Work done by a variable force

To calculate the work done when a variable force is applied to lift an object of some mass or weight, we'll use the formula

$$W=\int_a^b F(x)\ dx$$

where W is the work done, $F(x)$ is the equation of the variable force, and $[a,b]$ is the starting and ending height of the object.

If W is positive, it means that the force is doing work in the given interval. If W is negative, then work needs to be done on the interval. The answer to these types of work problems is usually given in Joules J.

Example

Find the work done to lift a 20 kg box from the floor to a height of 3 m when the variable force $F(x)$ is given in Newtons.

$$F(x)=4x^2-2x+3$$

Using the formula from this section, and defining the interval $[a,b]$ as $[0,3]$, we get

$$W=\int_0^3 4x^2-2x+3\ dx$$

$$W = \int_0^3 4x^2 \ dx + \int_0^3 -2x \ dx + \int_0^3 3 \ dx$$

$$W = 4\int_0^3 x^2 \ dx - 2\int_0^3 x \ dx + 3\int_0^3 1 \ dx$$

Integrating, we get

$$W = \left[4\left(\frac{x^3}{3}\right) - 2\left(\frac{x^2}{2}\right) + 3(x) \right] \Bigg|_0^3$$

$$W = \frac{4x^3}{3} - x^2 + 3x \Bigg|_0^3$$

Now we'll evaluate over the interval.

$$W = \left[\frac{4(3)^3}{3} - (3)^2 + 3(3) \right] - \left[\frac{4(0)^3}{3} - (0)^2 + 3(0) \right]$$

$$W = 36$$

36 J of force are required to lift a 20 kg box from the floor to a height of 3 m when the variable force applied is defined by $F(x) = 4x^2 - 2x + 3$.