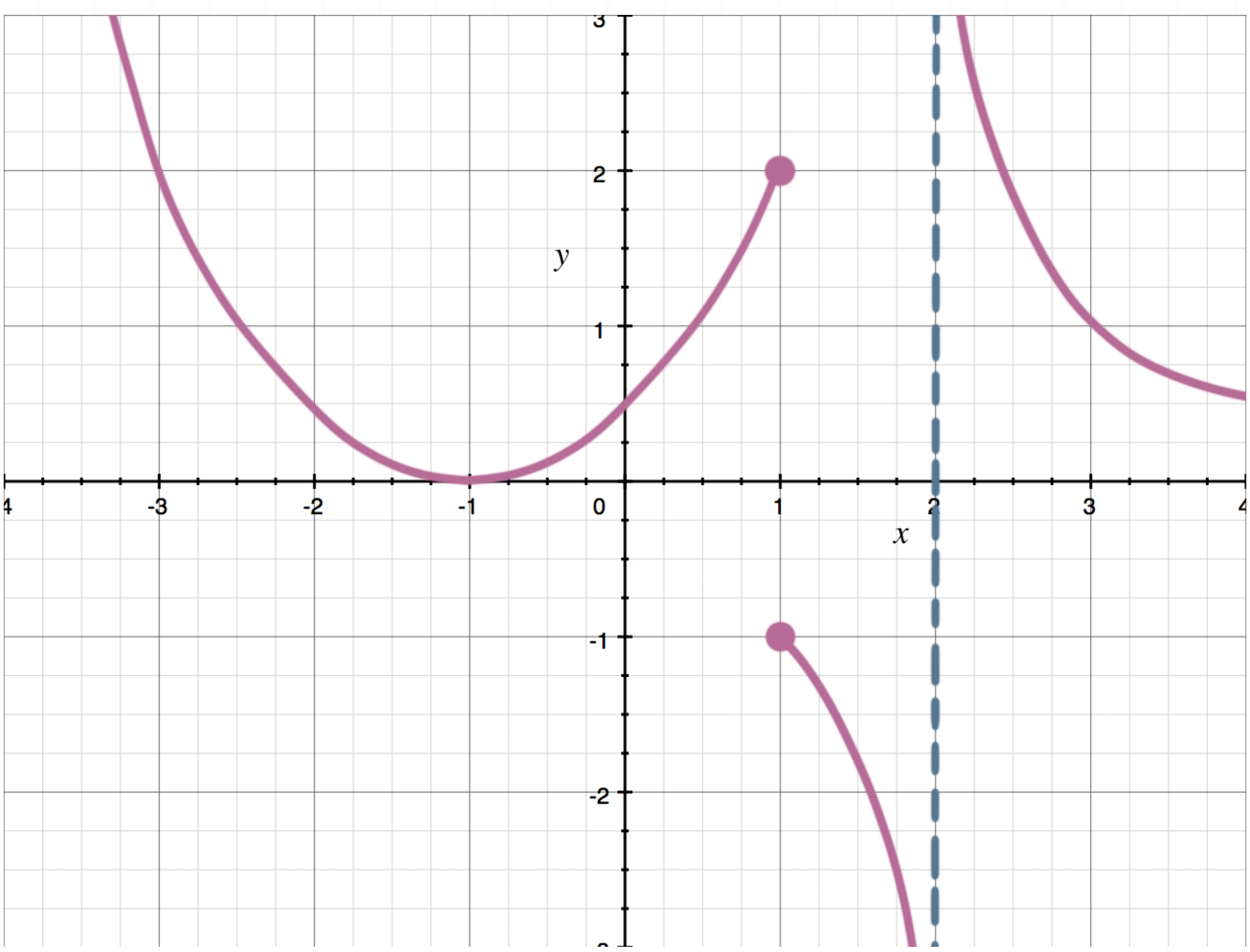
Point discontinuities

We already know that the general limit doesn't exist wherever the left- and right-hand limits are not equal.

The idea of continuity is exactly what it sounds like. If a function has continuity at a particular point, it means the function is continuous at that point, meaning that there are no holes, jumps, or asymptotes in the graph there.

For instance, this graph



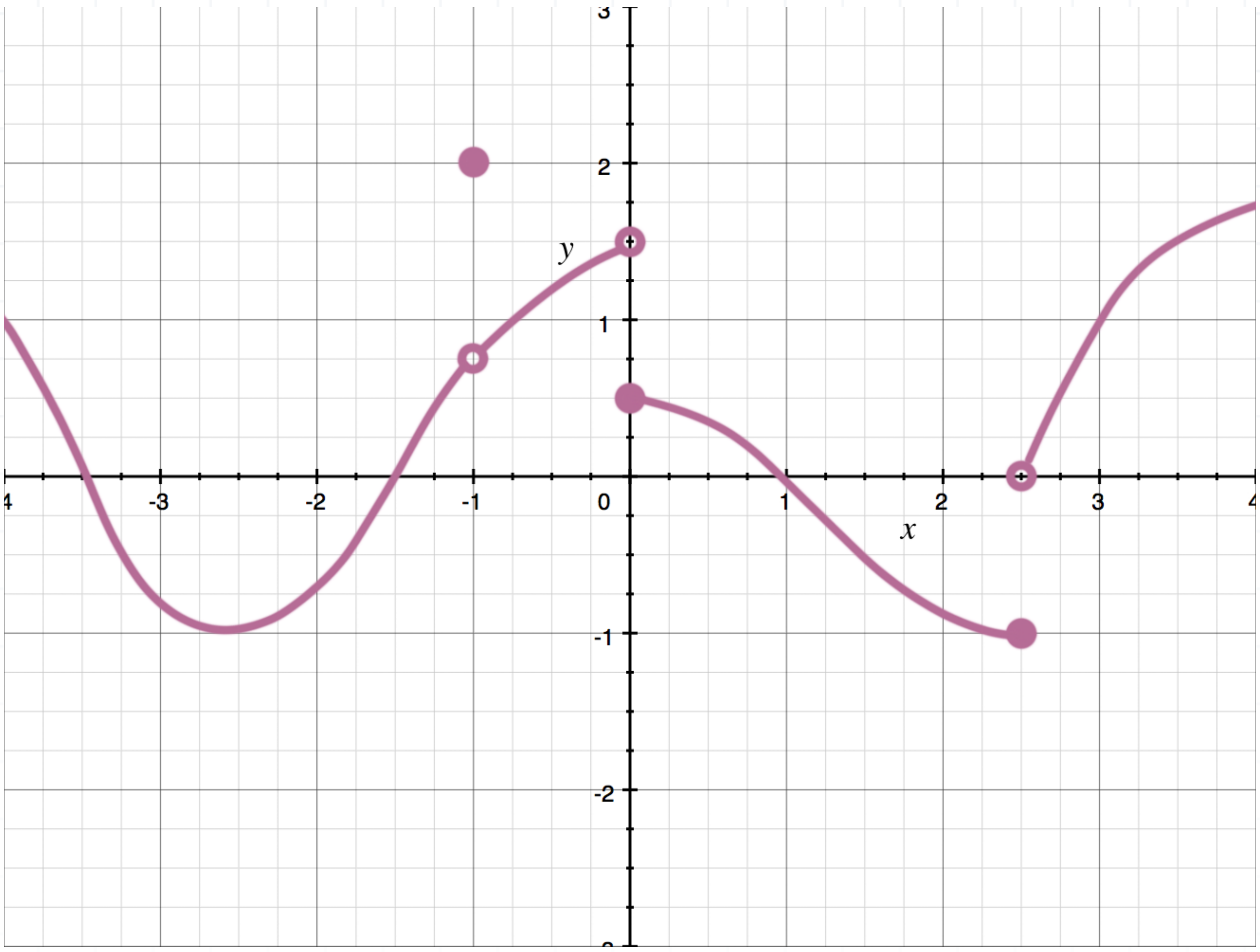
has a discontinuity at $x = 1$, because there's a jump there. The left piece of the graph has a value of 2 at $x = 1$, whereas the right piece of the graph has a value of -1 at $x = 1$. Because $2 \neq -1$, there's a jump in the graph at that point. The graph also has a discontinuity at $x = 2$, because there's a vertical asymptote there. It isn't continuous at the asymptote because the asymptote breaks the graph into two pieces.

If we can draw the graph without ever lifting our pencil off the paper as we sketch it out from left to right, then it's continuous everywhere. At any point where we have to lift our pencil off the paper in order to continue sketching it, the graph will have a discontinuity at that point.

There are different types of discontinuities, all of which mean different things for the value of the limit at the discontinuity.

Point (removable) discontinuities

A **point discontinuity** exists wherever there's a hole in the graph at one specific point. In this graph,



there's a point discontinuity at $x = -1$, which is shown by the empty hole in the graph there. When there's a point discontinuity, the function will look continuous and smooth around that point, but then have an empty hole in the graph at that exact spot.

We get this kind of discontinuity with rational functions (a rational function is a fraction in which the numerator and denominator are both polynomials) like this one:

$$f(x) = \frac{x^2 + 11x + 28}{x + 4}$$

It looks like this function has a vertical asymptote at $x = -4$, because $x = -4$ makes the denominator 0. But if we factor the numerator,

$$f(x) = \frac{(x + 4)(x + 7)}{x + 4}$$

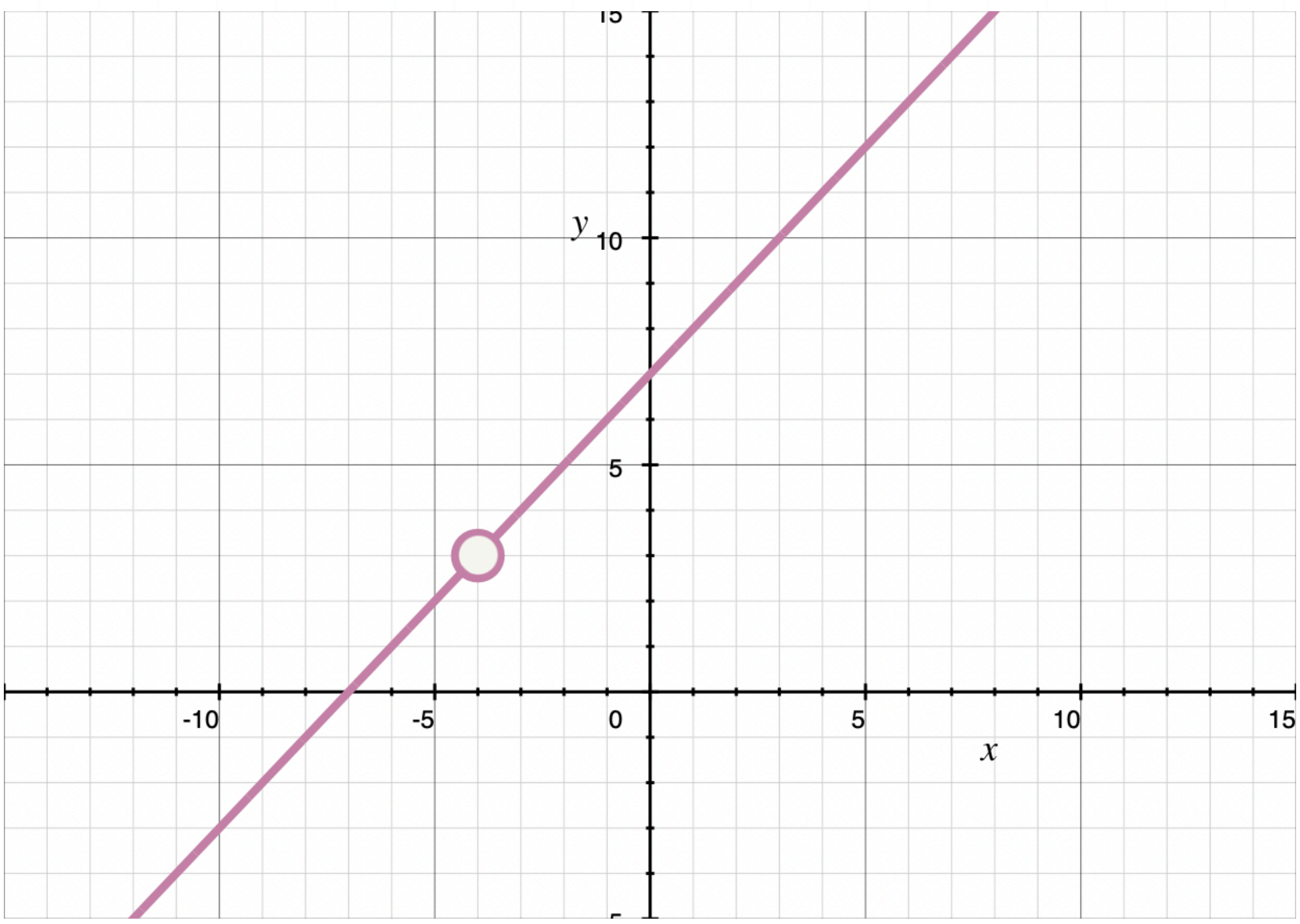
we can see that a factor of $x+4$ will cancel from both the numerator and denominator.

$$f(x)=\frac{x+4}{x+4}(x+7)$$

$$f(x)=1(x+7)$$

$$f(x)=x+7$$

Because we were able to eliminate the denominator, we know the function actually does <u>not</u> have a vertical asymptote at $x=-4$. Instead, the original function $f(x)$ follows the curve $f(x)=x+7$, but it has a removable discontinuity at $x=-4$, like this:



Notice how, if we changed the function to

$$f(x) = \begin{cases} \frac{x^2 + 11x + 28}{x + 4} & x \neq -4 \\ 3 & x = -4 \end{cases}$$

we would remove the hole in the graph at $(-4,3)$. Functions that are written this way are called "piecewise functions" or "piecewise-defined functions," because they're defined in pieces.

The first "piece" of this piecewise function defines the function everywhere except at $x = -4$. The second "piece" steps in to plug the hole and define the function as having a value of 3 when $x = -4$ specifically.

That's why point discontinuities are also called **removable discontinuities**: because we can "remove" the discontinuity just be redefining the function as a piecewise function.

Let's look at an example where we find the point discontinuities of a function.

Example

Find any point discontinuities in the graph of the function.

$$f(x) = \frac{x - 2}{x^2 + x - 6}$$

If we factor the denominator of the function,

$$f(x) = \frac{x - 2}{(x - 2)(x + 3)}$$

it looks as if there are discontinuities in the function at $x = 2$ and $x = -3$, because those values both make the denominator equal to 0. But we realize that we can cancel a factor of $x - 2$, leaving

$$f(x) = \frac{1}{x+3}$$

Therefore, we can say there's a point (removable) discontinuity at $x = 2$. As a side note, because $x = -3$ still makes the denominator equal to 0, even after we've simplified the function, the function has a vertical asymptote at $x = -3$.

Let's work through another example of how to remove a removable discontinuity by redefining the function as a piecewise function.

Example

Redefine the function as a piecewise function in order to remove the discontinuity.

$$f(x) = \frac{x^2 - 16}{x - 4}$$

We can see that the function is discontinuous at $x = 4$, because $x = 4$ makes the denominator equal to 0. But we can factor and then simplify the function.

$$f(x) = \frac{(x-4)(x+4)}{x-4}$$

$$f(x) = x + 4$$

Evaluate $f(x)$ at $x = 4$.

$$f(4) = 4 + 4$$

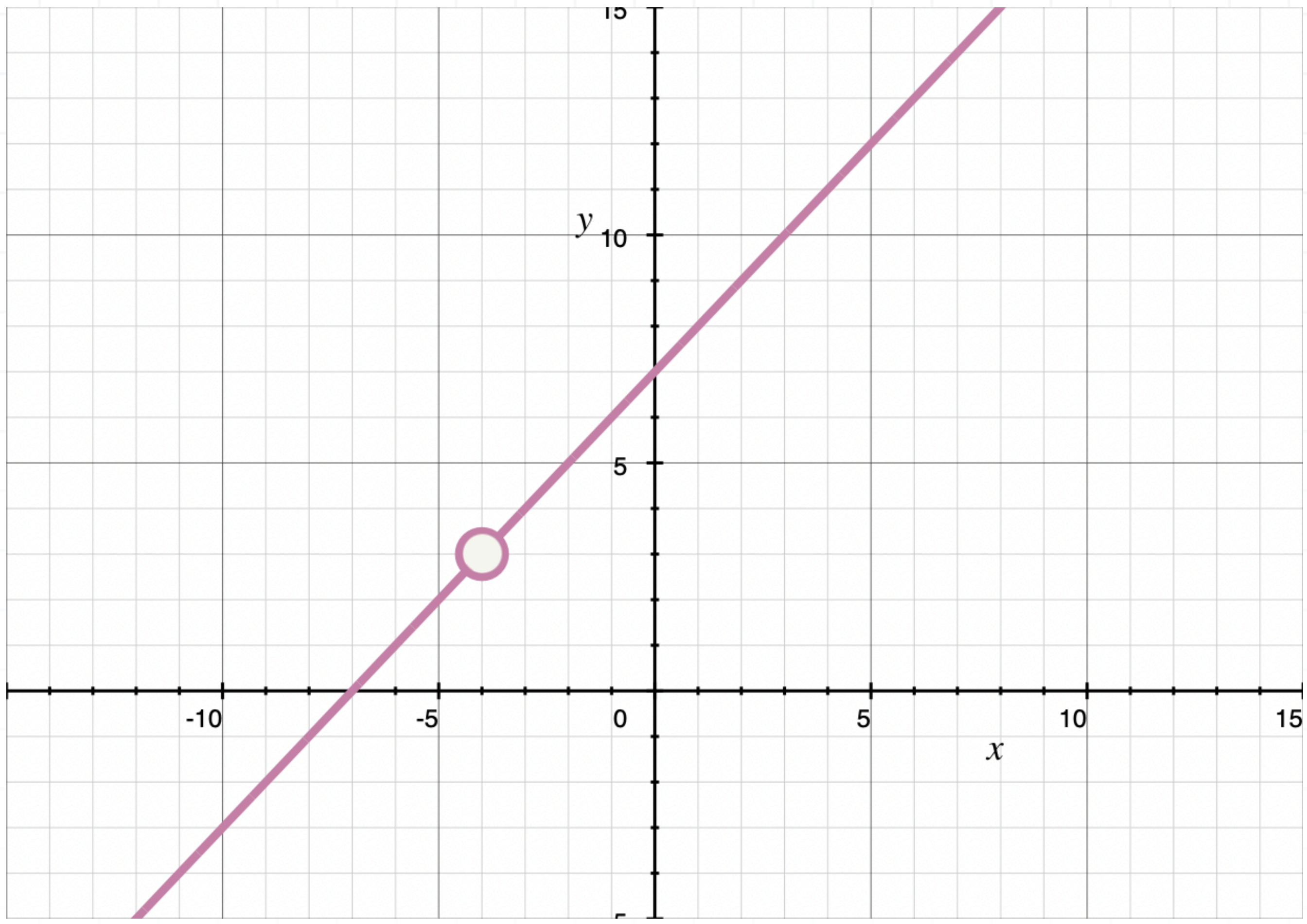
$$f(4) = 8$$

Therefore, we can make the function continuous if we redefine it as

$$f(x) = \begin{cases} \frac{x^2-16}{x-4} & x \neq 4 \\ 8 & x = 4 \end{cases}$$

Does the limit exist at a point discontinuity?

Keep in mind that the general limit always exists at a point/removable discontinuity. That's because the left-hand limit and right-hand limit both exist, and those one-sided limits are equal. For instance, in the graph



the left-hand limit as $x \to -4$ is 3, and the right-hand limit as $x \to -4$ is 3. Because both one-sided limits are 3, the general limit exists and is equal to 3. Even though the function is discontinuous at $x = -4$, the graph is approaching a value of 3 from both sides of $x = -4$ as we get really close to $x = -4$, so the general limit there is 3.