Topic: Alternating series estimation theorem

Question: Approximate the sum of the alternating series to three decimal places, then find the remainder of the approximation.

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}n}{10^n}$$

Answer choices:

A $\quad s_3 \approx 0.083 \qquad \left| R_3 \right| \le 0.004$

B $\quad s_3 \approx 0.083 \qquad \left| R_3 \right| \le 0.0004$

C $\quad s_3 \approx 0.083 \qquad \left| R_3 \right| \le 0.0002$

D $\quad s_3 \approx 0.083 \qquad \left| R_3 \right| \le 0.0020$

Solution: B

The alternating series estimation theorem gives us a way to approximate the sum of an alternating series and calculate the error in our approximation. We can only use the theorem if

$$b_{n+1} \leq b_n$$

and

$$\lim_{n\to\infty} b_n = 0$$

where b_n is the alternating series.

If the alternating series meets these two conditions, then the error in the estimation of the sum is

$$\left|R_n\right| = \left|s - s_n\right| \leq b_{n+1}$$

We've been asked to use the alternating series estimation theorem to approximate the sum of the given series to three decimal places.

We'll calculate the first few terms of the series.

$n = 1$ $\qquad a_1 = \dfrac{(-1)^{1-1}(1)}{10^1}$ $\qquad a_1 = 0.1$

$n = 2$ $\qquad a_2 = \dfrac{(-1)^{2-1}(2)}{10^2}$ $\qquad a_2 = -0.02$

$n = 3$ $\qquad a_3 = \dfrac{(-1)^{3-1}(3)}{10^3}$ $\qquad a_3 = 0.003$

$n = 4$ $\qquad a_4 = \frac{(-1)^{4-1}(4)}{10^4}$ $\qquad a_4 = -0.0004$

$n = 5$ $\qquad a_5 = \frac{(-1)^{5-1}(5)}{10^5}$ $\qquad a_5 = 0.00005$

Next, we need to sum these terms until we can see that the third decimal place isn't changing.

Adding the first two terms together, we get

$$a_1 + a_2 = 0.1 + (-0.02)$$

$$a_1 + a_2 = 0.1 - 0.02$$

$$a_1 + a_2 = 0.08$$

$$s_2 = 0.08$$

Since we're not to three decimal places, we'll add another term to the sum.

$$a_1 + a_2 + a_3 = 0.1 + (-0.02) + 0.003$$

$$a_1 + a_2 + a_3 = 0.1 - 0.02 + 0.003$$

$$a_1 + a_2 + a_3 = 0.083$$

$$s_3 = 0.083$$

We've made it to three decimal places, but we need to make sure that the third decimal place doesn't change, or that the fourth decimal place won't cause the third decimal place to round up.

$$a_1 + a_2 + a_3 + a_4 = 0.1 + (-0.02) + 0.003 + (-0.0004)$$

$$a_1 + a_2 + a_3 + a_4 = 0.1 - 0.02 + 0.003 - 0.0004$$

$$a_1 + a_2 + a_3 + a_4 = 0.0826$$

$$s_4 = 0.0826$$

Now we know that the fourth decimal place is going to cause us to round up the third decimal place, and our approximation to three decimal places is

$$s_3 \approx 0.083$$

We've got an estimation of the sum of the alternating series, so our next step is to calculate the error in our estimation. If we want to use the alternating series estimation theorem, we'll need to verify that

$$b_{n+1} \le b_n$$

and

$$\lim_{n \to \infty} b_n = 0$$

Since the sum of an alternating series is

$$\sum_{n=1}^{\infty} (-1)^{n-1} b_n$$

and the given series is

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}n}{10^n}$$

we can say that

$$b_n = \frac{n}{10^n}$$

So

$$b_{n+1} = \frac{n+1}{10^{n+1}}$$

Now we can calculate the first three terms for both b_n and b_{n+1}.

		b_n		b_{n+1}
$n = 1$	$\frac{1}{10^1}$	$\frac{1}{10}$	$\frac{1+1}{10^{1+1}}$	$\frac{1}{50}$
$n = 2$	$\frac{2}{10^2}$	$\frac{1}{50}$	$\frac{2+1}{10^{2+1}}$	$\frac{3}{1,000}$
$n = 3$	$\frac{3}{10^3}$	$\frac{3}{1,000}$	$\frac{3+1}{10^{3+1}}$	$\frac{1}{2,500}$

Looking at our results we can verify that $b_{n+1} \le b_n$.

Next, we need to show that $\lim_{n\to\infty} b_n = 0$.

$$\lim_{n\to\infty} \frac{n}{10^n}$$

When we evaluate b_n as it approaches infinity, we can see that the denominator will increase much more quickly than the numerator, so

$$\lim_{n\to\infty} \frac{n}{10^n} = 0$$

Since both rules are true for our series, we can use

$$\left|R_n\right| = \left|s - s_n\right| \le b_{n+1}$$

to estimate the error in our approximation of the sum of the series. n will be taken from the approximate sum we already calculated ($s_3 \approx 0.083$).

$$\left|R_3\right| = \left|s - s_3\right| \le b_{3+1}$$

We're looking for the remainder, so we'll use

$$\left|R_3\right| \le b_4$$

$$\left|R_3\right| \le \frac{4}{10^4}$$

$$\left|R_3\right| \le \frac{1}{2{,}500}$$

$$\left|R_3\right| \le 0.0004$$

We can now say that our approximation of the sum of the alternating series ($s_3 \approx 0.083$) has an error of $\left|R_3\right| \le 0.0004$.

Topic: Alternating series estimation theorem

Question: Approximate the sum of the alternating series to three decimal places, then find the remainder of the approximation.

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}n^2}{10^n}$$

Answer choices:

A	$s_3 \approx 0.086$	$\lvert R_5 \rvert \leq 0.000036$
B	$s_3 \approx 0.068$	$\lvert R_5 \rvert \leq 0.00036$
C	$s_5 \approx 0.086$	$\lvert R_5 \rvert \leq 0.00036$
D	$s_5 \approx 0.068$	$\lvert R_5 \rvert \leq 0.000036$

Solution: D

The alternating series estimation theorem gives us a way to approximate the sum of an alternating series and calculate the error in our approximation. We can only use the theorem if

$$b_{n+1} \leq b_n$$

and

$$\lim_{n \to \infty} b_n = 0$$

where b_n is the alternating series.

If the alternating series meets these two conditions, then the error in the estimation of the sum is

$$\left| R_n \right| = \left| s - s_n \right| \leq b_{n+1}$$

We've been asked to use the alternating series estimation theorem to approximate the sum of the given series to three decimal places.

We'll calculate the first few terms of the series.

$$n = 1 \qquad a_1 = \frac{(-1)^{1-1}(1)^2}{10^1} \qquad a_1 = 0.1$$

$$n = 2 \qquad a_2 = \frac{(-1)^{2-1}(2)^2}{10^2} \qquad a_2 = -0.04$$

$$n = 3 \qquad a_3 = \frac{(-1)^{3-1}(3)^2}{10^3} \qquad a_3 = 0.009$$

$n = 4$ $\qquad a_4 = \frac{(-1)^{4-1}(4)^2}{10^4}$ $\qquad a_4 = -0.0016$

$n = 5$ $\qquad a_5 = \frac{(-1)^{5-1}(5)^2}{10^5}$ $\qquad a_5 = 0.00025$

$n = 6$ $\qquad a_6 = \frac{(-1)^{6-1}(6)^2}{10^6}$ $\qquad a_6 = -0.000036$

Next, we need to sum these terms until we can see that the third decimal place isn't changing.

Adding the first two terms together, we get

$$a_1 + a_2 = 0.1 + (-0.04)$$

$$a_1 + a_2 = 0.1 - 0.04$$

$$a_1 + a_2 = 0.06$$

$$s_2 = 0.06$$

Since we're not to three decimal places, we'll add another term to the sum.

$$a_1 + a_2 + a_3 = 0.1 + (-0.04) + 0.009$$

$$a_1 + a_2 + a_3 = 0.1 - 0.04 + 0.009$$

$$a_1 + a_2 + a_3 = 0.069$$

$$s_3 = 0.069$$

We've made it to three decimal places, but we need to make sure that the third decimal place doesn't change, or that the fourth decimal place won't cause the third decimal place to round up.

$$a_1 + a_2 + a_3 + a_4 = 0.1 + (-0.04) + 0.009 + (-0.0016)$$

$$a_1 + a_2 + a_3 + a_4 = 0.1 - 0.04 + 0.009 - 0.0016$$

$$a_1 + a_2 + a_3 + a_4 = 0.0674$$

$$s_4 = 0.0674$$

The fourth decimal place didn't cause the third decimal place to round up, but the fourth term changed the third decimal place, and we need to find a consistent answer to three decimal places.

$$a_1 + a_2 + a_3 + a_4 + a_5 = 0.1 + (-0.04) + 0.009 + (-0.0016) + 0.00025$$

$$a_1 + a_2 + a_3 + a_4 + a_5 = 0.1 - 0.04 + 0.009 - 0.0016 + 0.00025$$

$$a_1 + a_2 + a_3 + a_4 + a_5 = 0.06765$$

$$s_5 = 0.06765$$

Now we know that the fourth decimal place is going to cause us to round up the third decimal place, and our approximation to three decimal places is

$$s_5 \approx 0.068$$

We've got an estimation of the sum of the alternating series, so our next step is to calculate the error in our estimation. If we want to use the alternating series estimation theorem, we'll need to verify that

$$b_{n+1} \leq b_n$$

and

$$\lim_{n \to \infty} b_n = 0$$

Since the sum of an alternating series is

$$\sum_{n=1}^{\infty} (-1)^{n-1} b_n$$

and the given series is

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1} n^2}{10^n}$$

we can say that

$$b_n = \frac{n^2}{10^n}$$

So

$$b_{n+1} = \frac{(n+1)^2}{10^{n+1}}$$

Now we can calculate the first three terms for both b_n and b_{n+1}.

b_n b_{n+1}

$n = 1$	$\frac{1^2}{10^1}$	$\frac{1}{10}$	$\frac{(1+1)^2}{10^{1+1}}$	$\frac{1}{25}$
$n = 2$	$\frac{2^2}{10^2}$	$\frac{1}{25}$	$\frac{(2+1)^2}{10^{2+1}}$	$\frac{9}{1,000}$
$n = 3$	$\frac{3^2}{10^3}$	$\frac{9}{1,000}$	$\frac{(3+1)^2}{10^{3+1}}$	$\frac{1}{625}$

Looking at our results we can verify that $b_{n+1} \leq b_n$.

Next, we need to show that $\lim_{n\to\infty} b_n = 0$.

$$\lim_{n\to\infty} \frac{n^2}{10^n}$$

When we evaluate b_n as it approaches infinity, we can see that the denominator will increase much more quickly than the numerator, so

$$\lim_{n\to\infty} \frac{n^2}{10^n} = 0$$

Since both rules are true for our series, we can use

$$\left|R_n\right| = \left|s - s_n\right| \leq b_{n+1}$$

to estimate the error in our approximation of the sum of the series. n will be taken from the approximate sum we already calculated ($s_5 \approx 0.068$).

$$\left|R_5\right| = \left|s - s_5\right| \leq b_{5+1}$$

We're looking for the remainder, so we'll use

$$|R_5| \le b_6$$

$$|R_5| \le \frac{6^2}{10^6}$$

$$|R_5| \le \frac{36}{1,000,000}$$

$$|R_5| \le 0.000036$$

We can now say that our approximation of the sum of the alternating series ($s_5 \approx 0.068$) has an error of $|R_5| \le 0.000036$.

Topic: Alternating series estimation theorem

Question: Approximate the sum of the alternating series to three decimal places, then find the remainder of the approximation.

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1} n^2}{4^n}$$

Answer choices:

A	$s_8 \approx 0.096$	$\lvert R_8 \rvert \leq 0.00031$
B	$s_8 \approx 0.099$	$\lvert R_8 \rvert \leq 0.00031$
C	$s_3 \approx 0.096$	$\lvert R_8 \rvert \leq 0.00013$
D	$s_3 \approx 0.069$	$\lvert R_8 \rvert \leq 0.00031$

Solution: A

The alternating series estimation theorem gives us a way to approximate the sum of an alternating series and calculate the error in our approximation. We can only use the theorem if

$$b_{n+1} \leq b_n$$

and

$$\lim_{n \to \infty} b_n = 0$$

where b_n is the alternating series.

If the alternating series meets these two conditions, then the error in the estimation of the sum is

$$\left| R_n \right| = \left| s - s_n \right| \leq b_{n+1}$$

We've been asked to use the alternating series estimation theorem to approximate the sum of the given series to three decimal places.

We'll calculate the first few terms of the series.

$$n = 1 \qquad a_1 = \frac{(-1)^{1-1}(1)^2}{4^1} \qquad a_1 = 0.25$$

$$n = 2 \qquad a_2 = \frac{(-1)^{2-1}(2)^2}{4^2} \qquad a_2 = -0.25$$

$$n = 3 \qquad a_3 = \frac{(-1)^{3-1}(3)^2}{4^3} \qquad a_3 = 0.1406$$

$n = 4$ $\quad a_4 = \frac{(-1)^{4-1}(4)^2}{4^4}$ $\quad a_4 = -0.0625$

$n = 5$ $\quad a_5 = \frac{(-1)^{5-1}(5)^2}{4^5}$ $\quad a_5 = 0.0244$

$n = 6$ $\quad a_6 = \frac{(-1)^{6-1}(6)^2}{4^6}$ $\quad a_6 = -0.0088$

$n = 7$ $\quad a_7 = \frac{(-1)^{7-1}(7)^2}{4^7}$ $\quad a_7 = 0.0030$

$n = 8$ $\quad a_8 = \frac{(-1)^{8-1}(8)^2}{4^8}$ $\quad a_8 = -0.0010$

$n = 9$ $\quad a_9 = \frac{(-1)^{9-1}(9)^2}{4^9}$ $\quad a_9 = 0.0003$

Next, we need to sum these terms until we can see that the third decimal place isn't changing.

Adding the first two terms together, we get

$$a_1 + a_2 = 0.25 + (-0.25)$$

$$a_1 + a_2 = 0.25 - 0.25$$

$$a_1 + a_2 = 0$$

$$s_2 = 0$$

Since we're not to three decimal places, we'll add another term to the sum.

$$a_1 + a_2 + a_3 = 0.25 + (-0.25) + 0.1406$$

$$a_1 + a_2 + a_3 = 0.25 - 0.25 + 0.1406$$

$$a_1 + a_2 + a_3 = 0.1406$$

$$s_3 = 0.1406$$

We've made it to three decimal places, but we need to make sure that the third decimal place doesn't change, or that the fourth decimal place won't cause the third decimal place to round up.

$$a_1 + a_2 + a_3 + a_4 = 0.25 + (-0.25) + 0.1406 + (-0.0625)$$

$$a_1 + a_2 + a_3 + a_4 = 0.25 - 0.25 + 0.1406 - 0.0625$$

$$a_1 + a_2 + a_3 + a_4 = 0.0781$$

$$s_4 = 0.0781$$

We need to keep going until the first three decimal places aren't changing.

For s_5:

$$s_5 = 0.0781 + 0.0244$$

$$s_5 = 0.1025$$

For s_6:

$$s_6 = 0.1025 + (-0.0088)$$

$$s_6 = 0.1025 - 0.0088$$

$$s_6 = 0.0937$$

For s_7:

$$s_7 = 0.0937 + 0.0030$$

$$s_7 = 0.0967$$

For s_8:

$$s_8 = 0.0967 + (-0.0010)$$

$$s_8 = 0.0967 - 0.0010$$

$$s_8 = 0.0957$$

For s_9:

$$s_9 = 0.0957 + 0.0003$$

$$s_9 = 0.0960$$

Now we know that the fourth decimal place isn't going to cause us to round up the third decimal place, and our approximation to three decimal places is

$$s_8 \approx 0.096$$

We've got an estimation of the sum of the alternating series, so our next step is to calculate the error in our estimation. If we want to use the alternating series estimation theorem, we'll need to verify that

$$b_{n+1} \leq b_n$$

and

$$\lim_{n\to\infty} b_n = 0$$

Since the sum of an alternating series is

$$\sum_{n=1}^{\infty} (-1)^{n-1} b_n$$

and the given series is

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1} n^2}{4^n}$$

we can say that

$$b_n = \frac{n^2}{4^n}$$

So

$$b_{n+1} = \frac{(n+1)^2}{4^{n+1}}$$

Now we can calculate the first three terms for both b_n and b_{n+1}.

		b_n		b_{n+1}
$n=1$	$\frac{1^2}{4^1}$	$\frac{1}{4}$	$\frac{(1+1)^2}{4^{1+1}}$	$\frac{1}{4}$
$n=2$	$\frac{2^2}{4^2}$	$\frac{1}{4}$	$\frac{(2+1)^2}{4^{2+1}}$	$\frac{9}{64}$
$n=3$	$\frac{3^2}{4^3}$	$\frac{9}{64}$	$\frac{(3+1)^2}{4^{3+1}}$	$\frac{1}{16}$

Looking at our results we can verify that $b_{n+1} \leq b_n$.

Next, we need to show that $\lim_{n\to\infty} b_n = 0$.

$$\lim_{n\to\infty} \frac{n^2}{4^n}$$

When we evaluate b_n as it approaches infinity, we can see that the denominator will increase much more quickly than the numerator, so

$$\lim_{n\to\infty} \frac{n^2}{4^n} = 0$$

Since both rules are true for our series, we can use

$$\left| R_n \right| = \left| s - s_n \right| \leq b_{n+1}$$

to estimate the error in our approximation of the sum of the series. n will be taken from the approximate sum we already calculated ($s_8 \approx 0.096$).

$$\left| R_8 \right| = \left| s - s_8 \right| \leq b_{8+1}$$

We're looking for the remainder, so we'll use

$$\left| R_8 \right| \leq b_9$$

$$\left| R_8 \right| \leq \frac{9^2}{4^9}$$

$$\left| R_8 \right| \leq \frac{81}{262{,}144}$$

$$\left| R_8 \right| \leq 0.00031$$

We can now say that our approximation of the sum of the alternating series ($s_8 \approx 0.096$) has an error of $\left| R_8 \right| \leq 0.00031$.