Rationalizing substitution

Sometimes we need to use partial fractions to evaluate an integral, but the integral isn't in a form that's ready for partial fractions decomposition. Remember that, in order to use partial fractions, the function has to be a proper rational function, which means that it's the quotient of two polynomials where the degree of the denominator is greater than the degree of the numerator.

If the function is rational (the quotient of two polynomials), but not proper (the degree of the denominator is not greater than the degree of the numerator), then you just need to perform polynomial long division to make it proper.

If, on the other hand, the function isn't rational, you may be able to use what's called a "rationalizing substitution" to make it rational. From there, you can check to make sure it's proper, and then once it's proper, use a partial fractions decomposition to evaluate the integral.

In other words, follow this process:

1. Make sure the function is a **rational** function. If it isn't, try a rationalizing substitution.

2. If the function is rational, make sure it's **proper**. If it isn't, perform polynomial long division to make it proper.

3. If the function is rational and proper, use **partial fractions** to evaluate it.

Example

Evaluate the integral.

$$\int \frac{\sqrt{x+4}}{2x}\ dx$$

We can't evaluate the integral as-is. We'll try making a substitution that will rationalize the function, letting

$$u = \sqrt{x+4}$$

$$du = \frac{1}{2\sqrt{x+4}}\ dx$$

$$dx = 2\sqrt{x+4}\ du$$

We'll plug these values into our integral and get

$$\int \frac{u}{2x} 2\sqrt{x+4}\ du$$

Substituting again, remembering that $u = \sqrt{x+4}$, the integral simplifies to

$$\int \frac{u}{2x} 2u\ du$$

$$\int \frac{u^2}{x}\ du$$

We need to replace the x in the denominator with a function in terms of u. We'll rearrange $u = \sqrt{x+4}$, solving it for x to get $x = u^2 - 4$, and then we'll plug it in for x.

$$\int \frac{u^2}{u^2 - 4} \ du$$

Now that we have a rational function, we can use partial fractions to evaluate the integral, we just need to make sure it's proper a proper rational function before we do.

We'll compare the degree of the numerator to the degree of the denominator. The degrees of the numerator and denominator are both 2. Since the degrees are equal, we'll need to use polynomial long division to make the function proper.

When we divide $u^2 - 4$ into u^2, we get an answer of 1 and a remainder of 4, so the integral simplifies to

$$\int 1 + \frac{4}{u^2 - 4} \ du$$

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To evaluate the now proper rational function in the second integral, we'll use partial fractions. We'll start by factoring the denominator.

$$\int 1 \ du + \int \frac{4}{(u-2)(u+2)} \ du$$

The partial fractions decomposition gives us

$$\int 1\ du + \int \frac{1}{u-2}\ du + \int \frac{-1}{u+2}\ du$$

$$\int 1\ du + \int \frac{1}{u-2}\ du - \int \frac{1}{u+2}\ du$$

Integrating, we get

$$u + \ln|u-2| - \ln|u+2| + C$$

Plugging back in for u, remembering that $u = \sqrt{x+4}$, the answer becomes

$$\sqrt{x+4} + \ln\left|\sqrt{x+4} - 2\right| - \ln\left|\sqrt{x+4} + 2\right| + C$$

$$\sqrt{x+4} + \ln\left|\frac{\sqrt{x+4}-2}{\sqrt{x+4}+2}\right| + C$$