Power series representation

We can convert functions into a power series using the standard form of a power series

$$\frac{1}{1-x}=\sum_{n=0}^{\infty}x^n$$

The interval of convergence of the series

$$a<x<b$$

will be the set of values for which the series is converging. Remember, even if we can find an interval of convergence for a series, it doesn't mean that the entire series is converging, only that the series is converging in the specific interval.

The radius of convergence of the series

$$R=\frac{b-a}{2}$$

will always be half of the interval of convergence. You can remember this if you think about the interval of convergence as the diameter of a circle, and the radius of convergence as the half of the diameter.

Example

Find the power series representation, then find the radius and interval of convergence.

$$f(x) = \frac{x}{8 + x^2}$$

Power series representation

In order to find a power series representation, we need to manipulate the function into the form

$$\frac{1}{1 - x}$$

First, we'll factor the x out of the numerator.

$$\frac{1}{1 - x} = (x)\frac{1}{8 + x^2}$$

Next we'll remove the 8 from the denominator.

$$\frac{1}{1 - x} = (x)\frac{1}{8\left(1 + \frac{x^2}{8}\right)}$$

$$\frac{1}{1 - x} = \left(\frac{x}{8}\right)\frac{1}{1 + \frac{x^2}{8}}$$

Now we'll make the sign in between the terms in the denominator negative.

$$\frac{1}{1 - x} = \left(\frac{x}{8}\right)\frac{1}{1 - \left(-\frac{x^2}{8}\right)}$$

We can see that $-x^2/8$ is the value of x from the standard form of the power series, so we'll plug that into the power series formula.

$$\sum_{n=0}^{\infty}\left(-\frac{x^2}{8}\right)^n$$

We can't forget that our power series is also multiplied by the $x/8$ that we factored out, and we'll need to multiply our sum by this term.

$$\frac{x}{8}\sum_{n=0}^{\infty}\left(-\frac{x^2}{8}\right)^n$$

Now we can simplify.

$$\frac{x}{8}\sum_{n=0}^{\infty}\left(-\frac{x^2}{8}\right)^n$$

$$\sum_{n=0}^{\infty}\frac{x}{8}\left(-\frac{x^2}{8}\right)^n$$

$$\sum_{n=0}^{\infty}\frac{x^1}{8^1}\left[\frac{(-1)x^2}{8}\right]^n$$

$$\sum_{n=0}^{\infty}\frac{x^1(-1)^n}{8^1}\left(\frac{x^{2n}}{8^n}\right)$$

$$\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{8^{n+1}}$$

This is the power series representation of the function.

Radius of convergence

To find the radius of convergence, we need to identify a_n from the power series representation we just found.

$$a_n = \frac{(-1)^n x^{2n+1}}{8^{n+1}}$$

Using a_n, we need to generate a_{n+1}.

$$a_{n+1} = \frac{(-1)^{n+1} x^{2(n+1)+1}}{8^{(n+1)+1}}$$

$$a_{n+1} = \frac{(-1)^{n+1} x^{2n+2+1}}{8^{n+1+1}}$$

$$a_{n+1} = \frac{(-1)^{n+1} x^{2n+3}}{8^{n+2}}$$

We can use the ratio test to say whether or not the series converges. The ratio test tells us that

If $L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$, then

the series converges absolutely if $L < 1$.

the series diverges if $L > 1$ or if L is infinite.

the test is inconclusive if $L=1$.

Plugging a_n and a_{n+1} into the formula for L from the ratio test, we get

$$L=\lim_{n\to\infty}\left|\frac{\frac{(-1)^{n+1}x^{2n+3}}{8^{n+2}}}{\frac{(-1)^n x^{2n+1}}{8^{n+1}}}\right|$$

$$L=\lim_{n\to\infty}\left|\frac{(-1)^{n+1}x^{2n+3}}{8^{n+2}}\left(\frac{8^{n+1}}{(-1)^n x^{2n+1}}\right)\right|$$

$$L=\lim_{n\to\infty}\left|\frac{(-1)^{n+1}}{(-1)^n}\cdot\frac{8^{n+1}}{8^{n+2}}\cdot\frac{x^{2n+3}}{x^{2n+1}}\right|$$

$$L=\lim_{n\to\infty}\left|(-1)^{n+1-n}\cdot 8^{n+1-(n+2)}\cdot x^{2n+3-(2n+1)}\right|$$

$$L=\lim_{n\to\infty}\left|(-1)^1\cdot 8^{n+1-n-2}\cdot x^{2n+3-2n-1}\right|$$

$$L=\lim_{n\to\infty}\left|(-1)\cdot 8^{-1}\cdot x^2\right|$$

$$L=\lim_{n\to\infty}\left|\frac{(-1)x^2}{8}\right|$$

Since it's inside absolute brackets, we can drop the -1.

$$L=\lim_{n\to\infty}\left|\frac{x^2}{8}\right|$$

We no longer have n in this function, which means the limit as $n \to \infty$ will have no effect, so we can remove it.

$$L = \left| \frac{x^2}{8} \right|$$

The ratio test tells us that L will converge when $L < 1$, so we generate the following inequality.

$$\left| \frac{x^2}{8} \right| < 1$$

$$\left| x^2 \right| < 8$$

$$-\sqrt{8} < x < \sqrt{8}$$

Based on this inequality, the radius of convergence is $R = \sqrt{8}$.

Interval of convergence

The interval of convergence is given by the inequality $-\sqrt{8} < x < \sqrt{8}$, but we still need to test the endpoints of the interval to say whether the series converges at one or both of them.

We'll plug the endpoints back into the original series and then test each of them for convergence.

Let's start by testing $x = -\sqrt{8}$.

$$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{8^{n+1}}$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n \left(-\sqrt{8}\right)^{2n+1}}{8^{n+1}}$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n \left[-(8)^{\frac{1}{2}}\right]^{2n} \left[-(8)^{\frac{1}{2}}\right]^{1}}{8^n 8^1}$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n (8)^n \left[-(8)^{\frac{1}{2}}\right]}{8^n 8^1}$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n \left(-\sqrt{8}\right)}{8}$$

$$\sum_{n=0}^{\infty} -\frac{\sqrt{8}}{8}(-1)^n$$

$$-\frac{\sqrt{8}}{8} \sum_{n=0}^{\infty} (-1)^n$$

$$-\frac{\sqrt{8}}{8} \left[(-1)^0 + (-1)^1 + (-1)^2 + (-1)^3 + (-1)^4 + \dots\right]$$

$$-\frac{\sqrt{8}}{8} (1 - 1 + 1 - 1 + 1 - \dots)$$

This is a geometric series with $r = -1$. The geometric series test tells us that the series will converge if $|r| < 1$.

$$|-1| < 1$$

$$1 < 1$$

Since 1 is not less than 1, the series diverges, which means it's divergent at the endpoint $x = -\sqrt{8}$.

Now we'll test $x = \sqrt{8}$.

$$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{8^{n+1}}$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n \left(\sqrt{8}\right)^{2n+1}}{8^{n+1}}$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n \left[(8)^{\frac{1}{2}}\right]^{2n+1}}{8^{n+1}}$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n 8^{n+\frac{1}{2}}}{8^{n+1}}$$

$$\sum_{n=0}^{\infty} (-1)^n 8^{n+\frac{1}{2}-(n+1)}$$

$$\sum_{n=0}^{\infty} (-1)^n 8^{n+\frac{1}{2}-n-1}$$

$$\sum_{n=0}^{\infty} (-1)^n 8^{-\frac{1}{2}}$$

$$\sum_{n=0}^{\infty} \frac{1}{\sqrt{8}}(-1)^n$$

$$\frac{1}{\sqrt{8}} \sum_{n=0}^{\infty} (-1)^n$$

We already know from testing the other endpoint that this is a divergent geometric series. Since the series is divergent at both endpoints, the interval of convergence is $-\sqrt{8} < x < \sqrt{8}$.