Topic: Trigonometric substitution with sine

Question: Use trigonometric substitution to evaluate the integral.

$$\int_{0}^{\frac{\sqrt{2}}{2}} \frac{1}{\sqrt{1-x^2}}\ dx$$

Answer choices:

A π

B $\frac{\pi}{2}$

C $\frac{\pi}{3}$

D $\frac{\pi}{4}$

Solution: D

We can use trigonometric substitution to evaluate the integral. Recognizing that

$$a^2 - u^2 = 1 - x^2$$

we get

$$u = x$$

$$a = 1$$

Knowing that

$$u = a \sin \theta$$

is the substitution we use for $a^2 - u^2$, we get

$$x = 1 \sin \theta$$

$$x = \sin \theta$$

$$dx = \cos \theta \ d\theta$$

$$\theta = \sin^{-1} x$$

Plugging these into the integral we get

$$\int_0^{\frac{\sqrt{2}}{2}} \frac{1}{\sqrt{1 - x^2}} \ dx$$

$$\int_{x=0}^{x=\frac{\sqrt{2}}{2}} \frac{1}{\sqrt{1-\left(\sin\theta\right)^2}}\cos\theta\ d\theta$$

$$\int_{x=0}^{x=\frac{\sqrt{2}}{2}} \frac{\cos\theta}{\sqrt{1-\sin^2\theta}}\ d\theta$$

We know that $1-\sin^2 x=\cos^2 x$, so we'll make a substitution to simplify the integral.

$$\int_{x=0}^{x=\frac{\sqrt{2}}{2}} \frac{\cos\theta}{\sqrt{\cos^2\theta}}\ d\theta$$

$$\int_{x=0}^{x=\frac{\sqrt{2}}{2}} \frac{\cos\theta}{\cos\theta}\ d\theta$$

$$\int_{x=0}^{x=\frac{\sqrt{2}}{2}} \ d\theta$$

$$\theta\Big|_{x=0}^{x=\frac{\sqrt{2}}{2}}$$

Back-substituting for x before we evaluate over the interval, we get

$$\sin^{-1}x\Big|_0^{\frac{\sqrt{2}}{2}}$$

$$\sin^{-1}\left(\frac{\sqrt{2}}{2}\right)-\sin^{-1}(0)$$

$$\frac{\pi}{4} - 0$$

$$\frac{\pi}{4}$$

Topic: Trigonometric substitution with sine

Question: Use trigonometric substitution to evaluate the integral.

$$\int \frac{\sqrt{1-x^2}}{x}\ dx$$

Answer choices:

A $\quad \ln\left|\frac{1-\sqrt{1-x^2}}{x}\right| + \sqrt{1-x^2} + C$

B $\quad -\ln\left|\frac{\sqrt{1-x^2}}{x}\right| + \sqrt{1-x^2} + C$

C $\quad \ln\left|\frac{1+\sqrt{1-x^2}}{x}\right| + \sqrt{1-x^2} + C$

D $\quad \ln\left|\frac{\sqrt{1-x^2}}{x}\right| + \sqrt{1-x^2} + C$

Solution: A

We can use trigonometric substitution to evaluate the integral. Recognizing that

$$a^2 - u^2 = 1 - x^2$$

we get

$$u = x$$

$$a = 1$$

Knowing that

$$u = a\sin\theta$$

is the substitution we use for $a^2 - u^2$, we get

$$x = 1\sin\theta$$

$$x = \sin\theta$$

$$dx = \cos\theta\ d\theta$$

$$\theta = \sin^{-1}x$$

Plugging these into the integral we get

$$\int \frac{\sqrt{1 - x^2}}{x}\ dx$$

$$\int \frac{\sqrt{1 - (\sin\theta)^2}}{\sin\theta}\cos\theta\ d\theta$$

$$\int \frac{\cos\theta\sqrt{1-\sin^2\theta}}{\sin\theta}\ d\theta$$

We know that $1-\sin^2 x = \cos^2 x$, so we'll make a substitution to simplify the integral.

$$\int \frac{\cos\theta\sqrt{\cos^2\theta}}{\sin\theta}\ d\theta$$

$$\int \frac{\cos^2\theta}{\sin\theta}\ d\theta$$

$$\int \frac{1-\sin^2\theta}{\sin\theta}\ d\theta$$

$$\int \frac{1}{\sin\theta}-\frac{\sin^2\theta}{\sin\theta}\ d\theta$$

$$\int \csc\theta-\sin\theta\ d\theta$$

Knowing that

$$\int \csc\theta\ d\theta = \ln\left|\csc\theta-\cot\theta\right|+C$$

the integral becomes

$$\ln\left|\csc\theta-\cot\theta\right|-(-\cos\theta)+C$$

$$\ln\left|\csc\theta-\cot\theta\right|+\cos\theta+C$$

Back-substituting for x, we get

$$\ln \left| \csc\left(\sin^{-1} x\right) - \cot\left(\sin^{-1} x\right) \right| + \cos\left(\sin^{-1} x\right) + C$$

$$\ln \left| \frac{1}{x} - \frac{\sqrt{1-x^2}}{x} \right| + \sqrt{1-x^2} + C$$

$$\ln \left| \frac{1-\sqrt{1-x^2}}{x} \right| + \sqrt{1-x^2} + C$$

Topic: Trigonometric substitution with sine

Question: Use trigonometric substitution to evaluate the integral.

$$\int \frac{x^2}{\sqrt{9-4x^2}}\ dx$$

Answer choices:

A $\quad \dfrac{9}{8}\arcsin\dfrac{2x}{3} - \dfrac{x\sqrt{9-4x^2}}{8} + C$

B $\quad \dfrac{9}{8}\arcsin\dfrac{2x}{3} + \dfrac{x\sqrt{9-4x^2}}{8} + C$

C $\quad \dfrac{9}{16}\arcsin\dfrac{2x}{3} - \dfrac{x\sqrt{9-4x^2}}{8} + C$

D $\quad \dfrac{9}{16}\arcsin\dfrac{2x}{3} + \dfrac{x\sqrt{9-4x^2}}{8} + C$

Solution: C

We can use trigonometric substitution to evaluate the integral. Recognizing that

$$a^2 - u^2 = 9 - 4x^2$$

we get

$$u = 2x$$

$$a = 3$$

Knowing that

$$u = a \sin \theta$$

is the substitution we use for $a^2 - u^2$, we get

$$2x = 3 \sin \theta$$

$$x = \frac{3}{2} \sin \theta$$

$$dx = \frac{3}{2} \cos \theta \ d\theta$$

$$\frac{2x}{3} = \sin \theta$$

$$\theta = \sin^{-1} \frac{2x}{3}$$

Plugging these into the integral we get

$$\int \frac{x^2}{\sqrt{9-4x^2}}\ dx$$

$$\int \frac{\left(\frac{3}{2}\sin\theta\right)^2}{\sqrt{9-4\left(\frac{3}{2}\sin\theta\right)^2}}\left(\frac{3}{2}\cos\theta\ d\theta\right)$$

$$\int \frac{\frac{9}{4}\sin^2\theta}{\sqrt{9-4\left(\frac{9}{4}\sin^2\theta\right)}}\left(\frac{3}{2}\cos\theta\ d\theta\right)$$

$$\frac{27}{8}\int \frac{\sin^2\theta\cos\theta}{\sqrt{9-9\sin^2\theta}}\ d\theta$$

$$\frac{27}{8}\int \frac{\sin^2\theta\cos\theta}{\sqrt{9\left(1-\sin^2\theta\right)}}\ d\theta$$

We know that $1-\sin^2 x = \cos^2 x$, so we'll make a substitution to simplify the integral.

$$\frac{27}{8}\int \frac{\sin^2\theta\cos\theta}{\sqrt{9\cos^2\theta}}\ d\theta$$

$$\frac{27}{8}\int \frac{\sin^2\theta\cos\theta}{3\cos\theta}\ d\theta$$

$$\frac{9}{8}\int \sin^2\theta\ d\theta$$

$$\frac{9}{8}\int 1-\cos^2\theta \ d\theta$$

Because we know the trigonometric identity

$$\cos^2\theta=\frac{1}{2}+\frac{1}{2}\cos 2\theta$$

we can substitute into the integral.

$$\frac{9}{8}\int 1-\left(\frac{1}{2}+\frac{1}{2}\cos 2\theta\right) \ d\theta$$

$$\frac{9}{8}\int 1-\frac{1}{2}-\frac{1}{2}\cos 2\theta \ d\theta$$

$$\frac{9}{8}\int \frac{1}{2}-\frac{1}{2}\cos 2\theta \ d\theta$$

$$\frac{9}{8}\int \frac{1}{2}(1-\cos 2\theta) \ d\theta$$

$$\frac{9}{16}\int 1-\cos 2\theta \ d\theta$$

Integrate.

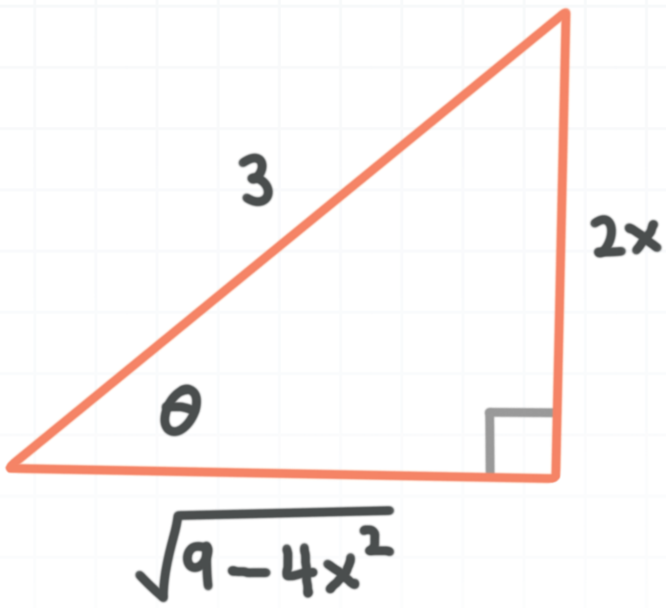
$$\frac{9}{16}\left(\theta-\frac{1}{2}\sin 2\theta\right)+C$$

The trigonometric identity $\sin(2x)=2\sin x\cos x$ lets us rewrite this value.

$$\frac{9}{16}\left(\theta-\frac{1}{2}(2\sin\theta\cos\theta)\right)+C$$

$$\frac{9}{16}\left(\theta - \sin\theta\cos\theta\right) + C$$

We need to remember the reference triangle



Because

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}}$$

$$\sin\theta = \frac{2x}{3}$$

and

$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}}$$

$$\cos\theta = \frac{\sqrt{9-4x^2}}{3}$$

and because we already know from earlier that

$$\theta = \sin^{-1} \frac{2x}{3}$$

we can make substitutions to put the value back in terms of x.

$$\frac{9}{16} \left[\sin^{-1} \frac{2x}{3} - \left(\frac{2x}{3} \right) \left(\frac{\sqrt{9-4x^2}}{3} \right) \right] + C$$

$$\frac{9}{16} \arcsin \frac{2x}{3} - \frac{9}{16} \cdot \frac{2x\sqrt{9-4x^2}}{9} + C$$

$$\frac{9}{16} \arcsin \frac{2x}{3} - \frac{x\sqrt{9-4x^2}}{8} + C$$