Quadratic functions

Quadratic functions are functions in the form

$$ax^2 + bx + c = 0$$

Integrating functions that include a quadratic can sometimes be a little difficult. Most often, we'll see an integral problem in the form

$$\int \frac{Ax + B}{ax^2 + bx + c} \ dx$$

There are three methods we'll use to evaluate quadratic integrals:

- Substitution
- Partial fractions
- Trigonometric substitution

You should try using these techniques in the order listed above, because substitution is the easiest and fastest, and trigonometric substitution is the longest and most difficult.

Substitution

Let's look at how to solve a quadratic integral using substitution.

Example

Evaluate the integral.

$$\int \frac{6x}{3x^2-1}\ dx$$

We'll try substitution, letting

$$u = 3x^2 - 1$$

$$du = 6x\ dx$$

Plugging these back into the integral, we get

$$\int \frac{1}{u}\ du$$

$$\ln|u| + C$$

$$\ln\left|3x^2-1\right| + C$$

Sometimes substitution doesn't work, and we need to use partial fractions instead to evaluate the integral. In order to use partial fractions, we must be able to factor the quadratic.

Partial fractions

Example

Evaluate the integral.

$$\int \frac{9x+8}{3x^2+10x-8}\ dx$$

We notice that the quadratic function in the denominator of the fraction can be factored.

$$\int \frac{9x+8}{(x+4)(3x-2)}\ dx$$

Here we'll use a partial fractions decomposition to split the integral in two. We can always double-check this step by finding a common denominator to bring our separated fractions back together again.

$$\int \frac{2}{x+4}\ dx + \int \frac{3}{3x-2}\ dx$$

$$2\int \frac{1}{x+4}\ dx + 3\int \frac{1}{3x-2}\ dx$$

Now we integrate to get the final answer.

$$2\ln|x+4| + \ln|3x-2| + C$$

If substitution and partial fractions don't work, you might need to use trigonometric substitution.

Trigonometric substitution

Remember that the general formulas for trigonometric substitution are

$$\int \sqrt{b^2x^2 - a^2}\ dx \quad \text{uses the substitution} \quad x = \frac{a}{b}\sec\theta$$

$$\int \sqrt{a^2 - b^2x^2}\ dx \quad \text{uses the substitution} \quad x = \frac{a}{b}\sin\theta$$

$$\int \sqrt{a^2 + b^2x^2}\ dx \quad \text{uses the substitution} \quad x = \frac{a}{b}\tan\theta$$

We won't always find our function already in this format, and sometimes we might have to alter it by completing the square before we can use trigonometric substitution.

Example

Evaluate the integral.

$$\int \sqrt{x^2 + 4x + 5}\ dx$$

First, we notice our equation is not already in the form for trigonometric substitution, so let's try completing the square to see if we can get it into the right format.

To complete the square we take

$$x^2 + 4x + 5 = \left(x^2 + 4x + 4\right) + 5 - 4$$

$$x^2 + 4x + 4 + 5 - 4 = (x + 2)^2 + 1$$

We can put this back into the integral to get

$$\int \sqrt{(x + 2)^2 + 1}\ dx$$

Now we can see that the trigonometric substitution is

$$x + 2 = \tan\theta$$

$$x = \tan\theta - 2$$

$$dx = \sec^2\theta\ d\theta$$

Making the substitution, we get

$$\int \sqrt{(\tan\theta)^2 + 1} \cdot \sec^2\theta\ d\theta$$

$$\int \sqrt{\sec^2\theta} \cdot \sec^2\theta\ d\theta$$

$$\int \sec\theta \cdot \sec^2\theta\ d\theta$$

$$\int \sec^3\theta\ d\theta$$

Integrating, we get

$$\frac{1}{2}\left(\sec\theta\tan\theta + \ln\left|\sec\theta + \tan\theta\right|\right) + C$$

To finish this problem, we need to get the answer back in terms of x instead of θ, so we'll back-substitute.

$$\frac{1}{2}\left[\left(\sqrt{x^2+4x+5}\right)(x+2)+\ln\left|\left(\sqrt{x^2+4x+5}\right)+(x+2)\right|\right]+C$$

$$\frac{1}{2}\left[(x+2)\sqrt{x^2+4x+5}+\ln\left|x+2+\sqrt{x^2+4x+5}\right|\right]+C$$