Calculus 2 Workbook Solutions

Parametric curves

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MATH

ELIMINATING THE PARAMETER

■ 1. Eliminate the parameter.

$$x = t^2 - 2$$

$$y = 8 - 3t$$

$$t \geq 0$$

Solution:

Solve $x = t^2 - 2$ for t and substitute the value of t into $y = 8 - 3t$.

$$x = t^2 - 2$$

$$x + 2 = t^2$$

$$t = \sqrt{x + 2}$$

Then for $t \geq 0$,

$$y = 8 - 3t$$

$$y = 8 - 3\sqrt{x + 2}$$

DERIVATIVES OF PARAMETRIC CURVES

■ 1. Find the derivative of the parametric curve.

$$x = 3 + \sqrt{t}$$

$$y = t^2 - 5t$$

Solution:

Find the derivatives of x and y with respect to t.

$$\frac{dy}{dt} = \frac{d}{dt}(t^2 - 5t) = 2t - 5$$

$$\frac{dx}{dt} = \frac{d}{dt}\left(3 + \sqrt{t}\right) = \frac{1}{2\sqrt{t}}$$

So the derivative of the parametric curve is

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{2t - 5}{\frac{1}{2\sqrt{t}}} = 2\sqrt{t}(2t - 5)$$

■ 2. Find the derivative of the parametric curve.

$$x = 4\cos t$$

$$y = t - 5\sin t$$

Solution:

Find the derivatives of x and y with respect to t.

$$\frac{dy}{dt} = \frac{d}{dt}(t - 5\sin t) = 1 - 5\cos t$$

$$\frac{dx}{dt} = \frac{d}{dt}(4\cos t) = -4\sin t$$

So the derivative of the parametric curve is

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{1 - 5\cos t}{-4\sin t} = \frac{5\cos t - 1}{4\sin t}$$

You could leave the answer this way, or rewrite it.

$$\frac{dy}{dx} = \frac{5\cos t}{4\sin t} - \frac{1}{4\sin t}$$

$$\frac{dy}{dx} = \frac{5}{4}\cot t - \frac{1}{4}\csc t$$

$$\frac{dy}{dx} = \frac{1}{4}(5\cot t - \csc t)$$

■ 3. Find the derivative of the parametric curve.

$$x = 7\cos t$$

$$y = 3t^2 - t$$

Solution:

Find the derivatives of x and y with respect to t.

$$\frac{dy}{dt} = \frac{d}{dt}(3t^2 - t) = 6t - 1$$

$$\frac{dx}{dt} = \frac{d}{dt}(7\cos t) = -7\sin t$$

So the derivative of the parametric curve is

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{6t - 1}{-7\sin t} = \frac{1 - 6t}{7\sin t}$$

■ 4. Find the derivative of the parametric curve.

$$x = e^t - 3t$$

$$y = e^{-t} + 2t$$

Solution:

Find the derivatives of x and y with respect to t.

$$\frac{dy}{dt} = \frac{d}{dt}(e^{-t} + 2t) = -e^{-t} + 2$$

$$\frac{dx}{dt} = \frac{d}{dt}(e^t - 3t) = e^t - 3$$

So the derivative of the parametric curve is

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{-e^{-t} + 2}{e^t - 3} = \frac{2 - e^{-t}}{e^t - 3}$$

5. Find the derivative of the parametric curve.

$$x = 7t - 4$$

$$y = 5t^2 + 9t$$

Solution:

Find the derivatives of x and y with respect to t.

$$\frac{dy}{dt} = \frac{d}{dt}(5t^2 + 9t) = 10t + 9$$

$$\frac{dx}{dt} = \frac{d}{dt}(7t - 4) = 7$$

So the derivative of the parametric curve is

$$\frac{dy}{dx}=\frac{\frac{dy}{dt}}{\frac{dx}{dt}}=\frac{10t+9}{7}$$

SECOND DERIVATIVES OF PARAMETRIC CURVES

■ 1. Find the second derivative of the parametric curve.

$$x = 1 - \cos^2 t$$

$$y = \sin t$$

Solution:

Find the derivatives of x and y with respect to t.

$$\frac{dy}{dt} = \frac{d}{dt}(\sin t) = \cos t$$

$$\frac{dx}{dt} = \frac{d}{dt}(1 - \cos^2 t) = -2\cos t \cdot (-\sin t) = 2\cos t \sin t$$

So the derivative of the parametric curve is

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\cos t}{2\cos t \cdot \sin t} = \frac{1}{2\sin t} = \frac{1}{2}\csc t$$

Take the derivative of dy/dx.

$$\frac{d}{dt}\left(\frac{dy}{dx}\right) = \frac{d}{dt}\left(\frac{1}{2}\csc t\right) = -\frac{1}{2}\csc t \cot t$$

Then the second derivative of the parametric curve is

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}} = \frac{-\frac{1}{2}\csc t \cot t}{2\cos t \sin t}$$

$$\frac{d^2y}{dx^2} = -\frac{\frac{1}{\sin t}\cdot\frac{\cos t}{\sin t}}{4\cos t \sin t}$$

$$\frac{d^2y}{dx^2} = -\frac{\cos t}{\sin^2 t}\cdot\frac{1}{4\cos t \sin t}$$

$$\frac{d^2y}{dx^2} = -\frac{1}{4\sin^3 t}$$

$$\frac{d^2y}{dx^2} = -\frac{1}{4}\csc^3 t$$

■ 2. Find the second derivative of the parametric curve.

$$x = e^{-3t}$$

$$y = e^{2t^2}$$

Solution:

Find the derivatives of x and y with respect to t.

$$\frac{dy}{dt} = \frac{d}{dt}(e^{2t^2}) = 4te^{2t^2}$$

$$\frac{dx}{dt} = \frac{d}{dt}(e^{-3t}) = -3e^{-3t}$$

So the derivative of the parametric curve is

$$\frac{dy}{dx}=\frac{\frac{dy}{dt}}{\frac{dx}{dt}}=\frac{4te^{2t^2}}{-3e^{-3t}}=-\frac{4te^{2t^2}}{3e^{-3t}}$$

Take the derivative of dy/dx.

$$\frac{d}{dt}\left(\frac{dy}{dx}\right)=\frac{d}{dt}\left(-\frac{4te^{2t^2}}{3e^{-3t}}\right)$$

$$\frac{d}{dt}\left(\frac{dy}{dx}\right)=-\frac{3e^{-3t}\left(16t^2e^{2t^2}+4e^{2t^2}\right)-4te^{2t^2}\cdot-9e^{-3t}}{9e^{-6t}}$$

$$\frac{d}{dt}\left(\frac{dy}{dx}\right)=-\frac{3e^{-3t}\left(16t^2e^{2t^2}+4e^{2t^2}\right)+36te^{2t^2}e^{-3t}}{9e^{-6t}}$$

$$\frac{d}{dt}\left(\frac{dy}{dx}\right)=-\frac{16t^2e^{2t^2}+4e^{2t^2}+12te^{2t^2}}{3e^{-3t}}$$

Then the second derivative of the parametric curve is

$$\frac{d^2y}{dx^2}=\frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}}=\frac{-\frac{16t^2e^{2t^2}+4e^{2t^2}+12te^{2t^2}}{3e^{-3t}}}{-3e^{-3t}}$$

$$\frac{d^2y}{dx^2}=-\frac{16t^2e^{2t^2}+4e^{2t^2}+12te^{2t^2}}{-9e^{-6t}}$$

$$\frac{d^2y}{dx^2}=-\frac{4e^{2t^2}\left(4t^2+3t+1\right)}{-9e^{-6t}}$$

3. Find the second derivative of the parametric curve.

$$x = t^2 + 2t + 1$$

$$y = 3t + 4$$

Solution:

Find the derivatives of x and y with respect to t.

$$\frac{dy}{dt} = \frac{d}{dt}(3t + 4) = 3$$

$$\frac{dx}{dt} = \frac{d}{dt}(t^2 + 2t + 1) = 2t + 2$$

So the derivative of the parametric curve is

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{3}{2t + 2}$$

Take the derivative of dy/dx.

$$\frac{d}{dt}\left(\frac{dy}{dx}\right) = \frac{d}{dt}\left(\frac{3}{2t + 2}\right) = \frac{(2t + 2)(0) - 3(2)}{(2t + 2)^2} = -\frac{6}{(2t + 2)^2}$$

Then the second derivative of the parametric curve is

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}} = \frac{-\frac{6}{(2t+2)^2}}{2t+2} = -\frac{6}{(2t+2)^3}$$

SKETCHING PARAMETRIC CURVES BY PLOTTING POINTS

■ 1. The graph of the parametric equation on the interval $0 \leq t \leq 2$ is a segment. What is the Cartesian equation in x and y? Find the left and right endpoints of the segment.

$$x = 2t + 3$$

$$y = 4t + 5$$

Solution:

To find the equation in cartesian coordinates, eliminate the parameter. First, solve one of the equations for t.

$$x = 2t + 3$$

$$x - 3 = 2t$$

$$t = \frac{x - 3}{2}$$

Then,

$$y = 4t + 5$$

$$y = 4\left(\frac{x - 3}{2}\right) + 5$$

$$y = 2(x - 3) + 5$$

$$y = 2x - 6 + 5$$

$$y = 2x - 1$$

The equation in x and y is $y = 2x - 1$.

To find the endpoints, substitute the endpoints of the domain of t into the parametric equation. Plugging in $t = 0$ gives

$$x = 2(0) + 3 = 3$$

$$y = 4(0) + 5 = 5$$

Then the left endpoint is $(x, y) = (3,5)$. Plugging in $t = 2$ gives

$$x = 2(2) + 3 = 7$$

$$y = 4(2) + 5 = 13$$

Then the right endpoint is $(x, y) = (7,13)$.

2. What are the points on the curve for the parameter values $t = 1,\ 2,\ 3,$ and 4?

$$x = t^2 + t$$

$$y = t^2 - t$$

Solution:

To find the points, substitute the values of t into the parametric equation.

For $t = 1$:

$$x(1) = 1^2 + 1 = 2$$

$$y(1) = 1^2 - 1 = 0$$

$$(x, y) = (2,0)$$

For $t = 2$:

$$x(2) = 2^2 + 2 = 6$$

$$y(2) = 2^2 - 2 = 2$$

$$(x, y) = (6,2)$$

For $t = 3$:

$$x(3) = 3^2 + 3 = 12$$

$$y(3) = 3^2 - 3 = 6$$

$$(x, y) = (12,6)$$

For $t = 4$:

$$x(4) = 4^2 + 4 = 20$$

$$y(2) = 4^2 - 4 = 12$$

$$(x, y) = (20,12)$$

3. What are the points on the curve for the parameter values $t = 0, 1, 2,$ and 3?

$$x = 3t^2 - 5$$

$$y = 2t^3 + 1$$

Solution:

To find the points, substitute the values of t into the parametric equation.

For $t = 0$:

$$x(0) = 3(0)^2 - 5 = -5$$

$$y(0) = 2(0)^3 + 1 = 1$$

$$(x, y) = (-5,1)$$

For $t = 1$:

$$x(1) = 3(1)^2 - 5 = -2$$

$$y(1) = 2(1)^3 + 1 = 3$$

$$(x, y) = (-2,3)$$

For $t = 2$:

$$x(2) = 3(2)^2 - 5 = 7$$

$$y(2) = 2(2)^3 + 1 = 17$$

$$(x, y) = (7,17)$$

For $t = 3$:

$$x(3) = 3(3)^2 - 5 = 22$$

$$y(3) = 2(3)^3 + 1 = 55$$

$$(x, y) = (22,55)$$

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