Calculus 2 Workbook Solutions

Taylor series

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TAYLOR SERIES

1. Find the third-degree Taylor polynomial and use it to approximate $f(5)$.

$$f(x) = 3\sqrt{x+1}$$

$n = 3$ and $a = 3$

Solution:

The formula for the Taylor polynomial of $f(x)$ at a is

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots + \frac{f^n(a)}{n!}(x-a)^n$$

Use the original function and the first three derivatives.

$$f(x) = 3(x+1)^{\frac{1}{2}} \qquad f(3) = 3(3+1)^{\frac{1}{2}} \qquad f(3) = 6$$

$$f'(x) = \frac{3}{2(x+1)^{\frac{1}{2}}} \qquad f'(3) = \frac{3}{2(3+1)^{\frac{1}{2}}} \qquad f'(3) = \frac{3}{4}$$

$$f''(x) = -\frac{3}{4(x+1)^{\frac{3}{2}}} \qquad f''(3) = -\frac{3}{4(3+1)^{\frac{3}{2}}} \qquad f''(3) = -\frac{3}{32}$$

$$f'''(x) = \frac{9}{8(x+1)^{\frac{5}{2}}} \qquad f'''(3) = \frac{9}{8(3+1)^{\frac{5}{2}}} \qquad f'''(3) = \frac{9}{256}$$

So the third-degree Taylor polynomial is

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$$f^{(3)}(x) = 6 + \frac{3}{4}(x-3) - \frac{3}{32 \cdot 2!}(x-3)^2 + \frac{9}{256 \cdot 3!}(x-3)^3$$

$$f^{(3)}(x) = 6 + \frac{3}{4}(x-3) - \frac{3}{64}(x-3)^2 + \frac{3}{512}(x-3)^3$$

Use the polynomial to estimate $f(5)$.

$$f^{(3)}(5) \approx 6 + \frac{3}{4}(5-3) - \frac{3}{64}(5-3)^2 + \frac{3}{512}(5-3)^3$$

$$f^{(3)}(5) \approx 6 + \frac{3}{4}(2) - \frac{3}{64}(4) + \frac{3}{512}(8)$$

$$f^{(3)}(5) \approx 6 + \frac{3}{2} - \frac{3}{16} + \frac{3}{64}$$

$$f^{(3)}(5) \approx 7.359$$

■ 2. Find the third-degree Taylor polynomial and use it to approximate $f(4)$.

$$f(x) = e^{2x} + 9$$

$n = 3$ and $a = 2$

Solution:

The formula for the Taylor polynomial of $f(x)$ at a is

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots + \frac{f^n(a)}{n!}(x-a)^n$$

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Use the original function and the first three derivatives.

$$f(x) = e^{2x} + 9 \qquad f(2) = e^{2(2)} + 9 \qquad f(2) = e^4 + 9$$

$$f'(x) = 2e^{2x} \qquad f'(2) = 2e^{2(2)} \qquad f'(2) = 2e^4$$

$$f''(x) = 4e^{2x} \qquad f''(2) = 4e^{2(2)} \qquad f''(2) = 4e^4$$

$$f'''(x) = 8e^{2x} \qquad f'''(2) = 8e^{2(2)} \qquad f'''(2) = 8e^4$$

So the third-degree Taylor polynomial is

$$f^{(3)}(x) = e^4 + 9 + 2e^4(x-2) + \frac{4e^4}{2!}(x-2)^2 + \frac{8e^4}{3!}(x-2)^3$$

$$f^{(3)}(x) = e^4 + 9 + 2e^4(x-2) + \frac{4e^4}{2}(x-2)^2 + \frac{8e^4}{6}(x-2)^3$$

$$f^{(3)}(x) = e^4 + 9 + 2e^4(x-2) + 2e^4(x-2)^2 + \frac{4e^4}{3}(x-2)^3$$

Use the polynomial to estimate $f(4)$.

$$f^{(3)}(4) = e^4 + 9 + 2e^4(4-2) + 2e^4(4-2)^2 + \frac{4e^4}{3}(4-2)^3$$

$$f^{(3)}(4) = e^4 + 9 + 2e^4(2) + 2e^4(2)^2 + \frac{4}{3}e^4(2)^3$$

$$f^{(3)}(4) = e^4 + 9 + 4e^4 + 8e^4 + \frac{32}{3}e^4$$

$$f^{(3)}(4) \approx 1{,}301.156$$

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3. Find the fourth-degree Taylor polynomial and use it to approximate $f(\pi/24)$.

$$f(x) = \sin(6x) + 5$$

$n = 4$ and $a = \dfrac{\pi}{12}$

Solution:

The formula for the Taylor polynomial of $f(x)$ at a is

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots + \frac{f^n(a)}{n!}(x-a)^n$$

Use the original function and the first four derivatives.

$$f(x) = \sin 6x + 5 \qquad f\left(\frac{\pi}{12}\right) = \sin\left(6\cdot\frac{\pi}{12}\right) + 5 \qquad f\left(\frac{\pi}{12}\right) = 6$$

$$f'(x) = 6\cos 6x \qquad f'\left(\frac{\pi}{12}\right) = 6\cos\left(6\cdot\frac{\pi}{12}\right) \qquad f'\left(\frac{\pi}{12}\right) = 0$$

$$f''(x) = -36\sin 6x \qquad f''\left(\frac{\pi}{12}\right) = -36\sin\left(6\cdot\frac{\pi}{12}\right) \qquad f''\left(\frac{\pi}{12}\right) = -36$$

$$f'''(x) = -216\cos 6x \qquad f'''\left(\frac{\pi}{12}\right) = -216\cos\left(6\cdot\frac{\pi}{12}\right) \qquad f'''\left(\frac{\pi}{12}\right) = 0$$

$$f''''(x) = 1{,}296\sin 6x \qquad f''''\left(\frac{\pi}{12}\right) = 1{,}296\sin\left(6\cdot\frac{\pi}{12}\right) \qquad f''''\left(\frac{\pi}{12}\right) = 1{,}296$$

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So the fourth-degree Taylor polynomial is

$$f^{(4)}(x) = 6 + 0\left(x - \frac{\pi}{12}\right) - \frac{36}{2!}\left(x - \frac{\pi}{12}\right)^2 + \frac{0}{3!}\left(x - \frac{\pi}{12}\right)^3 + \frac{1{,}296}{4!}\left(x - \frac{\pi}{12}\right)^4$$

$$f^{(4)}(x) = 6 + 0 - \frac{36}{2}\left(x - \frac{\pi}{12}\right)^2 + 0 + \frac{1{,}296}{24}\left(x - \frac{\pi}{12}\right)^4$$

$$f^{(4)}(x) = 6 + 0 - 18\left(x - \frac{\pi}{12}\right)^2 + 0 + 54\left(x - \frac{\pi}{12}\right)^4$$

$$f^{(4)}(x) = 6 - 18\left(x - \frac{\pi}{12}\right)^2 + 54\left(x - \frac{\pi}{12}\right)^4$$

Use the polynomial to estimate $f(\pi/24)$.

$$f^{(4)}\left(\frac{\pi}{24}\right) = 6 - 18\left(\frac{\pi}{24} - \frac{\pi}{12}\right)^2 + 54\left(\frac{\pi}{24} - \frac{\pi}{12}\right)^4$$

$$f^{(4)}\left(\frac{\pi}{24}\right) = 6 - 18\left(-\frac{\pi}{24}\right)^2 + 54\left(-\frac{\pi}{24}\right)^4$$

$$f^{(4)}\left(\frac{\pi}{24}\right) = 6 - 18\left(\frac{\pi^2}{576}\right) + 54\left(\frac{\pi^4}{331{,}776}\right)$$

$$f^{(4)}\left(\frac{\pi}{24}\right) \approx 6 - 0.30843 + 0.01585$$

$$f^{(4)}\left(\frac{\pi}{24}\right) \approx 5.707$$

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RADIUS AND INTERVAL OF CONVERGENCE OF A TAYLOR SERIES

1. Find the radius of convergence of the Taylor polynomial.

$$P_{(3)}(x) = 1 + 2(x - 3) + 4(x - 3)^2 + 8(x - 3)^3$$

Solution:

Rewrite the Taylor polynomial

$$f(x) = f(a) + f'(a)(x - a) + \frac{f''(a)}{2!}(x - a)^2 + \frac{f'''(a)}{3!}(x - a)^3 + \cdots + \frac{f^n(a)}{n!}(x - a)^n$$

as a power series in the form

$$f(x) = \sum_{n=1}^{\infty} \frac{f^n(a)}{n!}(x - a)^n = \sum_{n=0}^{\infty} \frac{f^{n-1}(a)}{(n - 1)!}(x - a)^{n-1}$$

Rewrite the Taylor polynomial.

$$P_{(3)}(x) = 1 + 2(x - 3) + 4(x - 3)^2 + 8(x - 3)^3$$

$$P_{(3)}(x) = 1(x - 3)^0 + 2(x - 3)^1 + 4(x - 3)^2 + 8(x - 3)^3$$

$$P_{(3)}(x) = 2^0(x - 3)^0 + 2^1(x - 3)^1 + 2^2(x - 3)^2 + 2^3(x - 3)^3$$

Then its power series representation is

$$P(x) = 2^0(x - 3)^0 + 2^1(x - 3)^1 + 2^2(x - 3)^2 + 2^3(x - 3)^3 + \cdots = \sum_{n=0}^{\infty} 2^n(x - 3)^n$$

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Apply the ratio test.

$$L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$$

$$L = \lim_{n\to\infty} \left| \frac{2^{n+1}(x-3)^{n+1}}{2^n(x-3)^n} \right|$$

$$L = \lim_{n\to\infty} \left| 2^{n+1-n}(x-3)^{n+1-n} \right|$$

$$L = \lim_{n\to\infty} \left| 2(x-3) \right|$$

$$L = \left| 2(x-3) \right|$$

Then set up an inequality.

$$\left| 2(x-3) \right| < 1$$

$$-1 < 2(x-3) < 1$$

$$-\frac{1}{2} < x - 3 < \frac{1}{2}$$

$$\frac{5}{2} < x < \frac{7}{2}$$

The interval of convergence spans from $5/2$ to $7/2$, which is a distance of 1 unit. The radius of convergence is will be half that distance, so the radius of convergence is $1/2$.

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2. Find the radius of convergence of the Taylor polynomial.

$$P_{(3)}(x) = 4 - 4(x-5) + 16(x-5)^2 - 64(x-5)^3$$

Solution:

Rewrite the Taylor polynomial

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots + \frac{f^n(a)}{n!}(x-a)^n$$

as a power series in the form

$$f(x) = \sum_{n=1}^{\infty} \frac{f^n(a)}{n!}(x-a)^n = \sum_{n=0}^{\infty} \frac{f^{n-1}(a)}{(n-1)!}(x-a)^{n-1}$$

Rewrite the Taylor polynomial.

$$P_{(3)}(x) = 4 - 4(x-5) + 16(x-5)^2 - 64(x-5)^3$$

$$P_{(3)}(x) = 4^1(x-5)^0 - 4^1(x-5)^1 + 4(4)(x-5)^2 - 4(4)^2(x-5)^3$$

Then its power series representation is

$$P(x) = 4 - 4^1(x-5)^1 + 4(4)(x-5)^2 - 4(4)^2(x-5)^3 + \cdots = 4 + \sum_{n=1}^{\infty} (-4)^n(x-5)^n$$

Apply the ratio test.

$$L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$$

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$$L = \lim_{n\to\infty} \left| \frac{(-4)^{n+1}(x-5)^{n+1}}{(-4)^n(x-5)^n} \right|$$

$$L = \lim_{n\to\infty} \left| (-4)^{n+1-n}(x-5)^{n+1-n} \right|$$

$$L = \lim_{n\to\infty} \left| (4)(x-5) \right|$$

$$L = \left| 4(x-5) \right|$$

Then set up an inequality.

$$\left| 4(x-5) \right| < 1$$

$$\left| 4x-20 \right| < 1$$

$$-1 < 4x - 20 < 1$$

$$19 < 4x < 21$$

$$\frac{19}{4} < x < \frac{21}{4}$$

The interval of convergence spans from $19/4$ to $21/4$, which is a distance of $2/4 = 1/2$ unit. The radius of convergence is will be half that distance, so the radius of convergence is $1/4$.

■ 3. Find the radius of convergence of the Taylor polynomial.

$$P_{(3)}(x) = \frac{1}{4} - \frac{1}{4}(x-4) + \frac{1}{8}(x-4)^2 - \frac{1}{24}(x-4)^3$$

Solution:

Rewrite the Taylor polynomial

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots + \frac{f^n(a)}{n!}(x-a)^n$$

as a power series in the form

$$f(x) = \sum_{n=1}^{\infty} \frac{f^n(a)}{n!}(x-a)^n = \sum_{n=0}^{\infty} \frac{f^{n-1}(a)}{(n-1)!}(x-a)^{n-1}$$

Rewrite the Taylor polynomial.

$$P_{(3)}(x) = \frac{1}{4} - \frac{1}{4}(x-4) + \frac{1}{8}(x-4)^2 - \frac{1}{24}(x-4)^3$$

$$P_{(3)}(x) = \frac{1}{4 \cdot 1}(x-4)^0 - \frac{1}{4 \cdot 1}(x-4)^1 + \frac{1}{4 \cdot 2}(x-4)^2 - \frac{1}{4 \cdot 6}(x-4)^3$$

$$P_{(3)}(x) = \frac{(-1)^0}{4 \cdot 0!}(x-4)^0 + \frac{(-1)^1}{4 \cdot 1!}(x-4)^1 + \frac{(-1)^2}{4 \cdot 2!}(x-4)^2 + \frac{(-1)^3}{4 \cdot 3!}(x-4)^3$$

Then its power series representation is

$$P(x) = \frac{(-1)^0}{4 \cdot 0!}(x-4)^0 + \frac{(-1)^1}{4 \cdot 1!}(x-4)^1 + \frac{(-1)^2}{4 \cdot 2!}(x-4)^2 + \frac{(-1)^3}{4 \cdot 3!}(x-4)^3 + \cdots$$

$$P(x) = \sum_{n=0}^{\infty} \frac{(-1)^n(x-4)^n}{4 \cdot n!}$$

Apply the ratio test.

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$$L = \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right|$$

$$L = \lim_{n \to \infty} \left| \frac{\frac{(-1)^{n+1}(x-4)^{n+1}}{4 \cdot (n+1)!}}{\frac{(-1)^n(x-4)^n}{4 \cdot n!}} \right|$$

$$L = \lim_{n \to \infty} \left| \frac{\frac{(x-4)^{n+1}}{4 \cdot (n+1)!}}{\frac{(x-4)^n}{4 \cdot n!}} \right|$$

$$L = \lim_{n \to \infty} \left| \frac{(x-4)^{n+1}}{4 \cdot (n+1)!} \cdot \frac{4 \cdot n!}{(x-4)^n} \right|$$

$$L = \lim_{n \to \infty} \left| \frac{x-4}{(n+1)n!} \cdot \frac{n!}{1} \right|$$

$$L = \lim_{n \to \infty} \left| \frac{x-4}{(n+1)} \cdot \frac{1}{1} \right|$$

$$L = \left| x-4 \right| \lim_{n \to \infty} \left| \frac{1}{(n+1)} \right|$$

$$L = \left| x-4 \right| \cdot 0$$

$$L = 0$$

Then set up an inequality.

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$$0 < 1$$

This inequality is true everywhere, so the interval of convergence is infinite, which means the radius of convergence is, too. The radius of convergence is ∞.

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TAYLOR'S INEQUALITY

■ 1. Find Taylor's inequality for the function.

$$f(x) = 5\cos x$$

Solution:

Modify Taylor's inequality to state that

If $\left|f^{n+1}(x)\right| \leq M$ for $|x| \leq d$, then $\left|R_n(x)\right| \leq \dfrac{M}{(n+1)!}|x|^{n+1}$ for $|x| \leq d$.

The function $y = \cos x$ is represented by

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

So $f(x) = 5\cos x$ is represented by

$$5\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n 5x^{2n}}{(2n)!}$$

Take the first few derivatives of $f(x) = 5\cos x$ in order to find a value for $f^{(n+1)}(x)$.

$$n = 0 \qquad f^{n+1}(x) = f^{0+1}(x) = f^1(x) = f'(x) = -5\sin x$$

$$n = 1 \qquad f^{n+1}(x) = f^{1+1}(x) = f^2(x) = f''(x) = -5\cos x$$

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$n = 2$ $\qquad f^{n+1}(x) = f^{2+1}(x) = f^3(x) = f'''(x) = 5\sin x$

$n = 3$ $\qquad f^{n+1}(x) = f^{3+1}(x) = f^4(x) = f''''(x) = 5\cos x$

$n = 4$ $\qquad f^{n+1}(x) = f^{4+1}(x) = f^5(x) = f'''''(x) = -5\sin x$

Then

$\left| f^{(n+1)}(x) \right| \le 5\cos x$ or $\left| f^{(n+1)}(x) \right| \le 5\sin x$

Since both $\cos x$ and $\sin x$ exist only between -1 and 1, both $5\cos x$ and $5\sin x$ exist only between -5 and 5. So

$$-5 \le \left| f^{(n+1)}(x) \right| \le 5$$

But the absolute value in the inequality requires only positive values, so

$$0 \le \left| f^{(n+1)}(x) \right| \le 5$$

■ 2. Find Taylor's inequality for the function.

$$f(x) = 3\sin x$$

Solution:

Modify Taylor's inequality to state that

If $\left| f^{n+1}(x) \right| \le M$ for $|x| \le d$, then $\left| R_n(x) \right| \le \dfrac{M}{(n+1)!}|x|^{n+1}$ for $|x| \le d$.

The function $y = \sin x$ is represented by

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

So $f(x) = 3\sin x$ is represented by

$$3\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n 3x^{2n+1}}{(2n+1)!}$$

Take the first few derivatives of $f(x) = 5\cos x$ in order to find a value for $f^{(n+1)}(x)$.

$$n = 0 \qquad f^{n+1}(x) = f^{0+1}(x) = f^{1}(x) = f'(x) = 3\cos x$$

$$n = 1 \qquad f^{n+1}(x) = f^{1+1}(x) = f^{2}(x) = f''(x) = -3\sin x$$

$$n = 2 \qquad f^{n+1}(x) = f^{2+1}(x) = f^{3}(x) = f'''(x) = -3\cos x$$

$$n = 3 \qquad f^{n+1}(x) = f^{3+1}(x) = f^{4}(x) = f''''(x) = 3\sin x$$

$$n = 4 \qquad f^{n+1}(x) = f^{4+1}(x) = f^{5}(x) = f'''''(x) = 3\cos x$$

Then

$\left| f^{(n+1)}(x) \right| \le 3\cos x$ or $\left| f^{(n+1)}(x) \right| \le 3\sin x$

Since both $\cos x$ and $\sin x$ exist only between -1 and 1, both $3\cos x$ and $3\sin x$ exist only between -3 and 3.

$$-3 \le \left| f^{(n+1)}(x) \right| \le 3$$

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But the absolute value in the inequality requires only positive values, so

$$0 \le \left| f^{(n+1)}(x) \right| \le 3$$

3. Find Taylor's inequality for the function.

$$f(x) = 7\sin x + 5$$

Solution:

Modify Taylor's inequality to state that

If $\left| f^{n+1}(x) \right| \le M$ for $|x| \le d$, then $\left| R_n(x) \right| \le \dfrac{M}{(n+1)!}|x|^{n+1}$ for $|x| \le d$.

The function $y = \sin x$ is represented by

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

So $f(x) = 7\sin x + 5$ is represented by

$$7\sin x + 5 = 5 + \sum_{n=0}^{\infty} \frac{(-1)^n 7x^{2n+1}}{(2n+1)!}$$

Take the first few derivatives of $f(x) = 7\sin x + 5$ in order to find a value for $f^{(n+1)}(x)$.

$$n = 0 \qquad f^{n+1}(x) = f^{0+1}(x) = f^{1}(x) = f'(x) = 7\cos x$$

$$n = 1 \qquad f^{n+1}(x) = f^{1+1}(x) = f^2(x) = f''(x) = -7\sin x$$

$$n = 2 \qquad f^{n+1}(x) = f^{2+1}(x) = f^3(x) = f'''(x) = -7\cos x$$

$$n = 3 \qquad f^{n+1}(x) = f^{3+1}(x) = f^4(x) = f''''(x) = 7\sin x$$

$$n = 4 \qquad f^{n+1}(x) = f^{4+1}(x) = f^5(x) = f'''''(x) = 7\cos x$$

Then

$$\left| f^{(n+1)}(x) \right| \le 7\cos x \text{ or } \left| f^{(n+1)}(x) \right| \le 7\sin x$$

Since both $\cos x$ and $\sin x$ exist only between -1 and 1, both $7\cos x$ and $7\sin x$ exist only between -7 and 7.

$$-7 \le \left| f^{(n+1)}(x) \right| \le 7$$

But the absolute value in the inequality requires only positive values, so

$$0 \le \left| f^{(n+1)}(x) \right| \le 7$$

■ 4. Find Taylor's inequality for the function.

$$f(x) = \pi \cos x$$

Solution:

Modify Taylor's inequality to state that

If $\left|f^{n+1}(x)\right| \leq M$ for $|x| \leq d$, then $\left|R_n(x)\right| \leq \frac{M}{(n+1)!}|x|^{n+1}$ for $|x| \leq d$.

The function $y = \cos x$ is represented by

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

So $f(x) = \pi \cos x$ is represented by

$$\pi \cos x = \sum_{n=0}^{\infty} \frac{(-1)^n \pi x^{2n}}{(2n)!}$$

Take the first few derivatives of $f(x) = \pi \cos x$ in order to find a value for $f^{(n+1)}(x)$.

$$n = 0 \qquad f^{n+1}(x) = f^{0+1}(x) = f^1(x) = f'(x) = -\pi \sin x$$

$$n = 1 \qquad f^{n+1}(x) = f^{1+1}(x) = f^2(x) = f''(x) = -\pi \cos x$$

$$n = 2 \qquad f^{n+1}(x) = f^{2+1}(x) = f^3(x) = f'''(x) = \pi \sin x$$

$$n = 3 \qquad f^{n+1}(x) = f^{3+1}(x) = f^4(x) = f''''(x) = \pi \cos x$$

$$n = 4 \qquad f^{n+1}(x) = f^{4+1}(x) = f^5(x) = f'''''(x) = -\pi \sin x$$

Then

$\left|f^{(n+1)}(x)\right| \leq \pi \cos x$ or $\left|f^{(n+1)}(x)\right| \leq \pi \sin x$

Since both $\cos x$ and $\sin x$ exist only between -1 and 1, both $\pi \cos x$ and $\pi \sin x$ exist only between $-\pi$ and π.

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$$-\pi \leq \left| f^{(n+1)}(x) \right| \leq \pi$$

But the absolute value in the inequality requires only positive values, so

$$0 \leq \left| f^{(n+1)}(x) \right| \leq \pi$$

5. Find Taylor's inequality for the function.

$$f(x) = e \sin x$$

Solution:

Modify Taylor's inequality to state that

If $\left| f^{n+1}(x) \right| \leq M$ for $|x| \leq d$, then $\left| R_n(x) \right| \leq \dfrac{M}{(n+1)!} |x|^{n+1}$ for $|x| \leq d$.

The function $y = \sin x$ is represented by

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

So $f(x) = e \sin x$ is represented by

$$e \sin x = \sum_{n=0}^{\infty} \frac{(-1)^n e x^{2n+1}}{(2n+1)!}$$

Take the first few derivatives of $f(x) = e \sin x$ in order to find a value for $f^{(n+1)}(x)$.

$$n = 0 \qquad f^{n+1}(x) = f^{0+1}(x) = f^1(x) = f'(x) = e\cos x$$

$$n = 1 \qquad f^{n+1}(x) = f^{1+1}(x) = f^2(x) = f''(x) = -e\sin x$$

$$n = 2 \qquad f^{n+1}(x) = f^{2+1}(x) = f^3(x) = f'''(x) = -e\cos x$$

$$n = 3 \qquad f^{n+1}(x) = f^{3+1}(x) = f^4(x) = f''''(x) = e\sin x$$

$$n = 4 \qquad f^{n+1}(x) = f^{4+1}(x) = f^5(x) = f'''''(x) = e\cos x$$

Then

$$\left| f^{(n+1)}(x) \right| \le e\cos x \text{ or } \left| f^{(n+1)}(x) \right| \le e\sin x$$

Since both $\cos x$ and $\sin x$ exist only between -1 and 1, both $e\cos x$ and $e\sin x$ exist only between $-e$ and e.

$$-e \le \left| f^{(n+1)}(x) \right| \le e$$

But the absolute value in the inequality requires only positive values, so

$$0 \le \left| f^{(n+1)}(x) \right| \le e$$

W W W . K R I S T A K I N G M A T H . C O M