Topic: Improper integrals, case 6

Question: Evaluate the improper integral.

$$\int_0^6 \frac{dx}{(x-1)^{\frac{2}{3}}}$$

Answer choices:

A $\quad -3+3\sqrt[3]{5}$

B $\quad 3-3\sqrt[3]{5}$

C $\quad \infty$

D $\quad 3+3\sqrt[3]{5}$

Solution: D

The integral in this problem is considered to be an improper integral, case 6, because the integrand is undefined not at either endpoint of the interval, but instead at a point inside the interval. Evaluating this type of improper integral requires us to split the integral at the point of discontinuity into two separate integrals.

$$\int_a^b f(x)\ dx = \int_a^c f(x)\ dx + \int_c^b f(x)\ dx$$

But there's a discontinuity at $x = c$, so we rewrite the integral as

$$\int_a^b f(x)\ dx = \lim_{x \to c^-} \int_a^c f(x)\ dx + \lim_{x \to c^+} \int_c^b f(x)\ dx$$

We'll start by rewriting the given integral as

$$\int_0^6 \frac{dx}{(x-1)^{\frac{2}{3}}} = \int_0^6 (x-1)^{-\frac{2}{3}}\ dx$$

The integrand is undefined at $x = 1$, which is between the integration limits. Therefore, we'll re-write the integral as two integrals, separating them at $x = 1$.

$$\int_0^1 (x-1)^{-\frac{2}{3}}\ dx + \int_1^6 (x-1)^{-\frac{2}{3}}\ dx$$

Since the integrand has a vertical asymptote at $x = 1$, both integrals are still improper integrals. Therefore, we'll re-write the integrals as limits, as shown in the above rule.

$$\lim_{b\to 1^-}\int_0^b (x-1)^{-\frac{2}{3}}\ dx + \lim_{a\to 1^+}\int_a^6 (x-1)^{-\frac{2}{3}}\ dx$$

Integrate.

$$\lim_{b\to 1^-}\left[\frac{(x-1)^{\frac{1}{3}}}{\frac{1}{3}}\right]_0^b + \lim_{a\to 1^+}\left[\frac{(x-1)^{\frac{1}{3}}}{\frac{1}{3}}\right]_a^6$$

$$\lim_{b\to 1^-}\left[3(x-1)^{\frac{1}{3}}\right]_0^b + \lim_{a\to 1^+}\left[3(x-1)^{\frac{1}{3}}\right]_a^6$$

Evaluate over the interval.

$$\lim_{b\to 1^-}\left[3(b-1)^{\frac{1}{3}} - 3(0-1)^{\frac{1}{3}}\right] + \lim_{a\to 1^+}\left[3(6-1)^{\frac{1}{3}} - 3(a-1)^{\frac{1}{3}}\right]$$

$$\lim_{b\to 1^-}\left[3(b-1)^{\frac{1}{3}} + 3\right] + \lim_{a\to 1^+}\left[3(5)^{\frac{1}{3}} - 3(a-1)^{\frac{1}{3}}\right]$$

Take the limit.

$$\left[3(1-1)^{\frac{1}{3}} + 3\right] + \left[3(5)^{\frac{1}{3}} - 3(1-1)^{\frac{1}{3}}\right]$$

$$(0+3) + \left[3(5)^{\frac{1}{3}}\right]$$

$$3 + 3(5)^{\frac{1}{3}}$$

$$3 + 3\sqrt[3]{5}$$

Topic: Improper integrals, case 6

Question: Evaluate the improper integral.

$$\int_{-3}^{3} \frac{7}{x^6}\, dx$$

Answer choices:

A $-\infty$

B ∞

C 0

D $8 \ln 6$

Solution: B

The integral in this problem is considered to be an improper integral, case 6, because the integrand is undefined not at either endpoint of the interval, but instead at a point inside the interval. Evaluating this type of improper integral requires us to split the integral at the point of discontinuity into two separate integrals.

$$\int_a^b f(x)\ dx = \int_a^c f(x)\ dx + \int_c^b f(x)\ dx$$

But there's a discontinuity at $x = c$, so we rewrite the integral as

$$\int_a^b f(x)\ dx = \lim_{x \to c^-} \int_a^c f(x)\ dx + \lim_{x \to c^+} \int_c^b f(x)\ dx$$

We'll start by rewriting the given integral as

$$\int_{-3}^3 \frac{7}{x^6}\ dx = \int_{-3}^3 7x^{-6}\ dx$$

The integrand is undefined at $x = 0$, which is between the integration limits. Therefore, we'll re-write the integral as two integrals, separating them at $x = 0$.

$$\int_{-3}^0 7x^{-6}\ dx + \int_0^3 7x^{-6}\ dx$$

Since the integrand has a vertical asymptote at $x = 0$, both integrals are still improper integrals. Therefore, we'll re-write the integrals as limits, as shown in the above rule.

$$\lim_{c\to 0^-}\int_{-3}^{c} 7x^{-6}\ dx+\lim_{c\to 0^+}\int_{c}^{3} 7x^{-6}\ dx$$

Integrate.

$$\lim_{c\to 0^-}\left(-\frac{7}{5}x^{-5}\right)\Bigg|_{-3}^{c}+\lim_{c\to 0^+}\left(-\frac{7}{5}x^{-5}\right)\Bigg|_{c}^{3}$$

$$\lim_{c\to 0^-}\left(-\frac{7}{5x^5}\right)\Bigg|_{-3}^{c}+\lim_{c\to 0^+}\left(-\frac{7}{5x^5}\right)\Bigg|_{c}^{3}$$

Evaluate over the interval.

$$\lim_{c\to 0^-}\left(-\frac{7}{5c^5}+\frac{7}{5(-3)^5}\right)+\lim_{c\to 0^+}\left(-\frac{7}{5(3)^5}+\frac{7}{5c^5}\right)$$

$$\frac{7}{5}\lim_{c\to 0^-}\left[\frac{1}{(-3)^5}-\frac{1}{c^5}\right]+\frac{7}{5}\lim_{c\to 0^+}\left[\frac{1}{c^5}-\frac{1}{(3)^5}\right]$$

Take the limit.

$$\frac{7}{5}\left[\frac{1}{(-3)^5}+\infty\right]+\frac{7}{5}\left[\infty-\frac{1}{(3)^5}\right]$$

$$\infty$$

Topic: Improper integrals, case 6

Question: Evaluate the improper integral.

$$\int_0^4 \frac{x^2}{x^3-8}\,dx$$

Answer choices:

A $3\ln 2 + 3\ln 4$

B 0

C $-3\ln 2$

D The integral diverges

Solution: D

The integral in this problem is considered to be an improper integral, case 6, because the integrand is undefined not at either endpoint of the interval, but instead at a point inside the interval. Evaluating this type of improper integral requires us to split the integral at the point of discontinuity into two separate integrals.

$$\int_a^b f(x)\ dx = \int_a^c f(x)\ dx + \int_c^b f(x)\ dx$$

But there's a discontinuity at $x = c$, so we rewrite the integral as

$$\int_a^b f(x)\ dx = \lim_{x\to c^-} \int_a^c f(x)\ dx + \lim_{x\to c^+} \int_c^b f(x)\ dx$$

The given integral is

$$\int_0^4 \frac{x^2}{x^3 - 8}\ dx$$

The integrand is undefined at $x = 2$, which is between the integration limits. Therefore, we'll re-write the integral as two integrals, separating them at $x = 2$.

$$\int_0^2 \frac{x^2}{x^3 - 8}\ dx + \int_2^4 \frac{x^2}{x^3 - 8}\ dx$$

Since the integrand has a vertical asymptote at $x = 2$, both integrals are still improper integrals. Therefore, we'll re-write the integrals as limits, as shown in the above rule.

$$\lim_{c \to 2^-} \int_0^c \frac{x^2}{x^3 - 8} \ dx + \lim_{c \to 2^+} \int_c^4 \frac{x^2}{x^3 - 8} \ dx$$

Use a u-substitution to integrate.

$$u = x^3 - 8$$

$$du = 3x^2 \ dx$$

$$dx = \frac{du}{3x^2}$$

Make substitutions into the integral.

$$\lim_{c \to 2^-} \int_{x=0}^{x=c} \frac{x^2}{u} \left(\frac{du}{3x^2} \right) + \lim_{c \to 2^+} \int_{x=c}^{x=4} \frac{x^2}{u} \left(\frac{du}{3x^2} \right)$$

$$\frac{1}{3} \lim_{c \to 2^-} \int_{x=0}^{x=c} \frac{1}{u} \ du + \frac{1}{3} \lim_{c \to 2^+} \int_{x=c}^{x=4} \frac{1}{u} \ du$$

Integrate and then back substitute.

$$\frac{1}{3} \lim_{c \to 2^-} \ln|u| \Big|_{x=0}^{x=c} + \frac{1}{3} \lim_{c \to 2^+} \ln|u| \Big|_{x=c}^{x=4}$$

$$\frac{1}{3} \lim_{c \to 2^-} \ln \left| x^3 - 8 \right| \Big|_0^c + \frac{1}{3} \lim_{c \to 2^+} \ln \left| x^3 - 8 \right| \Big|_c^4$$

Evaluate over the interval.

$$\frac{1}{3} \lim_{c \to 2^-} \left[\ln \left| c^3 - 8 \right| - \ln \left| 0^3 - 8 \right| \right] + \frac{1}{3} \lim_{c \to 2^+} \left[\ln \left| 4^3 - 8 \right| - \ln \left| c^3 - 8 \right| \right]$$

$$\frac{1}{3} \lim_{c \to 2^-} \left[\ln \left| c^3 - 8 \right| - \ln 8 \right] + \frac{1}{3} \lim_{c \to 2^+} \left[\ln 56 - \ln \left| c^3 - 8 \right| \right]$$

For the first limit, when $c \to 2^-$, $\ln|c^3 - 8|$ takes on a value of $-\infty$, so that first limit becomes

$$\frac{1}{3}\left[-\infty - \ln 8\right]$$

$$\frac{1}{3}[-\infty]$$

$$-\infty$$

Because the first integral diverges, the whole integral diverges.