Tangent line to the parametric curve

We'll use the same point-slope formula to define the equation of the tangent line to the parametric curve that we used to define the tangent line to a cartesian curve, which is

$$y - y_1 = m(x - x_1)$$

where m is the slope and (x_1, y_1) is the point where the tangent line intersects the curve.

To find the slope m, we'll use the formula for the derivative of a parametric curve.

$$m = \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

Once we find the derivative of the parametric curve using this formula, we'll plug the given point into the derivative to find the slope at that particular point.

Then we'll plug the slope and the given point into the point-slope formula for the equation of a line and simplify to get our tangent line equation.

Example

Find the tangent line(s) to the parametric curve.

$$x = t^5 - 4t^3$$

$$y = t^2$$

at $(0,4)$

To find the derivative of the parametric curve, we'll first need to calculate dy/dt and dx/dt.

$$x = t^5 - 4t^3$$

$$\frac{dx}{dt} = 5t^4 - 12t^2$$

and

$$y = t^2$$

$$\frac{dy}{dt} = 2t$$

Plugging these into the derivative formula, we get

$$\frac{dy}{dx} = \frac{2t}{5t^4 - 12t^2}$$

We need to plug the given point into the derivative we just found, but the given point is a cartesian point, and we only have t in our derivative equation. Therefore, in order to plug the given point into the derivative, we need to convert it from a cartesian point into a parameter value for t. To do this, we'll plug the given point into both of the original parametric equations, and look for matching solutions.

$$x = t^5 - 4t^3$$

$$0 = t^5 - 4t^3$$

$$0 = t^3\left(t^2 - 4\right)$$

$$t = 0, \ \ \pm 2$$

and

$$y = t^2$$

$$4 = t^2$$

$$t = \pm 2$$

In order for the parameter value to be valid, it has to be a solution in both equations, which means the parameter value we're interested in are $t = \pm 2$. Since we got two solutions, we're going to have two tangent lines.

To solve for the slope of each tangent line, we'll plug $t = \pm 2$ into the derivative equation we found above.

For $t = 2$:

$$f(2) = \frac{2(2)}{5(2)^4 - 12(2)^2}$$

$$f(2) = \frac{4}{80 - 48}$$

$$f(2) = \frac{1}{8}$$

For $t = -2$:

$$f(-2) = \frac{2(-2)}{5(-2)^4 - 12(-2)^2}$$

$$f(-2) = \frac{-4}{80 - 48}$$

$$f(-2) = -\frac{1}{8}$$

Finally, plugging the slopes we found and the given point $(0,4)$ into the point-slope formula for the equation of a line, we get the following two tangent lines.

$$y - y_1 = m(x - x_1)$$

$$y - 4 = \frac{1}{8}(x - 0)$$

$$y = \frac{1}{8}x + 4$$

and

$$y - y_1 = m(x - x_1)$$

$$y - 4 = -\frac{1}{8}(x - 0)$$

$$y = -\frac{1}{8}x + 4$$

The equations can also be manipulated into this form:

$$8y = x + 32$$

$$-x + 8y = 32$$

and

$$8y = -x + 32$$

$$x + 8y = 32$$