Area inside a polar curve

The area inside a polar curve is given by

$$A = \int_{\alpha}^{\beta} \frac{1}{2} r^2 \ d\theta$$

where $[\alpha, \beta]$ is the interval

where r is the equation of the polar curve

Example

Find the area inside the polar curve.

$$r = 4 \sin \theta$$

We need to find the interval, which we'll do by setting $r = 0$ and solving for any values of θ.

$$r = 4 \sin \theta$$

$$0 = \sin \theta$$

$\theta = \pi$ and $\theta = 2\pi$

$\alpha = \pi$ and $\beta = 2\pi$

Now we can plug the interval we found and the given polar equation into the formula for the area inside a polar curve.

$$A = \int_{\pi}^{2\pi} \frac{1}{2} \left(4 \sin \theta\right)^2 \ d\theta$$

$$A = \int_{\pi}^{2\pi} \frac{1}{2} \cdot 16 \sin^2 \theta \ d\theta$$

$$A = 8 \int_{\pi}^{2\pi} \sin^2 \theta \ d\theta$$

Since $\sin^2 \theta = \frac{1}{2} \left[1 - \cos(2\theta)\right]$, we get

$$A = 8 \int_{\pi}^{2\pi} \frac{1}{2} \left[1 - \cos(2\theta)\right] \ d\theta$$

$$A = 4 \int_{\pi}^{2\pi} 1 - \cos(2\theta) \ d\theta$$

$$A = 4 \left(\theta - \frac{\sin(2\theta)}{2} \right) \Bigg|_{\pi}^{2\pi}$$

$$A = 4 \left[2\pi - \frac{\sin\left(2(2\pi)\right)}{2} - \left(\pi - \frac{\sin(2\pi)}{2} \right) \right]$$

$$A = 4 \left[2\pi - \frac{\sin(4\pi)}{2} - \pi + \frac{\sin(2\pi)}{2} \right]$$

$$A = 4 \left(2\pi - \frac{0}{2} - \pi + \frac{0}{2} \right)$$

$$A = 4 \left(2\pi - \pi\right)$$

$$A = 4\pi$$