Topic: Binomial series

Question: Use the binomial series to expand the function as a power series.

$$f(x) = (2 + x)^4$$

Answer choices:

A $\quad 1 + \sum_{n=1}^{\infty} \frac{4 \cdot 3 \cdot 2 \cdot 1 \cdot \ldots \cdot (5 - n)}{n!}(x + 1)^n$

B $\quad 1 + \sum_{n=1}^{\infty} \frac{4 \cdot 3 \cdot 2 \cdot 1 \cdot \ldots \cdot (4 - n)}{n!}(x + 1)^n$

C $\quad 1 + \sum_{n=0}^{\infty} \frac{4 \cdot 3 \cdot 2 \cdot 1 \cdot \ldots \cdot (5 - n)}{n!}(x + 1)^n$

D $\quad 1 + \sum_{n=0}^{\infty} \frac{4 \cdot 3 \cdot 2 \cdot 1 \cdot \ldots \cdot (4 - n)}{n!}(x + 1)^n$

Solution: A

We'll start with the binomial series.

$$(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n = 1 + kx + \frac{k(k-1)}{2!}x^2 + \frac{k(k-1)(k-2)}{3!}x^3 + \dots$$

To make the left side match our function, we'll replace x with $x+1$ and k with 4.

$$(1+x+1)^k = \sum_{n=0}^{\infty} \binom{k}{n} (x+1)^n = 1 + k(x+1) + \frac{k(k-1)}{2!}(x+1)^2 + \frac{k(k-1)(k-2)}{3!}(x+1)^3 + \dots$$

$$(2+x)^4 = \sum_{n=0}^{\infty} \binom{4}{n} (x+1)^n = 1 + 4(x+1) + \frac{4(4-1)}{2!}(x+1)^2 + \frac{4(4-1)(4-2)}{3!}(x+1)^3 + \dots$$

Now that the left side matches the given function, we can use the series expansion on the right side to find its power series representation. We just have to find the pattern in the expansion. We'll identify the pattern by rewriting the expansion as

$$1 + 4(x+1) + \frac{4(4-1)}{2!}(x+1)^2 + \frac{4(4-1)(4-2)}{3!}(x+1)^3 + \dots$$

$$1(x+1)^0 + 4(x+1)^1 + \frac{4(4-1)}{2!}(x+1)^2 + \frac{4(4-1)(4-2)}{3!}(x+1)^3 + \dots$$

$$\frac{1}{0!}(x+1)^0 + \frac{4}{1!}(x+1)^1 + \frac{4 \cdot 3}{2!}(x+1)^2 + \frac{4 \cdot 3 \cdot 2}{3!}(x+1)^3 + \dots$$

When we match up these terms with their corresponding n-values, we get

$n = 0 \qquad \frac{1}{0!}(x+1)^0$

$n = 1$ $\qquad\qquad$ $\frac{4}{1!}(x+1)^1$

$n = 2$ $\qquad\qquad$ $\frac{4 \cdot 3}{2!}(x+1)^2$

$n = 3$ $\qquad\qquad$ $\frac{4 \cdot 3 \cdot 2}{3!}(x+1)^3$

The $n = 0$ term doesn't follow the same pattern as the rest of the series, since it's not multiplied by 4, so we'll pull it out in front of the power series representation and shift the index from $n = 0$ to $n = 1$. For all the terms starting with $n = 1$, we can see that the pattern is

$$\frac{4 \cdot 3 \cdot 2 \cdot 1 \cdot \ldots \cdot \left[4 - (n-1)\right]}{n!}(x+1)^n$$

$$\frac{4 \cdot 3 \cdot 2 \cdot 1 \cdot \ldots \cdot (4 - n + 1)}{n!}(x+1)^n$$

$$\frac{4 \cdot 3 \cdot 2 \cdot 1 \cdot \ldots \cdot (5 - n)}{n!}(x+1)^n$$

So the power series representation of the function is

$$1 + \sum_{n=1}^{\infty} \frac{4 \cdot 3 \cdot 2 \cdot 1 \cdot \ldots \cdot (5 - n)}{n!}(x+1)^n$$

Topic: Binomial series

Question: Use the binomial series to expand the function as a power series.

$$f(x) = (1 + 2x)^2$$

Answer choices:

A $\quad 1 + \sum_{n=1}^{\infty} \frac{2 \cdot 1 \cdot 0 \cdot \ldots \cdot (2-n)}{n!}(2x)^n$

B $\quad 1 + \sum_{n=1}^{\infty} \frac{2 \cdot 1 \cdot 0 \cdot \ldots \cdot (3-n)}{n!}(2x)^n$

C $\quad 1 + \sum_{n=0}^{\infty} \frac{2 \cdot 1 \cdot 0 \cdot \ldots \cdot (3-n)}{n!}(2x)^n$

D $\quad 1 + \sum_{n=0}^{\infty} \frac{2 \cdot 1 \cdot 0 \cdot \ldots \cdot (2-n)}{n!}(2x)^n$

Solution: B

We'll start with the binomial series.

$$(1+x)^k=\sum_{n=0}^{\infty}\binom{k}{n}x^n=1+kx+\frac{k(k-1)}{2!}x^2+\frac{k(k-1)(k-2)}{3!}x^3+\ldots$$

To make the left side match our function, we'll replace x with $2x$ and k with 2.

$$(1+2x)^k=\sum_{n=0}^{\infty}\binom{k}{n}(2x)^n=1+k(2x)+\frac{k(k-1)}{2!}(2x)^2+\frac{k(k-1)(k-2)}{3!}(2x)^3+\ldots$$

$$(1+2x)^2=\sum_{n=0}^{\infty}\binom{2}{n}(2x)^n=1+2(2x)+\frac{2(2-1)}{2!}(2x)^2+\frac{2(2-1)(2-2)}{3!}(2x)^3+\ldots$$

Now that the left side matches the given function, we can use the series expansion on the right side to find its power series representation. We just have to find the pattern in the expansion. We'll identify the pattern by rewriting the expansion as

$$1+2(2x)+\frac{2(2-1)}{2!}(2x)^2+\frac{2(2-1)(2-2)}{3!}(2x)^3+\ldots$$

$$1(2x)^0+2(2x)^1+\frac{2(2-1)}{2!}(2x)^2+\frac{2(2-1)(2-2)}{3!}(2x)^3+\ldots$$

$$\frac{1}{0!}(2x)^0+\frac{2}{1!}(2x)^1+\frac{2(2-1)}{2!}(2x)^2+\frac{2(2-1)(2-2)}{3!}(2x)^3+\ldots$$

When we match up these terms with their corresponding n-values, we get

$n=0$ $\qquad\qquad$ $\frac{1}{0!}(2x)^0$

$n=1$ $\qquad \dfrac{2}{1!}(2x)^1$

$n=2$ $\qquad \dfrac{2(2-1)}{2!}(2x)^2$

$n=3$ $\qquad \dfrac{2(2-1)(2-2)}{3!}(2x)^3$

The $n=0$ term doesn't follow the same pattern as the rest of the series, since it's not multiplied by 2, so we'll pull it out in front of the power series representation and shift the index from $n=0$ to $n=1$. For all the terms starting with $n=1$, we can see that the pattern is

$$\frac{2\cdot 1\cdot 0\cdot \ldots \cdot \left[2-(n-1)\right]}{n!}(2x)^n$$

$$\frac{2\cdot 1\cdot 0\cdot \ldots \cdot (2-n+1)}{n!}(2x)^n$$

$$\frac{2\cdot 1\cdot 0\cdot \ldots \cdot (3-n)}{n!}(2x)^n$$

So the power series representation of the function is

$$1+\sum_{n=1}^{\infty}\frac{2\cdot 1\cdot 0\cdot \ldots \cdot (3-n)}{n!}(2x)^n$$

Topic: Binomial series

Question: Use the binomial series to expand the function as a power series.

$$f(x) = (1+3x)^{\frac{1}{2}}$$

Answer choices:

A $\quad 1 + \sum_{n=0}^{\infty} \frac{\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\ldots\left(\frac{1}{2}-n\right)}{n!}(3x)^n$

B $\quad 1 + \sum_{n=0}^{\infty} \frac{\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\ldots\left(\frac{3}{2}-n\right)}{n!}(3x)^n$

C $\quad 1 + \sum_{n=1}^{\infty} \frac{\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\ldots\left(\frac{1}{2}-n\right)}{n!}(3x)^n$

D $\quad 1 + \sum_{n=1}^{\infty} \frac{\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\ldots\left(\frac{3}{2}-n\right)}{n!}(3x)^n$

Solution: D

We'll start with the binomial series.

$$(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n = 1 + kx + \frac{k(k-1)}{2!}x^2 + \frac{k(k-1)(k-2)}{3!}x^3 + \dots$$

To make the left side match our function, we'll replace x with $3x$ and k with $1/2$.

$$(1+3x)^k = \sum_{n=0}^{\infty} \binom{k}{n} (3x)^n = 1 + k(3x) + \frac{k(k-1)}{2!}(3x)^2 + \frac{k(k-1)(k-2)}{3!}(3x)^3 + \dots$$

$$(1+3x)^{\frac{1}{2}} = \sum_{n=0}^{\infty} \binom{\frac{1}{2}}{n} (3x)^n = 1 + \frac{1}{2}(3x) + \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2!}(3x)^2 + \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)}{3!}(3x)^3 + \dots$$

Now that the left side matches the given function, we can use the series expansion on the right side to find its power series representation. We just have to find the pattern in the expansion. We'll identify the pattern by rewriting the expansion as

$$1 + \frac{1}{2}(3x) + \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2!}(3x)^2 + \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)}{3!}(3x)^3 + \dots$$

$$1(3x)^0 + \frac{1}{2}(3x)^1 + \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2!}(3x)^2 + \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)}{3!}(3x)^3 + \dots$$

$$\frac{1}{0!}(3x)^0 + \frac{\frac{1}{2}}{1!}(3x)^1 + \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2!}(3x)^2 + \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)}{3!}(3x)^3 + \dots$$

When we match up these terms with their corresponding n-values, we get

$n = 0$ $\frac{1}{0!}(3x)^0$

$n = 1$ $\frac{\frac{1}{2}}{1!}(3x)^1$

$n = 2$ $\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2!}(3x)^2$

$n = 3$ $\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)}{3!}(3x)^3$

The $n = 0$ term doesn't follow the same pattern as the rest of the series, since it's not multiplied by $1/2$, so we'll pull it out in front of the power series representation and shift the index from $n = 0$ to $n = 1$. For all the terms starting with $n = 1$, we can see that the pattern is

$$\frac{\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\ldots\left[\frac{1}{2}-(n-1)\right]}{n!}(3x)^n$$

$$\frac{\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\ldots\left(\frac{1}{2}-n+1\right)}{n!}(3x)^n$$

$$\frac{\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\ldots\left(\frac{3}{2}-n\right)}{n!}(3x)^n$$

So the power series representation of the function is

$$1+\sum_{n=1}^{\infty}\frac{\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\ldots\left(\frac{3}{2}-n\right)}{n!}(3x)^n$$