Topic: Making the function continuous

Question: Determine whether the function is continuous at $x = 1/2$. If it's discontinuous, redefine the function to make it continuous.

$$f(x) = \begin{cases} |2x-1| & x < \frac{1}{2} \\ \frac{3}{4} & x = \frac{1}{2} \\ \frac{2x-1}{2} & x > \frac{1}{2} \end{cases}$$

Answer choices:

A The function is continuous at $x = 1/2$.

B The function is discontinuous at $x = 1/2$ and the discontinuity is non-removable.

C The function is discontinuous at $x = 1/2$. The discontinuity can be removed by redefining the function as

$$f(x) = \begin{cases} |2x-1| & x < \frac{1}{2} \\ \frac{1}{2} & x = \frac{1}{2} \\ \frac{2x-1}{2} & x > \frac{1}{2} \end{cases}$$

D The function is discontinuous at $x = 1/2$. The discontinuity can be removed by redefining the function as

$$f(x) = \begin{cases} |2x-1| & x < \frac{1}{2} \\ 0 & x = \frac{1}{2} \\ \frac{2x-1}{2} & x > \frac{1}{2} \end{cases}$$

Solution: D

In order for the function to be continuous at $x = 1/2$,

- the left-hand limit must exist,
- the right-hand limit must exist,
- the left- and right-hand limits must be equal to one another, and
- the general limit must be equal to the value of the function at $x = 1/2$.

We already know that the value of the function at $x = 1/2$ is $3/4$, because that's the second "piece" of the piecewise-defined function we were given.

Now we just need to show that the left- and right-hand limits both exist and that they're both equal to $3/4$. If they exist but aren't equal to $3/4$, then we'll have to "plug the hole" and remove the discontinuity by redefining the function at $x = 1/2$.

To look at the left-hand limit, we'll use the first "piece" of our piecewise-defined function, because it defines the function to the left of $x = 1/2$ (the domain of that piece is $x < 1/2$).

$$\lim_{x \to (1/2)^-} |2x - 1|$$

Since the domain of $|2x - 1|$ is $x < 1/2$, we know that no matter what value in the domain we plug in for, we're always going to get a negative value for $2x - 1$. That means we can take away the absolute value bars as long as we put a negative sign in front of $2x - 1$.

$$\lim_{x \to (1/2)^-} -(2x - 1)$$

$$\lim_{x \to (1/2)^-} 1 - 2x$$

$$1 - 2\left(\frac{1}{2}\right)$$

$$1 - 1$$

$$0$$

We know now that the left-hand limit exists, and that it's equal to 0. Let's look at the right-hand limit by using the third "piece" of the piecewise-defined function, since it defines the function to the right of $x = 1/2$ (the domain of that piece is $x > 1/2$).

$$\lim_{x \to (1/2)^+} \frac{2x - 1}{2}$$

$$\frac{2\left(\frac{1}{2}\right)-1}{2}$$

$$\frac{1-1}{2}$$

$$0$$

We know now that the right-hand limit exists, and that it's equal to 0.

Since the left-hand limit exists at $x=1/2$, the right-hand limit exists at $x=1/2$, and the left- and right-hand limits are equal and therefore the general limit exists at $x=1/2$, but the general limit at $x=1/2$ isn't equal to the value of the function at $x=1/2$, the function is discontinuous and we need to redefine the function in order to make it continuous at $x=1/2$.

So we just redefine the value of the function at $x=1/2$ to be equal to the general limit at $x=1/2$ that we found earlier by taking the left- and right-hand limits at $x=1/2$.

$$f(x)=\begin{cases} |2x-1| & x<\frac{1}{2} \\ 0 & x=\frac{1}{2} \\ \frac{2x-1}{2} & x>\frac{1}{2} \end{cases}$$

Topic: Making the function continuous

Question: Determine whether the function is continuous at $x = 0$. If it's discontinuous, redefine the function to make it continuous.

$$f(x) = \begin{cases} \dfrac{\sqrt{4x+4}-2}{2x} & x > 0 \\ -2 & x = 0 \\ \dfrac{x}{x^2+2x} & x < 0 \end{cases}$$

Answer choices:

A The function is continuous at $x = 0$.

B The function is discontinuous at $x = 0$ and the discontinuity is non-removable.

C The function is discontinuous at $x = 0$. The discontinuity can be removed by redefining the function as

$$f(x) = \begin{cases} \dfrac{\sqrt{4x+4}-2}{2x} & x > 0 \\ \dfrac{1}{2} & x = 0 \\ \dfrac{x}{x^2+2x} & x < 0 \end{cases}$$

D The function is discontinuous at $x = 0$. The discontinuity can be removed by redefining the function as

$$f(x) = \begin{cases} \dfrac{\sqrt{4x+4}-2}{2x} & x > 0 \\ 0 & x = 0 \\ \dfrac{x}{x^2+2x} & x < 0 \end{cases}$$

Solution: C

In order for the function to be continuous at $x = 0$,

- the left-hand limit must exist,
- the right-hand limit must exist,
- the left- and right-hand limits must be equal to one another, and
- the general limit must be equal to the value of the function at $x = 0$.

We already know that the value of the function at $x = 0$ is -2, because that's the second "piece" of the piecewise-defined function we were given.

Now we just need to show that the left- and right-hand limits both exist and that they're both equal to -2. If they exist but aren't equal to -2, then we'll have to "plug the hole" and remove the discontinuity by redefining the function at $x = 0$.

To look at the left-hand limit, we'll use the third "piece" of the piecewise-defined function, because it defines the function to the left of $x = 0$ (the domain of that piece is $x < 0$).

$$\lim_{x \to 0^-} \frac{x}{x^2 + 2x}$$

$$\lim_{x \to 0^-} \frac{x}{x(x + 2)}$$

$$\lim_{x \to 0^-} \frac{1}{x + 2}$$

$$\frac{1}{0 + 2}$$

$$\frac{1}{2}$$

We know now that the left-hand limit exists, and that it's equal to $1/2$. Let's look at the right-hand limit by using the first "piece" of our piecewise-defined function, since it defines the function to the right of $x = 0$ (the domain of that piece is $x > 0$).

$$\lim_{x \to 0^+} \frac{\sqrt{4x + 4} - 2}{2x}$$

$$\lim_{x \to 0^+} \frac{\sqrt{4x + 4} - 2}{2x} \left(\frac{\sqrt{4x + 4} + 2}{\sqrt{4x + 4} + 2} \right)$$

$$\lim_{x \to 0^+} \frac{4x + 4 - 4}{2x(\sqrt{4x + 4} + 2)}$$

$$\lim_{x \to 0^+} \frac{2}{\sqrt{4x + 4} + 2}$$

$$\lim_{x\to 0^+} \frac{2}{\sqrt{4(x+1)}+2}$$

$$\lim_{x\to 0^+} \frac{1}{\sqrt{x+1}+1}$$

$$\frac{1}{\sqrt{0+1}+1}$$

$$\frac{1}{1+1}$$

$$\frac{1}{2}$$

We know now that the right-hand limit exists, and that it’s equal to $1/2$.

Since the left-hand limit exists at $x = 0$, the right-hand limit exists at $x = 0$, and the left- and right-hand limits are equal and therefore the general limit exists at $x = 0$, but the general limit at $x = 0$ isn’t equal to the value of the function at $x = 0$, the function is discontinuous and we need to redefine the function in order to make it continuous at $x = 0$.

So we just redefine the value of the function at $x = 0$ to be equal to the general limit at $x = 0$ that we found earlier by taking the left- and right-hand limits at $x = 0$.

$$f(x) = \begin{cases} \frac{\sqrt{4x+4}-2}{2x} & x > 0 \\ \frac{1}{2} & x = 0 \\ \frac{x}{x^2+2x} & x < 0 \end{cases}$$

Topic: Making the function continuous

Question: Determine whether the function is continuous at $x = 0$. If it's discontinuous, redefine the function to make it continuous.

$$f(x) = \begin{cases} \dfrac{\sqrt{\cos x} - 1}{\sin^2 x} & x > 0 \\ 0 & x = 0 \\ \dfrac{4 - x}{x^2 - 2x - 8} & x < 0 \end{cases}$$

Answer choices:

A The function is continuous at $x = 0$.

B The function is discontinuous at $x = 0$ and the discontinuity is non-removable.

C The function is discontinuous at $x = 0$. The discontinuity can be removed by redefining the function as

$$f(x) = \begin{cases} \dfrac{\sqrt{\cos x} - 1}{\sin^2 x} & x > 0 \\ \dfrac{1}{2} & x = 0 \\ \dfrac{4 - x}{x^2 - 2x - 8} & x < 0 \end{cases}$$

D The function is discontinuous at $x = 0$. The discontinuity can be removed by redefining the function as

$$f(x) = \begin{cases} \frac{\sqrt{\cos x} - 1}{\sin^2 x} & x > 0 \\ -\frac{1}{2} & x = 0 \\ \frac{4 - x}{x^2 - 2x - 8} & x < 0 \end{cases}$$

Solution: B

In order for the function to be continuous at $x = 0$,

- the left-hand limit must exist,
- the right-hand limit must exist,
- the left- and right-hand limits must be equal to one another, and
- the general limit must be equal to the value of the function at $x = 0$.

We already know that the value of the function at $x = 0$ is 0, because that's the second "piece" of the piecewise-defined function we were given.

Now we just need to show that the left- and right-hand limits both exist and that they're both equal to 0. If they exist but aren't equal to 0, then we'll have to "plug the hole" and remove the discontinuity by redefining the function at $x = 0$.

To look at the left-hand limit, we'll use the third "piece" of the piecewise-defined function, because it defines the function to the left of $x = 0$ (the domain of that piece is $x < 0$).

$$\lim_{x\to 0^-} \frac{4-x}{x^2-2x-8}$$

$$\lim_{x\to 0^-} \frac{4-x}{(x-4)(x+2)}$$

$$\lim_{x\to 0^-} -\frac{x-4}{(x-4)(x+2)}$$

$$\lim_{x\to 0^-} -\frac{1}{x+2}$$

$$-\frac{1}{0+2}$$

$$-\frac{1}{2}$$

We know now that the left-hand limit exists, and that it's equal to $-1/2$. Let's look at the right-hand limit by using the first "piece" of the piecewise-defined function, since it defines the function to the right of $x=0$ (the domain of that piece is $x>0$). Substitute using a Pythagorean identity.

$$\lim_{x\to 0^+} \frac{\sqrt{\cos x}-1}{\sin^2 x}$$

$$\lim_{x\to 0^+} \frac{\sqrt{\cos x}-1}{1-\cos^2 x}$$

Factor the denominator in order to simplify the fraction and then evaluate the limit.

$$\lim_{x\to 0^+} \frac{\sqrt{\cos x}-1}{(1+\cos x)(1-\cos x)}$$

$$\lim_{x\to 0^+} \frac{\sqrt{\cos x}-1}{(1+\cos x)\left[(1+\sqrt{\cos x})(1-\sqrt{\cos x})\right]}$$

$$\lim_{x\to 0^+} -\frac{1-\sqrt{\cos x}}{(1+\cos x)(1+\sqrt{\cos x})(1-\sqrt{\cos x})}$$

$$\lim_{x\to 0^+} -\frac{1}{(1+\cos x)(1+\sqrt{\cos x})}$$

$$-\frac{1}{(1+\cos(0))(1+\sqrt{\cos(0)})}$$

$$-\frac{1}{(1+1)(1+1)}$$

$$-\frac{1}{4}$$

We know now that the right-hand limit exists, and that it's equal to $-1/4$.

Since the left-hand limit exists at $x=0$, the right-hand limit exists at $x=0$, but the left- and right-hand limits are not equal to another, that means the general limit does not exist at $x=0$. Furthermore, because the one-sided limits are unequal, it means the discontinuity is non-removable.