Integral test

The integral test for convergence is only valid for series that are

Positive: all of the terms in the series are positive

Decreasing: every term is less than the one before it, $a_{n-1} > a_n$

Continuous: the series is defined everywhere in its domain

If the given series meets these three criteria, then we can use the integral test for convergence to integrate the series and say whether the series is converging or diverging.

Given the series

$$\sum_{n=1}^{\infty} a_n$$

we set $f(x) = a_n$ and evaluate the integral

$$\int_1^{\infty} f(x)\ dx$$

According to the integral test, the series and the integral always have the same result, meaning that they either both converge or they both diverge. This means that if the value of the of the integral

converges to a **real number**, then the series also **converges**

diverges to **infinity**, then the series also **diverges**

Example

Use the integral test to say whether or not the series converges.

$$\sum_{n=1}^{\infty} \frac{3}{n^2}$$

Before we apply the integral test, we need to confirm that the series is positive, decreasing, and continuous.

We'll find the first few terms of the series using $n = 1$, $n = 2$, $n = 3$ and $n = 4$.

$$n = 1 \qquad \frac{3}{(1)^2} \qquad 3$$

$$n = 2 \qquad \frac{3}{(2)^2} \qquad \frac{3}{4}$$

$$n = 3 \qquad \frac{3}{(3)^2} \qquad \frac{1}{3}$$

$$n = 4 \qquad \frac{3}{(4)^2} \qquad \frac{3}{16}$$

Looking at the first four terms, we can already see that our terms will always be positive. There's no positive value of n that will make its term negative, so we know that our series is positive.

We can also tell by looking at the first four terms that our series is decreasing. The larger the value of n, the larger the denominator becomes, and the smaller the terms become. Therefore, every term in the series will be smaller than the one before it, $a_{n-1} > a_n$. We can also prove it this way:

$$\frac{3}{(n-1)^2} > \frac{3}{n^2}$$

We can flip both fractions if we flip the inequality.

$$\frac{(n-1)^2}{3} < \frac{n^2}{3}$$

$$(n-1)^2 < n^2$$

$$(n-1)^2 - n^2 < 0$$

$$n^2 - 2n + 1 - n^2 < 0$$

$$-2n + 1 < 0$$

$$-2n < -1$$

$$n > \frac{1}{2}$$

For all values $n > 1/2$, the series is decreasing. Since this series starts at $n = 1$, that means the series is decreasing everywhere in its domain.

Finally, we need to confirm that our series is continuous. The series is defined from 1 to ∞, so in order for the series to be discontinuous, the denominator would have to be equal to 0. Since $n^2 \neq 0$ for $1 \leq n \leq \infty$, we know that the series is continuous.

Now that we know the series is positive, decreasing, and continuous, we can use the integral test to say whether the series converges or diverges.

Plugging the given series into the integral, we get

$$\int_1^\infty \frac{3}{x^2}\ dx$$

$$\int_1^\infty 3x^{-2}\ dx$$

$$\frac{3x^{-1}}{-1}\Bigg|_1^\infty$$

$$-3x^{-1}\Big|_1^\infty$$

$$-\frac{3}{x}\Bigg|_1^\infty$$

$$-\frac{3}{\infty}-\left(-\frac{3}{1}\right)$$

$$0+3$$

$$3$$

Since the integral converges to a real number, we know that series also converges.

Note: The value of the integral is not necessarily the value to which the series converges. Don't confuse the value you find for the integral with the limit or the sum of the corresponding series. They’re not necessarily the same.