Topic: Improper integrals, case 3

Question: Evaluate the improper integral.

$$\int_{-\infty}^{\infty} \frac{1}{x^2+2x+5}\ dx$$

Answer choices:

A $\frac{\pi}{2}$

B $\frac{\pi}{3}$

C $\frac{\pi}{4}$

D $\frac{\pi}{6}$

Solution: A

First, rewrite the integral.

$$\int_{-\infty}^{\infty} \frac{1}{x^2+2x+5}\ dx = \int_{-\infty}^{\infty} \frac{dx}{(x+1)^2+2^2}$$

Since both limits of integration are infinite, we'll split the interval at $x = 0$ and rewrite the integral.

$$\int_{-\infty}^{0} \frac{dx}{(x+1)^2+2^2} + \int_{0}^{\infty} \frac{dx}{(x+1)^2+2^2}$$

Using arbitrary variables a and b, take the limit of the first integral as $a \to -\infty$ and the second integral as $b \to \infty$.

$$\lim_{a \to -\infty} \int_{a}^{0} \frac{dx}{(x+1)^2+2^2} + \lim_{b \to \infty} \int_{0}^{b} \frac{dx}{(x+1)^2+2^2}$$

To integrate, we need to use trigonometric substitution. We recognize that in the denominator of the function, we have a variable term $(x+1)^2$ plus a constant term 2^2. So we'll go through the setup process for trigonometric substitution.

$u^2 = (x+1)^2$ so $u = x + 1$

$a^2 = 2^2$ so $a = 2$

$x + 1 = 2\tan\theta$

$$\tan\theta = \frac{x+1}{2}$$

$$\theta = \arctan \frac{x+1}{2}$$

$$x = 2\tan\theta - 1$$

$$\frac{dx}{d\theta} = 2\sec^2\theta \text{ so } dx = 2\sec^2\theta \ d\theta$$

Make substitutions into the integral.

$$\lim_{a\to-\infty} \int_{x=a}^{x=0} \frac{2\sec^2\theta \ d\theta}{(2\tan\theta)^2 + 2^2} + \lim_{b\to\infty} \int_{x=0}^{x=b} \frac{2\sec^2\theta \ d\theta}{(2\tan\theta)^2 + 2^2}$$

$$\lim_{a\to-\infty} \int_{x=a}^{x=0} \frac{2\sec^2\theta \ d\theta}{4\tan^2\theta + 4} + \lim_{b\to\infty} \int_{x=0}^{x=b} \frac{2\sec^2\theta \ d\theta}{4\tan^2\theta + 4}$$

$$\lim_{a\to-\infty} \int_{x=a}^{x=0} \frac{2\sec^2\theta \ d\theta}{4\left(\tan^2\theta + 1\right)} + \lim_{b\to\infty} \int_{x=0}^{x=b} \frac{2\sec^2\theta \ d\theta}{4\left(\tan^2\theta + 1\right)}$$

Knowing that $\tan^2\theta + 1 = \sec^2\theta$, we get

$$\lim_{a\to-\infty} \int_{x=a}^{x=0} \frac{2\sec^2\theta \ d\theta}{4\sec^2\theta} + \lim_{b\to\infty} \int_{x=0}^{x=b} \frac{2\sec^2\theta \ d\theta}{4\sec^2\theta}$$

$$\lim_{a\to-\infty} \int_{x=a}^{x=0} \frac{1}{2} \ d\theta + \lim_{b\to\infty} \int_{x=0}^{x=b} \frac{1}{2} \ d\theta$$

$$\lim_{a\to-\infty} \frac{1}{2}\theta \Big|_{x=a}^{x=0} + \lim_{b\to\infty} \frac{1}{2}\theta \Big|_{x=0}^{x=b}$$

Back-substitute for θ.

$$\lim_{a\to-\infty}\frac{1}{2}\arctan\frac{x+1}{2}\Bigg|_a^0+\lim_{b\to\infty}\frac{1}{2}\arctan\frac{x+1}{2}\Bigg|_0^b$$

$$\lim_{a\to-\infty}\left(\frac{1}{2}\arctan\frac{0+1}{2}-\frac{1}{2}\arctan\frac{a+1}{2}\right)+\lim_{b\to\infty}\left(\frac{1}{2}\arctan\frac{b+1}{2}-\frac{1}{2}\arctan\frac{0+1}{2}\right)$$

$$\frac{1}{2}\arctan\frac{1}{2}-\frac{1}{2}\arctan\frac{-\infty+1}{2}+\frac{1}{2}\arctan\frac{\infty+1}{2}-\frac{1}{2}\arctan\frac{1}{2}$$

$$\frac{1}{2}\arctan\frac{\infty+1}{2}-\frac{1}{2}\arctan\frac{-\infty+1}{2}$$

$$\frac{1}{2}\arctan(\infty)-\frac{1}{2}\arctan(-\infty)$$

$$\frac{1}{2}\left(\frac{\pi}{2}\right)-\frac{1}{2}\left(-\frac{\pi}{2}\right)$$

$$\frac{\pi}{4}+\frac{\pi}{4}$$

$$\frac{\pi}{2}$$

Topic: Improper integrals, case 3

Question: Evaluate the improper integral.

$$\int_{-\infty}^{\infty} \frac{4}{9+x^2}\,dx$$

Answer choices:

A $\frac{2\pi}{3}$

B $\frac{\pi}{2}$

C $\frac{4\pi}{3}$

D ∞

Solution: C

The integral in this problem is considered to be an improper integral, case 3, because the lower limit of integration is $-\infty$ and the upper limit is ∞. Evaluating this type of improper integral follows this general rule:

$$\int_{-\infty}^{\infty} f(x)\ dx = \lim_{a \to -\infty} \int_{a}^{c} f(x)\ dx + \lim_{b \to \infty} \int_{c}^{b} f(x)\ dx$$

We basically ignore both limits of integration by replacing them with a and b and by using a limit process instead. Then, once we integrate, finding the anti-derivative, we use the limits to finish the evaluation. Let's begin by rewriting the integral as a limit.

$$\int_{-\infty}^{\infty} \frac{4}{9+x^2}\ dx = \lim_{a \to -\infty} \int_{a}^{c} \frac{4}{9+x^2}\ dx + \lim_{b \to \infty} \int_{c}^{b} \frac{4}{9+x^2}\ dx$$

$$\lim_{a \to -\infty} \int_{a}^{c} \frac{\frac{4}{9}}{\frac{9}{9}+\frac{x^2}{9}}\ dx + \lim_{b \to \infty} \int_{c}^{b} \frac{\frac{4}{9}}{\frac{9}{9}+\frac{x^2}{9}}\ dx$$

$$\frac{4}{9} \lim_{a \to -\infty} \int_{a}^{c} \frac{1}{1+\frac{x^2}{9}}\ dx + \frac{4}{9} \lim_{b \to \infty} \int_{c}^{b} \frac{1}{1+\frac{x^2}{9}}\ dx$$

$$\frac{4}{9} \lim_{a \to -\infty} \int_{a}^{c} \frac{1}{1+\left(\frac{x}{3}\right)^2}\ dx + \frac{4}{9} \lim_{b \to \infty} \int_{c}^{b} \frac{1}{1+\left(\frac{x}{3}\right)^2}\ dx$$

Integrate.

$$\frac{4}{9}\lim_{a\to-\infty}3\arctan\frac{x}{3}\Big|_a^c+\frac{4}{9}\lim_{b\to\infty}3\arctan\frac{x}{3}\Big|_c^b$$

$$\frac{4}{3}\lim_{a\to-\infty}\arctan\frac{x}{3}\Big|_a^c+\frac{4}{3}\lim_{b\to\infty}\arctan\frac{x}{3}\Big|_c^b$$

Evaluate over the interval.

$$\frac{4}{3}\lim_{a\to-\infty}\left(\arctan\frac{c}{3}-\arctan\frac{a}{3}\right)+\frac{4}{3}\lim_{b\to\infty}\left(\arctan\frac{b}{3}-\arctan\frac{c}{3}\right)$$

$$\frac{4}{3}\left[\arctan\frac{c}{3}-\left(-\frac{\pi}{2}\right)\right]+\frac{4}{3}\left(\frac{\pi}{2}-\arctan\frac{c}{3}\right)$$

$$\frac{4}{3}\left(\arctan\frac{c}{3}+\frac{\pi}{2}\right)+\frac{4}{3}\left(\frac{\pi}{2}-\arctan\frac{c}{3}\right)$$

$$\frac{4}{3}\arctan\frac{c}{3}+\frac{2\pi}{3}+\frac{2\pi}{3}-\frac{4}{3}\arctan\frac{c}{3}$$

$$\frac{2\pi}{3}+\frac{2\pi}{3}$$

$$\frac{4\pi}{3}$$

Topic: Improper integrals, case 3

Question: Evaluate the improper integral.

$$\int_{-\infty}^{\infty} xe^{x^2}\,dx$$

Answer choices:

A The integral diverges

B 0

C $\frac{8}{3}$

D $\frac{1}{2}$

Solution: A

The integral in this problem is considered to be an improper integral, case 3, because the lower limit of integration is $-\infty$ and the upper limit is ∞. Evaluating this type of improper integral follows this general rule:

$$\int_{-\infty}^{\infty} f(x)\ dx = \lim_{a \to -\infty} \int_{a}^{c} f(x)\ dx + \lim_{b \to \infty} \int_{c}^{b} f(x)\ dx$$

We basically ignore both limits of integration by replacing them with a and b and by using a limit process instead. Then, once we integrate, finding the anti-derivative, we use the limits to finish the evaluation. Let's begin by rewriting the integral as a limit.

$$\int_{-\infty}^{\infty} xe^{x^2}\ dx = \lim_{a \to -\infty} \int_{a}^{c} xe^{x^2}\ dx + \lim_{b \to \infty} \int_{c}^{b} xe^{x^2}\ dx$$

Use a u-substitution.

$$u = x^2$$

$$du = 2x\ dx$$

$$dx = \frac{du}{2x}$$

Substitute into each integral.

$$\lim_{a \to -\infty} \int_{x=a}^{x=c} xe^{u} \left(\frac{du}{2x}\right) + \lim_{b \to \infty} \int_{x=c}^{x=b} xe^{u} \left(\frac{du}{2x}\right)$$

$$\frac{1}{2}\lim_{a\to-\infty}\int_{x=a}^{x=c} e^u\ du + \frac{1}{2}\lim_{b\to\infty}\int_{x=c}^{x=b} e^u\ du$$

Integrate and then back-substitute.

$$\frac{1}{2}\lim_{a\to-\infty} e^u\Big|_{x=a}^{x=c} + \frac{1}{2}\lim_{b\to\infty} e^u\Big|_{x=c}^{x=b}$$

$$\frac{1}{2}\lim_{a\to-\infty} e^{x^2}\Big|_{a}^{c} + \frac{1}{2}\lim_{b\to\infty} e^{x^2}\Big|_{c}^{b}$$

Evaluate over the interval.

$$\frac{1}{2}\lim_{a\to-\infty} e^{c^2} - e^{a^2} + \frac{1}{2}\lim_{b\to\infty} e^{b^2} - e^{c^2}$$

$$\frac{1}{2}\left(e^{c^2} - \infty\right) + \frac{1}{2}\left(\infty - e^{c^2}\right)$$

$$\frac{1}{2}e^{c^2} - \frac{1}{2}\infty + \frac{1}{2}\infty - \frac{1}{2}e^{c^2}$$

$$-\infty + \infty$$

This doesn't converge to a real-number value, so the integral diverges.