Calculus 2 Workbook Solutions

Comparison tests

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MATH

COMPARISON TEST

1. Use the comparison test to say whether or not the series converges.

$$\sum_{n=0}^{\infty} \frac{4}{3^n + n}$$

Solution:

Identify the comparison series.

$$a_n = \frac{4}{3^n + n} \qquad\qquad b_n = \frac{4}{3^n}$$

For all $n \geq 0$,

$$\frac{4}{3^n + n} \leq \frac{4}{3^n}$$

Rewrite the comparison series.

$$\sum_{n=0}^{\infty} \frac{4}{3^n}$$

$$\sum_{n=1}^{\infty} 4\left(\frac{1}{3}\right)^{n-1}$$

The comparison series is a geometric series with $a = 4$ and $r = 1/3$. The sum of the geometric comparison series is

$$\sum_{n=0}^{\infty} \frac{4}{3^n} = \frac{a}{1-r} = \frac{4}{1-\frac{1}{3}} = \frac{4}{\frac{2}{3}} = \frac{4}{1} \cdot \frac{3}{2} = 6$$

Because we know the original series is always less than or equal to the comparison series, we can also say

$$\sum_{n=0}^{\infty} \frac{4}{3^n + n} \le \sum_{n=0}^{\infty} \frac{4}{3^n}$$

$$\sum_{n=0}^{\infty} \frac{4}{3^n + n} \le 6$$

Therefore, the series converges.

■ 2. Use the comparison test to say whether or not the series converges.

$$\sum_{n=1}^{\infty} \frac{n}{n^4 + 7}$$

Solution:

Identify the comparison series.

$$a_n = \frac{n}{n^4 + 7} \qquad b_n = \frac{n}{n^4} = \frac{1}{n^3}$$

The original series a_n will be less than the comparison series b_n,

$$\frac{n}{n^4+7} \le \frac{1}{n^3}$$

$$n \le \frac{n^4+7}{n^3}$$

$$n \le \frac{n^4}{n^3} + \frac{7}{n^3}$$

$$n \le n + \frac{7}{n^3}$$

$$0 \le \frac{7}{n^3}$$

for any $n > 0$. The comparison series is a p-series with $p = 3 > 1$, which means the comparison series converges, and because we know the original series is always less than or equal to the comparison series, we can say that the original series also converges.

■ 3. Use the comparison test to say whether or not the series converges.

$$\sum_{n=2}^{\infty} \frac{5}{\ln n}$$

Solution:

Identify the comparison series.

$$a_n = \frac{5}{\ln n} \qquad\qquad b_n = \frac{5}{n}$$

For all $n \geq 2$,

$$\frac{5}{\ln n} \geq \frac{5}{n}$$

The comparison series is

$$\sum_{n=2}^{\infty} \frac{5}{n}$$

The comparison series is a p-series with $p = 1 \leq 1$, which means the comparison series diverges.

Therefore, because b_n diverges, a_n also diverges.

LIMIT COMPARISON TEST

1. Use the limit comparison test to say whether or not the series converges.

$$\sum_{n=1}^{\infty} \frac{3n+2}{(2n-1)^4}$$

Solution:

Let the comparison series be

$$\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n^3}$$

Use the limit comparison test.

$$\lim_{b\to\infty} \frac{a_n}{b_n} = \lim_{b\to\infty} \frac{3n+2}{(2n-1)^4} \cdot \frac{n^3}{1} = \lim_{b\to\infty} \frac{3n^4+2n^3}{(2n-1)^4} = \lim_{b\to\infty} \frac{\frac{3n^4}{n^4}+\frac{2n^3}{n^4}}{\frac{(2n-1)^4}{n^4}}$$

$$= \lim_{b\to\infty} \frac{3+\frac{2}{n}}{\left(2-\frac{1}{n}\right)^4} = \frac{3+\frac{2}{\infty}}{\left(2-\frac{1}{\infty}\right)^4} = \frac{3+0}{(2-0)^4} = \frac{3}{16}$$

So the value of L is $L = 3/16 > 0$. We know also that the comparison series converges by the p-series test since for that series $p = 3 > 1$. Therefore, because b_n converges, a_n also converges.

2. Use the limit comparison test to say whether or not the series converges.

$$\sum_{n=1}^{\infty} \frac{12n^2+5}{n^3-7}$$

Solution:

Let the comparison series be

$$\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{n^2}{n^3} = \sum_{n=1}^{\infty} \frac{1}{n}$$

Use the limit comparison test.

$$\lim_{b\to\infty} \frac{a_n}{b_n} = \lim_{b\to\infty} \frac{12n^2+5}{n^3-7} \cdot \frac{n}{1} = \lim_{b\to\infty} \frac{12n^3+5n}{n^3-7} = \lim_{b\to\infty} \frac{\frac{12n^3}{n^3}+\frac{5n}{n^3}}{\frac{n^3}{n^3}-\frac{7}{n^3}}$$

$$= \lim_{b\to\infty} \frac{12+\frac{5}{n^2}}{1-\frac{7}{n^3}} = \frac{12+\frac{5}{\infty}}{1-\frac{7}{\infty}} = \frac{12+0}{1-0} = 12$$

So the value of L is $L=12>0$. We know also that the comparison series diverges by the p-series test since for that series $p=1\le 1$. Therefore, because b_n diverges, a_n also diverges.

3. Use the limit comparison test to say whether or not the series converges.

$$\sum_{n=1}^{\infty} \frac{n^4 + 3n^2}{7n^6 + 3n^4}$$

Solution:

Let the comparison series be

$$\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{n^4}{n^6} = \sum_{n=1}^{\infty} \frac{1}{n^2}$$

Use the limit comparison test.

$$\lim_{b \to \infty} \frac{a_n}{b_n} = \lim_{b \to \infty} \frac{n^4 + 3n^2}{7n^6 + 3n^4} \cdot \frac{n^2}{1} = \lim_{b \to \infty} \frac{n^6 + 3n^4}{7n^6 + 3n^4} = \lim_{b \to \infty} \frac{\frac{n^6}{n^6} + \frac{3n^4}{n^6}}{\frac{7n^6}{n^6} + \frac{3n^4}{n^6}}$$

$$= \lim_{b \to \infty} \frac{1 + \frac{3}{n^2}}{7 + \frac{3}{n^2}} = \frac{1 + \frac{3}{\infty}}{7 + \frac{3}{\infty}} = \frac{1 + 0}{7 + 0} = \frac{1}{7}$$

So the value of L is $L = 1/7 > 0$. We know also that the comparison series converges by the p-series test since for that series $p = 2 > 1$. Therefore, because b_n converges, a_n also converges.

ERROR OR REMAINDER OF A SERIES

1. Estimate the remainder of the series using the first three terms.

$$\sum_{n=1}^{\infty} \frac{3}{7n^3 + 2n^2 + 3}$$

Solution:

To find the remainder, estimate the total sum by calculating a partial sum for the series, determine whether the series converges or diverges using the comparison test, and use the integral test to solve for the remainder.

$$n = 1 \qquad a_1 = \frac{3}{7(1)^3 + 2(1)^2 + 3} = \frac{3}{12} = \frac{1}{4}$$

$$s_1 = a_1 = \frac{1}{4} = 0.25$$

$$n = 2 \qquad a_2 = \frac{3}{7(2)^3 + 2(2)^2 + 3} = \frac{3}{67}$$

$$s_2 = a_1 + a_2 = \frac{1}{4} + \frac{3}{67} = \frac{79}{268} \approx 0.295$$

$$n = 3 \qquad a_3 = \frac{3}{7(3)^3 + 2(3)^2 + 3} = \frac{3}{210} = \frac{1}{70}$$

$$s_3 = a_1 + a_2 + a_3 = \frac{1}{4} + \frac{3}{67} + \frac{1}{70} = \frac{2{,}899}{9{,}380} \approx 0.309$$

Use the comparison test to determine convergence or divergence, using the comparison series

$$\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{3}{n^3}$$

Apply the comparison test.

$$\lim_{b\to\infty} \frac{a_n}{b_n} = \lim_{b\to\infty} \frac{3}{7n^3+2n^2+3} \cdot \frac{n^3}{3} = \lim_{b\to\infty} \frac{3n^3}{21n^3+6n^2+9} = \lim_{b\to\infty} \frac{\frac{3n^3}{n^3}}{\frac{21n^3}{n^3}+\frac{6n^2}{n^3}+\frac{9}{n^3}}$$

$$= \lim_{b\to\infty} \frac{3}{21+\frac{6}{n}+\frac{9}{n^3}} = \frac{3}{21+\frac{6}{\infty}+\frac{9}{\infty}} = \frac{3}{21+0+0} = \frac{1}{7}$$

This value for L is $L = 1/7 > 0$. Now check convergence or divergence of the comparison series.

$$\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{3}{n^3} = 3\sum_{n=1}^{\infty} \frac{1}{n^3}$$

This is a p-series with $p = 3 > 1$, which means the comparison series converges. Which means the original series a_n also converges.

Use the integral test to find the remainder of a_n after the first three terms.

$$R_3 \le T_3 \le \int_3^{\infty} b_n \ dx = \int_3^{\infty} f(x) \ dx = \int_3^{\infty} \frac{3}{x^3} \ dx = \int_3^{\infty} 3x^{-3} \ dx$$

$$R_3 \le \int_3^{\infty} 3x^{-3} \ dx$$

Integrate, then evaluate over the interval.

$$R_3 \le \lim_{b\to\infty} \frac{3x^{-2}}{-2}\Bigg|_3^b$$

$$R_3 \le \lim_{b\to\infty} -\frac{3}{2x^2}\Bigg|_3^b$$

$$R_3 \le \lim_{b\to\infty} -\frac{3}{2b^2} - \left(-\frac{3}{2(3)^2}\right)$$

$$R_3 \le \lim_{b\to\infty} -\frac{3}{2b^2} + \frac{1}{6}$$

$$R_3 \le -0 + \frac{1}{6}$$

$$R_3 \le \frac{1}{6}$$

$$R_3 \le 0.167$$

The third partial sum of the series a_n is $s_3 \approx 0.309$, with error $R_3 \le 0.167$.

■ 2. Estimate the remainder of the series using the first three terms.

$$\sum_{n=1}^{\infty} \frac{5}{\sqrt{n^4+6}}$$

Solution:

To find the remainder, estimate the total sum by calculating a partial sum for the series, determine whether the series converges or diverges using the comparison test, and use the integral test to solve for the remainder.

$$n=1 \qquad a_1=\frac{5}{\sqrt{1^4+6}}=\frac{5}{\sqrt{7}}$$

$$s_1=a_1=\frac{5}{\sqrt{7}}\approx 1.890$$

$$n=2 \qquad a_2=\frac{5}{\sqrt{2^4+6}}=\frac{5}{\sqrt{22}}\approx 1.066$$

$$s_2=a_1+a_2=1.890+1.066\approx 2.956$$

$$n=3 \qquad a_3=\frac{5}{\sqrt{3^4+6}}=\frac{5}{\sqrt{87}}\approx 0.836$$

$$s_3=a_1+a_2+a_3=1.890+1.066+0.536\approx 3.492$$

Use the comparison test to determine convergence or divergence, using the comparison series

$$\sum_{n=1}^{\infty} b_n=\sum_{n=1}^{\infty}\frac{5}{\sqrt{n^4}}=\sum_{n=1}^{\infty}\frac{5}{n^2}$$

Apply the comparison test.

$$\lim_{b\to\infty}\frac{a_n}{b_n}=\lim_{b\to\infty}\frac{5}{\sqrt{n^4+6}}\cdot\frac{n^2}{5}=\lim_{b\to\infty}\frac{5n^2}{5\sqrt{n^4+6}}=\lim_{b\to\infty}\frac{\frac{5n^2}{n^2}}{5\frac{\sqrt{n^4+6}}{n^2}}$$

$$= \lim_{b\to\infty} \frac{\frac{5n^2}{n^2}}{5\sqrt{\frac{n^4}{n^4}+\frac{6}{n^4}}} = \lim_{b\to\infty} \frac{5}{5\sqrt{1+\frac{6}{n^4}}} = \frac{5}{5\sqrt{1+\frac{6}{\infty}}}$$

$$= \frac{5}{5\sqrt{1+0}} = \frac{5}{5} = 1$$

This value for L is $L = 1 > 0$. Now check convergence or divergence of the comparison series.

$$\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{5}{n^2} = 5\sum_{n=1}^{\infty} \frac{1}{n^2}$$

This is a p-series with $p = 2 > 1$, which means the comparison series converges. Which means the original series a_n also converges.

Use the integral test to find the remainder of a_n after the first three terms.

$$R_3 \le T_3 \le \int_3^{\infty} b_n\ dx = \int_3^{\infty} f(x)\ dx = \int_3^{\infty} \frac{5}{x^2}\ dx = \int_3^{\infty} 5x^{-2}\ dx$$

$$R_3 \le \int_3^{\infty} 5x^{-2}\ dx$$

Integrate, then evaluate over the interval.

$$R_3 \le \lim_{b\to\infty} \frac{5x^{-1}}{-1}\Bigg|_3^b$$

$$R_3 \le \lim_{b\to\infty} -\frac{5}{x}\Bigg|_3^b$$

$$R_3 \le \lim_{b\to\infty} -\frac{5}{b} - \left(-\frac{5}{3}\right)$$

$$R_3 \le \lim_{b\to\infty} -\frac{5}{b} + \frac{5}{3}$$

$$R_3 \le -0 + \frac{5}{3}$$

$$R_3 \le \frac{5}{3}$$

$$R_3 \le 0.167$$

The third partial sum of the series a_n is $s_3 \approx 3.492$, with error $R_3 \le 0.167$.

■ 3. Estimate the remainder of the series using the first three terms.

$$\sum_{n=1}^{\infty} \frac{4n^2}{n^5 + n^2 + 3}$$

Solution:

To find the remainder, estimate the total sum by calculating a partial sum for the series, determine whether the series converges or diverges using the comparison test, and use the integral test to solve for the remainder.

$$n = 1 \qquad a_1 = \frac{4(1)^2}{1^5 + 1^2 + 3} = \frac{4}{5}$$

$$s_1 = a_1 = \frac{4}{5} = 0.8$$

$n = 2$

$$a_2 = \frac{4(2)^2}{2^5 + 2^2 + 3} = \frac{16}{39} \approx 0.410$$

$$s_2 = a_1 + a_2 = 0.8 + 0.410 \approx 1.210$$

$n = 3$

$$a_3 = \frac{4(3)^2}{3^5 + 3^2 + 3} = \frac{36}{255} = \frac{12}{85} \approx 0.141$$

$$s_3 = a_1 + a_2 + a_3 = 0.8 + 0.410 + 0.141 \approx 1.351$$

Use the comparison test to determine convergence or divergence, using the comparison series

$$\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{4n^2}{n^5} = \sum_{n=1}^{\infty} \frac{4}{n^3}$$

Apply the comparison test.

$$\lim_{b\to\infty} \frac{a_n}{b_n} = \lim_{b\to\infty} \frac{4n^2}{n^5 + n^2 + 3} \cdot \frac{n^3}{4} = \lim_{b\to\infty} \frac{n^5}{n^5 + n^2 + 3} = \lim_{b\to\infty} \frac{\frac{n^5}{n^5}}{\frac{n^5}{n^5} + \frac{n^2}{n^5} + \frac{3}{n^5}}$$

$$= \lim_{b\to\infty} \frac{1}{1 + \frac{1}{n^3} + \frac{3}{n^5}} = \frac{1}{1 + \frac{1}{\infty} + \frac{3}{\infty}} = \frac{1}{1 + 0 + 0} = 1$$

This value for L is $L = 1 > 0$. Now check convergence or divergence of the comparison series.

$$\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{4}{n^3} = 4\sum_{n=1}^{\infty} \frac{1}{n^3}$$

This is a p-series with $p = 3 > 1$, which means the comparison series converges. Which means the original series a_n also converges.

Use the integral test to find the remainder of a_n after the first three terms.

$$R_3 \leq T_3 \leq \int_3^\infty b_n \ dx = \int_3^\infty f(x) \ dx = \int_3^\infty \frac{4}{x^3} \ dx = \int_3^\infty 4x^{-3} \ dx$$

$$R_3 \leq \int_3^\infty 4x^{-3} \ dx$$

Integrate, then evaluate over the interval.

$$R_3 \leq \lim_{b \to \infty} \frac{4x^{-2}}{-2} \Bigg|_3^b$$

$$R_3 \leq \lim_{b \to \infty} -\frac{2}{x^2} \Bigg|_3^b$$

$$R_3 \leq \lim_{b \to \infty} -\frac{2}{b^2} - \left(-\frac{2}{3^2}\right)$$

$$R_3 \leq \lim_{b \to \infty} -\frac{2}{b^2} + \frac{2}{9}$$

$$R_3 \leq -0 + \frac{2}{9}$$

$$R_3 \leq \frac{2}{9}$$

$$R_3 \leq 0.222$$

The third partial sum of the series a_n is $s_3 \approx 1.351$, with error $R_3 \leq 0.222$.

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