Topic: Theorem of Pappus

Question: Use the Theorem of Pappus to find the volume of a right circular cone with radius 4 feet and height 8 feet.

Answer choices:

A $\quad \frac{128}{3}\pi$ square feet

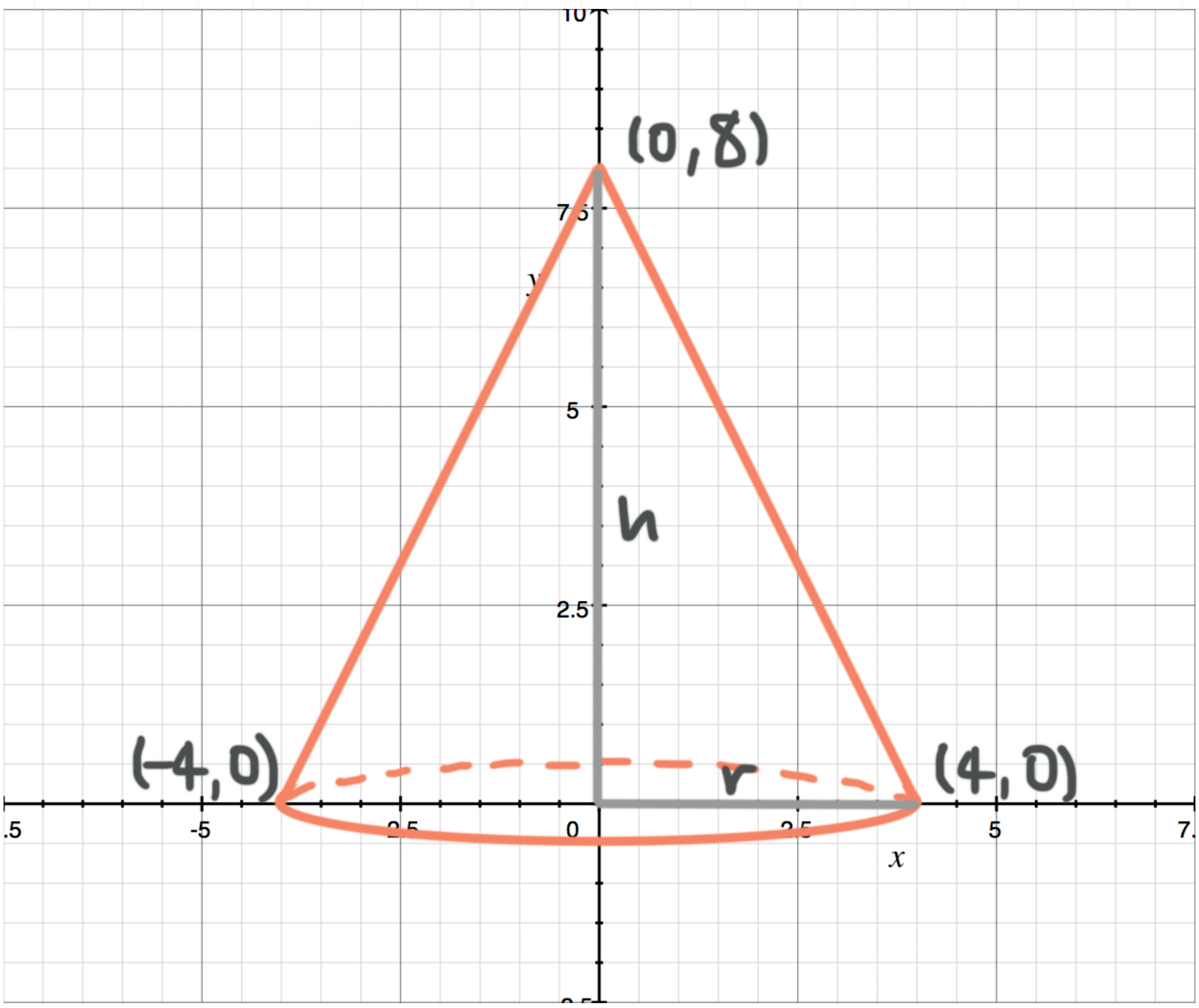
B $\quad \frac{128}{3}$ cubic feet

C $\quad \frac{128}{3}\pi$ cubic feet

D $\quad 128\pi$ cubic feet

Solution: C

It might be helpful to visualize the right circular cone that we have in this problem, drawn with the center of the base at the origin in the coordinate plane.



The cross section that the Theorem of Pappus uses is the area of a triangle drawn from the vertex of the cone to the center of the base, and then to the edge of the cone. The area of this cross section, as in any triangle is

$$A = \frac{1}{2}bh$$

where b is the radius of the base of the cone, and h is the height of the cone. The Theorem of Pappus calculates the volume of the cone with the formula

$$V = Ad$$

where V is volume, A is area of the cross section, and d is the distance traveled by the x-value of the centroid of the cross section during an integration.

Find the area of the triangular cross section. For this cone, $b = r = 4$ and $h = 8$. So

$$A = \frac{1}{2}(4)(8) = 16$$

Next, we need to find the function in the first quadrant that contains the lateral edge of the cone. We will use it to find the x-value of the centroid of the cross section of the cone. We will begin by identifying two points, from the graph above, and write an equation of the line that contains those two points. The two points are $(0,8)$ and $(4,0)$. We will use these two points to calculate the slope of the line first.

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{0 - 8}{4 - 0} = -\frac{8}{4} = -2$$

Use the point $(4,0)$ and the slope $m = -2$ we just found to write the equation of the lateral edge.

$$y - y_1 = m(x - x_1)$$

$$y - 0 = -2(x - 4)$$

$$y = -2x + 8$$

Now we'll find the x-value of the centroid of the cross section, which is $\bar{x}$.

$$\bar{x} = \frac{1}{A}\int_a^b xf(x)\ dx$$

We already found that $A = 16$, $a = 0$, $b = 4$, and $y = -2x + 8$. Substitute these values/expressions into the integral formula.

$$\bar{x} = \frac{1}{16}\int_0^4 -2x^2 + 8x\ dx$$

$$\bar{x} = \frac{1}{16}\left(-\frac{2}{3}x^3 + 4x^2\right)\Big|_0^4$$

$$\bar{x} = \frac{1}{16}\left(-\frac{2}{3}(4)^3 + 4(4)^2\right) - \frac{1}{16}\left(-\frac{2}{3}(0)^3 + 4(0)^2\right)$$

$$\bar{x} = \frac{1}{16}\left(-\frac{128}{3} + 64\right)$$

$$\bar{x} = \frac{4}{3}$$

Now we'll find the distance traveled by the x-value of the centroid. This is given by the formula

$$d = 2\pi\bar{x}$$

$$d = 2\pi\left(\frac{4}{3}\right)$$

$$d=\frac{8}{3}\pi$$

Now we can find volume using

$$V=Ad$$

$$V=16\left(\frac{8}{3}\pi\right)$$

$$V=\frac{128}{3}\pi$$

Topic: Theorem of Pappus

Question: Use the Theorem of Pappus to find the volume of a right circular cone with radius 5 inches and height 15 inches.

Answer choices:

A 125π square inches

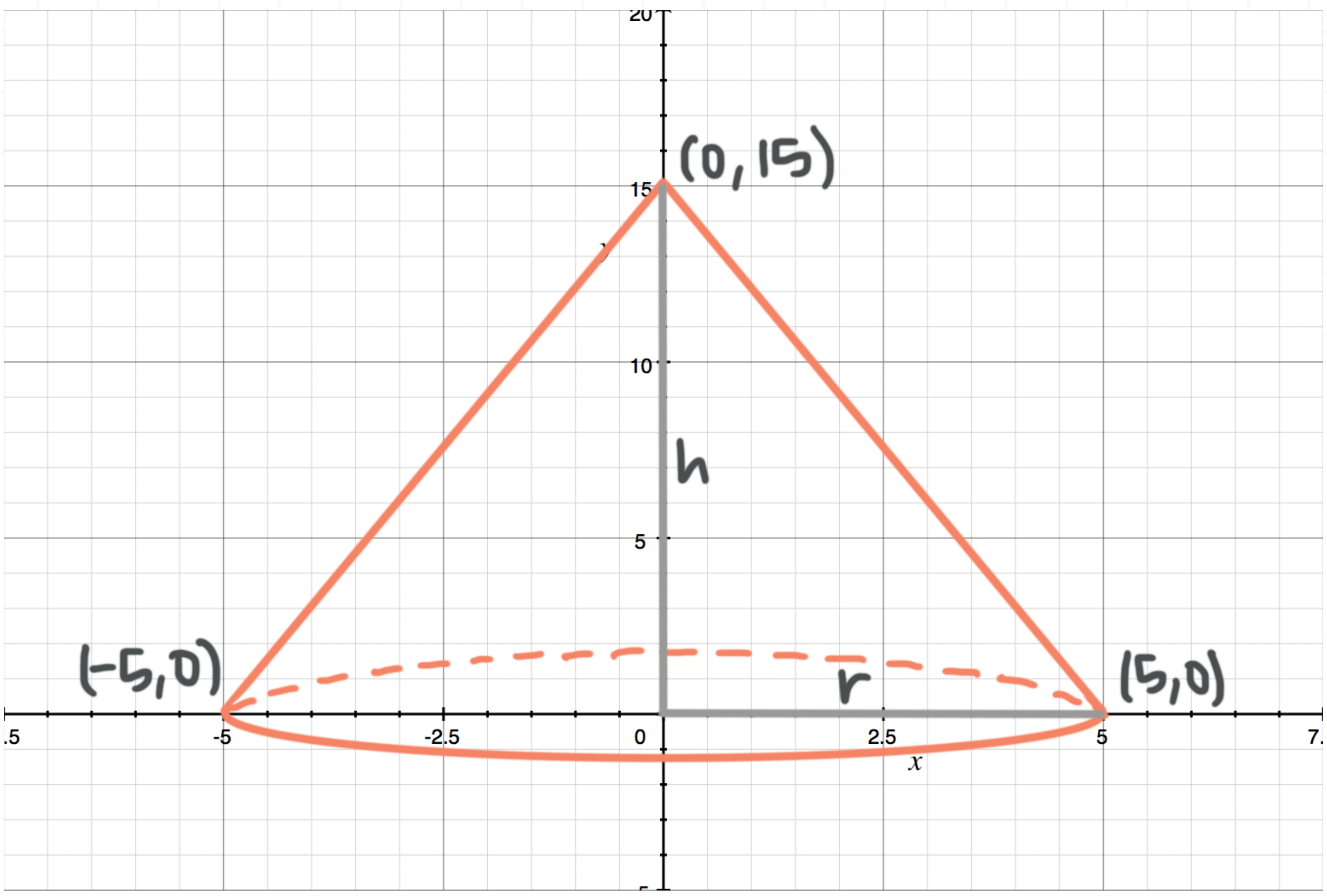
B 125π cubic inches

C $\frac{125}{3}\pi$ cubic inches

D $\frac{125}{3}\pi$ square inches

Solution: B

It might be helpful to visualize the right circular cone that we have in this problem, drawn with the center of the base at the origin in the coordinate plane.



The cross section that the Theorem of Pappus uses is the area of a triangle drawn from the vertex of the cone to the center of the base, and then to the edge of the cone. The area of this cross section, as in any triangle is

$$A = \frac{1}{2}bh$$

where b is the radius of the base of the cone, and h is the height of the cone. The Theorem of Pappus calculates the volume of the cone with the formula

$$V = Ad$$

where V is volume, A is area of the cross section, and d is the distance traveled by the x-value of the centroid of the cross section during an integration.

Find the area of the triangular cross section. For this cone, $b = r = 5$ and $h = 15$. So

$$A = \frac{1}{2}(5)(15) = \frac{75}{2}$$

Next, we need to find the function in the first quadrant that contains the lateral edge of the cone. We will use it to find the x-value of the centroid of the cross section of the cone. We will begin by identifying two points, from the graph above, and write an equation of the line that contains those two points. The two points are $(0,15)$ and $(5,0)$. We will use these two points to calculate the slope of the line first.

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{0 - 15}{5 - 0} = -\frac{15}{5} = -3$$

Use the point $(5,0)$ and the slope $m = -3$ we just found to write the equation of the lateral edge.

$$y - y_1 = m(x - x_1)$$

$$y - 0 = -3(x - 5)$$

$$y = -3x + 15$$

Now we'll find the x-value of the centroid of the cross section, which is $\bar{x}$.

$$\bar{x} = \frac{1}{A} \int_a^b xf(x)\ dx$$

We already found that $A = 75/2$, $a = 0$, $b = 5$, and $y = -3x + 15$. Substitute these values/expressions into the integral formula.

$$\bar{x} = \frac{2}{75} \int_0^5 x(-3x + 15)\ dx$$

$$\bar{x} = \frac{2}{75} \int_0^5 -3x^2 + 15x\ dx$$

$$\bar{x} = \frac{2}{75} \left(-x^3 + \frac{15}{2}x^2 \right) \Bigg|_0^5$$

$$\bar{x} = \frac{2}{75} \left(-(5)^3 + \frac{15}{2}(5)^2 \right) - \frac{2}{75} \left(-(0)^3 + \frac{15}{2}(0)^2 \right)$$

$$\bar{x} = -\frac{250}{75} + \frac{750}{150}$$

$$\bar{x} = \frac{5}{3}$$

Now we'll find the distance traveled by the x-value of the centroid. This is given by the formula

$$d = 2\pi\bar{x}$$

$$d = 2\pi \left(\frac{5}{3} \right)$$

$$d = \frac{10}{3}\pi$$

Now we can find volume using

$$V = Ad$$

$$V = \frac{75}{2}\left(\frac{10}{3}\pi\right)$$

$$V = 125\pi$$

Topic: Theorem of Pappus

Question: Use the Theorem of Pappus to find the volume of a right circular cone with radius 3 meters and height 12 meters.

Answer choices:

A 36π cubic meters

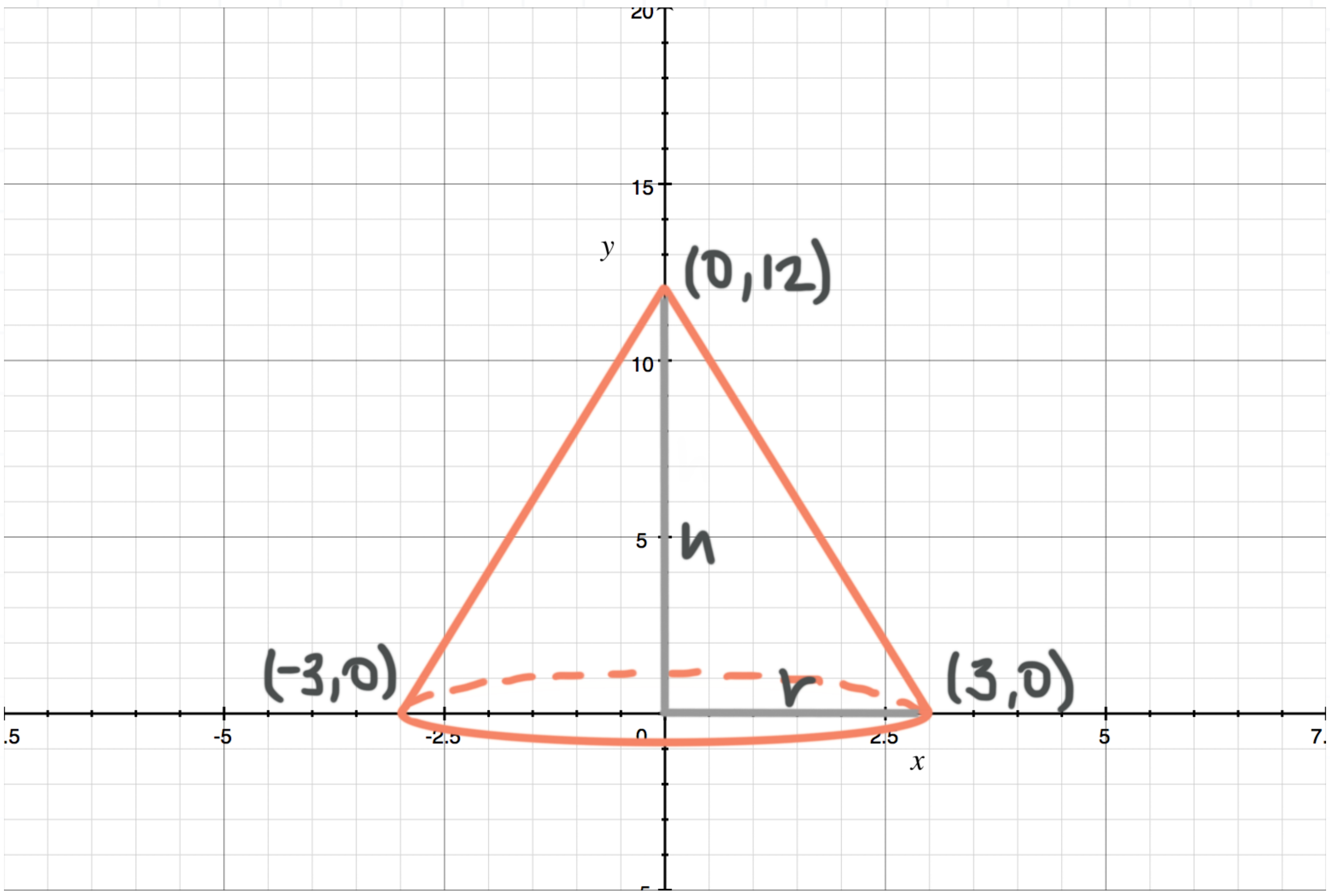
B 36π square meters

C 18π cubic meters

D 18π square meters

Solution: A

It might be helpful to visualize the right circular cone that we have in this problem, drawn with the center of the base at the origin in the coordinate plane.



The cross section that the Theorem of Pappus uses is the area of a triangle drawn from the vertex of the cone to the center of the base, and then to the edge of the cone. The area of this cross section, as in any triangle is

$$A = \frac{1}{2}bh$$

where b is the radius of the base of the cone, and h is the height of the cone. The Theorem of Pappus calculates the volume of the cone with the formula

$$V = Ad$$

where V is volume, A is area of the cross section, and d is the distance traveled by the x-value of the centroid of the cross section during an integration.

Find the area of the triangular cross section. For this cone, $b = r = 3$ and $h = 12$. So

$$A = \frac{1}{2}(3)(12) = 18$$

Next, we need to find the function in the first quadrant that contains the lateral edge of the cone. We will use it to find the x-value of the centroid of the cross section of the cone. We will begin by identifying two points, from the graph above, and write an equation of the line that contains those two points. The two points are $(0,12)$ and $(3,0)$. We will use these two points to calculate the slope of the line first.

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{0 - 12}{3 - 0} = -\frac{12}{3} = -4$$

Use the point $(3,0)$ and the slope $m = -4$ we just found to write the equation of the lateral edge.

$$y - y_1 = m(x - x_1)$$

$$y - 0 = -4(x - 3)$$

$$y = -4x + 12$$

Now we'll find the x-value of the centroid of the cross section, which is $\bar{x}$.

$$\bar{x} = \frac{1}{A} \int_a^b x f(x) \ dx$$

We already found that $A = 18$, $a = 0$, $b = 3$, and $y = -4x + 12$. Substitute these values/expressions into the integral formula.

$$\bar{x} = \frac{1}{18} \int_0^3 x(-4x + 12) \ dx$$

$$\bar{x} = \frac{1}{18} \left(-\frac{4}{3}x^3 + 6x^2 \right) \Big|_0^3$$

$$\bar{x} = \frac{1}{18} \left(-\frac{4}{3}(3)^3 + 6(3)^2 \right) - \frac{1}{18} \left(-\frac{4}{3}(0)^3 + 6(0)^2 \right)$$

$$\bar{x} = \frac{1}{18}(-36 + 54)$$

$$\bar{x} = \frac{1}{18}(18)$$

$$\bar{x} = 1$$

Now we'll find the distance traveled by the x-value of the centroid. This is given by the formula

$$d = 2\pi\bar{x}$$

$$d = 2\pi(1)$$

$$d = 2\pi$$

Now we can find volume using

$$V = Ad$$

$$V = 18(2\pi)$$

$$V = 36\pi$$