Topic: Area inside a polar curve

Question: Find the area bounded by the polar curve on the given interval.

$$r = 6\theta$$

$$0 \le \theta \le \pi$$

Answer choices:

A $6\pi^3$

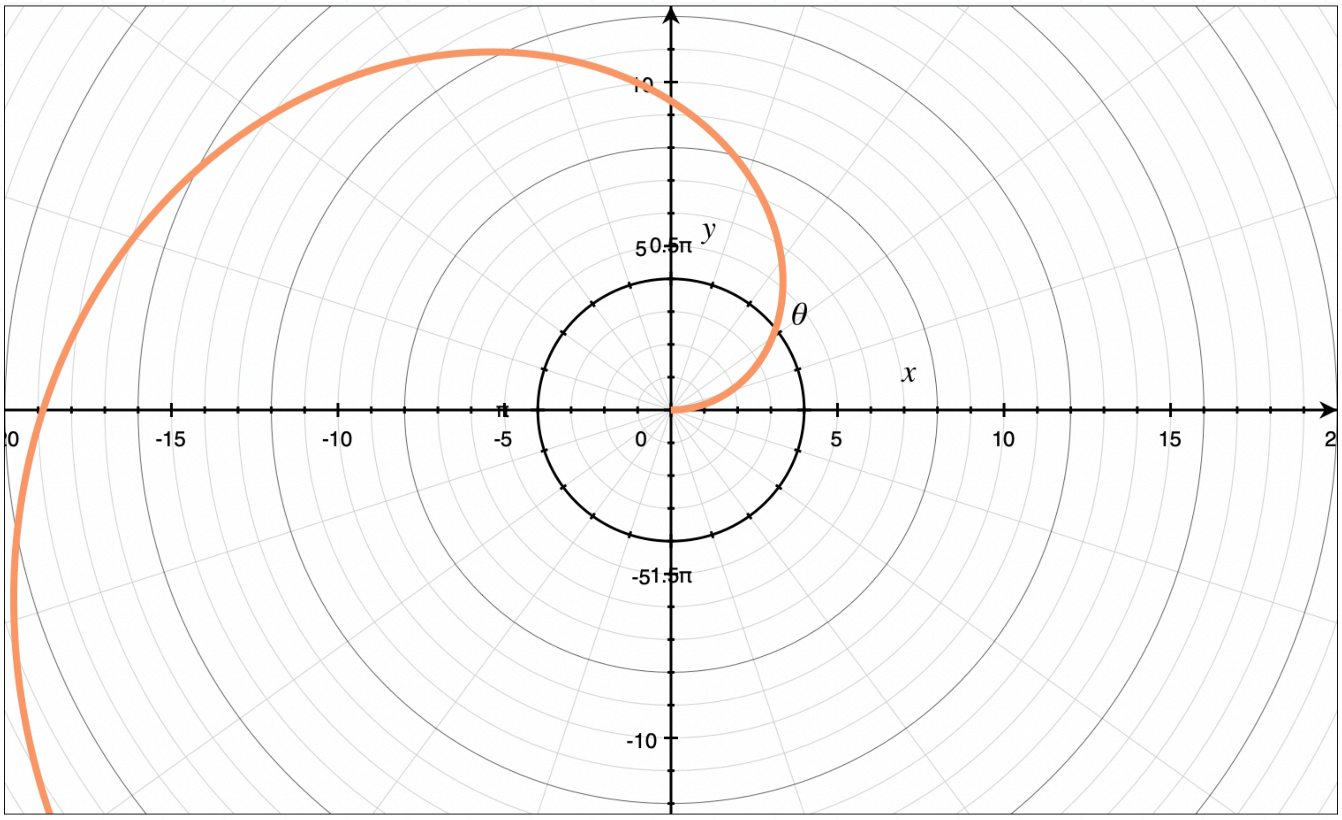
B $6\pi^2$

C $\frac{6}{5}\pi$

D $\frac{6}{5}\pi^3$

Solution: A

The graph of the polar curve looks like this:



Given the interval, the region in question is bounded by the spiral and the x-axis. The area formula is

$$A = \frac{1}{2}\int_{\alpha}^{\beta} r^2 \ d\theta$$

where $\alpha = 0$ and $\beta = \pi$. Therefore, the area bounded by the polar curve is

$$A = \frac{1}{2}\int_{0}^{\pi} (6\theta)^2 \ d\theta$$

$$A = 18\int_{0}^{\pi} \theta^2 \ d\theta$$

$$A = \frac{18}{3}\theta^3 \Big|_0^{\pi}$$

$$A = \frac{18}{3}\pi^3 - \frac{18}{3}(0)^3$$

$$A = 6\pi^3$$

Topic: Area inside a polar curve

Question: Find the area bounded by the polar curve on the given interval.

$$r = 5 - 5\sin\theta$$

Answer choices:

A 15π

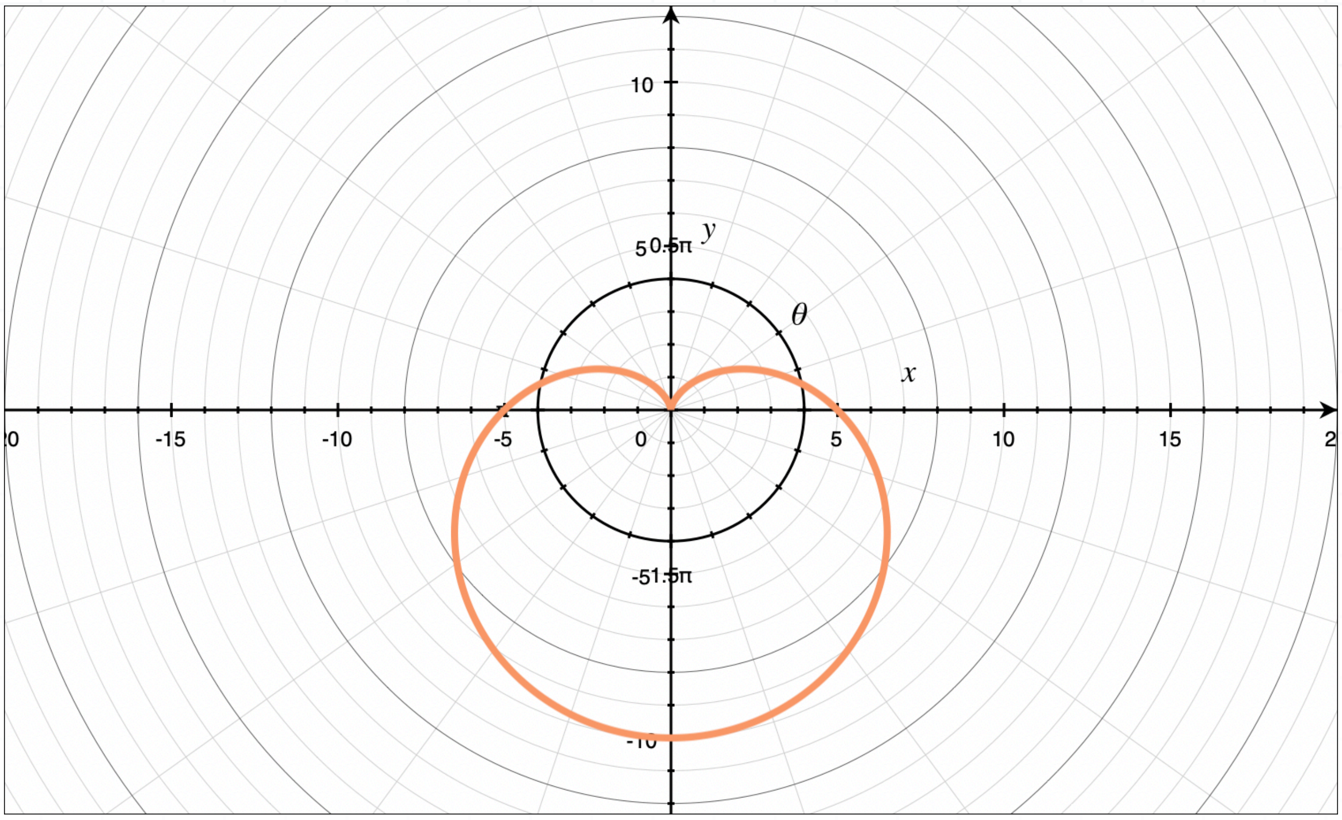
B $\dfrac{75\pi}{12}$

C $\dfrac{75\pi}{2}$

D $\dfrac{75}{2}$

Solution: C

The graph of the polar curve looks like this:



The graph of the polar curve is symmetric about the y-axis since $\sin\theta = \sin(\pi - \theta)$. Therefore, the area of the bounded region can be determined by doubling the integral from $\pi/2$ to $3\pi/2$. The area is given by

$$A = 2\left(\frac{1}{2}\int_{\alpha}^{\beta} r^2\ d\theta\right) = \int_{\alpha}^{\beta} r^2\ d\theta$$

where $\alpha = \pi/2$ and $\beta = 3\pi/2$.

$$A = \int_{\pi/2}^{3\pi/2} (5 - 5\sin\theta)^2\ d\theta$$

$$A = \int_{\pi/2}^{3\pi/2} 25 - 50\sin\theta + 25\sin^2\theta\ d\theta$$

$$A = \int_{\pi/2}^{3\pi/2} 25\left(1 - 2\sin\theta + \sin^2\theta\right)\ d\theta$$

$$A = 25\int_{\pi/2}^{3\pi/2} 1 - 2\sin\theta + \sin^2\theta\ d\theta$$

Using the power reduction formula

$$\sin^2\theta = \frac{1}{2} - \frac{1}{2}\cos 2\theta$$

we get

$$A = 25\int_{\pi/2}^{3\pi/2} 1 - 2\sin\theta + \frac{1}{2} - \frac{1}{2}\cos 2\theta\ d\theta$$

$$A = 25\int_{\pi/2}^{3\pi/2} \frac{3}{2} - 2\sin\theta - \frac{1}{2}\cos 2\theta\ d\theta$$

$$A = 25\left(\frac{3}{2}\theta + 2\cos\theta - \frac{1}{4}\sin 2\theta\right)\Bigg|_{\pi/2}^{3\pi/2}$$

$$A = 25\left[\left(\frac{3}{2}\left(\frac{3\pi}{2}\right) + 2\cos\left(\frac{3\pi}{2}\right) - \frac{1}{4}\sin 2\left(\frac{3\pi}{2}\right)\right) - \left(\frac{3}{2}\left(\frac{\pi}{2}\right) + 2\cos\left(\frac{\pi}{2}\right) - \frac{1}{4}\sin 2\left(\frac{\pi}{2}\right)\right)\right]$$

$$A = 25\left[\left(\frac{9\pi}{4} + 2(0) - \frac{1}{4}(0)\right) - \left(\frac{3\pi}{4} + 2(0) - \frac{1}{4}(0)\right)\right]$$

$$A = 25\left(\frac{9\pi}{4} - \frac{3\pi}{4}\right)$$

$$A = \frac{75\pi}{2}$$

Topic: Area inside a polar curve

Question: The x-axis is the line of symmetry for the cardioids $r = 2(1 + \cos\theta)$ and $r = 4(1 + \cos\theta)$. Assume that A_1 is the area of the first cardioid and A_2 is the area of the second cardioid. What is the ratio of A_1 to A_2?

Answer choices:

A $\frac{1}{2}$

B $\frac{1}{4}$

C 2

D 4

Solution: B

We'll first find the area A_1 of the cardioid $r = 2(1 + \cos\theta)$ by plugging into the formula for polar area. Because both cardioids are symmetric about the x-axis, we can integrate over the interval $[0,\pi]$ (representing the area above the x-axis), and then multiply the integral formula by 2 to get the full area.

$$A_1 = 2 \times \frac{1}{2}\int_0^{\pi} 4(1 + \cos\theta)^2 \ d\theta$$

$$A_1 = 4\int_0^{\pi} (1 + \cos\theta)^2 \ d\theta$$

$$A_1 = 4\int_0^{\pi} 1 + 2\cos\theta + \cos^2\theta \ d\theta$$

$$A_1 = 4\int_0^{\pi} 1 + 2\cos\theta + \frac{1}{2} + \frac{1}{2}\cos 2\theta \ d\theta$$

$$A_1 = 4\int_0^{\pi} \frac{3}{2} + 2\cos\theta + \frac{1}{2}\cos 2\theta \ d\theta$$

$$A_1 = 4\left(\frac{3}{2}\theta + 2\sin\theta + \frac{1}{4}\sin 2\theta\right)\Bigg|_0^{\pi}$$

$$A_1 = 4\left(\frac{3}{2}\pi + 2\sin\pi + \frac{1}{4}\sin 2\pi\right) - 4\left(\frac{3}{2}(0) + 2\sin(0) + \frac{1}{4}\sin 2(0)\right)$$

$$A_1 = 4\left(\frac{3}{2}\pi + 0 + 0\right) - 4(0 + 0 + 0)$$

$$A_1 = 6\pi$$

and

$$A_2 = 2 \times \frac{1}{2} \int_0^{\pi} 16\,(1 + \cos\theta)^2 \ \ d\theta$$

$$A_2 = 16 \int_0^{\pi} (1 + \cos\theta)^2 \ \ d\theta$$

$$A_2 = 16 \int_0^{\pi} 1 + 2\cos\theta + \cos^2\theta \ d\theta$$

$$A_2 = 16 \int_0^{\pi} 1 + 2\cos\theta + \frac{1}{2} + \frac{1}{2}\cos 2\theta \ d\theta$$

$$A_2 = 16 \int_0^{\pi} \frac{3}{2} + 2\cos\theta + \frac{1}{2}\cos 2\theta \ d\theta$$

$$A_2 = 16 \left(\frac{3}{2}\theta + 2\sin\theta + \frac{1}{4}\sin 2\theta \right) \Bigg|_0^{\pi}$$

$$A_2 = 16 \left(\frac{3}{2}\pi + 2\sin\pi + \frac{1}{4}\sin 2\pi \right) - 4 \left(\frac{3}{2}(0) + 2\sin(0) + \frac{1}{4}\sin 2(0) \right)$$

$$A_2 = 16 \left(\frac{3}{2}\pi + 0 + 0 \right) - 4(0 + 0 + 0)$$

$$A_2 = 24\pi$$

The ratio of the areas is therefore

$$\frac{A_1}{A_2} = \frac{6\pi}{24\pi} = \frac{1}{4}$$