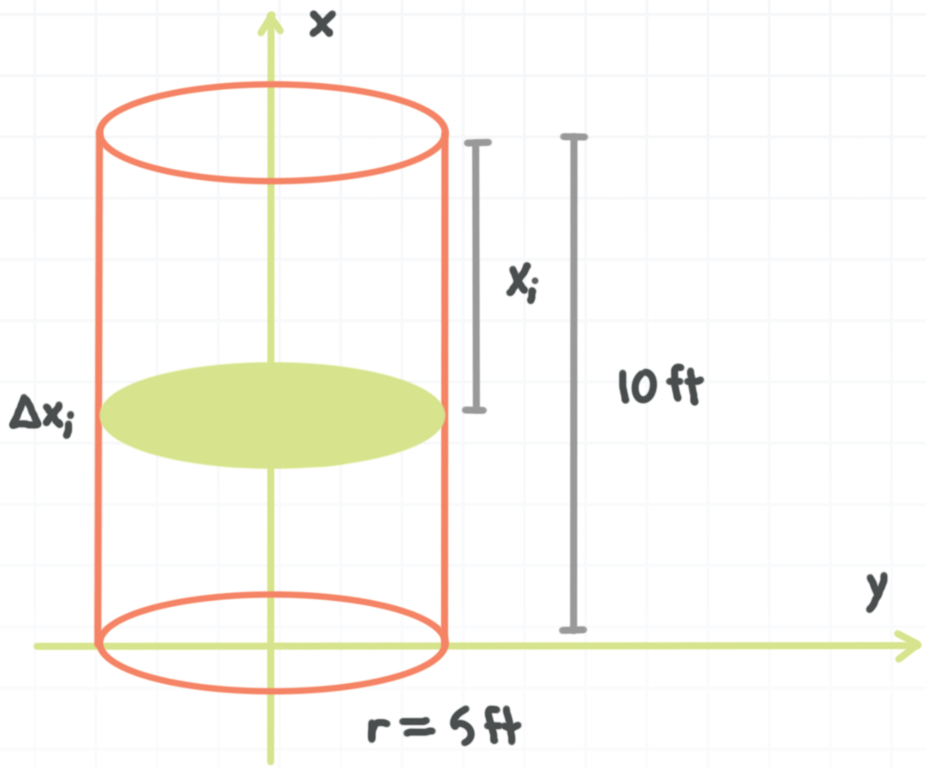
D $\quad 60{,}500$ ft-lbs

Solution: B

We can place the tank in a coordinate system, with the center of the base of the tank at the origin, and the top of the tank extending out along the x-axis. Notice that we've flipped the axes to accomplish this. The reason we sketch it this way is so that we can keep the whole problem in terms of x, instead of y.



If we divide the water in the tank into cross sections and we sketch in one cross section (which we'll call the ith cross section, where "ith" is just representing the 1st, 2nd, 3rd, 4th, cross section, etc.) then we can label the height of the cross section along the x-axis as Δx_i and the distance between the cross section and the top of the tank as x_i.



We know that the formula for work is $W = Fd$, where W is the work done to pump out the water, F is the force required to lift the water, and d is distance traveled by the water to get out of the tank.

Since the measurements of the tank were given in feet, we'll be solving for work in terms of foot-pounds (ft-lbs), which means we need to look at the weight of the water in terms of pounds. "Weight" already accounts for gravity, so we don't need to factor gravity into our calculations.

Remember that if the measurements of the tank had been given in meters instead, we'd be looking for a value for work in terms of Newtons, and we'd therefore need to look at the mass of the water in terms of kilograms, and we'd need to factor in gravity separately.

Starting from the equation for work $W = Fd$, we can say that force F is the same as weight, and we know that weight is the product of density and volume. The weight of the oil in terms of feet and pounds was given as

$$\rho = 50\ \frac{\text{lb}}{\text{ft}^3}$$

Now we need to find the volume of the ith cross section. Since the cross section is a circular slice, and since the tank is 10 feet tall, we can say

$$V = \pi r^2 \cdot \Delta x_i$$

$$V = \pi(5)^2 \cdot \Delta x_i$$

$$V = 25\pi \cdot \Delta x_i$$

Now that we have volume, and we know that weight is the product of volume and density, we can say that the weight of the ith cross section is

$$\text{weight}_i = 50\left(25\pi\Delta x_i\right)$$

$$\text{weight}_i = 1{,}250\pi\Delta x_i$$

Because weight already accounts for gravity, we can make a direct substitution of weight into the work equation for force F.

$$W = Fd$$

$$W_i = \left(1{,}250\pi\Delta x_i\right) d_i$$

According to the diagram we drew earlier, we know that the distance the ith cross section must move to get to the top of the tank is x_i, which means that the ith cross section has to travel a distance of $d_i = x_i$.

Therefore,

$$W_i = \left(1{,}250\pi\Delta x_i\right)(x_i)$$

Δx plays a special role here, so if it isn't already, we want to factor it out and move it to the end of the equation. We'll also simplify the equation as much as we can.

$$W_i = 1{,}250\pi \left(x_i\right) \Delta x_i$$

This is the equation that represents the work required to pump the water in the ith cross section out of the tank. But of course we're not just interested in this *one* cross section. We want an equation that will give us the work required to pump *all* of the water out of the tank.

The way that we do this is by pretending that we have an infinite number of cross sections, each of which is infinitely thin. We'll use a Riemann sum to add up the work required to pump out an infinite number n of infinitely thin cross sections, and we'll get

$$W = \lim_{n\to\infty} \sum_{i=1}^{n} 1{,}250\pi \left(x_i\right) \Delta x_i$$

The easy way to think about this is that you're just taking the equation for W_i and putting

$$W = \lim_{n\to\infty} \sum_{i=1}^{n}$$

in front of it. Then we realize that the summation notation becomes integral notation, and that Δx_i becomes dx. When we transition from summation notation to integral notation, all of the other x variables lose their $_i$ as well.

$$W = \lim_{n \to \infty} \sum_{i=1}^{n} 1{,}250\pi \left(x_i\right) \Delta x_i$$

$$W = \lim_{n \to \infty} \sum_{i=1}^{n} \left[1{,}250\pi \left(x_i\right)\right] \Delta x_i$$

$$W = \int 1{,}250\pi \, (x) \ dx$$

$$W = 1{,}250\pi \int x \ dx$$

Notice that all we really did here was take the equation for W_i, put it inside an integral, change Δx_i to dx and get rid of all the $_i$s. If it's easier, you can just skip the whole part with the summation notation and to straight to the integral once you find W_i.

Now we just need limits of integration for the integral. The limits of integration have to represent the distance the water has to travel to get out of the tank. If we refer back to our diagram, we can see that the water at the bottom of the tank has to travel 10 feet to get from the bottom of the tank to the top. On the other hand, the water at the top of the tank has to travel 0 feet. So the shortest distance the water travels is 0 feet; the longest distance the water travels is 10 feet. Therefore the interval we'll use for the limits of integration is $[0,10]$.

$$W = 1{,}250\pi \int_0^{10} x \ dx$$

$$W = 1{,}250\pi \left(\frac{1}{2}x^2\right) \Bigg|_0^{10}$$

Evaluating over the interval gives

$$W = 1{,}250\pi\left[\left(\frac{1}{2}(10)^2\right) - \left(\frac{1}{2}(0)^2\right)\right]$$

$$W = 1{,}250\pi(50 - 0)$$

$$W = 62{,}500\pi \text{ ft-lbs}$$

Topic: Work done to empty a tank

Question: A cylindrical tank has a height of 8 meters and a radius of 2 meters. The depth of the water in the tank is 6 meters. Find the work required to pump the water up to the level of the top of the tank and out of the tank. Use $1{,}000$ kg/m^3 for the density of the water and 9.8 m/sec^2 for acceleration due to gravity.

Answer choices:

A $W = 1{,}176{,}000$ Joules

B $W = 1{,}176{,}000\pi$ foot-pounds

C $W = 1{,}176{,}000\pi$ Joules

D $W = 1{,}176{,}000$ foot-pounds

Solution: C

We find work by integrating force times distance. Force includes the density of the water times the acceleration due to gravity times the volume of water being pumped. Distance is the distance the water is being pumped.

The formula for finding the work in this problem is

$$W = \int_a^b pgA(y)D(y)\ dy$$

In this formula,

1. W means work.

2. p is the density of the water being pumped. In this problem, the density of the water is $1{,}000$ kg/m³.

3. g is the acceleration due to gravity. The problem states that acceleration due to gravity is 9.8 m/sec².

4. $A(y)$ is the surface area of the water being pumped. The problem states that the tank is shaped like a cylinder, with a radius of 2 meters, so $A(y) = \pi r^2 = \pi(2)^2 = 4\pi$.

5. $D(y)$ is the distance the water is being pumped. The problem states that the tank is 8 meters tall, and filled to a depth of 6 meters. The top layer of water needs to be moved up 2 meters, and the bottom layer of water needs to be moved up 8 meters, so the limits of integration are $[2,8]$.

6. dy is the thickness of the slice of the cross section of the water being pumped, and includes the variable of integration.

The integration limits are found using the beginning depth of the water (6 meters) and the ending depth of the water (0 meters).

Plug everything into the integral.

$$W = \int_a^b pgA(y)D(y)\ dy$$

$$W = \int_0^6 (1{,}000\ \text{kg/m}^3)(9.8\ \text{m/sec}^2)(4\pi\ \text{m}^2)\left[(8-y)\ \text{m}\right]\left(dy\ \text{m}\right)$$

$$W = 39{,}200\pi \frac{\text{kg m m}^2\ \text{m m}}{\text{m}^3\ \text{sec}^2} \int_0^6 8 - y\ dy$$

$$W = 39{,}200\pi \frac{\text{kg m}^2}{\text{sec}^2} \int_0^6 8 - y\ dy$$

$$W = 39{,}200\pi\ \text{J} \int_0^6 8 - y\ dy$$

Integrate, then evaluate over the interval.

$$W = 39{,}200\pi\ \text{J} \left(8y - \frac{1}{2}y^2\right)\Bigg|_0^6$$

$$W = 39{,}200\pi\ \text{J} \left(8(6) - \frac{1}{2}(6)^2\right) - 39{,}200\pi\ \text{J} \left(8(0) - \frac{1}{2}(0)^2\right)$$

$$W = 39{,}200\pi\ \text{J}\,(48 - 18)$$

$$W = 39{,}200\pi \text{ J}(30)$$

$$W = 1{,}176{,}000\pi \text{ J}$$