Work done on elastic springs

To calculate the work done when we stretch or compress an elastic spring, we'll use the formula

$$W = \int_a^b F(x)\ dx$$

where W is the work done, $F(x)$ is the force equation, and $[a, b]$ is the distance over which the spring is stretched or compressed.

Every spring has its own spring constant k. This spring constant is part of Hooke's Law, which states that

$$F(x) = kx$$

where $F(x)$ is the force required to stretch or compress the spring, k is the spring constant, and x is the difference between the natural length and the stretched or compressed length. Since k is unique to each spring, we'll need to calculate it prior to determining work, unless it's given in the problem.

Keep in mind that we'll want to find work in terms of Joules J, which is the same as Newton-meters N-m.

Example

A spring has a natural length of 30 cm. A 50 N force is required to stretch and hold the spring at a length of 40 cm.

1. How much work is done to stretch the spring from 42 cm to 48 cm?

2. How much work is done to compress the spring from 30 cm to 25 cm?

We'll use Hooke's Law to find $F(x)$, but first we need to find k.

Since we know that a 50 N force is required to stretch and hold the spring at a length of 40 cm, from its natural length of 30 cm, we'll set $F(x) = 50$ and $x = 0.10$ m, which is the difference between the natural length and the stretched length, converted from cm to m. Remember that we'll be finding work in terms of Newtons and meters, which is why we converted 10 cm to 0.10 m.

$$50 = 0.10k$$

$$k = 500$$

With k, we can develop a generic equation for our spring using Hooke's Law.

$$F(x) = kx$$

$$F(x) = 500x$$

Work done to stretch the spring

To calculate the work required to stretch the spring from 42 cm to 48 cm, we pretend that the spring at its natural length of 30 cm ends at the origin, which means that stretching it to 42 cm means we've stretched it to 12, because $42-30=12$. Stretching it to 48 cm means we've stretched it from the origin to 18, because $48-30=18$.

Therefore, the work equation would be

$$W=\int_a^b F(x)\ dx$$

$$W=\int_{12}^{18} 500x\ dx$$

But we need to convert the units from cm to m, so the interval becomes 0.12 m to 0.18 m.

$$W=\int_{0.12}^{0.18} 500x\ dx$$

$$W=250x^2\Big|_{0.12}^{0.18}$$

$$W=250(0.18)^2-250(0.12)^2$$

$$W=4.5$$

The work done to stretch a spring with natural length 30 cm and spring constant $k=500$ from 42 cm to 48 cm is 4.5 J.

Work done to compress the spring

To calculate the work required to compress the spring from 30 cm to 25 cm, we pretend that the spring ends at the origin, which means that compressing it to 25 cm means we've compressed it to -5, because $25-30=-5$.

Therefore, the work equation would be

$$W=\int_a^b F(x)\ dx$$

$$W=\int_0^{-5} 500x\ dx$$

But we need to convert the units from cm to m, so the interval becomes 0 m to -0.05 m.

$$W=\int_0^{-0.05} 500x\ dx$$

$$W=250x^2\Big|_0^{-0.05}$$

$$W=250(-0.05)^2-250(0)^2$$

$$W=0.625$$

The work done to compress a spring with natural length 30 cm and spring constant $k=500$ from 30 cm to 25 cm is 0.625 J.