Binomial series

Just as we did with Maclaurin series, we can use the binomial series

$$(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n = 1 + kx + \frac{k(k-1)}{2!}x^2 + \frac{k(k-1)(k-2)}{3!}x^3 + \dots$$

to find the power series representation of functions that are similar to $(1+x)^k$. We'll just try to manipulate the binomial series until it matches the function we've been given.

Once we have the new, manipulated binomial series, we'll expand it through its first few terms and then use the pattern that we see in those terms to find a power series representation for the original function.

With the power series representation in hand, we'll be able to find the radius and interval of convergence of the series.

Example

Use the binomial series to expand the function as a power series, and then find the radius and interval of convergence.

$$f(x) = (1-x)^{-2}$$

We'll start with the binomial series.

$$(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n = 1 + kx + \frac{k(k-1)}{2!}x^2 + \frac{k(k-1)(k-2)}{3!}x^3 + \dots$$

To make the left-hand side match our function, we'll replace x with $-x$ and k with -2.

$$[1+(-x)]^{-2}=\sum_{n=0}^{\infty}\binom{-2}{n}(-x)^n=1-2(-x)+\frac{-2(-2-1)}{2!}(-x)^2+\frac{-2(-2-1)(-2-2)}{3!}(-x)^3+\dots$$

$$(1-x)^{-2}=\sum_{n=0}^{\infty}\binom{-2}{n}(-x)^n=1+2x+\frac{6}{2!}x^2+\frac{24}{3!}x^3+\dots$$

Now that the left side matches the given function, we can use the series expansion on the right side to find its power series representation. We just have to find the pattern in the expansion. We'll identify the pattern by rewriting the expansion as

$$1+2x+\frac{6}{2!}x^2+\frac{24}{3!}x^3+\dots$$

$$1x^0+2x^1+\frac{6}{2!}x^2+\frac{24}{3!}x^3+\dots$$

$$\frac{1}{0!}x^0+\frac{2}{1!}x^1+\frac{6}{2!}x^2+\frac{24}{3!}x^3+\dots$$

$$\frac{1}{1}x^0+\frac{2\cdot 1}{1}x^1+\frac{3\cdot 2\cdot 1}{2\cdot 1}x^2+\frac{4\cdot 3\cdot 2\cdot 1}{3\cdot 2\cdot 1}x^3+\dots$$

$$1x^0+2x^1+3x^2+4x^3+\dots$$

When we match up these terms with their corresponding n-values, we get

$n=0$ $\qquad$ $1x^0$

$n=1$ $\qquad$ $2x^1$

$n=2$	$3x^2$
$n=3$	$4x^3$

We can see that the coefficients can all be represented by $n+1$, and that the exponents can all be represented by n. Therefore, the power series representation of the function is

$$(1-x)^{-2}=\sum_{n=0}^{\infty}(n+1)x^n$$

If we want to find the radius of convergence of this power series, we first identify that

$$a_n=(n+1)x^n$$

Then we generate a_{n+1}.

$$a_{n+1}=(n+1+1)x^{n+1}$$

$$a_{n+1}=(n+2)x^{n+1}$$

We plug both a_n and a_{n+1} into

$$L=\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|$$

and then we'll set $L<1$, since the ratio test tells us that the series converges absolutely if $L<1$.

$$L=\lim_{n\to\infty}\left|\frac{(n+2)x^{n+1}}{(n+1)x^n}\right|$$

$$L = \lim_{n \to \infty} \left| \frac{n+2}{n+1} \cdot \frac{x^{n+1}}{x^n} \right|$$

$$L = \lim_{n \to \infty} \left| \frac{n+2}{n+1} \cdot x^{n+1-n} \right|$$

$$L = \lim_{n \to \infty} \left| \frac{n+2}{n+1} \cdot x \right|$$

The limit only effects n, which means we can pull x out of the limit, as long as we keep it inside absolute value bars.

$$L = |x| \lim_{n \to \infty} \left| \frac{n+2}{n+1} \right|$$

Since in our series n starts at 0, it's impossible for either the numerator or the denominator of the fraction to be negative, which means we can drop its absolute value bars.

$$L = |x| \lim_{n \to \infty} \frac{n+2}{n+1}$$

$$L = |x| \lim_{n \to \infty} \frac{n+2}{n+1} \left(\frac{\frac{1}{n}}{\frac{1}{n}} \right)$$

$$L = |x| \lim_{n \to \infty} \frac{\frac{n}{n} + \frac{2}{n}}{\frac{n}{n} + \frac{1}{n}}$$

$$L = |x| \lim_{n \to \infty} \frac{1 + \frac{2}{n}}{1 + \frac{1}{n}}$$

$$L = |x| \left(\frac{1 + \frac{2}{\infty}}{1 + \frac{1}{\infty}} \right)$$

$$L = |x| \left(\frac{1 + 0}{1 + 0} \right)$$

$$L = |x|(1)$$

$$L = |x|$$

Now we can set $L < 1$.

$$|x| < 1$$

The result is already in the form $|x - a| < R$, as

$$|x - 0| < 1$$

We can see from this equation that the radius of convergence is $R = 1$. That means that the interval of convergence is

$$|x| < 1$$

$$-1 < x < 1$$

Remember though that we always have to test the endpoints of the interval of convergence to say whether or not the series converges there.

We'll do this by plugging the endpoints back into the power series representation.

For $x = -1$:

$$\sum_{n=0}^{\infty} (n+1)(-1)^n$$

$$\sum_{n=0}^{\infty} (-1)^n(n+1)$$

By the divergence test (nth-term test), this series diverges.

For $x = 1$:

$$\sum_{n=0}^{\infty} (n+1)(1)^n$$

$$\sum_{n=0}^{\infty} n+1$$

By the divergence test (nth-term test), this series diverges.

Therefore, we can confirm that the interval of convergence is still

$$-1 < x < 1$$

If we summarize our results, we can say that the power series representation is $(1-x)^{-2} = \sum_{n=0}^{\infty} (n+1)x^n$

the radius of convergence is $R = 1$

the interval of convergence is $-1 < x < 1$