Ratio test

The ratio test for convergence lets us determine the convergence or divergence of a series a_n using the limit

$$L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$$

Once we find a value for L, we can say that

the series converges absolutely if $L < 1$.

the series diverges if $L > 1$ or if L is infinite.

the test is inconclusive if $L = 1$.

The ratio test is used most often when our series includes a factorial or something raised to the nth power.

Example

Use the ratio test to say whether the series converges or diverges.

$$\sum_{n=1}^{\infty} \frac{n^3}{4^n}$$

To use the ratio test, we need to solve for the limit

$$L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$$

and then evaluate the value of L.

$$L = \lim_{n\to\infty} \left| \frac{\frac{(n+1)^3}{4^{n+1}}}{\frac{n^3}{4^n}} \right|$$

We can drop the absolute value bars since all of our terms will be positive.

$$L = \lim_{n\to\infty} \frac{\frac{(n+1)^3}{4^{n+1}}}{\frac{n^3}{4^n}}$$

$$L = \lim_{n\to\infty} \frac{(n+1)^3}{4^{n+1}} \left(\frac{4^n}{n^3} \right)$$

Grouping like bases together, we get

$$L = \lim_{n\to\infty} \frac{(n+1)^3}{n^3} \left(\frac{4^n}{4^{n+1}} \right)$$

$$L = \lim_{n\to\infty} \frac{(n+1)^3}{n^3} \left(4^{n-(n+1)} \right)$$

$$L = \lim_{n\to\infty} \frac{(n+1)^3}{n^3} \left(4^{-1} \right)$$

$$L = \lim_{n\to\infty} \frac{(n+1)^3}{n^3} \left(\frac{1}{4} \right)$$

$$L = \frac{1}{4} \lim_{n \to \infty} \frac{(n+1)^3}{n^3}$$

$$L = \frac{1}{4} \lim_{n \to \infty} \frac{n^3 + 3n^2 + 3n + 1}{n^3}$$

$$L = \frac{1}{4} \lim_{n \to \infty} \frac{n^3 + 3n^2 + 3n + 1}{n^3} \left(\frac{\frac{1}{n^3}}{\frac{1}{n^3}} \right)$$

$$L = \frac{1}{4} \lim_{n \to \infty} \frac{\frac{n^3}{n^3} + \frac{3n^2}{n^3} + \frac{3n}{n^3} + \frac{1}{n^3}}{\frac{n^3}{n^3}}$$

$$L = \frac{1}{4} \lim_{n \to \infty} \frac{1 + \frac{3}{n} + \frac{3}{n^2} + \frac{1}{n^3}}{1}$$

$$L = \left(\frac{1}{4} \right) \frac{1 + \frac{3}{\infty} + \frac{3}{\infty} + \frac{1}{\infty}}{1}$$

$$L = \left(\frac{1}{4} \right) \frac{1 + 0 + 0 + 0}{1}$$

$$L = \frac{1}{4}$$

Since $L < 1$, we can say that the original series a_n converges absolutely.
