**Topic**: Improper integrals, case 1

**Question**: Evaluate the improper integral.

$$\int_2^\infty \frac{dx}{x(\ln x)^2}$$

**Answer choices**:

A $0$

B $\ln 2$

C $-1$

D $\frac{1}{\ln 2}$

**Solution**: D

Using an arbitrary variable $b$, first take the limit of the integral as $b \to \infty$.

$$\int_2^\infty \frac{dx}{x(\ln x)^2} = \lim_{b\to\infty} \int_2^b \frac{dx}{x(\ln x)^2}$$

Let

$$u = \ln x$$

$$du = \frac{1}{x}\ dx$$

$$dx = x\ du$$

Plugging these values into the integral, we get

$$\lim_{b\to\infty} \int_{x=2}^{x=b} \frac{x\ du}{xu^2}$$

$$\lim_{b\to\infty} \int_{x=2}^{x=b} u^{-2}\ du$$

$$\lim_{b\to\infty} \left(-u^{-1}\right)\Big|_{x=2}^{x=b}$$

Back-substituting for $x$ before we evaluate over the interval, we get

$$\lim_{b\to\infty} \left[-(\ln x)^{-1}\right]\Big|_2^b$$

$$\lim_{b\to\infty}\left(-\frac{1}{\ln x}\right)\Bigg|_2^b$$

Evaluating over the interval, we get

$$\lim_{b\to\infty}\left(-\frac{1}{\ln b}+\frac{1}{\ln 2}\right)$$

$$-\frac{1}{\ln\infty}+\frac{1}{\ln 2}$$

$$\frac{1}{\ln 2}$$

**Topic**: Improper integrals, case 1

**Question**: Evaluate the improper integral.

$$\int_{3}^{\infty} 5x^{-7}\, dx$$

**Answer choices**:

A $\quad \dfrac{5}{4{,}374}$

B $\quad -\dfrac{5}{4{,}374}$

C $\quad \dfrac{1}{729}$

D $\quad -\dfrac{1}{729}$

**Solution**: A

The integral in this problem is considered to be an improper integral, case 1, because the lower limit of integration is a constant and the upper limit is $\infty$. Evaluating this type of improper integral follows this general rule:

$$\int_a^{\infty} f(x)\ dx = \lim_{b\to\infty} \int_a^b f(x)\ dx$$

We basically ignore the upper limit by replacing it with $b$ and using a limit process. Then, once we integrate, finding the anti-derivative, we use the limit to finish the evaluation. Let's begin by rewriting the integral as a limit.

$$\int_3^{\infty} 5x^{-7}\ dx = \lim_{b\to\infty} \int_3^b 5x^{-7}\ dx$$

$$5 \lim_{b\to\infty} \int_3^b x^{-7}\ dx$$

$$\left[5 \lim_{b\to\infty} \frac{x^{-6}}{-6}\right]_3^b$$

$$-\frac{5}{6} \lim_{b\to\infty} \left(b^{-6} - 3^{-6}\right)$$

$$-\frac{5}{6} \lim_{b\to\infty} \left(\frac{1}{b^6} - \frac{1}{3^6}\right)$$

$$-\frac{5}{6} \lim_{b\to\infty} \left(\frac{1}{b^6} - \frac{1}{729}\right)$$

When we take the limit, $1/b^6$ becomes $0$.

$$-\frac{5}{6}\left(0-\frac{1}{729}\right)$$

$$\frac{5}{6}\left(\frac{1}{729}\right)$$

$$\frac{5}{4{,}374}$$

**Topic**: Improper integrals, case 1

**Question**: Evaluate the improper integral.

$$\int_{9}^{\infty} \frac{2x-5}{x^2-5x-7}\, dx$$

**Answer choices**:

A $-\infty$

B $0$

C $\infty$

D $\ln\left(\frac{13}{29}\right)$

**Solution**: C

The integral in this problem is considered to be an improper integral, case 1, because the lower limit of integration is a constant and the upper limit is $\infty$. Evaluating this type of improper integral follows this general rule:

$$\int_a^\infty f(x)\ dx = \lim_{b\to\infty}\int_a^b f(x)\ dx$$

We basically ignore the upper limit by replacing it with $b$ and using a limit process. Then, once we integrate, finding the anti-derivative, we use the limit to finish the evaluation. Let's begin by rewriting the integral as a limit.

$$\int_9^\infty \frac{2x-5}{x^2-5x-7}\ dx = \lim_{b\to\infty}\int_9^b \frac{2x-5}{x^2-5x-7}\ dx$$

Now we'll change the integral using u-substitution.

$$u = x^2-5x-7$$

$$du = (2x-5)\ dx$$

$$dx = \frac{du}{2x-5}$$

Substitute into the integral.

$$\lim_{b\to\infty}\int_{x=9}^{x=b}\frac{2x-5}{u}\left(\frac{du}{2x-5}\right)$$

$$\lim_{b\to\infty}\int_{x=9}^{x=b}\frac{1}{u}\ du$$

Integrate.

$$\lim_{b\to\infty} \ln|u| \Big|_{x=9}^{x=b}$$

Back-substitute to get the value in terms of $x$.

$$\lim_{b\to\infty} \ln\left|x^2-5x-7\right| \Big|_9^b$$

$$\lim_{b\to\infty} \left[\ln\left|b^2-5b-7\right| - \ln\left|(9)^2-5(9)-7\right|\right]$$

$$\lim_{b\to\infty} \left[\ln\left|b^2-5b-7\right| - \ln 29\right]$$

$$\lim_{b\to\infty} \ln \frac{\left|b^2-5b-7\right|}{29}$$

When we take the limit, the numerator becomes $\infty$. Therefore, the value of the whole limit will be $\infty$.