Volume of revolution of a parametric curve

In the same way that we could find the volume of a three-dimensional object generated by rotating a two-dimensional area around in axis when we studied applications of integrals, we can find the volume of revolution generated by revolving the area enclosed by two parametric curves.

The formulas we use to find the volume of revolution for a parametric curve are

rotation around the x-axis $\quad V_x = \int_{\alpha}^{\beta} \pi y^2 \frac{dx}{dt}\ dt$

rotation around the y-axis $\quad V_y = \int_{\alpha}^{\beta} \pi x^2 \frac{dy}{dt}\ dt$

Example

Find the volume of revolution of the parametric curve.

$x = 3t^2 + 4$

$y = t^4$

for $0 \le t \le 2$, rotated around the y-axis

Since we're rotating around the y-axis, we'll use the formula

$$V_y = \int_{\alpha}^{\beta} \pi x^2 \frac{dy}{dt}\ dt$$

The problem gave the interval $0 \le t \le 2$, so $\alpha = 0$ and $\beta = 2$. Now we need to find dy/dt so that we can plug it into the volume formula.

$$y = t^4$$

$$\frac{dy}{dt} = 4t^3$$

Plugging everything into the volume formula, we get

$$V_y = \int_0^2 \pi\left(3t^2 + 4\right)^2\left(4t^3\right)\ dt$$

$$V_y = 4\pi\int_0^2 t^3\left(3t^2 + 4\right)^2\ dt$$

$$V_y = 4\pi\int_0^2 t^3\left(9t^4 + 24t^2 + 16\right)\ dt$$

$$V_y = 4\pi\int_0^2 9t^7 + 24t^5 + 16t^3\ dt$$

$$V_y = 4\pi\left(\frac{9t^8}{8} + \frac{24t^6}{6} + \frac{16t^4}{4}\right)\Bigg|_0^2$$

$$V_y = 4\pi\left(\frac{9t^8}{8} + 4t^6 + 4t^4\right)\Bigg|_0^2$$

$$V_y = 4\pi \left[\frac{9(2)^8}{8} + 4(2)^6 + 4(2)^4 \right] - 4\pi \left[\frac{9(0)^8}{8} + 4(0)^6 + 4(0)^4 \right]$$

$$V_y = 4\pi \left[9(2)^5 + 256 + 64 \right]$$

$$V_y = 2{,}432\pi$$