Sum of the Maclaurin series

To find the sum of a Maclaurin series, we'll try to use a common Maclaurin series for which we already know the sum, manipulating the given series until it matches the standard series.

Example

Find the sum of the Maclaurin series.

$$\sum_{n=0}^{\infty} \frac{(-1)^n 4^n \pi^{2n}}{(2n)!}$$

From a table of standard Maclaurin series, we already know that the sum of the Maclaurin series of $\cos x$ is

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

Since this series is really similar to the series we're given in this problem, we want to try to manipulate our series until it matches the form of this standard series.

We'll start by changing the 4-based term so that its exponent becomes $2n$, like the exponent in the standard series.

$$\sum_{n=0}^{\infty} \frac{(-1)^n 4^n \pi^{2n}}{(2n)!}$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n \left(2^2\right)^n \pi^{2n}}{(2n)!}$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n} \pi^{2n}}{(2n)!}$$

Since they have the same exponent, we can combine the 2-based term with the π-based term.

$$\sum_{n=0}^{\infty} \frac{(-1)^n (2\pi)^{2n}}{(2n)!}$$

With the changes we've made, the given series now matches the standard series for $\cos x$, except that $x = 2\pi$. Knowing that $x = 2\pi$, we can make the substitution on the left-hand side of the formula for the sum of the Maclaurin series of $\cos x$.

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

$$\cos(2\pi) = \sum_{n=0}^{\infty} \frac{(-1)^n (2\pi)^{2n}}{(2n)!}$$

We know that $\cos(2\pi) = 1$, so

$$1 = \sum_{n=0}^{\infty} \frac{(-1)^n (2\pi)^{2n}}{(2n)!}$$

Since the right side of this equation is equal to the sum of the given series, we can say that the sum of the given series is 1.