Alternating series estimation theorem

The alternating series estimation theorem gives us a way to approximate the sum of an alternating series with a remainder or error that we can calculate. To use this theorem, our series must follow two rules:

1. The series must be decreasing, $b_n \geq b_{n+1}$

2. The limit of the series must be zero, $\lim_{n \to \infty} b_n = 0$

Once we confirm that our alternating series meets these two conditions, we can calculate the error using

$$|R_n| = |s - s_n| \leq b_{n+1}$$

Example

Approximate the sum of the series to three decimal places.

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1} n}{10^n}$$

We'll calculate the first few terms of the series until we have a stable answer to three decimal places.

$$n = 1 \qquad a_1 = \frac{(-1)^{1-1}(1)}{10^1} \qquad a_1 = 0.1$$

$$n = 2 \qquad a_2 = \frac{(-1)^{2-1}(2)}{10^2} \qquad a_2 = -0.02$$

$$n = 3 \qquad a_3 = \frac{(-1)^{3-1}(3)}{10^3} \qquad a_3 = 0.003$$

$$n = 4 \qquad a_4 = \frac{(-1)^{4-1}(4)}{10^4} \qquad a_4 = -0.0004$$

$$n = 5 \qquad a_5 = \frac{(-1)^{5-1}(5)}{10^5} \qquad a_5 = 0.00005$$

Next, we need to sum these terms until we can see that the third decimal place isn't changing.

Adding the first two terms together, we get

$$a_1 + a_2 = 0.1 + (-0.02)$$

$$a_1 + a_2 = 0.1 - 0.02$$

$$a_1 + a_2 = 0.08$$

$$s_2 = 0.08$$

Since we're not to three decimal places, we'll add another term to the sum

$$a_1 + a_2 + a_3 = 0.1 + (-0.02) + 0.003$$

$$a_1 + a_2 + a_3 = 0.1 - 0.02 + 0.003$$

$$a_1 + a_2 + a_3 = 0.083$$

$$s_3 = 0.083$$

We've made it to three decimal places, but we need to make sure that the fourth decimal place won't cause the third decimal place to round up.

$$a_1 + a_2 + a_3 + a_4 = 0.1 + (-0.02) + 0.003 + (-0.0004)$$

$$a_1 + a_2 + a_3 + a_4 = 0.1 - 0.02 + 0.003 - 0.0004$$

$$a_1 + a_2 + a_3 + a_4 = 0.0826$$

$$s_4 = 0.0826$$

Now we know that the fourth decimal place is going to cause us to round up the third decimal place, and our approximation to three decimal places is

$$s_3 \approx 0.083$$

In order to use the alternating series estimation theorem, we need to show that the series is decreasing, $b_n \geq b_{n+1}$. Pulling out b_n from the given series, we get

$$b_n = \frac{n}{10^n}$$

Which means that

$$b_{n+1} = \frac{n+1}{10^{n+1}}$$

Now we can calculate the first three terms for both b_n and b_{n+1}.

	b_n	b_{n+1}

$n=1$	$\frac{1}{10^1}$	$\frac{1}{10}$	$\frac{1+1}{10^{1+1}}$	$\frac{1}{50}$
$n=2$	$\frac{2}{10^2}$	$\frac{1}{50}$	$\frac{2+1}{10^{2+1}}$	$\frac{3}{1,000}$
$n=3$	$\frac{3}{10^3}$	$\frac{3}{1,000}$	$\frac{3+1}{10^{3+1}}$	$\frac{1}{2,500}$

Looking at these results, we can see that $b_n \geq b_{n+1}$, so b_n is decreasing.

Next, we need to show that $\lim_{n\to\infty} b_n = 0$.

$$\lim_{n\to\infty} \frac{n}{10^n}$$

When we evaluate b_n as it approaches infinity, we can see that the denominator will increase much faster than the numerator. This means that the fraction will approach 0.

$$\lim_{n\to\infty} \frac{n}{10^n} = 0$$

Now that we've shown that our series meets the two criteria, we can use the alternating series estimation theorem. We'll use the inequality

$$|R_n| = |s - s_n| \leq b_{n+1}$$

Plugging in the values we have, we get

$$|R_3| = |s - s_3| \leq b_{3+1}$$

$$|R_3| \leq b_4$$

$$|R_3| \le \frac{4}{10^4}$$

$$|R_3| \le \frac{1}{2,500}$$

$$|R_3| \le 0.0004$$

The approximate sum of the series to three decimal places is 0.083 with an error of $|R_3| \le 0.0004$.