Limits of composites

Think of a **composite function** as a "function of a function."

For instance, assume that $f(x)=x+1$ and $g(x)=x^2-4$. If we find the composite $f(g(x))$, it means we're plugging $g(x)=x^2-4$ into $f(x)=x+1$. That means we replace every x in $f(x)$ with x^2-4.

$$f(x)=x+1$$

$$f(g(x))=x^2-4+1$$

$$f(g(x))=x^2-3$$

Alternatively, we could find the composite $g(f(x))$, in which case, we'd be plugging $f(x)=x+1$ into $g(x)=x^2-4$. That means we replace every x in $g(x)$ with $x+1$.

$$g(x)=x^2-4$$

$$g(f(x))=(x+1)^2-4$$

$$g(f(x))=(x+1)(x+1)-4$$

$$g(f(x))=x^2+x+x+1-4$$

$$g(f(x))=x^2+2x-3$$

To find the limit of a composite function, we'll find the composite first, and then take the limit of the composite. Let's finish this example so that we can see how to find the limits of both composites.

Example

If $f(x) = x + 1$ and $g(x) = x^2 - 4$, find each limit.

$$\lim_{x \to -1} f(g(x))$$

$$\lim_{x \to -1} g(f(x))$$

First, find the composite $f(g(x))$.

$$f(x) = x + 1$$

$$f(g(x)) = x^2 - 4 + 1$$

$$f(g(x)) = x^2 - 3$$

Next, find the limit of $f(g(x))$.

$$\lim_{x \to -1} f(g(x))$$

$$\lim_{x \to -1} (x^2 - 3)$$

$$(-1)^2 - 3$$

$$-2$$

Now find the composite $g(f(x))$.

$$g(x) = x^2 - 4$$

$$g(f(x)) = (x+1)^2 - 4$$

$$g(f(x)) = (x+1)(x+1) - 4$$

$$g(f(x)) = x^2 + x + x + 1 - 4$$

$$g(f(x)) = x^2 + 2x - 3$$

Next, find the limit of $g(f(x))$.

$$\lim_{x \to -1} g(f(x))$$

$$\lim_{x \to -1} (x^2 + 2x - 3)$$

$$(-1)^2 + 2(-1) - 3$$

$$1 - 2 - 3$$

$$-4$$

So the limits of the composite functions are

$$\lim_{x \to -1} f(g(x)) = -2$$

$$\lim_{x \to -1} g(f(x)) = -4$$

Two ways to evaluate the limit of the composite

Formally, the theorem for the limit of a composite tells us that, if f is continuous at b, and the limit as $x \to a$ of $g(x)$ is b,

$$\lim_{x \to a} g(x) = b$$

then the limit as $x \to a$ of $f(g(x))$ will be $f(b)$

$$\lim_{x \to a} f(g(x)) = f(b)$$

Therefore, we have the following property for the limit of composite functions,

$$\lim_{x \to a} f(g(x)) = f(\lim_{x \to a} g(x))$$

as long as f continuous at b. This equation shows us that we actually have two options when it comes to evaluating the limit of a composite:

1. The left side of the equation shows us taking the composite function $f(g(x))$ first, and then finding the limit as $x \to a$ of the composite.

2. The right side of the equation shows us taking the limit as $x \to a$ of the inner function first, and then plugging that resulting value into the outer function.

Let's continue with the previous example so that we can compare the process for evaluating the limit in both ways.

Example (cont'd)

If $f(x) = x + 1$ and $g(x) = x^2 - 4$, use the theorem for the limit of a composite to evaluate the limit in two ways.

$$\lim_{x \to -1} f(g(x))$$

This is the first part of the last example, and we already solved the limit one way. Previously, we found the composite $f(g(x))$

$$f(x) = x + 1$$

$$f(g(x)) = x^2 - 4 + 1$$

$$f(g(x)) = x^2 - 3$$

and then we took the limit of $f(g(x))$.

$$\lim_{x \to -1} f(g(x))$$

$$\lim_{x \to -1} (x^2 - 3)$$

$$(-1)^2 - 3$$

$$-2$$

Now we want to show that we can solve it a second way. This time, we'll take the limit of the inner function first,

$$\lim_{x \to -1} (x^2 - 4)$$

$$(-1)^2 - 4$$

$$1 - 4$$

$$-3$$

and then we'll evaluate the outer function at this resulting value.

$$f(x) = x + 1$$

$$f(-3) = -3 + 1$$

$$f(-3) = -2$$

We end up with -2 using both methods, whether we find the composite first and then take the limit of the composite function, or whether we take the limit of the inner function and then evaluate the outer function at that resulting value.