Calculus 2 Workbook Solutions

Calculus with polar curves

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TANGENT LINE TO THE POLAR CURVE

■ 1. Find the tangent line to the polar curve at $\theta = 2\pi/3$.

$$r = 3\cos\theta$$

Solution:

The slope of the tangent line m is given by

$$m = \frac{dy}{dx} = \frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta}$$

Calculate $dr/d\theta$.

$$\frac{dr}{d\theta} = \frac{d}{d\theta}(3\cos\theta) = -3\sin\theta$$

Then m is

$$m = \frac{-3\sin\theta\sin\theta + 3\cos\theta\cos\theta}{-3\sin\theta\cos\theta - 3\cos\theta\sin\theta}$$

$$m = \frac{-\sin^2\theta + \cos^2\theta}{-2\sin\theta\cos\theta}$$

$$m = \frac{-\sin^2\left(\frac{2\pi}{3}\right) + \cos^2\left(\frac{2\pi}{3}\right)}{-2\sin\left(\frac{2\pi}{3}\right)\cos\left(\frac{2\pi}{3}\right)}$$

$$m=\frac{-\left(\frac{\sqrt{3}}{2}\right)^2+\left(-\frac{1}{2}\right)^2}{-2\left(\frac{\sqrt{3}}{2}\right)\left(-\frac{1}{2}\right)}$$

$$m=\frac{-\frac{3}{4}+\frac{1}{4}}{\frac{\sqrt{3}}{2}}=\frac{-\frac{1}{2}}{\frac{\sqrt{3}}{2}}=-\frac{1}{\sqrt{3}}=-\frac{\sqrt{3}}{3}$$

Use $\theta=2\pi/3$ and the conversion equations $x=r\cos\theta$ and $y=r\sin\theta$ to find a point on the tangent line.

$$x=r\cos\theta$$

$$x=3\cos\theta\cos\theta$$

$$x_1=3\cos^2\left(\frac{3\pi}{2}\right)$$

$$x_1=3\left(-\frac{1}{2}\right)^2$$

$$x_1=\frac{3}{4}$$

and

$$y=r\sin\theta$$

$$y=3\cos\theta\sin\theta$$

$$y_1=3\cos\left(\frac{3\pi}{2}\right)\sin\left(\frac{3\pi}{2}\right)$$

$$y_1 = 3\left(-\frac{1}{2}\right)\left(\frac{\sqrt{3}}{2}\right)$$

$$y_1 = -\frac{3\sqrt{3}}{4}$$

Therefore, the equation of the tangent line is

$$y - y_1 = m(x - x_1)$$

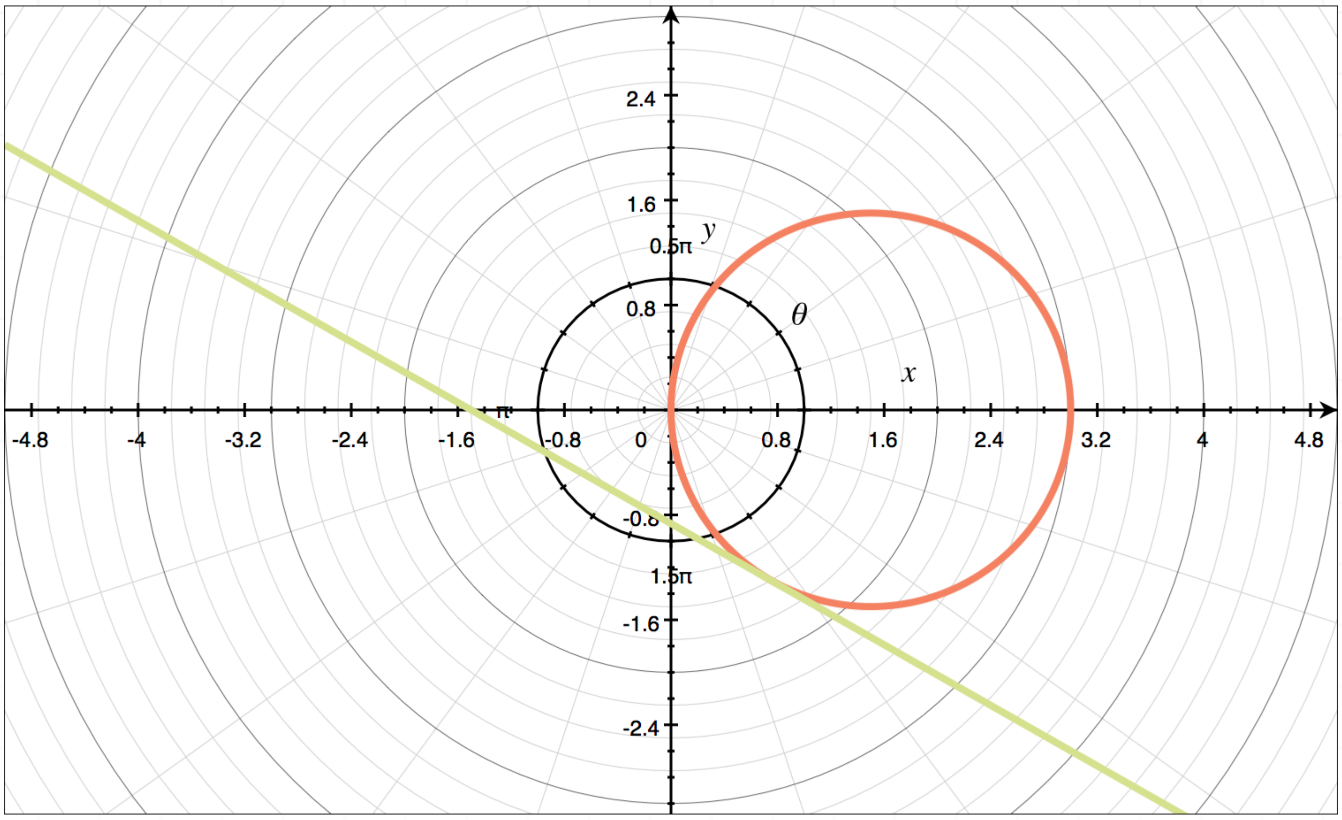
$$y + \frac{3\sqrt{3}}{4} = -\frac{\sqrt{3}}{3}\left(x - \frac{3}{4}\right)$$

$$y = -\frac{\sqrt{3}}{3}\left(x - \frac{3}{4}\right) - \frac{3\sqrt{3}}{4}$$

$$y = -\frac{\sqrt{3}}{3}x + \frac{\sqrt{3}}{4} - \frac{3\sqrt{3}}{4}$$

$$y = -\frac{\sqrt{3}}{3}x - \frac{\sqrt{3}}{2}$$

The graph shows the polar curve and the tangent line.



■ 2. Find the tangent line to the polar curve at $\theta = \pi/3$.

$$r = 5\sin\theta$$

Solution:

The slope of the tangent line m is given by

$$m = \frac{dy}{dx} = \frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta}$$

Calculate $dr/d\theta$.

$$\frac{dr}{d\theta} = \frac{d}{d\theta}(5\sin\theta) = 5\cos\theta$$

Then m is

$$m=\frac{5\cos\theta\sin\theta+5\sin\theta\cos\theta}{5\cos\theta\cos\theta-5\sin\theta\sin\theta}$$

$$m=\frac{\cos\theta\sin\theta+\sin\theta\cos\theta}{\cos^2\theta-\sin^2\theta}$$

$$m=\frac{2\cos\theta\sin\theta}{\cos^2\theta-\sin^2\theta}$$

$$m=\frac{2\cos\left(\frac{\pi}{3}\right)\sin\left(\frac{\pi}{3}\right)}{\cos^2\left(\frac{\pi}{3}\right)-\sin^2\left(\frac{\pi}{3}\right)}$$

$$m=\frac{2\left(\frac{1}{2}\right)\left(\frac{\sqrt{3}}{2}\right)}{\left(\frac{1}{2}\right)^2-\left(\frac{\sqrt{3}}{2}\right)^2}$$

$$m=\frac{\frac{\sqrt{3}}{2}}{\frac{1}{4}-\frac{3}{4}}=-\frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}}=-\frac{\sqrt{3}}{1}=-\sqrt{3}$$

Use $\theta=\pi/3$ and the conversion equations $x=r\cos\theta$ and $y=r\sin\theta$ to find a point on the tangent line.

$$x=r\cos\theta$$

$$x=5\sin\theta\cos\theta$$

$$x_1=5\sin\left(\frac{\pi}{3}\right)\cos\left(\frac{\pi}{3}\right)$$

$$x_1 = 5\left(\frac{\sqrt{3}}{2}\right)\left(\frac{1}{2}\right)$$

$$x_1 = \frac{5\sqrt{3}}{4}$$

and

$$y = r\sin\theta$$

$$y = 5\sin\theta\sin\theta$$

$$y_1 = 5\sin^2\left(\frac{\pi}{3}\right)$$

$$y_1 = 5\left(\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{3}}{2}\right)$$

$$y_1 = \frac{15}{4}$$

Therefore, the equation of the tangent line is

$$y - y_1 = m(x - x_1)$$

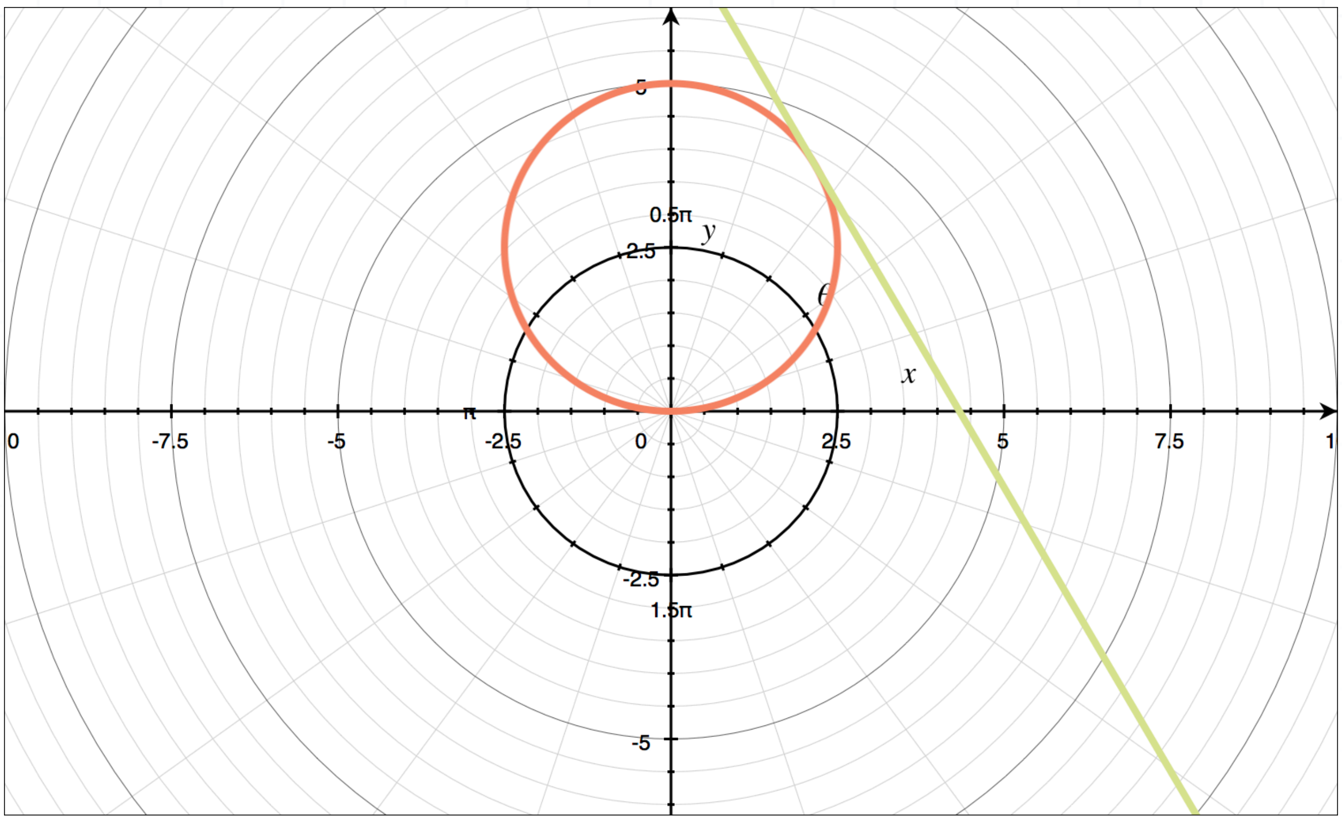
$$y - \frac{15}{4} = -\sqrt{3}\left(x - \frac{5\sqrt{3}}{4}\right)$$

$$y = -\sqrt{3}\left(x - \frac{5\sqrt{3}}{4}\right) + \frac{15}{4}$$

$$y = -\sqrt{3}x + \frac{15}{4} + \frac{15}{4}$$

$$y = -\sqrt{3}x + \frac{15}{2}$$

The graph shows the polar curve and the tangent line.



3. Find the tangent line to the polar curve at $\theta = \pi/4$.

$$r = 4 - 2\cos\theta$$

Solution:

The slope of the tangent line m is given by

$$m = \frac{dy}{dx} = \frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta}$$

Calculate $dr/d\theta$.

$$\frac{dr}{d\theta} = \frac{d}{d\theta}(4 - 2\cos\theta) = 2\sin\theta$$

Then m is

$$m = \frac{2\sin\theta\sin\theta + (4 - 2\cos\theta)\cos\theta}{2\sin\theta\cos\theta - (4 - 2\cos\theta)\sin\theta}$$

$$m = \frac{2\sin\theta\sin\theta + 4\cos\theta - 2\cos^2\theta}{2\sin\theta\cos\theta - 4\sin\theta + 2\cos\theta\sin\theta}$$

$$m = \frac{\sin\theta\sin\theta + 2\cos\theta - \cos^2\theta}{\sin\theta\cos\theta - 2\sin\theta + \cos\theta\sin\theta}$$

$$m = \frac{\sin\left(\frac{\pi}{4}\right)\sin\left(\frac{\pi}{4}\right) + 2\cos\left(\frac{\pi}{4}\right) - \cos^2\left(\frac{\pi}{4}\right)}{\sin\left(\frac{\pi}{4}\right)\cos\left(\frac{\pi}{4}\right) - 2\sin\left(\frac{\pi}{4}\right) + \cos\left(\frac{\pi}{4}\right)\sin\left(\frac{\pi}{4}\right)}$$

$$m = \frac{\left(\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{2}}{2}\right) + 2\left(\frac{\sqrt{2}}{2}\right) - \left(\frac{\sqrt{2}}{2}\right)^2}{\left(\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{2}}{2}\right) - 2\left(\frac{\sqrt{2}}{2}\right) + \left(\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{2}}{2}\right)}$$

$$m = \frac{\frac{1}{2} + \sqrt{2} - \frac{1}{2}}{\frac{1}{2} - \sqrt{2} + \frac{1}{2}} = \frac{\sqrt{2}}{1 - \sqrt{2}} = \frac{\sqrt{2}\left(1 + \sqrt{2}\right)}{\left(1 - \sqrt{2}\right)\left(1 + \sqrt{2}\right)} = \frac{2 + \sqrt{2}}{1 - 2} = -2 - \sqrt{2}$$

Use $\theta = \pi/4$ and the conversion equations $x = r\cos\theta$ and $y = r\sin\theta$ to find a point on the tangent line.

$$x = r\cos\theta$$

$$x = (4 - 2\cos\theta)\cos\theta$$

$$x_1 = 4\cos\left(\frac{\pi}{4}\right) - 2\cos^2\left(\frac{\pi}{4}\right)$$

$$x_1 = 4\left(\frac{\sqrt{2}}{2}\right) - 2\left(\frac{\sqrt{2}}{2}\right)^2$$

$$x_1 = 2\sqrt{2} - 1$$

and

$$y = r\sin\theta$$

$$y = (4 - 2\cos\theta)\sin\theta$$

$$y_1 = 4\sin\left(\frac{\pi}{4}\right) - 2\cos\left(\frac{\pi}{4}\right)\sin\left(\frac{\pi}{4}\right)$$

$$y_1 = 4\left(\frac{\sqrt{2}}{2}\right) - 2\left(\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{2}}{2}\right)$$

$$y_1 = 2\sqrt{2} - 1$$

Therefore, the equation of the tangent line is

$$y - y_1 = m(x - x_1)$$

$$y-\left(2\sqrt{2}-1\right)=\left(-2-\sqrt{2}\right)\left(x-\left(2\sqrt{2}-1\right)\right)$$

$$y=\left(-2-\sqrt{2}\right)\left(x-\left(2\sqrt{2}-1\right)\right)+\left(2\sqrt{2}-1\right)$$

$$y=-\left(2+\sqrt{2}\right)\left(x-2\sqrt{2}+1\right)+2\sqrt{2}-1$$

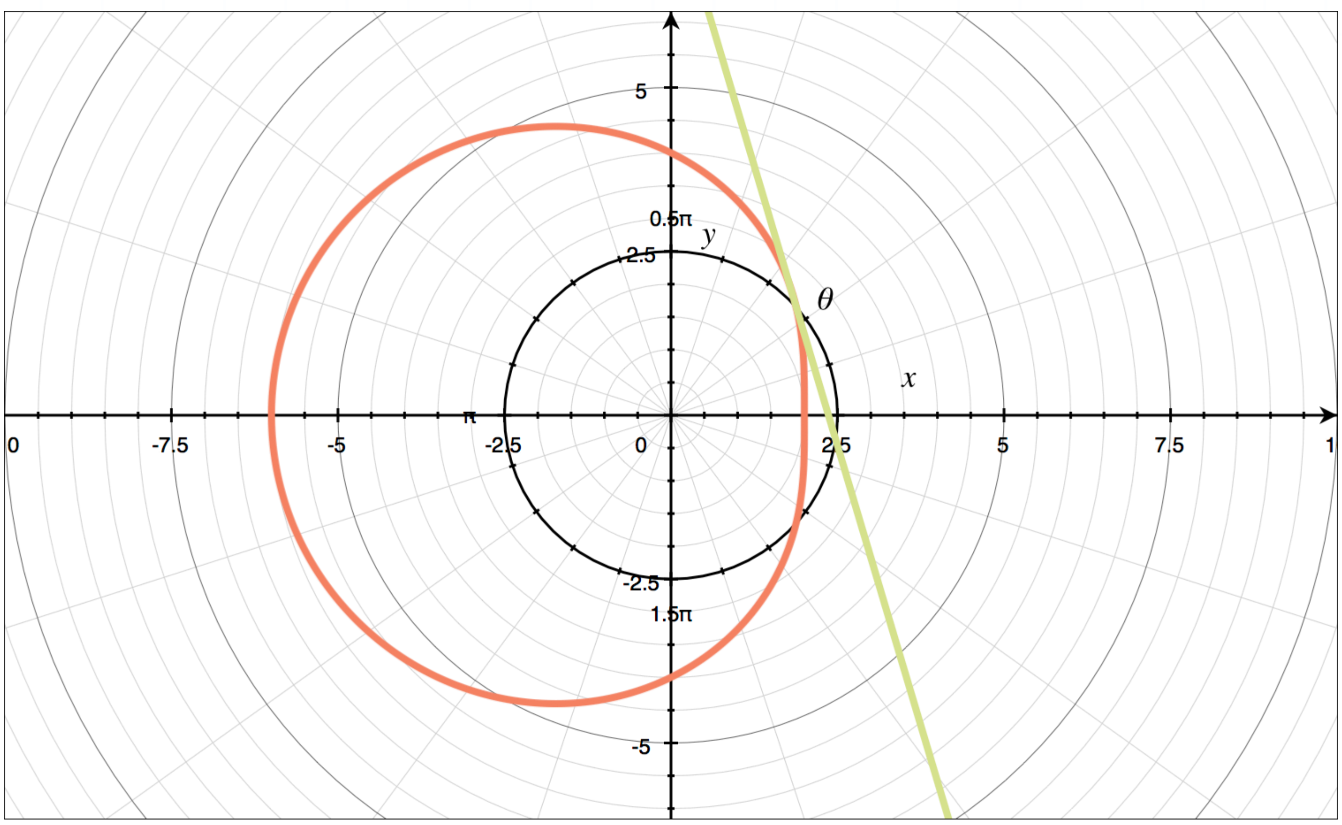
$$y=-\left(2x-4\sqrt{2}+2+\sqrt{2}x-4+\sqrt{2}\right)+2\sqrt{2}-1$$

$$y=-2x+4\sqrt{2}-2-\sqrt{2}x+4-\sqrt{2}+2\sqrt{2}-1$$

$$y=-2x-\sqrt{2}x+5\sqrt{2}+1$$

$$y=\left(-2-\sqrt{2}\right)x+5\sqrt{2}+1$$

The graph shows the polar curve and the tangent line.



4. Find the tangent line to the polar curve at $\theta = \pi$.

$$r = 8 - 5\sin\theta$$

Solution:

The slope of the tangent line m is given by

$$m = \frac{dy}{dx} = \frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta}$$

Calculate $dr/d\theta$.

$$\frac{dr}{d\theta} = \frac{d}{d\theta}(8 - 5\sin\theta) = -5\cos\theta$$

Then m is

$$m = \frac{-5\cos\theta\sin\theta + (8 - 5\sin\theta)\cos\theta}{-5\cos\theta\cos\theta - (8 - 5\sin\theta)\sin\theta}$$

$$m = \frac{-5\cos\theta\sin\theta + 8\cos\theta - 5\sin\theta\cos\theta}{-5\cos\theta\cos\theta - 8\sin\theta + 5\sin^2\theta}$$

$$m = \frac{-5\cos\pi\sin\pi + 8\cos\pi - 5\sin\pi\cos\pi}{-5\cos\pi\cos\pi - 8\sin\pi + 5\sin^2\pi}$$

$$m = \frac{-5(-1)(0) + 8(-1) - 5(0)(-1)}{-5(-1)(-1) - 8(0) + 5(0)^2}$$

$$m = \frac{0 - 8 + 0}{-5 - 0 + 0} = \frac{-8}{-5} = \frac{8}{5}$$

Use $\theta = \pi$ and the conversion equations $x = r\cos\theta$ and $y = r\sin\theta$ to find a point on the tangent line.

$$x = r\cos\theta$$

$$x = (8 - 5\sin\theta)\cos\theta$$

$$x_1 = (8 - 5\sin\pi)\cos\pi$$

$$x_1 = (8 - 5(0))(-1)$$

$$x_1 = 8(-1)$$

$$x_1 = -8$$

and

$$y = r\sin\theta$$

$$y = (8 - 5\sin\theta)\sin\theta$$

$$y_1 = (8 - 5\sin\pi)\sin\pi$$

$$y_1 = (8 - 5(0))(0)$$

$$y_1 = 0$$

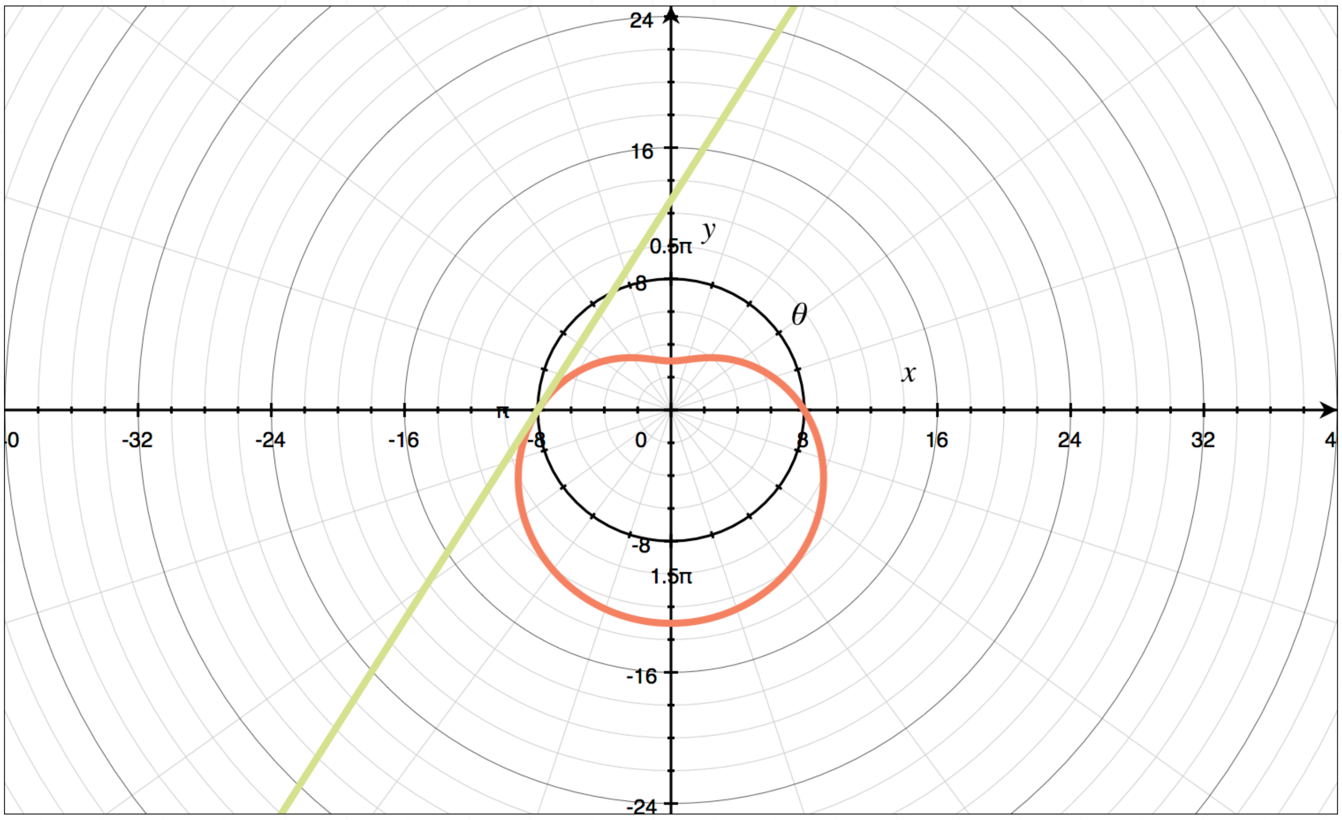
Therefore, the equation of the tangent line is

$$y - y_1 = m(x - x_1)$$

$$y - 0 = \frac{8}{5}(x + 8)$$

$$y = \frac{8}{5}x + \frac{64}{5}$$

The graph shows the polar curve and the tangent line.



5. Find the tangent line to the polar curve at $\theta = \pi/2$.

$$r = 7 - 6\cos\theta$$

Solution:

The slope of the tangent line m is given by

$$m = \frac{dy}{dx} = \frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta}$$

Calculate $dr/d\theta$.

$$\frac{dr}{d\theta} = \frac{d}{d\theta}(7 - 6\cos\theta) = 6\sin\theta$$

Then m is

$$m = \frac{6\sin\theta\sin\theta + \left(7 - 6\cos\theta\right)\cos\theta}{6\sin\theta\cos\theta - \left(7 - 6\cos\theta\right)\sin\theta}$$

$$m = \frac{6\sin\theta\sin\theta + 7\cos\theta - 6\cos^2\theta}{6\sin\theta\cos\theta - 7\sin\theta + 6\cos\theta\sin\theta}$$

$$m = \frac{6\sin\left(\frac{\pi}{2}\right)\sin\left(\frac{\pi}{2}\right) + 7\cos\left(\frac{\pi}{2}\right) - 6\cos^2\left(\frac{\pi}{2}\right)}{6\sin\left(\frac{\pi}{2}\right)\cos\left(\frac{\pi}{2}\right) - 7\sin\left(\frac{\pi}{2}\right) + 6\cos\left(\frac{\pi}{2}\right)\sin\left(\frac{\pi}{2}\right)}$$

$$m = \frac{6(1)(1) + 7(0) - 6(0)^2}{6(1)(0) - 7(1) + 6(0)(1)} = \frac{6 + 0 - 0}{0 - 7 + 0} = -\frac{6}{7}$$

Use $\theta = \pi/2$ and the conversion equations $x = r\cos\theta$ and $y = r\sin\theta$ to find a point on the tangent line.

$$x = r\cos\theta$$

$$x = (7 - 6\cos\theta)\cos\theta$$

$$x_1 = \left(7 - 6\cos\left(\frac{\pi}{2}\right)\right)\cos\left(\frac{\pi}{2}\right)$$

$$x_1 = \left(7 - 6(0)\right)(0)$$

$$x_1 = 0$$

and

$$y = r\sin\theta$$

$$y = (7 - 6\cos\theta)\sin\theta$$

$$y_1 = \left(7 - 6\cos\left(\frac{\pi}{2}\right)\right)\sin\left(\frac{\pi}{2}\right)$$

$$y_1 = (7 - 6(0))(1)$$

$$y_1 = 7$$

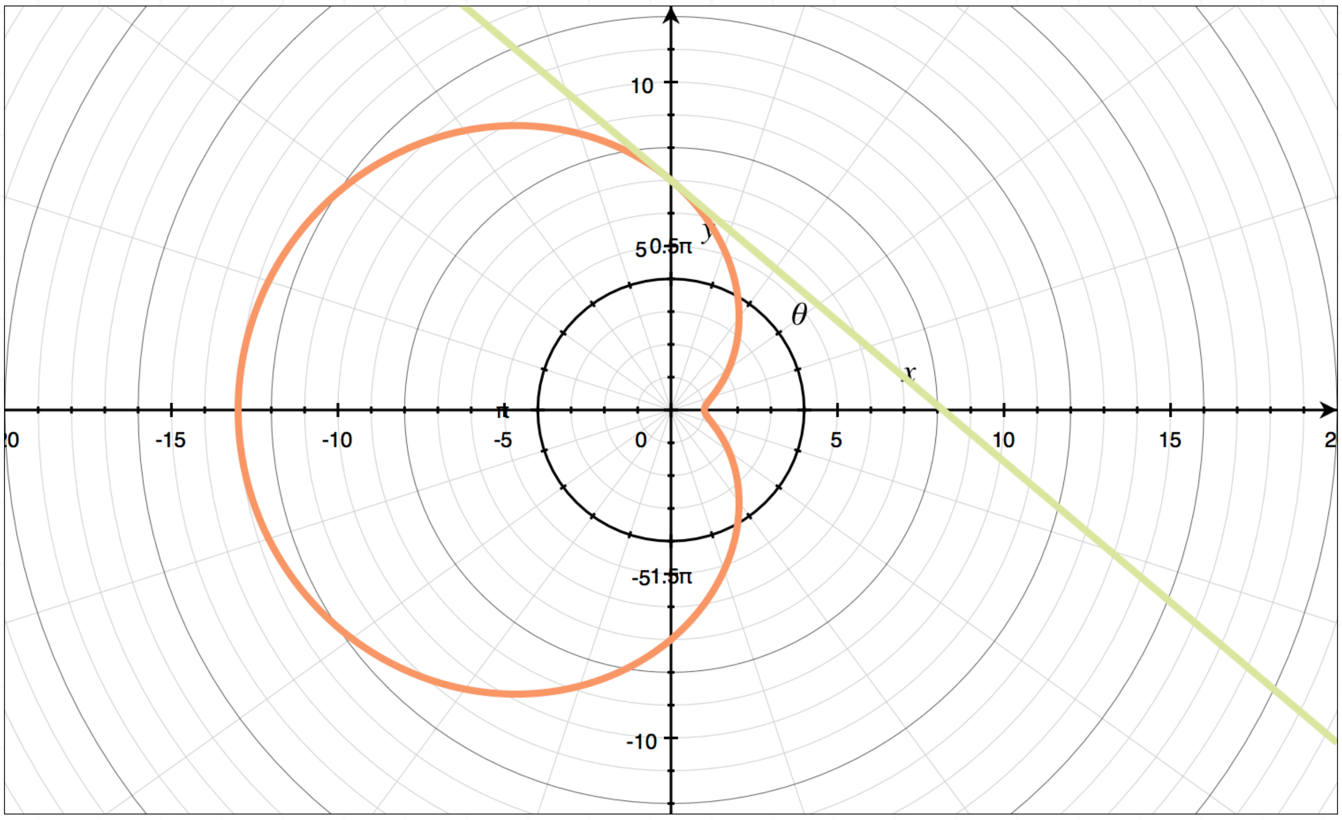
Therefore, the equation of the tangent line is

$$y - y_1 = m(x - x_1)$$

$$y - 7 = -\frac{6}{7}(x - 0)$$

$$y = -\frac{6}{7}x + 7$$

The graph shows the polar curve and the tangent line.


10
5
0.5π
y
θ
x
π
-15
-10
-5
0
5
10
15
-5
1.5π
-10

VERTICAL AND HORIZONTAL TANGENT LINES TO THE POLAR CURVE

1. At which points does the polar curve have horizontal tangent lines?

$$r = 4 - 4\sin\theta$$

Solution:

Use the conversion equations $x = r\cos\theta$ and $y = r\sin\theta$ and the polar equation into each of them.

$$x = r\cos\theta$$

$$x = (4 - 4\sin\theta)\cos\theta$$

$$x = 4\cos\theta - 4\sin\theta\cos\theta$$

and

$$y = r\sin\theta$$

$$y = (4 - 4\sin\theta)\sin\theta$$

$$y = 4\sin\theta - 4\sin^2\theta$$

Take the derivative of each equation.

$$\frac{dx}{d\theta} = \frac{d}{d\theta}(4\cos\theta - 4\sin\theta\cos\theta)$$

$$\frac{dx}{d\theta} = -4\sin\theta - 4(\cos\theta\cos\theta - \sin\theta\sin\theta)$$

$$\frac{dx}{d\theta} = 4\sin^2\theta - 4\cos^2\theta - 4\sin\theta$$

and

$$\frac{dy}{d\theta} = \frac{d}{d\theta}(4\sin\theta - 4\sin^2\theta)$$

$$\frac{dy}{d\theta} = 4\cos\theta - 8\sin\theta\cos\theta$$

Put these together to get dy/dx.

$$\frac{dy}{dx} = \frac{4\cos\theta - 8\sin\theta\cos\theta}{4\sin^2\theta - 4\cos^2\theta - 4\sin\theta}$$

$$\frac{dy}{dx} = \frac{\cos\theta - 2\sin\theta\cos\theta}{\sin^2\theta - \cos^2\theta - \sin\theta}$$

Horizontal tangent lines exist where the numerator is 0.

$$\cos\theta - 2\sin\theta\cos\theta = 0$$

$$\cos\theta(1 - 2\sin\theta) = 0$$

The solutions are

$\cos\theta = 0$ gives $\theta = \dfrac{\pi}{2}, \dfrac{3\pi}{2}$

$1 - 2\sin\theta = 0$ gives $\dfrac{\pi}{6}, \dfrac{5\pi}{6}$

Plug each of these angles into the equations for x and y. For $\theta = \pi/2$, we get

$$x = \left(4 - 4\sin\left(\frac{\pi}{2}\right)\right)\cos\left(\frac{\pi}{2}\right) \qquad y = \left(4 - 4\sin\left(\frac{\pi}{2}\right)\right)\sin\left(\frac{\pi}{2}\right)$$

$$x = (4 - 4(1))(0) \qquad y = (4 - 4(1))(1)$$

$$x = 0 \qquad y = 0$$

Plug $\theta = 3\pi/2$ into the equations for x and y.

$$x = \left(4 - 4\sin\left(\frac{3\pi}{2}\right)\right)\cos\left(\frac{3\pi}{2}\right) \qquad y = \left(4 - 4\sin\left(\frac{3\pi}{2}\right)\right)\sin\left(\frac{3\pi}{2}\right)$$

$$x = (4 - 4(-1))(0) \qquad y = (4 - 4(-1))(-1)$$

$$x = 0 \qquad y = -8$$

Plug $\theta = \pi/6$ into the equations for x and y.

$$x = \left(4 - 4\sin\left(\frac{\pi}{6}\right)\right)\cos\left(\frac{\pi}{6}\right) \qquad y = \left(4 - 4\sin\left(\frac{\pi}{6}\right)\right)\sin\left(\frac{\pi}{6}\right)$$

$$x = \left(4 - 4\left(\frac{1}{2}\right)\right)\left(\frac{\sqrt{3}}{2}\right) \qquad y = \left(4 - 4\left(\frac{1}{2}\right)\right)\left(\frac{1}{2}\right)$$

$$x = \sqrt{3} \qquad y = 1$$

Plug $\theta = 5\pi/6$ into the equations for x and y.

$$x = \left(4 - 4\sin\left(\frac{5\pi}{6}\right)\right)\cos\left(\frac{5\pi}{6}\right) \qquad y = \left(4 - 4\sin\left(\frac{5\pi}{6}\right)\right)\sin\left(\frac{5\pi}{6}\right)$$

$$x = \left(4 - 4\left(\frac{1}{2}\right)\right)\left(-\frac{\sqrt{3}}{2}\right) \qquad y = \left(4 - 4\left(\frac{1}{2}\right)\right)\left(\frac{1}{2}\right)$$

$$x = -\sqrt{3} \qquad y = 1$$

Therefore, the curve has horizontal tangent lines at

$$(0,0), (0,-8), (\sqrt{3},1), (-\sqrt{3},1)$$

■ 2. At which points does the polar curve have vertical tangent lines?

$$r = 6 - 6\cos\theta$$

Solution:

Use the conversion equations $x = r\cos\theta$ and $y = r\sin\theta$ and the polar equation into each of them.

$$x = r\cos\theta$$

$$x = (6 - 6\cos\theta)\cos\theta$$

$$x = 6\cos\theta - 6\cos^2\theta$$

and

$$y = r\sin\theta$$

$$y = (6 - 6\cos\theta)\sin\theta$$

$$y = 6\sin\theta - 6\cos\theta\sin\theta$$

Take the derivative of each equation.

$$\frac{dx}{d\theta} = \frac{d}{d\theta}(6\cos\theta - 6\cos^2\theta)$$

$$\frac{dx}{d\theta} = -6\sin\theta + 12\cos\theta\sin\theta$$

and

$$\frac{dy}{d\theta} = \frac{d}{d\theta}(6\sin\theta - 6\cos\theta\sin\theta)$$

$$\frac{dy}{d\theta} = 6\cos\theta - 6(\cos^2\theta - \sin^2\theta)$$

$$\frac{dy}{d\theta} = 6\cos\theta - 6\cos(2\theta)$$

Put these together to get dy/dx.

$$\frac{dy}{dx} = \frac{6\cos\theta - 6\cos(2\theta)}{-6\sin\theta + 12\cos\theta\sin\theta}$$

$$\frac{dy}{dx} = \frac{\cos\theta - \cos(2\theta)}{2\cos\theta\sin\theta - \sin\theta}$$

Vertical tangent lines exist where the denominator is 0.

$$2\cos\theta\sin\theta - \sin\theta = 0$$

$$\sin\theta\,(2\cos\theta - 1) = 0$$

The solutions are

$\sin\theta = 0$ gives $\theta = 0, \pi$

$2\cos\theta - 1 = 0$ gives $\theta = \frac{\pi}{3}, \frac{5\pi}{3}$

Plug $\theta = 0$ into the equations for x and y.

$$x = (6 - 6\cos(0))\cos(0) \qquad y = (6 - 6\sin(0))\sin(0)$$

$$x = (6 - 6(1))(1) \qquad y = (6 - 6(1))(0)$$

$$x = 0 \qquad y = 0$$

Plug $\theta = \pi$ into the equations for x and y.

$$x = (6 - 6\cos(\pi))\cos(\pi) \qquad y = (6 - 6\sin(\pi))\sin(\pi)$$

$$x = (6 - 6(-1))(-1) \qquad y = (6 - 6(0))(0)$$

$$x = -12 \qquad y = 0$$

Plug $\theta = \pi/3$ into the equations for x and y.

$$x = \left(6 - 6\cos\left(\frac{\pi}{3}\right)\right)\cos\left(\frac{\pi}{3}\right) \qquad y = \left(6 - 6\cos\left(\frac{\pi}{3}\right)\right)\sin\left(\frac{\pi}{3}\right)$$

$$x = \left(6 - 6\left(\frac{1}{2}\right)\right)\left(\frac{1}{2}\right) \qquad y = \left(6 - 6\left(\frac{1}{2}\right)\right)\left(\frac{\sqrt{3}}{2}\right)$$

$$x = \frac{3}{2} \qquad\qquad y = \frac{3\sqrt{3}}{2}$$

Plug $\theta = 5\pi/3$ into the equations for x and y.

$$x = \left(6 - 6\cos\left(\frac{5\pi}{3}\right)\right)\cos\left(\frac{5\pi}{3}\right) \qquad x = \left(6 - 6\cos\left(\frac{5\pi}{3}\right)\right)\sin\left(\frac{5\pi}{3}\right)$$

$$x = \left(6 - 6\left(\frac{1}{2}\right)\right)\left(\frac{1}{2}\right) \qquad y = \left(6 - 6\left(\frac{1}{2}\right)\right)\left(-\frac{\sqrt{3}}{2}\right)$$

$$x = \frac{3}{2} \qquad\qquad y = -\frac{3\sqrt{3}}{2}$$

Therefore, the curve has vertical tangent lines at

$$(0,0),\ (-12,0),\ \left(\frac{3}{2}, \frac{3\sqrt{3}}{2}\right),\ \left(\frac{3}{2}, -\frac{3\sqrt{3}}{2}\right)$$

■ 3. At which points does the polar curve have horizontal tangent lines?

$$r = 8 - 2\sin\theta$$

Solution:

Use the conversion equations $x = r\cos\theta$ and $y = r\sin\theta$ and the polar equation into each of them.

$$x = r\cos\theta$$

$$x = (8 - 2\sin\theta)\cos\theta$$

$$x = 8\cos\theta - 2\sin\theta\cos\theta$$

and

$$y = r\sin\theta$$

$$y = (8 - 2\sin\theta)\sin\theta$$

$$y = 8\sin\theta - 2\sin^2\theta$$

Take the derivative of each equation.

$$\frac{dx}{d\theta} = \frac{d}{d\theta}(8\cos\theta - 2\sin\theta\cos\theta)$$

$$\frac{dx}{d\theta} = -8\sin\theta - 2(\cos^2\theta - \sin^2\theta)$$

$$\frac{dx}{d\theta} = -8\sin\theta - 2\cos(2\theta)$$

and

$$\frac{dy}{d\theta} = \frac{d}{d\theta}(8\sin\theta - 2\sin^2\theta)$$

$$\frac{dy}{d\theta} = 8\cos\theta - 4\sin\theta\cos\theta$$

Put these together to get dy/dx.

$$\frac{dy}{dx} = \frac{8\cos\theta - 4\sin\theta\cos\theta}{-8\sin\theta - 2\cos(2\theta)}$$

$$\frac{dy}{dx} = \frac{4\cos\theta - 2\sin\theta\cos\theta}{-4\sin\theta - \cos(2\theta)}$$

Horizontal tangent lines exist where the numerator is 0.

$$4\cos\theta - 2\sin\theta\cos\theta = 0$$

$$2\cos\theta(2 - \sin\theta) = 0$$

$$\theta = \frac{\pi}{2}, \frac{3\pi}{2}$$

Plug $\theta = \pi/2$ into the equations for x and y.

$$x = \left(8 - 2\sin\left(\frac{\pi}{2}\right)\right)\cos\left(\frac{\pi}{2}\right) \qquad y = \left(8 - 2\sin\left(\frac{\pi}{2}\right)\right)\sin\left(\frac{\pi}{2}\right)$$

$$x = (8 - 2(1))(0) \qquad y = (8 - 2(1))(1)$$

$$x = 0 \qquad y = 6$$

Plug $\theta = 3\pi/2$ into the equations for x and y.

$$x = \left(8 - 2\sin\left(\frac{3\pi}{2}\right)\right)\cos\left(\frac{3\pi}{2}\right) \qquad y = \left(8 - 2\sin\left(\frac{3\pi}{2}\right)\right)\sin\left(\frac{3\pi}{2}\right)$$

$$x = (8 - 2(-1))(0) \qquad y = (8 - 2(-1))(-1)$$

$$x = 0 \qquad y = -10$$

Therefore, when $\theta = \pi/2$, the curve has a horizontal tangent line at $(0,6)$. And when $\theta = 3\pi/2$, the curve has a horizontal tangent at $(0, -10)$.

INTERSECTION OF THE POLAR CURVES

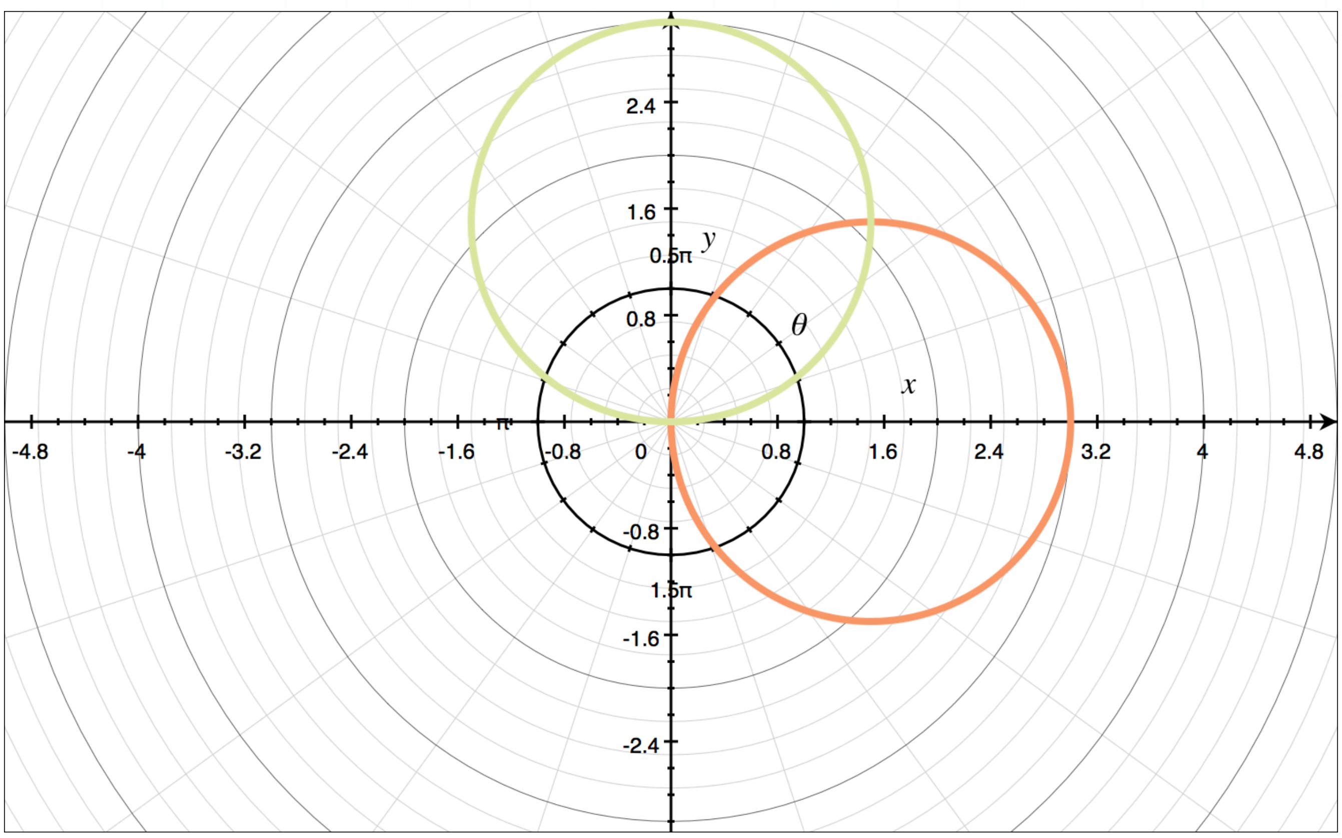
1. Find the rectangular points of intersection of the polar curves.

$$r = 3\cos\theta$$

$$r = 3\sin\theta$$

Solution:

A sketch of the polar curves is



Set the two equations equal to each other and solve for θ.

$$3\cos\theta = 3\sin\theta$$

$$\cos\theta = \sin\theta$$

$$\theta = \frac{\pi}{4}, \frac{5\pi}{4}$$

Plugging these values of θ back into $r = 3\cos\theta$ gives the polar points of intersection as

$$r = 3\cos\theta = 3\cos\frac{\pi}{4} = 3\left(\frac{\sqrt{2}}{2}\right) = \frac{3\sqrt{2}}{2}$$

$$r = 3\cos\theta = 3\cos\frac{5\pi}{4} = 3\left(-\frac{\sqrt{2}}{2}\right) = -\frac{3\sqrt{2}}{2}$$

The polar points of intersection are therefore

$$\left(\frac{3\sqrt{2}}{2}, \frac{\pi}{4}\right) \text{ and } \left(-\frac{3\sqrt{2}}{2}, \frac{5\pi}{4}\right)$$

If we convert these to rectangular points, we get

$$x = r\cos\theta \qquad\qquad y = r\sin\theta$$

$$x = \frac{3\sqrt{2}}{2}\cos\frac{\pi}{4} \qquad\qquad y = \frac{3\sqrt{2}}{2}\sin\frac{\pi}{4}$$

$$x = \frac{3\sqrt{2}}{2}\left(\frac{\sqrt{2}}{2}\right) = \frac{3(2)}{4} = \frac{3}{2} \qquad\qquad y = \frac{3\sqrt{2}}{2}\left(\frac{\sqrt{2}}{2}\right) = \frac{3(2)}{4} = \frac{3}{2}$$

and

$$x = r\cos\theta \qquad\qquad y = r\sin\theta$$

$$x = -\frac{3\sqrt{2}}{2}\cos\frac{5\pi}{4} \qquad y = -\frac{3\sqrt{2}}{2}\sin\frac{5\pi}{4}$$

$$x = -\frac{3\sqrt{2}}{2}\left(-\frac{\sqrt{2}}{2}\right) = \frac{3(2)}{4} = \frac{3}{2} \qquad y = -\frac{3\sqrt{2}}{2}\left(-\frac{\sqrt{2}}{2}\right) = \frac{3(2)}{4} = \frac{3}{2}$$

Both cases give the rectangular intersection point

$$\left(\frac{3}{2},\frac{3}{2}\right)$$

Notice though that from the sketch the curves also intersect at the pole $(0,0)$. Which means the rectangular points of intersection are

$$(0,0) \text{ and } \left(\frac{3}{2},\frac{3}{2}\right)$$

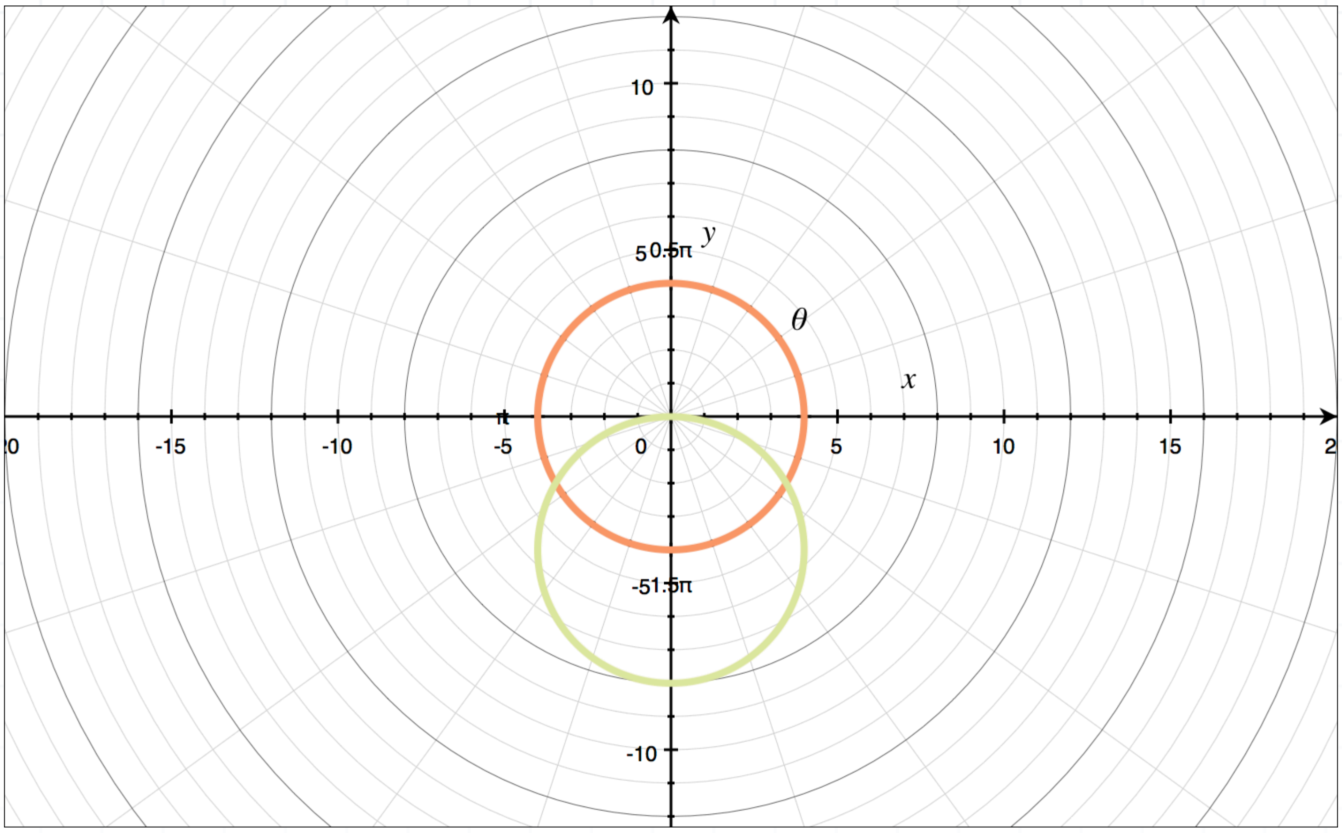
■ 2. Find the polar points of intersection of the polar curves.

$$r = 4$$

$$r = -8\sin\theta$$

Solution:

A sketch of the polar curves is



To find points of intersection, set the two equations equal to each other and solve for θ.

$$4 = -8\sin\theta$$

$$-\frac{1}{2} = \sin\theta$$

$$\theta = \sin^{-1}\left(-\frac{1}{2}\right)$$

$$\theta = \frac{7\pi}{6}, \frac{11\pi}{6}$$

Plugging these values of θ back into $r = 4$ gives the polar points of intersection as

$\left(4,\frac{7\pi}{6}\right)$ and $\left(4,\frac{11\pi}{6}\right)$

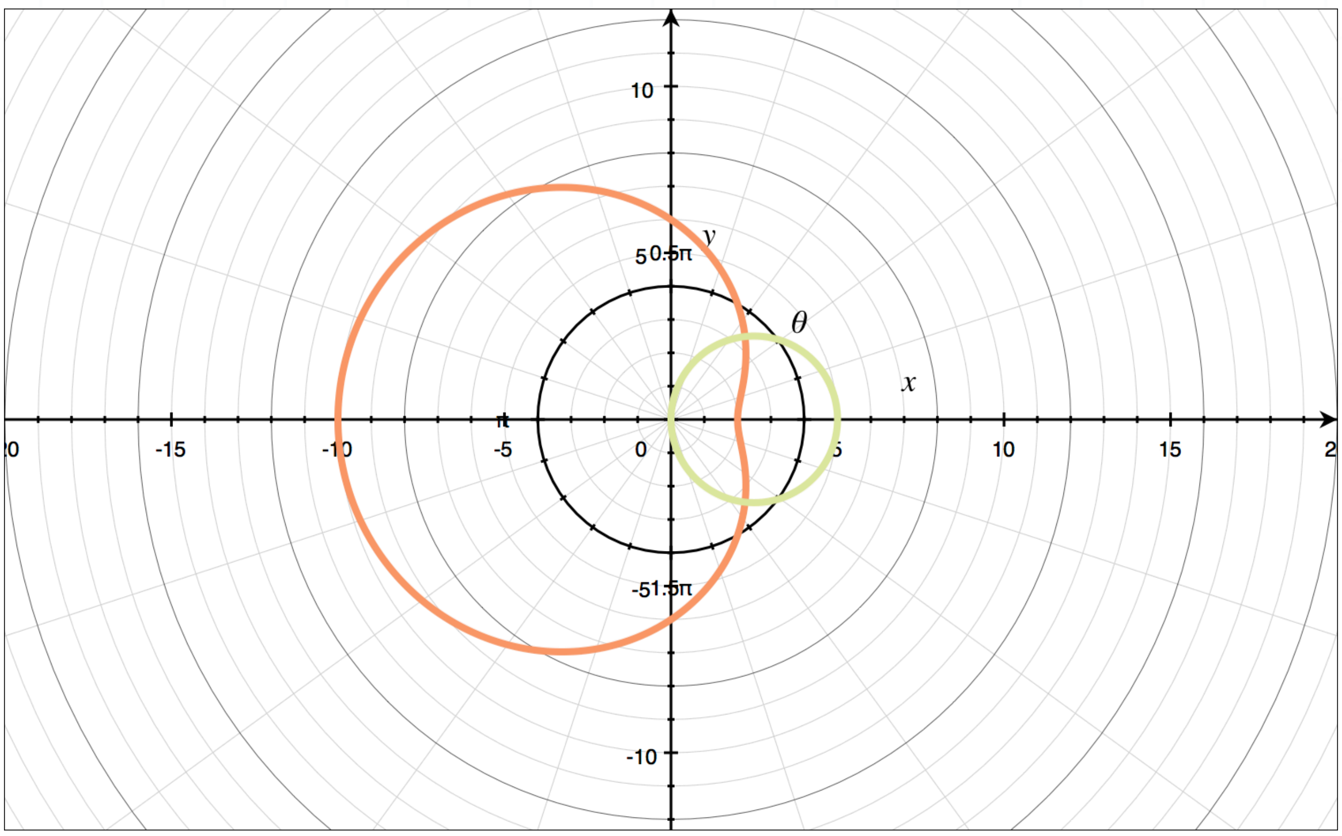
3. Find the rectangular points of intersection of the polar curves.

$$r = 6 - 4\cos\theta$$

$$r = 5\cos\theta$$

Solution:

A sketch of the polar curves is



To find points of intersection, set the two equations equal to each other and solve for θ.

$$6 - 4\cos\theta = 5\cos\theta$$

$$6 = 9\cos\theta$$

$$\frac{6}{9} = \cos\theta$$

$$\theta = \cos^{-1}\left(\frac{2}{3}\right)$$

$$\theta = \pm\, 0.841069$$

Plugging these values of θ back into $r = 5\cos\theta$ gives the polar points of intersection as

$$r = 5\cos(0.841069)$$

$$r = 5\left(\frac{2}{3}\right)$$

$$r = \frac{10}{3}$$

and

$$r = 5\cos(-0.841069)$$

$$r = 5\left(\frac{2}{3}\right)$$

$$r = \frac{10}{3}$$

So the polar points of intersection are

$$\left(\frac{10}{3}, 0.841069\right) \text{ and } \left(\frac{10}{3}, -0.841069\right)$$

Convert these to rectangular points. For x we get

$$x = r\cos\theta \qquad x = r\cos\theta$$

$$x_1 = \frac{10}{3}\cos(0.841069) \qquad x_2 = \frac{10}{3}\cos(-0.841069)$$

$$x_1 = \frac{10}{3}\left(\frac{2}{3}\right) \qquad x_2 = \frac{10}{3}\left(\frac{2}{3}\right)$$

$$x_1 = \frac{20}{9} \qquad x_2 = \frac{20}{9}$$

And for y we get

$$y = r\sin\theta \qquad y = r\sin\theta$$

$$y_1 = \frac{10}{3}\sin(0.841069) \qquad y_2 = \frac{10}{3}\sin(-0.841069)$$

$$y_1 = \frac{10}{3}(0.7453562121) \qquad y_2 = \frac{10}{3}(-0.7453562121)$$

$$y_1 = 2.48 \qquad y_2 = -2.48$$

So the rectangular points of intersection are

$\left(\frac{20}{9}, 2.48\right)$ and $\left(\frac{20}{9}, -2.48\right)$

$(2.22, 2.48)$ and $(2.22, -2.48)$

AREA INSIDE A POLAR CURVE

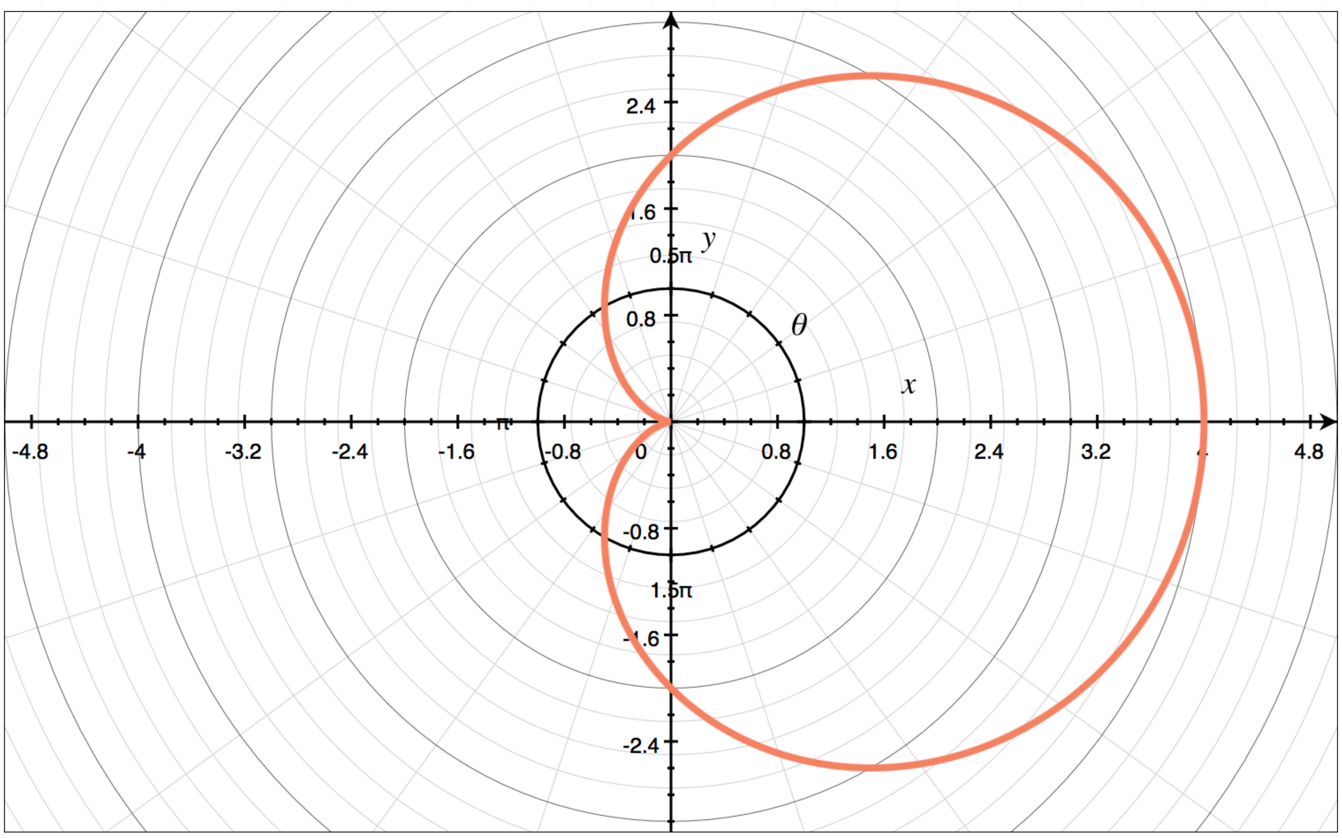
1. Find the area bounded by the polar curve over the interval.

$$r = 2 + 2\cos\theta$$

$$0 \le \theta \le 2\pi$$

Solution:

A sketch of the polar curve is



The area bounded by the curve over the interval is

$$A = \int_a^b \frac{1}{2}\left[f(\theta)\right]^2\ d\theta$$

$$A=\int_0^{2\pi}\frac{1}{2}(2+2\cos\theta)^2\ d\theta$$

$$A=\frac{1}{2}\int_0^{2\pi}4+8\cos\theta+4\cos^2\theta\ d\theta$$

Use the trig identity

$$\cos^2\theta=\frac{1}{2}(1+\cos(2\theta))$$

to substitute.

$$A=\frac{1}{2}\int_0^{2\pi}4+8\cos\theta+4\cdot\frac{1}{2}(1+\cos(2\theta))\ d\theta$$

$$A=\frac{1}{2}\int_0^{2\pi}4+8\cos\theta+2+2\cos(2\theta)\ d\theta$$

$$A=\int_0^{2\pi}3+4\cos\theta+\cos(2\theta)\ d\theta$$

Integrate, then evaluate over the interval.

$$A=3\theta+4\sin\theta+\frac{1}{2}\sin(2\theta)\Big|_0^{2\pi}$$

$$A=3(2\pi)+4\sin(2\pi)+\frac{1}{2}\sin(2\cdot 2\pi)-\left(3(0)+4\sin(0)+\frac{1}{2}\sin(2\cdot 0)\right)$$

$$A=6\pi+4(0)+\frac{1}{2}(0)$$

$$A = 6\pi$$

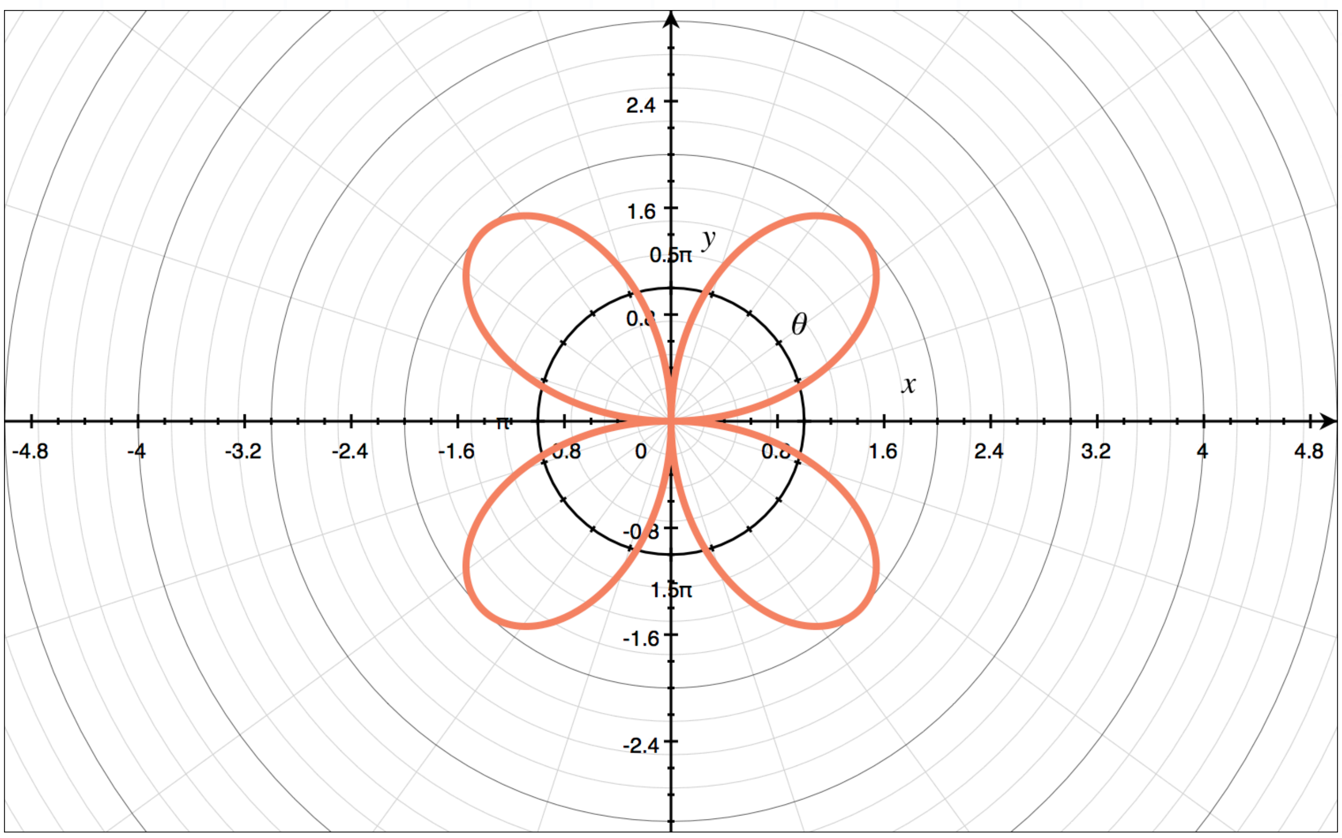
2. Find the area bounded by the polar curve over the interval.

$$r = 2\sin 2\theta$$

$$0 \le \theta \le 2\pi$$

Solution:

A sketch of the polar curve is



The area bounded by the curve over the interval is

$$A = \int_a^b \frac{1}{2}\left[f(\theta)\right]^2 \ d\theta$$

$$A=\int_0^{2\pi}\frac{1}{2}(2\sin(2\theta))^2\ d\theta$$

$$A=\int_0^{2\pi}\frac{1}{2}(4\sin^2(2\theta))\ d\theta$$

$$A=2\int_0^{2\pi}\sin^2(2\theta)\ d\theta$$

Use the trig identity

$$\sin^2\theta=\frac{1}{2}(1-\cos(2\theta))$$

$$\sin^2(2\theta)=\frac{1}{2}(1-\cos(4\theta))$$

to substitute.

$$A=2\int_0^{2\pi}\frac{1}{2}(1-\cos(4\theta))\ d\theta$$

$$A=\int_0^{2\pi}1-\cos(4\theta)\ d\theta$$

Integrate, then evaluate over the interval.

$$A=\theta-\frac{1}{4}\sin(4\theta)\Big|_0^{2\pi}$$

$$A=2\pi-\frac{1}{4}\sin(4\cdot 2\pi)-\left(0-\frac{1}{4}\sin(4\cdot 0)\right)$$

$$A = 2\pi - \frac{1}{4}\sin(8\pi)$$

$$A = 2\pi - \frac{1}{4}(0)$$

$$A = 2\pi$$

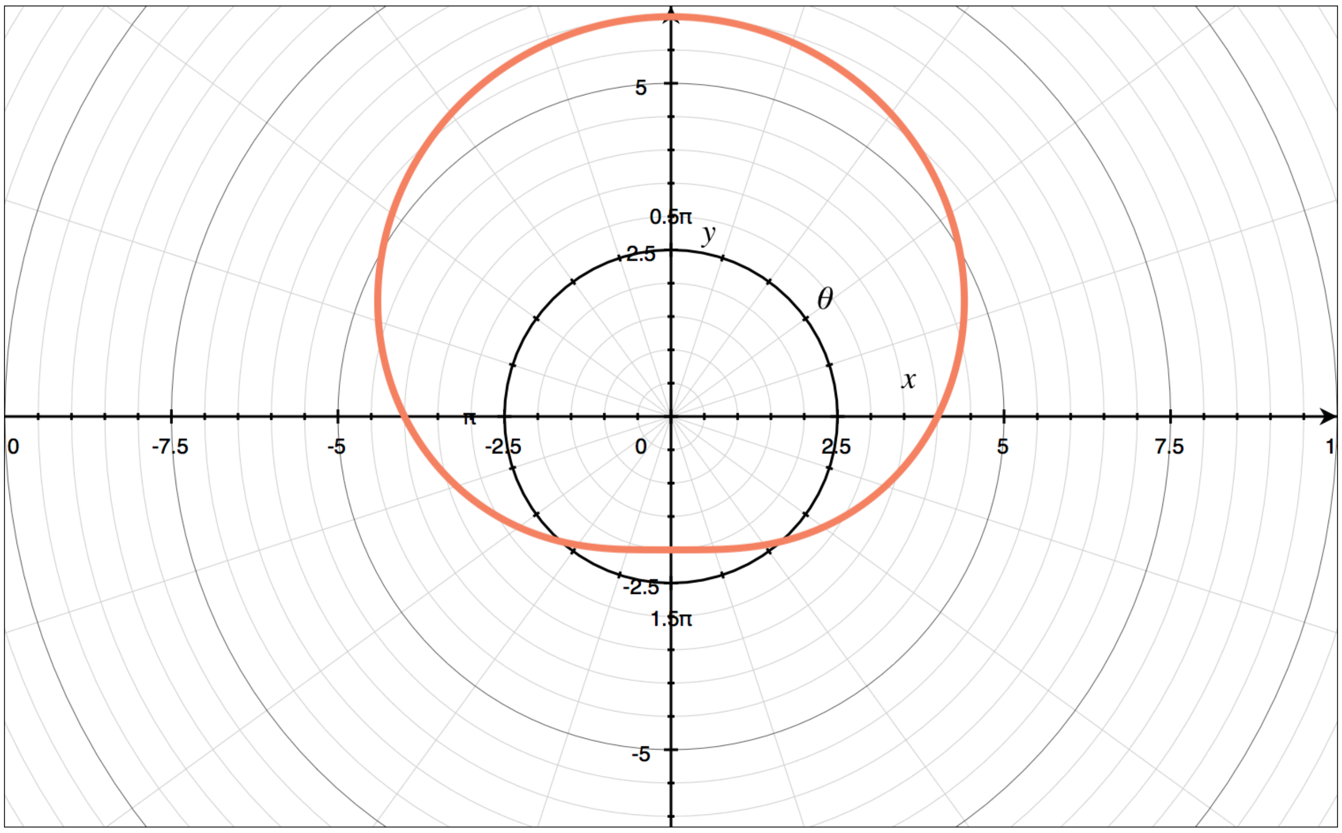
3. Find the area bounded by the polar curve over the interval.

$$r = 4 + 2\sin\theta$$

$$0 \le \theta \le 2\pi$$

Solution:

A sketch of the polar curve is



The area bounded by the curve over the interval is

$$A=\int_{a}^{b}\frac{1}{2}\left[f(\theta)\right]^{2}\ d\theta$$

$$A=\int_{0}^{2\pi}\frac{1}{2}(4+2\sin\theta)^{2}\ d\theta$$

$$A=\frac{1}{2}\int_{0}^{2\pi}16+16\sin\theta+4\sin^{2}\theta\ d\theta$$

Use the trig identity

$$\sin^{2}\theta=\frac{1}{2}(1-\cos(2\theta))$$

to substitute.

$$A=\frac{1}{2}\int_{0}^{2\pi}16+16\sin\theta+4\cdot\frac{1}{2}(1-\cos(2\theta))\ d\theta$$

$$A=\frac{1}{2}\int_{0}^{2\pi}16+16\sin\theta+2-2\cos(2\theta)\ d\theta$$

$$A=\int_{0}^{2\pi}9+8\sin\theta-\cos(2\theta)\ d\theta$$

Integrate, then evaluate over the interval.

$$A=9\theta-8\cos\theta-\frac{1}{2}\sin(2\theta)\Big|_{0}^{2\pi}$$

$$A = 9(2\pi) - 8\cos(2\pi) - \frac{1}{2}\sin(2 \cdot 2\pi) - \left(9(0) - 8\cos(0) - \frac{1}{2}\sin(2 \cdot 0)\right)$$

$$A = 18\pi - 8(1) - \frac{1}{2}(0) + 8(1)$$

$$A = 18\pi - 8 + 8$$

$$A = 18\pi$$

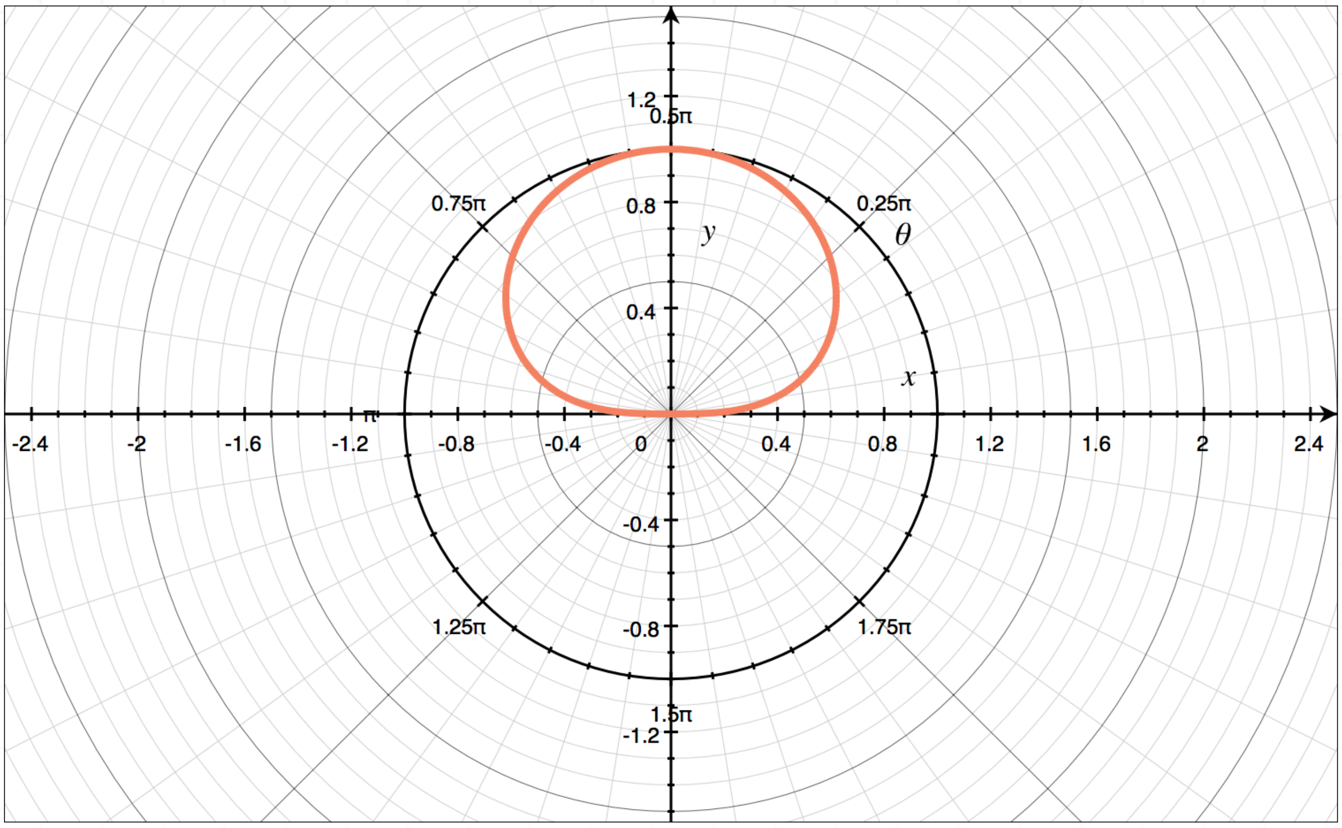
■ 4. Find the area bounded by the polar curve over the interval.

$$r^2 = \sin\theta$$

$$0 \le \theta \le \pi$$

Solution:

A sketch of the polar curve is



Since

$$r^2 = \sin\theta$$

$$r = \pm\sqrt{\sin\theta}$$

$r = \sqrt{\sin\theta}$ and $r = -\sqrt{\sin\theta}$

The area bounded by the curve over the interval is

$$A = \int_a^b \frac{1}{2}\left[f(\theta)\right]^2 \ d\theta$$

$$A = \int_0^\pi \frac{1}{2}\left(\sqrt{\sin\theta}\right)^2 \ d\theta + \int_0^\pi \frac{1}{2}\left(-\sqrt{\sin\theta}\right)^2 \ d\theta$$

$$A = \frac{1}{2}\int_0^\pi \sin\theta \ d\theta + \frac{1}{2}\int_0^\pi \sin\theta \ d\theta$$

$$A = \int_0^{\pi} \sin\theta \ d\theta$$

Integrate, then evaluate over the interval.

$$A = -\cos\theta \Big|_0^{\pi}$$

$$A = -\cos\pi - (-\cos(0))$$

$$A = -(-1) + 1$$

$$A = 1 + 1$$

$$A = 2$$

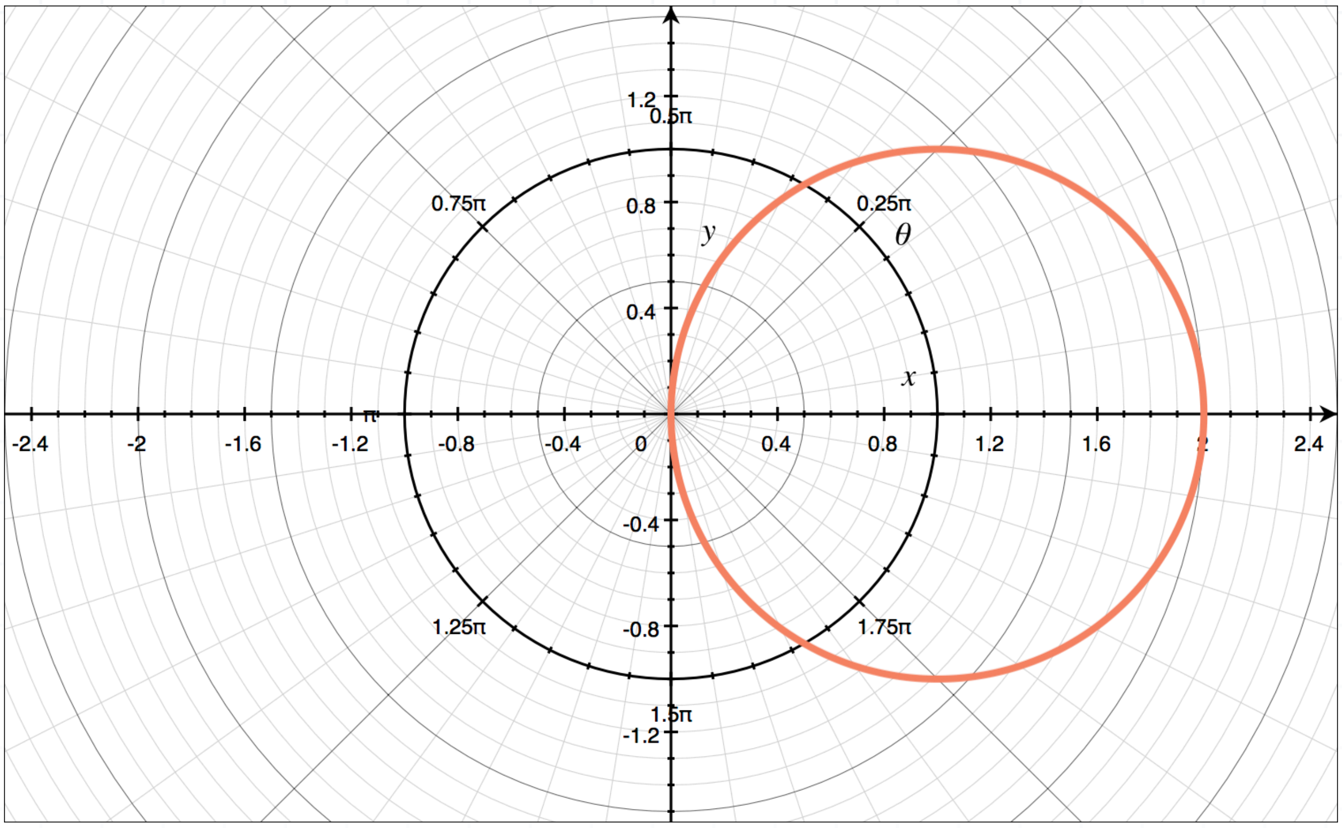
■ 5. Find the area bounded by the polar curve over the interval.

$$r = 2\cos\theta$$

$$-\frac{\pi}{4} \le \theta \le \frac{\pi}{4}$$

Solution:

The graph of the polar region is



The area bounded by the curve over the interval is

$$A=\int_{a}^{b}\frac{1}{2}\left[f(\theta)\right]^2\ d\theta$$

$$A=\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\frac{1}{2}(2\cos\theta)^2\ d\theta$$

$$A=\frac{1}{2}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}4\cos^2\theta\ d\theta$$

$$A=2\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\cos^2\theta\ d\theta$$

Use the trig identity

$$\cos^2\theta=\frac{1}{2}(1+\cos(2\theta))$$

to substitute.

$$A=2\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\frac{1}{2}(1+\cos(2\theta))\ d\theta$$

$$A=\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}1+\cos(2\theta)\ d\theta$$

Integrate, then evaluate over the interval.

$$A=\theta+\frac{1}{2}\sin(2\theta)\Big|_{-\frac{\pi}{4}}^{\frac{\pi}{4}}$$

$$A=\frac{\pi}{4}+\frac{1}{2}\sin\left(2\cdot\frac{\pi}{4}\right)-\left(-\frac{\pi}{4}+\frac{1}{2}\sin\left(2\cdot-\frac{\pi}{4}\right)\right)$$

$$A=\frac{\pi}{4}+\frac{1}{2}\sin\frac{\pi}{2}+\frac{\pi}{4}-\frac{1}{2}\sin\left(-\frac{\pi}{2}\right)$$

$$A=\frac{\pi}{2}+\frac{1}{2}\sin\frac{\pi}{2}-\frac{1}{2}\sin\left(-\frac{\pi}{2}\right)$$

$$A=\frac{\pi}{2}+\frac{1}{2}(1)-\frac{1}{2}(-1)$$

$$A=\frac{\pi}{2}+1$$

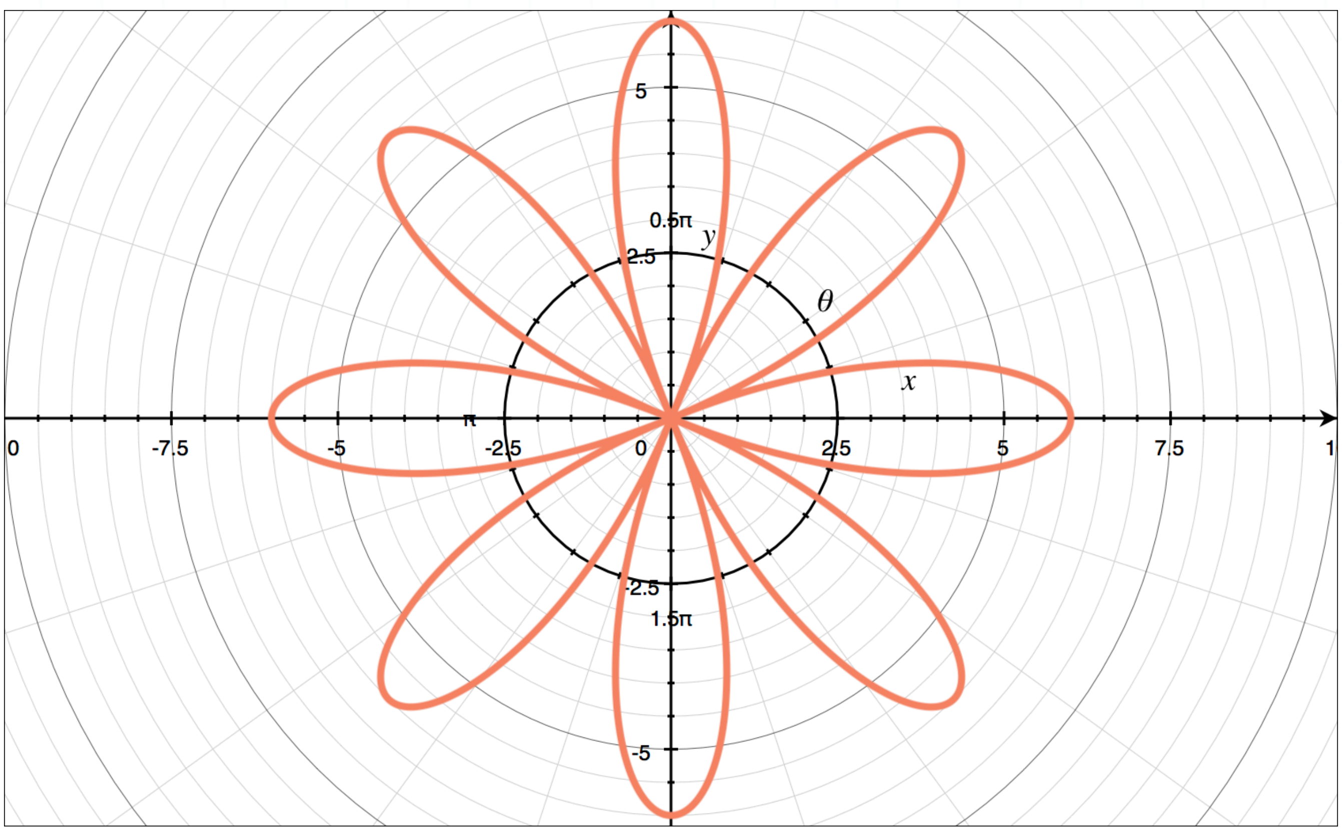
AREA BOUNDED BY ONE LOOP OF A POLAR CURVE

1. Find the area of one loop of the polar curve.

$$r = 6\cos(4\theta)$$

Solution:

A sketch of the polar curve is



Setting $4\theta = \pi/2$ gives $\theta = \pi/8$.

At $\theta = 0$, $r = 6\cos(4(0)) = 6$

At $\theta = \pi/8$, $r = 6\cos(4(\pi/8)) = 0$

So these angles define the top half of the loop that straddles the positive side of the horizontal axis. We'll use these limits of integration and then double the integral to get the area of the full loop.

$$A = 2\int_a^b \frac{1}{2}r^2\ d\theta$$

$$A = \int_a^b r^2\ d\theta$$

$$A = \int_0^{\frac{\pi}{8}} (6\cos(4\theta))^2\ d\theta$$

$$A = \int_0^{\frac{\pi}{8}} 36\cos^2(4\theta)\ d\theta$$

Use the trig identity

$$\cos^2\theta = \frac{1}{2}(1+\cos(2\theta))$$

$$\cos^2(4\theta) = \frac{1}{2}(1+\cos(8\theta))$$

to substitute.

$$A = \int_0^{\frac{\pi}{8}} 36\cdot\frac{1}{2}(1+\cos(8\theta))\ d\theta$$

$$A = 18\int_0^{\frac{\pi}{8}} 1+\cos(8\theta)\ d\theta$$

Integrate, then evaluate over the interval.

$$A = 18\left(\theta + \frac{1}{8}\sin(8\theta)\right)\Big|_0^{\frac{\pi}{8}}$$

$$A = 18\left(\frac{\pi}{8} + \frac{1}{8}\sin\left(8 \cdot \frac{\pi}{8}\right)\right) - 18\left(0 + \frac{1}{8}\sin(8 \cdot 0)\right)$$

$$A = 18\left(\frac{\pi}{8} + \frac{1}{8}\sin\pi\right)$$

$$A = \frac{9}{4}(\pi + 0)$$

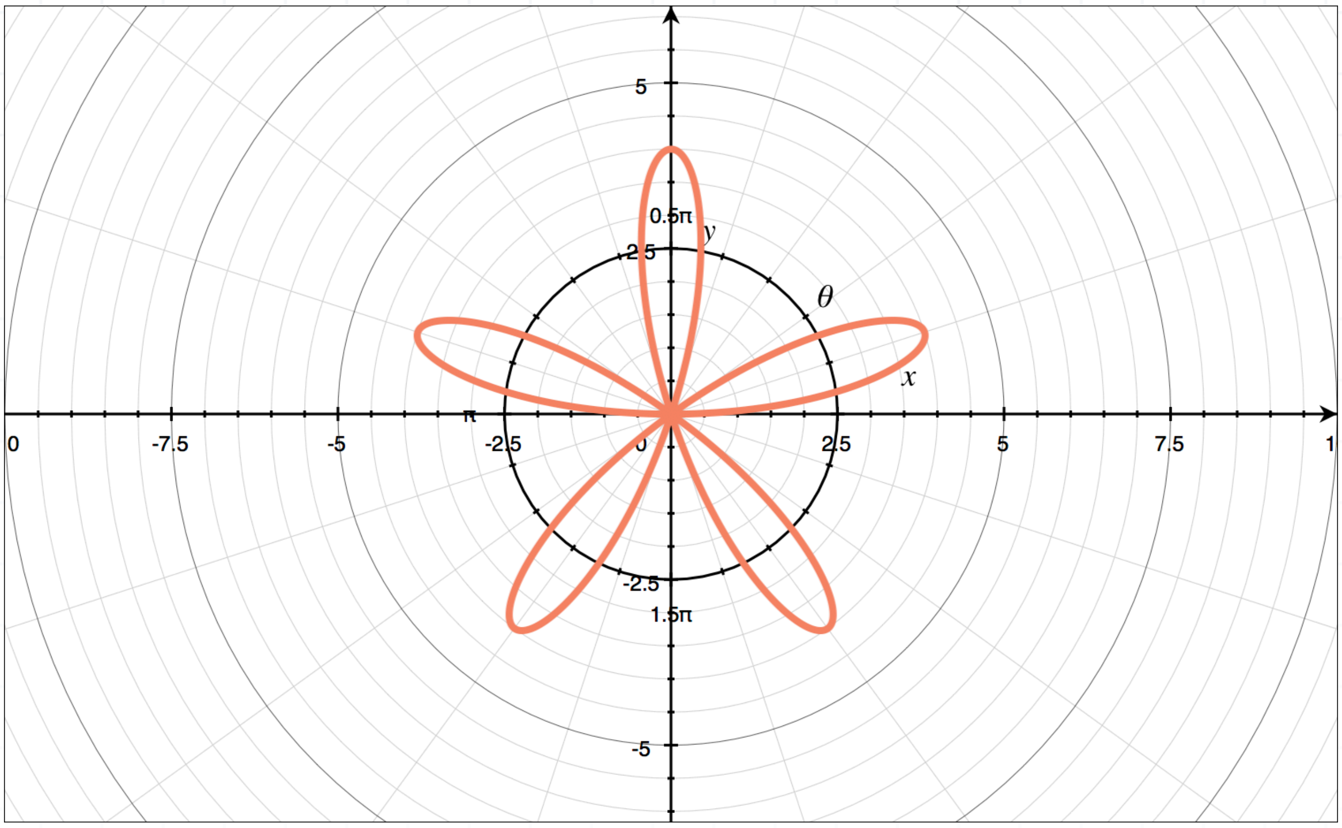
$$A = \frac{9\pi}{4}$$

■ 2. Find the area of one loop of the polar curve.

$$r = 4\sin(5\theta)$$

Solution:

A sketch of the polar curve is



Setting $5\theta = \pi/2$ gives $\theta = \pi/10$.

At $\theta = 0$, $r = 4\sin(5(0)) = 0$

At $\theta = \pi/10$, $r = 4\sin(5(\pi/10)) = 4$

At $\theta = \pi/5$, $r = 4\sin(5(\pi/5)) = 0$

So $\theta = 0$ and $\theta = \pi/5$ define the first loop in the first quadrant.

$$A = \int_a^b \frac{1}{2}r^2\ d\theta$$

$$A = \int_0^{\frac{\pi}{5}} \frac{1}{2}(4\sin(5\theta))^2\ d\theta$$

$$A = \int_0^{\frac{\pi}{5}} \frac{1}{2}(16\sin^2(5\theta))\ d\theta$$

$$A = 8\int_0^{\frac{\pi}{5}} \sin^2(5\theta)\ d\theta$$

Use the trig identity

$$\sin^2\theta = \frac{1}{2}(1 - \cos(2\theta))$$

$$\sin^2(5\theta) = \frac{1}{2}(1 - \cos(10\theta))$$

to substitute.

$$A = 8\int_0^{\frac{\pi}{5}} \frac{1}{2}(1 - \cos(10\theta))\ d\theta$$

$$A = 4\int_0^{\frac{\pi}{5}} 1 - \cos(10\theta)\ d\theta$$

Integrate, then evaluate over the interval.

$$A = 4\left(\theta - \frac{1}{10}\sin(10\theta)\right)\Bigg|_0^{\frac{\pi}{5}}$$

$$A = 4\left(\frac{\pi}{5} - \frac{1}{10}\sin\left(10 \cdot \frac{\pi}{5}\right)\right) - 4\left(0 - \frac{1}{10}\sin(10 \cdot 0)\right)$$

$$A = 4\left(\frac{\pi}{5} - \frac{1}{10}\sin(2\pi)\right)$$

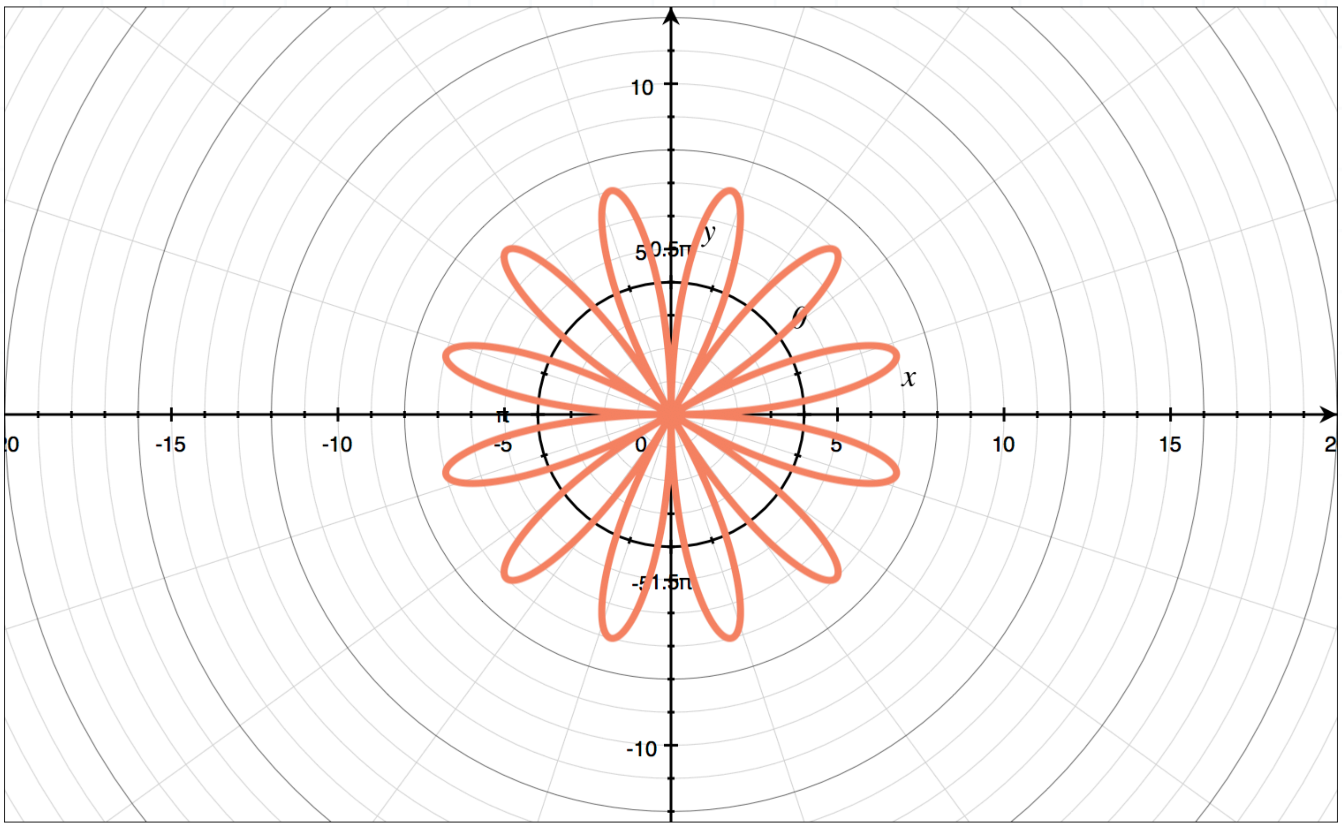
$$A = \frac{4\pi}{5}$$

3. Find the area of one loop of the polar curve.

$$r = 7\sin(6\theta)$$

Solution:

A sketch of the polar curve is



Setting $6\theta = \pi/2$ gives $\theta = \pi/12$.

At $\theta = 0$, $r = 7\sin(6(0)) = 0$

At $\theta = \pi/12$, $r = 7\sin(6(\pi/12)) = 7$

At $\theta = \pi/6$, $r = 7\sin(6(\pi/6)) = 0$

So $\theta = 0$ and $\theta = \pi/6$ define the first loop in the first quadrant.

$$A=\int_a^b \frac{1}{2}r^2\ d\theta$$

$$A=\frac{1}{2}\int_0^{\frac{\pi}{6}} (7\sin(6\theta))^2\ d\theta$$

$$A=\frac{1}{2}\int_0^{\frac{\pi}{6}} 49\sin^2(6\theta)\ d\theta$$

$$A=\frac{49}{2}\int_0^{\frac{\pi}{6}} \sin^2(6\theta)\ d\theta$$

Use the trig identity

$$\sin^2\theta=\frac{1}{2}(1-\cos(2\theta))$$

$$\sin^2(6\theta)=\frac{1}{2}(1-\cos(12\theta))$$

to substitute.

$$A=\frac{49}{2}\int_0^{\frac{\pi}{6}} \frac{1}{2}(1-\cos(12\theta))\ d\theta$$

$$A=\frac{49}{4}\int_0^{\frac{\pi}{6}} 1-\cos(12\theta)\ d\theta$$

Integrate, then evaluate over the interval.

$$A=\frac{49}{4}\left(\theta-\frac{1}{12}\sin(12\theta)\right)\Bigg|_0^{\frac{\pi}{6}}$$

$$A = \frac{49}{4}\left(\frac{\pi}{6} - \frac{1}{12}\sin\left(12 \cdot \frac{\pi}{6}\right)\right) - \frac{49}{4}\left(0 - \frac{1}{12}\sin(12 \cdot 0)\right)$$

$$A = \frac{49}{4}\left(\frac{\pi}{6} - \frac{1}{12}\sin(2\pi)\right)$$

$$A = \frac{49}{4}\left(\frac{\pi}{6}\right)$$

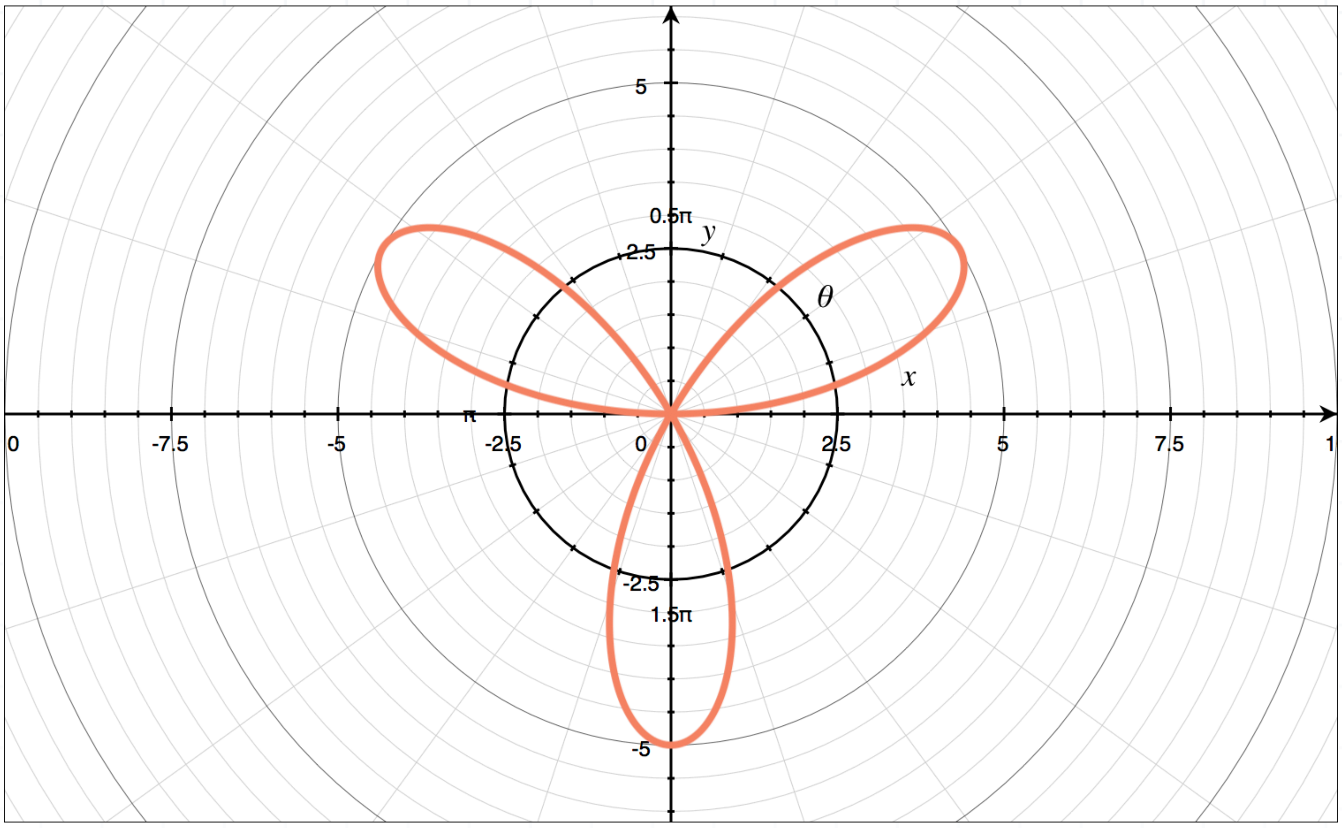
$$A = \frac{49\pi}{24}$$

■ 4. Find the area of one loop of the polar curve.

$$r = 5\sin(3\theta)$$

Solution:

A sketch of the polar curve is



Setting $3\theta = \pi/2$ gives $\theta = \pi/6$.

At $\theta = 0$, $r = 5\sin(3(0)) = 0$

At $\theta = \pi/6$, $r = 5\sin(3(\pi/6)) = 5$

At $\theta = \pi/3$, $r = 5\sin(3(\pi/3)) = 0$

So $\theta = 0$ and $\theta = \pi/3$ define the first loop in the first quadrant.

$$A = \int_a^b \frac{1}{2} r^2 \ d\theta$$

$$A = \frac{1}{2}\int_0^{\frac{\pi}{3}} (5\sin(3\theta))^2 \ d\theta$$

$$A = \frac{25}{2}\int_0^{\frac{\pi}{3}} \sin^2(3\theta) \ d\theta$$

Use the trig identity

$$\sin^2\theta = \frac{1}{2}(1 - \cos(2\theta))$$

$$\sin^2(3\theta) = \frac{1}{2}(1 - \cos(6\theta))$$

to substitute.

$$A = \frac{25}{2}\int_0^{\frac{\pi}{3}} \frac{1}{2}(1 - \cos(6\theta))\ d\theta$$

$$A = \frac{25}{4}\int_0^{\frac{\pi}{3}} 1 - \cos(6\theta)\ d\theta$$

Integrate, then evaluate over the interval.

$$A = \frac{25}{4}\left(\theta - \frac{1}{6}\sin(6\theta)\right)\Big|_0^{\frac{\pi}{3}}$$

$$A = \frac{25}{4}\left(\frac{\pi}{3} - \frac{1}{6}\sin\left(6 \cdot \frac{\pi}{3}\right)\right) - \frac{25}{4}\left(0 - \frac{1}{6}\sin(6 \cdot 0)\right)$$

$$A = \frac{25}{4}\left(\frac{\pi}{3} - \frac{1}{6}\sin(2\pi)\right)$$

$$A = \frac{25}{4}\left(\frac{\pi}{3}\right)$$

$$A = \frac{25\pi}{12}$$

AREA BETWEEN POLAR CURVES

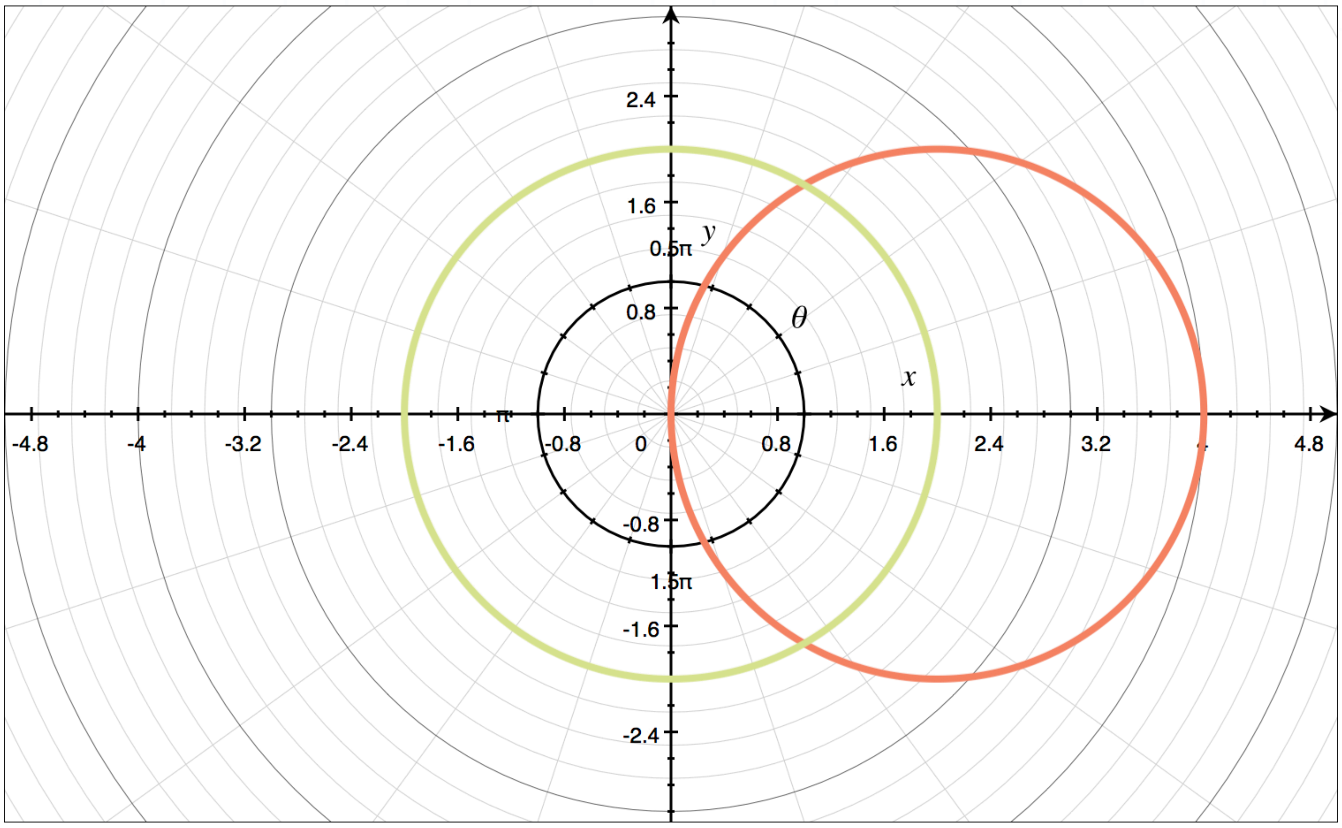
1. Find the area of the region that inside both polar curves.

$$r = 4\cos\theta$$

$$r = 2$$

Solution:

A sketch of the curves is



Find the intersection points of the curves to get the bounds of integration.

$$2 = 4\cos\theta$$

$$\frac{1}{2} = \cos\theta$$

$$\theta = \cos^{-1}\left(\frac{1}{2}\right)$$

$$\theta = \frac{\pi}{3} \text{ and } \theta = -\frac{\pi}{3}$$

Looking at the sketch, we can see that $r = 2$ is outside of $r = 4\cos\theta$ on $[-\pi/3,\pi/3]$.

$$A = \frac{1}{2}\int_a^b r^2_{\text{outside}} - r^2_{\text{inside}}\ d\theta$$

$$A = \frac{1}{2}\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} (4\cos\theta)^2 - (2)^2\ d\theta$$

$$A = \frac{1}{2}\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} 16\cos^2\theta - 4\ d\theta$$

$$A = 2\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} 4\cos^2\theta - 1\ d\theta$$

Use the trig identity

$$\cos^2\theta = \frac{1}{2}(1 + \cos(2\theta))$$

to substitute.

$$A = 2\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} 4 \cdot \frac{1}{2}(1 + \cos(2\theta)) - 1\ d\theta$$

$$A=2\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}}2+2\cos(2\theta)-1\ d\theta$$

$$A=2\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}}1+2\cos(2\theta)\ d\theta$$

Integrate, then evaluate over the interval.

$$A=2(\theta+\sin(2\theta))\Big|_{-\frac{\pi}{3}}^{\frac{\pi}{3}}$$

$$A=2\theta+2\sin(2\theta)\Big|_{-\frac{\pi}{3}}^{\frac{\pi}{3}}$$

$$A=2\cdot\frac{\pi}{3}+2\sin\left(2\cdot\frac{\pi}{3}\right)-\left(2\left(-\frac{\pi}{3}\right)+2\sin\left(2\left(-\frac{\pi}{3}\right)\right)\right)$$

$$A=\frac{2\pi}{3}+2\sin\frac{2\pi}{3}-\left(-\frac{2\pi}{3}+2\sin\left(-\frac{2\pi}{3}\right)\right)$$

$$A=\frac{2\pi}{3}+2\cdot\frac{\sqrt{3}}{2}+\frac{2\pi}{3}-2\left(-\frac{\sqrt{3}}{2}\right)$$

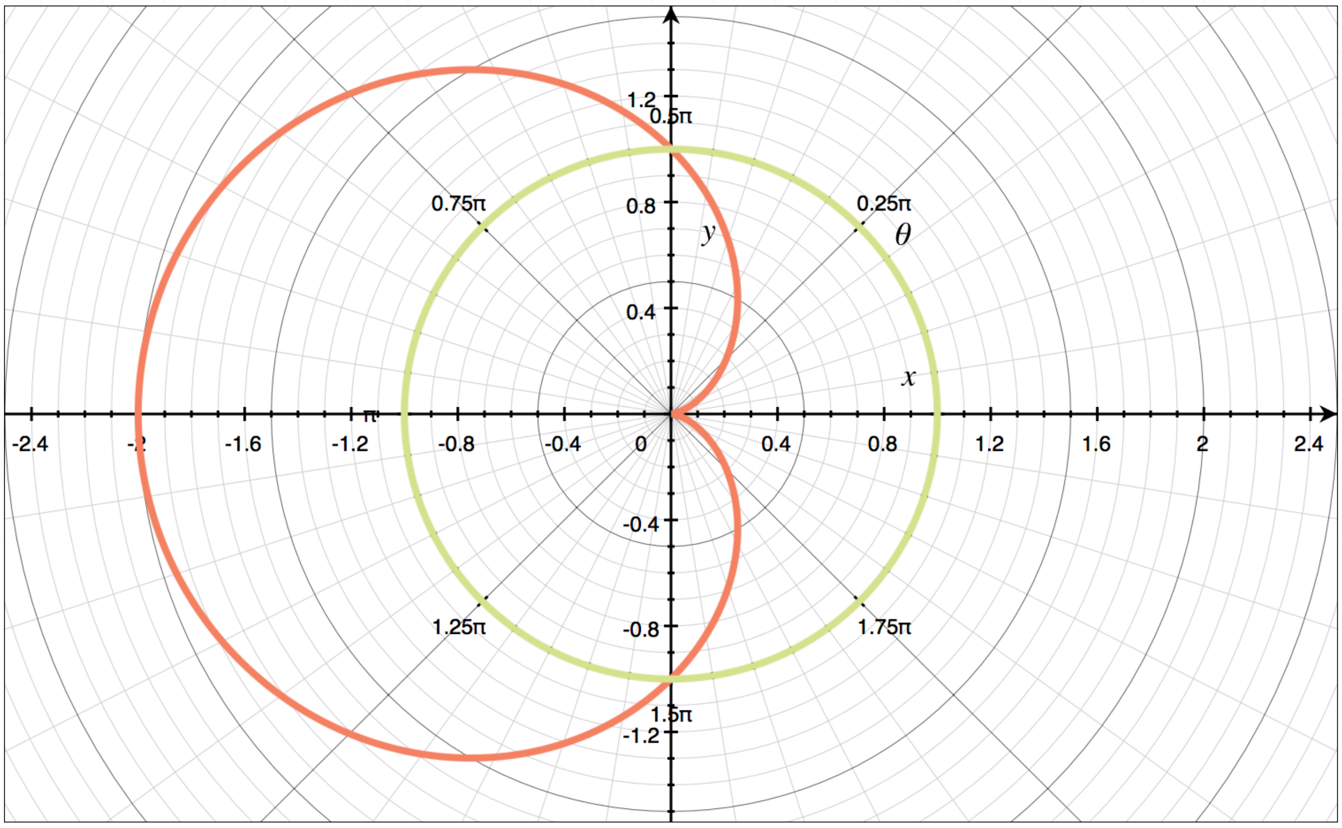
$$A=\frac{4\pi}{3}+\sqrt{3}+\sqrt{3}$$

$$A=\frac{4\pi}{3}+2\sqrt{3}$$

2. Find the area of the region inside $r = 1 - \cos\theta$ but outside $r = 1$.

Solution:

A sketch of the curves is



Find the intersection points of the curves to get the bounds of integration.

$$1 - \cos\theta = 1$$

$$-\cos\theta = 0$$

$$\cos\theta = 0$$

$$\theta = \cos^{-1}(0)$$

$$\theta = \frac{\pi}{2} \text{ and } \theta = -\frac{\pi}{2}$$

Looking at the sketch, we can see that $r = 1 - \cos\theta$ is outside of $r = 1$ on $[\pi/2, 3\pi/2]$.

$$A = \frac{1}{2}\int_a^b r_{\text{outside}}^2 - r_{\text{inside}}^2\ d\theta$$

$$A = \frac{1}{2}\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} (1 - \cos\theta)^2 - (1)^2\ d\theta$$

$$A = \frac{1}{2}\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} 1 - 2\cos\theta + \cos^2\theta - 1\ d\theta$$

$$A = \frac{1}{2}\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \cos^2\theta - 2\cos\theta\ d\theta$$

Use the trig identity

$$\cos^2\theta = \frac{1}{2}(1 + \cos(2\theta))$$

to substitute.

$$A = \frac{1}{2}\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \frac{1}{2}(1 + \cos(2\theta)) - 2\cos\theta\ d\theta$$

$$A = \frac{1}{2}\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \frac{1}{2} + \frac{1}{2}\cos(2\theta) - 2\cos\theta\ d\theta$$

Integrate, then evaluate over the interval.

$$A=\frac{1}{2}\left(\frac{1}{2}\theta+\frac{1}{4}\sin(2\theta)-2\sin\theta\right)\Bigg|_{\frac{\pi}{2}}^{\frac{3\pi}{2}}$$

$$A=\frac{1}{2}\left(\frac{1}{2}\cdot\frac{3\pi}{2}+\frac{1}{4}\sin\left(2\cdot\frac{3\pi}{2}\right)-2\sin\frac{3\pi}{2}\right)-\frac{1}{2}\left(\frac{1}{2}\cdot\frac{\pi}{2}+\frac{1}{4}\sin\left(2\cdot\frac{\pi}{2}\right)-2\sin\frac{\pi}{2}\right)$$

$$A=\frac{1}{2}\left(\frac{3\pi}{4}+\frac{1}{4}\sin 3\pi-2\sin\frac{3\pi}{2}\right)-\frac{1}{2}\left(\frac{\pi}{4}+\frac{1}{4}\sin\pi-2\sin\frac{\pi}{2}\right)$$

$$A=\frac{1}{2}\left(\frac{3\pi}{4}+\frac{1}{4}(0)-2(-1)\right)-\frac{1}{2}\left(\frac{\pi}{4}+\frac{1}{4}(0)-2(1)\right)$$

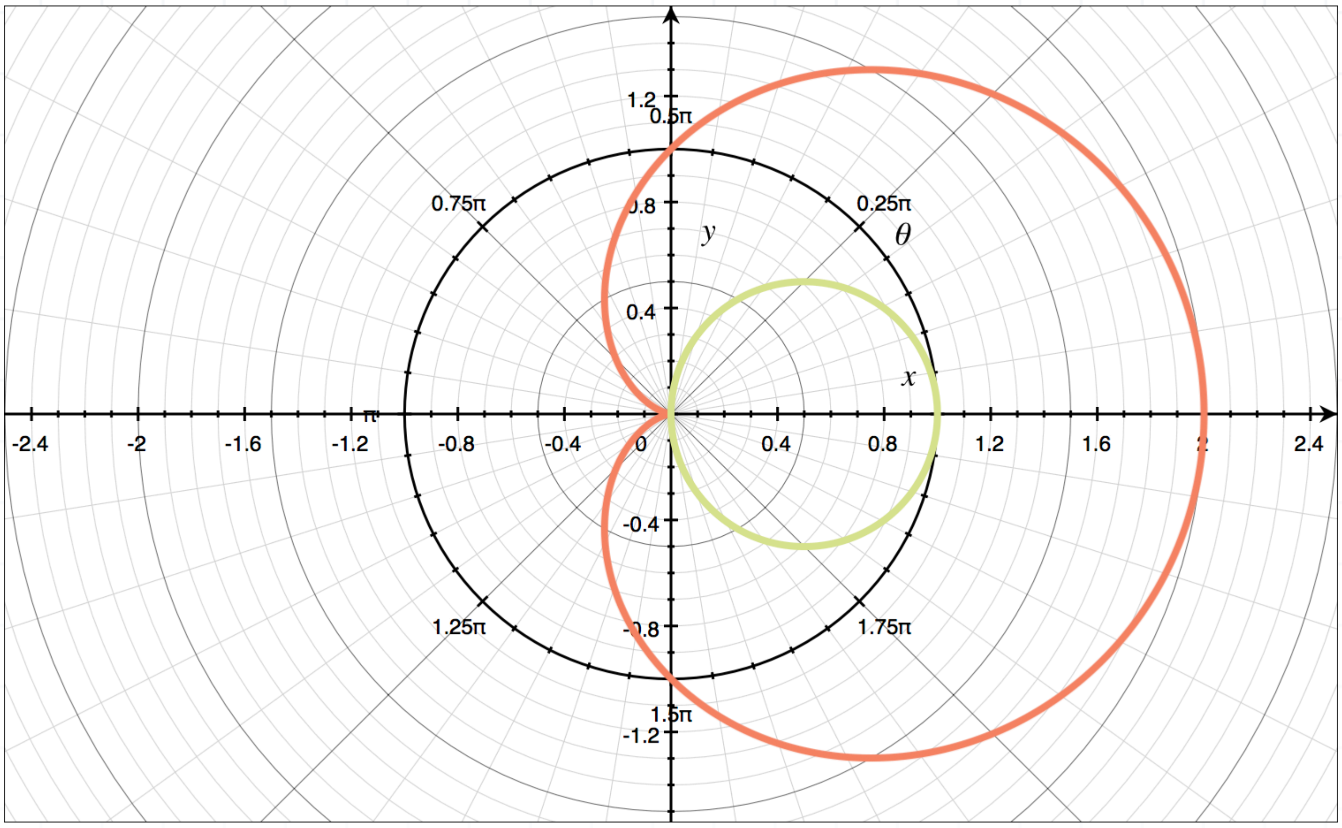
$$A=\frac{1}{2}\left(\frac{3\pi}{4}+2-\frac{\pi}{4}+2\right)$$

$$A=\frac{\pi}{4}+2$$

■ 3. Find the area of the region inside $r=1+\cos\theta$ but outside the circle $r=\cos\theta$.

Solution:

A sketch of the curves is



Since $r=1+\cos\theta$ is outside $r=\cos\theta$ everywhere, we can find the area between the curves by integrating the difference of the curves over $[0,2\pi]$.

$$A=\frac{1}{2}\int_0^{2\pi}(1+\cos\theta)^2-(\cos\theta)^2\ d\theta$$

$$A=\frac{1}{2}\int_0^{2\pi}(1+2\cos\theta+\cos^2\theta)-\cos^2\theta\ d\theta$$

$$A=\frac{1}{2}\int_0^{2\pi}1+2\cos\theta\ d\theta$$

Integrate, then evaluate over each interval.

$$A=\frac{1}{2}\left(\theta+2\sin\theta\right)\Big|_0^{2\pi}$$

$$A=\frac{1}{2}\theta+\sin\theta\Big|_0^{2\pi}$$

$$A = \frac{1}{2}(2\pi) + \sin(2\pi) - \left(\frac{1}{2}(0) + \sin(0)\right)$$

$$A = \pi + \sin(2\pi)$$

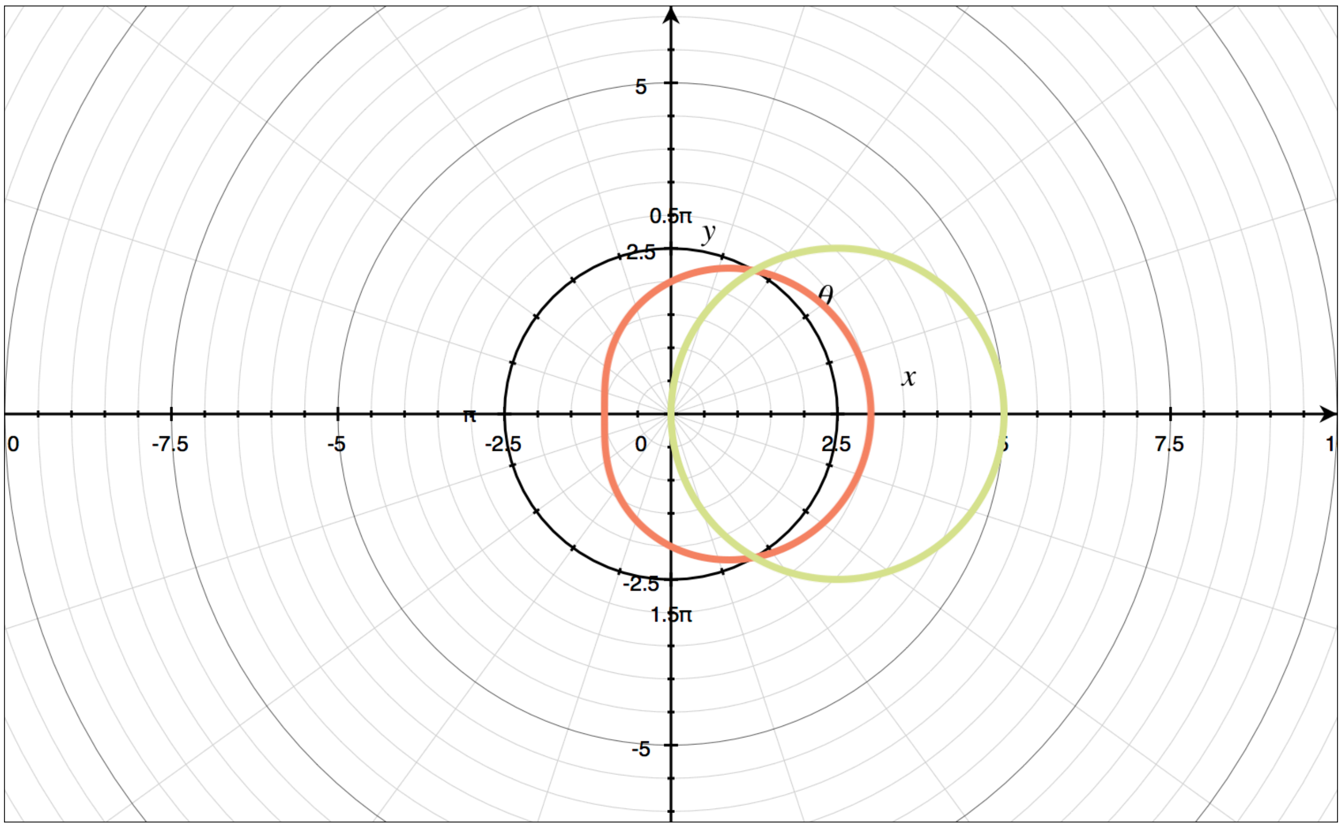
$$A = \pi + 0$$

$$A = \pi$$

4. Find the area of the region inside $r = 2 + \cos\theta$ but outside the circle $r = 5\cos\theta$.

Solution:

A sketch of the curves is



Find the intersection points of the curves to get the bounds of integration.

$$5\cos\theta = 2 + \cos\theta$$

$$4\cos\theta = 2$$

$$\cos\theta = \frac{1}{2}$$

$$\theta = \cos^{-1}\left(\frac{1}{2}\right)$$

$$\theta = \frac{\pi}{3} \text{ and } \frac{5\pi}{3}$$

Looking at the sketch, we can see that $r = 2 + \cos\theta$ is outside of $r = 5\cos\theta$ on $[\pi/3, 5\pi/3]$.

$$A = \frac{1}{2}\int_a^b r^2_{\text{outside}} - r^2_{\text{inside}}\ d\theta$$

$$A = \frac{1}{2}\int_{\frac{\pi}{3}}^{\frac{5\pi}{3}} (2 + \cos\theta)^2 - (5\cos\theta)^2\ d\theta$$

$$A = \frac{1}{2}\int_{\frac{\pi}{3}}^{\frac{5\pi}{3}} 4 + 4\cos\theta + \cos^2\theta - 25\cos^2\theta\ d\theta$$

$$A = \frac{1}{2}\int_{\frac{\pi}{3}}^{\frac{5\pi}{3}} 4 + 4\cos\theta - 24\cos^2\theta\ d\theta$$

$$A = 2\int_{\frac{\pi}{3}}^{\frac{5\pi}{3}} 1 + \cos\theta - 6\cos^2\theta\ d\theta$$

Use the trig identity

$$\cos^2\theta = \frac{1}{2}(1+\cos(2\theta))$$

to substitute.

$$A = 2\int_{\frac{\pi}{3}}^{\frac{5\pi}{3}} 1+\cos\theta - 6\cdot\frac{1}{2}(1+\cos(2\theta))\ d\theta$$

$$A = 2\int_{\frac{\pi}{3}}^{\frac{5\pi}{3}} 1+\cos\theta - 3 - 3\cos(2\theta)\ d\theta$$

$$A = 2\int_{\frac{\pi}{3}}^{\frac{5\pi}{3}} \cos\theta - 3\cos(2\theta) - 2\ d\theta$$

Integrate, then evaluate over the interval.

$$A = 2\left(\sin\theta - \frac{3}{2}\sin(2\theta) - 2\theta\right)\Bigg|_{\frac{\pi}{3}}^{\frac{5\pi}{3}}$$

$$A = 2\sin\theta - 3\sin(2\theta) - 4\theta\Bigg|_{\frac{\pi}{3}}^{\frac{5\pi}{3}}$$

$$A = 2\sin\frac{5\pi}{3} - 3\sin\left(2\cdot\frac{5\pi}{3}\right) - 4\cdot\frac{5\pi}{3} - \left(2\sin\frac{\pi}{3} - 3\sin\left(2\cdot\frac{\pi}{3}\right) - 4\cdot\frac{\pi}{3}\right)$$

$$A = -2\frac{\sqrt{3}}{2} - 3\sin\frac{10\pi}{3} - \frac{20\pi}{3} - 2\frac{\sqrt{3}}{2} + 3\sin\frac{2\pi}{3} + \frac{4\pi}{3}$$

$$A = -\sqrt{3} + 3\frac{\sqrt{3}}{2} - \frac{16\pi}{3} - \sqrt{3} + 3\frac{\sqrt{3}}{2}$$

$$A = -2\sqrt{3} + 3\sqrt{3} - \frac{16\pi}{3}$$

$$A = \sqrt{3} - \frac{16\pi}{3}$$

AREA INSIDE BOTH POLAR CURVES

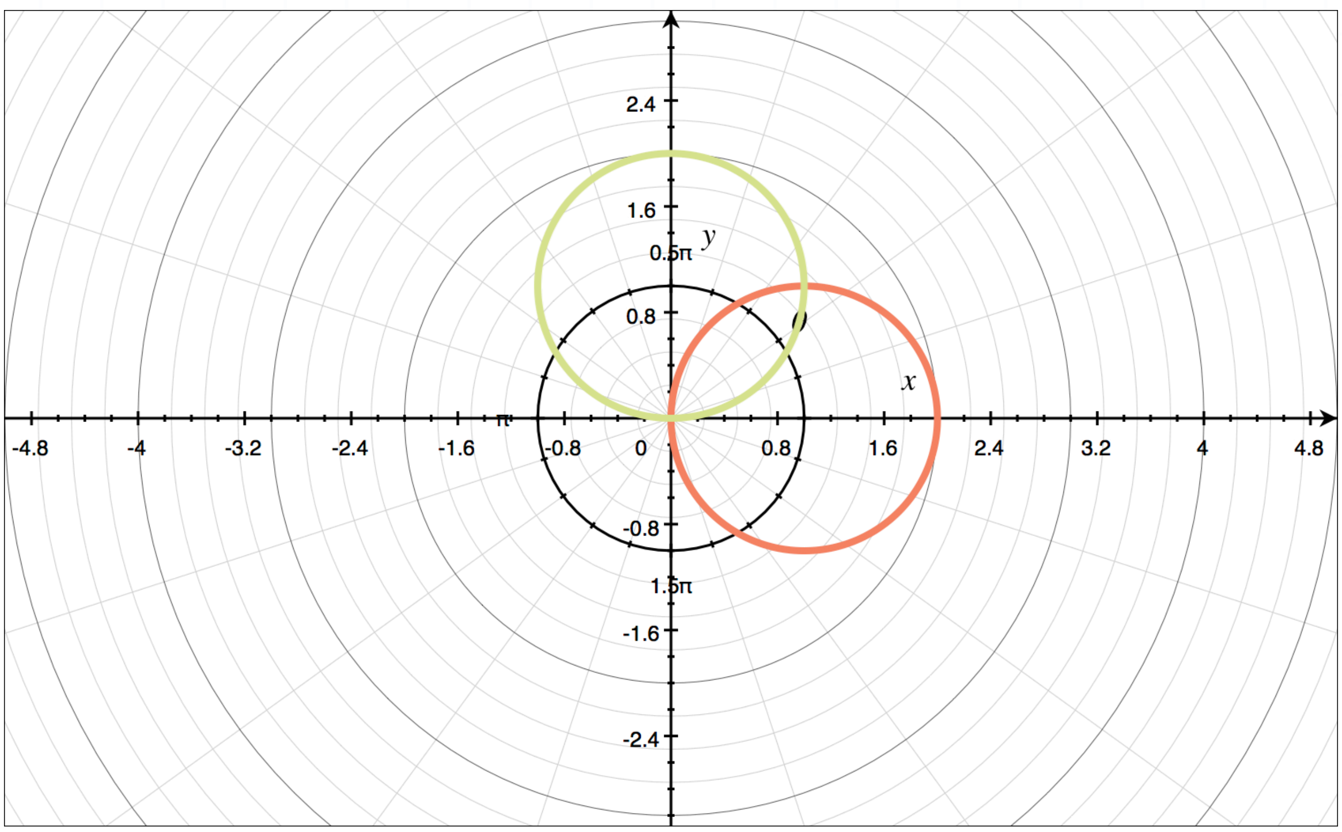
1. Find the area of the region that's inside both polar curves.

$$r = 2\cos\theta$$

$$r = 2\sin\theta$$

Solution:

A sketch of the curves is



Find points of intersection by setting the curves equal to each other.

$$2\cos\theta = 2\sin\theta$$

$$\cos\theta = \sin\theta$$

$$\theta = \frac{\pi}{4}$$

Integrating the $r = 2\sin\theta$ curve on $[0,\pi/4]$ will give the lower half of the area inside both curves, so we'll integrate the sine curve on that interval, and then double the result to get the total area inside both curves.

$$A = 2\left(\frac{1}{2}\int_0^{\frac{\pi}{4}} (2\sin\theta)^2\ d\theta\right)$$

$$A = \int_0^{\frac{\pi}{4}} 4\sin^2\theta\ d\theta$$

Use a double-angle identity to rewrite the integral.

$$A = \int_0^{\frac{\pi}{4}} 4\left(\frac{1}{2}(1-\cos(2\theta))\right)\ d\theta$$

$$A = \int_0^{\frac{\pi}{4}} 2 - 2\cos(2\theta)\ d\theta$$

Integrate, then evaluate over the interval.

$$A = 2\theta - \sin(2\theta)\Big|_0^{\frac{\pi}{4}}$$

$$A = 2\left(\frac{\pi}{4}\right) - \sin\left(2\cdot\frac{\pi}{4}\right) - (2(0) - \sin(2(0)))$$

$$A = \frac{\pi}{2} - 1$$

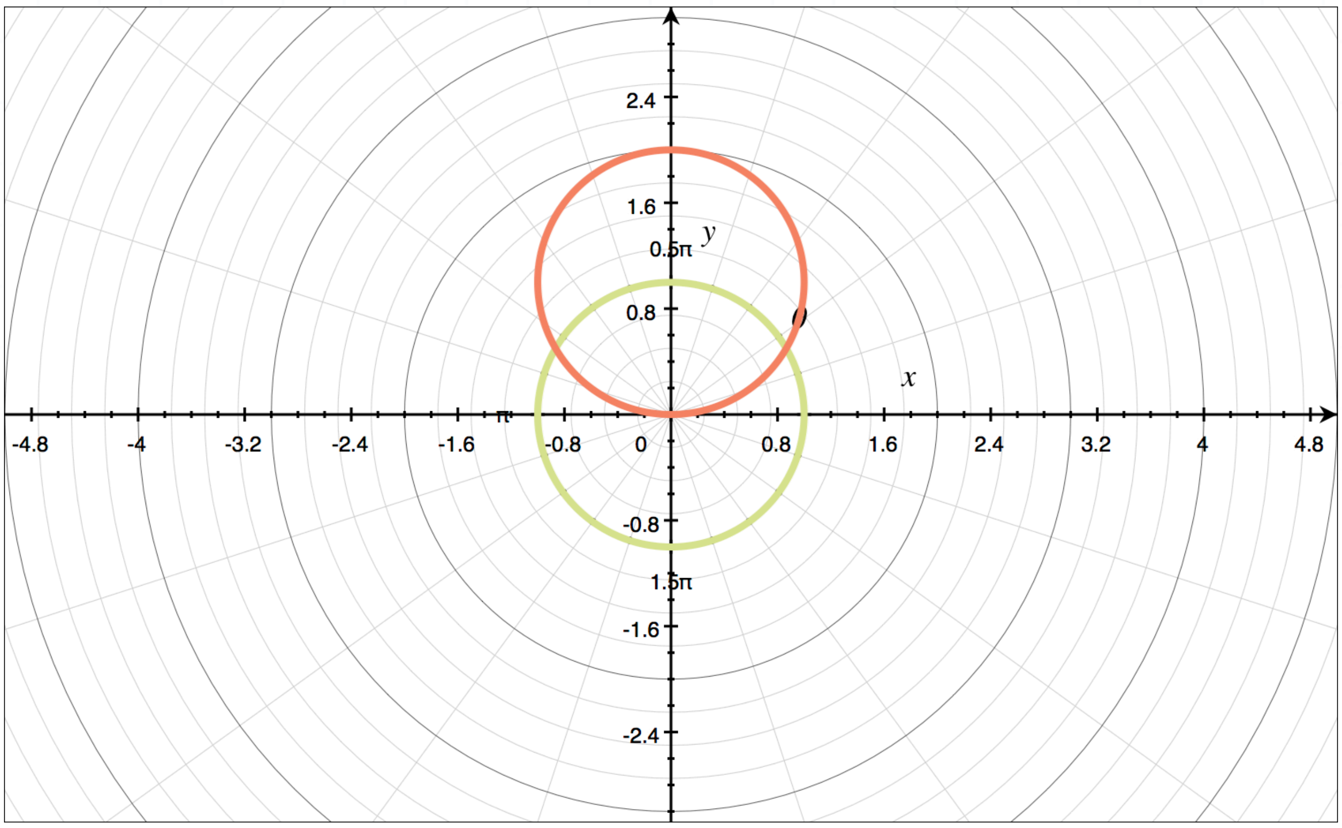
■ 2. Find the area of the region that's inside both polar curves.

$$r = 2\sin\theta$$

$$r = 1$$

Solution:

A sketch of the curves is



Find points of intersection by setting the curves equal to each other.

$$2\sin\theta = 1$$

$$\theta = \sin^{-1}\left(\frac{1}{2}\right)$$

$$\theta = \frac{\pi}{6}, \frac{5\pi}{6}$$

The overlapping area is given by the area inside $r=1$ between the points of intersection, plus the two slivers of area inside $r=2\sin\theta$ on $[0,\pi/6]$ and $[5\pi/6,\pi]$. Since the two slivers contain equal area, we can just double the area given on the interval $[0,\pi/6]$.

$$A=\frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}}(1)^2\ d\theta+2\left(\frac{1}{2}\int_0^{\frac{\pi}{6}}(2\sin\theta)^2\ d\theta\right)$$

$$A=\frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}}\ d\theta+\int_0^{\frac{\pi}{6}}4\sin^2\theta\ d\theta$$

Use a double-angle identity to rewrite the integral.

$$A=\frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}}\ d\theta+\int_0^{\frac{\pi}{6}}4\left(\frac{1}{2}(1-\cos(2\theta))\right)\ d\theta$$

$$A=\frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}}\ d\theta+2\int_0^{\frac{\pi}{6}}1-\cos(2\theta)\ d\theta$$

Integrate, then evaluate over the interval.

$$A=\frac{1}{2}\theta\Big|_{\frac{\pi}{6}}^{\frac{5\pi}{6}}+2\left(\theta-\frac{1}{2}\sin(2\theta)\right)\Big|_0^{\frac{\pi}{6}}$$

$$A=\frac{1}{2}\left(\frac{5\pi}{6}\right)-\frac{1}{2}\left(\frac{\pi}{6}\right)+2\left(\frac{\pi}{6}\right)-\sin\left(2\cdot\frac{\pi}{6}\right)-2(0)-\sin(2\cdot 0)$$

$$A=\frac{1}{2}\left(\frac{5\pi}{6}-\frac{\pi}{6}\right)+\frac{\pi}{3}-\sin\left(\frac{\pi}{3}\right)-\sin(0)$$

$$A = \frac{\pi}{3} + \frac{\pi}{3} - \frac{\sqrt{3}}{2}$$

$$A = \frac{2\pi}{3} - \frac{\sqrt{3}}{2}$$

$$A = \frac{4\pi - 3\sqrt{3}}{6}$$

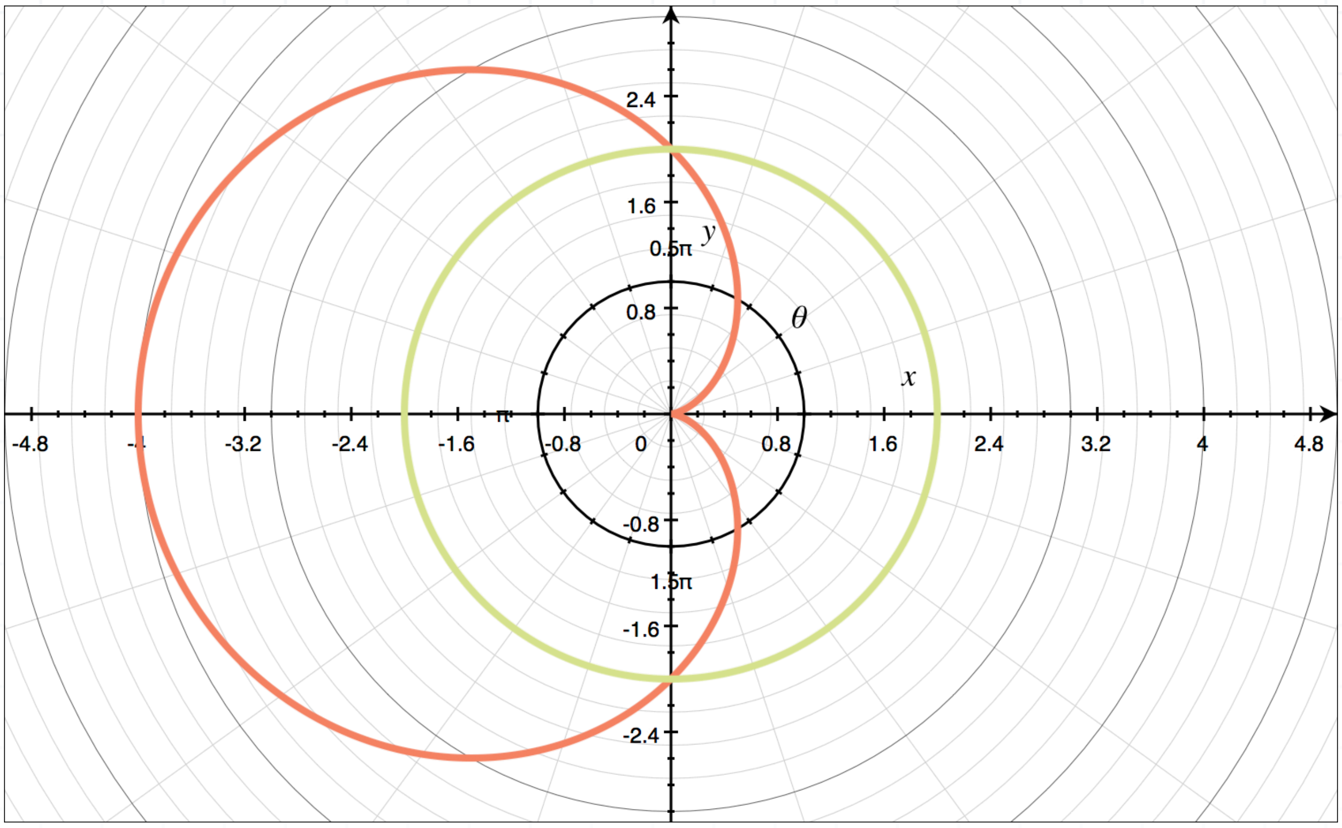
■ 3. Find the area of the region that's inside both polar curves.

$$r = 2(1 - \cos\theta)$$

$$r = 2$$

Solution:

A sketch of the curves is



Find points of intersection by setting the curves equal to each other.

$$2-2\cos\theta=2$$

$$-2\cos\theta=0$$

$$\cos\theta=0$$

$$\theta=\frac{\pi}{2},\frac{3\pi}{2}$$

The overlapping area is given by the area inside $r=2$ between the points of intersection, plus the two slivers of area inside $r=2(1-\cos\theta)$ on $[0,\pi/2]$ and $[3\pi/2,2\pi]$. Since the two slivers contain equal area, we can just double the area given on the interval $[0,\pi/2]$.

$$A=\frac{1}{2}\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}}(2)^2\ d\theta+2\left(\frac{1}{2}\int_{0}^{\frac{\pi}{2}}(2-2\cos\theta)^2\ d\theta\right)$$

$$A = \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} 2\ d\theta + \int_{0}^{\frac{\pi}{2}} 4 - 8\cos\theta + 4\cos^2\theta\ d\theta$$

Use a double-angle identity to rewrite the integral.

$$A = \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} 2\ d\theta + \int_{0}^{\frac{\pi}{2}} 4 - 8\cos\theta + 4\left(\frac{1}{2}(1+\cos(2\theta))\right)\ d\theta$$

$$A = \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} 2\ d\theta + \int_{0}^{\frac{\pi}{2}} 4 - 8\cos\theta + 2 + 2\cos(2\theta)\ d\theta$$

$$A = \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} 2\ d\theta + \int_{0}^{\frac{\pi}{2}} 6 - 8\cos\theta + 2\cos(2\theta)\ d\theta$$

Integrate, then evaluate over the interval.

$$A = 2\theta\Big|_{\frac{\pi}{2}}^{\frac{3\pi}{2}} + 6\theta - 8\sin\theta + \sin(2\theta)\Big|_{0}^{\frac{\pi}{2}}$$

$$A = 2\left(\frac{3\pi}{2}\right) - 2\left(\frac{\pi}{2}\right) + 6\left(\frac{\pi}{2}\right) - 8\sin\left(\frac{\pi}{2}\right) + \sin\left(2\cdot\frac{\pi}{2}\right)$$

$$-(6(0) - 8\sin(0) + \sin(2(0)))$$

$$A = 3\pi - \pi + 3\pi - 8$$

$$A = 5\pi - 8$$

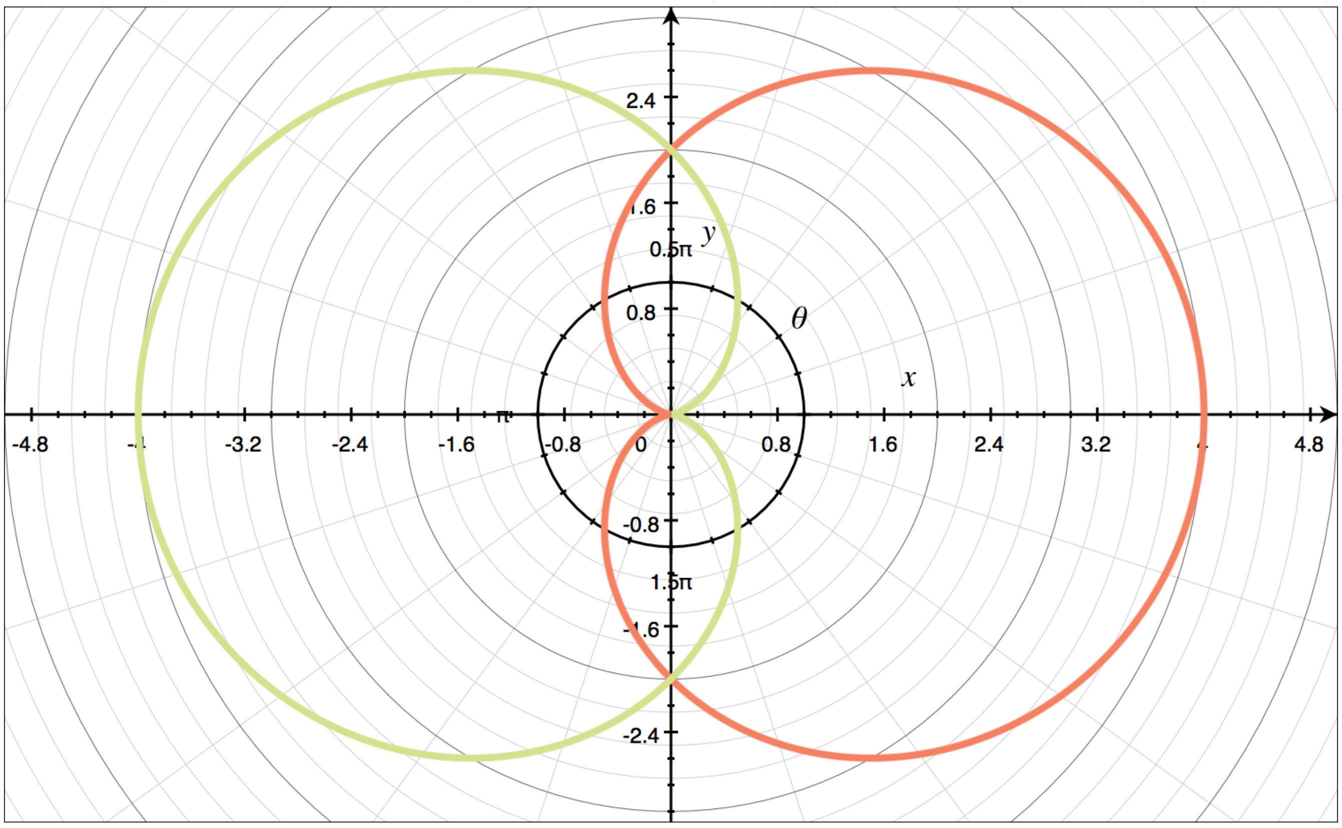
■ 4. Find the area of the region that's inside both polar curves.

$$r = 2(1 + \cos\theta)$$

$$r = 2(1 - \cos\theta)$$

Solution:

A sketch of the curves is



Find points of intersection by setting the curves equal to each other.

$$2(1 + \cos\theta) = 2(1 - \cos\theta)$$

$$2 + 2\cos\theta = 2 - 2\cos\theta$$

$$\cos\theta = -\cos\theta$$

$$\theta = \frac{\pi}{2}, \frac{3\pi}{2}$$

The overlapping area is given by the area inside $r = 2(1 - \cos\theta)$ on $[0,\pi/2]$, as long as we multiply that area by 4, which will give us the total area inside both curves.

$$A = 4\left(\frac{1}{2}\int_0^{\frac{\pi}{2}} (2(1-\cos\theta))^2\ d\theta\right)$$

$$A = 2\int_0^{\frac{\pi}{2}} 4(1-\cos\theta)^2\ d\theta$$

$$A = 8\int_0^{\frac{\pi}{2}} 1 - 2\cos\theta + \cos^2\theta\ d\theta$$

Use a double-angle identity to rewrite the integral.

$$A = 8\int_0^{\frac{\pi}{2}} 1 - 2\cos\theta + \frac{1}{2}(1+\cos(2\theta))\ d\theta$$

$$A = 8\int_0^{\frac{\pi}{2}} \frac{3}{2} - 2\cos\theta + \frac{1}{2}\cos(2\theta)\ d\theta$$

Integrate, then evaluate over the interval.

$$A = 8\left(\frac{3}{2}\theta - 2\sin\theta + \frac{1}{4}\sin(2\theta)\right)\Bigg|_0^{\frac{\pi}{2}}$$

$$A = 8\left(\frac{3}{2}\left(\frac{\pi}{2}\right) - 2\sin\left(\frac{\pi}{2}\right) + \frac{1}{4}\sin\left(2\cdot\frac{\pi}{2}\right)\right)$$

$$-8\left(\frac{3}{2}(0) - 2\sin(0) + \frac{1}{4}\sin(2(0))\right)$$

$$A = 8\left(\frac{3\pi}{4} - 2\right)$$

$$A = 6\pi - 16$$

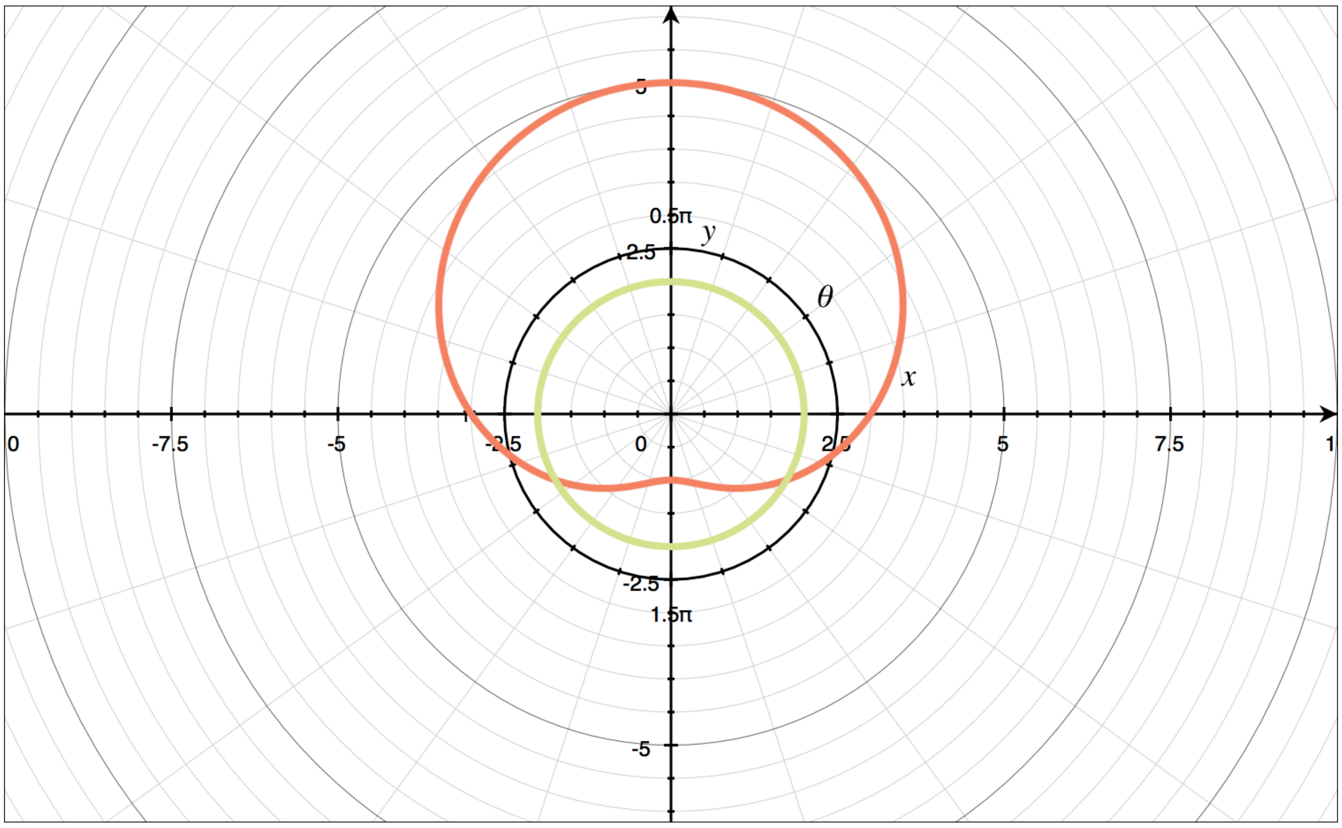
■ 5. Find the area of the region that's inside both polar curves.

$$r = 3 + 2\sin\theta$$

$$r = 2$$

Solution:

A sketch of the curves is



Find points of intersection by setting the curves equal to each other.

$$3+2\sin\theta=2$$

$$2\sin\theta=-1$$

$$\theta=\frac{7\pi}{6},\ \frac{11\pi}{6}$$

The overlapping area is given by the area inside $r=2$ on $[0,2\pi]$ (the area of the full circle), minus the area between $r=2$ and $r=3+2\sin\theta$ on the interval $[7\pi/6,11\pi/6]$.

$$A=\frac{1}{2}\int_0^{2\pi}(2)^2\ d\theta-\frac{1}{2}\int_{\frac{7\pi}{6}}^{\frac{11\pi}{6}}2^2-(3+2\sin\theta)^2\ d\theta$$

$$A=2\int_0^{2\pi}\ d\theta-\frac{1}{2}\int_{\frac{7\pi}{6}}^{\frac{11\pi}{6}}4-(9+12\sin\theta+4\sin^2\theta)\ d\theta$$

$$A=2\int_0^{2\pi}\ d\theta+\frac{1}{2}\int_{\frac{7\pi}{6}}^{\frac{11\pi}{6}}5+12\sin\theta+4\sin^2\theta\ d\theta$$

Use a double-angle identity to rewrite the integral.

$$A=2\int_0^{2\pi}\ d\theta+\frac{1}{2}\int_{\frac{7\pi}{6}}^{\frac{11\pi}{6}}5+12\sin\theta+4\left(\frac{1}{2}(1-\cos(2\theta))\right)\ d\theta$$

$$A=2\int_0^{2\pi}\ d\theta+\frac{1}{2}\int_{\frac{7\pi}{6}}^{\frac{11\pi}{6}}5+12\sin\theta+2-2\cos(2\theta)\ d\theta$$

$$A=2\int_0^{2\pi}\ d\theta+\frac{1}{2}\int_{\frac{7\pi}{6}}^{\frac{11\pi}{6}}7+12\sin\theta-2\cos(2\theta)\ d\theta$$

Integrate, then evaluate over the interval.

$$A=2\theta\Big|_0^{2\pi}+\frac{1}{2}\left(7\theta-12\cos\theta-\sin(2\theta)\right)\Big|_{\frac{7\pi}{6}}^{\frac{11\pi}{6}}$$

$$A=2(2\pi)-2(0)+\frac{1}{2}\left(7\left(\frac{11\pi}{6}\right)-12\cos\left(\frac{11\pi}{6}\right)-\sin\left(2\cdot\frac{11\pi}{6}\right)\right)$$

$$-\frac{1}{2}\left(7\left(\frac{7\pi}{6}\right)-12\cos\left(\frac{7\pi}{6}\right)-\sin\left(2\cdot\frac{7\pi}{6}\right)\right)$$

$$A=4\pi+\frac{77\pi}{12}-6\cos\left(\frac{11\pi}{6}\right)-\frac{1}{2}\sin\left(\frac{22\pi}{6}\right)$$

$$-\frac{49\pi}{12}+6\cos\left(\frac{7\pi}{6}\right)+\frac{1}{2}\sin\left(\frac{7\pi}{3}\right)$$

$$A=4\pi+\frac{77\pi}{12}-6\left(\frac{\sqrt{3}}{2}\right)-\frac{1}{2}\left(-\frac{\sqrt{3}}{2}\right)-\frac{49\pi}{12}+6\left(-\frac{\sqrt{3}}{2}\right)+\frac{1}{2}\left(\frac{\sqrt{3}}{2}\right)$$

$$A=\frac{19\pi}{3}-\frac{11\sqrt{3}}{2}$$

SURFACE AREA OF REVOLUTION OF A POLAR CURVE

■ 1. Find the surface area generated by revolving the polar curve about the y-axis over the interval $0 \le \theta \le \pi$.

$$r = 2\cos\theta$$

Solution:

The derivative of $r = 2\cos\theta$ is

$$\frac{dr}{d\theta} = -2\sin\theta$$

So the surface area of revolution will be

$$S_y = \int_{\alpha}^{\beta} 2\pi x \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\ d\theta$$

$$S_y = \int_{0}^{\pi} 2\pi(2\cos\theta)\cos\theta\sqrt{(2\cos\theta)^2 + (-2\sin\theta)^2}\ d\theta$$

$$S_y = \int_{0}^{\pi} 4\pi\cos^2\theta\sqrt{4\cos^2\theta + 4\sin^2\theta}\ d\theta$$

$$S_y = 4\pi\int_{0}^{\pi} \cos^2\theta\sqrt{4(\cos^2\theta + \sin^2\theta)}\ d\theta$$

$$S_y = 4\pi \int_0^{\pi} \cos^2\theta \sqrt{4(1)}\ d\theta$$

$$S_y = 4\pi \int_0^{\pi} \cos^2\theta \sqrt{4}\ d\theta$$

$$S_y = 8\pi \int_0^{\pi} \cos^2\theta\ d\theta$$

Use the trig identity

$$\cos^2\theta = \frac{1}{2}(1 + \cos(2\theta))$$

to substitute.

$$S_y = 8\pi \int_0^{\pi} \frac{1}{2}(1 + \cos(2\theta))\ d\theta$$

$$S_y = 4\pi \int_0^{\pi} 1 + \cos(2\theta)\ d\theta$$

Integrate, then evaluate over the interval.

$$S_y = 4\pi \left(\theta + \frac{1}{2}\sin(2\theta)\right)\Bigg|_0^{\pi}$$

$$S_y = 4\pi\theta + 2\pi\sin(2\theta)\Bigg|_0^{\pi}$$

$$S_y = 4\pi^2 + 2\pi\sin(2\pi) - (4\pi(0) + 2\pi\sin(2(0)))$$

$$S_y = 4\pi^2 + 2\pi(0) - 4\pi(0) - 2\pi(0)$$

$$S_y = 4\pi^2$$

■ 2. Find the surface area generated by revolving the polar curve about the x-axis over the interval $0 \leq \theta \leq \pi/2$.

$$r = 4\cos\theta$$

Solution:

The derivative of $r = 4\cos\theta$ is

$$\frac{dr}{d\theta} = -4\sin\theta$$

So the surface area of revolution will be

$$S_x = \int_{\alpha}^{\beta} 2\pi y \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\ d\theta$$

$$S_x = \int_0^{\frac{\pi}{2}} 2\pi(4\cos\theta)\sin\theta\sqrt{(4\cos\theta)^2 + (-4\sin\theta)^2}\ d\theta$$

$$S_x = 8\pi\int_0^{\frac{\pi}{2}} \cos\theta\sin\theta\sqrt{16\cos^2\theta + 16\sin^2\theta}\ d\theta$$

$$S_x = 8\pi\int_0^{\frac{\pi}{2}} \cos\theta\sin\theta\sqrt{16(\cos^2\theta + \sin^2\theta)}\ d\theta$$

$$S_x = 8\pi \int_0^{\frac{\pi}{2}} \cos\theta \sin\theta \sqrt{16(1)}\ d\theta$$

$$S_x = 8\pi \int_0^{\frac{\pi}{2}} \cos\theta \sin\theta \sqrt{16}\ d\theta$$

$$S_x = 32\pi \int_0^{\frac{\pi}{2}} \cos\theta \sin\theta\ d\theta$$

Integrate, then evaluate over the interval.

$$S_x = 32\pi \left(-\frac{\cos^2\theta}{2} \right) \Bigg|_0^{\frac{\pi}{2}}$$

$$S_x = -16\pi(\cos^2\theta) \Bigg|_0^{\frac{\pi}{2}}$$

$$S_x = -16\pi \left(\cos^2 \frac{\pi}{2} \right) + 16\pi(\cos^2(0))$$

$$S_x = -16\pi(0^2) + 16\pi(1^2)$$

$$S_x = 16\pi$$

■ 3. Find the surface area generated by revolving the polar curve about the y-axis over the interval $0 \le \theta \le \pi/2$.

$$r = 8\sin\theta$$

Solution:

The derivative of $r = 8\sin\theta$ is

$$\frac{dr}{d\theta} = 8\cos\theta$$

So the surface area of revolution will be

$$S_y = \int_{\alpha}^{\beta} 2\pi x\sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\ d\theta$$

$$S_y = \int_0^{\frac{\pi}{2}} 2\pi(8\sin\theta)\cos\theta\sqrt{(8\sin\theta)^2 + (8\cos\theta)^2}\ d\theta$$

$$S_y = 16\pi\int_0^{\frac{\pi}{2}} \sin\theta\cos\theta\sqrt{64\sin^2\theta + 64\cos^2\theta}\ d\theta$$

$$S_y = 16\pi\int_0^{\frac{\pi}{2}} \sin\theta\cos\theta\sqrt{64(\sin^2\theta + \cos^2\theta)}\ d\theta$$

$$S_y = 16\pi\int_0^{\frac{\pi}{2}} \sin\theta\cos\theta\sqrt{64(1)}\ d\theta$$

$$S_y = 16\pi\int_0^{\frac{\pi}{2}} \sin\theta\cos\theta\sqrt{64}\ d\theta$$

$$S_y = 128\pi\int_0^{\frac{\pi}{2}} \sin\theta\cos\theta\ d\theta$$

Integrate, then evaluate over the interval.

$$S_y = 128\pi \left(\frac{\sin^2 \theta}{2} \right) \Bigg|_0^{\frac{\pi}{2}}$$

$$S_y = 64\pi \sin^2 \theta \Bigg|_0^{\frac{\pi}{2}}$$

$$S_y = 64\pi \sin^2 \frac{\pi}{2} - 64\pi \sin^2(0)$$

$$S_y = 64\pi(1)^2 - 64\pi(0)^2$$

$$S_y = 64\pi(1)$$

$$S_y = 64\pi$$

■ 4. Find the surface area generated by revolving the polar curve about the x-axis over the interval $0 \leq \theta \leq \pi$.

$$r = 7 \sin \theta$$

Solution:

The derivative of $r = 7 \sin \theta$ is

$$\frac{dr}{d\theta} = 7 \cos \theta$$

So the surface area of revolution will be

$$S_x = \int_{\alpha}^{\beta} 2\pi y \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\ d\theta$$

$$S_x = \int_{0}^{\pi} 2\pi(7\sin\theta)\sin\theta\sqrt{(7\sin\theta)^2 + (7\cos\theta)^2}\ d\theta$$

$$S_x = 14\pi\int_{0}^{\pi} \sin\theta\sin\theta\sqrt{49\sin^2\theta + 49\cos^2\theta}\ d\theta$$

$$S_x = 14\pi\int_{0}^{\pi} \sin\theta\sin\theta\sqrt{49(\sin^2\theta + \cos^2\theta)}\ d\theta$$

$$S_x = 14\pi\int_{0}^{\pi} \sin\theta\sin\theta\sqrt{49(1)}\ d\theta$$

$$S_x = 14\pi\int_{0}^{\pi} \sin\theta\sin\theta\sqrt{49}\ d\theta$$

$$S_x = 98\pi\int_{0}^{\pi} \sin^2\theta\ d\theta$$

Use the trig identities

$$\sin^2\theta = \frac{1}{2}(1 - \cos(2\theta))$$

to substitute.

$$S_x = 98\pi\int_{0}^{\pi} \frac{1}{2}(1 - \cos(2\theta))\ d\theta$$

$$S_x = 49\pi\int_{0}^{\pi} 1 - \cos(2\theta)\ d\theta$$

Integrate, then evaluate over the interval.

$$S_x = 49\pi \left(\theta - \frac{1}{2}\sin(2\theta) \right) \Big|_0^{\pi}$$

$$S_x = 49\pi^2 - \frac{49}{2}\pi \sin(2\pi) - \left(49\pi(0) - \frac{49}{2}\pi \sin(2(0)) \right)$$

$$S_x = 49\pi^2 - \frac{49}{2}\pi(0) - 49\pi(0) + \frac{49}{2}\pi(0)$$

$$S_x = 49\pi^2 - 49\pi(0)$$

$$S_x = 49\pi^2$$

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