Bounded sequences

Only monotonic sequences can be bounded, because bounded sequences must be either increasing or decreasing, and monotonic sequences are sequences that are always increasing or always decreasing. Bounded sequences can be

bounded above by the largest value of the sequence

bounded below by the smallest value of the sequence

bounded both above and below

The smallest value of an increasing monotonic sequence will be its first term, where $n = 1$. In this case, $a_n \geq a_1$, so we know that **increasing monotonic sequences are bounded below**.

The largest value of a decreasing monotonic sequence will be its first term, where $n = 1$. In this case, $a_n \leq a_1$, so we know that **decreasing monotonic sequences are bounded above**.

To determine if the end of the monotonic sequence is bounded, we'll need to take the limit of the sequence as $n \to \infty$. If we obtain a real-number answer for the limit, then the sequence is bounded at the end as well as at the beginning.

Because we're using the limit as $n \to \infty$ to solve for any possible end of sequence bounding, our end bounds will be in the form $a_n < a_\infty$ for an increasing sequence and $a_n > a_\infty$ for a decreasing sequence if end bounds exist.

Example

Say whether or not the sequence is bounded, and if it is, find its bounds.

$$a_n = \frac{n^2+6}{3n^2-1}$$

In order for a sequence to be bounded the sequence needs to be monotonic (either increasing or decreasing). Let's assess our sequence to see if it's monotonic. We can do this by calculating the first few terms of the sequence. Let's calculate $n = 1$, $n = 2$, $n = 3$ and $n = 4$.

When $n = 1$ $\quad a_1 = \dfrac{(1)^2+6}{3(1)^2-1}$ so $a_1 = \dfrac{7}{2}$

When $n = 2$ $\quad a_2 = \dfrac{(2)^2+6}{3(2)^2-1}$ so $a_2 = \dfrac{10}{11}$

When $n = 3$ $\quad a_3 = \dfrac{(3)^2+6}{3(3)^2-1}$ so $a_3 = \dfrac{15}{26}$

When $n = 4$ $\quad a_4 = \dfrac{(4)^2+6}{3(4)^2-1}$ so $a_4 = \dfrac{22}{47}$

The first four terms of the sequence are

$$\left\{\frac{7}{2}, \frac{10}{11}, \frac{15}{26}, \frac{22}{47}\right\}$$

Looking at the first four terms we can see that the sequence is decreasing, which means it's also monotonic. Since the sequence is decreasing and monotonic, it means it'll also be bounded above.

Now we need to find the bounds of our sequence. In the case of a decreasing sequence, the first term of the sequence $n = 1$ will be the largest term of the sequence. In this case, $a_1 = 7/2$ is our first term. We can say that our sequence is bounded above at $a_n \le 7/2$.

Now, we can check to see if our sequence is also bounded below. To do this we'll need to take the limit of our sequence as $n \to \infty$.

$$\lim_{n\to\infty} \frac{n^2+6}{3n^2-1}$$

We can divide each term in the numerator and denominator by the highest-degree variable, n^2.

$$\lim_{n\to\infty} \frac{\frac{n^2}{n^2}+\frac{6}{n^2}}{\frac{3n^2}{n^2}-\frac{1}{n^2}}$$

$$\lim_{n\to\infty} \frac{1+\frac{6}{n^2}}{3-\frac{1}{n^2}}$$

Now we can evaluate the limit.

$$\frac{1+\frac{6}{\infty}}{3-\frac{1}{\infty}}$$

$$\frac{1+0}{3-0}$$

$$\frac{1}{3}$$

Since we got a real answer for our limit, we know our sequence is also bounded below. In this case our sequence is bounded below at $a_n > 1/3$. Remember, we calculated this answer by taking the limit of our sequence as it approaches infinity, so our sequence will be greater then the bounded limit but not equal to it. Therefore, the sequence

$$a_n = \frac{n^2+6}{3n^2-1}$$

is bounded above at $a_n \leq 7/2$ and bounded below at $a_n > 1/3$.