Simpson's rule

Simpson's rule is a method we can use to approximate the area under a function over a given interval. If it's difficult to find area exactly using an integral, we can use Simpson's rule instead to estimate the integral.

In order to use Simpson's rule to get an estimate of the area, we need to know the interval over which we're calculating area so that we can divide the area into n subintervals. That'll allow us to calculate the width of each subinterval, Δx. The larger the value of n, the smaller the value of Δx and the more accurate our final answer will be.

In order to use Simpson's rule, n **must** be an even number.

The interval over which we want to find area is $[a, b]$, and the sub-intervals are $[x_0, x_1]$, $[x_1, x_2]$, ... , $[x_{n-1}, x_n]$ where

$$x_0 = a$$

$$x_1 = a + \Delta x$$

$$x_2 = x_1 + \Delta x$$

...

$$x_{n-1} = x_{n-2} + \Delta x$$

$$x_n = x_{n-1} + \Delta x$$

To find Δx, we use

$$\Delta x = \frac{b-a}{n}$$

Putting it all together, we get the formula for the Simpson's rule, which is

$$\int_a^b f(x)\ dx \approx \frac{\Delta x}{3}\left[f(x_0) + 4f(x_1) + 2f(x_2) + \dots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)\right]$$

This formula is a little tricky. We have to remember that

- the odd subscripts (like $f(x_1)$) are multiplied by 4 and
- the even subscripts (like $f(x_4)$) are multiplied by 2, except
- the first and last terms ($f(x_0)$ and $f(x_n)$), which have no multiplier.

Example

Using $n = 4$ and Simpson's rule, approximate the value of the integral.

$$\int_0^3 e^{x^2}\ dx$$

First we'll find the width of the subintervals. We'll use $\Delta x = \dfrac{b-a}{n}$ where $a = 0$, $b = 3$, and $n = 4$.

$$\Delta x = \frac{3-0}{4}$$

$$\Delta x = \frac{3}{4}$$

This means that each sub-interval is $3/4$ units wide. Now we can solve for our subintervals using $[x_0, x_1]$, $[x_1, x_2]$, ... , $[x_{n-1}, x_n]$, where $x_0 = 0$, $x_1 = 3/4$, $x_2 = 3/2$, $x_3 = 9/4$, and $x_4 = 3$.

$$\left[0, \frac{3}{4}\right], \left[\frac{3}{4}, \frac{3}{2}\right], \left[\frac{3}{2}, \frac{9}{4}\right], \left[\frac{9}{4}, 3\right]$$

Now we can plug all of this information into the Simpson's rule formula. Remember that with this method, we'll end up with $n+1$ terms since we include the endpoints of the interval.

$$\int_a^b f(x)\ dx \approx \frac{\Delta x}{3}\left[f(x_0) + 4f(x_1) + 2f(x_2) + \dots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)\right]$$

$$\int_0^3 e^{x^2}\ dx \approx \frac{\frac{3}{4}}{3}\left[e^{(0)^2} + 4e^{\left(\frac{3}{4}\right)^2} + 2e^{\left(\frac{3}{2}\right)^2} + 4e^{\left(\frac{9}{4}\right)^2} + e^{(3)^2}\right]$$

$$\int_0^3 e^{x^2}\ dx \approx \frac{1}{4}\left(e^0 + 4e^{\frac{9}{16}} + 2e^{\frac{9}{4}} + 4e^{\frac{81}{16}} + e^9\right)$$

$$\int_0^3 e^{x^2}\ dx \approx \frac{1}{4}\left(1 + 4e^{\frac{9}{16}} + 2e^{\frac{9}{4}} + 4e^{\frac{81}{16}} + e^9\right)$$

$$\int_0^3 e^{x^2}\ dx \approx 2{,}191$$

Using Simpson's rule, the approximate area is $2{,}191$ square units.