Calculus 2 Workbook Solutions

Calculus with parametric curves

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TANGENT LINES OF PARAMETRIC CURVES

1. Find the equation of the tangent line to the parametric curve at $t = 3$.

$$x = 3t + 5$$

$$y = 7t - 2$$

Solution:

Use the formula of the tangent line as $y - y_1 = m(x - x_1)$ and transform the equation into the form $y = mx + b$.

At $t = 3$, the slope of the parametric equation is

$$m = \frac{dy}{dx} = \frac{\frac{d}{dt}(7t - 2)}{\frac{d}{dt}(3t + 5)} = \frac{7}{3}$$

At $t = 3$, the parametric equation has the values

$$x(3) = 3(3) + 5 = 14$$

$$y(3) = 7(3) - 2 = 19$$

Putting these values together gives the coordinate point $(x, y) = (14,19)$. Plug everything into the equation of the tangent line.

$$y - y_1 = m(x - x_1)$$

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$$y - 19 = \frac{7}{3}(x - 14)$$

$$y = \frac{7}{3}x - \frac{98}{3} + \frac{57}{3}$$

$$y = \frac{7}{3}x - \frac{41}{3}$$

■ 2. Find the equation of the tangent line to the parametric curve at $t = 4$.

$$x = 3t^2 - 12$$

$$y = 2t^3 + 6$$

Solution:

Use the formula of the tangent line as $y - y_1 = m(x - x_1)$ and transform the equation into the form $y = mx + b$.

At $t = 4$, the slope of the parametric equation is

$$m = \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\frac{d}{dt}(2t^3 + 6)}{\frac{d}{dt}(3t^2 - 12)} = \frac{6t^2}{6t} = t = 4$$

At $t = 4$, the parametric equation has the values

$$x(4) = 3(4)^2 - 12 = 36$$

$$y(4) = 2(4)^3 + 6 = 134$$

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Putting these values together gives the coordinate point $(x, y) = (36,134)$. Plug everything into the equation of the tangent line.

$$y - y_1 = m(x - x_1)$$

$$y - 134 = 4(x - 36)$$

$$y = 4x - 144 + 134$$

$$y = 4x - 10$$

■ 3. Find the equation of the tangent line to the parametric curve at $t = \pi/3$.

$$x = \cos^2 t$$

$$y = \sin^2 t$$

Solution:

Use the formula of the tangent line as $y - y_1 = m(x - x_1)$ and transform the equation into the form $y = mx + b$.

At $t = 4$, the slope of the parametric equation is

$$m = \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\frac{d}{dt}(\sin^2 t)}{\frac{d}{dt}(\cos^2 t)} = \frac{2\sin t\cos t}{-2\cos t\sin t} = -1$$

At $t = 4$, the parametric equation has the values

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$$x\left(\frac{\pi}{3}\right)=\cos^2\left(\frac{\pi}{3}\right)=\left(\frac{1}{2}\right)^2=\frac{1}{4}$$

$$y\left(\frac{\pi}{3}\right)=\sin^2\left(\frac{\pi}{3}\right)=\left(\frac{\sqrt{3}}{2}\right)^2=\frac{3}{4}$$

Putting these values together gives the coordinate point $(x, y)=(1/4,3/4)$. Plug everything into the equation of the tangent line.

$$y-y_1=m(x-x_1)$$

$$y-\frac{3}{4}=-1\left(x-\frac{1}{4}\right)$$

$$y=-x+\frac{1}{4}+\frac{3}{4}$$

$$y=-x+1$$

■ 4. Find the equation of the tangent line to the parametric curve at $t=4$.

$$x=t^2+t+3$$

$$y=t^2-3t+2$$

Solution:

Use the formula of the tangent line as $y-y_1=m(x-x_1)$ and transform the equation into the form $y=mx+b$.

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At $t = 4$, the slope of the parametric equation is

$$m = \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\frac{d}{dt}(t^2 - 3t + 2)}{\frac{d}{dt}(t^2 + t + 3)} = \frac{2t - 3}{2t + 1} = \frac{2(4) - 3}{2(4) + 1} = \frac{5}{9}$$

At $t = 4$, the parametric equation has the values

$$x(4) = 4^2 + 4 + 3 = 23$$

$$y(4) = 4^2 - 3(4) + 2 = 6$$

Putting these values together gives the coordinate point $(x, y) = (23,6)$. Plug everything into the equation of the tangent line.

$$y - y_1 = m(x - x_1)$$

$$y - 6 = \frac{5}{9}(x - 23)$$

$$y = \frac{5}{9}x - \frac{115}{9} + \frac{54}{9}$$

$$y = \frac{5}{9}x - \frac{61}{9}$$

■ 5. Find the equation of the tangent line to the parametric curve at $t = 9$.

$$x = 3\sqrt{t}$$

$$y = 5t\sqrt{t}$$

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Solution:

Use the formula of the tangent line as $y - y_1 = m(x - x_1)$ and transform the equation into the form $y = mx + b$.

At $t = 9$, the slope of the parametric equation is

$$m = \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\frac{d}{dt}\left(5t\sqrt{t}\right)}{\frac{d}{dt}\left(3\sqrt{t}\right)} = \frac{\frac{15}{2}\sqrt{t}}{\frac{3}{2\sqrt{t}}} = \frac{15}{2}\sqrt{t} \cdot \frac{2\sqrt{t}}{3} = 5t = 5(9) = 45$$

At $t = 9$, the parametric equation has the values

$$x(9) = 3\sqrt{9} = 3 \cdot 3 = 9$$

$$y(9) = 5 \cdot 9\sqrt{9} = 45 \cdot 3 = 135$$

Putting these values together gives the coordinate point $(x, y) = (9,135)$. Plug everything into the equation of the tangent line.

$$y - y_1 = m(x - x_1)$$

$$y - 135 = 45(x - 9)$$

$$y = 45x - 405 + 135$$

$$y = 45x - 270$$

AREA UNDER A PARAMETRIC CURVE

1. Find the area under the parametric curve.

$$x(t) = 3t^2$$

$$y(t) = t + 2$$

$$0 \le t \le 3$$

Solution:

Find the area under the curve on $a \le t \le b$ using the integral formula.

$$A = \int_a^b y(t)x'(t)\ dt$$

$$A = \int_0^3 (t+2)(6t)\ dt$$

$$A = \int_0^3 6t^2 + 12t\ dt$$

Integrate and evaluate over the interval.

$$A = \frac{6t^3}{3} + \frac{12t^2}{2}\Big|_0^3$$

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$$A = 2t^3 + 6t^2 \Big|_0^3$$

$$A = \left(2(3)^3 + 6(3)^2\right) - \left(2(0)^3 + 6(0)^2\right)$$

$$A = 54 + 54$$

$$A = 108$$

■ 2. Find the area under the parametric curve.

$$x(t) = 5t^2 - 3t + 4$$

$$y(t) = 6t - 1$$

$$0 \le t \le 5$$

Solution:

Find the area under the curve on $a \le t \le b$ using the integral formula.

$$A = \int_a^b y(t)x'(t)\ dt$$

$$A = \int_0^5 (6t - 1)(10t - 3)\ dt$$

$$A = \int_0^5 60t^2 - 28t + 3\ dt$$

Integrate and evaluate over the interval.

$$A = \frac{60t^3}{3} - \frac{28t^2}{2} + 3t \Big|_0^5$$

$$A = 20t^3 - 14t^2 + 3t \Big|_0^5$$

$$A = \left(20(5)^3 - 14(5)^2 + 3(5)\right) - \left(20(0)^3 - 14(0)^2 + 3(0)\right)$$

$$A = 2{,}500 - 350 + 15$$

$$A = 2{,}165$$

■ 3. Find the area under the parametric curve.

$$x(t) = t + \sin t$$

$$y(t) = 4 + \cos t$$

$$0 \le t \le 2\pi$$

Solution:

Find the area under the curve on $a \le t \le b$ using the integral formula.

$$A = \int_a^b y(t)x'(t)\ dt$$

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$$A = \int_0^{2\pi} (4 + \cos t)(1 + \cos t)\ dt$$

$$A = \int_0^{2\pi} 4 + 5\cos t + \cos^2 t\ dt$$

Use the trig identity to make a substitution.

$$\cos^2 t = \frac{1}{2} + \frac{1}{2}\cos(2t)$$

Substitute.

$$A = \int_0^{2\pi} 4 + 5\cos t + \frac{1}{2} + \frac{1}{2}\cos(2t)\ dt$$

$$A = \int_0^{2\pi} \frac{9}{2} + 5\cos t + \frac{1}{2}\cos(2t)\ dt$$

Integrate and evaluate over the interval.

$$A = \frac{9}{2}t + 5\sin t + \frac{1}{4}\sin(2t)\Big|_0^{2\pi}$$

$$A = \frac{9}{2}(2\pi) + 5\sin(2\pi) + \frac{1}{4}\sin(2(2\pi)) - \left(\frac{9}{2}(0) + 5\sin(0) + \frac{1}{4}\sin(2(0))\right)$$

$$A = 9\pi + 5(0) + \frac{1}{4}(0)$$

$$A = 9\pi$$

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4. Find the area under the parametric curve.

$$x(t) = t^2 + 5t - 8$$

$$y(t) = t^2 + 4t + 2$$

$$0 \le t \le 2$$

Solution:

Find the area under the curve on $a \le t \le b$ using the integral formula.

$$A = \int_a^b y(t)x'(t)\ dt$$

$$A = \int_0^2 (t^2 + 4t + 2)(2t + 5)\ dt$$

$$A = \int_0^2 2t^3 + 13t^2 + 24t + 10\ dt$$

Integrate and evaluate over the interval.

$$A = \frac{2t^4}{4} + \frac{13t^3}{3} + \frac{24t^2}{2} - 10t\Big|_0^2$$

$$A = \frac{t^4}{2} + \frac{13t^3}{3} + 12t^2 + 10t\Big|_0^2$$

$$A = \left(\frac{2^4}{2} + \frac{13(2)^3}{3} + 12(2)^2 + 10(2)\right) - \left(\frac{0^4}{2} + \frac{13(0)^3}{3} + 12(0)^2 + 10(0)\right)$$

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$$A = 8 + \frac{104}{3} + 48 + 20$$

$$A = \frac{104}{3} + 76$$

$$A = \frac{104}{3} + \frac{228}{3}$$

$$A = \frac{332}{3}$$

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AREA UNDER ONE ARC OR LOOP

1. Find the area in one loop of the parametric curve.

$$x(\theta) = 2\cos(2\theta)$$

$$y(\theta) = 4 + \sin(2\theta)$$

$$0 \le \theta \le \pi$$

Solution:

Plug the parametric equation and the given interval into the integral formula for the area under one arc or in one loop of the parametric curve.

$$A = \int_a^b y(\theta) \cdot x'(t)\ d\theta$$

$$A = \int_0^\pi (4 + \sin(2\theta))(-4\sin(2\theta))\ d\theta$$

$$A = \int_0^\pi -16\sin(2\theta) - 4\sin^2(2\theta)\ d\theta$$

Use the reduction formula

$$\sin^2(2\theta) = \frac{1}{2}(1 - \cos(4\theta)) = \frac{1}{2} - \frac{1}{2}\cos(4\theta)$$

Substitute that into the integral.

$$A=\int_0^{\pi}-16\sin(2\theta)-4\left(\frac{1}{2}-\frac{1}{2}\cos(4\theta)\right)\ d\theta$$

$$A=\int_0^{\pi}-16\sin(2\theta)-2+2\cos(4\theta)\ d\theta$$

Integrate, then evaluate over the interval.

$$A=-16\left(-\frac{\cos(2\theta)}{2}\right)-2\theta+2\left(\frac{\sin(4\theta)}{4}\right)\Bigg|_0^{\pi}$$

$$A=8\cos(2\theta)-2\theta+\frac{1}{2}\sin(4\theta)\Bigg|_0^{\pi}$$

$$A=8\cos(2\pi)-2\pi+\frac{1}{2}\sin(4\pi)-\left(8\cos(2(0))-2(0)+\frac{1}{2}\sin(4(0))\right)$$

$$A=8(1)-2\pi+\frac{1}{2}(0)-8(1)+0-\frac{1}{2}(0)$$

$$A=8-2\pi-8$$

$$A=-2\pi$$

$$A=\left|-2\pi\right|$$

$$A=2\pi$$

■ 2. Find the area in one loop of the parametric curve.

$$x(\theta)=2\sin\theta$$

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$$y(\theta) = 5 + \cos\theta$$

$$0 \le \theta \le 2\pi$$

Solution:

Plug the parametric equation and the given interval into the integral formula for the area under one arc or in one loop of the parametric curve.

$$A = \int_a^b y(\theta) \cdot x'(t)\ d\theta$$

$$A = \int_0^{2\pi} (5 + \cos\theta)(2\cos\theta)\ d\theta$$

$$A = \int_0^{2\pi} 10\cos\theta + 2\cos^2\theta\ d\theta$$

Use the reduction formula

$$\cos^2\theta = \frac{1}{2}(1 + \cos(2\theta)) = \frac{1}{2} + \frac{1}{2}\cos(2\theta)$$

Substitute that into the integral.

$$A = \int_0^{2\pi} 10\cos\theta + 2\left(\frac{1}{2} + \frac{1}{2}\cos(2\theta)\right)\ d\theta$$

$$A = \int_0^{2\pi} 10\cos\theta + 1 + \cos(2\theta)\ d\theta$$

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Integrate, then evaluate over the interval.

$$A = 10\sin\theta + \theta + \frac{\sin(2\theta)}{2}\Big|_0^{2\pi}$$

$$A = 10\sin(2\pi) + 2\pi + \frac{\sin(2(2\pi))}{2} - \left(10\sin(0) + 0 + \frac{\sin(2(0))}{2}\right)$$

$$A = 10(0) + 2\pi + \frac{0}{2} - 10(0) - 0 - \frac{0}{2}$$

$$A = 2\pi$$

3. Find the area in one loop of the parametric curve.

$$x(\theta) = 8 + 3\cos\theta$$

$$y(\theta) = 9 - 2\sin\theta$$

$$0 \le \theta \le 2\pi$$

Solution:

Plug the parametric equation and the given interval into the integral formula for the area under one arc or in one loop of the parametric curve.

$$A = \int_a^b y(\theta) \cdot x'(t)\ d\theta$$

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$$A=\int_0^{2\pi}(9-2\sin\theta)(-3\sin\theta)\ d\theta$$

$$A=\int_0^{2\pi}-27\sin\theta+6\sin^2\theta\ d\theta$$

Use the reduction formula

$$\sin^2(2\theta)=\frac{1}{2}(1-\cos(4\theta))=\frac{1}{2}-\frac{1}{2}\cos(4\theta)$$

Substitute that into the integral.

$$A=\int_0^{2\pi}-27\sin\theta+6\left(\frac{1}{2}-\frac{1}{2}\cos(2\theta)\right)\ d\theta$$

$$A=\int_0^{2\pi}-27\sin\theta+3-3\cos(2\theta)\ d\theta$$

Integrate, then evaluate over the interval.

$$A=27\cos\theta+3\theta-\frac{3}{2}\sin(2\theta)\Big|_0^{2\pi}$$

$$A=27\cos(2\pi)+3(2\pi)-\frac{3}{2}\sin(2(2\pi))-\left(27\cos(0)+3(0)-\frac{3}{2}\sin(2(0))\right)$$

$$A=27(1)+6\pi-\frac{3}{2}(0)-27(1)-0+\frac{3}{2}(0)$$

$$A=27+6\pi-27$$

$$A=6\pi$$

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4. Find the area in one loop of the parametric curve.

$$x(\theta) = 12 + 6\sin\theta$$

$$y(\theta) = 12 - 6\cos\theta$$

$$0 \le \theta \le 2\pi$$

Solution:

Plug the parametric equation and the given interval into the integral formula for the area under one arc or in one loop of the parametric curve.

$$A = \int_a^b y(\theta) \cdot x'(t)\ d\theta$$

$$A = \int_0^{2\pi} (12 - 6\cos\theta)(6\cos\theta)\ d\theta$$

$$A = \int_0^{2\pi} 72\cos\theta - 36\cos^2\theta\ d\theta$$

Use the reduction formula

$$\cos^2\theta = \frac{1}{2}(1 + \cos(2\theta)) = \frac{1}{2} + \frac{1}{2}\cos(2\theta)$$

Substitute that into the integral.

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$$A=\int_0^{2\pi} 72\cos\theta-36\left(\frac{1}{2}+\frac{1}{2}\cos(2\theta)\right)\ d\theta$$

$$A=\int_0^{2\pi} 72\cos\theta-18-18\cos(2\theta)\ d\theta$$

Integrate, then evaluate over the interval.

$$A=72\sin\theta-18\theta-9\sin(2\theta)\Big|_0^{2\pi}$$

$$A=72\sin(2\pi)-18(2\pi)-9\sin(2(2\pi))-\big(72\sin(0)-18(0)-9\sin(2(0))\big)$$

$$A=72(0)-36\pi-9(0)-72(0)+0+9(0)$$

$$A=-36\pi$$

$$A=\left|-36\pi\right|$$

$$A=36\pi$$

■ 5. Find the area in one loop of the parametric curve.

$$x(\theta)=15-5\cos\theta$$

$$y(\theta)=5+15\sin\theta$$

$$0\le\theta\le 2\pi$$

Solution:

Plug the parametric equation and the given interval into the integral formula for the area under one arc or in one loop of the parametric curve.

$$A=\int_a^b y(\theta)\cdot x'(t)\ d\theta$$

$$A=\int_0^{2\pi}(5+15\sin\theta)(5\sin\theta)\ d\theta$$

$$A=\int_0^{2\pi}25\sin\theta+75\sin^2\theta\ d\theta$$

Use the reduction formula

$$\sin^2(2\theta)=\frac{1}{2}(1-\cos(4\theta))=\frac{1}{2}-\frac{1}{2}\cos(4\theta)$$

Substitute that into the integral.

$$A=\int_0^{2\pi}25\sin\theta+75\left(\frac{1}{2}-\frac{1}{2}\cos(2\theta)\right)\ d\theta$$

$$A=\int_0^{2\pi}25\sin\theta+\frac{75}{2}-\frac{75}{2}\cos(2\theta)\ d\theta$$

Integrate, then evaluate over the interval.

$$A=-25\cos\theta+\frac{75}{2}\theta-\frac{75}{4}\sin(2\theta)\Big|_0^{2\pi}$$

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$$A = -25\cos(2\pi) + \frac{75}{2}(2\pi) - \frac{75}{4}\sin(2(2\pi)) - \left(-25\cos(0) + \frac{75}{2}(0) - \frac{75}{4}\sin(2(0))\right)$$

$$A = -25(1) + 75\pi - \frac{75}{4}(0) + 25(1) - 0 + \frac{75}{4}(0)$$

$$A = -25 + 75\pi + 25$$

$$A = 75\pi$$

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ARC LENGTH OF PARAMETRIC CURVES

1. Find the length of the parametric curve on the given interval.

$$x(t) = 7 - 3t$$

$$y(t) = 5 + 8t$$

$$-1 \le t \le 4$$

Solution:

Plug the derivatives of $x(t)$ and $y(t)$ and the given interval into the integral formula for arc length.

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\ dt$$

$$L = \int_{-1}^{4} \sqrt{(-3)^2 + (8)^2}\ dt$$

$$L = \int_{-1}^{4} \sqrt{9 + 64}\ dt$$

$$L = \int_{-1}^{4} \sqrt{73}\ dt$$

Integrate, then evaluate over the interval.

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$$L = \sqrt{73}t \Big|_{-1}^{4}$$

$$L = \sqrt{73}(4) - \sqrt{73}(-1)$$

$$L = 4\sqrt{73} + \sqrt{73}$$

$$L = 5\sqrt{73}$$

■ 2. Find the length of the parametric curve on the given interval.

$$x(t) = \cos^3 t$$

$$y(t) = \sin^3 t$$

$$0 \le t \le \frac{3\pi}{4}$$

Solution:

Plug the derivatives of $x(t)$ and $y(t)$ and the given interval into the integral formula for arc length.

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\ dt$$

$$L = \int_0^{\frac{3\pi}{4}} \sqrt{\left(3\cos^2 t(-\sin t)\right)^2 + \left(3\sin^2 t\cos t\right)^2}\ dt$$

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$$L=\int_0^{\frac{3\pi}{4}}\sqrt{9\cos^4 t\sin^2 t+9\sin^4 t\cos^2 t}\ dt$$

$$L=\int_0^{\frac{3\pi}{4}}\sqrt{9\sin^2 t\cos^2 t(\cos^2 t+\sin^2 t)}\ dt$$

$$L=\int_0^{\frac{3\pi}{4}}\sqrt{9\sin^2 t\cos^2 t(1)}\ dt$$

$$L=\int_0^{\frac{3\pi}{4}}\sqrt{9\sin^2 t\cos^2 t}\ dt$$

$$L=\int_0^{\frac{3\pi}{4}}3\sin t\cos t\ dt$$

Use u-substitution.

$$u=\sin t$$

$du=\cos t\ dt$, so $dt=\dfrac{du}{\cos t}$

Substitute.

$$L=\int_{t=0}^{t=\frac{3\pi}{4}}3u\cos t\left(\frac{du}{\cos t}\right)$$

$$L=\int_{t=0}^{t=\frac{3\pi}{4}}3u\ du$$

Integrate, back-substitute, then evaluate over the interval.

$$L=\frac{3}{2}u^2\Big|_{t=0}^{t=\frac{3\pi}{4}}$$

$$L=\frac{3}{2}\sin^2 t\Big|_0^{\frac{3\pi}{4}}$$

$$L=\frac{3}{2}\sin^2\frac{3\pi}{4}-\frac{3}{2}\sin^2(0)$$

$$L=\frac{3}{2}\left(\frac{\sqrt{2}}{2}\right)^2-\frac{3}{2}(0)^2$$

$$L=\frac{3}{2}\left(\frac{2}{4}\right)$$

$$L=\frac{3}{4}$$

■ 3. Find the length of the parametric curve on the given interval.

$$x(t)=5t-5\sin t$$

$$y(t)=-5\cos t$$

$$0\le t\le 2\pi$$

Solution:

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Plug the derivatives of $x(t)$ and $y(t)$ and the given interval into the integral formula for arc length.

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\ dt$$

$$L = \int_0^{2\pi} \sqrt{(5 - 5\cos t)^2 + (5\sin t)^2}\ dt$$

$$L = \int_0^{2\pi} \sqrt{25 - 50\cos t + 25\cos^2 t + 25\sin^2 t}\ dt$$

$$L = \int_0^{2\pi} \sqrt{25 - 50\cos t + 25(\cos^2 t + \sin^2 t)}\ dt$$

$$L = \int_0^{2\pi} \sqrt{25 - 50\cos t + 25(1)}\ dt$$

$$L = \int_0^{2\pi} \sqrt{50 - 50\cos t}\ dt$$

$$L = \int_0^{2\pi} \sqrt{50(1 - \cos t)}\ dt$$

$$L = \int_0^{2\pi} \sqrt{100 \cdot \frac{1}{2}(1 - \cos t)}\ dt$$

Use the reduction formula

$$\sin^2\theta = \frac{1}{2}(1 - \cos(2\theta)) = \frac{1}{2} - \frac{1}{2}\cos(2\theta)$$

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Substitute.

$$L = \int_0^{2\pi} \sqrt{100 \sin^2 \frac{t}{2}}\ dt$$

$$L = \int_0^{2\pi} 10 \sin \frac{t}{2}\ dt$$

Integrate, then evaluate over the interval.

$$L = -20 \cos \frac{t}{2} \Big|_0^{2\pi}$$

$$L = -20 \cos \frac{2\pi}{2} - \left(-20 \cos \frac{0}{2}\right)$$

$$L = -20 \cos \pi - (-20 \cos 0)$$

$$L = -20(-1) + 20(1)$$

$$L = 20 + 20$$

$$L = 40$$

■ 4. Find the length of the parametric curve on the given interval.

$$x(t) = \cos t$$

$$y(t) = t + \sin t$$

$$0 \le t \le \pi$$

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Solution:

Plug the derivatives of $x(t)$ and $y(t)$ and the given interval into the integral formula for arc length.

$$L=\int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\ dt$$

$$L=\int_0^\pi \sqrt{(-\sin t)^2+(1+\cos t)^2}\ dt$$

$$L=\int_0^\pi \sqrt{\sin^2 t+1+2\cos t+\cos^2 t}\ dt$$

$$L=\int_0^\pi \sqrt{1+2\cos t+(\sin^2 t+\cos^2 t)}\ dt$$

$$L=\int_0^\pi \sqrt{1+2\cos t+(1)}\ dt$$

$$L=\int_0^\pi \sqrt{2+2\cos t}\ dt$$

$$L=\int_0^\pi \sqrt{2(1+\cos t)}\ dt$$

$$L=\int_0^\pi \sqrt{4\cdot\frac{1}{2}(1+\cos t)}\ dt$$

Use the reduction formula

$$\cos^2\theta = \frac{1}{2}(1+\cos(2\theta)) = \frac{1}{2} + \frac{1}{2}\cos(2\theta)$$

Substitute.

$$L = \int_0^{\pi} \sqrt{4\cos^2\frac{t}{2}}\ dt$$

$$L = \int_0^{\pi} 2\cos\frac{t}{2}\ dt$$

Integrate, then evaluate over the interval.

$$L = 4\sin\frac{t}{2}\Big|_0^{\pi}$$

$$L = 4\sin\frac{\pi}{2} - 4\sin\frac{0}{2}$$

$$L = 4(1) - 4(0)$$

$$L = 4$$

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SURFACE AREA OF REVOLUTION, HORIZONTAL AXIS

■ 1. Find the surface area of revolution of the parametric curve on the interval $0 \le t \le 3$, rotated about the x-axis.

$$x = \frac{5}{3}t$$

$$y = 4t + 6$$

Solution:

Plug the derivatives and the interval into the integral formula formula for the surface area of revolution for a parametric curve about the x-axis.

$$S = \int_a^b 2\pi y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\ dt$$

$$S = \int_0^3 2\pi(4t+6)\sqrt{\left(\frac{5}{3}\right)^2 + (4)^2}\ dt$$

$$S = 4\pi \int_0^3 (2t+3)\sqrt{\frac{25}{9} + 16}\ dt$$

$$S = 4\pi \int_0^3 (2t+3)\sqrt{\frac{25}{9} + \frac{144}{9}}\ dt$$

$$S = 4\pi \int_0^3 (2t+3)\sqrt{\frac{169}{9}}\ dt$$

$$S = 4\pi \int_0^3 (2t+3)\frac{13}{3}\ dt$$

$$S = \frac{52\pi}{3} \int_0^3 2t+3\ dt$$

Integrate, then evaluate over the interval.

$$S = \frac{52\pi}{3}(t^2+3t)\Big|_0^3$$

$$S = \frac{52\pi}{3}(3^2+3(3)) - \frac{52\pi}{3}(0^2+3(0))$$

$$S = \frac{52\pi}{3}(18) - \frac{52\pi}{3}(0)$$

$$S = 312\pi$$

■ 2. Find the surface area of revolution of the parametric curve on the interval $0 \le t \le 2\pi$, rotated about the x-axis.

$$x = 3 + \cos t$$

$$y = 4 + \sin t$$

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Solution:

Plug the derivatives and the interval into the integral formula formula for the surface area of revolution for a parametric curve about the x-axis.

$$S = \int_a^b 2\pi y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\ dt$$

$$S = \int_0^{2\pi} 2\pi(4 + \sin t)\sqrt{(-\sin t)^2 + (\cos t)^2}\ dt$$

$$S = \int_0^{2\pi} 2\pi(4 + \sin t)\sqrt{\sin^2 t + \cos^2 t}\ dt$$

$$S = 2\pi \int_0^{2\pi} (4 + \sin t)\sqrt{1}\ dt$$

$$S = 2\pi \int_0^{2\pi} 4 + \sin t\ dt$$

Integrate, then evaluate over the interval.

$$S = 2\pi(4t - \cos t)\Big|_0^{2\pi}$$

$$S = 2\pi(4(2\pi) - \cos(2\pi)) - 2\pi(4(0) - \cos(0))$$

$$S = 2\pi(8\pi - 1) - 2\pi(0 - 1)$$

$$S = 16\pi^2 - 2\pi + 2\pi$$

$$S = 16\pi^2$$

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3. Find the surface area of revolution of the parametric curve on the interval $0 \leq t \leq 2\pi$, rotated about the x-axis.

$$x = 7 - 3\sin t$$

$$y = 6 + 3\cos t$$

Solution:

Plug the derivatives and the interval into the integral formula formula for the surface area of revolution for a parametric curve about the x-axis.

$$S = \int_a^b 2\pi y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\ dt$$

$$S = \int_0^{2\pi} 2\pi(6 + 3\cos t)\sqrt{(-3\cos t)^2 + (-3\sin t)^2}\ dt$$

$$S = 6\pi \int_0^{2\pi} (2 + \cos t)\sqrt{9\cos^2 t + 9\sin^2 t}\ dt$$

$$S = 6\pi \int_0^{2\pi} (2 + \cos t)\sqrt{9(\cos^2 t + \sin^2 t)}\ dt$$

$$S = 6\pi \int_0^{2\pi} (2 + \cos t)\sqrt{9(1)}\ dt$$

$$S = 6\pi \int_0^{2\pi} (2 + \cos t)(3)\ dt$$

$$S = 18\pi \int_0^{2\pi} 2 + \cos t\ dt$$

Integrate, then evaluate over the interval.

$$S = 18\pi(2t + \sin t) \Big|_0^{2\pi}$$

$$S = 18\pi(2(2\pi) + \sin(2\pi)) - 18\pi(2(0) + \sin(0))$$

$$S = 18\pi(4\pi + 0) - 18\pi(0 + 0)$$

$$S = 18\pi(4\pi)$$

$$S = 72\pi^2$$

■ 4. Find the surface area of revolution of the parametric curve on the interval $0 \le t \le \pi$, rotated about the x-axis.

$$x = 5 - \cos(2t)$$

$$y = 3 + \sin(2t)$$

Solution:

Plug the derivatives and the interval into the integral formula formula for the surface area of revolution for a parametric curve about the x-axis.

$$S=\int_a^b 2\pi y\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\ dt$$

$$S=\int_0^\pi 2\pi(3+\sin(2t))\sqrt{(2\sin(2t))^2+(2\cos(2t))^2}\ dt$$

$$S=2\pi\int_0^\pi (3+\sin(2t))\sqrt{4\sin^2(2t)+4\cos^2(2t)}\ dt$$

$$S=2\pi\int_0^\pi (3+\sin(2t))\sqrt{4(\sin^2(2t)+\cos^2(2t))}\ dt$$

$$S=2\pi\int_0^\pi (3+\sin(2t))\sqrt{4(1)}\ dt$$

$$S=4\pi\int_0^\pi 3+\sin(2t)\ dt$$

Integrate, then evaluate over the interval.

$$S=4\pi\left(3t-\frac{1}{2}\cos(2t)\right)\Big|_0^\pi$$

$$S=4\pi\left(3\pi-\frac{1}{2}\cos(2\pi)\right)-4\pi\left(3(0)-\frac{1}{2}\cos(2(0))\right)$$

$$S=4\pi\left(3\pi-\frac{1}{2}(1)\right)-4\pi\left(0-\frac{1}{2}(1)\right)$$

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$$S = 12\pi^2 - 2\pi + 2\pi$$

$$S = 12\pi^2$$

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SURFACE AREA OF REVOLUTION, VERTICAL AXIS

1. Find the surface area of revolution of the parametric curve on the interval $0 \le t \le \pi/3$, rotated about the y-axis.

$$x = 8 + \sin(6t)$$

$$y = 7 - \cos(6t)$$

Solution:

Plug the derivatives and the interval into the integral formula formula for the surface area of revolution for a parametric curve about the y-axis.

$$S = \int_a^b 2\pi x \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\ dt$$

$$S = \int_0^{\frac{\pi}{3}} 2\pi(8 + \sin(6t))\sqrt{(6\cos(6t))^2 + (6\sin(6t))^2}\ dt$$

$$S = 2\pi \int_0^{\frac{\pi}{3}} (8 + \sin(6t))\sqrt{36\cos^2(6t) + 36\sin^2(6t)}\ dt$$

$$S = 2\pi \int_0^{\frac{\pi}{3}} (8 + \sin(6t))\sqrt{36(\cos^2(6t) + \sin^2(6t))}\ dt$$

$$S = 2\pi \int_0^{\frac{\pi}{3}} (8 + \sin(6t))\sqrt{36(1)}\ dt$$

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$$S = 12\pi \int_0^{\frac{\pi}{3}} 8 + \sin(6t)\ dt$$

Integrate, then evaluate over the interval.

$$S = 12\pi \left(8t - \frac{1}{6}\cos(6t) \right) \Big|_0^{\frac{\pi}{3}}$$

$$S = 12\pi \left(8 \cdot \frac{\pi}{3} - \frac{1}{6}\cos\left(6 \cdot \frac{\pi}{3}\right) \right) - 12\pi \left(8(0) - \frac{1}{6}\cos(6(0)) \right)$$

$$S = 12\pi \left(\frac{8\pi}{3} - \frac{1}{6}\cos(2\pi) \right) - 12\pi \left(0 - \frac{1}{6}\cos(0) \right)$$

$$S = 12\pi \left(\frac{8\pi}{3} - \frac{1}{6}(1) \right) - 12\pi \left(0 - \frac{1}{6}(1) \right)$$

$$S = 32\pi^2 - 2\pi + 2\pi$$

$$S = 32\pi^2$$

■ 2. Find the surface area of revolution of the parametric curve on the interval $0 \le t \le 2\pi$, rotated about the y-axis.

$$x = 5 + 4\sin(t)$$

$$y = 5 + 4\cos(t)$$

Solution:

Plug the derivatives and the interval into the integral formula formula for the surface area of revolution for a parametric curve about the y-axis.

$$S=\int_a^b 2\pi x\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\ dt$$

$$S=\int_0^{2\pi} 2\pi(5+4\sin t)\sqrt{(4\cos t)^2+(-4\sin t)^2}\ dt$$

$$S=2\pi\int_0^{2\pi} (5+4\sin t)\sqrt{16\cos^2 t+16\sin^2 t}\ dt$$

$$S=2\pi\int_0^{2\pi} (5+4\sin t)\sqrt{16(\cos^2 t+16\sin^2 t)}\ dt$$

$$S=2\pi\int_0^{2\pi} (5+4\sin t)\sqrt{16(1)}\ dt$$

$$S=8\pi\int_0^{2\pi} 5+4\sin t\ dt$$

Integrate, then evaluate over the interval.

$$S=8\pi(5t-4\cos t)\Big|_0^{2\pi}$$

$$S=8\pi(5(2\pi)-4\cos(2\pi))-8\pi(5(0)-4\cos(0))$$

$$S=8\pi(10\pi-4(1))-8\pi(0-4(1))$$

$$S=80\pi^2-32\pi+32\pi$$

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$$S = 80\pi^2$$

■ 3. Find the surface area of revolution of the parametric curve on the interval $0 \le t \le 2\pi$, rotated about the y-axis.

$$x = 12 - \sin t$$

$$y = 2 + \cos t$$

Solution:

Plug the derivatives and the interval into the integral formula formula for the surface area of revolution for a parametric curve about the y-axis.

$$S = \int_a^b 2\pi x \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \ dt$$

$$S = \int_0^{2\pi} 2\pi(12 - \sin t)\sqrt{(-\cos t)^2 + (-\sin t)^2} \ dt$$

$$S = \int_0^{2\pi} 2\pi(12 - \sin t)\sqrt{\cos^2 t + \sin^2 t} \ dt$$

$$S = 2\pi \int_0^{2\pi} (12 - \sin t)\sqrt{1} \ dt$$

$$S = 2\pi \int_0^{2\pi} 12 - \sin t \ dt$$

Integrate, then evaluate over the interval.

$$S = 2\pi(12t + \cos t)\Big|_0^{2\pi}$$

$$S = 2\pi(12(2\pi) + \cos(2\pi)) - 2\pi(12(0) + \cos(0))$$

$$S = 2\pi(24\pi + 1) - 2\pi(0 + 1)$$

$$S = 48\pi^2 + 2\pi - 2\pi$$

$$S = 48\pi^2$$

■ 4. Find the surface area of revolution of the parametric curve on the interval $0 \le t \le \pi$, rotated about the y-axis.

$$x = 4 - 3\sin(2t)$$

$$y = 4 - 3\cos(2t)$$

Solution:

Plug the derivatives and the interval into the integral formula formula for the surface area of revolution for a parametric curve about the y-axis.

$$S = \int_a^b 2\pi x\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\ dt$$

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$$S=\int_0^{\pi}2\pi(4-3\sin(2t))\sqrt{\left(-6\cos(2t)\right)^2+\left(6\sin(2t)\right)^2}\ dt$$

$$S=2\pi\int_0^{\pi}(4-3\sin(2t))\sqrt{36\cos^2(2t)+36\sin^2(2t)}\ dt$$

$$S=2\pi\int_0^{\pi}(4-3\sin(2t))\sqrt{36(\cos^2(2t)+\sin^2(2t))}\ dt$$

$$S=2\pi\int_0^{\pi}(4-3\sin(2t))\sqrt{36(1)}\ dt$$

$$S=12\pi\int_0^{\pi}4-3\sin(2t)\ dt$$

Integrate, then evaluate over the interval.

$$S=12\pi\left(4t+\frac{3}{2}\cos(2t)\right)\Big|_0^{\pi}$$

$$S=12\pi\left(4\pi+\frac{3}{2}\cos(2\pi)\right)-12\pi\left(4(0)+\frac{3}{2}\cos(2(0))\right)$$

$$S=12\pi\left(4\pi+\frac{3}{2}(1)\right)-12\pi\left(0+\frac{3}{2}(1)\right)$$

$$S=12\pi\left(4\pi+\frac{3}{2}\right)-12\pi\left(\frac{3}{2}\right)$$

$$S=48\pi^2+18\pi-18\pi$$

$$S=48\pi^2$$

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5. Find the surface area of revolution of the parametric curve on the interval $0 \leq t \leq 4$, rotated about the y-axis.

$$x = 6t + 5$$

$$y = 8t + 7$$

Solution:

Plug the derivatives and the interval into the integral formula formula for the surface area of revolution for a parametric curve about the y-axis.

$$S = \int_a^b 2\pi x \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\ dt$$

$$S = \int_0^4 2\pi(6t + 5)\sqrt{(6)^2 + (8)^2}\ dt$$

$$S = 2\pi \int_0^4 (6t + 5)\sqrt{36 + 64}\ dt$$

$$S = 2\pi \int_0^4 (6t + 5)\sqrt{100}\ dt$$

$$S = 20\pi \int_0^4 6t + 5\ dt$$

Integrate, then evaluate over the interval.

$$S = 20\pi(3t^2 + 5t)\Big|_0^4$$

$$S = 20\pi(3(4)^2 + 5(4)) - 20\pi(3(0)^2 + 5(0))$$

$$S = 20\pi(3(16) + 20) - 20\pi(0 + 0)$$

$$S = 20\pi(68)$$

$$S = 1{,}360\pi$$

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VOLUME OF REVOLUTION, PARAMETRIC CURVES

■ 1. Find the volume of revolution of the parametric curve, rotated about the x-axis, over the interval $1 \le t \le 2$.

$$x(t) = 2t^2$$

$$y(t) = 4t^2$$

Solution:

Plug the interval and the parametric equation into the integral formula for volume of revolution for a parametric equation around the x-axis.

$$V_x = \int_a^b \pi y^2 \frac{dx}{dt}\ dt$$

$$V_x = \int_1^2 \pi(4t^2)^2(4t)\ dt$$

$$V_x = 64\pi \int_1^2 t^5\ dt$$

Integrate, then evaluate over the interval.

$$V_x = 64\pi \left(\frac{1}{6}t^6\right)\Bigg|_1^2$$

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$$V_x = \frac{32\pi}{3}t^6\Big|_1^2$$

$$V_x = \frac{32\pi}{3}(2)^6 - \frac{32\pi}{3}(1)^6$$

$$V_x = \frac{32\pi}{3}(64) - \frac{32\pi}{3}$$

$$V_x = \frac{32\pi}{3}(64 - 1)$$

$$V_x = \frac{32\pi}{3}(63)$$

$$V_x = 32\pi(21)$$

$$V_x = 672\pi$$

■ 2. Find the volume of revolution of the parametric curve, rotated about the y-axis, over the interval $1 \leq t \leq 3$.

$$x(t) = 3t$$

$$y(t) = 4t^2$$

Solution:

Plug the interval and the parametric equation into the integral formula for volume of revolution for a parametric equation around the y-axis.

$$V_y = \int_a^b \pi x^2 \frac{dy}{dt}\ dt$$

$$V_y = \int_1^3 \pi(3t)^2(8t)\ dt$$

$$V_y = 72\pi \int_1^3 t^3\ dt$$

Integrate, then evaluate over the interval.

$$V_y = 72\pi \left(\frac{1}{4}t^4\right)\Big|_1^3$$

$$V_y = \frac{72\pi}{4}t^4\Big|_1^3$$

$$V_y = \frac{72\pi}{4}(3)^4 - \frac{72\pi}{4}(1)^4$$

$$V_y = \frac{72\pi}{4}(81 - 1)$$

$$V_y = \frac{72\pi}{4}(80)$$

$$V_y = 72\pi(20)$$

$$V_y = 1{,}440\pi$$

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3. Find the volume of revolution of the parametric curve, rotated about the x-axis, over the interval $1 \leq t \leq 3$.

$$x(t) = 2e^{2t} - 4t$$

$$y(t) = 6e^{\frac{5t}{2}}$$

Solution:

Plug the interval and the parametric equation into the integral formula for volume of revolution for a parametric equation around the x-axis.

$$V_x = \int_a^b \pi y^2 \frac{dx}{dt}\ dt$$

$$V_x = \int_1^3 \pi(6e^{\frac{5t}{2}})^2(4e^{2t} - 4)\ dt$$

$$V_x = 4\pi \int_1^3 (36e^{5t})(e^{2t} - 1)\ dt$$

$$V_x = 144\pi \int_1^3 e^{5t}e^{2t} - e^{5t}\ dt$$

$$V_x = 144\pi \int_1^3 e^{7t} - e^{5t}\ dt$$

Integrate, then evaluate over the interval.

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$$V_x = 144\pi \left(\frac{1}{7}e^{7t} - \frac{1}{5}e^{5t} \right) \Big|_1^3$$

$$V_x = 144\pi \left(\frac{1}{7}e^{7(3)} - \frac{1}{5}e^{5(3)} \right) - 144\pi \left(\frac{1}{7}e^{7(1)} - \frac{1}{5}e^{5(1)} \right)$$

$$V_x = 144\pi \left(\frac{1}{7}e^{21} - \frac{1}{5}e^{15} \right) - 144\pi \left(\frac{1}{7}e^{7} - \frac{1}{5}e^{5} \right)$$

$$V_x = 144\pi \left(\frac{1}{7}e^{21} - \frac{1}{5}e^{15} - \frac{1}{7}e^{7} + \frac{1}{5}e^{5} \right)$$

$$V_x = 144\pi \left(\frac{e^{21} - e^{7}}{7} - \frac{e^{15} - e^{5}}{5} \right)$$

■ 4. Find the volume of revolution of the parametric curve, rotated about the y-axis, over the interval $0 \le t \le 1$.

$$x(t) = 3e^t$$

$$y(t) = e^t$$

Solution:

Plug the interval and the parametric equation into the integral formula for volume of revolution for a parametric equation around the y-axis.

$$V_y = \int_a^b \pi x^2 \frac{dy}{dt} \ dt$$

$$V_y = \int_0^1 \pi(3e^t)^2(e^t)\ dt$$

$$V_y = 9\pi \int_0^1 (e^{2t})(e^t)\ dt$$

$$V_y = 9\pi \int_0^1 e^{3t}\ dt$$

Integrate, then evaluate over the interval.

$$V_y = 9\pi \left(\frac{1}{3}e^{3t}\right)\Bigg|_0^1$$

$$V_y = 3\pi \left(e^{3t}\right)\Big|_0^1$$

$$V_y = 3\pi \left(e^{3(1)}\right) - 3\pi \left(e^{3(0)}\right)$$

$$V_y = 3\pi(e^3) - 3\pi(1)$$

$$V_y = 3\pi(e^3 - 1)$$

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