Inflection points and the second derivative test

In the last lesson, we saw that the first derivative allowed us to determine

- where the function has critical points, which told us
- where the function changed direction from increasing to decreasing, or vice versa, and therefore
- where the function had local maxima and/or local minima.

As it turns out, the second derivative can be used in a similar way, except that the second derivative allows us to determine

- where the function has inflection points, which tell us
- where the graph changes concavity from concave up to concave down, or vice versa.

Inflection points

In other words, an **inflection point** is a point at which the function changes from concave up to concave down, or from concave down to concave up.

In the same way that we found critical points by setting the first derivative equal to 0, we'll find inflection points by setting the second derivative equal to 0. Inflection points will exist wherever the second derivative is equal to 0 (or possibly at points where the second derivative is undefined).

Let's continue on with the same example we were using in the last lesson, and walk through how to find the function's inflection points.

Example

Find the inflection points of the function whose first derivative is given.

$$f'(x) = 1 - \frac{4}{x^2}$$

In the previous lesson, we were working with the function

$$f(x) = x + \frac{4}{x}$$

and we'd already found that its first derivative was

$$f'(x) = 1 - \frac{4}{x^2}$$

$$f'(x) = 1 - 4x^{-2}$$

So we'll take the derivative of this first derivative in order to find the function's second derivative.

$$f''(x) = 0 + 8x^{-3}$$

$$f''(x) = \frac{8}{x^3}$$

To find inflection points, we set this second derivative equal to 0.

$$0 = \frac{8}{x^3}$$

There's no solution to this equation. If we multiply both sides by x^3, we'll get $0 = 8$, which is nonsensical. We can also see that the second derivative is undefined at $x = 0$, since an $x = 0$ value would make the denominator of the fraction 0, which is always undefined. So $x = 0$ is the only possible inflection point.

Let's do another example.

Example

Find the inflection points of the function whose first derivative is given.

$$f'(x) = 3x^2 + x - 10$$

The first and second derivatives are

$$f'(x) = 3x^2 + x - 10$$

$$f''(x) = 6x + 1$$

To find inflection points, set the second derivative equal to 0.

$$6x + 1 = 0$$

$$6x = -1$$

$$x = -\frac{1}{6}$$

So $x = -1/6$ is the only possible inflection point.

It's possible that the function will have no inflection points. Lines (first-degree functions) and parabolas (second-degree) functions, will never have inflection points, because there concavity never changes.

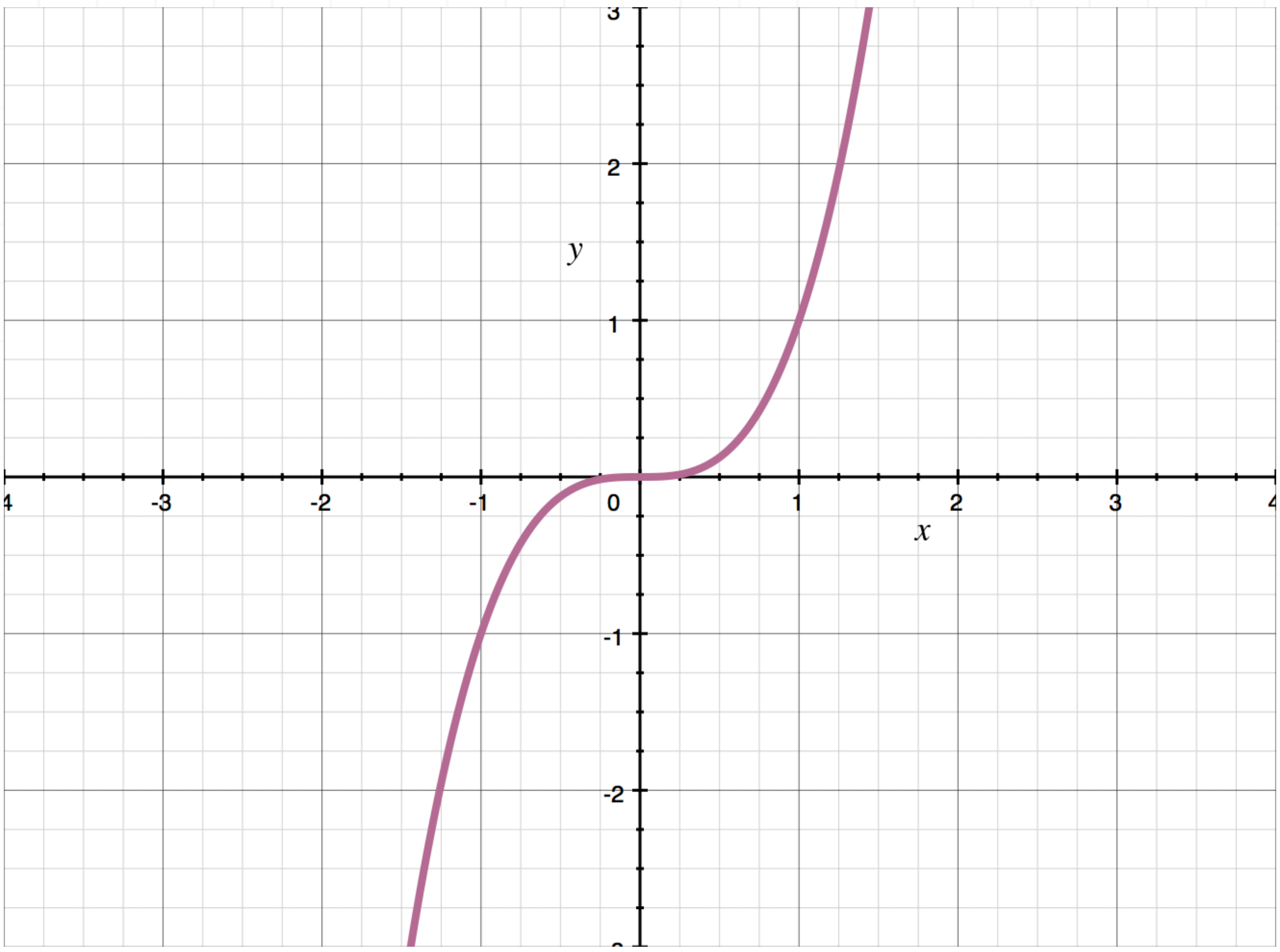
Concave up and concave down

Because the inflection points are the points at which the function changes concavity, from concave up to concave down or from concave down to concave up, the next step is to investigate the behavior in between the inflection points.

- Where the second derivative is positive, the function is concave up. A function is **concave up** when it's "scooping" upwards, like a bowl or a cup.

- Where the second derivative is negative, the function is concave down. A function is **concave down** when it's "scooping" downwards, like a hat or a dome.

As an example, let's look at the graph below. From $-\infty < x < 0$, the graph is concave down. We can think "concave down looks like a frown." The inflection point at which the graph changes concavity is at $x = 0$. On the interval $0 < x < \infty$, the graph is concave up, and it looks like a smile.



To test the sign of the second derivative, we'll simply pick a value between each pair of inflection points, and plug that test value into the second derivative to see whether we get a positive result or a negative result. If the test value gives a positive result, it means the function is concave up on that interval, and if the test value gives a negative result, it means the function is concave down on that interval.

If we find one inflection point for the function, then we just need to look at the sign of the second derivative on the left side and right side of that one inflection point.

But if we find multiple inflection points, then we need to find the sign of the second derivative to the left of the left-most inflection point, to the right of the right-most inflection point, and between each inflection point.

Let's continue with the same example we used to find inflection points, looking at the sign of the second derivative around that point.

Example

The only potential inflection point of the function is $x = 0$. Where is the function concave up and where is it concave down?

$$f(x) = x + \frac{4}{x}$$

$$f'(x) = 1 - \frac{4}{x^2}$$

$$f''(x) = \frac{8}{x^3}$$

Previously, we used the second derivative to find that the function had a potential inflection point at $x = 0$. Once we have the inflection point(s), it's helpful to plot it along a number line from least to greatest, left to right.

From this diagram, we can see that we have to test two intervals.

$$-\infty < x < 0$$

$$0 < x < \infty$$

To test $-\infty < x < 0$, we'll plug a test value of $x = -1$ into the second derivative, since $x = -1$ is a value in that interval. We could have picked any other value to use instead, as long as it fell in the interval $-\infty < x < 0$.

$$f''(x) = \frac{8}{x^3}$$

$$f''(-1) = \frac{8}{(-1)^3}$$

$$f''(-1) = \frac{8}{-1}$$

$$f''(-1) = -8 < 0$$

To test $0 < x < \infty$, we'll plug $x = 1$ into the derivative.

$$f''(x) = \frac{8}{x^3}$$

$$f''(1) = \frac{8}{(1)^3}$$

$$f''(1) = \frac{8}{1}$$

$$f''(1) = 8 > 0$$

The second derivative was negative on the first interval and positive on the second interval. Plot these signs on the inflection point diagram we drew earlier.

Remember that the original function $f(x)$ is concave up where we found a positive result, and concave down where we found a negative result. So we can say

- $f(x)$ is concave down on $-\infty < x < 0$

- $f(x)$ is concave up on $0 < x < \infty$

Let's look at the concavity of the function we were looking at before, $f'(x) = 3x^2 + x - 10$.

Example

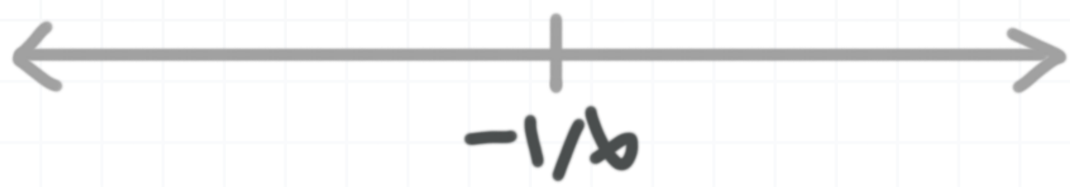
If the only potential inflection point of the function is $x = -1/6$, where is the function concave up and where is it concave down?

$$f(x) = x^3 + \frac{1}{2}x^2 - 10x - 5$$

$$f'(x) = 3x^2 + x - 10$$

$$f''(x) = 6x + 1$$

Previously, we used the second derivative to find that the function had a potential inflection point at $x = -1/6$. Once we have the inflection point(s), it's helpful to plot it along a number line from least to greatest, left to right.



From this diagram, we can see that we have to test two intervals.

$$-\infty < x < -\frac{1}{6}$$

$$-\frac{1}{6} < x < \infty$$

To test $-\infty < x < -1/6$, we'll plug a test value of $x = -1$ into the second derivative, since $x = -1$ is a value in that interval. We could have picked any other value to use instead, as long as it falls in the interval $-\infty < x < -1/6$.

$$f''(x) = 6x + 1$$

$$f''(-1) = 6(-1) + 1$$

$$f''(-1) = -6 + 1$$

$$f''(-1) = -5 < 0$$

To test $-1/6 < x < \infty$, we'll plug $x = 1$ into the derivative.

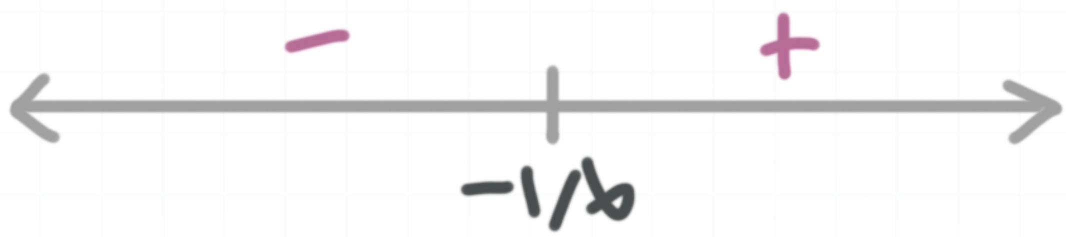
$$f''(x) = 6x + 1$$

$$f''(1) = 6(1) + 1$$

$$f''(1) = 6 + 1$$

$$f''(1) = 7 > 0$$

The second derivative was negative on the first interval and positive on the second interval. Plot these signs on the inflection point diagram we drew earlier.



Remember that the original function $f(x)$ is concave up when we find a positive result, and concave down when we find a negative result. So we can say

- $f(x)$ is concave down on $-\infty < x < -1/6$

- $f(x)$ is concave up on $-1/6 < x < \infty$

Second derivative test

Surprisingly, the second derivative test actually isn't related to inflection points or concavity. Like the first derivative test, the **second derivative test** is just another way of classifying the function's local extrema.

To use the second derivative test, we just plug any critical points we've found (not the inflection points) into the second derivative. If the result is negative, there's a local maximum at that critical point. If the result is positive, there's a local minimum at that critical point. If the result is zero, then the second derivative test is inconclusive at that critical point.

Be careful! Notice here how the results are opposite what we might think they would be.

- If $f''(x) > 0$ at a critical point, there's a local minimum there.
- If $f''(x) < 0$ at a critical point, there's a local maximum there.

Let's keep going with the same example we've been working through to see how to use the second derivative test to classify critical points.

Example

The function has critical points at $x = \pm 2$. Use the second derivative test to find local extrema.

$$f(x) = x + \frac{4}{x}$$

$$f'(x) = 1 - \frac{4}{x^2}$$

$$f''(x) = \frac{8}{x^3}$$

We already found in the last example that the second derivative of the original function $f(x)$ was

$$f''(x) = \frac{8}{x^3}$$

To find the function's local maxima and minima, we'll plug both critical points, $x = \pm 2$, into the second derivative.

$$f''(-2) = \frac{8}{(-2)^3}$$

$$f''(-2) = \frac{8}{-8}$$

$$f''(-2) = -1 < 0$$

and

$$f''(2) = \frac{8}{2^3}$$

$$f''(2) = \frac{8}{8}$$

$$f''(2) = 1 > 0$$

Since the second derivative is negative at $x = -2$, the function has a local maximum at that point, and since the second derivative is positive at $x = 2$, the function has a local minimum at that point.

Notice how these are the same results we got from using the first derivative test in the last lesson.

Let's do another example.

Example

The function has critical points at $x = -2, 5/3$. Use the second derivative test to find local extrema.

$$f(x) = x^3 + \frac{1}{2}x^2 - 10x - 5$$

$$f'(x) = 3x^2 + x - 10$$

We already found that the second derivative was

$$f''(x) = 6x + 1$$

To find the function's local maxima and minima, we'll plug both critical points, $x = -2$ and $x = 5/3$, into the second derivative.

$$f''(-2) = 6(-2) + 1$$

$$f''(-2) = -12 + 1$$

$$f''(-2) = -11 < 0$$

and

$$f''\left(\frac{5}{3}\right) = 6\left(\frac{5}{3}\right) + 1$$

$$f''\left(\frac{5}{3}\right) = 10 + 1$$

$$f''\left(\frac{5}{3}\right) = 11 > 0$$

Since the second derivative is negative at $x = -2$, the function has a local maximum at that point, and since the second derivative is positive at $x = 5/3$, the function has a local minimum at that point.

Notice how these are the same results we got from using the first derivative test previously.