Area bounded by one loop of a polar curve

When we need to find the area bounded by a single loop of the polar curve, we'll use the same formula we used to find area inside the polar curve in general.

$$A = \int_{\alpha}^{\beta} \frac{1}{2} r^2 \, d\theta$$

where $[\alpha, \beta]$ is the interval

where r is the equation of the polar curve

The best way to find the interval that defines one loop of the curve is to graph the curve.

Example

Find the area bounded by one loop of the the polar curve.

$$r = 3\sin(2\theta)$$

We'll start by finding points that we can use to graph the curve. In order to do so, we'll take the value inside the trigonometric function, set it equal to $\pi/2$, and solve for θ.

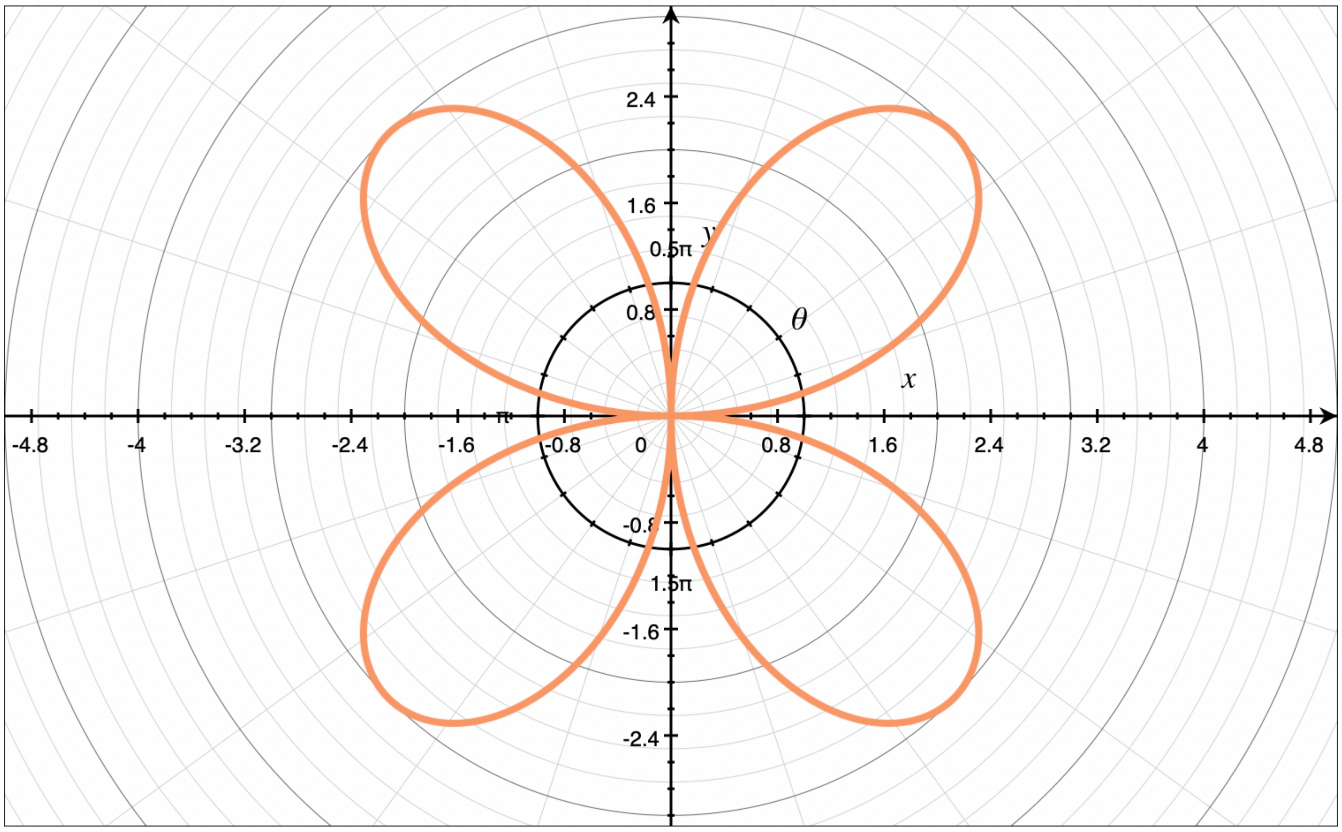
$$2\theta = \frac{\pi}{2}$$

$$\theta = \frac{\pi}{4}$$

We need to find coordinate points for multiples of $\pi/4$ in the interval $0 \le \theta \le 2\pi$.

θ	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π	$\frac{5\pi}{4}$	$\frac{3\pi}{2}$	$\frac{7\pi}{4}$	2π
$r = 3\sin(2\theta)$	0	3	0	-3	0	3	0	-3	0

Plotting these points on polar axes, we get



From the graph, we can see that the curve starts at $(0,0)$, goes out to 3 at an angle $\pi/4$, then curves back to the origin at the angle $\pi/2$. Plugging this into the area formula, we get

$$A = \int_0^{\frac{\pi}{2}} \frac{1}{2} \left[3\sin(2\theta)\right]^2 \ d\theta$$

$$A = \int_0^{\frac{\pi}{2}} \frac{1}{2} \left[9 \sin^2(2\theta)\right] \ d\theta$$

$$A = \frac{9}{2} \int_0^{\frac{\pi}{2}} \sin^2(2\theta) \ d\theta$$

We'll use u-substitution, letting

$$u = 2\theta$$

$$du = 2 \ d\theta$$

$$d\theta = \frac{du}{2}$$

We'll substitute into the integral.

$$A = \frac{9}{2} \int_{\theta=0}^{\theta=\frac{\pi}{2}} \sin^2 u \ \frac{du}{2}$$

$$A = \frac{9}{4} \int_{\theta=0}^{\theta=\frac{\pi}{2}} \sin^2 u \ du$$

Since $\sin^2 u = \frac{1}{2} \left[1 - \cos(2u)\right]$, we get

$$A = \frac{9}{4} \int_{\theta=0}^{\theta=\frac{\pi}{2}} \frac{1}{2} \left[1 - \cos(2u)\right] \ du$$

$$A = \frac{9}{4} \int_{\theta=0}^{\theta=\frac{\pi}{2}} \frac{1}{2} - \frac{1}{2} \cos(2u) \ du$$

$$A = \frac{9}{4}\left[\frac{1}{2}u - \frac{1}{4}\sin(2u)\right]\Bigg|_{\theta=0}^{\theta=\frac{\pi}{2}}$$

Back-substituting for u, we get

$$A = \frac{9}{4}\left[\frac{1}{2}(2\theta) - \frac{1}{4}\sin(2(2\theta))\right]\Bigg|_{0}^{\frac{\pi}{2}}$$

$$A = \frac{9}{4}\left[\theta - \frac{1}{4}\sin(4\theta)\right]\Bigg|_{0}^{\frac{\pi}{2}}$$

$$A = \frac{9}{4}\left[\frac{\pi}{2} - \frac{1}{4}\sin\left(4\cdot\frac{\pi}{2}\right) - \left(0 - \frac{1}{4}\sin(4\cdot 0)\right)\right]$$

$$A = \frac{9}{4}\left(\frac{\pi}{2} - \frac{1}{4}\sin 2\pi + \frac{1}{4}\sin 0\right)$$

$$A = \frac{9}{4}\left(\frac{\pi}{2} - \frac{1}{4}(0) + \frac{1}{4}(0)\right)$$

$$A = \frac{9}{4}\left(\frac{\pi}{2}\right)$$

$$A = \frac{9\pi}{8}$$