Topic: Absolute and conditional convergence

Question: Determine the convergence (absolute or conditional) of the series.

$$\sum_{n=1}^{\infty}\left(\frac{n}{2n+1}\right)^{n}$$

Answer choices:

A The series converges absolutely

B The series converges conditionally

C The series diverges

D The test was inconclusive

Solution: A

Both the ratio and root tests can determine absolute vs. conditional convergence of a series.

The series converges absolutely if $a_n = |a_n|$ for all possible values of n

The series converges conditionally if $a_n \neq |a_n|$ for all possible values of n

Since all terms in the given series

$$\sum_{n=1}^{\infty} \left(\frac{n}{2n+1} \right)^n$$

are raised to the power of n, we should use the root test to determine convergence.

Let

$$a_n = \left(\frac{n}{2n+1} \right)^n$$

Then by the root test,

$$R = \lim_{n \to \infty} \left| \left(\frac{n}{2n+1} \right)^n \right|^{\frac{1}{n}}$$

$$R = \lim_{n \to \infty} \left| \left(\frac{n}{2n+1} \right)^{n \cdot \frac{1}{n}} \right|$$

$$R = \lim_{n\to\infty} \left| \frac{n}{2n+1} \right|$$

$$R = \lim_{n\to\infty} \left| \frac{n}{2n+1} \left(\frac{\frac{1}{n}}{\frac{1}{n}} \right) \right|$$

$$R = \lim_{n\to\infty} \left| \frac{\frac{n}{n}}{\frac{2n}{n} + \frac{1}{n}} \right|$$

$$R = \lim_{n\to\infty} \left| \frac{1}{2 + \frac{1}{n}} \right|$$

$$R = \left| \frac{1}{2 + \frac{1}{\infty}} \right|$$

$$R = \left| \frac{1}{2+0} \right|$$

$$R = \left| \frac{1}{2} \right|$$

$$R = \frac{1}{2}$$

Since

$$R = \frac{1}{2} < 1$$

the series converges absolutely.

Topic: Absolute and conditional convergence

Question: Use the ratio test to determine the convergence (absolute or conditional) of the series.

$$\sum_{n=1}^{\infty} \frac{n+1}{2^n}$$

Answer choices:

A The series converges absolutely

B The series converges conditionally

C The series diverges

D The test was inconclusive

Solution: A

Both the ratio and root tests can determine absolute vs. conditional convergence of a series.

The series converges absolutely if $a_n = |a_n|$ for all possible values of n

The series converges conditionally if $a_n \neq |a_n|$ for all possible values of n

Since the given series

$$\sum_{n=1}^{\infty} \frac{n+1}{2^n}$$

would be easier to evaluate with the ratio test than the root test, and the ratio test for convergence lets us calculate L as

$$L = \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right|$$

and then says that the series

converges if $L < 1$

diverges if $L > 1$

we'll find L, by starting with a_n and a_{n+1}.

$$a_n = \frac{n+1}{2^n}$$

$$a_{n+1} = \frac{n+1+1}{2^{n+1}} = \frac{n+2}{2^{n+1}}$$

Plugging these into the formula for L from the ratio test, we get

$$L = \lim_{n \to \infty} \left| \frac{\frac{n+2}{2^{n+1}}}{\frac{n+1}{2^n}} \right|$$

$$L = \lim_{n \to \infty} \left| \frac{n+2}{2^{n+1}} \cdot \frac{2^n}{n+1} \right|$$

Pairing similar numerators and denominators together, we get

$$L = \lim_{n \to \infty} \left| \frac{2^n}{2^{n+1}} \cdot \frac{n+2}{n+1} \right|$$

$$L = \lim_{n \to \infty} \left| 2^{n-(n+1)} \cdot \frac{n+2}{n+1} \right|$$

$$L = \lim_{n \to \infty} \left| 2^{n-n-1} \cdot \frac{n+2}{n+1} \right|$$

$$L = \lim_{n \to \infty} \left| 2^{-1} \cdot \frac{n+2}{n+1} \right|$$

$$L = \lim_{n \to \infty} \left| \frac{1}{2} \cdot \frac{n+2}{n+1} \right|$$

$$L = \frac{1}{2} \lim_{n \to \infty} \left| \frac{n+2}{n+1} \right|$$

$$L = \frac{1}{2} \lim_{n\to\infty} \left| \frac{n+2}{n+1} \left(\frac{\frac{1}{n}}{\frac{1}{n}} \right) \right|$$

$$L = \frac{1}{2} \lim_{n\to\infty} \left| \frac{\frac{n}{n} + \frac{2}{n}}{\frac{n}{n} + \frac{1}{n}} \right|$$

$$L = \frac{1}{2} \lim_{n\to\infty} \left| \frac{1 + \frac{2}{n}}{1 + \frac{1}{n}} \right|$$

$$L = \frac{1}{2} \left| \frac{1 + \frac{2}{\infty}}{1 + \frac{1}{\infty}} \right|$$

$$L = \frac{1}{2} \left| \frac{1+0}{1+0} \right|$$

$$L = \frac{1}{2} \left| 1 \right|$$

$$L = \frac{1}{2}$$

or

$$L = \frac{1}{2} < 1$$

Therefore, the series converges absolutely for all $x \in R$.

Topic: Absolute and conditional convergence

Question: Determine the convergence (absolute or conditional) of the series.

$$\sum_{n=1}^{\infty} \frac{1}{n^3}$$

Answer choices:

A The series converges absolutely

B The series converges conditionally

C The series diverges

D The test was inconclusive

Solution: A

The given series

$$\sum_{n=1}^{\infty} \frac{1}{n^3}$$

is a p-series with $p = 3$.

The p-series test for convergence tells us that the series will

converge when $p > 1$

diverge when $p \leq 1$

Since $3 > 1$, the given series converges by the p-series test.