Calculus 1 Workbook Solutions

Definition of the limit

krista king MATH

IDEA OF THE LIMIT

1. The table below shows some values of a function $g(x)$. What does the table show for the value of $\lim_{x\to 4} g(x)$?

x	$g(x)$
3.9	1.9748
3.99	1.9975
3.999	1.9997
4.001	2.0002
4.01	2.0025
4.1	2.0248

Solution:

We see that when x approaches 4 both from the left and right sides, $g(x)$ approaches 2. Then $\lim_{x\to 4} g(x) = 2$.

2. How would we express, mathematically, the limit of the function $f(x) = x^2 - x + 2$ as x approaches 3?

Solution:

When a is the value that x approaches, and $f(x)$ is the given function, the limit is written as

$$\lim_{x\to a} f(x)$$

In this case x approaches 3 so $a=3$, and the function is $f(x)=x^2-x+2$. So we'd write the limit as

$$\lim_{x\to 3}(x^2-x+2)$$

■ 3. How would you write the limit of $g(x)$ as x approaches ∞, using correct mathematical notation?

$$g(x)=\frac{5x^2-7}{3x^2+8}$$

Solution:

When a is the value that x approaches, and $g(x)$ is the given function, the limit is written as

$$\lim_{x\to a} g(x)$$

In this case x approaches ∞ so $a=\infty$, and the function is

$$g(x)=\frac{5x^2-7}{3x^2+8}$$

So we'd write the limit as

$$\lim_{x\to\infty} \frac{5x^2 - 7}{3x^2 + 8}$$

4. Explain what is meant by the equation.

$$\lim_{x\to-2} (x^3 + 2) = -6$$

Solution:

Break down the given limit into its component parts.

- x approaches -2
- the function is $f(x) = x^3 + 2$
- the value of the limit is -6

Putting these pieces together gives a full statement about the limit:

"The limit as x approaches -2 of the function $f(x) = x^3 + 2$ is equal to -6."

5. Evaluate the limit.

$$\lim_{x\to-1} \frac{-x^2 + 3x - 1}{5}$$

Solution:

To evaluate the limit,

$$\lim_{x \to -1} \frac{-x^2 + 3x - 1}{5}$$

plug the value that's being approached into the function, then simplify the result.

$$\frac{-(-1)^2 + 3(-1) - 1}{5}$$

$$\frac{-1 - 3 - 1}{5}$$

$$-\frac{5}{5}$$

$$-1$$

■ 6. Evaluate the limit.

$$\lim_{x \to 0} \frac{x^2 - 5}{2}$$

Solution:

To evaluate the limit,

$$\lim_{x \to 0} \frac{x^2 - 5}{2}$$

plug the value that's being approached into the function, then simplify the answer.

$$\frac{0^2 - 5}{2}$$

$$\frac{-5}{2}$$

$$-\frac{5}{2}$$

ONE-SIDED LIMITS

1. Find the limit.

$$\lim_{x\to -7^+} x^2\sqrt{x+7}$$

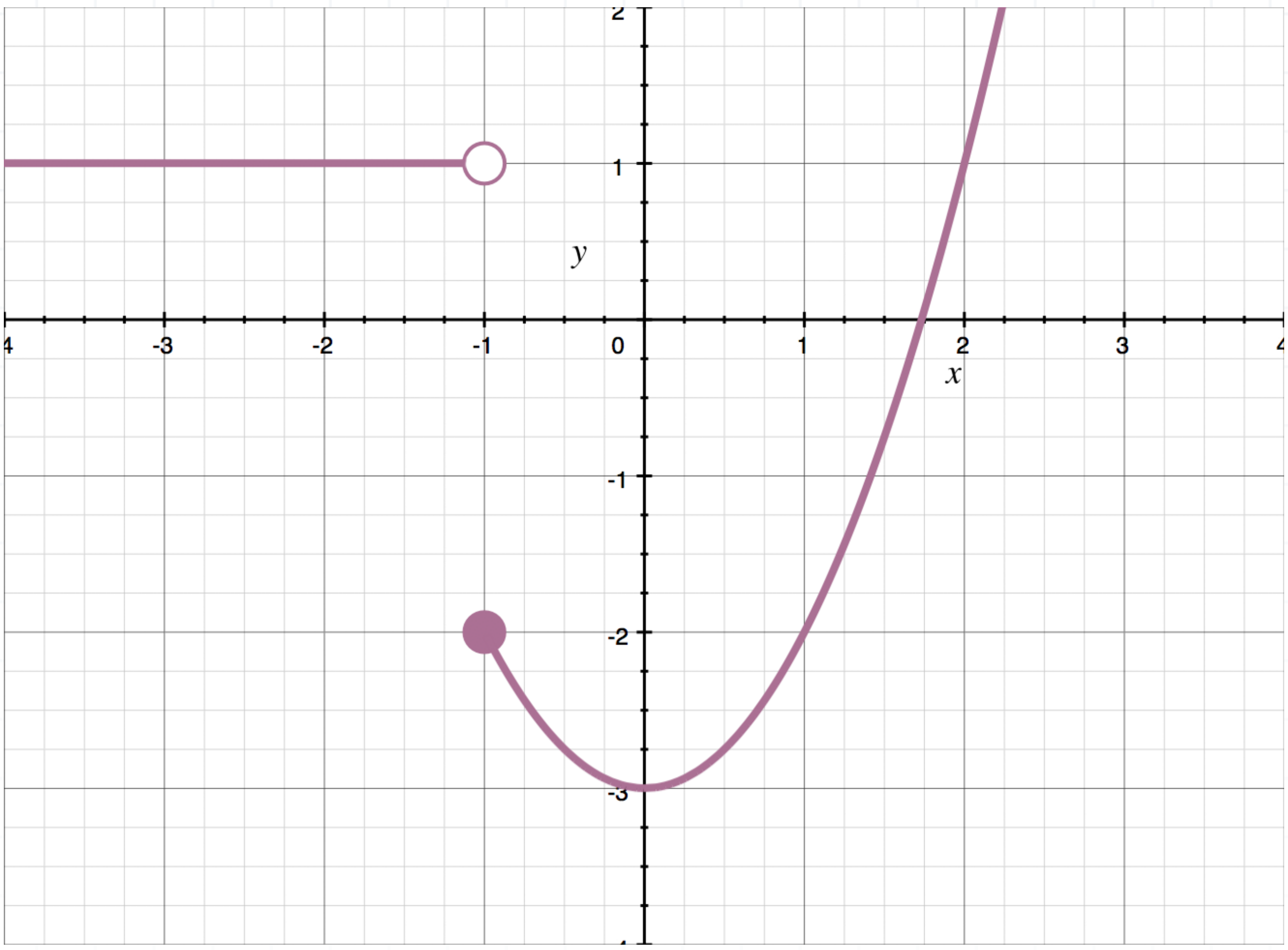
Solution:

The value of the limit is 0.

x	-6.09	-6.9	-6.99	-6.999	-6.9999	-7
Value	35.38	15.056	4.886	1.5481	0.48999	0

We see that as x approaches -7 from the right, the value of the function approaches 0. Then $\lim_{x\to -7^+} x^2\sqrt{x+7}=0$. We could also graph the function to visually analyze its limit.

2. What does the graph of $f(x)$ say about the value of $\lim_{x\to -1^+} f(x)$?



Solution:

The positive sign after the -1 indicates that we're talking about the limit as we approach -1 from the positive, or right side of -1. From the graph, we see that the limit is

$$\lim_{x \to -1^+} f(x) = -2$$

■ 3. The table shows values of $k(x)$. What is $\lim_{x \to -5^-} k(x)$?

x	-5.1	-5.01	-5.0001	-5	-4.999	-4.99	-4.9
k(x)	-392.1	-3,812	-38,012	?	37,988	3,788	368.1

Solution:

The negative sign after the -5 indicates that we're talking about the limit as we approach -5 from the negative, or left side. From the table, we see that as we get very close to $x=-5$ on the left side, the function's value is trending toward $-\infty$, but as we get very close to $x=-5$ on the right side, the function's value is trending toward ∞. So the left-hand limit is

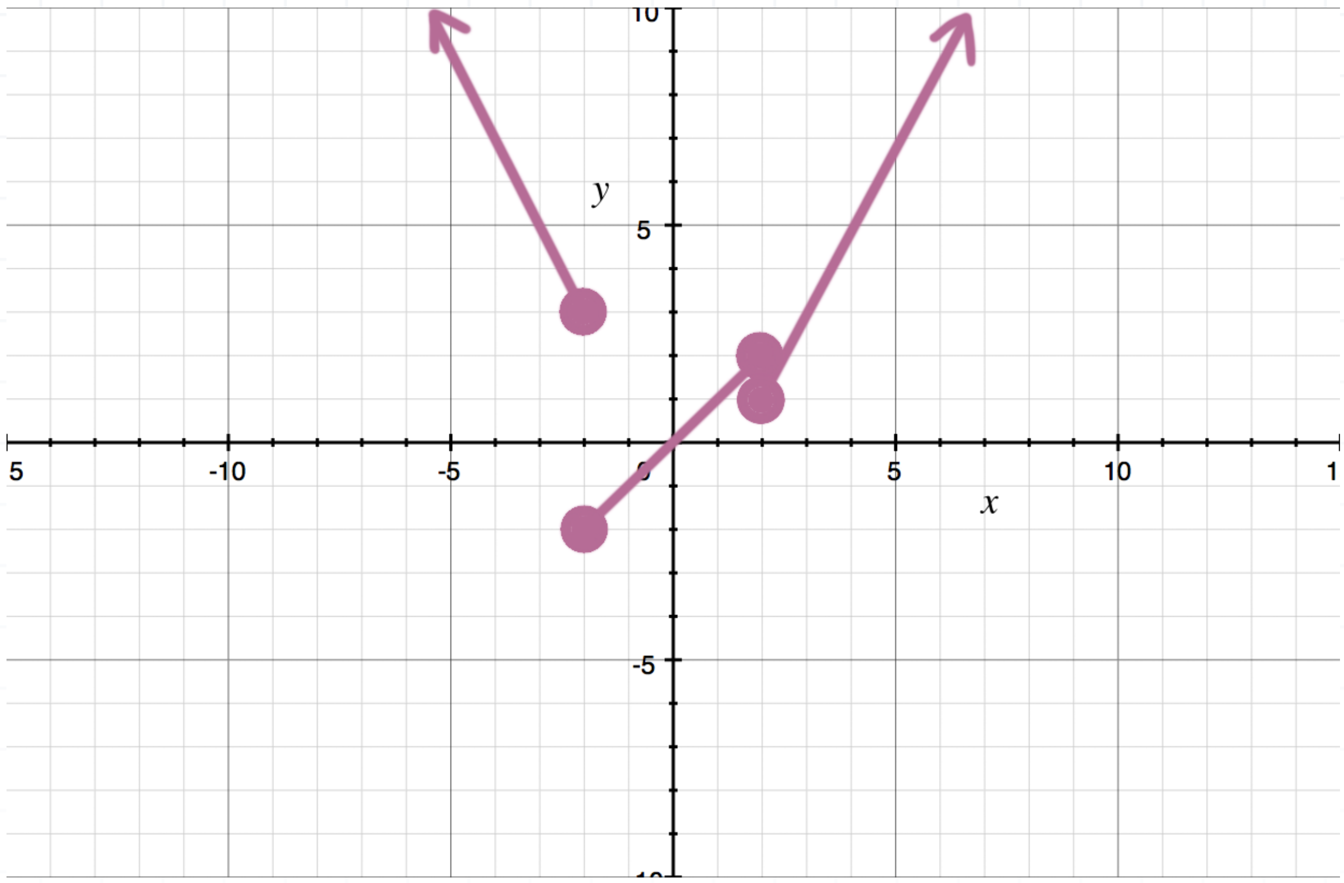
$$\lim_{x\to -5^-} k(x) = -\infty$$

■ 4. What is $\lim_{x\to -2^-} h(x)$?

$$h(x)=\begin{cases} -2x-1 & x<-2 \\ x & -2\le x<2 \\ 2x-3 & x\ge 2 \end{cases}$$

Solution:

The graph of $h(x)$ is



Based on the graph, the limit is 3. Or we could plug into the first piece of the function, which is the piece that approaches $x=-2$ from the left side.

$$\lim_{x\to -2^-} h(x) = \left[-2(-2)-1\right] = 3$$

■ 5. What is $\lim_{x\to 6^+} g(x)$?

$$g(x) = \frac{x^2+x-42}{x-6}$$

Solution:

We could tell that the limit is 13 by making a table,

x	6	6.001	6.01	6.1
g(x)	?	13.001	13.01	13.1

Alternatively, we could have factored the numerator, canceled like terms, and then evaluated at the limit.

$$g(x) = \frac{x^2 + x - 42}{x - 6}$$

$$g(x) = \frac{(x + 7)(x - 6)}{x - 6}$$

$$g(x) = x + 7$$

Then the limit is

$$\lim_{x \to 6^+} x + 7$$

$$6 + 7$$

$$13$$

■ 6. Find the left- and right-hand limits of the function at $x = 3$.

$$f(x) = \frac{|x - 3|}{x - 3}$$

Solution:

This function includes $|x-3|$, which is the absolute value of $x-3$. When $x<3$, $|x-3|=-(x-3)$, so the left-hand limit is

$$\lim_{x\to 3^-}\frac{-(x-3)}{x-3}$$

$$\frac{-1}{1}$$

$$-1$$

When $x>3$, $|x-3|=x-3$, so the right-hand limit is

$$\lim_{x\to 3^+}\frac{x-3}{x-3}$$

$$1$$

PROVING THAT THE LIMIT DOES NOT EXIST

1. Prove that the limit does not exist.

$$\lim_{x \to 0} \frac{-2|3x|}{3x}$$

Solution:

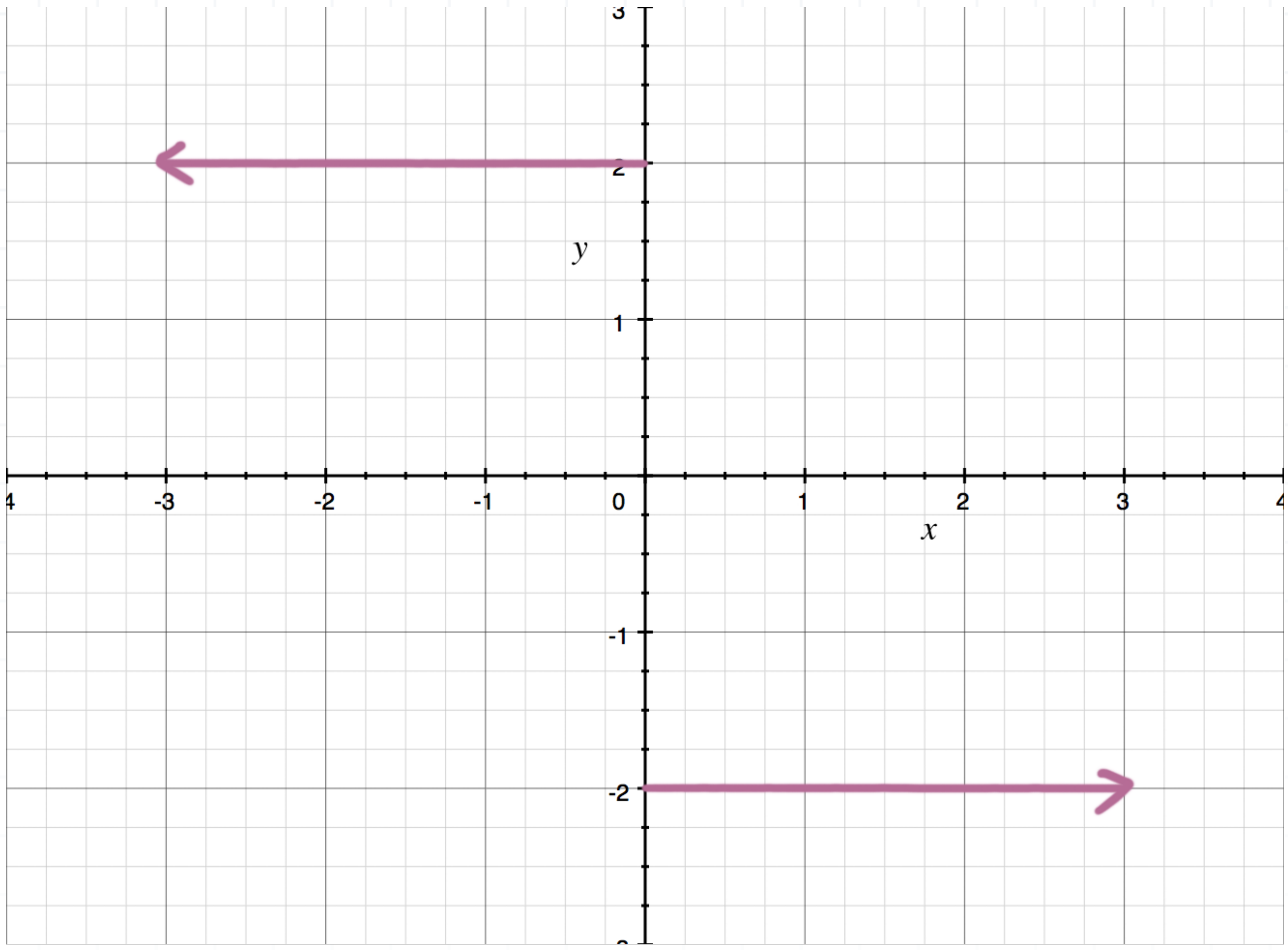
The left-hand limit is

$$\lim_{x \to 0^-} \frac{-2|3x|}{3x} = \lim_{x \to 0^-} \frac{-2(-3x)}{3x} = \frac{6x}{3x} = 2$$

The right-hand limit is

$$\lim_{x \to 0^+} \frac{-2|3x|}{3x} = \lim_{x \to 0^+} \frac{-2(3x)}{3x} = \frac{-6x}{3x} = -2$$

Since the left- and right-hand limits aren't equal, the limit does not exist. The graph of the function would also prove that the limit doesn't exist.



2. Prove that the limit does not exist.

$$\lim_{x \to -5} \frac{x^2 + 7x + 9}{x^2 - 25}$$

Solution:

The left-hand limit is

$$\lim_{x \to -5.001} \frac{x^2 + 7x + 9}{x^2 - 25} = \frac{(-5.001)^2 + 7(-5.001) + 9}{(-5.001)^2 - 25} = -99.69$$

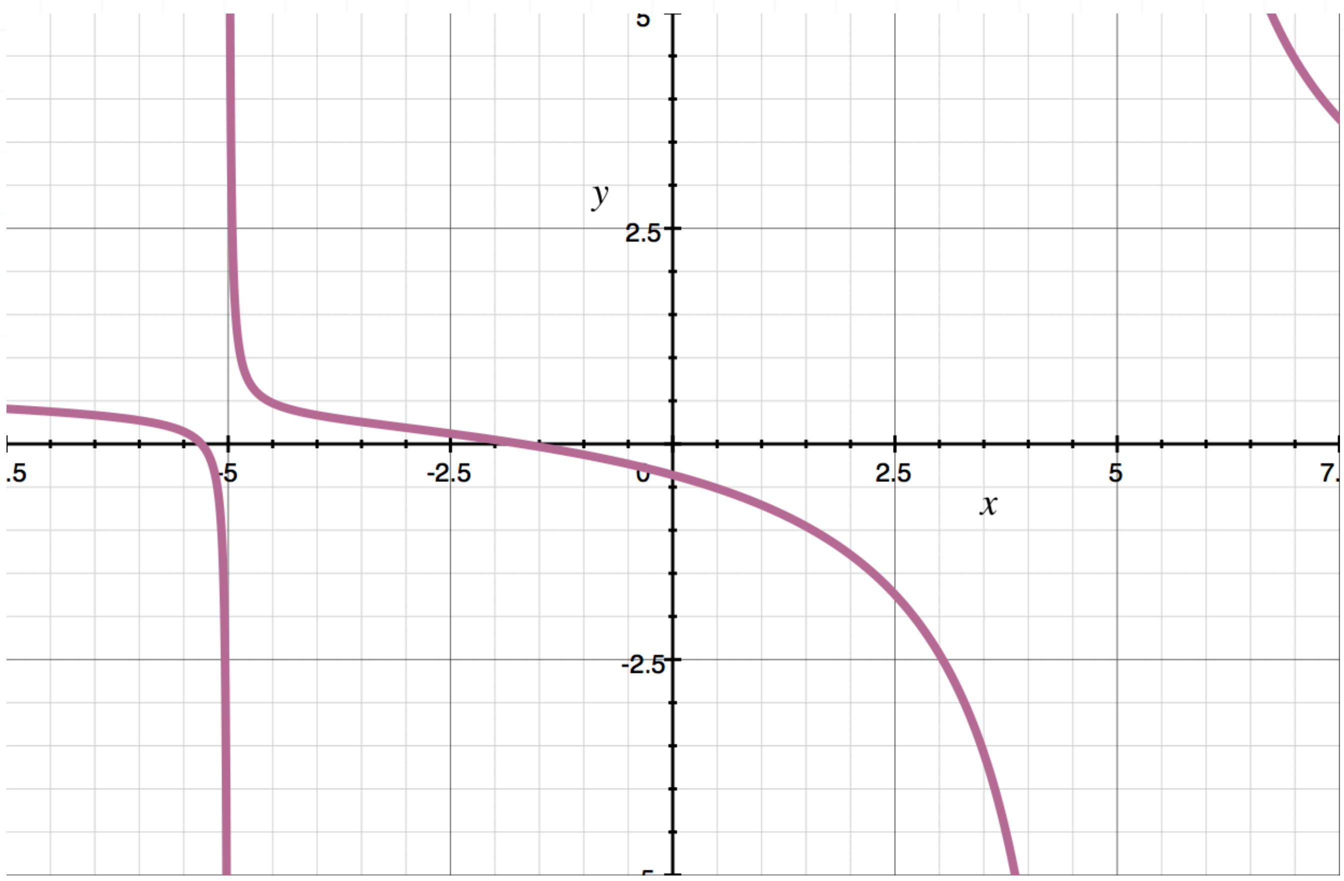
$$\lim_{x \to -5^-} \frac{x^2 + 7x + 9}{x^2 - 25} = -\infty$$

The right-hand limit is

$$\lim_{x\to -4.999}\frac{x^2+7x+9}{x^2-25}=\frac{(-4.999)^2+7(-4.999)+9}{(-4.999)^2-25}=100.31$$

$$\lim_{x\to -5^+}\frac{x^2+7x+9}{x^2-25}=\infty$$

Since the left- and right-hand limits aren't equal, the limit does not exist. The graph of the function would also prove that the limit doesn't exist.



■ 3. Prove that $\lim_{x\to 1} f(x)$ does not exist.

$$f(x)=\begin{cases}-3x+2 & x<1\\ 3x-2 & x\geq 1\end{cases}$$

Solution:

The left-hand limit is

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} (-3x + 2) = \left[-3(1) + 2\right] = -1$$

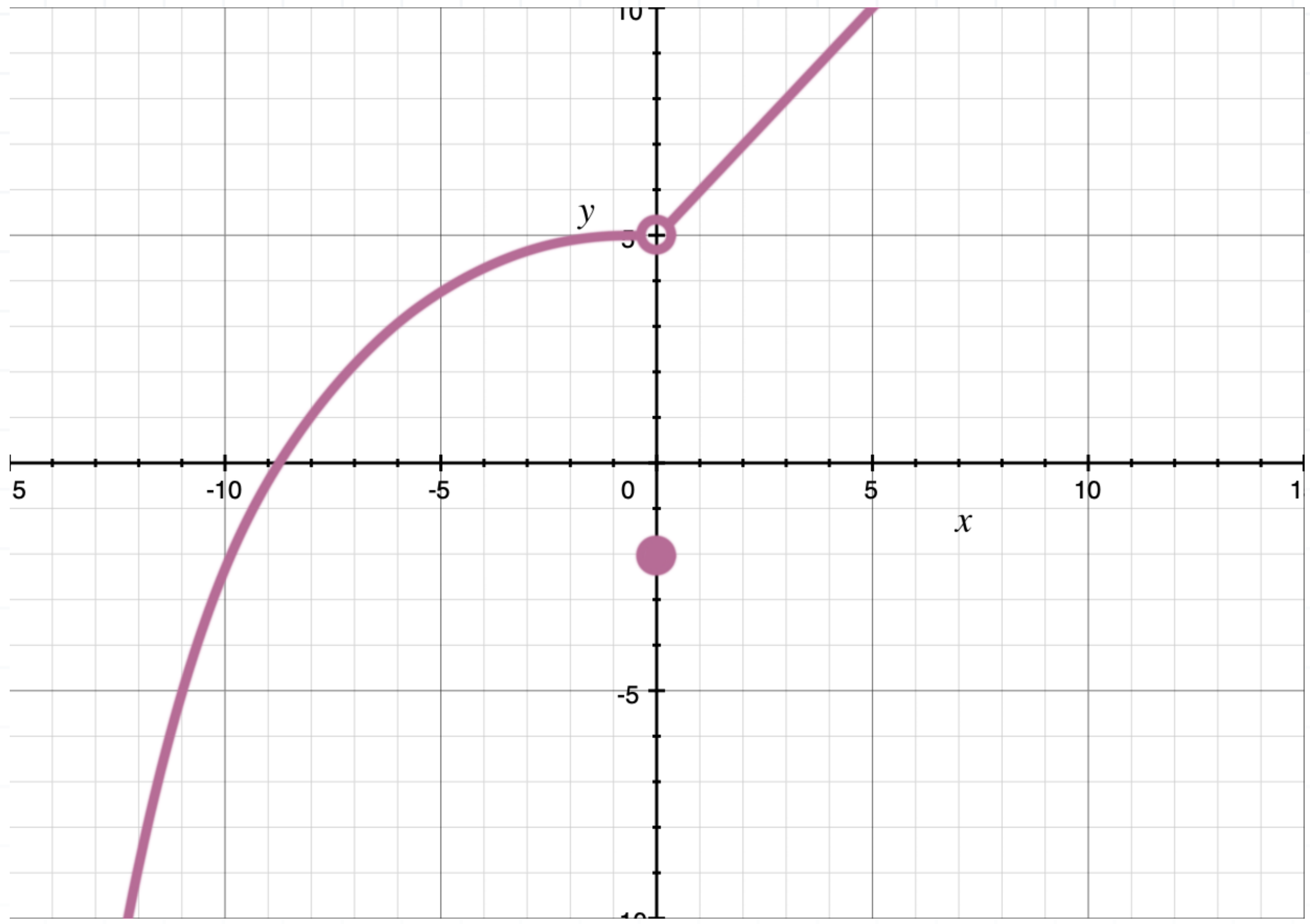
The right-hand limit is

$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} (3x - 2) = \left[3(1) - 2\right] = 1$$

Because the left- and right-hand limits aren't equal, the limit does not exist.

$$\lim_{x \to 1^-} f(x) \neq \lim_{x \to 1^+} f(x)$$

■ 4. Use the graph to determine whether or not the limit exists at $x = 0$.



Solution:

At $x = 0$, the function is approaching 5 from the left side and approaching 5 from the right side. So if we say that the graph represents the function $f(x)$, then the one-sided limits are

$$\lim_{x \to 0^-} f(x) = 5$$

$$\lim_{x \to 0^+} f(x) = 5$$

Because the left- and right-hand limits are equal, we've proven that the general limit of the function exists at $x = 0$ and is equal to 5.

$$\lim_{x \to 0} f(x) = 5$$

■ 5. Suppose we know that $\lim_{x\to 5} f(x) = 12$. If possible, determine the values of the one-sided limits.

$$\lim_{x\to 5^-} f(x)$$

$$\lim_{x\to 5^+} f(x)$$

Solution:

If the general limit exists at a point $x = c$, then the left- and right-hand limits exist at $x = c$ and are equal to one another. Because the general limit exists, we know that the one-sided limits also exist, and they must both be equal to the value of the general limit. Therefore,

$$\lim_{x\to 5^-} f(x) = \lim_{x\to 5^+} f(x) = 12$$

■ 6. Prove that the limit does not exist.

$$\lim_{x\to -2} \frac{x^2 - 4}{(x+2)^2}$$

Solution:

The left-hand limit is

$$\lim_{x \to -2.001} \frac{x^2 - 4}{(x+2)^2} = \frac{(-2.001)^2 - 4}{(-2.001+2)^2} = 4{,}001$$

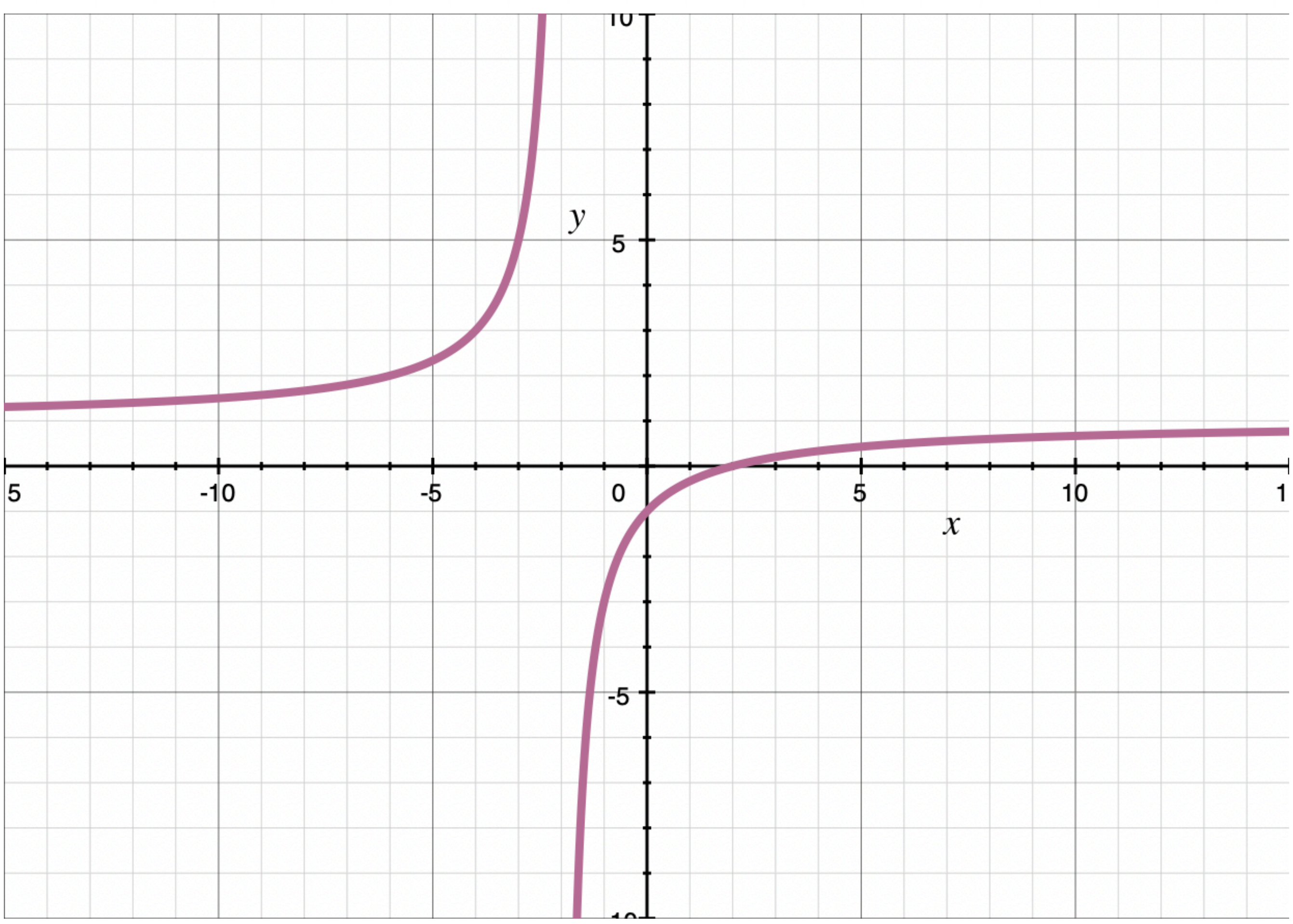
$$\lim_{x \to -2^-} \frac{x^2 - 4}{(x+2)^2} = \infty$$

The right-hand limit is

$$\lim_{x \to -1.999} \frac{x^2 - 4}{(x+2)^2} = \frac{(-1.999)^2 - 4}{(-1.999+2)^2} = -\,3{,}999$$

$$\lim_{x \to -2^+} \frac{x^2 - 4}{(x+2)^2} = -\,\infty$$

Since the left- and right-hand limits aren't equal, the limit does not exist. Graphing the function shows the unequal one-sided limits.



PRECISE DEFINITION OF THE LIMIT

■ 1. Use the precise definition of the limit to prove the value of the limit.

$$\lim_{x\to 4}(5x-16)=4$$

Solution:

If $0<|x-4|<\delta$, then $|(5x-16)-4|<\epsilon$. So,

$$|5x-20|<\epsilon$$

$$|5(x-4)|<\epsilon$$

$$|5|\cdot|x-4|<\epsilon$$

$$5\cdot|x-4|<\epsilon$$

$$|x-4|<\frac{\epsilon}{5}$$

Now if $|x-4|<\epsilon/5$ and $0<|x-4|<\delta$, then if $\epsilon>0$ then $\delta=\epsilon/5$. Therefore, the limit equation is true.

■ 2. Use the precise definition of the limit to prove the value of the limit.

$$\lim_{x\to -7}(-2x+15)=29$$

Solution:

If $0 < |x - (-7)| < \delta$ then $|-2x + 15 - 29| < \epsilon$. Or we could rewrite this as $0 < |x + 7| < \delta$ and $|-2x - 14| < \epsilon$. So,

$$|(-2)(x + 7)| < \epsilon$$

$$|-2| \cdot |x + 7| < \epsilon$$

$$2 \cdot |x + 7| < \epsilon$$

$$|x + 7| < \frac{\epsilon}{2}$$

Now if $|x + 7| < \epsilon/2$ and $0 < |x + 7| < \delta$, then if $\epsilon > 0$ then $\delta = \epsilon/2$. Therefore, the limit equation is true.

■ 3. Use the precise definition of the limit to prove the value of the limit.

$$\lim_{x \to 16} \left(\frac{2}{5}x - \frac{17}{5} \right) = 3$$

Solution:

If $0 < |x - 16| < \delta$ then $\left| \left((2/5)x - (17/5) \right) - 3 \right| < \epsilon$. Or we could rewrite this as $0 < |x - 16| < \delta$ and

$$\left|\left(\frac{2}{5}x-\frac{17}{5}\right)-\frac{15}{5}\right|<\epsilon$$

$$\left|\frac{2}{5}x-\frac{32}{5}\right|<\epsilon$$

$$\left|\frac{2}{5}(x-16)\right|<\epsilon$$

$$\left|\frac{2}{5}\right||x-16|<\epsilon$$

$$|x-16|<\frac{5}{2}\epsilon$$

Now if $|x-16|<(5/2)\epsilon$ and $0<|x-16|<\delta$, then if $\epsilon>0$, then $\delta=(5/2)\epsilon$. Therefore, the limit equation is true.

■ 4. Use the precise definition of the limit to prove the value of the limit.

$$\lim_{x\to 7}\frac{x^2-15x+56}{x-7}=-1$$

Solution:

We'll apply the precise definition to the given limit.

If $0 < |x-7| < \delta$, then $\left|\left(\frac{x^2 - 15x + 56}{x-7}\right) - (-1)\right| < \epsilon$.

If $0 < |x-7| < \delta$, then $\left|\left(\frac{x^2 - 15x + 56}{x-7}\right) - \frac{-1(x-7)}{x-7}\right| < \epsilon$.

So,

$$\left|\left(\frac{x^2 - 15x + 56}{x-7}\right) + \frac{x-7}{x-7}\right| < \epsilon$$

$$\left|\frac{x^2 - 14x + 49}{x-7}\right| < \epsilon$$

$$\left|\frac{(x-7)(x-7)}{x-7}\right| < \epsilon$$

$$|x-7| < \varepsilon$$

Now, if $|x-7| < \epsilon$ and $0 < |x-7| < \delta$, then if $\epsilon > 0$ and $\delta = \epsilon$. Therefore, the limit equation is true.

■ 5. Find δ when $f(x) = 2x - 5$, such that if $0 < |x-1| < \delta$ then $|f(x) + 3| < 0.1$.

Solution:

We want to use the value for ϵ to determine the δ value by remembering from the precise definition of the limit that

if $0 < |x - a| < \delta$ then $|f(x) - L| < \epsilon$

If $0 < |x - 1| < \delta$ then $|2x - 5 + 3| < \epsilon = 0.1$, and we can rewrite this second inequality as

$$|2x - 2| < 0.1$$

$$|2| \cdot |x - 1| < 0.1$$

$$2 \cdot |x - 1| < 0.1$$

$$|x - 1| < \frac{0.1}{2}$$

$$|x - 1| < 0.05$$

So,

$$\delta = 0.05$$

■ 6. Find a value of δ given $\epsilon = 0.04$.

$$\lim_{x \to 2} (x - 2)^2 = 0$$

Solution:

We want to use the value for ϵ to determine the δ value by remembering from the precise definition of the limit that

if $0 < |x - a| < \delta$ then $|f(x) - L| < \epsilon$

If $0 < |x - 2| < \delta$, then $|(x - 2)^2 - 0| < \epsilon$, and we can rewrite this second inequality as

$$|(x - 2)^2| < \epsilon$$

$$|x - 2| < \sqrt{\epsilon}$$

So,

$$\delta = \sqrt{\epsilon} = \sqrt{0.04} = 0.2$$

W W W . K R I S T A K I N G M A T H . C O M