Part 2 of the FTC

Part 2 of the Fundamental Theorem of Calculus states that

$$\int_a^b f(x)\ dx = F(x)\Big|_a^b = F(b) - F(a)$$

if $f(x)$ is a continuous function on $[a, b]$ and $F(x)$ is the anti-derivative of $f(x)$.

Part 2 of the FTC tells us that we can figure out the exact value of an indefinite integral (area under the curve) when we know the interval over which to evaluate (in this case the interval $[a, b]$).

There are rules to keep in mind. For instance, the function $f(x)$ must be continuous over the interval $[a, b]$ (no holes, breaks, or jumps), and the interval must be closed, which means that both limits of integration must be constants (real numbers only, no infinity allowed).

Example

Use Part 2 of the Fundamental Theorem of Calculus to find the value of the integral.

$$F(x) = \int_1^3 x^3\ dx$$

First, we perform the integration.

$$F(x) = \frac{x^4}{4}\Big|_1^3$$

Next, we plug in the upper and lower limits, subtracting the value at the lower bound from the value at the upper bound.

$$F = \frac{3^4}{4} - \frac{1^4}{4}$$

$$F = \frac{81}{4} - \frac{1}{4}$$

$$F = \frac{80}{4}$$

Let's double check that this satisfies Part 2 of the FTC. If we break the equation into parts,

$$F(b) = \int x^3\ dx \text{ where } b = 3 \text{ and } F(a) = \int x^3\ dx \text{ where } a = 1$$

and evaluate the two equations separately, we can double check our answer. First we integrate as an indefinite integral.

$$F(x) = \int x^3\ dx$$

$$F(x) = \frac{x^4}{4} + C$$

Next we plug in $b = 3$ and $a = 1$.

$$F(3) = \frac{3^4}{4} + C$$

$$F(1) = \frac{1^4}{4} + C$$

Finally, we find $F(b) - F(a)$.

$$F(3) - F(1) = \frac{3^4}{4} + C - \left(\frac{1^4}{4} + C\right)$$

$$F(3) - F(1) = \frac{3^4}{4} + C - \frac{1^4}{4} - C$$

$$F(3) - F(1) = \frac{80}{4}$$

As you can see, we've verified that value of F that we found earlier. This answer is what we expected and it confirms Part 2 of the FTC.

Example

Integrate using Part 2 of the FTC.

$$F(t) = \int_2^x t^3 + 2t^4 \ dt$$

When we integrate we get

$$F(t) = \left(\frac{t^4}{4} + \frac{2t^5}{5} + C\right)\Bigg|_2^x$$

Evaluating over the interval, we get

$$F(x)=\frac{x^4}{4}+\frac{2x^5}{5}+C-\left(\frac{2^4}{4}+\frac{2(2)^5}{5}+C\right)$$

$$F(x)=\frac{x^4}{4}+\frac{2x^5}{5}+C-\left(4+\frac{64}{5}+C\right)$$

$$F(x)=\frac{x^4}{4}+\frac{2x^5}{5}+C-\left(\frac{20}{5}+\frac{64}{5}+C\right)$$

$$F(x)=\frac{x^4}{4}+\frac{2x^5}{5}+C-\frac{84}{5}-C$$

$$F(x)=\frac{x^4}{4}+\frac{2x^5}{5}-\frac{84}{5}$$

For this last example, we could use Part 1 of the FTC to confirm.

$$\frac{dF(x)}{dx}=\frac{d\left(\frac{x^4}{4}+\frac{2x^5}{5}-\frac{84}{5}\right)}{dx}$$

$$\frac{dF(x)}{dx}=\left(\frac{4x^3}{4}+\frac{10x^4}{5}-0\right)$$

$$\frac{dF(x)}{dx}=x^3+2x^4$$

We know that

$$r(t)=t^3+2t^4$$

So by substituting x for t we get

$$f(x) = x^3 + 2x^4$$

We can see that $\frac{dF(x)}{dx} = f(x)$.

$$\frac{dF(x)}{dx} = x^3 + 2x^4 = f(x)$$