Topic: Centroids of plane regions

Question: Find the centroid of the plane region bounded by $y = 2x + 1$, $y = -x + 7$, and $x = 8$.

Answer choices:

A $(6,6)$

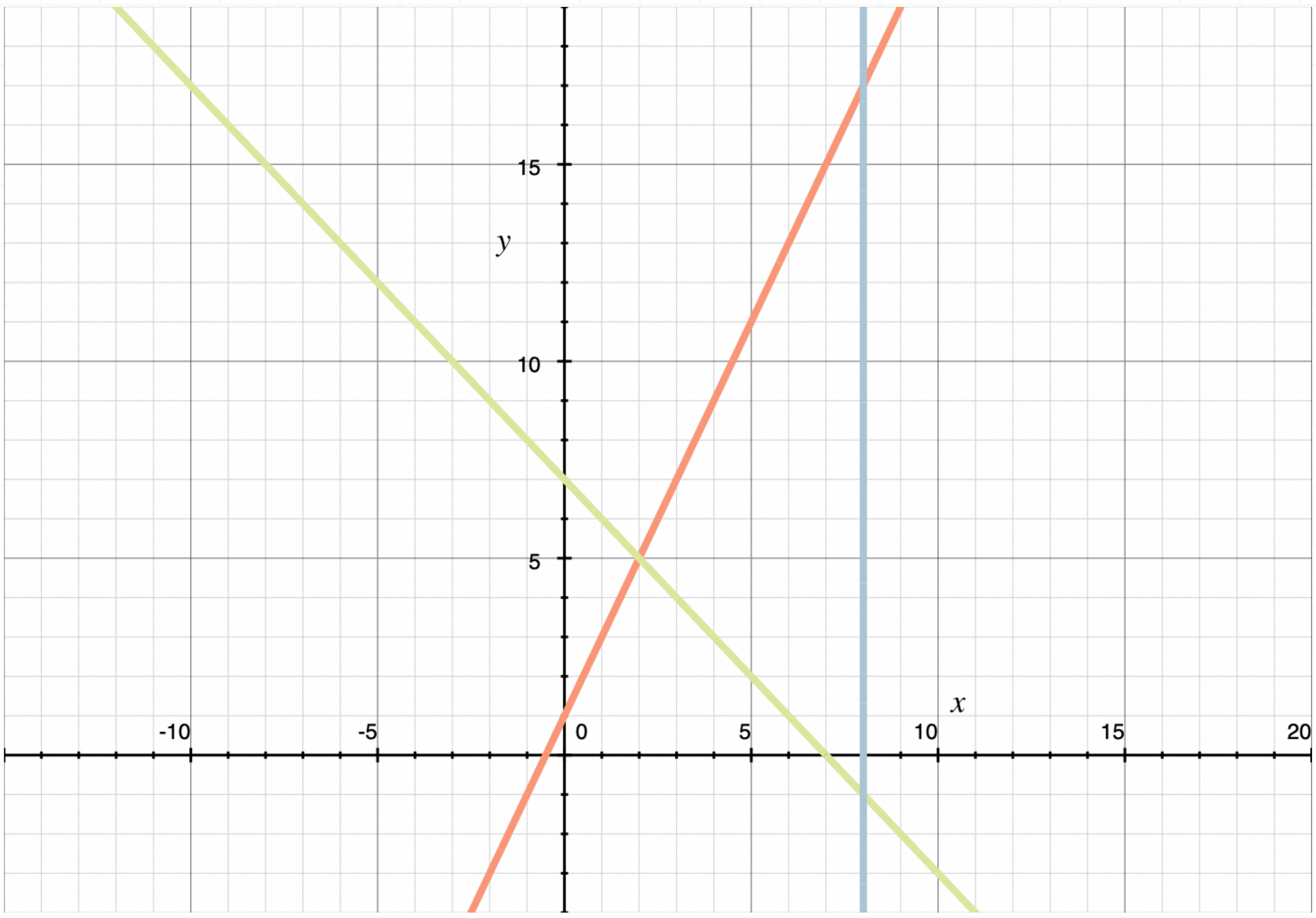
B $(6,7)$

C $(7,6)$

D $(7,7)$

Solution: B

A sketch of all three lines is



The limits of integration will be $x = [2,8]$, and the region is bounded above by $f(x) = 2x + 1$ and below by $g(x) = -x + 7$. Then the area of the region is

$$A = \int_a^b f(x) - g(x)\ dx$$

$$A = \int_2^8 2x + 1 - (-x + 7)\ dx$$

$$A = \int_2^8 3x - 6\ dx$$

$$A = \frac{3}{2}x^2 - 6x\Big|_2^8$$

$$A=\frac{3}{2}(8)^2-6(8)-\left[\frac{3}{2}(2)^2-6(2)\right]$$

$$A=96-48-(6-12)$$

$$A=54$$

So the coordinates of the centroid are

$$\bar{x}=\frac{1}{A}\int_a^b x(f(x)-g(x))\ dx$$

$$\bar{x}=\frac{1}{54}\int_2^8 x(3x-6)\ dx$$

$$\bar{x}=\frac{1}{54}\int_2^8 3x^2-6x\ dx$$

$$\bar{x}=\frac{1}{54}(x^3-3x^2)\Big|_2^8$$

$$\bar{x}=\frac{1}{54}\left[((8)^3-3(8)^2)-((2)^3-3(2)^2)\right]$$

$$\bar{x}=\frac{1}{54}\left[(512-192)-(8-12)\right]$$

$$\bar{x}=\frac{1}{54}(324)$$

$$\bar{x}=6$$

and

$$\bar{y}=\frac{1}{A}\int_a^b \frac{1}{2}[(f(x))^2-(g(x))^2]\ dx$$

$$\bar{y}=\frac{1}{54}\int_2^8 \frac{1}{2}\left[(2x+1)^2-(-x+7)^2\right]\ dx$$

$$\bar{y}=\frac{1}{108}\int_2^8 4x^2+4x+1-(x^2-14x+49)\ dx$$

$$\bar{y}=\frac{1}{108}\int_2^8 4x^2+4x+1-x^2+14x-49\ dx$$

$$\bar{y}=\frac{1}{108}\int_2^8 3x^2+18x-48\ dx$$

$$\bar{y}=\frac{1}{108}(x^3+9x^2-48x)\Big|_2^8$$

$$\bar{y}=\frac{1}{108}\left[(8^3+9(8)^2-48(8))-(2^3+9(2)^2-48(2))\right]$$

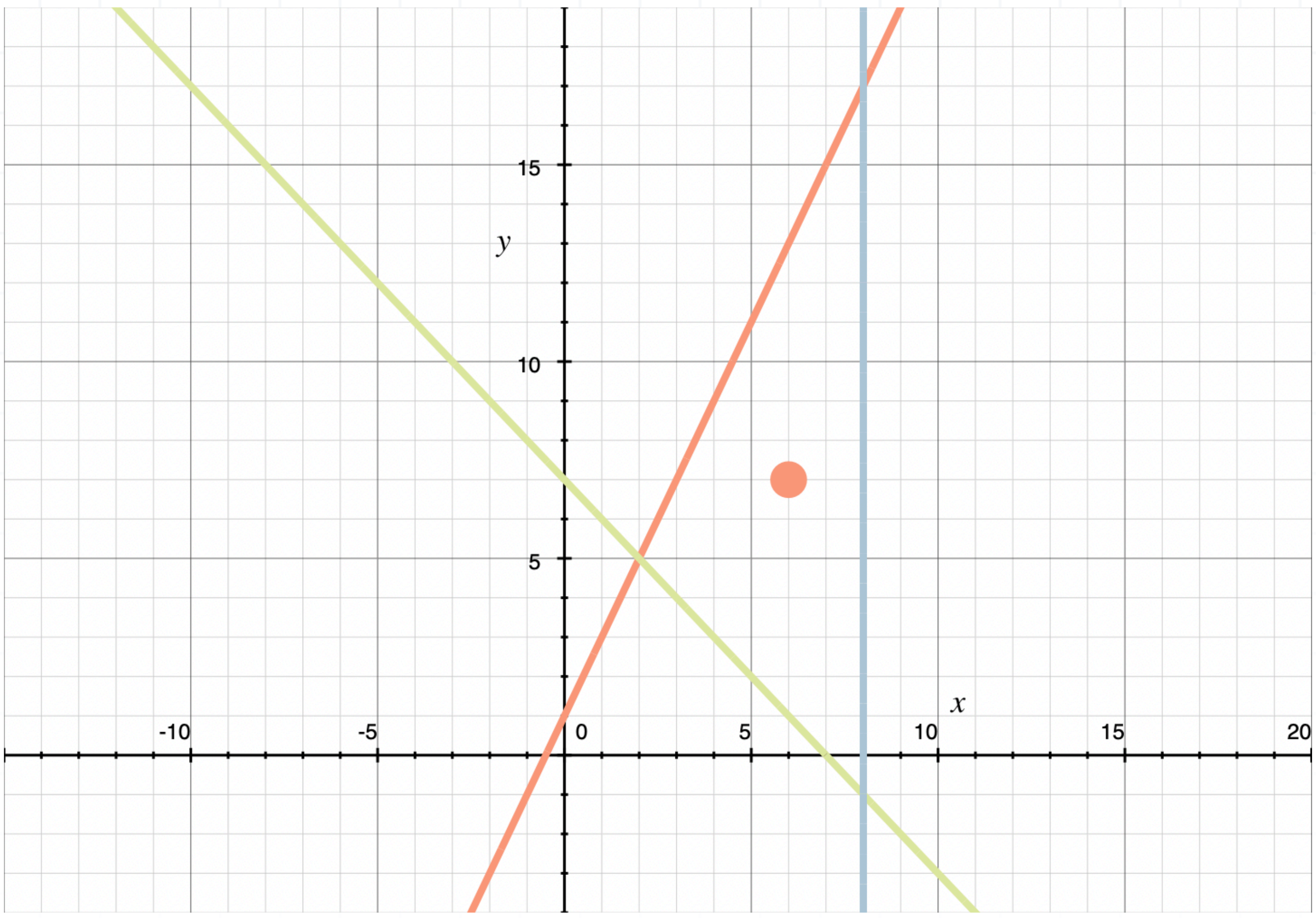
$$\bar{y}=\frac{1}{108}\left[(512+576-384)-(8+36-96)\right]$$

$$\bar{y}=\frac{1}{108}(512+576-384-8-36+96)$$

$$\bar{y}=\frac{1}{108}(756)$$

$$\bar{y}=7$$

Therefore, the centroid of the region is at $(6,7)$. We can confirm this visually by graphing it in the region.



Topic: Centroids of plane regions

Question: Find the centroid of the plane region bounded by $y = 4 - x^2$ and $y = 0$.

Answer choices:

A $\left(0, \frac{8}{5}\right)$

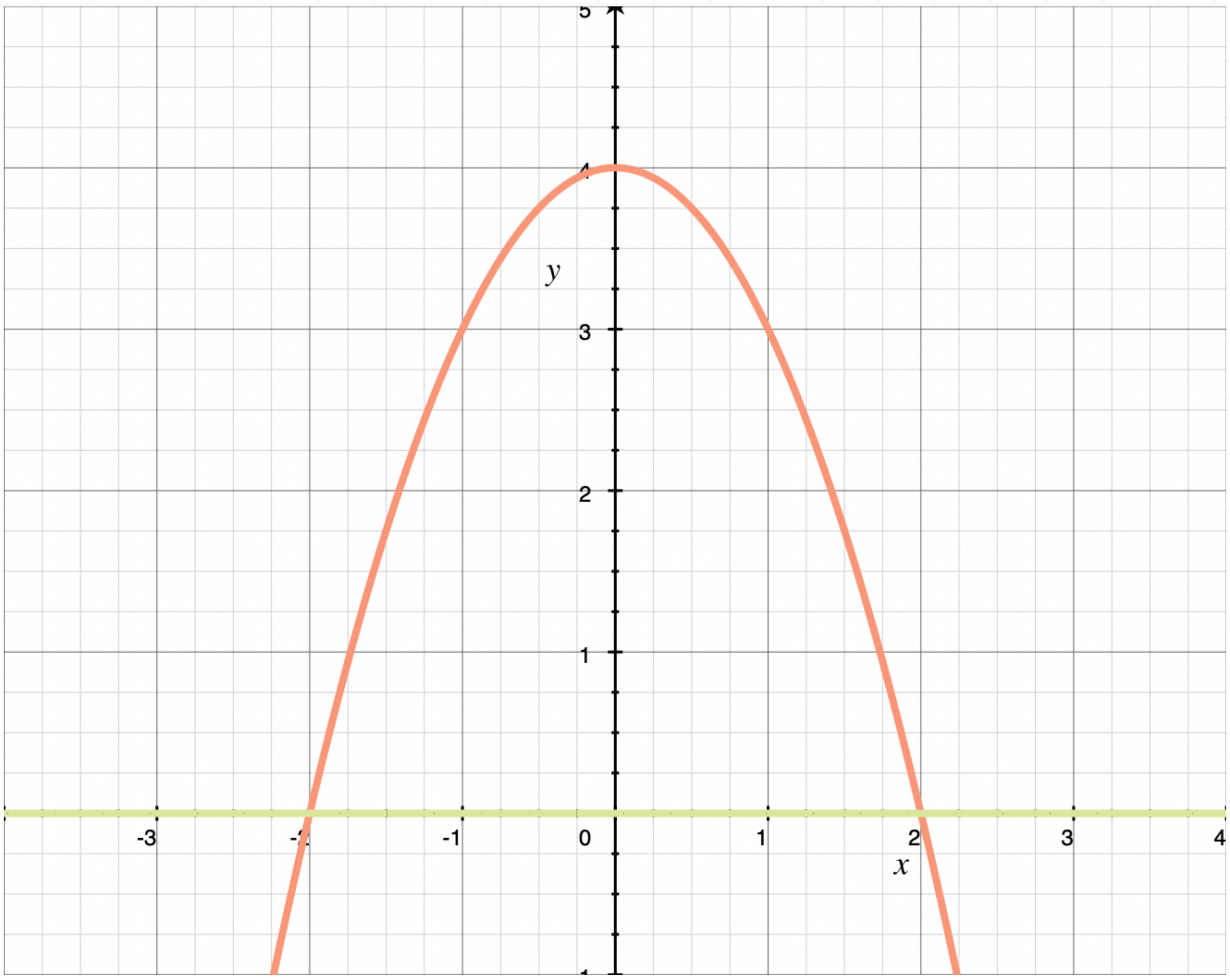
B $(0,0)$

C $\left(\frac{8}{5}, 0\right)$

D $\left(0, \frac{5}{8}\right)$

Solution: A

A sketch of both curves is



Because the region is symmetric about the y-axis, $\bar{x} = 0$. To find $\bar{y}$, we'll first find the area of the region.

$$A = \int_a^b f(x) - g(x)\ dx$$

$$A = \int_{-2}^{2} 4 - x^2 - 0\ dx$$

$$A = 4x - \frac{1}{3}x^3 \Big|_{-2}^{2}$$

$$A=\left[4(2)-\frac{1}{3}(2)^3\right]-\left[4(-2)-\frac{1}{3}(-2)^3\right]$$

$$A=\left(8-\frac{8}{3}\right)-\left(-8+\frac{8}{3}\right)$$

$$A=8-\frac{8}{3}+8-\frac{8}{3}$$

$$A=16-\frac{16}{3}$$

$$A=\frac{48}{3}-\frac{16}{3}$$

$$A=\frac{32}{3}$$

Then $\bar{y}$ is

$$\bar{y}=\frac{1}{A}\int_a^b \frac{1}{2}[(f(x))^2-(g(x))^2]\ dx$$

$$\bar{y}=\frac{1}{\frac{32}{3}}\int_{-2}^{2}\frac{1}{2}(4-x^2)^2-0^2\ dx$$

$$\bar{y}=\frac{3}{32}\cdot\frac{1}{2}\int_{-2}^{2}16-8x^2+x^4\ dx$$

$$\bar{y}=\frac{3}{64}\left(16x-\frac{8}{3}x^3+\frac{1}{5}x^5\right)\Bigg|_{-2}^{2}$$

$$\bar{y} = \frac{3}{64}\left[\left(16(2) - \frac{8}{3}(2)^3 + \frac{1}{5}(2)^5\right) - \left(16(-2) - \frac{8}{3}(-2)^3 + \frac{1}{5}(-2)^5\right)\right]$$

$$\bar{y} = \frac{3}{64}\left[\left(32 - \frac{64}{3} + \frac{32}{5}\right) - \left(-32 + \frac{64}{3} - \frac{32}{5}\right)\right]$$

$$\bar{y} = \frac{3}{64}\left(32 - \frac{64}{3} + \frac{32}{5} + 32 - \frac{64}{3} + \frac{32}{5}\right)$$

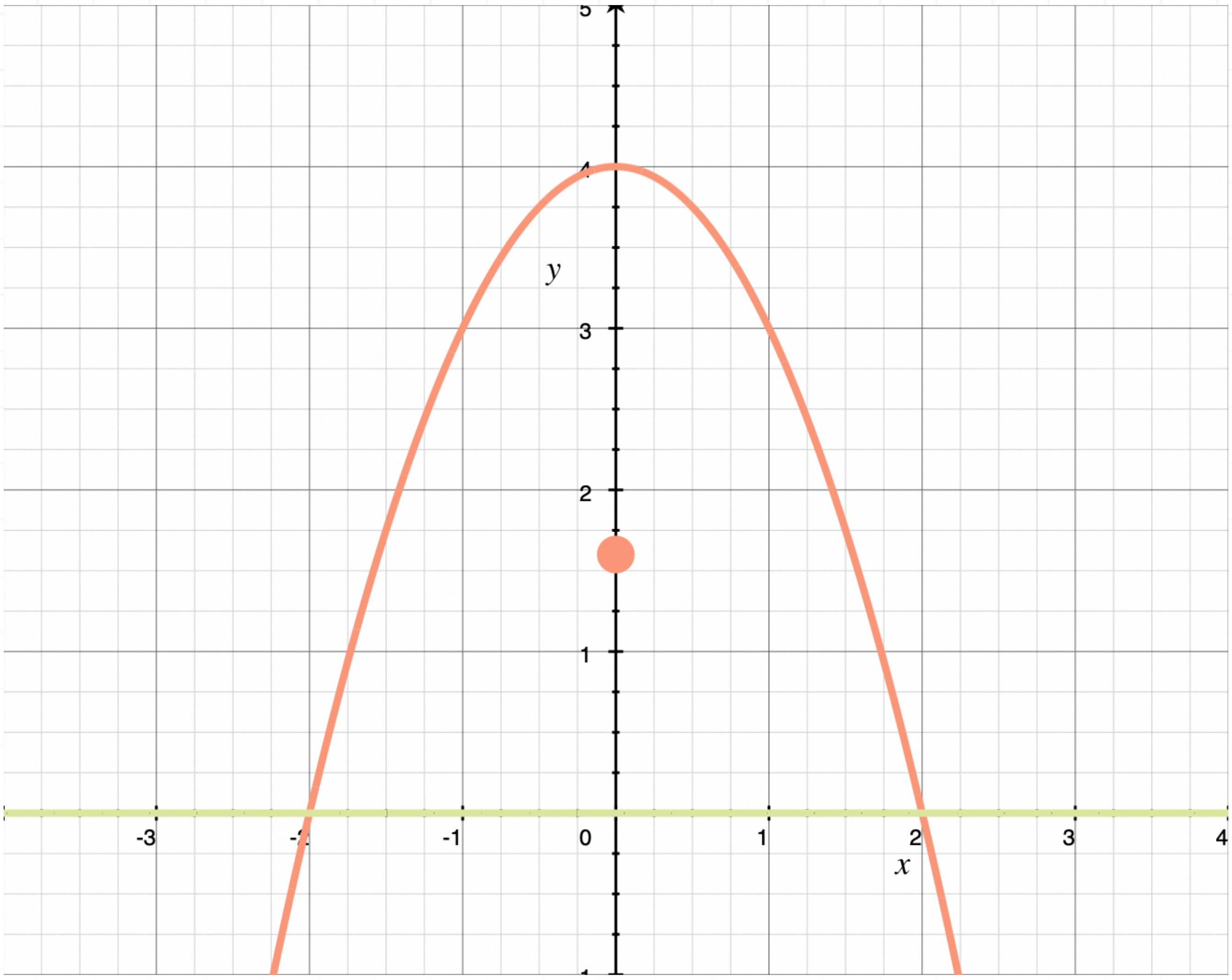
$$\bar{y} = \frac{3}{64}\left(64 - \frac{128}{3} + \frac{64}{5}\right)$$

$$\bar{y} = 3\left(1 - \frac{2}{3} + \frac{1}{5}\right)$$

$$\bar{y} = 3\left(\frac{15}{15} - \frac{10}{15} + \frac{3}{15}\right)$$

$$\bar{y} = \frac{8}{5}$$

Therefore, the centroid of the region is at $(0,8/5)$. We can confirm this visually by graphing it in the region.



y
x
5
4
3
2
1
-3
-2
-1
0
1
2
3
4

Topic: Centroids of plane regions

Question: Find the centroid of the plane region bounded by $y=2x+4$, $y=0$, $x=3$, and $x=9$.

Answer choices:

A $\left(\frac{51}{8},\frac{67}{8}\right)$

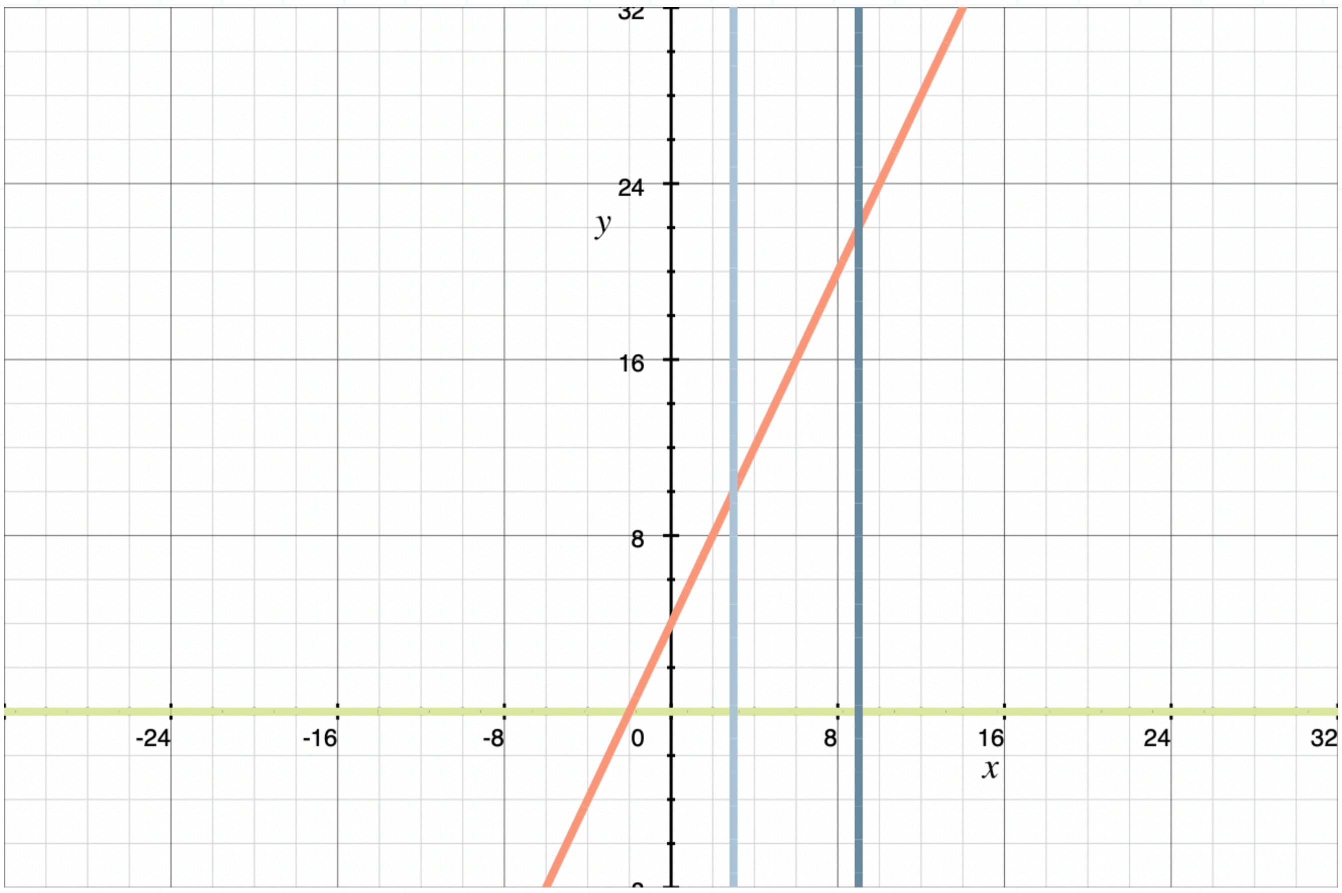
B $\left(\frac{51}{8},\frac{67}{4}\right)$

C $\left(\frac{67}{8},\frac{51}{8}\right)$

D $\left(\frac{51}{4},\frac{67}{4}\right)$

Solution: A

A sketch of all the curves is



The limits of integration will be $x = [3,9]$, and the region is bounded above by $f(x) = 2x + 4$ and below by $g(x) = 0$. Then the area of the region is

$$A = \int_a^b f(x) - g(x)\ dx$$

$$A = \int_3^9 2x + 4 - 0\ dx$$

$$A = x^2 + 4x \Big|_3^9$$

$$A = 9^2 + 4(9) - (3^2 + 4(3))$$

$$A = 81 + 36 - 9 - 12$$

$$A = 96$$

So the coordinates of the centroid are

$$\bar{x} = \frac{1}{A}\int_a^b x(f(x) - g(x))\ dx$$

$$\bar{x} = \frac{1}{96}\int_3^9 x(2x + 4 - 0)\ dx$$

$$\bar{x} = \frac{1}{96}\int_3^9 2x^2 + 4x\ dx$$

$$\bar{x} = \frac{1}{96}\left(\frac{2}{3}x^3 + 2x^2\right)\Bigg|_3^9$$

$$\bar{x} = \frac{1}{96}\left(\frac{2}{3}(9)^3 + 2(9)^2\right) - \frac{1}{96}\left(\frac{2}{3}(3)^3 + 2(3)^2\right)$$

$$\bar{x} = \frac{1}{96}(486 + 162) - \frac{1}{96}(18 + 18)$$

$$\bar{x} = \frac{51}{8}$$

and

$$\bar{y} = \frac{1}{A}\int_a^b \frac{1}{2}[(f(x))^2 - (g(x))^2]\ dx$$

$$\bar{y} = \frac{1}{96}\int_3^9 \frac{1}{2}(2x + 4)^2\ dx$$

$$\bar{y}=\frac{1}{192}\int_{3}^{9}4x^2+16x+16\ dx$$

$$\bar{y}=\frac{1}{192}\left(\frac{4}{3}x^3+8x^2+16x\right)\Big|_{3}^{9}$$

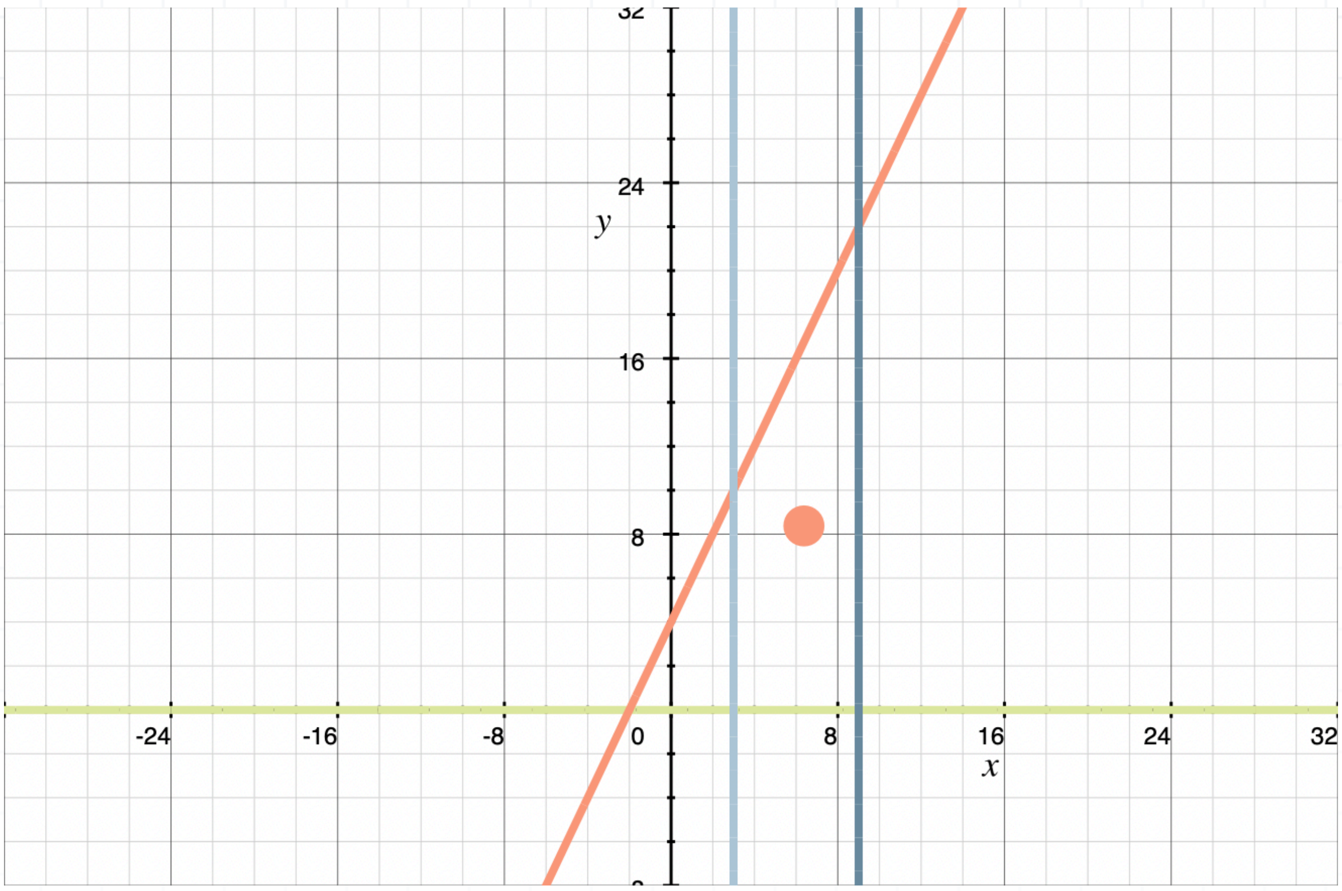
$$\bar{y}=\frac{1}{192}\left(\frac{4}{3}(9)^3+8(9)^2+16(9)\right)-\frac{1}{192}\left(\frac{4}{3}(3)^3+8(3)^2+16(3)\right)$$

$$\bar{y}=\frac{1}{192}(972+648+144)-\frac{1}{192}(36+72+48)$$

$$\bar{y}=\frac{1{,}764}{192}-\frac{156}{192}$$

$$\bar{y}=\frac{67}{8}$$

Therefore, the centroid of the region is at $(51/8,67/8)$. We can confirm this visually by graphing it in the region.


32
24
y
16
8
-24
-16
-8
0
8
16
24
32
x