Mean value theorem for integrals

The mean value theorem for integrals tells us that, for a continuous function $f(x)$, there's at least one point c inside the interval $[a, b]$ at which the value of the function will be equal to the average value of the function over that interval.

This means we can equate the average value of the function over the interval to the value of the function at the single point.

In other words,

$$\frac{1}{b-a}\int_a^b f(x)\ dx = f(c)$$

The equation above sets the average value of the function over the interval $[a, b]$ (on the left), equal to the value of the function at the point c (on the right). If we multiply both sides by $(b-a)$, we get the mean value theorem for integrals:

$$\int_a^b f(x)\ dx = f(c)(b-a)$$

Example

Find the point c that satisfies the mean value theorem for integrals on the interval $[1,4]$.

$$f(x) = 3x^2 - 2x$$

Looking at the equation we can see that it is a polynomial and is therefore continuous. This means that we can go ahead and use the mean value theorem for integrals.

$$\int_a^b f(x)\ dx = f(c)(b-a)$$

$$\int_1^4 3x^2 - 2x\ dx = \left(3c^2 - 2c\right)(4-1)$$

$$\int_1^4 3x^2 - 2x\ dx = 9c^2 - 6c$$

Now we can break up the integral to make it easier to solve.

$$\int_1^4 3x^2\ dx + \int_1^4 -2x\ dx = 9c^2 - 6c$$

$$3\int_1^4 x^2\ dx - 2\int_1^4 x\ dx = 9c^2 - 6c$$

Integrate.

$$\left[3\left(\frac{x^3}{3}\right) - 2\left(\frac{x^2}{2}\right)\right]\Bigg|_1^4 = 9c^2 - 6c$$

$$\left(x^3 - x^2\right)\Big|_1^4 = 9c^2 - 6c$$

Now we can evaluate over the interval.

$$\left[(4)^3-(4)^2\right]-\left[(1)^3-(1)^2\right]=9c^2-6c$$

$$48=9c^2-6c$$

$$0=9c^2-6c-48$$

Now we need to solve for c.

$$0=3\left(3c^2-2c-16\right)$$

$$0=3c^2-2c-16$$

$$0=(3c-8)(c+2)$$

Setting each of the factors equal to 0 individually to solve for c, we get

$$3c-8=0$$

$$c=\frac{8}{3}$$

and

$$c+2=0$$

$$c=-2$$

Only one of these values, $c=8/3$, falls in the interval $[1,4]$, which means it's the only solution.

It's possible to find more than one valid answer for c, but in the last example, there's only one point, $c=8/3$, at which the value of the function is equal to the average value of the function over the interval.