Calculus 2 Workbook Solutions

Antiderivatives and indefinite integrals

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MATH

INDEFINITE INTEGRALS

1. Evaluate the indefinite integral.

$$\int 5x^4 - 4x^3 + 6x^2 - 2x + 1\ dx$$

Solution:

Take the integral one term at a time using the integration rule for basic power functions.

$$\int x^a\ dx = \frac{x^{a+1}}{a+1} + C$$

$$\int 5x^4 - 4x^3 + 6x^2 - 2x + 1\ dx$$

$$\frac{5x^{4+1}}{4+1} - \frac{4x^{3+1}}{3+1} + \frac{6x^{2+1}}{2+1} - \frac{2x^{1+1}}{1+1} + \frac{x^{0+1}}{0+1} + C$$

$$\frac{5x^5}{5} - \frac{4x^4}{4} + \frac{6x^3}{3} - \frac{2x^2}{2} + \frac{x}{1} + C$$

$$x^5 - x^4 + 2x^3 - x^2 + x + C$$

2. Evaluate the indefinite integral.

$$\int \frac{3x^3 + x^2 - 12x - 4}{x^2 - 4} \ dx$$

Solution:

Take the integral one term at a time using the integration rule for basic power functions.

$$\int x^a \ dx = \frac{x^{a+1}}{a+1} + C$$

$$\int \frac{3x^3 + x^2 - 12x - 4}{x^2 - 4} \ dx$$

$$\int \frac{(3x+1)(x+2)(x-2)}{(x+2)(x-2)} \ dx$$

$$\int 3x + 1 \ dx$$

$$\frac{3x^{1+1}}{1+1} + \frac{x^{0+1}}{0+1} + C$$

$$\frac{3}{2}x^2 + x + C$$

3. Evaluate the indefinite integral.

$$\int (5x - 7)(3x + 2) \ dx$$

Solution:

Take the integral one term at a time using the integration rule for basic power functions. First, rewrite the function by multiplying the binomials.

$$\int (5x-7)(3x+2)\ dx$$

$$\int 15x^2 - 11x - 14\ dx$$

$$\frac{15x^{2+1}}{2+1} - \frac{11x^{1+1}}{1+1} - \frac{14x^{0+1}}{0+1} + C$$

$$\frac{15x^3}{3} - \frac{11x^2}{2} - \frac{14x}{1} + C$$

$$5x^3 - \frac{11}{2}x^2 - 14x + C$$

■ 4. Evaluate the indefinite integral.

$$\int \frac{x^3 - 3x + 2}{x^3}\ dx$$

Solution:

Take the integral one term at a time using the integration rule for basic power functions. First, rewrite the function by separating the fraction and then bringing the power functions to the numerator.

$$\int \frac{x^3 - 3x + 2}{x^3}\ dx$$

$$\int \frac{x^3}{x^3} - \frac{3x}{x^3} + \frac{2}{x^3}\ dx$$

$$\int 1 - \frac{3}{x^2} + \frac{2}{x^3}\ dx$$

$$\int 1 - 3x^{-2} + 2x^{-3}\ dx$$

Then the integrated value is

$$\frac{x^{0+1}}{0+1} - \frac{3x^{-2+1}}{-2+1} + \frac{2x^{-3+1}}{-3+1} + C$$

$$\frac{x^1}{1} - \frac{3x^{-1}}{-1} + \frac{2x^{-2}}{-2} + C$$

$$x + \frac{3}{x} - \frac{1}{x^2} + C$$

PROPERTIES OF INTEGRALS

1. Given the value of each of these integrals,

$$\int_0^3 f(x)\ dx = 7 \qquad \int_3^6 f(x)\ dx = 9 \qquad \int_0^3 g(x)\ dx = 2 \qquad \int_3^6 g(x)\ dx = 5$$

what is the value of of the following integral?

$$\int_0^6 \left[2f(x) + 3g(x)\right]\ dx$$

Solution:

If

$$\int_0^3 f(x)\ dx = 7 \text{ and } \int_3^6 f(x)\ dx = 9,$$

then

$$\int_0^6 f(x)\ dx = 7 + 9 = 16$$

Then,

$$\int_0^6 f(x)\ dx = 16 \int_0^6 2f(x)\ dx = 2\int_0^6 f(x)\ dx = 2 \cdot 16 = 32$$

Similarly, if

$$\int_0^3 g(x)\ dx = 2 \text{ and } \int_3^6 g(x)\ dx = 5$$

then

$$\int_0^6 g(x)\ dx = 2 + 5 = 7$$

Then, if

$$\int_0^6 g(x)\ dx = 7 \int_0^6 3g(x)\ dx = 3\int_0^6 g(x)\ dx = 3 \cdot 7 = 21$$

Therefore,

$$\int_0^6 \left[2f(x) + 3g(x)\right]\ dx = 32 + 21 = 53$$

FIND F GIVEN F''

1. Find $f(x)$ from its second derivative.

$$f''(x) = 3x^2 + 4x - 7$$

Solution:

Given the second derivative, the first derivative is

$$f'(x) = \int 3x^2 + 4x - 7\ dx$$

$$f'(x) = \frac{3x^3}{3} + \frac{4x^2}{2} - 7x + C_1$$

$$f'(x) = x^3 + 2x^2 - 7x + C_1$$

Then $f(x)$ is

$$f(x) = \int x^3 + 2x^2 - 7x + C_1\ dx$$

$$f(x) = \frac{x^4}{4} + \frac{2x^3}{3} - \frac{7x^2}{2} + C_1x + C_2$$

2. Find $g(x)$ from its second derivative.

$$g''(x) = \frac{x^4 - 4x^2 + 4}{x^2 - 2}$$

Solution:

Given the second derivative, the first derivative is

$$g'(x) = \int \frac{x^4 - 4x^2 + 4}{x^2 - 2} \ dx$$

$$g'(x) = \int \frac{(x^2 - 2)(x^2 - 2)}{x^2 - 2} \ dx$$

$$g'(x) = \int x^2 - 2 \ dx$$

$$g'(x) = \frac{x^3}{3} - 2x + C_1$$

Then $g(x)$ is

$$g(x) = \int \frac{x^3}{3} - 2x + C_1 \ dx$$

$$g(x) = \frac{x^4}{12} - x^2 + C_1x + C_2$$

■ 3. Find $h(x)$ from its second derivative.

$$h''(x) = \frac{8x^3 - 9x^2 + 6x}{x^7}$$

Solution:

First, simplify the second derivative function.

$$h''(x) = \frac{8x^3 - 9x^2 + 6x}{x^7}$$

$$h''(x) = \frac{8x^2 - 9x + 6}{x^6}$$

Given the second derivative, the first derivative is

$$h'(x) = \int \frac{8x^2 - 9x + 6}{x^6}\ dx$$

$$h'(x) = \int \frac{8x^2}{x^6} - \frac{9x}{x^6} + \frac{6}{x^6}\ dx$$

$$h'(x) = \int 8x^{-4} - 9x^{-5} + 6x^{-6}\ dx$$

$$h'(x) = \frac{8x^{-4+1}}{-4+1} - \frac{9x^{-5+1}}{-5+1} + \frac{6x^{-6+1}}{-6+1} + C_1$$

$$h'(x) = -\frac{8}{3}x^{-3} + \frac{9}{4}x^{-4} - \frac{6}{5}x^{-5} + C_1$$

Then $h(x)$ is

$$h(x) = \int -\frac{8}{3}x^{-3} + \frac{9}{4}x^{-4} - \frac{6}{5}x^{-5} + C_1\ dx$$

$$h(x) = \frac{-8x^{-2}}{3 \cdot -2} + \frac{9x^{-3}}{4 \cdot -3} - \frac{6x^{-4}}{5 \cdot -4} + C_1x + C_2$$

$$h(x) = \frac{4x^{-2}}{3} - \frac{3x^{-3}}{4} + \frac{3x^{-4}}{10} + C_1x + C_2$$

$$h(x) = \frac{4}{3x^2} - \frac{3}{4x^3} + \frac{3}{10x^4} + C_1x + C_2$$

FIND F GIVEN F'''

1. Find $f(x)$ given its third derivative.

$$f'''(x) = 2x + 3$$

Solution:

Given the third derivative, the second derivative is

$$f''(x) = \int 2x + 3 \ dx$$

$$f''(x) = x^2 + 3x + C_1$$

From the second derivative, the first derivative is

$$f'(x) = \int x^2 + 3x + C_1 \ dx$$

$$f'(x) = \frac{x^3}{3} + \frac{3x^2}{2} + C_1x + C_2$$

Then $f(x)$ is

$$f(x) = \int \frac{x^3}{3} + \frac{3x^2}{2} + C_1x + C_2 \ dx$$

$$f(x) = \frac{x^4}{3 \cdot 4} + \frac{3x^3}{2 \cdot 3} + \frac{C_1x^2}{2} + C_2x + C_3$$

$$f(x) = \frac{1}{12}x^4 + \frac{1}{2}x^3 + C_1x^2 + C_2x + C_3$$

■ 2. Find $g(x)$ given its third derivative.

$$g'''(x) = 4x^3 + x^2 - 3$$

Solution:

Given the third derivative, the second derivative is

$$g''(x) = \int 4x^3 + x^2 - 3 \ dx$$

$$g''(x) = x^4 + \frac{x^3}{3} - 3x + C_1$$

From the second derivative, the first derivative is

$$g'(x) = \int x^4 + \frac{x^3}{3} - 3x + C_1 \ dx$$

$$g'(x) = \frac{x^5}{5} + \frac{x^4}{12} - \frac{3x^2}{2} + C_1x + C_2$$

Then $g(x)$ is

$$g(x) = \int \frac{x^5}{5} + \frac{x^4}{12} - \frac{3x^2}{2} + C_1x + C_2 \ dx$$

$$g(x) = \frac{x^6}{5 \cdot 6} + \frac{x^5}{12 \cdot 5} - \frac{3x^3}{2 \cdot 3} + \frac{C_1x^2}{2} + C_2x + C_3$$

$$g(x) = \frac{1}{30}x^6 + \frac{1}{60}x^5 - \frac{1}{2}x^3 + C_1x^2 + C_2x + C_3$$

■ 3. Find $h(x)$ given its third derivative.

$$h'''(x) = \frac{3}{x^5} - \frac{2}{x^4} + 4$$

Solution:

Given the third derivative, the second derivative is

$$h''(x) = \int \frac{3}{x^5} - \frac{2}{x^4} + 4 \ dx$$

$$h''(x) = \int 3x^{-5} - 2x^{-4} + 4 \ dx$$

$$h''(x) = \frac{3x^{-4}}{-4} - \frac{2x^{-3}}{-3} + 4x + C_1$$

$$h''(x) = -\frac{3x^{-4}}{4} + \frac{2x^{-3}}{3} + 4x + C_1$$

From the second derivative, the first derivative is

$$h'(x) = \int -\frac{3x^{-4}}{4} + \frac{2x^{-3}}{3} + 4x + C_1 \ dx$$

$$h'(x) = -\frac{3x^{-3}}{4 \cdot -3} + \frac{2x^{-2}}{3 \cdot -2} + \frac{4x^2}{2} + C_1x + C_2$$

$$h'(x) = \frac{x^{-3}}{4} - \frac{x^{-2}}{3} + 2x^2 + C_1x + C_2$$

Then $h(x)$ is

$$h(x) = \int \frac{x^{-3}}{4} - \frac{x^{-2}}{3} + 2x^2 + C_1x + C_2 \ dx$$

$$h(x) = \frac{x^{-2}}{4 \cdot -2} - \frac{x^{-1}}{3 \cdot -1} + \frac{2x^3}{3} + \frac{C_1x^2}{2} + C_2x + C_3$$

$$h(x) = -\frac{1}{8x^2} + \frac{1}{3x} + \frac{2}{3}x^3 + C_1x^2 + C_2x + C_3$$

INITIAL VALUE PROBLEMS

1. Find $f(x)$ if $f'(x) = 7x - 5$ and $f(4) = 24$.

Solution:

Given $f'(x) = 7x - 5$, then

$$f(x) = \int 7x - 5\ dx$$

$$f(x) = \frac{7x^2}{2} - 5x + C$$

If $f(4) = 24$, then

$$24 = \frac{7(4)^2}{2} - 5(4) + C$$

$$48 = 7(4)^2 - 2(5)(4) + 2C$$

$$48 = 112 - 40 + 2C$$

$$-24 = 2C$$

$$C = -12$$

Therefore,

$$f(x) = \frac{7}{2}x^2 - 5x - 12$$

■ 2. Find $g(x)$ if $g'(x) = 2x^2 + 5x - 9$ and $g(-4) = 34$.

Solution:

Given $g'(x) = 2x^2 + 5x - 9$, then

$$g(x) = \int 2x^2 + 5x - 9 \ dx$$

$$g(x) = \frac{2x^3}{3} + \frac{5x^2}{2} - 9x + C$$

If $g(-4) = 34$, then

$$34 = \frac{2(-4)^3}{3} + \frac{5(-4)^2}{2} - 9(-4) + C$$

$$34 = -\frac{128}{3} + 40 + 36 + C$$

$$102 = -128 + 120 + 108 + 3C$$

$$2 = 3C$$

$$C = \frac{2}{3}$$

Therefore,

$$g(x) = \frac{2}{3}x^3 + \frac{5}{2}x^2 - 9x + \frac{2}{3}$$

■ 3. Find $h(x)$ if $h'(x) = 3x^2 + 8x + 1$ and $h(2) = 31$.

Solution:

Given $h'(x) = 3x^2 + 8x + 1$, then

$$h(x) = \int 3x^2 + 8x + 1 \ dx$$

$$h(x) = x^3 + 4x^2 + x + C$$

If $h(2) = 31$, then

$$31 = 2^3 + 4(2)^2 + 2 + C$$

$$31 = 8 + 16 + 2 + C$$

$$C = 5$$

Therefore,

$$h(x) = x^3 + 4x^2 + x + 5$$

■ 4. Find $f(x)$ if $f'(x) = x^3 + 4x + 3$ and $f(-2) = 15$.

Solution:

Given $f'(x) = x^3 + 4x + 3$, then

$$f(x) = \int x^3 + 4x + 3 \ dx$$

$$f(x) = \frac{x^4}{4} + \frac{4x^2}{2} + 3x + C$$

$$f(x) = \frac{x^4}{4} + 2x^2 + 3x + C$$

If $f(-2) = 15$, then

$$15 = \frac{(-2)^4}{4} + 2(-2)^2 + 3(-2) + C$$

$$15 = 4 + 8 - 6 + C$$

$$15 = 6 + C$$

$$C = 9$$

Therefore,

$$g(x) = \frac{1}{4}x^4 + 2x^2 + 3x + 9$$

FIND F GIVEN F'' AND INITIAL CONDITIONS

1. Find $g(x)$ if $g''(x) = 2x + 1$, $g'(1) = 5$, and $g(1) = 4$.

Solution:

Given $g''(x) = 2x + 1$, then

$$g'(x) = \int 2x + 1 \ dx$$

$$g'(x) = x^2 + x + C$$

If $g'(1) = 5$, then

$$5 = 1^2 + 1 + C$$

$$5 = 2 + C$$

$$C = 3$$

and $g'(x) = x^2 + x + 3$. Then $g(x)$ is

$$g(x) = \int x^2 + x + 3 \ dx$$

$$g(x) = \frac{x^3}{3} + \frac{x^2}{2} + 3x + C$$

If $g(1) = 4$, then

$$4 = \frac{1^3}{3} + \frac{1^2}{2} + 3(1) + C$$

$$4 = \frac{1}{3} + \frac{1}{2} + 3 + C$$

$$24 = 2 + 3 + 18 + 6C$$

$$1 = 6C$$

$$C = \frac{1}{6}$$

Therefore,

$$g(x) = \frac{x^3}{3} + \frac{x^2}{2} + 3x + \frac{1}{6}$$

2. Find $h(x)$ if $h''(x) = 2x - 7$, $h'(3) = -20$, and $h(6) = -98$.

Solution:

Given $h''(x) = 2x - 7$, then

$$h'(x) = \int 2x - 7\ dx$$

$$h'(x) = x^2 - 7x + C$$

If $h'(3) = -20$, then

$$-20 = 3^2 - 7(3) + C$$

$$-20 = 9 - 21 + C$$

$$C = -8$$

and $h'(x) = x^2 - 7x - 8$. Then $h(x)$ is

$$h(x) = \int x^2 - 7x - 8 \ dx$$

$$h(x) = \frac{x^3}{3} - \frac{7x^2}{2} - 8x + C$$

If $h(6) = -98$, then

$$-98 = \frac{6^3}{3} - \frac{7(6)^2}{2} - 8(6) + C$$

$$-98 = 72 - 7(18) - 48 + C$$

$$C = 4$$

Therefore,

$$h(x) = \frac{1}{3}x^3 - \frac{7}{2}x^2 - 8x + 4$$

■ 3. Find $f(x)$ if $f''(x) = 3x - 6$, $f'(2) = 2$, and $f(2) = 15$.

Solution:

Given $f''(x) = 3x - 6$, then

$$f'(x) = \int 3x - 6\ dx$$

$$f'(x) = \frac{3x^2}{2} - 6x + C$$

If $f'(2) = 2$, then

$$2 = \frac{3(2)^2}{2} - 6(2) + C$$

$$2 = 6 - 12 + C$$

$$C = 8$$

and $f'(x) = (3/2)x^2 - 6x + 8$. Then $f(x)$ is

$$f(x) = \int \frac{3}{2}x^2 - 6x + 8\ dx$$

$$f(x) = \frac{1}{2}x^3 - 3x^2 + 8x + C$$

If $f(2) = 15$, then

$$15 = \frac{1}{2}(2)^3 - 3(2)^2 + 8(2) + C$$

$$15 = 4 - 12 + 16 + C$$

$$C = 7$$

Therefore,

$$f(x) = \frac{1}{2}x^3 - 3x^2 + 8x + 7$$

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