Calculus 2 Workbook Solutions

Basic convergence tests

krista king MATH

LIMIT VS. SUM OF THE SERIES

1. Find the limit of the series, and if it converges, find its sum.

$$\sum_{n=1}^{\infty} 3e^{-n} + 2^{-n}$$

Solution:

The limit of the series is given by

$$\lim_{n\to\infty} 3e^{-n} + \frac{1}{2^n} = \lim_{n\to\infty} \frac{3}{e^n} + \lim_{n\to\infty} \frac{1}{2^n}$$

Notice that the denominator of both expressions gets bigger and bigger, but the numerator is a constant. So the value of each fraction approaches 0.

$$\lim_{n\to\infty} \frac{3}{e^n} + \lim_{n\to\infty} \frac{1}{2^n} = \frac{3}{\infty} + \frac{1}{\infty} = 0 + 0 = 0$$

To find the sum of the series, rewrite it as

$$\sum_{n=1}^{\infty} 3e^{-n} + 2^{-n}$$

$$\sum_{n=1}^{\infty} \frac{3}{e^n} + \frac{1}{2^n}$$

The first few terms of the series are

$$a_1 = \frac{3}{e^1} + \frac{1}{2^1} = \frac{3}{e} + \frac{1}{2}$$

$$a_2 = \frac{3}{e^2} + \frac{1}{2^2} = \frac{3}{e^2} + \frac{1}{4}$$

$$a_3 = \frac{3}{e^3} + \frac{1}{2^3} = \frac{3}{e^3} + \frac{1}{8}$$

$$a_4 = \frac{3}{e^4} + \frac{1}{2^4} = \frac{3}{e^4} + \frac{1}{16}$$

Then the sum of the series is

$$S = \frac{3}{e} + \frac{1}{2} + \frac{3}{e^2} + \frac{1}{4} + \frac{3}{e^3} + \frac{1}{8} + \frac{3}{e^4} + \frac{1}{16} + \dots + \frac{3}{e^n} + \frac{1}{2^n} + \dots$$

Split this into two separate sums.

$$S_1 = \frac{3}{e} + \frac{3}{e^2} + \frac{3}{e^3} + \frac{3}{e^4} + \dots + \frac{3}{e^n} + \dots$$

$$S_2 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \dots + \frac{1}{2^n} + \dots$$

S_1 is a geometric series with $a = 3/e$ and $r = 1/e$. So S_1 is

$$S_1 = \frac{a}{1-r} = \frac{\frac{3}{e}}{1-\frac{1}{e}} = \frac{\frac{3}{e}}{\frac{e}{e}-\frac{1}{e}} = \frac{\frac{3}{e}}{\frac{e-1}{e}} = \frac{3}{e-1}$$

S_2 is a geometric series with $a = 1/2$ and $r = 1/2$. So S_2 is

$$S_2 = \frac{a}{1-r} = \frac{\frac{1}{2}}{1-\frac{1}{2}} = \frac{\frac{1}{2}}{\frac{1}{2}} = 1$$

So the sum of the series and limit of the series are

$$S_1 + S_2 = \frac{3}{e-1} + 1$$

$$\lim_{n\to\infty} \frac{3}{e^n} + \frac{1}{2^n} = 0$$

■ 2. Find the limit of the series, and if it converges, find its sum.

$$\sum_{n=1}^{\infty} \frac{3^n + 2^n}{6^n}$$

Solution:

The limit of the series is given by

$$\lim_{n\to\infty} \frac{3^n + 2^n}{6^n}$$

$$\lim_{n\to\infty} \frac{3^n}{6^n} + \lim_{n\to\infty} \frac{2^n}{6^n}$$

$$\lim_{n\to\infty} \left(\frac{3}{6}\right)^n + \lim_{n\to\infty} \left(\frac{2}{6}\right)^n$$

$$\lim_{n\to\infty}\left(\frac{1}{2}\right)^n+\lim_{n\to\infty}\left(\frac{1}{3}\right)^n$$

$$\lim_{n\to\infty}\frac{1}{2^n}+\lim_{n\to\infty}\frac{1}{3^n}$$

Notice that the denominator of both expressions gets bigger and bigger, but the numerator is a constant. So the value of each fraction approaches 0.

$$\lim_{n\to\infty}\frac{1}{2^n}+\lim_{n\to\infty}\frac{1}{3^n}=\frac{1}{\infty}+\frac{1}{\infty}=0+0=0$$

To find the sum of the series, rewrite it as

$$\sum_{n=1}^{\infty}\frac{3^n+2^n}{6^n}$$

$$\sum_{n=1}^{\infty}\frac{1}{2^n}+\frac{1}{3^n}$$

The first few terms of the series are

$$a_1=\frac{1}{2^1}+\frac{1}{3^1}=\frac{1}{2}+\frac{1}{3}$$

$$a_2=\frac{1}{2^2}+\frac{1}{3^2}=\frac{1}{4}+\frac{1}{9}$$

$$a_3=\frac{1}{2^3}+\frac{1}{3^3}=\frac{1}{8}+\frac{1}{27}$$

$$a_4=\frac{1}{2^4}+\frac{1}{3^4}=\frac{1}{16}+\frac{1}{81}$$

Then the sum of the series is

$$S = \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{9} + \frac{1}{8} + \frac{1}{27} + \frac{1}{16} + \frac{1}{81} + \dots + \frac{1}{2^n} + \frac{1}{3^n} + \dots$$

Split this into two separate sums.

$$S_1 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \dots + \frac{1}{2^n} + \dots$$

$$S_2 = \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \frac{1}{81} + \dots + \frac{1}{3^n} + \dots$$

S_1 is a geometric series with $a = 1/2$ and $r = 1/2$. So S_1 is

$$S_1 = \frac{a}{1-r} = \frac{\frac{1}{2}}{1-\frac{1}{2}} = \frac{\frac{1}{2}}{\frac{1}{2}} = 1$$

S_2 is a geometric series with $a = 1/3$ and $r = 1/3$. So S_2 is

$$S_2 = \frac{a}{1-r} = \frac{\frac{1}{3}}{1-\frac{1}{3}} = \frac{\frac{1}{3}}{\frac{2}{3}} = \frac{1}{2}$$

So the sum of the series and limit of the series are

$$S_1 + S_2 = 1 + \frac{1}{2} = \frac{3}{2}$$

$$\lim_{n\to\infty} \frac{1}{2^n} + \frac{1}{3^n} = 0$$

■ 3. Find the limit of the series, and if it converges, find its sum.

$$\sum_{n=1}^{\infty} \frac{3}{5^n} + \frac{2}{n}$$

Solution:

The limit of the series is given by

$$\lim_{n\to\infty} \frac{3}{5^n} + \frac{2}{n} = \lim_{n\to\infty} \frac{3}{5^n} + \lim_{n\to\infty} \frac{2}{n}$$

Notice that the denominator of both expressions gets bigger and bigger, but the numerator is a constant. So the value of each fraction approaches 0.

$$\lim_{n\to\infty} \frac{1}{2^n} + \lim_{n\to\infty} \frac{1}{3^n} = \frac{1}{\infty} + \frac{1}{\infty} = 0 + 0 = 0$$

To find the sum of the series, rewrite it as

$$\sum_{n=1}^{\infty} \frac{3}{5^n} + \frac{2}{n}$$

$$\sum_{n=1}^{\infty} \frac{3}{5^n} + \sum_{n=1}^{\infty} \frac{2}{n}$$

The first few terms of the first series are

$$a_1 = \frac{3}{5^1} = \frac{3}{5}$$

$$a_2 = \frac{3}{5^2} = \frac{3}{25}$$

$$a_3 = \frac{3}{5^3} = \frac{3}{125}$$

$$a_4 = \frac{3}{5^4} = \frac{3}{625}$$

Then the sum of the series is

$$S_1 = \frac{3}{5} + \frac{3}{25} + \frac{3}{125} + \frac{3}{625} + \ldots + \frac{3}{5^n} + \ldots$$

S_1 is a geometric series with $a = 3/5$ and $r = 1/5$. So S_1 is

$$S_1 = \frac{a}{1-r} = \frac{\frac{3}{5}}{1 - \frac{1}{5}} = \frac{\frac{3}{5}}{\frac{4}{5}} = \frac{3}{4}$$

The first few terms of the second series are

$$a_1 = \frac{2}{1}$$

$$a_2 = \frac{2}{2}$$

$$a_3 = \frac{2}{3}$$

$$a_4 = \frac{2}{4}$$

Notice that this series is a p-series with $p = 1$, which means the series diverges and has no sum. Since part of the given series has no sum, the whole series has no sum.

INTEGRAL TEST

1. Use the integral test to say whether the series converges or diverges. If it converges, give the value to which it converges.

$$\sum_{n=1}^{\infty} \frac{7}{n^{\frac{3}{2}}}$$

Solution:

Every term of the series is positive, every term is less than the preceding term, and the series is defined for every term because $n \geq 1$, so the integral test will apply.

So express the series as a function.

$$f(x) = \frac{7}{x^{\frac{3}{2}}} = 7x^{-\frac{3}{2}}$$

Then set up the integral.

$$\int_1^{\infty} f(x)\ dx = \lim_{b \to \infty} \int_1^b 7x^{-\frac{3}{2}}\ dx = 7 \lim_{b \to \infty} \int_1^b x^{-\frac{3}{2}}\ dx$$

Integrate, then evaluate over the interval.

$$7 \lim_{b \to \infty} -2x^{-\frac{1}{2}} \Big|_1^b$$

$$7\lim_{b\to\infty} -\frac{2}{\sqrt{x}}\Bigg|_1^b$$

$$7\lim_{b\to\infty} -\frac{2}{\sqrt{b}} - \left(-\frac{2}{\sqrt{1}}\right)$$

$$7\lim_{b\to\infty} -\frac{2}{\sqrt{b}} + 2$$

$$7(0+2)$$

$$14$$

The integral converges to a real number value, which means the series also converges. To find the value to which the series converges, we'll take the limit of the series as $n \to \infty$.

$$\lim_{n\to\infty} \frac{7}{n^{\frac{3}{2}}}$$

$$\lim_{n\to\infty} \frac{7}{\sqrt{n^3}}$$

$$0$$

Therefore, the series is convergent, and converges to 0.

■ 2. Use the integral test to say whether the series converges or diverges. If it converges, give the value to which it converges.

$$\sum_{n=1}^{\infty} \frac{9}{n+1}$$

Solution:

Every term of the series is positive, every term is less than the preceding term, and the series is defined for every term because $n \geq 1$, so the integral test will apply.

So express the series as a function.

$$f(x) = \frac{9}{x+1}$$

Then set up the integral.

$$\int_1^{\infty} f(x)\ dx = \lim_{b \to \infty} \int_1^b \frac{9}{x+1}\ dx = 9 \lim_{b \to \infty} \int_1^b \frac{1}{x+1}\ dx$$

Integrate, then evaluate over the interval.

$$9 \lim_{b \to \infty} \ln|x+1| \Big|_1^b$$

$$9 \lim_{b \to \infty} \ln|b+1| - \ln|1+1|$$

$$9(\infty - \ln 2)$$

$$\infty$$

The integral diverges, which means the series also diverges.

3. Use the integral test to say whether the series converges or diverges. If it converges, give the value to which it converges.

$$\sum_{n=1}^{\infty} \frac{9}{7n-2}$$

Solution:

Every term of the series is positive, every term is less than the preceding term, and the series is defined for every term because $n \geq 1$, so the integral test will apply.

So express the series as a function.

$$f(x) = \frac{9}{7x-2}$$

Then set up the integral.

$$\int_{1}^{\infty} f(x)\ dx = \lim_{b \to \infty} \int_{1}^{b} \frac{9}{7x-2}\ dx = 9 \lim_{b \to \infty} \int_{1}^{b} \frac{1}{7x-2}\ dx$$

Integrate, then evaluate over the interval.

$$\frac{9}{7} \lim_{b \to \infty} \ln|7x-2| \Big|_{1}^{b}$$

$$\frac{9}{7} \lim_{b \to \infty} \ln|7b-2| - \ln|7(1)-2|$$

$$\frac{9}{7}(\infty - \ln 5)$$

$$\infty$$

The integral diverges, which means the series also diverges.

P-SERIES TEST

1. Use the p-series test to say whether the series converges of diverges.

$$\sum_{n=1}^{\infty} \frac{23}{4\sqrt[3]{n}}$$

Solution:

Rewrite the series as

$$\sum_{n=1}^{\infty} \frac{23}{4\sqrt[3]{n}} = \frac{23}{4} \sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n}} = \frac{23}{4} \sum_{n=1}^{\infty} \frac{1}{n^{\frac{1}{3}}}$$

Now that the series is in standard form for a p-series, and $p = 1/3 \leq 1$, we know the series diverges.

2. Use the p-series test to say whether the series converges of diverges.

$$\sum_{n=1}^{\infty} \frac{7}{5n^3}$$

Solution:

Rewrite the series as

$$\sum_{n=1}^{\infty} \frac{7}{5n^3} = \frac{7}{5} \sum_{n=1}^{\infty} \frac{1}{n^3}$$

Now that the series is in standard form for a p-series, and $p = 3 > 1$, we know the series converges.

■ 3. Use the p-series test to say whether the series converges of diverges.

$$\sum_{n=1}^{\infty} \frac{6n^2 + 2n}{9n^4}$$

Solution:

Rewrite the series as

$$\sum_{n=1}^{\infty} \frac{6n^2 + 2n}{9n^4} = \sum_{n=1}^{\infty} \frac{6n^2}{9n^4} + \frac{2n}{9n^4} = \sum_{n=1}^{\infty} \frac{2}{3n^2} + \sum_{n=1}^{\infty} \frac{2}{9n^3} = \frac{2}{3} \sum_{n=1}^{\infty} \frac{1}{n^2} + \frac{2}{9} \sum_{n=1}^{\infty} \frac{1}{n^3}$$

In both series, $1/n^2$ and $1/n^3$, $p > 1$ (because p is 2 in the first series and 3 in the second series), which means both series converge, which means the series in general converges.

NTH TERM TEST

1. Use the nth term test to say whether the series diverges, or whether the nth term test is inconclusive.

$$\sum_{n=1}^{\infty} \frac{1}{2n-1}$$

Solution:

First find the limit of the series.

$$\lim_{n\to\infty} \frac{1}{2n-1} = \frac{1}{2\cdot\infty-1} = \frac{1}{\infty} = 0$$

Because the limit is 0, the nth term test is inconclusive.

2. Use the nth term test to say whether the series diverges, or whether the nth term test is inconclusive.

$$\sum_{n=1}^{\infty} a_n = 8 + 2 + \frac{1}{2} + \frac{1}{8} + \frac{1}{32} + \frac{1}{128} + \dots$$

Solution:

First find the limit of the series.

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} 8 \cdot \left(\frac{1}{4}\right)^{n-1} = \lim_{n\to\infty} \frac{8}{4^{n-1}} = \frac{8}{4^{\infty-1}} = \frac{8}{4^{\infty}} = \frac{8}{\infty} = 0$$

Because the limit is 0, the nth term test is inconclusive.

■ 3. Use the nth term test to say whether the series diverges, or whether the nth term test is inconclusive.

$$\sum_{n=1}^{\infty} \frac{11^n}{10^n}$$

Solution:

First find the limit of the series.

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{11^n}{10^n} = \lim_{n\to\infty} \left(\frac{11}{10}\right)^n = \infty$$

Because the limit is not 0, the nth term test tells us the series will diverge.

■ 4. Use the nth term test to say whether the series diverges, or whether the nth term test is inconclusive.

$$\sum_{n=1}^{\infty} \frac{n}{n+1}$$

Solution:

First find the limit of the series.

$$\lim_{n\to\infty}\frac{n}{n+1}=\frac{\infty}{\infty+1}=\frac{\infty}{\infty}=1$$

Because the limit is not 0, the nth term test tells us the series will diverge.

W W W . K R I S T A K I N G M A T H . C O M