U-substitution

Finding derivatives of elementary functions was a relatively simple process, because taking the derivative only meant applying the right derivative rules.

This is not the case with integration. Unlike derivatives, it may not be immediately clear which integration rules to use, and every function is like a puzzle.

Most integrals need some work before you can even begin the integration. They have to be transformed or manipulated in order to reduce the function's form into some simpler form. U-substitution is the simplest tool we have to transform integrals.

When you use u-substitution, you'll define u as a differentiable function in terms of the variable in the integral, take the derivative of u to get du, and then substitute these values back into your integrals.

Unfortunately, there are no perfect rules for defining u. If you try a substitution that doesn't work, just try another one. With practice, you'll get faster at identifying the right value for u.

Here are some common substitutions you can try.

For integrals that contain power functions, try using the base of the power function as the substitution.

Example

Use u-substitution to evaluate the integral.

$$\int x\left(x^2+1\right)^4\ dx$$

Let

$$u=x^2+1$$

$$du=2x\ dx$$

$$dx=\frac{du}{2x}$$

Substituting back into the integral, we get

$$\int x(u)^4\ \frac{du}{2x}$$

$$\int u^4\ \frac{du}{2}$$

$$\frac{1}{2}\int u^4\ du$$

This is much simpler than our original integral, and something we can actually integrate.

$$\frac{1}{2}\left(\frac{1}{5}u^5\right)+C$$

$$\frac{1}{10}u^5+C$$

Now, back-substitute to put the answer back in terms of x instead of u.

$$\frac{1}{10}\left(x^2+1\right)^5+C$$

For integrals of rational functions, if the numerator is of equal or greater degree than the denominator, always perform division first. Otherwise, try using the denominator as a possible substitution.

Example

Use u-substitution to evaluate the integral.

$$\int \frac{x}{x^2+1}\ dx$$

Let

$$u=x^2+1$$

$$du=2x\ dx$$

$$dx=\frac{du}{2x}$$

Substituting back into the integral, we get

$$\int \frac{x}{u}\cdot\frac{du}{2x}$$

$$\int \frac{1}{u} \cdot \frac{du}{2}$$

$$\frac{1}{2} \int \frac{1}{u}\ du$$

This is much simpler than our original integral, and something we can actually integrate.

$$\frac{1}{2} \ln |u| + C$$

Now, back-substitute to put the answer back in terms of x instead of u.

$$\frac{1}{2} \ln \left| x^2 + 1 \right| + C$$

For integrals containing exponential functions, try using the power for the substitution.

Example

Use u-substitution to evaluate the integral.

$$\int e^{\sin x \cos x} \cos 2x\ dx$$

Let $u = \sin x \cos x$, and using the product rule to differentiate,

$$du = \left[\left(\frac{d}{dx}\sin x\right)\cos x + \sin x\left(\frac{d}{dx}\cos x\right)\right]\ dx$$

$$du = \left[\cos x \cdot \cos x + \sin x \cdot (-\sin x)\right]\ dx$$

$$du = \cos^2 x - \sin^2 x\ dx$$

$$du = \cos 2x\ dx$$

Substituting back into the integral, we get

$$\int e^u\ du$$

$$e^u + C$$

Now, back-substitute to put the answer back in terms of x instead of u.

$$e^{\sin x \cos x} + C$$

Integrals containing trigonometric functions can be more challenging to manipulate. Sometimes, the value of u isn't even part of the original integral. Therefore, the better you know your trigonometric identities, the better off you'll be.

Example

Use u-substitution to evaluate the integral.

$$\int \frac{\tan x}{\cos x}\ dx$$

Since

$$\tan x = \frac{\sin x}{\cos x}$$

we can rewrite the integral as

$$\int \frac{\frac{\sin x}{\cos x}}{\cos x}\ dx$$

$$\int \frac{\sin x}{\cos x} \cdot \frac{1}{\cos x}\ dx$$

$$\int \frac{\sin x}{\cos^2 x}\ dx$$

Let

$$u = \cos x$$

$$du = -\sin x\ dx$$

$$dx = -\frac{du}{\sin x}$$

Substituting back into the integral, we get

$$\int \frac{\sin x}{u^2} \cdot \left(-\frac{du}{\sin x}\right)$$

$$-\int \frac{1}{u^2}\ du$$

$$-\int u^{-2}\ du$$

$$-\frac{1}{-1}u^{-1}+C$$

$$u^{-1}+C$$

$$\frac{1}{u}+C$$

Now, back-substitute to put the answer back in terms of x instead of u.

$$\frac{1}{\cos x}+C$$