Topic: Power series representation

Question: Find the power series representation of the function.

$$f(x)=\frac{2x}{4+x^2}$$

Answer choices:

A $\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{4^{n+1}}$

B $\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{2^{2n+1}}$

C $\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{4^{2n+1}}$

D $\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{2^{n+1}}$

Solution: B

To find the power series representation of the given function

$$f(x) = \frac{2x}{4 + x^2}$$

we'll use the standard form of a power series

$$\frac{1}{1 - x} = \sum_{n=0}^{\infty} x^n$$

We'll manipulate the given series until it's in the form

$$\frac{1}{1 - x}$$

We'll get

$$\frac{1}{1 - x} = \frac{2x}{4 + x^2}$$

$$\frac{1}{1 - x} = (2x)\frac{1}{4 + x^2}$$

$$\frac{1}{1 - x} = (2x)\frac{1}{4\left(1 + \frac{x^2}{4}\right)}$$

$$\frac{1}{1 - x} = \left(\frac{2x}{4}\right)\frac{1}{1 + \frac{x^2}{4}}$$

$$\frac{1}{1 - x} = \left(\frac{x}{2}\right)\frac{1}{1 + \frac{x^2}{4}}$$

$$\frac{1}{1-x}=\left(\frac{x}{2}\right)\frac{1}{1-\left(-\frac{x^2}{4}\right)}$$

Matching the new form of the given series to $1/(1-x)$, we can see that $-\left(x^2\right)/4$ from the given series is going to represent x from the standard power series. Plugging this value into the standard form of a power series, we get

$$\sum_{n=0}^{\infty}\left(-\frac{x^2}{4}\right)^n$$

But don't forget about the $x/2$ that we factored out of the given series! We'll need to multiply the sum by this term.

$$\frac{x}{2}\sum_{n=0}^{\infty}\left(-\frac{x^2}{4}\right)^n$$

$$\sum_{n=0}^{\infty}\frac{x^1}{2^1}\left[\frac{(-1)x^2}{4}\right]^n$$

$$\sum_{n=0}^{\infty}\frac{x^1(-1)^n}{2^1}\left(\frac{x^{2n}}{4^n}\right)$$

$$\sum_{n=0}^{\infty}\frac{x^1(-1)^n}{2^1}\left(\frac{x^{2n}}{2^{2n}}\right)$$

$$\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{2^{2n+1}}$$

This is the power series representation of the given function.

Topic: Power series representation

Question: Which statement is true about the behavior of the series when $n \to \infty$?

$$\sum_{n=1}^{\infty} \frac{x^n}{\ln(n+1)}$$

Answer choices:

A Series diverges for $-1 \leq x < 1$.

B Series converges for $-1 \leq x < 1$.

C Series diverges for $0 \leq x < 2$.

D Series converges for $-3 \leq x < 0$.

Solution: B

Given the series

$$\sum_{n=1}^{\infty} \frac{x^n}{\ln(n+1)}$$

apply the ratio test,

$$L = \lim_{n \to \infty} \left| \frac{\frac{x^{n+1}}{\ln(n+1+1)}}{\frac{x^n}{\ln(n+1)}} \right|$$

then simplify the expression.

$$L = \lim_{n \to \infty} \left| \frac{x^{n+1}}{\ln(n+1+1)} \cdot \frac{\ln(n+1)}{x^n} \right|$$

$$L = \lim_{n \to \infty} \left| \frac{x^{n+1}}{x^n} \cdot \frac{\ln(n+1)}{\ln(n+1+1)} \right|$$

$$L = \lim_{n \to \infty} \left| x^{n+1-n} \cdot \frac{\ln(n+1)}{\ln(n+2)} \right|$$

$$L = \lim_{n \to \infty} \left| x \cdot \frac{\ln(n+1)}{\ln(n+2)} \right|$$

The limit affects n, not x, so we can pull $|x|$ out in front of the limit.

$$L = |x| \lim_{n \to \infty} \left| \frac{\ln(n+1)}{\ln(n+2)} \right|$$

Rewrite the fraction.

$$L = |x| \lim_{n\to\infty} \left| \frac{\ln\left(\frac{1}{n}(n+1)n\right)}{\ln\left(\frac{1}{n}(n+2)n\right)} \right|$$

$$L = |x| \lim_{n\to\infty} \left| \frac{\ln\left(\left(1+\frac{1}{n}\right)n\right)}{\ln\left(\left(1+\frac{2}{n}\right)n\right)} \right|$$

Use laws of logarithms to simplify.

$$L = |x| \lim_{n\to\infty} \left| \frac{\ln n + \ln\left(1+\frac{1}{n}\right)}{\ln n + \ln\left(1+\frac{2}{n}\right)} \right|$$

$$L = |x| \lim_{n\to\infty} \left| \frac{\ln n + \ln\left(1+\frac{1}{n}\right)}{\ln n + \ln\left(1+\frac{2}{n}\right)} \cdot \frac{\frac{1}{\ln n}}{\frac{1}{\ln n}} \right|$$

$$L = |x| \lim_{n\to\infty} \left| \frac{\frac{\ln n}{\ln n} + \frac{\ln\left(1+\frac{1}{n}\right)}{\ln n}}{\frac{\ln n}{\ln n} + \frac{\ln\left(1+\frac{2}{n}\right)}{\ln n}} \right|$$

$$L = |x| \lim_{n\to\infty} \left| \frac{1 + \frac{\ln\left(1+\frac{1}{n}\right)}{\ln n}}{1 + \frac{\ln\left(1+\frac{2}{n}\right)}{\ln n}} \right|$$

If we now evaluate the limit as $n \to \infty$, the $1/n$ in the numerator and the $2/n$ in the denominator both go to 0.

$$L = |x| \lim_{n \to \infty} \left| \frac{1 + \frac{\ln(1+0)}{\ln n}}{1 + \frac{\ln(1+0)}{\ln n}} \right|$$

$$L = |x| \lim_{n \to \infty} \left| \frac{1 + \frac{\ln 1}{\ln n}}{1 + \frac{\ln 1}{\ln n}} \right|$$

$$L = |x| \lim_{n \to \infty} \left| \frac{1 + \frac{0}{\ln n}}{1 + \frac{0}{\ln n}} \right|$$

$$L = |x| \lim_{n \to \infty} \left| \frac{1+0}{1+0} \right|$$

$$L = |x| \lim_{n \to \infty} |1|$$

$$L = |x|(1)$$

$$L = |x|$$

This will converge when $L < 1$, so

$$|x| < 1$$

$$-1 \le x \le 1$$

But if we test the endpoints of the interval, $x = -1$ and $x = 1$, we can see that the series converges at $x = -1$, but not at $x = 1$. For $x = -1$,

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{\ln(n+1)}$$

Given the (-1^n) in the numerator, this is an alternating series, so you want to use the alternating series test. Therefore, pull the (-1^n) out in front of the sum.

$$(-1)^n \sum_{n=1}^{\infty} \frac{1}{\ln(n+1)}$$

By the alternating series test, the remaining series, $a_n = 1/\ln(n+1)$ will converge if 1) a_n is decreasing and 2) $\lim_{n\to\infty} a_n = 0$. We can tell that the series is decreasing, because as n gets larger, the $n+1$ in the denominator gets larger, and as the argument of $\ln$ gets larger, the value for the log gets larger and larger. And as the denominator of a fraction gets larger, the value of the fraction in general gets smaller. So we know the series is decreasing. Now we just need to show that $\lim_{n\to\infty} a_n = 0$.

$$\lim_{n\to\infty} \frac{1}{\ln(n+1)} = 0$$

$$\frac{1}{\ln(\infty+1)} = 0$$

$$\frac{1}{\ln(\infty)} = 0$$

$$\frac{1}{\infty} = 0$$

$$0 = 0$$

We've now shown that $a_n = 1/\ln(n+1)$ is decreasing and that $\lim_{n\to\infty} a_n = 0$.

Therefore, we have proof that $a_n = 1/\ln(n+1)$ converges by the alternating series test, which means the series converges at $x = -1$.

For $x = 1$,

$$\sum_{n=1}^{\infty} \frac{(1)^n}{\ln(n+1)}$$

$$\sum_{n=1}^{\infty} \frac{1}{\ln(n+1)}$$

For this series $a_n = \dfrac{1}{\ln(n+1)}$, we can use the comparison test to determine convergence. We'll use the harmonic series $b_n = 1/n$, which we know diverges. Since we know it diverges, we want to see if we can prove that $a_n > b_n$. If a_n is always greater than b_n, and b_n diverges, then we'll know that a_n also diverges. Let's compare some values of n in both series.

For $n = 1$, $a_n = 1/\ln 2 = 1.44$ and $b_n = 1/1 = 1.00$

For $n = 2$, $a_n = 1/\ln 3 = 0.91$ and $b_n = 1/2 = 0.50$

For $n = 3$, $a_n = 1/\ln 4 = 0.72$ and $b_n = 1/3 = 0.33$

For $n = 4$, $a_n = 1/\ln 5 = 0.62$ and $b_n = 1/4 = 0.25$

For $n = 5$, $a_n = 1/\ln 6 = 0.56$ and $b_n = 1/5 = 0.20$

We can see that a_n is always going to be greater than b_n, and since we know that b_n diverges, this proves by the comparison test that a_n must also diverge.

Therefore, the original series in this problem converges at $x = -1$, but diverges at $x = 1$. So the series converges for $-1 \leq x < 1$.

Topic: Power series representation

Question: Which series represents the expression?

$$e^{\frac{1}{2x^2}}$$

Answer choices:

A $\displaystyle\sum_{n=0}^{\infty} \frac{1}{2^n n! x^{2n}}$

B $\displaystyle\sum_{n=0}^{\infty} \frac{1}{n! x^{2n-1}}$

C $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{n-1} n!}$

D $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n-1}}{2^n n!}$

Solution: A

We know that the power series expression of e^x is

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

Replacing e^x with $e^{\frac{1}{x}}$ gives us

$$e^{\frac{1}{x}} = \sum_{n=0}^{\infty} \frac{\left(\frac{1}{x}\right)^n}{n!}$$

$$e^{\frac{1}{x}} = \sum_{n=0}^{\infty} \frac{\frac{1^n}{x^n}}{n!}$$

$$e^{\frac{1}{x}} = \sum_{n=0}^{\infty} \frac{1}{x^n n!}$$

Now if we substitute $2x^2$ for x, we'll arrive at the power series representation for $e^{\frac{1}{2x^2}}$.

$$e^{\frac{1}{2x^2}} = \sum_{n=0}^{\infty} \frac{1}{(2x^2)^n n!}$$

$$e^{\frac{1}{2x^2}} = \sum_{n=0}^{\infty} \frac{1}{2^n n! x^{2n}}$$