Topic: Arc length of a polar curve

Question: Find the length of the polar curve on the given interval.

$$r = 5\theta^2$$

on the interval $0 \leq \theta \leq \sqrt{21}$

Answer choices:

A $\frac{585}{3}$

B $\frac{585\pi}{3}$

C 585

D 585π

Solution: A

The arc length for a polar curve is given by

$$L = \int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\ d\theta$$

where the limits of integration are $\alpha = 0$ and $\beta = \sqrt{21}$. Also, since $r = 5\theta^2$, then

$$\frac{dr}{d\theta} = 10\theta$$

So the length is

$$L = \int_0^{\sqrt{21}} \sqrt{\left(5\theta^2\right)^2 + (10\theta)^2}\ d\theta$$

$$L = \int_0^{\sqrt{21}} \sqrt{25\theta^4 + 100\theta^2}\ d\theta$$

$$L = \int_0^{\sqrt{21}} \sqrt{25\theta^2\left(\theta^2 + 4\right)}\ d\theta$$

$$L = 5\int_0^{\sqrt{21}} \theta\sqrt{\theta^2 + 4}\ d\theta$$

Letting

$$u = \theta^2 + 4$$

$$du = 2\theta\ d\theta$$

$$d\theta = \frac{du}{2\theta}$$

and making a substitution into our integral, we get

$$L = 5\int_{x=0}^{x=\sqrt{21}} \theta\sqrt{u}\ \frac{du}{2\theta}$$

$$L = \frac{5}{2}\int_{x=0}^{x=\sqrt{21}} \sqrt{u}\ du$$

$$L = \frac{5}{2}\left(\frac{2}{3}u^{\frac{3}{2}}\right)\Big|_{x=0}^{x=\sqrt{21}}$$

$$L = \frac{5}{3}(\theta^2+4)^{\frac{3}{2}}\Big|_{0}^{\sqrt{21}}$$

$$L = \frac{5}{3}\left[\left(\left(\sqrt{21}\right)^2+4\right)^{\frac{3}{2}} - ((0)^2+4)^{\frac{3}{2}}\right]$$

$$L = \frac{5}{3}\left[(25)^{\frac{3}{2}} - (4)^{\frac{3}{2}}\right]$$

$$L = \frac{5}{3}(125-8)$$

$$L = \frac{585}{3}$$

Topic: Arc length of a polar curve

Question: Find the length of the polar curve on the given interval.

$$r = \cos^3 \frac{\theta}{3}$$

on the interval $0 \leq \theta \leq \pi$

Answer choices:

A $\quad \pi + \dfrac{3\sqrt{3}}{8}$

B $\quad \dfrac{1}{2}\pi + \dfrac{3\sqrt{3}}{8}$

C $\quad \dfrac{1}{2}\pi - \dfrac{3\sqrt{3}}{8}$

D $\quad \pi - \dfrac{3\sqrt{3}}{8}$

Solution: B

The arc length of a polar curve on an interval is given by

$$L = \int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} \ d\theta$$

where $\alpha = 0$ and $\beta = \pi$. We'll find the derivative of the given polar equation so that we can plug it into the formula for arc length.

$$\frac{dr}{d\theta} = -\sin\frac{\theta}{3}\cos^2\frac{\theta}{3}$$

Plugging this into the arc length formula, we get

$$L = \int_0^{\pi} \sqrt{\left(\cos^3\frac{\theta}{3}\right)^2 + \left(-\sin\frac{\theta}{3}\cos^2\frac{\theta}{3}\right)^2} \ d\theta$$

$$L = \int_0^{\pi} \sqrt{\cos^6\frac{\theta}{3} + \sin^2\frac{\theta}{3}\cos^4\frac{\theta}{3}} \ d\theta$$

$$L = \int_0^{\pi} \sqrt{\cos^4\frac{\theta}{3}\left(\cos^2\frac{\theta}{3} + \sin^2\frac{\theta}{3}\right)} \ d\theta$$

Using the pythagorean identity

$$\sin^2 x + \cos^2 x = 1$$

we can simplify the integral to

$$L = \int_0^{\pi} \sqrt{\cos^4\frac{\theta}{3}(1)} \ d\theta$$

$$L=\int_0^{\pi}\cos^2\frac{\theta}{3}\ d\theta$$

Using the power reduction formula

$$\cos^2\theta=\frac{1}{2}+\frac{1}{2}\cos 2\theta$$

we get

$$L=\int_0^{\pi}\frac{1}{2}+\frac{1}{2}\cos\frac{2\theta}{3}\ d\theta$$

$$L=\frac{1}{2}\theta+\frac{3}{4}\sin\frac{2\theta}{3}\Bigg|_0^{\pi}$$

$$L=\left(\frac{1}{2}\pi+\frac{3}{4}\sin\frac{2\pi}{3}\right)-\left(\frac{1}{2}(0)+\frac{3}{4}\sin\frac{2(0)}{3}\right)$$

$$L=\frac{\pi}{2}+\frac{3}{4}\cdot\frac{\sqrt{3}}{2}$$

$$L=\frac{\pi}{2}+\frac{3\sqrt{3}}{8}$$

Topic: Arc length of a polar curve

Question: Find the length of the polar curve on the given interval.

$$r = 1 - \cos\theta$$

on the interval $0 \leq \theta \leq 2\pi$

Answer choices:

A 6

B 7

C 8

D 9

Solution: C

The arc length of a polar curve is given by

$$L = \int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\ d\theta$$

According to the question, $\alpha = 0$ and $\beta = 2\pi$. Let's find the derivative of the original equation so that we can plug everything into the arc length formula.

$$r = 1 - \cos\theta$$

$$\frac{dr}{d\theta} = \sin\theta$$

Plugging into the formula, we get

$$L = \int_0^{2\pi} \sqrt{(1 - \cos\theta)^2 + \left(\sin\theta\right)^2}\ d\theta$$

$$L = \int_0^{2\pi} \sqrt{1 - 2\cos\theta + \cos^2\theta + \sin^2\theta}\ d\theta$$

Knowing that $\sin^2 x + \cos^2 x = 1$, the integral simplifies to

$$L = \int_0^{2\pi} \sqrt{1 - 2\cos\theta + 1}\ d\theta$$

$$L = \int_0^{2\pi} \sqrt{2 - 2\cos\theta}\ d\theta$$

Using half-angle formulas, we can say that

$$2 - 2\cos\theta = 4\sin^2\frac{\theta}{2}$$

and therefore that

$$L = \int_0^{2\pi} \sqrt{4\sin^2\frac{\theta}{2}}\ d\theta$$

$$L = 2\int_0^{2\pi} \sin\frac{\theta}{2}\ d\theta$$

$$L = -4\cos\frac{\theta}{2}\Big|_0^{2\pi}$$

$$L = -4\cos\frac{2\pi}{2} - \left(-4\cos\frac{0}{2}\right)$$

$$L = -4\cos\pi + 4\cos 0$$

$$L = -4(-1) + 4(1)$$

$$L = 8$$