Tangent line to the polar curve

We'll find the equation of the tangent line to a polar curve in much the same way that we find the tangent line to a cartesian curve. We'll follow these steps:

1. Find the **slope** of the tangent line m, using the formula

$$m = \frac{dy}{dx} = \frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta}$$

   remembering to plug the value of θ at the tangent point into dy/dx to get a real-number value for the slope m.

2. **Find** x_1 **and** y_1 by plugging the value of θ at the tangent point into the conversion formulas

$$x = r\cos\theta$$

$$y = r\sin\theta$$

3. Plug the slope m and the point (x_1, y_1) into the **point-slope formula** for the equation of a line

$$y - y_1 = m(x - x_1)$$

Example

Find the tangent line to the polar curve at the given point.

$$r = 1 + 2\cos\theta$$

$$\text{at } \theta = \frac{\pi}{4}$$

We'll start by calculating $dr/d\theta$, the derivative of the given polar equation, so that we can plug it into the formula for the slope of the tangent line.

$$r = 1 + 2\cos\theta$$

$$\frac{dr}{d\theta} = -2\sin\theta$$

Plugging $dr/d\theta$ and the given polar equation $r = 1 + 2\cos\theta$ into the formula for dy/dx, we get

$$m = \frac{dy}{dx} = \frac{(-2\sin\theta)\sin\theta + (1 + 2\cos\theta)\cos\theta}{(-2\sin\theta)\cos\theta - (1 + 2\cos\theta)\sin\theta}$$

$$m = \frac{dy}{dx} = \frac{-2\sin^2\theta + \cos\theta + 2\cos^2\theta}{-2\sin\theta\cos\theta - \sin\theta - 2\sin\theta\cos\theta}$$

$$m = \frac{dy}{dx} = \frac{-2\sin^2\theta + \cos\theta + 2\cos^2\theta}{-4\sin\theta\cos\theta - \sin\theta}$$

Plugging the value of $\theta = \pi/4$ into the slope equation, we'll get a real-number value for the slope m.

$$m = \frac{dy}{dx} = \frac{-2\sin^2\frac{\pi}{4} + \cos\frac{\pi}{4} + 2\cos^2\frac{\pi}{4}}{-4\sin\frac{\pi}{4}\cos\frac{\pi}{4} - \sin\frac{\pi}{4}}$$

$$m=\frac{dy}{dx}=\frac{-2\left(\frac{\sqrt{2}}{2}\right)^2+\frac{\sqrt{2}}{2}+2\left(\frac{\sqrt{2}}{2}\right)^2}{-4\cdot\frac{\sqrt{2}}{2}\cdot\frac{\sqrt{2}}{2}-\frac{\sqrt{2}}{2}}$$

$$m=\frac{dy}{dx}=\frac{-2\left(\frac{2}{4}\right)+\frac{\sqrt{2}}{2}+2\left(\frac{2}{4}\right)}{-4\cdot\frac{2}{4}-\frac{\sqrt{2}}{2}}$$

$$m=\frac{dy}{dx}=\frac{-1+\frac{\sqrt{2}}{2}+1}{-2-\frac{\sqrt{2}}{2}}$$

$$m=\frac{dy}{dx}=\frac{\frac{\sqrt{2}}{2}}{-\frac{4}{2}-\frac{\sqrt{2}}{2}}$$

$$m=\frac{dy}{dx}=\frac{\frac{\sqrt{2}}{2}}{\frac{-4-\sqrt{2}}{2}}$$

$$m=\frac{dy}{dx}=\frac{\sqrt{2}}{2}\left(\frac{2}{-4-\sqrt{2}}\right)$$

$$m=\frac{dy}{dx}=\frac{\sqrt{2}}{-4-\sqrt{2}}$$

If we want to get rid of the square root in the denominator, we can multiply by the conjugate.

$$m=\frac{dy}{dx}=\frac{\sqrt{2}}{-4-\sqrt{2}}\left(\frac{-4+\sqrt{2}}{-4+\sqrt{2}}\right)$$

$$m=\frac{dy}{dx}=\frac{-4\sqrt{2}+2}{16-2}$$

$$m=\frac{dy}{dx}=\frac{-4\sqrt{2}+2}{14}$$

$$m=\frac{dy}{dx}=\frac{-2\sqrt{2}+1}{7}$$

$$m=\frac{dy}{dx}=\frac{1-2\sqrt{2}}{7}$$

Now we want to find x_1 and y_1 by plugging the value of θ at the tangent point and the given polar equation $r=1+2\cos\theta$ into the conversion formulas

$$x=r\cos\theta$$

$$x_1=\left(1+2\cos\frac{\pi}{4}\right)\cos\frac{\pi}{4}$$

$$x_1=\left[1+2\left(\frac{\sqrt{2}}{2}\right)\right]\frac{\sqrt{2}}{2}$$

$$x_1=\left(1+\sqrt{2}\right)\frac{\sqrt{2}}{2}$$

$$x_1 = \frac{\sqrt{2}+2}{2}$$

$$x_1 = \frac{2+\sqrt{2}}{2}$$

and

$$y = r\sin\theta$$

$$y_1 = \left(1 + 2\cos\frac{\pi}{4}\right)\sin\frac{\pi}{4}$$

$$y_1 = \left[1 + 2\left(\frac{\sqrt{2}}{2}\right)\right]\frac{\sqrt{2}}{2}$$

$$y_1 = \left(1+\sqrt{2}\right)\frac{\sqrt{2}}{2}$$

$$y_1 = \frac{\sqrt{2}+2}{2}$$

$$y_1 = \frac{2+\sqrt{2}}{2}$$

Plugging m and (x_1, y_1) into the point-slope formula for the equation of a line, we get

$$y - y_1 = m(x - x_1)$$

$$y - \frac{2+\sqrt{2}}{2} = \frac{1-2\sqrt{2}}{7}\left(x - \frac{2+\sqrt{2}}{2}\right)$$

$$y-\frac{2+\sqrt{2}}{2}=\frac{1-2\sqrt{2}}{7}x-\frac{2+\sqrt{2}-4\sqrt{2}-4}{14}$$

$$y-\frac{2+\sqrt{2}}{2}=\frac{1-2\sqrt{2}}{7}x-\frac{-3\sqrt{2}-2}{14}$$

$$y-\frac{2+\sqrt{2}}{2}=\frac{1-2\sqrt{2}}{7}x+\frac{3\sqrt{2}+2}{14}$$

$$2y-\left(2+\sqrt{2}\right)=\frac{2-4\sqrt{2}}{7}x+\frac{3\sqrt{2}+2}{7}$$

Eliminate the fractions by multiplying through by 7.

$$14y-7\left(2+\sqrt{2}\right)=\left(2-4\sqrt{2}\right)x+3\sqrt{2}+2$$

$$14y=\left(2-4\sqrt{2}\right)x+3\sqrt{2}+2+7\left(2+\sqrt{2}\right)$$

$$14y=\left(2-4\sqrt{2}\right)x+3\sqrt{2}+2+14+7\sqrt{2}$$

$$14y=\left(2-4\sqrt{2}\right)x+16+10\sqrt{2}$$

$$14y-\left(2-4\sqrt{2}\right)x=16+10\sqrt{2}$$

$$7y-\left(1-2\sqrt{2}\right)x=8+5\sqrt{2}$$

The equation of the tangent line is $7y-\left(1-2\sqrt{2}\right)x=8+5\sqrt{2}$.