Topic: Limit comparison test

Question: Use the limit comparison test to say whether or not the series converges.

$$\sum_{n=1}^{\infty} \frac{1}{4\sqrt{n} + \sqrt[3]{n}}$$

Answer choices:

A a_n converges

B a_n diverges

C b_n converges but the test is inconclusive

D b_n diverges but the test is inconclusive

Solution: B

The limit comparison test for convergence lets us determine the convergence or divergence of the given series a_n by *comparing* it to a similar, but simpler comparison series b_n.

We're usually trying to find a comparison series that's a geometric or p-series, since it's very easy to determine the convergence of a geometric or p-series.

We can use the limit comparison test to show that

the original series a_n is **diverging** if

$a_n > 0$ and $b_n > 0$,

$\lim_{n\to\infty} \frac{a_n}{b_n} = L > 0$, and

the comparison series b_n is diverging

the original series a_n is **converging** if

$a_n > 0$ and $b_n > 0$,

$\lim_{n\to\infty} \frac{a_n}{b_n} = L > 0$, and

the comparison series b_n is converging

Before we can use the limit comparison test with the series a_n that we're given in this problem, we need to create a similar, but simpler comparison series b_n.

We'll use the numerator from a_n for b_n, since the numerator is already pretty simple. In the denominator, the $4\sqrt{n}$ carries a lot more weight and will affect the series more than the $\sqrt[3]{n}$ when n gets very large, so we'll use only the $4\sqrt{n}$ in the denominator of the comparison series.

$$b_n = \frac{1}{4\sqrt{n}}$$

$$b_n = \frac{1}{4}\left(\frac{1}{\sqrt{n}}\right)$$

$$b_n = \frac{1}{4}\left(\frac{1}{n^{\frac{1}{2}}}\right)$$

The comparison series is a p-series. Since the p-series test tells us that the series will

converge when $p > 1$

diverge when $p \leq 1$

we can say that $1/2 \leq 1$ and therefore that b_n diverges.

Knowing that the comparison series diverges, we need to show that

$a_n > 0$ and $b_n > 0$, and

$$\lim_{n\to\infty} \frac{a_n}{b_n} = L > 0$$

in order to prove that the original series a_n is also diverging. If we can't verify these inequalities, then the limit comparison test will be inconclusive.

To verify that $a_n > 0$ and $b_n > 0$, we'll compare a few points from a_n and b_n. Since we've got a square root in a_n, let's use squares, like $n = 1$, $n = 4$ and $n = 9$.

	a_n	b_n
$n = 1$	$a_1 = \frac{1}{4\sqrt{1} + \sqrt[3]{1}} = \frac{1}{5}$	$b_1 = \frac{1}{4\sqrt{1}} = \frac{1}{4}$
$n = 4$	$a_4 = \frac{1}{4\sqrt{4} + \sqrt[3]{4}} = \frac{1}{8 + \sqrt[3]{4}}$	$b_4 = \frac{1}{4\sqrt{4}} = \frac{1}{8}$
$n = 9$	$a_9 = \frac{1}{4\sqrt{9} + \sqrt[3]{9}} = \frac{1}{12 + \sqrt[3]{9}}$	$b_9 = \frac{1}{4\sqrt{9}} = \frac{1}{12}$

Looking at just these few terms, we can see that $a_n > 0$ and $b_n > 0$ for all n, so our series satisfy the first two inequalities.

Now we need to verify that

$$\lim_{n \to \infty} \frac{a_n}{b_n} = L > 0$$

Plugging our series into this formula, we get

$$L = \lim_{n \to \infty} \frac{\frac{1}{4\sqrt{n} + \sqrt[3]{n}}}{\frac{1}{4\sqrt{n}}}$$

$$L = \lim_{n \to \infty} \frac{1}{4\sqrt{n} + \sqrt[3]{n}} \left(\frac{4\sqrt{n}}{1} \right)$$

$$L = \lim_{n\to\infty} \frac{4\sqrt{n}}{4\sqrt{n} + \sqrt[3]{n}}$$

$$L = \lim_{n\to\infty} \frac{4n^{\frac{1}{2}}}{4n^{\frac{1}{2}} + n^{\frac{1}{3}}}$$

$$L = \lim_{n\to\infty} \frac{4n^{\frac{1}{2}}}{4n^{\frac{1}{2}} + n^{\frac{1}{3}}} \left(\frac{\frac{1}{n^{\frac{1}{2}}}}{\frac{1}{n^{\frac{1}{2}}}} \right)$$

$$L = \lim_{n\to\infty} \frac{\frac{4n^{\frac{1}{2}}}{n^{\frac{1}{2}}}}{\frac{4n^{\frac{1}{2}}}{n^{\frac{1}{2}}} + \frac{n^{\frac{1}{3}}}{n^{\frac{1}{2}}}}$$

$$L = \lim_{n\to\infty} \frac{4}{4 + \frac{1}{n^{\frac{1}{2}-\frac{1}{3}}}}$$

$$L = \lim_{n\to\infty} \frac{4}{4 + \frac{1}{n^{\frac{1}{6}}}}$$

$$L = \frac{4}{4 + \frac{1}{\infty^{\frac{1}{6}}}}$$

$$L = \frac{4}{4 + \frac{1}{\infty}}$$

$$L = \frac{4}{4 + 0}$$

$$L = 1$$

Since

$$L = 1 > 0$$

we've shown that the limit comparison test is valid for this problem, and therefore that the original series a_n diverges since b_n diverges.

Topic: Limit comparison test

Question: Use the limit comparison test to say whether or not the series converges.

$$\sum_{n=1}^{\infty} \frac{3}{n\sqrt[n]{n}}$$

Answer choices:

A a_n converges

B a_n diverges

C b_n converges but the test is inconclusive

D b_n diverges but the test is inconclusive

Solution: B

The limit comparison test for convergence lets us determine the convergence or divergence of the given series a_n by *comparing* it to a similar, but simpler comparison series b_n.

We're usually trying to find a comparison series that's a geometric or p-series, since it's very easy to determine the convergence of a geometric or p-series.

We can use the limit comparison test to show that

the original series a_n is **diverging** if

$a_n > 0$ and $b_n > 0$,

$\lim_{n\to\infty} \frac{a_n}{b_n} = L > 0$, and

the comparison series b_n is diverging

the original series a_n is **converging** if

$a_n > 0$ and $b_n > 0$,

$\lim_{n\to\infty} \frac{a_n}{b_n} = L > 0$, and

the comparison series b_n is converging

Before we can use the limit comparison test with the series a_n that we're given in this problem, we need to create a similar, but simpler comparison series b_n.

We'll use the numerator from a_n for b_n, since the numerator is already pretty simple, but we'll leave out the 3 since it'll have less impact on our series as $n \to \infty$ (which means we'll just use 1). In the denominator, the n carries a lot more weight and will affect the series more than the $\sqrt[n]{n}$ (since $\sqrt[n]{n}$ approaches 1 when n gets very large), so we'll use only the n in the denominator of the comparison series.

$$b_n = \frac{1}{n}$$

$$b_n = \frac{1}{n^1}$$

The comparison series is a p-series (it's also the harmonic series). Since the p-series test tells us that the series will

converge when $p > 1$

diverge when $p \le 1$

we can say that $1 \le 1$ and therefore that b_n diverges.

Knowing that the comparison series diverges, we need to show that

$a_n > 0$ and $b_n > 0$, and

$$\lim_{n\to\infty} \frac{a_n}{b_n} = L > 0$$

in order to prove that the original series a_n is also diverging. If we can't verify these inequalities, then the limit comparison test will be inconclusive. To verify that $a_n > 0$ and $b_n > 0$, we'll compare a few points from a_n and b_n.

Since we've got a square root in a_n, let's use squares, like $n = 1$, $n = 4$ and $n = 9$.

	a_n	b_n
$n = 1$	$a_1 = \frac{3}{1\sqrt[1]{1}} = \frac{3}{1^{\frac{1}{1}}} = 3$	$b_1 = \frac{1}{1} = 1$
$n = 4$	$a_4 = \frac{3}{4\sqrt[4]{4}} = \frac{3}{4(4)^{\frac{1}{4}}} = \frac{3}{4^{\frac{5}{4}}}$	$b_4 = \frac{1}{4}$
$n = 9$	$a_9 = \frac{3}{9\sqrt[9]{9}} = \frac{3}{9(9)^{\frac{1}{9}}} = \frac{3}{9^{\frac{10}{9}}}$	$b_9 = \frac{1}{9}$

Looking at just these few terms, we can see that $a_n > 0$ and $b_n > 0$ for all n, so our series satisfy the first two inequalities.

Now we need to verify that

$$\lim_{n\to\infty} \frac{a_n}{b_n} = L > 0$$

Plugging our series into this formula, we get

$$L = \lim_{n\to\infty} \frac{\frac{3}{n\sqrt[n]{n}}}{\frac{1}{n}}$$

$$L = \lim_{n\to\infty} \frac{3}{n\sqrt[n]{n}}\left(\frac{n}{1}\right)$$

$$L = \lim_{n\to\infty} \frac{3}{\sqrt[n]{n}}$$

$$L = \lim_{n \to \infty} \frac{3}{n^{\frac{1}{n}}}$$

$$L = \frac{3}{\infty^{\frac{1}{\infty}}}$$

$$L = \frac{3}{\infty^0}$$

$$L = \frac{3}{1}$$

$$L = 3$$

Since

$$L = 3 > 0$$

we've shown that the limit comparison test is valid for this problem, and therefore that the original series a_n diverges since b_n diverges.

Topic: Limit comparison test

Question: Use the limit comparison test to say whether or not the series converges.

$$\sum_{n=1}^{\infty} \frac{2n}{3n^2+1}$$

Answer choices:

A $\quad a_n$ converges

B $\quad a_n$ diverges

C $\quad b_n$ converges but the test is inconclusive

D $\quad b_n$ diverges but the test is inconclusive

Solution: B

The limit comparison test for convergence lets us determine the convergence or divergence of the given series a_n by *comparing* it to a similar, but simpler comparison series b_n.

We're usually trying to find a comparison series that's a geometric or p-series, since it's very easy to determine the convergence of a geometric or p-series.

We can use the limit comparison test to show that

the original series a_n is **diverging** if

$a_n > 0$ and $b_n > 0$,

$$\lim_{n\to\infty} \frac{a_n}{b_n} = L > 0, \text{ and}$$

the comparison series b_n is diverging

the original series a_n is **converging** if

$a_n > 0$ and $b_n > 0$,

$$\lim_{n\to\infty} \frac{a_n}{b_n} = L > 0, \text{ and}$$

the comparison series b_n is converging

Before we can use the limit comparison test with the series a_n that we're given in this problem, we need to create a similar, but simpler comparison series b_n.

We'll use the numerator from a_n for b_n, since the numerator is already pretty simple, but we'll leave out the 6 since it'll have less impact on our series as $n \to \infty$. In the denominator, the $3n^2$ carries a lot more weight and will affect the series more than the 1, so we'll use only the $3n^2$ in the denominator of the comparison series, leaving out the 3 since it'll have less effect than the n^2.

$$b_n = \frac{n}{n^2}$$

$$b_n = \frac{1}{n}$$

$$b_n = \frac{1}{n^1}$$

The comparison series is a p-series. Since the p-series test tells us that the series will

converge when $p > 1$

diverge when $p \le 1$

we can say that $1 \le 1$ and therefore that b_n diverges.

Knowing that the comparison series diverges, we need to show that

$a_n > 0$ and $b_n > 0$, and

$$\lim_{n \to \infty} \frac{a_n}{b_n} = L > 0$$

in order to prove that the original series a_n is also diverging. If we can't verify these inequalities, then the limit comparison test will be inconclusive.

To verify that $a_n > 0$ and $b_n > 0$, we'll compare a few points from a_n and b_n. Let's use $n = 1$, $n = 2$ and $n = 3$.

	a_n	b_n
$n = 1$	$a_1 = \frac{2(1)}{3(1)^2 + 1} = \frac{1}{2}$	$b_1 = \frac{1}{1} = 1$
$n = 2$	$a_2 = \frac{2(2)}{3(2)^2 + 1} = \frac{4}{13}$	$b_2 = \frac{1}{2}$
$n = 3$	$a_3 = \frac{2(3)}{3(3)^2 + 1} = \frac{3}{14}$	$b_3 = \frac{1}{3}$

Looking at just these few terms, we can see that $a_n > 0$ and $b_n > 0$ for all n, so our series satisfy the first two inequalities.

Now we need to verify that

$$\lim_{n \to \infty} \frac{a_n}{b_n} = L > 0$$

Plugging our series into this formula, we get

$$L = \lim_{n \to \infty} \frac{\frac{2n}{3n^2 + 1}}{\frac{1}{n}}$$

$$L = \lim_{n \to \infty} \frac{2n}{3n^2 + 1} \left(\frac{n}{1} \right)$$

$$L = \lim_{n \to \infty} \frac{2n^2}{3n^2 + 1}$$

$$L = \lim_{n \to \infty} \frac{2n^2}{3n^2 + 1} \left(\frac{\frac{1}{n^2}}{\frac{1}{n^2}} \right)$$

$$L = \lim_{n \to \infty} \frac{\frac{2n^2}{n^2}}{\frac{3n^2}{n^2} + \frac{1}{n^2}}$$

$$L = \lim_{n \to \infty} \frac{2}{3 + \frac{1}{n^2}}$$

$$L = \frac{2}{3 + \frac{1}{\infty}}$$

$$L = \frac{2}{3 + 0}$$

$$L = \frac{2}{3}$$

Since

$$L = \frac{2}{3} > 0$$

we've shown that the limit comparison test is valid for this problem, and therefore that the original series a_n diverges since b_n diverges.