Topic: Area under one arc or loop of a parametric curve

Question: Find the area under one arc of the parametric curve.

$$x = \sin\theta$$

$$y = \cos\theta$$

Answer choices:

A $\frac{\pi}{2}$

B 2π

C π

D $-\pi$

Solution: C

To find the area under one arc or loop of a parametric curve, we will need to use the formula

$$A=\int_a^b y(t)x'(t)\ dt$$

where $[a,b]$ is the interval that contains the loop (typically $[0,2\pi]$), and $x'(t)$ is the derivative of $x(t)$.

First we'll find the bounds we need to use by setting up a table for θ, x, and y.

θ	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
x	0	1	0	-1	0
y	1	0	-1	0	1

Because $(x,y)=(0,1)$ at $\theta=0$, and we don't get back to $(0,1)$ until $\theta=2\pi$, we know the first loop of the parametric curve is closed by $\theta=[0,2\pi]$.

Before we can plug everything into our area formula, we'll need to find the derivative of $x(\theta)$.

$$x'(\theta)=\cos\theta$$

Plugging everything into the area formula, we get

$$A=\int_0^{2\pi}\cos\theta\cos\theta\ d\theta$$

$$A=\int_0^{2\pi}\cos^2\theta\ d\theta$$

Before we can integrate, we need to do a substitution for $\cos^2\theta$ using the identity

$$\cos^2\theta=\frac{1}{2}(1+\cos 2\theta)$$

We'll make the substitution.

$$A=\int_0^{2\pi}\frac{1}{2}(1+\cos(2\theta))\ d\theta$$

$$A=\frac{1}{2}\int_0^{2\pi}1+\cos(2\theta)\ d\theta$$

$$A=\frac{1}{2}\left[\theta+\frac{1}{2}\sin(2\theta)\right]\Bigg|_0^{2\pi}$$

$$A=\frac{1}{2}\left[2\pi+\frac{1}{2}\sin(4\pi)\right]-\frac{1}{2}\left[0+\frac{1}{2}\sin(0)\right]$$

$$A=\frac{1}{2}\left(2\pi+\frac{1}{2}(0)\right)-\frac{1}{2}\left(0+\frac{1}{2}(0)\right)$$

$$A=\pi$$

Topic: Area under one arc or loop of a parametric curve

Question: Find the area under one arc of the parametric curve.

$$x = \cos 2\theta$$

$$y = 6 + \sin 2\theta$$

Answer choices:

A $-\pi$

B $-\dfrac{\pi}{2}$

C π

D $\dfrac{\pi}{2}$

Solution: A

To find the area under one arc or loop of a parametric curve, we will need to use the formula

$$A=\int_{a}^{b} y(t)x'(t)\ dt$$

where $[a,b]$ is the interval that contains the loop (typically $[0,2\pi]$), and $x'(t)$ is the derivative of $x(t)$.

First we'll find the bounds we need to use by setting up a table for θ, x, and y.

θ	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π
x	1	0	-1	0	1
y	6	7	6	5	6

Because $(x,y)=(1,6)$ at $\theta=0$, and we don't get back to $(1,6)$ until $\theta=\pi$, we know the first loop of the parametric curve is closed by $\theta=[0,\pi]$.

Before we can plug everything into our area formula, we'll need to find the derivative of $x(\theta)$.

$$x'(\theta)=-2\sin(2\theta)$$

Plugging everything into the area formula, we get

$$A=\int_{0}^{\pi}\left[6+\sin(2\theta)\right]\left[-2\sin(2\theta)\right]\ d\theta$$

$$A=\int_{0}^{\pi}-12\sin(2\theta)-2\sin^2(2\theta)\ d\theta$$

$$A=-12\int_{0}^{\pi}\sin(2\theta)\ d\theta-2\int_{0}^{\pi}\sin^2(2\theta)\ d\theta$$

Before we can integrate, we need to do a substitution for $\sin^2\theta$ using the identity

$$\sin^2\theta=\frac{1}{2}\left[1-\cos(2\theta)\right]$$

We'll make the substitution.

$$A=-12\int_{0}^{\pi}\sin(2\theta)\ d\theta-2\int_{0}^{\pi}\frac{1}{2}\left[1-\cos(4\theta)\right]\ d\theta$$

$$A=-12\int_{0}^{\pi}\sin(2\theta)\ d\theta-\int_{0}^{\pi}1-\cos(4\theta)\ d\theta$$

Integrate.

$$A=-12\left(-\frac{1}{2}\cos(2\theta)\right)\Big|_{0}^{\pi}-\left(\theta-\frac{1}{4}\sin(4\theta)\right)\Big|_{0}^{\pi}$$

$$A=6\cos(2\theta)\Big|_{0}^{\pi}+\frac{1}{4}\sin(4\theta)-\theta\Big|_{0}^{\pi}$$

$$A=6\cos(2\theta)+\frac{1}{4}\sin(4\theta)-\theta\Big|_{0}^{\pi}$$

Evaluate over the interval.

$$A=6\cos(2\pi)+\frac{1}{4}\sin(4\pi)-\pi-\left(6\cos(2(0))+\frac{1}{4}\sin(4(0))-0\right)$$

$$A=6(1)+\frac{1}{4}(0)-\pi-\left(6(1)+\frac{1}{4}(0)\right)$$

$$A=6-\pi-6$$

$$A=-\pi$$

Topic: Area under one arc or loop of a parametric curve

Question: Find the area under one arc of the parametric curve.

$$x = 8 + \sin\theta$$

$$y = 8\cos\theta$$

Answer choices:

A -4π

B -8π

C 4π

D 8π

Solution: D

To find the area under one arc or loop of a parametric curve, we will need to use the formula

$$A=\int_a^b y(t)x'(t)\ dt$$

where $[a,b]$ is the interval that contains the loop (typically $[0,2\pi]$), and $x'(t)$ is the derivative of $x(t)$.

First we'll find the bounds we need to use by setting up a table for θ, x, and y.

θ	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
x	8	9	8	7	8
y	8	0	-8	0	8

Because $(x,y)=(8,8)$ at $\theta=0$, and we don't get back to $(8,8)$ until $\theta=2\pi$, we know the first loop of the parametric curve is closed by $\theta=[0,2\pi]$.

Before we can plug everything into our area formula, we'll need to find the derivative of $x(\theta)$.

$$x'(\theta)=\cos\theta$$

Plugging everything into the area formula, we get

$$A=\int_0^{2\pi}(8\cos\theta)(\cos\theta)\ d\theta$$

$$A = 8\int_0^{2\pi} \cos^2\theta \ d\theta$$

Before we can integrate, we need to do a substitution for $\cos^2\theta$ using the formula

$$\cos^2\theta = \frac{1}{2}(1 + \cos 2\theta)$$

We'll make the substitution.

$$A = 8\int_0^{2\pi} \frac{1}{2}(1 + \cos 2\theta) \ d\theta$$

$$A = 4\int_0^{2\pi} 1 + \cos(2\theta) \ d\theta$$

$$A = 4\left[\theta + \frac{1}{2}\sin(2\theta)\right]\Big|_0^{2\pi}$$

$$A = 4\left[2\pi + \frac{1}{2}\sin(4\pi)\right] - 4\left[0 + \frac{1}{2}\sin(0)\right]$$

$$A = 4\left[2\pi + \frac{1}{2}(0)\right] - 4\left[0 + \frac{1}{2}(0)\right]$$

$$A = 4(2\pi)$$

$$A = 8\pi$$