Surface area of revolution of a parametric curve, vertical axis

The surface area of the solid created by revolving a parametric curve around the y-axis is given by

$$S_y = \int_a^b 2\pi x\sqrt{[f'(t)]^2 + [g'(t)]^2}\ dt$$

where the curve is defined over the interval $[a, b]$, $f'(t)$ is the derivative of the curve $x = f(t)$, and $g'(t)$ is the derivative of the curve $y = g(t)$.

Let's do an example where we calculate the surface area of revolution around the y-axis over a specific interval.

Example

Find the surface area of revolution of the solid created when the parametric curve is rotated around the y-axis over $0 \le t \le 3$.

$$x = 2t^2$$

$$y = 2t^3$$

We'll call the parametric equations

$$f(t) = 2t^2$$

$$g(t) = 2t^3$$

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The limits of integration are defined in the problem, but we need to find both derivatives before we can plug into the formula.

$$f'(t) = 4t$$

$$g'(t) = 6t^2$$

Now we'll plug into the formula for the surface area of revolution.

$$S_y = \int_0^3 2\pi(2t^2)\sqrt{(4t)^2 + (6t^2)^2}\ dt$$

$$S_y = \int_0^3 4\pi t^2\sqrt{16t^2 + 36t^4}\ dt$$

$$S_y = \int_0^3 4\pi t^2\sqrt{4t^2(4 + 9t^2)}\ dt$$

$$S_y = \int_0^3 8\pi t^3\sqrt{4 + 9t^2}\ dt$$

$$S_y = 8\pi\int_0^3 t^3\sqrt{4 + 9t^2}\ dt$$

We'll use a substitution, letting $u = 4 + 9t^2$, $t^2 = (u - 4)/9$, and $dt = du/18t$.

$$S_y = 8\pi\int_{t=0}^{t=3} t^3\sqrt{u}\ \frac{du}{18t}$$

$$S_y = \frac{8\pi}{18}\int_{t=0}^{t=3} t^2\sqrt{u}\ du$$

$$S_y = \frac{8\pi}{18} \int_{t=0}^{t=3} \frac{u-4}{9} \sqrt{u} \ du$$

$$S_y = \frac{8\pi}{18} \int_{t=0}^{t=3} \left(\frac{u}{9} - \frac{4}{9} \right) u^{\frac{1}{2}} \ du$$

$$S_y = \frac{8\pi}{18} \int_{t=0}^{t=3} \frac{u^{\frac{3}{2}}}{9} - \frac{4u^{\frac{1}{2}}}{9} \ du$$

$$S_y = \frac{8\pi}{162} \int_{t=0}^{t=3} u^{\frac{3}{2}} - 4u^{\frac{1}{2}} \ du$$

$$S_y = \frac{4\pi}{81} \left(\frac{2}{5} u^{\frac{5}{2}} - \frac{8}{3} u^{\frac{3}{2}} \right) \Bigg|_{t=0}^{t=3}$$

Back-substituting for u, we get

$$S_y = \frac{4\pi}{81} \left(\frac{2}{5} (4+9t^2)^{\frac{5}{2}} - \frac{8}{3} (4+9t^2)^{\frac{3}{2}} \right) \Bigg|_0^3$$

Evaluate over the interval.

$$S_y = \frac{4\pi}{81} \left(\frac{2}{5} (4+9(3)^2)^{\frac{5}{2}} - \frac{8}{3} (4+9(3)^2)^{\frac{3}{2}} \right) - \frac{4\pi}{81} \left(\frac{2}{5} (4+9(0)^2)^{\frac{5}{2}} - \frac{8}{3} (4+9(0)^2)^{\frac{3}{2}} \right)$$

$$S_y = \frac{4\pi}{81} \left(\frac{2}{5} (4+81)^{\frac{5}{2}} - \frac{8}{3} (4+81)^{\frac{3}{2}} \right) - \frac{4\pi}{81} \left(\frac{2}{5} (4+0)^{\frac{5}{2}} - \frac{8}{3} (4+0)^{\frac{3}{2}} \right)$$

$$S_y = \frac{4\pi}{81} \left(\frac{2}{5} (85)^{\frac{5}{2}} - \frac{8}{3} (85)^{\frac{3}{2}} - \frac{2}{5} (4)^{\frac{5}{2}} + \frac{8}{3} (4)^{\frac{3}{2}} \right)$$

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$$S_y = \frac{4\pi}{81}\left(\frac{2}{5}[(85)^5]^{\frac{1}{2}} - \frac{8}{3}[(85)^3]^{\frac{1}{2}} - \frac{2}{5}[(4)^{\frac{1}{2}}]^5 + \frac{8}{3}[(4)^{\frac{1}{2}}]^3\right)$$

$$S_y = \frac{4\pi}{81}\left(\frac{2}{5}[85(85)^4]^{\frac{1}{2}} - \frac{8}{3}[85(85)^2]^{\frac{1}{2}} - \frac{2}{5}(2)^5 + \frac{8}{3}(2)^3\right)$$

$$S_y = \frac{4\pi}{81}\left(\frac{2}{5}[(85)^2\sqrt{85}] - \frac{8}{3}[85\sqrt{85}] - \frac{2}{5}(32) + \frac{8}{3}(8)\right)$$

$$S_y = \frac{4\pi}{81}\left(\frac{2(85)^2\sqrt{85}}{5} - \frac{680\sqrt{85}}{3} - \frac{64}{5} + \frac{64}{3}\right)$$

$$S_y = \frac{4\pi}{81}\left(\frac{2\cdot 5\cdot 5\cdot 17\cdot 17\cdot \sqrt{85}}{5} - \frac{680\sqrt{85}}{3} - \frac{64}{5} + \frac{64}{3}\right)$$

$$S_y = \frac{4\pi}{81}\left(2{,}890\sqrt{85} - \frac{680\sqrt{85}}{3} - \frac{64}{5} + \frac{64}{3}\right)$$

$$S_y = \frac{4\pi}{81}\left(2{,}890\sqrt{85} - \frac{64}{5} + \frac{64 - 680\sqrt{85}}{3}\right)$$

Find a common denominator.

$$S_y = \frac{4\pi}{81}\left(\frac{43{,}350\sqrt{85}}{15} - \frac{192}{15} + \frac{320 - 3{,}400\sqrt{85}}{15}\right)$$

$$S_y = \frac{4\pi}{81}\left(\frac{43{,}350\sqrt{85} - 3{,}400\sqrt{85} - 192 + 320}{15}\right)$$

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$$S_y = \frac{4\pi}{81}\left(\frac{39{,}950\sqrt{85} + 128}{15}\right)$$