# Calculus 2 Workbook Solutions

Surface area of revolution

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MATH

## SURFACE AREA OF REVOLUTION

1. Find the surface area of the object generated by revolving the curve around the $x$-axis on the interval $2 \le x \le 7$.

$$f(x) = \frac{1}{3}x + 4$$

*Solution:*

The surface area of an object formed by rotating the graph of a function $y = f(x)$ on the interval $[a, b]$ is given by

$$A = \int_a^b 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\ dx$$

If we plug in what we've been given, we get

$$A = \int_2^7 2\pi \left(\frac{1}{3}x + 4\right) \sqrt{1 + \left(\frac{1}{3}\right)^2}\ dx$$

$$A = 2\pi \int_2^7 \left(\frac{1}{3}x + 4\right) \sqrt{1 + \frac{1}{9}}\ dx$$

$$A = 2\pi \int_2^7 \left(\frac{1}{3}x + 4\right) \sqrt{\frac{10}{9}}\ dx$$

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$$A=\frac{2\sqrt{10}\pi}{3}\int_{2}^{7}\frac{1}{3}x+4\ dx$$

Integrate, then evaluate over the interval.

$$A=\frac{2\sqrt{10}\pi}{3}\left(\frac{x^2}{6}+4x\right)\Bigg|_{2}^{7}$$

$$A=\frac{2\sqrt{10}\pi}{3}\left(\frac{7^2}{6}+4(7)\right)-\frac{2\sqrt{10}\pi}{3}\left(\frac{2^2}{6}+4(2)\right)$$

$$A=\frac{2\sqrt{10}\pi}{3}\left(\frac{49}{6}+28\right)-\frac{2\sqrt{10}\pi}{3}\left(\frac{4}{6}+8\right)$$

$$A=\frac{2\sqrt{10}\pi}{3}\left(\frac{49}{6}+\frac{168}{6}\right)-\frac{2\sqrt{10}\pi}{3}\left(\frac{4}{6}+\frac{48}{6}\right)$$

$$A=\frac{2\sqrt{10}\pi}{3}\left(\frac{217}{6}\right)-\frac{2\sqrt{10}\pi}{3}\left(\frac{52}{6}\right)$$

$$A=\frac{434\sqrt{10}\pi}{18}-\frac{104\sqrt{10}\pi}{18}$$

Combine into one fraction.

$$A=\frac{434\sqrt{10}\pi-104\sqrt{10}\pi}{18}$$

$$A=\frac{330\sqrt{10}\pi}{18}$$

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$$A = \frac{55\sqrt{10}\pi}{3}$$

2. Find the surface area of the object generated by revolving the curve around the $x$-axis on the interval $1 \leq x \leq 5$.

$$g(x) = \frac{2}{3}x + 5$$

*Solution:*

The surface area of an object formed by rotating the graph of a function $y = g(x)$ on the interval $[a, b]$ is given by

$$A = \int_a^b 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\ dx$$

If we plug in what we've been given, we get

$$A = \int_1^5 2\pi \left(\frac{2}{3}x + 5\right) \sqrt{1 + \left(\frac{2}{3}\right)^2}\ dx$$

$$A = 2\pi \int_1^5 \left(\frac{2}{3}x + 5\right) \sqrt{1 + \frac{4}{9}}\ dx$$

$$A = 2\pi \int_1^5 \left(\frac{2}{3}x + 5\right) \sqrt{\frac{13}{9}}\ dx$$

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$$A = \frac{2\sqrt{13}\pi}{3} \int_1^5 \frac{2}{3}x + 5\ dx$$

Integrate, then evaluate over the interval.

$$A = \frac{2\sqrt{13}\pi}{3} \left( \frac{1}{3}x^2 + 5x \right) \Big|_1^5$$

$$A = \frac{2\sqrt{13}\pi}{3} \left( \frac{1}{3}(5)^2 + 5(5) \right) - \frac{2\sqrt{13}\pi}{3} \left( \frac{1}{3}(1)^2 + 5(1) \right)$$

$$A = \frac{2\sqrt{13}\pi}{3} \left( \frac{25}{3} + 25 \right) - \frac{2\sqrt{13}\pi}{3} \left( \frac{1}{3} + 5 \right)$$

$$A = \frac{2\sqrt{13}\pi}{3} \left( \frac{25}{3} + \frac{75}{3} \right) - \frac{2\sqrt{13}\pi}{3} \left( \frac{1}{3} + \frac{15}{3} \right)$$

$$A = \frac{2\sqrt{13}\pi}{3} \left( \frac{100}{3} \right) - \frac{2\sqrt{13}\pi}{3} \left( \frac{16}{3} \right)$$

$$A = \frac{200\sqrt{13}\pi}{9} - \frac{32\sqrt{13}\pi}{9}$$

Combine into one fraction.

$$A = \frac{168\sqrt{13}\pi}{9}$$

$$A = \frac{56\sqrt{13}\pi}{3}$$

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■ 3. Set up the integral that approximates the surface area of the object generated by revolving the curve around the $x$-axis on the interval $-3 \le x \le 3$. Do not evaluate the integral.

$$h(x) = x^2 + 3$$

*Solution:*

The surface area of an object formed by rotating the graph of a function $y = h(x)$ on the interval $[a, b]$ is given by

$$A = \int_a^b 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \ dx$$

If we plug in what we've been given, we get

$$A = \int_{-3}^{3} 2\pi(x^2 + 3)\sqrt{1 + (2x)^2} \ dx$$

$$A = 2\pi \int_{-3}^{3} (x^2 + 3)\sqrt{1 + 4x^2} \ dx$$

■ 4. Find the surface area of the object generated by revolving the curve around the line $y = -1$ on the interval $3 \le x \le 9$.

$$g(x) = 2\sqrt{2x + 7}$$

*Solution:*

The surface area of an object formed by rotating the graph of a function $y = g(x)$ on the interval $[a, b]$ is given by

$$A = \int_a^b 2\pi y\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\ dx$$

Since the curve is rotated around the line $y = -1$, which is $1$ unit below the $x$-axis, add $1$ to the function to get $g(x) = 2\sqrt{2}x + 8$ in the integral.

$$A = \int_3^9 2\pi\left(2\sqrt{2}x + 8\right)\sqrt{1 + (2\sqrt{2})^2}\ dx$$

$$A = 2\pi\int_3^9 \left(2\sqrt{2}x + 8\right)\sqrt{1 + 4(2)}\ dx$$

$$A = 2\sqrt{9}\pi\int_3^9 2\sqrt{2}x + 8\ dx$$

$$A = 6\pi\int_3^9 2\sqrt{2}x + 8\ dx$$

Integrate, then evaluate over the interval.

$$A = 6\pi\left(\sqrt{2}x^2 + 8x\right)\Big|_3^9$$

$$A = 6\pi\left(\sqrt{2}(9)^2 + 8(9)\right) - 6\pi\left(\sqrt{2}(3)^2 + 8(3)\right)$$

$$A = 6\pi\left(81\sqrt{2} + 72\right) - 6\pi\left(9\sqrt{2} + 24\right)$$

$$A = 6\pi\left(81\sqrt{2} + 72 - 9\sqrt{2} - 24\right)$$

$$A = 6\pi\left(72\sqrt{2} + 48\right)$$

$$A = 144\pi\left(3\sqrt{2} + 2\right)$$

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## SURFACE OF REVOLUTION EQUATION

1. Find an equation for the surface generated by revolving the curve around the $x$-axis.

$$3x^2 + 2y^2 = 8$$

*Solution:*

Pick a point $P(x, y, z)$ on the surface of the rotation. Then pick another point $Q(x, y_1, 0)$ with the same $x$-coordinate as point $P$.

Then for point $Q$, the equation is $3x^2 + 2{y_1}^2 = 8$. Since the distance from the $x$-axis to point $P$ is the same as the distance from the $x$-axis to point $Q$, the square of the distances are also equal.

$$d_P = \sqrt{y^2 + z^2} \qquad {d_P}^2 = y^2 + z^2$$

$$d_Q = \sqrt{{y_1}^2 + 0^2} \qquad {d_Q}^2 = {y_1}^2$$

So

$${y_1}^2 = y^2 + z^2$$

Substitute this expression into the original equation, simplify, and get an equation for the surface.

$$3x^2 + 2\left(y^2 + z^2\right) = 8$$

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$$3x^2 + 2y^2 + 2z^2 = 8$$

■ 2. Find an equation for the surface generated by revolving the curve around the $y$-axis.

$$5x^2 = 8y^2$$

*Solution:*

Pick a point $P(x, y, z)$ on the surface of the rotation. Then pick another point $Q(x, y_1, 0)$ with the same $y$-coordinate as point $P$.

Then for point $Q$, the equation is $5{x_1}^2 = 8y^2$. Since the distance from the $y$-axis to point $P$ is the same as the distance from the $y$-axis to point $Q$, the square of the distances are also equal.

$$d_P = \sqrt{y^2 + z^2} \qquad {d_P}^2 = y^2 + z^2$$

$$d_Q = \sqrt{{x_1}^2 + 0^2} \qquad {d_Q}^2 = {x_1}^2$$

So

$${x_1}^2 = x^2 + z^2$$

Substitute this expression into the original equation, simplify, and get an equation for the surface.

$$5(x^2 + z^2) = 8y^2$$

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$$5x^2 + 5z^2 = 8y^2$$

■ 3. Find an equation for the surface generated by revolving the curve around the $x$-axis.

$$9x^2 + 25y^2 = 36$$

*Solution:*

Pick a point $P(x, y, z)$ on the surface of the rotation. Then pick another point $Q(x, y_1, 0)$ with the same $x$-coordinate as point $P$.

Then for point $Q$, the equation is $9x^2 + 25{y_1}^2 = 36$. Since the distance from the $x$-axis to point $P$ is the same as the distance from the $x$-axis to point $Q$, the square of the distances are also equal.

$$d_P = \sqrt{y^2 + z^2} \qquad {d_P}^2 = y^2 + z^2$$

$$d_Q = \sqrt{{y_1}^2 + 0^2} \qquad {d_Q}^2 = {y_1}^2$$

So

$${y_1}^2 = y^2 + z^2$$

Substitute this expression into the original equation, simplify, and get an equation for the surface.

$$9x^2 + 25(y^2 + z^2) = 36$$

$$9x^2 + 25y^2 + 25z^2 = 36$$

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