Estimating a root

Very commonly, we'll use linear approximation to estimate the value of a root.

Linear approximation is particularly good at this. For instance, without a calculator, it's extremely difficult to find $\sqrt{82}$. At the same time, we know immediately that $\sqrt{81}=9$.

So to estimate $\sqrt{82}$, we'll instead consider the function $f(x)=\sqrt{x}$, and use $(a,f(a))=(81,9)$ as the point of tangency along $f(x)=\sqrt{81}$, in order to get an approximation for $\sqrt{82}$.

As always with linear approximations, we'll differentiate the function and evaluate the derivative at the point of tangency, and then substitute the slope and the point of tangency into the linear approximation equation.

Let's do an example with a fourth root.

Example

Use linear approximation to estimate $\sqrt[4]{17}$.

We certainly don't know the value of $\sqrt[4]{17}$, but we know that $\sqrt[4]{16}=2$. So instead of trying to calculate $\sqrt[4]{17}$ directly, let's use the function $f(x)=\sqrt[4]{x}$.

Differentiate the function.

$$f'(x) = \frac{1}{4}x^{-\frac{3}{4}}$$

$$f'(x) = \frac{1}{4x^{\frac{3}{4}}}$$

and then evaluate it at $a = 16$.

$$f'(16) = \frac{1}{4(16)^{\frac{3}{4}}}$$

$$f'(16) = \frac{1}{4(16^{\frac{1}{4}})^3}$$

$$f'(16) = \frac{1}{4(2)^3}$$

$$f'(16) = \frac{1}{4(8)}$$

$$f'(16) = \frac{1}{32}$$

So along the function $f(x) = \sqrt[4]{x}$, we have the point of tangency $(16,2)$ and the slope $m = 1/32$. Substitute these into the linear approximation equation.

$$L(x) = f(a) + f'(a)(x - a)$$

$$L(x) = 2 + \frac{1}{32}(x - 16)$$

$$L(x) = 2 + \frac{1}{32}x - \frac{16}{32}$$

$$L(x) = \frac{1}{32}x - \frac{1}{2} + 2$$

$$L(x) = \frac{1}{32}x + \frac{3}{2}$$

Now that we have the linear approximation equation, we can use it to estimate $\sqrt[4]{17}$. Substitute $x = 17$.

$$L(17) = \frac{1}{32}(17) + \frac{3}{2}$$

$$L(17) = \frac{17}{32} + \frac{3}{2}$$

$$L(17) = \frac{17}{32} + \frac{48}{32}$$

$$L(17) = \frac{65}{32}$$