Topic: Probability density functions

Question: Which of the following is a probability density function?

Answer choices:

A $f(x) = x^3$ $\quad -1 \le x \le 1$

B $f(x) = \frac{x^3}{5{,}000}(10 - x)$ $\quad 0 \le x \le 10$

C $f(x) = \frac{x^3}{5{,}000}(2 - x)$ $\quad -1 \le x \le 3$

D $f(x) = \frac{x^2}{2{,}000}(4 - x)$ $\quad 4 \le x \le 5$

Solution: B

In order for a function to be a probability density function, it must meet these two criteria:

1. The function must be greater than or equal to 0 in its entire domain.

2. The integral of the function must equal 1 in its entire domain.

To evaluate each of answer choices as a potential probability density function, we can assess each one to see if it meet the first criterion above. The easiest way to do this is to plug in the ends of the interval.

For answer A, when we plug in the interval endpoints $-1 \le x \le 1$ we get one answer that is less than 0. This means that A is not a probability density function.

For answer B, when we plug in the interval endpoints $0 \le x \le 10$ we get two answers that are both equal to 0. This means that B is potentially a probability density function.

For answer C, when we plug in the interval endpoints $-1 \le x \le 3$ we get one answer that is less than 0. This means that C is not a probability density function.

For answer D, when we plug in the interval endpoints $4 \le x \le 5$ we get one answer that is less than 0. This means that D is not a probability density function.

Answer B is the only possibility. To be sure it's a probability density function, we can check to make sure that it meets the second criterion.

$$\int_0^{10} \frac{x^3}{5{,}000}(10-x)\ dx = \int_0^{10} \frac{10}{5{,}000}x^3 - \frac{1}{5{,}000}x^4\ dx$$

$$\int_0^{10} \frac{x^3}{5{,}000}(10-x)\ dx = \frac{10}{20{,}000}x^4 - \frac{1}{25{,}000}x^5\Big|_0^{10}$$

$$\int_0^{10} \frac{x^3}{5{,}000}(10-x)\ dx = \left[\frac{10}{20{,}000}(10)^4 - \frac{1}{25{,}000}(10)^5\right] - \left[\frac{10}{20{,}000}(0)^4 - \frac{1}{25{,}000}(0)^5\right]$$

$$\int_0^{10} \frac{x^3}{5{,}000}(10-x)\ dx = 5-4$$

$$\int_0^{10} \frac{x^3}{5{,}000}(10-x)\ dx = 1$$

This result verifies the second criterion, so we can say that answer choice B is a probability density function.

Topic: Probability density functions

Question: Find the probability.

$$P(x \leq 1)$$

for $f(x) = \frac{1}{4}x^3$

on the interval $0 \leq x \leq 2$

Answer choices:

A 16

B $\frac{1}{4}$

C 1

D $\frac{1}{16}$

Solution: D

In order for a function to be a probability density function, it must meet these two criteria:

1. The function must be greater than or equal to 0 in its entire domain.

2. The integral of the function must equal 1 in its entire domain.

The question gives us the function

$$f(x) = \frac{1}{4}x^3$$

and defines the interval $0 \le x \le 2$. The question assumes it, but let's first prove to ourselves that this is a probability density function. The first thing we need to show is that $f(x) \ge 0$ on the given interval. If we plug in the endpoints of the interval, we get

$$f(0) = \frac{1}{4}(0)^3$$

$$f(0) = 0$$

and

$$f(2) = \frac{1}{4}(2)^3$$

$$f(2) = 2$$

We've shown that $f(x) \ge 0$ at the endpoints, but what about in between the endpoints of the interval? Well, since the function starts at 0 on the left side of the interval, and works its way up to 2 on the right side of the interval, we can prove that the function is always greater than or equal to 0 if we can show that it's always increasing. To do that, we'll take the derivative, set it equal to 0 to find critical points, and then test them to show where the function is increasing and decreasing.

$$f(x) = \frac{1}{4}x^3$$

$$f'(x) = \frac{3}{4}x^2$$

$$\frac{3}{4}x^2 = 0$$

$$x = 0$$

We have one possible critical point that divides the function into the intervals

$$(-\infty,0]$$

$$[0,\infty)$$

Since the interval we're given in the problem is $0 \leq x \leq 2$, we're only interested in the interval to the right of the potential critical point, $[0,\infty)$, since the given interval lies entirely inside $[0,\infty)$.

We'll test the function's behavior in $[0,\infty)$ by picking one point in the interval and plugging it into the derivative. If we get a positive result, it means the function is increasing in the interval. If we get a negative result, it means the function is decreasing in the interval.

For the interval $[0,\infty)$, we'll test $x = 1$.

$$f'(1) = \frac{3}{4}(1)^2$$

$$f'(1) = \frac{3}{4}$$

Since the result is positive, it means the function is increasing in the interval $[0,\infty)$, which means we can also say that it's increasing in $0 \leq x \leq 2$. Therefore, we can conclude the given function is greater than or equal to 0 it its domain, which means it meets the first criterion for a probability density function.

To show that it also meets the second criterion, we need to show that the integral of the function over the given interval is equal to 1.

$$\int_0^2 \frac{1}{4}x^3\ dx$$

$$\int_0^2 \frac{1}{4}x^3\ dx = \frac{1}{16}x^4\Big|_0^2$$

$$\int_0^2 \frac{1}{4}x^3\ dx = \frac{1}{16}(2)^4 - \frac{1}{16}(0)^4$$

$$\int_0^2 \frac{1}{4}x^3\ dx = 1$$

This function also meets the second criterion, so we've proven that it represents a probability density function over the given interval.

The question is asking us to find the probability that x exists as $x \leq 1$. Since we're dealing with probability, we write this as $P(x \leq 1)$. Since the given interval is defined as $0 \leq x \leq 2$, we want to know the probability that x exists on the interval $0 \leq x \leq 1$.

To calculate this probability, we integrate the probability density function on the interval $0 \leq x \leq 1$.

$$P(x \leq 1) = \int_0^1 \frac{1}{4}x^3 \ dx$$

$$P(x \leq 1) = \frac{1}{16}x^4 \Big|_0^1$$

$$P(x \leq 1) = \frac{1}{16}(1)^4 - \frac{1}{16}(0)^4$$

$$P(x \leq 1) = \frac{1}{16}$$

Topic: Probability density functions

Question: Find the probability.

$$P\left(x \le \frac{3}{2}\right)$$

for $f(x) = \dfrac{2}{x^2}$

on the interval $1 \le x \le 2$

Answer choices:

A $\dfrac{3}{2}$

B $\dfrac{8}{9}$

C $\dfrac{2}{3}$

D $\dfrac{9}{8}$

Solution: C

In order for a function to be a probability density function, it must meet these two criteria:

1. The function must be greater than or equal to 0 in its entire domain.

2. The integral of the function must equal 1 in its entire domain.

The question gives us the function

$$f(x) = \frac{2}{x^2}$$

and defines the interval $1 \leq x \leq 2$. The question assumes it, but let's first prove to ourselves that this is a probability density function. The first thing we need to show is that $f(x) \geq 0$ on the given interval. If we plug in the endpoints of the interval, we get

$$f(1) = \frac{2}{(1)^2}$$

$$f(1) = 2$$

and

$$f(2) = \frac{2}{(2)^2}$$

$$f(2) = \frac{1}{2}$$

We've shown that $f(x) \geq 0$ at the endpoints, but what about in between the endpoints of the interval? Well, since the function starts at 2 on the left side of the interval, and works its way down to $1/2$ on the right side of the interval, we can prove that the function is always greater than or equal to 0 if we can show that it's always decreasing. If the function were to dip below the x-axis inside the interval, it would have to increase again to get back up to $1/2$, which is why showing that it's decreasing in the entire interval will prove that it stays positive.

To do that, we'll take the derivative, set it equal to 0 to find critical points, and then test them to show where the function is increasing and decreasing.

$$f(x) = \frac{2}{x^2}$$

$$f'(x) = -\frac{4}{x^3}$$

$$-\frac{4}{x^3} = 0$$

We can't get this function equal to 0, but it's undefined when $x = 0$. That means that $x = 0$ is a potential critical point. Since that point is outside our interval, it means that the given interval $1 \leq x \leq 2$ is entirely to the right of the critical point, and therefore that the function is either increasing everywhere in the interval, or decreasing everywhere in the interval. But of course, looking at the endpoints of the interval that we calculated earlier, we know that the function is decreasing throughout the interval, and therefore, $f(x) \geq 0$ throughout the interval. The function meets the first criterion.

To show that it also meets the second criterion, we need to show that the integral of the function over the given interval is equal to 1.

$$\int_1^2 \frac{2}{x^2}\ dx$$

$$\int_1^2 \frac{2}{x^2}\ dx = -\frac{2}{x}\Big|_1^2$$

$$\int_1^2 \frac{2}{x^2}\ dx = -\frac{2}{2} - \left(-\frac{2}{1}\right)$$

$$\int_1^2 \frac{2}{x^2}\ dx = -1 + 2$$

$$\int_1^2 \frac{2}{x^2}\ dx = 1$$

This function also meets the second criterion, so we've proven that it represents a probability density function over the given interval.

The question is asking us to find the probability that x exists as $x \leq 3/2$. Since we're dealing with probability, we write this as $P(x \leq 3/2)$. Since the given interval is defined as $1 \leq x \leq 2$, we want to know the probability that x exists on the interval $1 \leq x \leq 3/2$.

To calculate this probability, we integrate the probability density function on the interval $1 \leq x \leq 3/2$.

$$P\left(x \leq \frac{3}{2}\right) = \int_1^{\frac{3}{2}} \frac{2}{x^2}\ dx$$

$$P\left(x \le \frac{3}{2}\right) = -\frac{2}{x}\Bigg|_1^{\frac{3}{2}}$$

$$P\left(x \le \frac{3}{2}\right) = -\frac{2}{\frac{3}{2}} - \left(-\frac{2}{1}\right)$$

$$P\left(x \le \frac{3}{2}\right) = -\frac{4}{3} + 2$$

$$P\left(x \le \frac{3}{2}\right) = \frac{2}{3}$$