Geometric series test

Before we can learn how to determine the convergence or divergence of a geometric series, we have to define a geometric series.

The general form of a geometric series is ar^{n-1} when the index of n begins at $n = 1$. Therefore, the sum of a convergent geometric series is given by

$$\sum_{n=1}^{\infty} ar^{n-1}$$

Sometimes you'll come across a geometric series with an index shift, where n starts at $n = 0$ instead of $n = 1$. In that case, the standard form of the geometric series is ar^n, and if it's convergent, its sum is given by

$$\sum_{n=0}^{\infty} ar^n$$

Both of these are valid geometric series. The important thing is that the exponent on r matches the index. So, if the index starts at $n = 1$, we want to make sure we have r^{n-1}. If the index begins at $n = 0$, we want to have r^n.

If we look at the expanded forms of both of these series by calculating the first few terms ($n = 1$, $n = 2$, $n = 3$ and $n = 4$, ...), we'll see that they're identical.

$$\sum_{n=1}^{\infty} ar^{n-1} = \qquad\qquad \sum_{n=0}^{\infty} ar^n =$$

$$\{ar^{1-1} + ar^{2-1} + ar^{3-1} + ar^{4-1} + \dots\}$$

$$\{ar^0 + ar^1 + ar^2 + ar^3 + \dots\} \qquad \{ar^0 + ar^1 + ar^2 + ar^3 + \dots\}$$

$$a\{r^0 + r^1 + r^2 + r^3 + \dots\} \qquad a\{r^0 + r^1 + r^2 + r^3 + \dots\}$$

$$a\{1 + r + r^2 + r^3 + \dots\} \qquad a\{1 + r + r^2 + r^3 + \dots\}$$

Which means that, regardless of the kind of geometric series we start with, ar^{n-1} with $n = 1$ or ar^n with $n = 0$, we can find the values of a and r in the same way: by expanding the series through its first few terms and then factoring out the a. Then a will be the coefficient we factored out of the series, and r will be the second term in the series, the term immediately following the 1.

$$\sum_{n=1}^{\infty} ar^{n-1} = a\{1 + r + r^2 + r^3 + \dots\}$$

$$\sum_{n=0}^{\infty} ar^{n} = a\{1 + r + r^2 + r^3 + \dots\}$$

Sometimes we won't even need to expand the series. If we can just make the form of the series match one of the standard forms of a geometric series given above, then we'll be able to prove that the series is geometric and identify a and r.

It's important to be able to find the values of a and r because we'll use r to say whether or not the geometric series is convergent or divergent. If we find that it's convergent, then we'll use a and r to find the sum of the series.

Convergence of a geometric series

We can use the value of r in the geometric series test for convergence to determine whether or not the geometric series converges.

The geometric series test says that

if $|r| < 1$ then the series converges

if $|r| \geq 1$ then the series diverges

Let's do an example where we use the geometric series test.

Example

Show that the series is a geometric series, then use the geometric series test to say whether the series converges or diverges.

$$\sum_{n=0}^{\infty} \frac{2^{n-1}}{3^n}$$

Since the index starts at $n = 0$, we need to get the series into the form ar^n, which we can do using simple exponent rules.

$$\sum_{n=0}^{\infty} \frac{2^{n-1}}{3^n}$$

$$\sum_{n=0}^{\infty} \frac{2^n 2^{-1}}{3^n}$$

$$\sum_{n=0}^{\infty} 2^{-1}\left(\frac{2^n}{3^n}\right)$$

$$\sum_{n=0}^{\infty} \frac{1}{2}\left(\frac{2}{3}\right)^n$$

Now that we have the series in the right form, we can say

$$\sum_{n=0}^{\infty} ar^n = \sum_{n=0}^{\infty} \frac{1}{2}\left(\frac{2}{3}\right)^n \text{ where}$$

$$a = \frac{1}{2}$$

$$r = \frac{2}{3}$$

The fact that we've been able to put the series in this form and identify values of a and r proves that it's a geometric series. Now we just need to say whether or not the series converges.

Remember that the geometric series test for converges tells us that

if $|r| < 1$ then the series converges

if $|r| \geq 1$ then the series diverges

Since

$$\left|\frac{2}{3}\right| = \frac{2}{3} < 1$$

we can say that $|r| < 1$ and therefore that the series converges.