Topic: Critical points and the first derivative test

Question: Find the critical point of the function.

$$f(x) = x^2 - 10x + 2$$

Answer choices:

A $x = \frac{1}{5}$

B $x = 5$

C $x = -5$

D $x = -\frac{1}{5}$

Solution: B

Take the derivative of the function.

$$f(x) = x^2 - 10x + 2$$

$$f'(x) = 2x - 10$$

This derivative exists everywhere. Set the derivative equal to 0 and solve for x.

$$2x - 10 = 0$$

$$2x = 10$$

$$x = 5$$

The function has one potential critical point at $x = 5$.

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Question: Where is the function increasing and decreasing?

$$f(x) = x^2$$

Answer choices:

A Increasing on $x < 1$ and decreasing on $x > 1$

B Increasing on $x < 0$ and decreasing on $x > 0$

C Increasing on $x > 0$ and decreasing on $x < 0$

D Increasing on $x > 1$ and decreasing on $x < 1$

Solution: C

Find the derivative.

$$f(x) = x^2$$

$$f'(x) = 2x$$

This derivative exists everywhere. Set the derivative equal to 0 and solve for x.

$$0 = 2x$$

$$x = 0$$

Investigate the critical point $x = 0$ by testing $x = -1$ and $x = 1$ in the first derivative.

$$f'(-1) = 2(-1)$$

$$f'(-1) = -2$$

and

$$f'(1) = 2(1)$$

$$f'(1) = 2$$

On the left side of $x = 0$ the derivative is negative so the function is decreasing. On the right side of $x = 0$ the derivative is positive so the function is increasing.

Topic: Critical points and the first derivative test

Question: Where is the function increasing and decreasing?

$$f(x) = x^4 - 4x^3 + 4x^2 - 7$$

Answer choices:

A Decreasing on $x < 0$ and $1/2 < x < 3/2$, increasing on $0 < x < 1/2$ and $x > 3/2$

B Decreasing on $0 < x < 1/2$ and $x > 3/2$, increasing on $x < 0$ and $1/2 < x < 3/2$

C Decreasing on $x < 0$ and $1 < x < 2$, increasing on $0 < x < 1$ and $x > 2$

D Decreasing on $0 < x < 1$ and $x > 2$, increasing on $x < 0$ and $1 < x < 2$

Solution: C

Take the first derivative of the function.

$$f(x) = x^4 - 4x^3 + 4x^2 - 7$$

$$f'(x) = 4x^3 - 12x^2 + 8x$$

$$f'(x) = 4x(x^2 - 3x + 2)$$

$$f'(x) = 4x(x - 2)(x - 1)$$

This derivative exists everywhere. Set the derivative equal to 0 and solve for x.

$$4x(x - 2)(x - 1) = 0$$

$$x = 0,\ 1,\ 2$$

Investigate each interval by evaluating the first derivative at $x = -1$, $x = 1/2$, $x = 3/2$, and $x = 3$.

$$f'(-1) = 4(-1)^3 - 12(-1)^2 + 8(-1)$$

$$f'(-1) = -4 - 12 - 8$$

$$f'(-1) = -24$$

and

$$f'\left(\frac{1}{2}\right) = 4\left(\frac{1}{2}\right)^3 - 12\left(\frac{1}{2}\right)^2 + 8\left(\frac{1}{2}\right)$$

$$f'\left(\frac{1}{2}\right) = 4\left(\frac{1}{8}\right) - 12\left(\frac{1}{4}\right) + 4$$

$$f'\left(\frac{1}{2}\right) = \frac{1}{2} - 3 + 4$$

$$f'\left(\frac{1}{2}\right) = \frac{3}{2}$$

and

$$f'\left(\frac{3}{2}\right) = 4\left(\frac{3}{2}\right)^3 - 12\left(\frac{3}{2}\right)^2 + 8\left(\frac{3}{2}\right)$$

$$f'\left(\frac{3}{2}\right) = 4\left(\frac{27}{8}\right) - 12\left(\frac{9}{4}\right) + 12$$

$$f'\left(\frac{3}{2}\right) = \frac{27}{2} - 27 + 12$$

$$f'\left(\frac{3}{2}\right) = -\frac{3}{2}$$

and

$$f'(3) = 4(3)^3 - 12(3)^2 + 8(3)$$

$$f'(3) = 4(27) - 12(9) + 24$$

$$f'(3) = 108 - 108 + 24$$

$$f'(3) = 24$$

To the left of $x = 0$ the derivative is negative so the function is decreasing. Between $x = 0$ and $x = 1$, the derivative is positive so the function is increasing. Between $x = 1$ and $x = 2$, the derivative is negative so the function is decreasing. To the right of $x = 2$ the derivative is positive so the function is increasing.

The function $f(x) = x^4 - 4x^3 + 4x^2 - 7$ is decreasing when $x < 0$, increasing between $x = 0$ and $x = 1$, decreasing between $x = 1$ and $x = 2$, and increasing when $x > 2$.