nth term test

When the terms of a series decrease toward 0, we say that the series is converging. Otherwise, the series is diverging.

The nth term test is inspired by this idea, and we can use it to show that a series is diverging. Ironically, even though the nth term test is one of the *convergence* tests that we learn when we study sequences and series, it can only test for *divergence*, it can never confirm convergence.

The nth term test says that

$$\text{if } \lim_{n\to\infty} a_n \neq 0$$

$$\text{then } \sum a_n \text{ diverges}$$

In other words,

> If we take the limit as $n \to \infty$ and the result is **non-zero**, then the series **diverges**
>
> If we take the limit as $n \to \infty$ and the result is **zero**, then the test is **inconclusive**

Notice that the only conclusion we can draw is that the series diverges. It's possible that the series we're testing converges, but we can't use the nth term test to show convergence. It can only be used to show divergence, and if it doesn't prove divergence, then the test is inconclusive.

Example

Use the nth term test to show whether the series diverges.

$$\sum_{n=1}^{\infty} \frac{4n^3 - 4}{3n^3 + 2}$$

To use the nth term test we'll take the limit of the series as it approaches ∞.

If the result is non-zero, then the series diverges

If the result is zero, then the test is inconclusive

Taking the limit, we get

$$\lim_{n\to\infty} \frac{4n^3 - 4}{3n^3 + 2}$$

We'll simplify the limit by dividing each term in the fraction by the variable of the highest degree, n^3.

$$\lim_{n\to\infty} \frac{4n^3 - 4}{3n^3 + 2} \left(\frac{\frac{1}{n^3}}{\frac{1}{n^3}} \right)$$

$$\lim_{n\to\infty} \frac{\frac{4n^3}{n^3} - \frac{4}{n^3}}{\frac{3n^3}{n^3} + \frac{2}{n^3}}$$

$$\lim_{n\to\infty} \frac{4 - \frac{4}{n^3}}{3 + \frac{2}{n^3}}$$

Evaluating the limit at ∞, we get

$$\frac{4-\frac{4}{\infty^3}}{3+\frac{2}{\infty^3}}$$

When we have a fraction in which the numerator is constant and the denominator is infinite, the whole fraction approaches 0.

$$\frac{4-0}{3+0}$$

$$\frac{4}{3}$$

Since our answer is non-zero, the nth term test proves that the series diverges.