Calculus 1 Workbook Solutions

Implicit differentiation

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IMPLICIT DIFFERENTIATION

■ 1. Use implicit differentiation to find dy/dx at $(3,4)$.

$$4x^3 - 3xy^2 + y^3 = 28$$

Solution:

Use implicit differentiation to take the derivative of both sides.

$$12x^2 - 3y^2 - 6xy\frac{dy}{dx} + 3y^2\frac{dy}{dx} = 0$$

$$(3y^2 - 6xy)\frac{dy}{dx} = 3y^2 - 12x^2$$

$$\frac{dy}{dx} = \frac{3y^2 - 12x^2}{3y^2 - 6xy}$$

$$\frac{dy}{dx} = \frac{y^2 - 4x^2}{y^2 - 2xy}$$

Evaluate dy/dx at $(3,4)$.

$$\frac{dy}{dx}(3,4) = \frac{(4)^2 - 4(3)^2}{(4)^2 - 2(3)(4)} = \frac{16 - 36}{16 - 24} = \frac{5}{2}$$

■ 2. Use implicit differentiation to find dy/dx.

$$5x^3 + xy^2 = 4x^3y^3$$

Solution:

Rearrange the function. We'll do this to get all the terms that include y on one side of the equation, which will make it easier to solve for dy/dx later on.

$$5x^3 + xy^2 = 4x^3y^3$$

$$xy^2 - 4x^3y^3 = -5x^3$$

Use implicit differentiation to take the derivative of both sides.

$$y^2 + 2xy\frac{dy}{dx} - 12x^2y^3 - 12x^3y^2\frac{dy}{dx} = -15x^2$$

$$2xy\frac{dy}{dx} - 12x^3y^2\frac{dy}{dx} = 12x^2y^3 - 15x^2 - y^2$$

$$(2xy - 12x^3y^2)\frac{dy}{dx} = 12x^2y^3 - 15x^2 - y^2$$

$$\frac{dy}{dx} = \frac{12x^2y^3 - 15x^2 - y^2}{2xy - 12x^3y^2}$$

■ 3. Use implicit differentiation to find dy/dx.

$$3x^2 = (3xy - 1)^2$$

Solution:

Rearrange the function. We'll do this to get all the terms that include y on one side of the equation, which will make it easier to solve for dy/dx later on.

$$3x^2 = (3xy - 1)^2$$

$$3x^2 = 9x^2y^2 - 6xy + 1$$

Use implicit differentiation to take the derivative of both sides.

$$6x = 18xy^2 + 18x^2y\frac{dy}{dx} - 6y - 6x\frac{dy}{dx}$$

$$6x - 18xy^2 + 6y = 18x^2y\frac{dy}{dx} - 6x\frac{dy}{dx}$$

$$6x - 18xy^2 + 6y = (18x^2y - 6x)\frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{6x - 18xy^2 + 6y}{18x^2y - 6x}$$

$$\frac{dy}{dx} = \frac{x - 3xy^2 + y}{3x^2y - x}$$

■ 4. Use implicit differentiation to find dy/dx.

$$\sin(2x + 5y) = \cos^2 x + \cos^2 y$$

Solution:

Use implicit differentiation to take the derivative of both sides.

$$\cos(2x+5y)\left(2+5\frac{dy}{dx}\right)=2\cos x(-\sin x)+2\cos y(-\sin y)\frac{dy}{dx}$$

$$2\cos(2x+5y)+5\cos(2x+5y)\frac{dy}{dx}=-2\sin x\cos x-2\sin y\cos y\frac{dy}{dx}$$

$$2\sin y\cos y\frac{dy}{dx}+5\cos(2x+5y)\frac{dy}{dx}=-2\sin x\cos x-2\cos(2x+5y)$$

$$(2\sin y\cos y+5\cos(2x+5y))\frac{dy}{dx}=-2\sin x\cos x-2\cos(2x+5y)$$

$$\frac{dy}{dx}=\frac{-2\sin x\cos x-2\cos(2x+5y)}{2\sin y\cos y+5\cos(2x+5y)}$$

$$\frac{dy}{dx}=-\frac{2\sin x\cos x+2\cos(2x+5y)}{2\sin y\cos y+5\cos(2x+5y)}$$

$$\frac{dy}{dx}=-\frac{\sin(2x)+2\cos(2x+5y)}{\sin(2y)+5\cos(2x+5y)}$$

■ 5. Use implicit differentiation to find dy/dx.

$$e^{2xy}=3x^3-\ln(xy^2)$$

Solution:

Rearrange the equation. We'll do this to get all the terms that include y on one side of the equation, which will make it easier to solve for dy/dx later on.

$$e^{2xy} = 3x^3 - \ln(xy^2)$$

$$e^{2xy} + \ln(xy^2) = 3x^3$$

Use implicit differentiation to take the derivative of both sides.

$$e^{2xy}\left(2y + 2x\frac{dy}{dx}\right) + \frac{1}{xy^2}\left(y^2 + 2xy\frac{dy}{dx}\right) = 9x^2$$

$$2ye^{2xy} + 2xe^{2xy}\frac{dy}{dx} + \frac{y^2}{xy^2} + \frac{2xy}{xy^2}\left(\frac{dy}{dx}\right) = 9x^2$$

$$2ye^{2xy} + 2xe^{2xy}\frac{dy}{dx} + \frac{1}{x} + \frac{2}{y}\left(\frac{dy}{dx}\right) = 9x^2$$

$$2xe^{2xy}\frac{dy}{dx} + \frac{2}{y}\left(\frac{dy}{dx}\right) = 9x^2 - 2ye^{2xy} - \frac{1}{x}$$

$$\left(2xe^{2xy} + \frac{2}{y}\right)\frac{dy}{dx} = 9x^2 - 2ye^{2xy} - \frac{1}{x}$$

$$\frac{dy}{dx} = \frac{9x^2 - 2ye^{2xy} - \frac{1}{x}}{2xe^{2xy} + \frac{2}{y}}$$

■ 6. Use implicit differentiation to find dy/dx at $(0, -5)$.

$$\frac{2x - y^3}{y + x^2} = 5x - 4$$

Solution:

Use implicit differentiation to take the derivative of both sides.

$$\frac{\left(2 - 3y^2\frac{dy}{dx}\right)(y + x^2) - (2x - y^3)\left(\frac{dy}{dx} + 2x\right)}{(y + x^2)^2} = 5$$

$$\frac{2y + 2x^2 - 3y^3\frac{dy}{dx} - 3x^2y^2\frac{dy}{dx} - \left(2x\frac{dy}{dx} + 4x^2 - y^3\frac{dy}{dx} - 2xy^3\right)}{(y + x^2)^2} = 5$$

$$\frac{2y + 2x^2 - 3y^3\frac{dy}{dx} - 3x^2y^2\frac{dy}{dx} - 2x\frac{dy}{dx} - 4x^2 + y^3\frac{dy}{dx} + 2xy^3}{(y + x^2)^2} = 5$$

$$2y + 2x^2 - 3y^3\frac{dy}{dx} - 3x^2y^2\frac{dy}{dx} - 2x\frac{dy}{dx} - 4x^2 + y^3\frac{dy}{dx} + 2xy^3 = 5(y + x^2)^2$$

$$-3y^3\frac{dy}{dx} - 3x^2y^2\frac{dy}{dx} - 2x\frac{dy}{dx} + y^3\frac{dy}{dx} = 5(y + x^2)^2 - 2y - 2x^2 + 4x^2 - 2xy^3$$

$$(-3y^3 - 3x^2y^2 - 2x + y^3)\frac{dy}{dx} = 5(y + x^2)^2 - 2y + 2x^2 - 2xy^3$$

$$(-2y^3 - 3x^2y^2 - 2x)\frac{dy}{dx} = 5(y + x^2)^2 - 2y + 2x^2 - 2xy^3$$

$$\frac{dy}{dx} = \frac{5(y + x^2)^2 - 2y + 2x^2 - 2xy^3}{-2y^3 - 3x^2y^2 - 2x}$$

$$\frac{dy}{dx} = -\frac{5(y+x^2)^2 - 2y + 2x^2 - 2xy^3}{2y^3 + 3x^2y^2 + 2x}$$

Evaluate dy/dx at $(0, -5)$.

$$\frac{dy}{dx}(0, -5) = -\frac{5(-5+0^2)^2 - 2(-5) + 2(0)^2 - 2(0)(-5)^3}{2(-5)^3 + 3(0)^2(-5)^2 + 2(0)}$$

$$\frac{dy}{dx}(0, -5) = -\frac{5(-5)^2 + 10}{2(-5)^3}$$

$$\frac{dy}{dx}(0, -5) = -\frac{5(25) + 10}{2(-125)}$$

$$\frac{dy}{dx}(0, -5) = -\frac{135}{-250}$$

$$\frac{dy}{dx}(0, -5) = \frac{27}{50}$$

EQUATION OF THE TANGENT LINE WITH IMPLICIT DIFFERENTIATION

■ 1. Use implicit differentiation to find the equation of the tangent line to $5y^2 = 2x^3 - 5y + 6$ at $(3,3)$.

Solution:

Rearrange the function.

$$5y^2 = 2x^3 - 5y + 6$$

$$5y^2 + 5y = 2x^3 + 6$$

Use implicit differentiation to take the derivative of both sides.

$$10y\frac{dy}{dx} + 5\frac{dy}{dx} = 6x^2$$

$$(10y + 5)\frac{dy}{dx} = 6x^2$$

$$\frac{dy}{dx} = \frac{6x^2}{10y + 5}$$

Evaluate dy/dx at $(3,3)$.

$$\frac{dy}{dx}(3,3) = \frac{6(3)^2}{10(3) + 5} = \frac{54}{35}$$

Then the equation of the tangent line is

$$y - y_1 = m(x - x_1)$$

$$y - 3 = \frac{54}{35}(x - 3)$$

$$y = \frac{54}{35}(x - 3) + 3$$

■ 2. Use implicit differentiation to find the equation of the tangent line to $5x^3 = -3xy + 4$ at $(2, -6)$.

Solution:

Use implicit differentiation to take the derivative of both sides.

$$15x^2 = -3y - 3x\frac{dy}{dx}$$

$$15x^2 + 3y = -3x\frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{15x^2 + 3y}{-3x}$$

$$\frac{dy}{dx} = -\frac{5x^2 + y}{x}$$

Evaluate dy/dx at $(2, -6)$.

$$\frac{dy}{dx}(2, -6) = -\frac{5(2)^2 + (-6)}{2} = -\frac{20 - 6}{2} = -7$$

Then the equation of the tangent line is

$$y - y_1 = m(x - x_1)$$

$$y + 6 = -7(x - 2)$$

$$y = -7(x - 2) - 6$$

$$y = -7x + 8$$

■ 3. Use implicit differentiation to find the equation of the tangent line to $4y^2 + 8 = 3x^2$ at $(6, -5)$.

Solution:

Use implicit differentiation to take the derivative of both sides.

$$8y\frac{dy}{dx} = 6x$$

$$\frac{dy}{dx} = \frac{6x}{8y} = \frac{3x}{4y}$$

Evaluate dy/dx at $(6, -5)$.

$$\frac{dy}{dx}(6, -5) = \frac{3(6)}{4(-5)} = -\frac{18}{20} = -\frac{9}{10}$$

Then the equation of the tangent line is

$$y - y_1 = m(x - x_1)$$

$$y + 5 = -\frac{9}{10}(x - 6)$$

$$y = -\frac{9}{10}(x - 6) - 5$$

$$y = -\frac{9}{10}x + \frac{54}{10} - 5$$

$$y = -\frac{9}{10}x + \frac{27}{5} - \frac{25}{5}$$

$$y = -\frac{9}{10}x + \frac{2}{5}$$

■ 4. Use implicit differentiation to find the equation of the tangent line to $2x + 3y - 5 = \ln(x^5 + y^5)$ at $(1,0)$.

Solution:

Use implicit differentiation to take the derivative of both sides.

$$2 + 3y' = \frac{1}{x^5 + y^5}(5x^4 + 5y^4y')$$

$$2 + 3y' = \frac{5x^4}{x^5 + y^5} + \frac{5y^4y'}{x^5 + y^5}$$

$$3y' - \frac{5y^4y'}{x^5+y^5} = \frac{5x^4}{x^5+y^5} - 2$$

$$y'\left(3 - \frac{5y^4}{x^5+y^5}\right) = \frac{5x^4}{x^5+y^5} - 2$$

$$y'\left(\frac{3(x^5+y^5) - 5y^4}{x^5+y^5}\right) = \frac{5x^4 - 2(x^5+y^5)}{x^5+y^5}$$

$$y' = \frac{5x^4 - 2(x^5+y^5)}{x^5+y^5}\left(\frac{x^5+y^5}{3(x^5+y^5) - 5y^4}\right)$$

$$y' = \frac{5x^4 - 2(x^5+y^5)}{3(x^5+y^5) - 5y^4}$$

Evaluate y' at $(1,0)$ to find the slope of the tangent line.

$$\frac{dy}{dx}(1,0) = \frac{5(1)^4 - 2(1^5+0^5)}{3(1^5+0^5) - 5(0)^4} = \frac{5(1) - 2(1+0)}{3(1+0) - 0} = \frac{5-2}{3} = 1$$

Then the equation of the tangent line is

$$y - y_1 = m(x - x_1)$$

$$y + 0 = 1(x - 1)$$

$$y = x - 1$$

■ 5. Use implicit differentiation to find the equations of the tangent and normal line to $\cos x = \sin(2y) + 9$ at $(\pi/2, \pi)$.

Solution:

Use implicit differentiation to take the derivative of both sides.

$$-\sin x = \cos(2y)(2y')$$

$$-\sin x = 2y' \cos(2y)$$

$$y' = -\frac{\sin x}{2\cos(2y)}$$

Evaluate y' at $(\pi/2,\pi)$ to find the slope of the tangent line.

$$y'\left(\frac{\pi}{2},\pi\right) = -\frac{\sin\frac{\pi}{2}}{2\cos(2\pi)} = -\frac{1}{2(1)} = -\frac{1}{2}$$

Then the equation of the tangent line is

$$y - y_1 = m(x - x_1)$$

$$y - \pi = -\frac{1}{2}\left(x - \frac{\pi}{2}\right)$$

$$y - \pi = -\frac{1}{2}x + \frac{\pi}{4}$$

$$y = -\frac{1}{2}x + \frac{5\pi}{4}$$

Since the normal line is perpendicular to the tangent line at the point of tangency, the slope of the normal line is 2 (the negative reciprocal of $-1/2$), so the equation of the normal line is

$$y - \pi = 2\left(x - \frac{\pi}{2}\right)$$

$$y - \pi = 2x - \pi$$

$$y = 2x$$

■ 6. Use implicit differentiation to find the equation of the tangent line to $4x^2 - xy + y^2 = 6$ at the points in the second and third quadrant when $x = -1$.

Solution:

First we need to find points on the curve in second and third quadrant at $x = -1$.

$$4(-1)^2 - (-1)y + y^2 = 6$$

$$4 + y + y^2 = 6$$

$$y^2 + y - 2 = 0$$

$$(y + 2)(y - 1) = 0$$

$y = -2$ and $y = 1$

Therefore, we get two points $(-1, -2)$ and $(-1,1)$.

Now use implicit differentiation to take the derivative of both sides of $4x^2 - xy + y^2 = 6$.

$$8x - y - xy' + 2yy' = 0$$

$$2yy' - xy' = y - 8x$$

$$y'(2y - x) = y - 8x$$

$$y' = \frac{y - 8x}{2y - x}$$

Evaluate y' at $(-1, -2)$.

$$y'(-1, -2) = \frac{-2 - 8(-1)}{2(-2) - (-1)} = \frac{-2 + 8}{-4 + 1} = \frac{6}{-3} = -2$$

Then the equation of the tangent line at $(-1, -2)$ is

$$y - y_1 = m(x - x_1)$$

$$y + 2 = -2(x + 1)$$

$$y + 2 = -2x - 2$$

$$y = -2x - 4$$

Evaluate y' at $(-1,1)$.

$$y'(-1,1) = \frac{1 - 8(-1)}{2(1) - (-1)} = \frac{1 + 8}{2 + 1} = \frac{9}{3} = 3$$

Then the equation of the tangent line at $(-1,1)$ is

$$y - y_1 = m(x - x_1)$$

$$y - 1 = 3(x + 1)$$

$$y - 1 = 3x + 3$$

$$y = 3x + 4$$

HIGHER-ORDER DERIVATIVES

1. Find the second and third derivatives of the function at $x = -1$.

$$y = 2x^5 - 3x^4 + x^3 + x^2 - 7$$

Solution:

Find the first derivative.

$$y = 2x^5 - 3x^4 + x^3 + x^2 - 7$$

$$y' = 10x^4 - 12x^3 + 3x^2 + 2x$$

Now we can find the second derivative by taking the derivative of the first derivative.

$$y'' = (y')' = (10x^4 - 12x^3 + 3x^2 + 2x)'$$

$$y'' = 40x^3 - 36x^2 + 6x + 2$$

Evaluate the second derivative at $x = -1$.

$$y''(-1) = 40(-1)^3 - 36(-1)^2 + 6(-1) + 2$$

$$y''(-1) = -40 - 36 - 6 + 2$$

$$y''(-1) = -80$$

Find the third derivative by taking the derivative of the second derivative.

$$y''' = (y'')' = (40x^3 - 36x^2 + 6x + 2)'$$

$$y''' = 120x^2 - 72x + 6$$

Evaluate the third derivative at $x = -1$.

$$y'''(-1) = 120(-1)^2 - 72(-1) + 6$$

$$y'''(-1) = 120 + 72 + 6$$

$$y'''(-1) = 198$$

■ 2. Find the second derivative of the function $y = -3x^{\frac{2}{3}} + x^{-\frac{1}{2}}$.

Solution:

Find the first derivative.

$$y = -3x^{\frac{2}{3}} + x^{-\frac{1}{2}}$$

$$y' = -3\left(\frac{2}{3}x^{-\frac{1}{3}}\right) - \frac{1}{2}x^{-\frac{3}{2}}$$

$$y' = -2x^{-\frac{1}{3}} - \frac{1}{2}x^{-\frac{3}{2}}$$

Now we can find the second derivative by taking the derivative of the first derivative.

$$y'' = (y')' = \left(-2x^{-\frac{1}{3}} - \frac{1}{2}x^{-\frac{3}{2}}\right)'$$

$$y'' = -2\left(-\frac{1}{3}\right)x^{-\frac{4}{3}} - \frac{1}{2}\left(-\frac{3}{2}\right)x^{-\frac{5}{2}}$$

$$y'' = \frac{2}{3}x^{-\frac{4}{3}} + \frac{3}{4}x^{-\frac{5}{2}}$$

3. Find the second derivative of the function.

$$y = -3x^7 \sin x$$

Solution:

Find the first derivative.

$$y = -3x^7 \sin x$$

$$y' = -21x^6 \sin x + (-3x^7 \cos x)$$

$$y' = -21x^6 \sin x - 3x^7 \cos x$$

Now we can find the second derivative by taking the derivative of the first derivative.

$$y'' = (y')' = (-21x^6 \sin x - 3x^7 \cos x)'$$

$$y'' = -126x^5 \sin x + (-21x^6 \cos x) - (21x^6 \cos x + 3x^7(-\sin x))$$

$$y'' = -126x^5 \sin x - 21x^6 \cos x - (21x^6 \cos x - 3x^7 \sin x)$$

$$y'' = -126x^5 \sin x - 21x^6 \cos x - 21x^6 \cos x + 3x^7 \sin x$$

$$y'' = -126x^5 \sin x - 42x^6 \cos x + 3x^7 \sin x$$

$$y'' = 3x^7 \sin x - 42x^6 \cos x - 126x^5 \sin x$$

$$y'' = 3x^5(x^2 \sin x - 14x \cos x - 42 \sin x)$$

■ 4. Find the second and the third derivatives of the function.

$$y = \ln(x^5\sqrt{x})$$

Solution:

Find the first derivative.

$$y = \ln(x^5\sqrt{x})$$

$$\frac{dy}{dx} = \frac{1}{x^5\sqrt{x}}\left(5x^4\sqrt{x} + x^5\frac{1}{2\sqrt{x}}\right)$$

$$\frac{dy}{dx} = \frac{5x^4\sqrt{x}}{x^5\sqrt{x}} + \frac{\frac{x^5}{2\sqrt{x}}}{x^5\sqrt{x}}$$

$$\frac{dy}{dx} = \frac{5}{x} + \frac{1}{2x}$$

$$\frac{dy}{dx} = \frac{10}{2x} + \frac{1}{2x}$$

$$\frac{dy}{dx} = \frac{11}{2x}$$

Now we can find the second derivative by taking the derivative of the first derivative.

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{11}{2x}\right)$$

$$\frac{d^2y}{dx^2} = \frac{11}{2}(-1)x^{-2}$$

$$\frac{d^2y}{dx^2} = -\frac{11}{2x^2}$$

Find the third derivative by taking the derivative of the second derivative.

$$\frac{d}{dx}\left(\frac{d^2y}{dx^2}\right) = \frac{d}{dx}\left(-\frac{11}{2x^2}\right)$$

$$\frac{d^3y}{dx^3} = -\frac{11}{2}(-2)x^{-3}$$

$$\frac{d^3y}{dx^3} = 11x^{-3}$$

$$\frac{d^3y}{dx^3} = \frac{11}{x^3}$$

■ 5. Find the second derivative of the function.

$$y = \frac{2x}{\sin(x^2)}$$

Solution:

Find the first derivative.

$$y = \frac{2x}{\sin(x^2)}$$

$$y' = \frac{2\sin(x^2) - 2x\cos(x^2)(2x)}{\sin^2(x^2)}$$

$$y' = \frac{2\sin(x^2) - 4x^2\cos(x^2)}{\sin^2(x^2)}$$

$$y' = \frac{2\sin(x^2)}{\sin^2(x^2)} - \frac{4x^2\cos(x^2)}{\sin^2(x^2)}$$

$$y' = \frac{2}{\sin(x^2)} - \frac{4x^2\cot(x^2)}{\sin(x^2)}$$

Now we can find the second derivative by taking the derivative of the first derivative.

$$y'' = (y')' = \left(\frac{2}{\sin(x^2)} - \frac{4x^2\cot(x^2)}{\sin(x^2)}\right)'$$

$$y'' = \frac{(0)\sin(x^2) - 2\cos(x^2)(2x)}{\sin^2(x^2)}$$

$$-\frac{(8x\cot(x^2) + 4x^2(-\csc^2(x^2))(2x))\sin(x^2) - 4x^2\cot(x^2)\cos(x^2)(2x)}{\sin^2(x^2)}$$

$$y'' = -\frac{4x\cos(x^2)}{\sin^2(x^2)} - \frac{(8x\cot(x^2) - 8x^3\csc^2(x^2))\sin(x^2) - 8x^3\cot(x^2)\cos(x^2)}{\sin^2(x^2)}$$

$$y'' = -\frac{4x\cot(x^2)}{\sin(x^2)} - \frac{8x\sin(x^2)\cot(x^2) - 8x^3\sin(x^2)\csc^2(x^2) - 8x^3\cos(x^2)\cot(x^2)}{\sin^2(x^2)}$$

$$y'' = -\frac{4x\cot(x^2)}{\sin(x^2)} - \frac{8x\sin(x^2)\cot(x^2)}{\sin^2(x^2)} + \frac{8x^3\sin(x^2)\csc^2(x^2)}{\sin^2(x^2)} + \frac{8x^3\cos(x^2)\cot(x^2)}{\sin^2(x^2)}$$

$$y'' = -\frac{4x\cot(x^2)}{\sin(x^2)} - \frac{8x\cot(x^2)}{\sin(x^2)} + \frac{8x^3\csc^2(x^2)}{\sin(x^2)} + \frac{8x^3\cot^2(x^2)}{\sin(x^2)}$$

$$y'' = -\frac{12x\cot(x^2)}{\sin(x^2)} + \frac{8x^3\csc^2(x^2)}{\sin(x^2)} + \frac{8x^3\cot^2(x^2)}{\sin(x^2)}$$

$$y'' = \frac{8x^3\cot^2(x^2) + 8x^3\csc^2(x^2) - 12x\cot(x^2)}{\sin(x^2)}$$

■ 6. Find the second derivative of the function at $x = 0$.

$$y = \frac{e^x}{4x - 9}$$

Solution:

Find the first derivative.

$$y = \frac{e^x}{4x-9}$$

$$\frac{dy}{dx} = \frac{e^x(4x-9) - e^x(4)}{(4x-9)^2}$$

$$\frac{dy}{dx} = \frac{e^x(4x-9-4)}{(4x-9)^2}$$

$$\frac{dy}{dx} = \frac{e^x(4x-13)}{(4x-9)^2}$$

Now we can find the second derivative by taking the derivative of the first derivative.

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{e^x(4x-13)}{(4x-9)^2}\right)$$

$$\frac{d^2y}{dx^2} = \frac{(e^x(4x-13)+e^x(4))(4x-9)^2 - e^x(4x-13)2(4x-9)(4)}{(4x-9)^4}$$

$$\frac{d^2y}{dx^2} = \frac{(e^x(4x-13)+4e^x)(4x-9)^2 - 8e^x(4x-13)(4x-9)}{(4x-9)^4}$$

$$\frac{d^2y}{dx^2} = \frac{(e^x(4x-13)+4e^x)(4x-9)^2}{(4x-9)^4} - \frac{8e^x(4x-13)(4x-9)}{(4x-9)^4}$$

$$\frac{d^2y}{dx^2} = \frac{e^x(4x-13)+4e^x}{(4x-9)^2} - \frac{8e^x(4x-13)}{(4x-9)^3}$$

$$\frac{d^2y}{dx^2} = \frac{e^x(4x-13+4)}{(4x-9)^2} - \frac{8e^x(4x-13)}{(4x-9)^3}$$

$$\frac{d^2y}{dx^2} = \frac{e^x(4x-9)}{(4x-9)^2} - \frac{8e^x(4x-13)}{(4x-9)^3}$$

$$\frac{d^2y}{dx^2} = \frac{e^x}{4x-9} - \frac{8e^x(4x-13)}{(4x-9)^3}$$

Then evaluate the derivative at $x = 0$.

$$\frac{d^2y}{dx^2}(0) = \frac{e^0}{4(0)-9} - \frac{8e^0(4(0)-13)}{(4(0)-9)^3}$$

$$\frac{d^2y}{dx^2}(0) = \frac{1}{-9} - \frac{8(-13)}{(-9)^3}$$

$$\frac{d^2y}{dx^2}(0) = -\frac{1}{9} - \frac{104}{729}$$

$$\frac{d^2y}{dx^2}(0) = -\frac{185}{729}$$

SECOND DERIVATIVES WITH IMPLICIT DIFFERENTIATION

■ 1. Use implicit differentiation to find d^2y/dx^2.

$$2x^3 = 2y^2 + 4$$

Solution:

Use implicit differentiation to take the derivative of both sides.

$$6x^2 = 4y\frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{6x^2}{4y}$$

$$\frac{dy}{dx} = \frac{3x^2}{2y}$$

Use implicit differentiation again on both sides to find the second derivative.

$$\frac{d^2y}{dx^2} = \frac{(6x)(2y) - (3x^2)\left(2 \cdot \frac{dy}{dx}\right)}{(2y)^2}$$

$$\frac{d^2y}{dx^2} = \frac{12xy - 6x^2\frac{dy}{dx}}{4y^2}$$

$$\frac{d^2y}{dx^2} = \frac{6xy - 3x^2 \frac{dy}{dx}}{2y^2}$$

Substitute the first derivative for dy/dx and then simplify.

$$\frac{d^2y}{dx^2} = \frac{6xy - 3x^2 \left(\frac{3x^2}{2y}\right)}{2y^2}$$

$$\frac{d^2y}{dx^2} = \frac{6xy - \frac{9x^4}{2y}}{2y^2}$$

$$\frac{d^2y}{dx^2} = \frac{12xy - \frac{9x^4}{y}}{4y^2}$$

$$\frac{d^2y}{dx^2} = \frac{12xy - \frac{9x^4}{y}}{4y^2}$$

Multiply through the numerator and denominator by y to get rid of the fraction in the numerator.

$$\frac{d^2y}{dx^2} = \frac{12xy^2 - 9x^4}{4y^3}$$

■ 2. Use implicit differentiation to find d^2y/dx^2.

$$4x^2 = 2y^3 + 4y - 2$$

Solution:

Use implicit differentiation to take the derivative of both sides.

$$8x = 6y^2\frac{dy}{dx} + 4\frac{dy}{dx}$$

$$8x = (6y^2 + 4)\frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{8x}{6y^2 + 4}$$

$$\frac{dy}{dx} = \frac{4x}{3y^2 + 2}$$

Use implicit differentiation again on both sides to find the second derivative.

$$\frac{d^2y}{dx^2} = \frac{(4)(3y^2 + 2) - (4x)\left(6y \cdot \frac{dy}{dx}\right)}{(3y^2 + 2)^2}$$

$$\frac{d^2y}{dx^2} = \frac{12y^2 + 8 - 24xy\frac{dy}{dx}}{(3y^2 + 2)^2}$$

Substitute the first derivative for dy/dx and then simplify.

$$\frac{d^2y}{dx^2} = \frac{12y^2 + 8 - 24xy\left(\frac{4x}{3y^2 + 2}\right)}{(3y^2 + 2)^2}$$

$$\frac{d^2y}{dx^2} = \frac{12y^2 + 8 - \frac{96x^2y}{3y^2 + 2}}{(3y^2 + 2)^2}$$

Multiply through the numerator and denominator by $3y^2+2$ to get rid of the fraction in the numerator.

$$\frac{d^2y}{dx^2}=\frac{12y^2(3y^2+2)+8(3y^2+2)-96x^2y}{(3y^2+2)^3}$$

$$\frac{d^2y}{dx^2}=\frac{(12y^2+8)(3y^2+2)-96x^2y}{(3y^2+2)^3}$$

$$\frac{d^2y}{dx^2}=\frac{4(3y^2+2)(3y^2+2)-96x^2y}{(3y^2+2)^3}$$

$$\frac{d^2y}{dx^2}=\frac{4(3y^2+2)^2-96x^2y}{(3y^2+2)^3}$$

■ 3. Use implicit differentiation to find d^2y/dx^2 at $(0,3)$.

$$3x^2+3y^2=27$$

Solution:

Rewrite the equation.

$$3x^2+3y^2=27$$

$$x^2+y^2=9$$

Use implicit differentiation to take the derivative of both sides.

$$2x + 2y\frac{dy}{dx} = 0$$

$$2y\frac{dy}{dx} = -2x$$

$$\frac{dy}{dx} = -\frac{2x}{2y}$$

$$\frac{dy}{dx} = -\frac{x}{y}$$

Use implicit differentiation again on both sides to find the second derivative.

$$\frac{d^2y}{dx^2} = -\frac{(1)(y) - (x)(1)\left(\frac{dy}{dx}\right)}{y^2}$$

$$\frac{d^2y}{dx^2} = -\frac{y - x\frac{dy}{dx}}{y^2}$$

Substitute the first derivative for dy/dx and then simplify.

$$\frac{d^2y}{dx^2} = -\frac{y - x\left(-\frac{x}{y}\right)}{y^2}$$

$$\frac{d^2y}{dx^2} = -\frac{y + \frac{x^2}{y}}{y^2}$$

Multiply through the numerator and denominator by y to get rid of the fraction in the numerator.

$$\frac{d^2y}{dx^2} = -\frac{y^2 + x^2}{y^3}$$

$$\frac{d^2y}{dx^2} = \frac{-x^2 - y^2}{y^3}$$

Evaluate the second derivative at $(0,3)$.

$$\frac{d^2y}{dx^2}(0,3) = \frac{-0^2 - 3^2}{3^3} = \frac{-9}{27} = -\frac{1}{3}$$

■ 4. Use implicit differentiation to find d^2y/dx^2 at $(2,1)$.

$$e^{x-2y} = 2x - y$$

Solution:

Use implicit differentiation to take the derivative of both sides.

$$e^{x-2y}\left(1 - 2\frac{dy}{dx}\right) = 2 - \frac{dy}{dx}$$

$$e^{x-2y} - 2e^{x-2y}\frac{dy}{dx} = 2 - \frac{dy}{dx}$$

$$\frac{dy}{dx} - 2e^{x-2y}\frac{dy}{dx} = 2 - e^{x-2y}$$

$$\frac{dy}{dx}(1 - 2e^{x-2y}) = 2 - e^{x-2y}$$

$$\frac{dy}{dx}=\frac{2-e^{x-2y}}{1-2e^{x-2y}}$$

Use implicit differentiation again on both sides to find the second derivative.

$$\frac{d^2y}{dx^2}=\frac{-e^{x-2y}\left(1-2\frac{dy}{dx}\right)(1-2e^{x-2y})-(2-e^{x-2y})(-2e^{x-2y})\left(1-2\frac{dy}{dx}\right)}{(1-2e^{x-2y})^2}$$

$$\frac{d^2y}{dx^2}=\frac{\left(1-2\frac{dy}{dx}\right)e^{x-2y}[-(1-2e^{x-2y})-(2-e^{x-2y})(-2)]}{(1-2e^{x-2y})^2}$$

$$\frac{d^2y}{dx^2}=\frac{\left(1-2\frac{dy}{dx}\right)e^{x-2y}(-1+2e^{x-2y}+4-2e^{x-2y})}{(1-2e^{x-2y})^2}$$

$$\frac{d^2y}{dx^2}=\frac{\left(1-2\frac{dy}{dx}\right)e^{x-2y}(3)}{(1-2e^{x-2y})^2}$$

$$\frac{d^2y}{dx^2}=\frac{3e^{x-2y}\left(1-2\frac{dy}{dx}\right)}{(1-2e^{x-2y})^2}$$

Substitute the first derivative for dy/dx and then simplify.

$$\frac{d^2y}{dx^2}=\frac{3e^{x-2y}\left(1-2\left(\frac{2-e^{x-2y}}{1-2e^{x-2y}}\right)\right)}{(1-2e^{x-2y})^2}$$

$$\frac{d^2y}{dx^2}=\frac{3e^{x-2y}\left(\frac{1-2e^{x-2y}-2(2-e^{x-2y})}{1-2e^{x-2y}}\right)}{(1-2e^{x-2y})^2}$$

$$\frac{d^2y}{dx^2} = \frac{3e^{x-2y}\left(\frac{1-2e^{x-2y}-4+2e^{x-2y}}{1-2e^{x-2y}}\right)}{(1-2e^{x-2y})^2}$$

$$\frac{d^2y}{dx^2} = \frac{3e^{x-2y}\left(\frac{-3}{1-2e^{x-2y}}\right)}{(1-2e^{x-2y})^2}$$

$$\frac{d^2y}{dx^2} = \frac{-\frac{9e^{x-2y}}{1-2e^{x-2y}}}{(1-2e^{x-2y})^2}$$

$$\frac{d^2y}{dx^2} = -\frac{9e^{x-2y}}{(1-2e^{x-2y})^3}$$

Evaluate the second derivative at $(2,1)$.

$$\frac{d^2y}{dx^2}(2,1) = -\frac{9e^{2-2(1)}}{(1-2e^{2-2(1)})^3} = -\frac{9e^0}{(1-2e^0)^3} = -\frac{9}{(1-2)^3} = -\frac{9}{-1} = 9$$

■ 5. Use implicit differentiation to find y''.

$$y \sin x = 7 - 2y^2$$

Solution:

Use implicit differentiation to take the derivative of both sides.

$$y' \sin x + y \cos x = -4yy'$$

$$y' \sin x + 4yy' = -y \cos x$$

$$y'(\sin x + 4y) = -y\cos x$$

$$y' = -\frac{y\cos x}{\sin x + 4y}$$

Use implicit differentiation again on both sides to find the second derivative.

$$y'' = -\frac{[y'\cos x + y(-\sin x)](\sin x + 4y) - (y\cos x)(\cos x + 4y')}{(\sin x + 4y)^2}$$

$$y'' = -\frac{(y'\cos x - y\sin x)(\sin x + 4y) - (y\cos x)(\cos x + 4y')}{(\sin x + 4y)^2}$$

$$y'' = -\frac{y'\sin x\cos x + 4yy'\cos x - y\sin^2 x - 4y^2\sin x - y\cos^2 x - 4yy'\cos x}{(\sin x + 4y)^2}$$

$$y'' = -\frac{y'\sin x\cos x - y(\sin^2 x + \cos^2 x) - 4y^2\sin x}{(\sin x + 4y)^2}$$

$$y'' = -\frac{y'\sin x\cos x - y - 4y^2\sin x}{(\sin x + 4y)^2}$$

Substitute the first derivative for y' and then simplify.

$$y'' = -\frac{\left(-\frac{y\cos x}{\sin x + 4y}\right)\sin x\cos x - y - 4y^2\sin x}{(\sin x + 4y)^2}$$

$$y'' = \frac{\left(\frac{y\cos x}{\sin x + 4y}\right)\sin x\cos x + y + 4y^2\sin x}{(\sin x + 4y)^2}$$

$$y'' = \frac{\left(\frac{y\cos x}{\sin x + 4y}\right)\sin x \cos x}{(\sin x + 4y)^2} + \frac{y + 4y^2 \sin x}{(\sin x + 4y)^2}$$

$$y'' = \frac{\frac{y \sin x \cos^2 x}{\sin x + 4y}}{(\sin x + 4y)^2} + \frac{y + 4y^2 \sin x}{(\sin x + 4y)^2}$$

$$y'' = \frac{y \sin x \cos^2 x}{(\sin x + 4y)^3} + \frac{y + 4y^2 \sin x}{(\sin x + 4y)^2}$$

■ 6. Use implicit differentiation to find y'' at $(0,3)$.

$$e^{2y} - 2x = y^4 - 2$$

Solution:

Use implicit differentiation to take the derivative of both sides.

$$e^{2y}(2y') - 2 = 4y^3y'$$

$$2e^{2y}y' - 4y^3y' = 2$$

$$y'(2e^{2y} - 4y^3) = 2$$

$$y' = \frac{2}{2e^{2y} - 4y^3}$$

$$y' = \frac{1}{e^{2y} - 2y^3}$$

Use implicit differentiation again on both sides to find the second derivative.

$$y'' = \frac{(0)(e^{2y} - 2y^3) - (1)(e^{2y}(2y') - 6y^2y')}{(e^{2y} - 2y^3)^2}$$

$$y'' = \frac{-2e^{2y}y' + 6y^2y'}{(e^{2y} - 2y^3)^2}$$

$$y'' = -\frac{2y'(e^{2y} - 3y^2)}{(e^{2y} - 2y^3)^2}$$

Substitute the first derivative for y' and then simplify.

$$y'' = -\frac{2\left(\frac{1}{e^{2y} - 2y^3}\right)(e^{2y} - 3y^2)}{(e^{2y} - 2y^3)^2}$$

$$y'' = -\frac{\left(\frac{2(e^{2y} - 3y^2)}{e^{2y} - 2y^3}\right)}{(e^{2y} - 2y^3)^2}$$

$$y'' = -\frac{2(e^{2y} - 3y^2)}{(e^{2y} - 2y^3)^3}$$

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