Volume of revolution, washer method

We can use integrals to find the volume of the three-dimensional object created by rotating a function around either the x-axis (or some other horizontal axis with the equation $y = b$) or around the y-axis (or some other vertical axis with the equation $x = a$).

We can do this using the disk method, the washer method, or using cylindrical shells. To use the washer method, the volume generated by rotating the function has to be a ring (like a washer, or a donut) with a hole in the middle.

In order to generate a volume like this, the region has to be bounded by two functions, either $y = f(x)$ and $y = g(x)$ or $x = f(y)$ and $x = g(y)$.

The washer method formulas we use to find volume of rotation are different depending on the form of the functions and the axis of rotation.

1. If the region is bounded by $y = f(x)$ and $y = g(x)$, and if $f(x) > g(x)$, and if we're rotating around the x-axis over the interval $[a, b]$, the formula for the volume of the solid is

$$V = \int_a^b \pi[f(x)]^2 - \pi[g(x)]^2 \ dx$$

2. If the region is bounded by $x = f(y)$ and $x = g(y)$, and if $f(y) > g(y)$, and if we're rotating around the y-axis over the interval $[c, d]$, the formula for the volume of the solid is

$$V = \int_c^d \pi[f(y)]^2 - \pi[g(y)]^2 \, dy$$

The table below will help guide you through how to solve a volume problem when you're using disks or washers to find the volume. Start in the first row of the table, and determine the line of rotation or revolution. The problem will usually tell you the line of rotation. If you're asked to rotate about the x-axis or some line defined for y in terms of x, then stay in the first column of the table. If you're asked to rotate about the y-axis or some line defined for x in terms of y, then stay in the second column of the table.

The best way to figure out whether you need to use disks or washers is to graph the functions and the axis of rotation and draw a picture of the rotated volume.

Axis	Disks	Washers	Shells
	$\int$ area width	$\int$ area width	$\int$ circumference height width

Axis of revolution: HORIZONTAL

x-axis	$\int_a^b \pi\left[f(x)\right]^2\ dx$	$\int_a^b \pi\left[f(x)\right]^2-\pi\left[g(x)\right]^2\ dx$	$\int_c^d 2\pi y\left[f(y)-g(y)\right]\ dy$
$y=-k$		$\int_a^b \pi\left[k+f(x)\right]^2-\pi\left[k+g(x)\right]^2\ dx$	$\int_c^d 2\pi(y+k)\left[f(y)-g(y)\right]\ dy$
$y=k$		$\int_a^b \pi\left[k-g(x)\right]^2-\pi\left[k-f(x)\right]^2\ dx$	$\int_c^d 2\pi(k-y)\left[f(y)-g(y)\right]\ dy$

Axis of revolution: VERTICAL

y-axis	$\int_c^d \pi\left[f(y)\right]^2\ dy$	$\int_c^d \pi\left[f(y)\right]^2-\pi\left[g(y)\right]^2\ dy$	$\int_a^b 2\pi x\left[f(x)-g(x)\right]\ dx$
$x=-k$		$\int_c^d \pi\left[k+f(y)\right]^2-\pi\left[k+g(y)\right]^2\ dy$	$\int_a^b 2\pi(x+k)\left[f(x)-g(x)\right]\ dx$
$x=k$		$\int_c^d \pi\left[k-g(y)\right]^2-\pi\left[k-f(y)\right]^2\ dy$	$\int_a^b 2\pi(k-x)\left[f(x)-g(x)\right]\ dx$

Example

Find the volume of the solid created by rotating the region bounded by the curves about the x-axis.

$$y=x^2+2$$

$$y = x + 4$$

Looking at the question, we can see that the functions that bound our region are in the form $y = f(x)$ and $y = g(x)$, so we'll use the formula

$$V = \int_a^b \pi[f(x)]^2 - \pi[g(x)]^2 \ dx$$

The problem doesn't give us an interval over which to integrate, but when we're using washer method the interval is defined by the points of intersection of the two curves. To find the points of intersection and generate the interval, we'll set the functions equal to each other and solve for x.

$$x^2 + 2 = x + 4$$

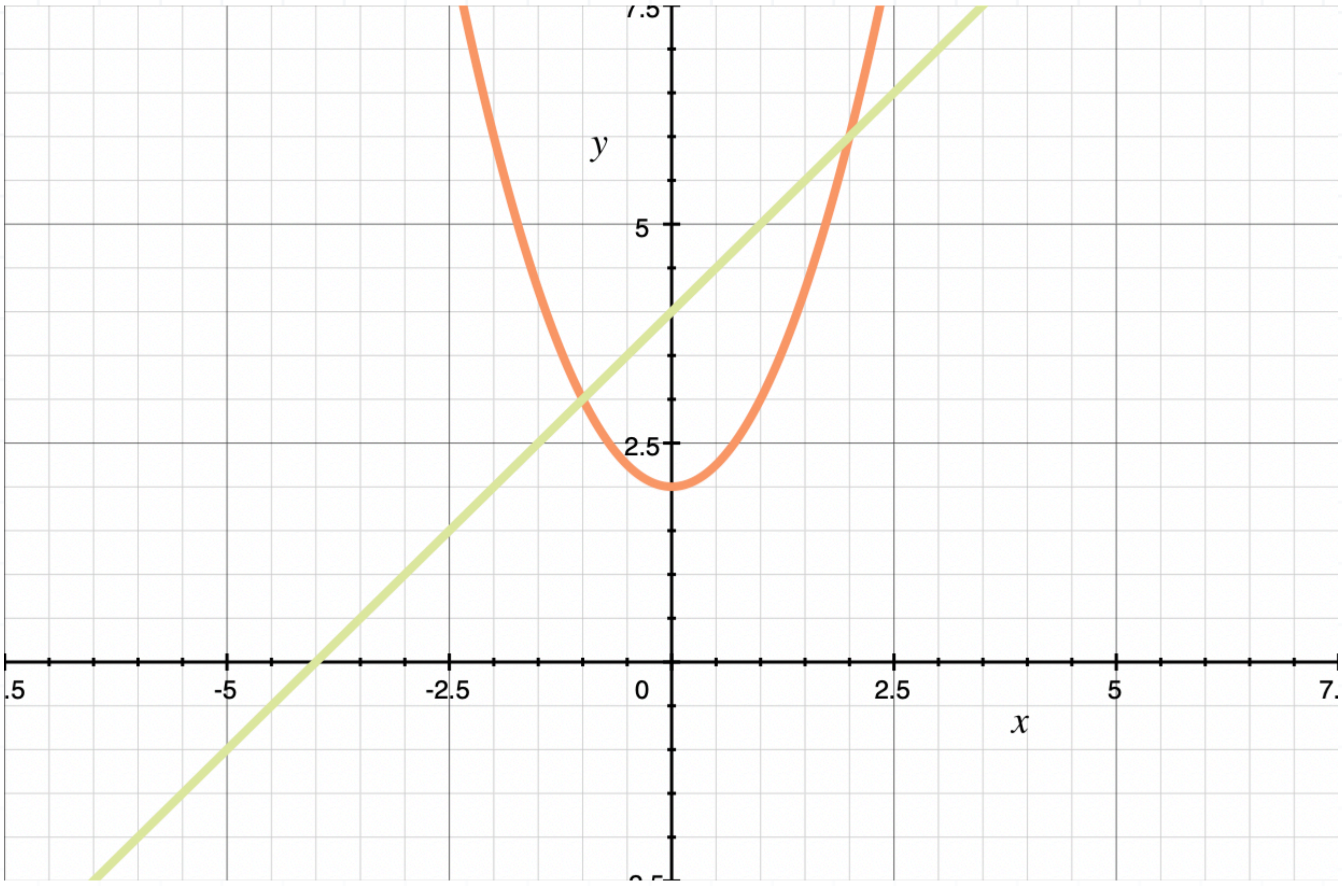
$$x^2 - x - 2 = 0$$

$$(x + 1)(x - 2) = 0$$

$x = -1$ and $x = 2$

Based on the values we found for x, the interval is $[-1,2]$.

The next step is deciding which of the functions will be $f(x)$ and which will be $g(x)$, remembering that $f(x) \geq g(x)$. The simplest way to see this is to graph the two equations. Remember, we are only interested in what is happening in the interval $[-1,2]$.



Based on the graph, $f(x) = x + 4$ and $g(x) = x^2 + 2$, because $y = x + 4$ is greater than $y = x^2 + 2$ over the interval $[-1,2]$.

If you can't graph the functions to see which one is greater than the other over the interval, you can always pick a value in the interval and plug it into both functions. For example, since 0 is in the interval $[-1,2]$, we can plug it into both of the functions and we get

$$y = x + 4$$

$$y = 0 + 4 = 4$$

and

$$y = x^2 + 2$$

$$y = (0)^2 + 2 = 2$$

Because $y = x + 4$ gives us a greater value back than $y = x^2 + 2$ ($4 > 2$), we know that $f(x) = x + 4$ and $g(x) = x^2 + 2$.

Remember also that, if you're dealing with a problem in which the functions are defined as $x = f(y)$ and $x = g(y)$, then you're plugging in a y-value to see which of the functions returns a greater x-value. The right-most function (the one with the larger x-value), is greater than the left-most function (the one with the smaller x-value).

Now we can solve for volume.

$$V = \int_a^b \pi \left([f(x)]^2 - [g(x)]^2 \right) \ dx$$

$$V = \int_{-1}^2 \pi \left([x+4]^2 - [x^2+2]^2 \right) \ dx$$

$$V = \int_{-1}^2 \pi \left(x^2 + 8x + 16 - [x^4 + 4x^2 + 4] \right) \ dx$$

$$V = \int_{-1}^2 \pi \left(-x^4 - 3x^2 + 8x + 12 \right) \ dx$$

$$V = \pi \int_{-1}^2 -x^4 - 3x^2 + 8x + 12 \ dx$$

Now we can break the integral into smaller parts.

$$V = \pi \int_{-1}^2 -x^4 \ dx + \pi \int_{-1}^2 -3x^2 \ dx + \pi \int_{-1}^2 8x \ dx + \pi \int_{-1}^2 12 \ dx$$

$$V = -\pi\int_{-1}^{2} x^4\ dx - 3\pi\int_{-1}^{2} x^2\ dx + 8\pi\int_{-1}^{2} x\ dx + 12\pi\int_{-1}^{2} 1\ dx$$

Integrating, we get

$$V = -\pi\left(\frac{x^5}{5}\right) - 3\pi\left(\frac{x^3}{3}\right) + 8\pi\left(\frac{x^2}{2}\right) + 12\pi(x)\Bigg|_{-1}^{2}$$

$$V = \frac{-\pi x^5}{5} - \pi x^3 + 4\pi x^2 + 12\pi x\Bigg|_{-1}^{2}$$

Evaluating over the interval, we get

$$V = \frac{-\pi(2)^5}{5} - \pi(2)^3 + 4\pi(2)^2 + 12\pi(2) - \left[\frac{-\pi(-1)^5}{5} - \pi(-1)^3 + 4\pi(-1)^2 + 12\pi(-1)\right]$$

$$V = \frac{162}{5}\pi$$

This is the volume of the solid object created by rotating the region bounded by $y = x^2 + 2$ and $y = x + 4$ about the x-axis.