Topic: Repeated quadratic factors

Question: Rewrite the integral using partial fractions. Do not solve it.

$$\int \frac{x^3 + 2x - 1}{\left(x^2 + 1\right)^2}\, dx$$

Answer choices:

A $\quad \displaystyle\int \frac{x - 1}{\left(x^2 + 1\right)^2} + \frac{x}{\left(x^2 + 1\right)}\, dx$

B $\quad \displaystyle\int \frac{x + 1}{\left(x^2 + 1\right)^2} + \frac{x}{\left(x^2 + 1\right)}\, dx$

C $\quad \displaystyle\int \frac{x - 1}{\left(x^2 + 1\right)^2} - \frac{x}{\left(x^2 + 1\right)}\, dx$

D $\quad \displaystyle\int \frac{x - 1}{\left(x^2 + 1\right)^2} + \frac{x}{\left(x^2 + 1\right)^2}\, dx$

Solution: A

First, factor the denominator. Since we have a repeated factor, we need to include all factors of a lesser degree.

$$\int \frac{x^3+2x-1}{\left(x^2+1\right)^2}\ dx = \int \frac{Ax+B}{\left(x^2+1\right)^2} + \frac{Cx+D}{\left(x^2+1\right)}\ dx$$

Using partial fractions decomposition containing a quadratic factor, we have

$$\frac{x^3+2x-1}{\left(x^2+1\right)^2} = \frac{Ax+B}{\left(x^2+1\right)^2} + \frac{Cx+D}{\left(x^2+1\right)}$$

Now we'll solve for constants.

$$\frac{\left(x^3+2x-1\right)\left(x^2+1\right)^2}{\left(x^2+1\right)^2} = \frac{(Ax+B)\left(x^2+1\right)^2}{\left(x^2+1\right)^2} + \frac{(Cx+D)\left(x^2+1\right)^2}{\left(x^2+1\right)}$$

$$x^3+2x-1 = Ax+B+(Cx+D)\left(x^2+1\right)$$

$$x^3+2x-1 = Ax+B+Cx^3+Cx+Dx^2+D$$

$$x^3+2x-1 = Cx^3+Dx^2+Ax+Cx+B+D$$

$$x^3+2x-1 = Cx^3+Dx^2+(A+C)x+(B+D)$$

Equating coefficients on both sides, we get

[1] $C=1$

[2] $D=0$

[3] $A+C=2$

[4] $B+D=-1$

We already know the value of C and D. Plugging [1] into [3] to solve for A, we get

$$A+1=2$$

$$A=1$$

Plugging [2] into [4] to solve for B, we get

$$B+0=-1$$

$$B=-1$$

Plugging the values for each of the four constants back into the partial fractions decomposition, and putting the decomposition back into the integral, we get

$$\int \frac{x^3+2x-1}{\left(x^2+1\right)^2}\ dx=\int \frac{1x+(-1)}{\left(x^2+1\right)^2}+\frac{1x+0}{\left(x^2+1\right)}\ dx$$

$$\int \frac{x-1}{\left(x^2+1\right)^2}+\frac{x}{\left(x^2+1\right)}\ dx$$

Topic: Repeated quadratic factors

Question: Rewrite the integral using partial fractions. Do not solve it.

$$\int \frac{x^3 + 4x^2 - 10}{x^2(x^2 - 1)} \, dx$$

Answer choices:

A $\quad \displaystyle\int \frac{1}{x^2} + \frac{7}{x+1} - \frac{5}{x-1} \, dx$

B $\quad \displaystyle\int \frac{1}{x^2} - \frac{7}{x+1} + \frac{5}{x-1} \, dx$

C $\quad \displaystyle\int \frac{10}{x^2} + \frac{\frac{7}{2}}{x+1} - \frac{\frac{5}{2}}{x-1} \, dx$

D $\quad \displaystyle\int \frac{10}{x^2} - \frac{\frac{7}{2}}{x+1} + \frac{\frac{5}{2}}{x-1} \, dx$

Solution: C

First, factor the denominator.

$$\int \frac{x^3+4x^2-10}{x^2(x+1)(x-1)}\ dx$$

Set up the partial fractions decomposition.

$$\frac{x^3+4x^2-10}{x^2(x+1)(x-1)}=\frac{Ax+B}{x^2}+\frac{C}{x+1}+\frac{D}{x-1}$$

Solve for the constants.

$$x^3+4x^2-10=(Ax+B)(x+1)(x-1)+C(x^2)(x-1)+D(x^2)(x+1)$$

$$x^3+4x^2-10=(Ax+B)(x^2-1)+Cx^2(x-1)+Dx^2(x+1)$$

$$x^3+4x^2-10=Ax^3-Ax+Bx^2-B+Cx^3-Cx^2+Dx^3+Dx^2$$

$$x^3+4x^2-10=(A+C+D)x^3+(B-C+D)x^2-Ax-B$$

Equating coefficients on both sides, we get

[1] $A+C+D=1$

[2] $B-C+D=4$

[3] $-A=0$

[4] $-B=-10$

From equation [3] we know $A=0$, and from equation [4] we know $B=10$. So equations [1] and [2] become

$$0+C+D=1$$

[5] $C+D=1$

and

$$10-C+D=4$$

$$-C+D=-6$$

$$C-D=6$$

Solve [5] for C to get $C=1-D$. Substituting $C=1-D$ into $C-D=6$ gives

$$1-D-D=6$$

$$1-2D=6$$

$$-2D=5$$

$$D=-\frac{5}{2}$$

Then

$$C=1-\left(-\frac{5}{2}\right)$$

$$C=1+\frac{5}{2}$$

$$C=\frac{7}{2}$$

Plugging the values for each of the four constants back into the partial fractions decomposition, and putting the decomposition back into the integral, we get

$$\int \frac{0x+10}{x^2}+\frac{\frac{7}{2}}{x+1}+\frac{-\frac{5}{2}}{x-1}\ dx$$

$$\int \frac{10}{x^2}+\frac{\frac{7}{2}}{x+1}-\frac{\frac{5}{2}}{x-1}\ dx$$

Topic: Repeated quadratic factors

Question: Rewrite the integral using partial fractions. Do not solve it.

$$\int \frac{2x^4+16}{x\left(x^2+2\right)^2}\ dx$$

Answer choices:

A $\quad \displaystyle\int \frac{4}{x}+\frac{12x}{\left(x^2+2\right)^2}+\frac{2x}{\left(x^2+2\right)}\ dx$

B $\quad \displaystyle\int \frac{4}{x}-\frac{12x}{\left(x^2+2\right)^2}-\frac{2x}{\left(x^2+2\right)}\ dx$

C $\quad \displaystyle\int \frac{4}{x}-\frac{-12x}{\left(x^2+2\right)^2}+\frac{-2x}{\left(x^2+2\right)}\ dx$

D $\quad \displaystyle\int \frac{4}{x}+\frac{-12x}{\left(x^2+2\right)}+\frac{-2x}{\left(x^2+2\right)^2}\ dx$

Solution: B

First, factor the denominator. Since we have a repeated factor, we need to include all factors of a lesser degree.

$$\int \frac{2x^4+16}{x\left(x^2+2\right)^2}\ dx=\int \frac{A}{x}+\frac{Bx+C}{\left(x^2+2\right)^2}+\frac{Dx+E}{\left(x^2+2\right)}\ dx$$

Using partial fractions decomposition containing a quadratic factor, we have

$$\frac{2x^4+16}{x\left(x^2+2\right)^2}=\frac{A}{x}+\frac{Bx+C}{\left(x^2+2\right)^2}+\frac{Dx+E}{\left(x^2+2\right)}$$

Now we'll solve for constants.

$$\frac{\left(2x^4+16\right)\left[x\left(x^2+2\right)^2\right]}{x\left(x^2+2\right)^2}=\frac{A\left[x\left(x^2+2\right)^2\right]}{x}+\frac{(Bx+C)\left[x\left(x^2+2\right)^2\right]}{\left(x^2+2\right)^2}+\frac{(Dx+E)\left[x\left(x^2+2\right)^2\right]}{\left(x^2+2\right)}$$

$$2x^4+16=A\left(x^2+2\right)^2+(Bx+C)(x)+(Dx+E)\left[x\left(x^2+2\right)\right]$$

$$2x^4+16=A\left(x^4+4x^2+4\right)+Bx^2+Cx+(Dx+E)\left(x^3+2x\right)$$

$$2x^4+16=Ax^4+4Ax^2+4A+Bx^2+Cx+Dx^4+2Dx^2+Ex^3+2Ex$$

$$2x^4+16=Ax^4+Dx^4+Ex^3+4Ax^2+Bx^2+2Dx^2+Cx+2Ex+4A$$

$$2x^4+16=(A+D)x^4+Ex^3+(4A+B+2D)x^2+(C+2E)x+4A$$

Equating coefficients on both sides, we get

[1] $A+D=2$

[2] $E=0$

[3] $4A+B+2D=0$

[4] $C+2E=0$

[5] $4A=16$

We already know the value of E. Plugging [2] into [4] to solve for C, we get

$$C+2(0)=0$$

$$C=0$$

Solving [5] for A, we get

$$4A=16$$

$$A=4$$

Plugging $A=4$ into [1] to solve for D, we get

$$4+D=2$$

$$D=-2$$

Plugging our values for A and D into [3] to solve for B, we get

$$4(4)+B+2(-2)=0$$

$$B=-12$$

Plugging the values for each of the five constants back into the partial fractions decomposition, and putting the decomposition back into the integral, we get

$$\int \frac{2x^4+16}{x\left(x^2+2\right)^2}\ dx = \int \frac{4}{x} + \frac{-12x+0}{\left(x^2+2\right)^2} + \frac{-2x+0}{\left(x^2+2\right)}\ dx$$

$$\int \frac{4}{x} - \frac{12x}{\left(x^2+2\right)^2} - \frac{2x}{\left(x^2+2\right)}\ dx$$