Topic: Bounded sequences

Question: Describe how the sequence is bounded.

$$a_n = \frac{n+2}{n^2}$$

Answer choices:

A The sequence is bounded below at 0 and not bounded above.

B The sequence is bounded below at 0 and bounded above at 3.

C The sequence is not bounded.

D The sequence is only bounded above at 3 and not bounded below.

Solution: B

Only monotonic sequences can be bounded, because bounded sequences must be either increasing or decreasing, and monotonic sequences are sequences that are always increasing or always decreasing. Bounded sequences can be

bounded above by the largest value of the sequence

bounded below by the smallest value of the sequence

bounded both above and below

The smallest value of an increasing monotonic sequence will be its first term, where $n = 1$. In this case, $a_n \geq a_1$, so we know that **increasing monotonic sequences are bounded below**.

The largest value of a decreasing monotonic sequence will be its first term, where $n = 1$. In this case, $a_n \leq a_1$, so we know that **decreasing monotonic sequences are bounded above**.

To determine if the end of the monotonic sequence is bounded, we'll need to take the limit of the sequence as $n \to \infty$. If we obtain a real-number answer for the limit, then the sequence is bounded at the end as well as at the beginning.

Because we're using the limit as $n \to \infty$ to solve for any possible end of sequence bounding, our end bounds will be in the form $a_n < a_\infty$ for an increasing sequence and $a_n > a_\infty$ for a decreasing sequence if end bounds exist.

Before we can say whether or not the given sequence is bounded, we need to first prove that it's monotonic, which we'll do by expanding the sequence through the first few terms.

$$n = 1 \qquad a_1 = \frac{1+2}{1^2} = 3$$

$$n = 2 \qquad a_2 = \frac{2+2}{2^2} = 1$$

$$n = 3 \qquad a_3 = \frac{3+2}{3^2} = \frac{5}{9}$$

$$n = 4 \qquad a_4 = \frac{4+2}{4^2} = \frac{3}{8}$$

We can see that the sequence is decreasing, which means it's also monotonic. Knowing that it's a decreasing monotonic sequence, we know that the sequence is bounded above by its first term, $a_1 = 3$.

To figure out whether or not the sequence is bounded below at its end, we'll take the limit of the sequence as $n \to \infty$.

If the answer is finite, then the sequence is bounded below by that value.

If the answer is infinite or doesn't exist, then the sequence isn't bounded below.

Evaluating the limit of the sequence as $n \to \infty$, we get

$$\lim_{n\to\infty} \frac{n+2}{n^2} = \frac{\infty}{\infty}$$

Since we get an indeterminate form, we'll back up a step and simplify the function.

$$\lim_{n\to\infty} \frac{n+2}{n^2}\left(\frac{\frac{1}{n^2}}{\frac{1}{n^2}}\right)$$

$$\lim_{n\to\infty} \frac{\frac{n}{n^2}+\frac{2}{n^2}}{\frac{n^2}{n^2}}$$

$$\lim_{n\to\infty} \frac{\frac{1}{n}+\frac{2}{n^2}}{1}$$

$$\frac{\frac{1}{\infty}+\frac{2}{\infty^2}}{1}$$

$$\frac{0+0}{1}$$

$$0$$

Since this is a finite answer, we can say that the sequence is bounded below by 0.

Pulling our conclusions together, we'll say that the given sequence is

decreasing and monotonic, and

bounded above by $a_n \leq 3$

bounded below by $a_n > 0$

Topic: Bounded sequences

Question: Describe how the sequence is bounded.

$$a_n = \frac{e^n}{n}$$

Answer choices:

A The sequence is bounded below at e and not bounded above.

B The sequence is bounded below at e and bounded above at ∞.

C The sequence is not bounded.

D The sequence is only bounded above at ∞ and not bounded below.

Solution: A

Only monotonic sequences can be bounded, because bounded sequences must be either increasing or decreasing, and monotonic sequences are sequences that are always increasing or always decreasing. Bounded sequences can be

bounded above by the largest value of the sequence

bounded below by the smallest value of the sequence

bounded both above and below

The smallest value of an increasing monotonic sequence will be its first term, where $n = 1$. In this case, $a_n \geq a_1$, so we know that **increasing monotonic sequences are bounded below**.

The largest value of a decreasing monotonic sequence will be its first term, where $n = 1$. In this case, $a_n \leq a_1$, so we know that **decreasing monotonic sequences are bounded above**.

To determine if the end of the monotonic sequence is bounded, we'll need to take the limit of the sequence as $n \to \infty$. If we obtain a real-number answer for the limit, then the sequence is bounded at the end as well as at the beginning.

Because we're using the limit as $n \to \infty$ to solve for any possible end of sequence bounding, our end bounds will be in the form $a_n < a_\infty$ for an increasing sequence and $a_n > a_\infty$ for a decreasing sequence if end bounds exist.

Before we can say whether or not the given sequence is bounded, we need to first prove that it's monotonic, which we'll do by expanding the sequence through the first few terms.

$$n = 1 \qquad a_1 = \frac{e^1}{1} = e$$

$$n = 2 \qquad a_2 = \frac{e^2}{2}$$

$$n = 3 \qquad a_3 = \frac{e^3}{3}$$

$$n = 4 \qquad a_4 = \frac{e^4}{4}$$

We can see that the sequence is increasing, which means it's also monotonic. Knowing that it's an increasing monotonic sequence, we know that the sequence is bounded below by its first term, $a_1 = e$.

To figure out whether or not the sequence is bounded above at its end, we'll take the limit of the sequence as $n \to \infty$.

> If the answer is finite, then the sequence is bounded above by that value.
>
> If the answer is infinite or doesn't exist, then the sequence isn't bounded above.

Evaluating the limit of the sequence as $n \to \infty$, we get

$$\lim_{n \to \infty} \frac{e^n}{n} = \frac{\infty}{\infty}$$

Since we get an indeterminate form, we'll back up a step and use L'Hospital's rule simplify the function by replacing the numerator and denominator with their derivatives.

$$\lim_{n\to\infty}\frac{e^n}{n}$$

$$\lim_{n\to\infty}\frac{e^n}{1}$$

$$e^{\infty}$$

$$\infty$$

Since this is an infinite answer, we can say that the sequence isn't bounded above.

Pulling our conclusions together, we'll say that the given sequence is

increasing and monotonic, and

bounded below by $a_n \geq e$

not bounded above

Topic: Bounded sequences

Question: Describe how the sequence is bounded.

$$a_n = (-1)^n \frac{3}{n^2}$$

Answer choices:

A The sequence is bounded below at -3 and not bounded above.

B The sequence is bounded below at -3 and bounded above at 0.

C The sequence is not bounded.

D The sequence is only bounded above at 0 and not bounded below.

Solution: C

Only monotonic sequences can be bounded, because bounded sequences must be either increasing or decreasing, and monotonic sequences are sequences that are always increasing or always decreasing. Bounded sequences can be

bounded above by the largest value of the sequence

bounded below by the smallest value of the sequence

bounded both above and below

The smallest value of an increasing monotonic sequence will be its first term, where $n = 1$. In this case, $a_n \geq a_1$, so we know that **increasing monotonic sequences are bounded below**.

The largest value of a decreasing monotonic sequence will be its first term, where $n = 1$. In this case, $a_n \leq a_1$, so we know that **decreasing monotonic sequences are bounded above**.

To determine if the end of the monotonic sequence is bounded, we'll need to take the limit of the sequence as $n \to \infty$. If we obtain a real-number answer for the limit, then the sequence is bounded at the end as well as at the beginning.

Because we're using the limit as $n \to \infty$ to solve for any possible end of sequence bounding, our end bounds will be in the form $a_n < a_\infty$ for an increasing sequence and $a_n > a_\infty$ for a decreasing sequence if end bounds exist.

Before we can say whether or not the given sequence is bounded, we need to first prove that it's monotonic, which we'll do by expanding the sequence through the first few terms.

$$n = 1 \qquad a_1 = (-1)^1 \frac{3}{1^2} = -3$$

$$n = 2 \qquad a_2 = (-1)^2 \frac{3}{2^2} = \frac{3}{4}$$

$$n = 3 \qquad a_3 = (-1)^3 \frac{3}{3^2} = -\frac{1}{3}$$

$$n = 4 \qquad a_4 = (-1)^4 \frac{3}{4^2} = \frac{3}{16}$$

We can see that the sequence is not consistently increasing or consistently decreasing, which means it's not monotonic.

Since the sequence is not monotonic, we can't say that it's bounded above or below.