# Logarithmic differentiation

Some problems are easiest to solve using logarithmic differentiation. **Logarithmic differentiation** is a problem-solving method in which we start by applying the natural log function to both sides of the equation. We use logarithmic differentiation when it's easier to differentiate the logarithm of a function than the function itself.

If we let $y = f(x)$, then we take the natural log of both sides, differentiate both sides using chain rule, and work toward rewriting the equation so that it's solved for $y'$.

$$\ln y = \ln f(x)$$

$$(\ln y)' = (\ln f(x))'$$

$$\frac{1}{y}y'(x) = (\ln(f(x)))'$$

$$y'(x) = y(\ln f(x))'$$

$$y'(x) = f(x)(\ln f(x))'$$

Oftentimes, we'll utilize laws of logarithms in order to simplify one or both sides of the equation. As a reminder, these are the laws of logs we'll want to use:

| **Laws of logs** | **Laws of natural logs** |
|---|---|
| $\log_a a^x = x$ | $\ln(e^x) = x$ |
| $a^{\log_a x} = x$ | $e^{\ln x} = x$ |

$$\log_a x^r = r\log_a x \qquad \ln(x^a) = a\ln x$$

$$\log_a(xy) = \log_a x + \log_a y \qquad \ln(xy) = \ln x + \ln y$$

$$\log_a \frac{x}{y} = \log_a x - \log_a y \qquad \ln\frac{x}{y} = \ln x - \ln y$$

The easiest way to illustrate this method is to work through an example.

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**Example**

Use logarithmic differentiation to find the derivative of the function.

$$y = \frac{(\ln x)^x}{2^{3x+1}}$$

To start, we'll apply the natural log to both sides of the equation.

$$\ln y = \ln\left(\frac{(\ln x)^x}{2^{3x+1}}\right)$$

Use laws of logs to rewrite the right-hand side.

$$\ln y = \ln((\ln x)^x) - \ln(2^{3x+1})$$

$$\ln y = x\ln(\ln x) - (3x+1)\ln 2$$

$$\ln y = x\ln(\ln x) - 3x\ln 2 - \ln 2$$

Now we'll take the derivative of both sides. We'll need to use product rule to differentiate $x\ln(\ln x)$.

$$\frac{1}{y}y' = \left[(1)(\ln(\ln x)) + (x)\left(\frac{1}{\ln x}\right)\left(\frac{1}{x}\right)\right] - 3\ln 2 - 0$$

$$\frac{1}{y}y' = \ln(\ln x) + \frac{1}{\ln x} - 3\ln 2$$

$$\frac{1}{y}y' = \ln(\ln x) + \frac{1}{\ln x} - \ln(2^3)$$

We want to solve for $y'$, so we'll multiply both sides by $y$ in order to get $y'$ by itself.

$$y' = y\left[\ln(\ln x) + \frac{1}{\ln x} - \ln 8\right]$$

Now we'll use the original equation to substitute for $y$.

$$y' = \frac{(\ln x)^x}{2^{3x+1}}\left[\ln(\ln x) + \frac{1}{\ln x} - \ln 8\right]$$

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Since these are a little tricky, let's do one more example.

**Example**

Use logarithmic differentiation to find the derivative of the function.

$$y = x^{(x^{(x^4)})}$$

To start, we'll apply the natural log to both sides of the equation.

$$\ln y = \ln(x^{(x^{(x^4)})})$$

Use laws of logs to rewrite the right-hand side.

$$\ln y = (x^{(x^4)})\ln x$$

Apply the natural log to both sides again.

$$\ln(\ln y) = \ln((x^{(x^4)})\ln x)$$

Use laws of logs to rewrite the right-hand side.

$$\ln(\ln y) = \ln(x^{(x^4)}) + \ln(\ln x)$$

$$\ln(\ln y) = x^4 \ln x + \ln(\ln x)$$

Now we'll take the derivative of both sides. We'll need to use product rule to differentiate $x^4 \ln x$.

$$\left(\frac{1}{\ln y}\right)\left(\frac{1}{y}\right)(y') = \left[(x^4)\left(\frac{1}{x}\right) + (4x^3)(\ln x)\right] + \left(\frac{1}{\ln x}\right)\left(\frac{1}{x}\right)$$

$$\frac{1}{y \ln y} y' = x^3 + 4x^3 \ln x + \frac{1}{x \ln x}$$

We want to solve for $y'$, so we'll multiply both sides by $y \ln y$ in order to get $y'$ by itself.

$$y' = y \ln y \left(x^3 + 4x^3 \ln x + \frac{1}{x \ln x}\right)$$

Now we'll use the original equation to substitute for $y$.

$$y' = x^{(x^{(x^4)})} \ln(x^{(x^{(x^4)})}) \left( x^3 + 4x^3 \ln x + \frac{1}{x \ln x} \right)$$

$$y' = x^{(x^{(x^4)})} (x^{(x^4)}) \ln x \left( x^3 + 4x^3 \ln x + \frac{1}{x \ln x} \right)$$

$$y' = x^{(x^{(x^4)} + x^4)} \ln x \left( x^3 + 4x^3 \ln x + \frac{1}{x \ln x} \right)$$

$$y' = x^{(x^{(x^4)} + x^4)} \left( x^3 \ln x + 4x^3 \ln^2 x + \frac{1}{x} \right)$$

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