Topic: Alternating series test

Question: Use the alternating series test to say whether the series converges or diverges.

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{2}{3n+1}$$

Answer choices:

A The series converges

B The series conditionally converges

C The series diverges

D The test was inconclusive

Solution: A

The alternating series test for convergence tells us that

an alternating series

$$\sum_{n=1}^{\infty} (-1)^{n-1} a_n \text{ where } a_n > 0$$

converges if

$0 < a_{n+1} < a_n$ for all values of n, and

$$\lim_{n \to \infty} a_n = 0$$

When we use the alternating series test, we need to make sure that we separate the series a_n from the $(-1)^n$ part that makes it alternating.

$$a_n = \frac{2}{3n+1}$$

Now we need to show that $0 < a_{n+1} < a_n$. Remembering that this series starts at $n = 1$, let's check the first few terms of the series to see if it looks like $0 < a_{n+1} < a_n$.

		a_n		a_{n+1}
$n = 1$	$\frac{2}{3(1)+1}$	$\frac{2}{4}$	$\frac{2}{3(2)+1}$	$\frac{2}{7}$
$n = 2$	$\frac{2}{3(2)+1}$	$\frac{2}{7}$	$\frac{2}{3(3)+1}$	$\frac{2}{10}$

$n = 3$	$\frac{2}{3(3)+1}$	$\frac{2}{10}$	$\frac{2}{3(4)+1}$	$\frac{2}{13}$
$n = 4$	$\frac{2}{3(4)+1}$	$\frac{2}{13}$	$\frac{2}{3(5)+1}$	$\frac{2}{16}$

We can see that the terms of a_n and a_{n+1} will always be positive, because there's no value of n, when $n \geq 1$, that will make either series negative. We can also see that a_{n+1} is always going to be smaller than a_n. If you're not convinced by their fractional values in the table, compute the decimal values on your calculator to be sure.

If you can't be sure that $0 < a_{n+1} < a_n$ just by looking at the table, you can always take the derivative of a_n to double-check. If the derivative is negative, then you know the series is decreasing, which means that a_{n+1} will always be less than a_n.

$$\frac{d}{dx}\left(\frac{2}{3x+1}\right)$$

Using the quotient rule, we get

$$\frac{0(3x+1)-2(3)}{(3x+1)^2}$$

$$\frac{-6}{(3x+1)^2}$$

Looking at the derivative, we can see that for all values of the series (remember, the series starts at $n = 1$), the derivative is negative because the numerator will be negative and the denominator will be positive. This confirms that the series is decreasing.

The final step is to verify that $\lim_{n\to\infty} a_n = 0$.

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{2}{3n+1}$$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{2}{3n+1} \cdot \left(\frac{\frac{1}{n}}{\frac{1}{n}}\right)$$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{\frac{2}{n}}{3+\frac{1}{n}}$$

$$\lim_{n\to\infty} a_n = \frac{0}{3+0}$$

$$\lim_{n\to\infty} a_n = 0$$

Since we've shown that $0 < a_{n+1} < a_n$ and that $\lim_{n\to\infty} a_n = 0$, we can say that the series converges.

Topic: Alternating series test

Question: Use the alternating series test to say whether the series converges or diverges.

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{\ln n}{n}$$

Answer choices:

A The series converges

B The series conditionally converges

C The series diverges

D The test was inconclusive

Solution: A

The alternating series test for convergence tells us that

an alternating series

$$\sum_{n=1}^{\infty} (-1)^{n-1} a_n \text{ where } a_n > 0$$

converges if

$0 < a_{n+1} < a_n$ for all values of n, and

$$\lim_{n \to \infty} a_n = 0$$

When we use the alternating series test, we need to make sure that we separate the series a_n from the $(-1)^n$ part that makes it alternating.

$$a_n = \frac{\ln n}{n}$$

Now we need to show that $0 < a_{n+1} < a_n$. Remembering that this series starts at $n = 1$, let's check the first few terms of the series to see if it looks like $0 < a_{n+1} < a_n$.

		a_n		a_{n+1}
$n = 1$	$\frac{\ln 1}{1}$	0	$\frac{\ln 2}{2}$	0.3465736
$n = 2$	$\frac{\ln 2}{2}$	0.3465736	$\frac{\ln 3}{3}$	0.3662041

$n = 3$	$\frac{\ln 3}{3}$	0.3662041	$\frac{\ln 4}{4}$	0.3465736
$n = 4$	$\frac{\ln 4}{4}$	0.3465736	$\frac{\ln 5}{5}$	0.3218876

We can see that the terms of a_n and a_{n+1} will always be positive, because there's no value of n, when $n \geq 1$, that will make either series negative. However, even when we look at the terms to eight decimal places, it's unclear whether or not $0 < a_{n+1} < a_n$. To double-check, we'll take the derivative of a_n. If the derivative is negative, then you know the series is decreasing, which means that a_{n+1} will always be less than a_n.

$$\frac{d}{dx}\left(\frac{\ln x}{x}\right)$$

Using the quotient rule, we get

$$\frac{\left(\frac{1}{x}\right)x - (\ln x)(1)}{x^2}$$

$$\frac{1 - \ln x}{x^2}$$

For $n > 2$, the derivative is negative and the series is decreasing. The final step is to verify that $\lim_{n \to \infty} a_n = 0$. Using L'Hospital's rule, we get

$$\lim_{n \to \infty} a_n = \lim_{n \to \infty} \frac{\ln n}{n}$$

$$\lim_{n \to \infty} a_n = \lim_{n \to \infty} \frac{\frac{d}{dn} \ln n}{\frac{d}{dn} n}$$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{\frac{1}{n}}{1}$$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n}$$

$$\lim_{n\to\infty} a_n = 0$$

Since we've shown that $0 < a_{n+1} < a_n$ and that $\lim_{n\to\infty} a_n = 0$, we can say that the series converges.

Topic: Alternating series test

Question: Use the alternating series test to say whether the series converges or diverges.

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n}$$

Answer choices:

A The series converges

B The series conditionally converges

C The series diverges

D The test was inconclusive

Solution: A

The alternating series test for convergence tells us that

an alternating series

$$\sum_{n=1}^{\infty} (-1)^{n-1} a_n \text{ where } a_n > 0$$

converges if

$0 < a_{n+1} < a_n$ for all values of n, and

$$\lim_{n \to \infty} a_n = 0$$

When we use the alternating series test, we need to make sure that we separate the series a_n from the $(-1)^n$ part that makes it alternating.

$$a_n = \frac{1}{n}$$

Now we need to show that $0 < a_{n+1} < a_n$. Remembering that this series starts at $n = 1$, let's check the first few terms of the series to see if it looks like $0 < a_{n+1} < a_n$.

		a_n		a_{n+1}
$n = 1$	$\frac{1}{1}$	1	$\frac{1}{1+1}$	$\frac{1}{2}$
$n = 2$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2+1}$	$\frac{1}{3}$

$n = 3$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3+1}$	$\frac{1}{4}$
$n = 4$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4+1}$	$\frac{1}{5}$

We can see that the terms of a_n and a_{n+1} will always be positive, because there's no value of n, when $n \geq 1$, that will make either series negative. We can also see that a_{n+1} is always going to be smaller than a_n. If you're not convinced by their fractional values in the table, compute the decimal values on your calculator to be sure.

If you can't be sure that $0 < a_{n+1} < a_n$ just by looking at the table, you can always take the derivative of a_n to double-check. If the derivative is negative, then you know the series is decreasing, which means that a_{n+1} will always be less than a_n.

$$\frac{d}{dx}\left(\frac{1}{x}\right)$$

$$\frac{d}{dx}\left(x^{-1}\right)$$

$$-x^{-2}$$

$$-\frac{1}{x^2}$$

Looking at the derivative, we can see that for all values of the series (remember, the series starts at $n = 1$), the derivative is negative because the numerator will be negative and the denominator will be positive. This confirms that the series is decreasing.

The final step is to verify that $\lim_{n\to\infty} a_n = 0$.

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n}$$

$$\lim_{n\to\infty} a_n = \frac{1}{\infty}$$

$$\lim_{n\to\infty} a_n = 0$$

Since we've shown that $0 < a_{n+1} < a_n$ and that $\lim_{n\to\infty} a_n = 0$, we can say that the series converges.