Topic: Power series differentiation

Question: Differentiate to find the power series representation of the function.

$$f(x) = \frac{1}{(2-x)^2}$$

Answer choices:

A $\sum_{n=0}^{\infty} (-1)^n n(x-1)^n$

B $\sum_{n=0}^{\infty} (-1)^n (n+1)x^n$

C $\sum_{n=0}^{\infty} (n+1)(x-1)^n$

D $\sum_{n=0}^{\infty} (-1)^n + 2(n+1)(x-1)^n$

Solution: C

When we use differentiation to find the power series representation of a function like the given function

$$f(x) = \frac{1}{(2-x)^2}$$

we use the standard form of a power series

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \ldots = \sum_{n=0}^{\infty} x^n$$

Our goal will be to start with this standard form, and then manipulate it until the value on the far left is equal to the integral of the given function. This way, when we take the derivative of the manipulated standard form, the far left will become the given function, and we can use the sum of the series on the far right as the power series representation of the given series.

To figure out what we need the far left side to become, we'll integrate the given function.

$$\int \frac{1}{(2-x)^2}\ dx$$

$$u = 2 - x$$

$du = -\,dx$, so $dx = -\,du$

$$\int \frac{1}{u^2}(-du)$$

$$\int -u^{-2}\ du$$

$$u^{-1} + C$$

$$\frac{1}{u} + C$$

$$\frac{1}{2-x} + C$$

This tells us that we need to get the far left side of the standard form of a power series to equal

$$\frac{1}{2-x}$$

We'll start to manipulate the standard form in this direction, but we have to remember that, in order to keep the equation balanced, whenever we make a change on the far left, we have to make the same change to the expanded form of the series in the middle and to the sum of the power series on the right.

To get the far left side of the standard form of a power series to

$$\frac{1}{2-x}$$

we'll substitute $x-1$ in for x. We need to make sure we make the same change to the middle and to the right.

$$\frac{1}{1-(x-1)} = 1 + (x-1) + (x-1)^2 + (x-1)^3 + \ldots = \sum_{n=0}^{\infty} (x-1)^n$$

$$\frac{1}{1-x+1} = 1 + (x-1) + (x-1)^2 + (x-1)^3 + \ldots = \sum_{n=0}^{\infty} (x-1)^n$$

$$\frac{1}{2-x} = 1 + (x-1) + (x-1)^2 + (x-1)^3 + \ldots = \sum_{n=0}^{\infty} (x-1)^n$$

Now, if we differentiate (take the derivative) the value on the far left, the result will be the given function, which is exactly what we want. So now we'll take the derivative of the left, middle, and right sides of this manipulated standard form.

Using quotient rule to take the derivative of the left side, power rule and chain rule to take the derivative of the middle, and power rule and chain rule to take the derivative of the right side, we get

$$\frac{(0)(2-x) - (1)(-1)}{(2-x)^2} = 1 + 2(x-1) + 3(x-1)^2 + \ldots = \sum_{n=0}^{\infty} n(x-1)^{n-1}$$

$$\frac{1}{(2-x)^2} = 1 + 2(x-1) + 3(x-1)^2 + \ldots = \sum_{n=0}^{\infty} n(x-1)^{n-1}$$

Since the left side is now equal to the given function, we'll use the right side as our power series representation. Whenever possible though, we want the power on the x variable to be n instead of $n-1$. To accomplish this, we'll do an index shift and substitute $n+1$ in for n.

$$\frac{1}{(2-x)^2} = \sum_{n+1=0}^{\infty} (n+1)(x-1)^{n+1-1}$$

$$\frac{1}{(2-x)^2} = \sum_{n=-1}^{\infty} (n+1)(x-1)^n$$

Since plugging in $n = -1$ yields a 0 term, we can shift the index back to $n = 0$ without effecting the value of the series.

$$\frac{1}{(2-x)^2} = \sum_{n=0}^{\infty} (n+1)(x-1)^n$$

This is the power series representation of the function.

Topic: Power series differentiation

Question: Differentiate to find the power series representation of the function.

$$f(x) = \frac{1}{(-2-x)^2}$$

Answer choices:

A $\sum_{n=0}^{\infty} (-1)^n (n+1)(x+2)^n$

B $\sum_{n=0}^{\infty} (n+1)(x+3)^n$

C $\sum_{n=0}^{\infty} n(x+3)^n$

D $\sum_{n=0}^{\infty} (-1)^n (n+1)(x-3)^n$

Solution: B

When we use differentiation to find the power series representation of a function like the given function

$$f(x) = \frac{1}{(-2-x)^2}$$

we use the standard form of a power series

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots = \sum_{n=0}^{\infty} x^n$$

Our goal will be to start with this standard form, and then manipulate it until the value on the far left is equal to the integral of the given function. This way, when we take the derivative of the manipulated standard form, the far left will become the given function, and we can use the sum of the series on the far right as the power series representation of the given series.

To figure out what we need the far left side to become, we'll integrate the given function.

$$\int \frac{1}{(-2-x)^2}\ dx$$

$$u = -2 - x$$

$du = -dx$, so $dx = -du$

$$\int \frac{1}{u^2}(-du)$$

$$\int -u^{-2}\ du$$

$$u^{-1} + C$$

$$\frac{1}{u} + C$$

$$\frac{1}{-2 - x} + C$$

This tells us that we need to get the far left side of the standard form of a power series to equal

$$\frac{1}{-2 - x}$$

We'll start to manipulate the standard form in this direction, but we have to remember that, in order to keep the equation balanced, whenever we make a change on the far left, we have to make the same change to the expanded form of the series in the middle and to the sum of the power series on the right.

To get the far left side of the standard form of a power series to

$$\frac{1}{-2 - x}$$

we'll substitute $x + 3$ in for x. We need to make sure we make the same change to the middle and to the right.

$$\frac{1}{1 - (x + 3)} = 1 + (x + 3) + (x + 3)^2 + (x + 3)^3 + \ldots = \sum_{n=0}^{\infty} (x + 3)^n$$

$$\frac{1}{1 - x - 3} = 1 + (x + 3) + (x + 3)^2 + (x + 3)^3 + \ldots = \sum_{n=0}^{\infty} (x + 3)^n$$

$$\frac{1}{-2-x} = 1 + (x+3) + (x+3)^2 + (x+3)^3 + \ldots = \sum_{n=0}^{\infty} (x+3)^n$$

Now, if we differentiate (take the derivative) the value on the far left, the result will be the given function, which is exactly what we want. So now we'll take the derivative of the left, middle, and right sides of this manipulated standard form.

Using quotient rule to take the derivative of the left side, power rule and chain rule to take the derivative of the middle, and power rule and chain rule to take the derivative of the right side, we get

$$\frac{(0)(-2-x)-(1)(-1)}{(-2-x)^2} = 1 + 2(x+3) + 3(x+3)^2 + \ldots = \sum_{n=0}^{\infty} n(x+3)^{n-1}$$

$$\frac{1}{(-2-x)^2} = 1 + 2(x+3) + 3(x+3)^2 + \ldots = \sum_{n=0}^{\infty} n(x+3)^{n-1}$$

Since the left side is now equal to the given function, we'll use the right side as our power series representation. Whenever possible though, we want the power on the x variable to be n instead of $n-1$. To accomplish this, we'll do an index shift and substitute $n+1$ in for n.

$$\frac{1}{(-2-x)^2} = \sum_{n+1=0}^{\infty} (n+1)(x+3)^{n+1-1}$$

$$\frac{1}{(-2-x)^2} = \sum_{n=-1}^{\infty} (n+1)(x+3)^{n}$$

Since plugging in $n = -1$ yields a 0 term, we can shift the index back to $n = 0$ without effecting the value of the series.

$$\frac{1}{(-2-x)^2} = \sum_{n=0}^{\infty} (n+1)(x+3)^n$$

This is the power series representation of the function.

Topic: Power series differentiation

Question: Differentiate to find the power series representation of the function.

$$f(x) = \frac{1}{(4-x)^2}$$

Answer choices:

A $\displaystyle\sum_{n=0}^{\infty} (-1)^n(n+1)(x-4)^n$

B $\displaystyle\sum_{n=0}^{\infty} (-1)^n(n+1)(x+3)^n$

C $\displaystyle\sum_{n=0}^{\infty} (-1)^n n(x+3)^n$

D $\displaystyle\sum_{n=0}^{\infty} (n+1)(x-3)^n$

Solution: D

When we use differentiation to find the power series representation of a function like the given function

$$f(x) = \frac{1}{(4-x)^2}$$

we use the standard form of a power series

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots = \sum_{n=0}^{\infty} x^n$$

Our goal will be to start with this standard form, and then manipulate it until the value on the far left is equal to the integral of the given function. This way, when we take the derivative of the manipulated standard form, the far left will become the given function, and we can use the sum of the series on the far right as the power series representation of the given series.

To figure out what we need the far left side to become, we'll integrate the given function.

$$\int \frac{1}{(4-x)^2} \ dx$$

$$u = 4 - x$$

$du = -\,dx$, so $dx = -\,du$

$$\int \frac{1}{u^2}(-du)$$

$$\int -u^{-2} \ du$$

$$u^{-1} + C$$

$$\frac{1}{u} + C$$

$$\frac{1}{4-x} + C$$

This tells us that we need to get the far left side of the standard form of a power series to equal

$$\frac{1}{4-x}$$

We'll start to manipulate the standard form in this direction, but we have to remember that, in order to keep the equation balanced, whenever we make a change on the far left, we have to make the same change to the expanded form of the series in the middle and to the sum of the power series on the right.

To get the far left side of the standard form of a power series to

$$\frac{1}{4-x}$$

we'll substitute $x-3$ in for x. We need to make sure we make the same change to the middle and to the right.

$$\frac{1}{1-(x-3)} = 1 + (x-3) + (x-3)^2 + (x-3)^3 + \dots = \sum_{n=0}^{\infty} (x-3)^n$$

$$\frac{1}{1-x+3} = 1 + (x-3) + (x-3)^2 + (x-3)^3 + \dots = \sum_{n=0}^{\infty} (x-3)^n$$

$$\frac{1}{4-x} = 1 + (x-3) + (x-3)^2 + (x-3)^3 + \ldots = \sum_{n=0}^{\infty} (x-3)^n$$

Now, if we differentiate (take the derivative) the value on the far left, the result will be the given function, which is exactly what we want. So now we'll take the derivative of the left, middle, and right sides of this manipulated standard form.

Using quotient rule to take the derivative of the left side, power rule and chain rule to take the derivative of the middle, and power rule and chain rule to take the derivative of the right side, we get

$$\frac{(0)(4-x) - (1)(-1)}{(4-x)^2} = 1 + 2(x-3) + 3(x-3)^2 + \ldots = \sum_{n=0}^{\infty} n(x-3)^{n-1}$$

$$\frac{1}{(4-x)^2} = 1 + 2(x-3) + 3(x-3)^2 + \ldots = \sum_{n=0}^{\infty} n(x-3)^{n-1}$$

Since the left side is now equal to the given function, we'll use the right side as our power series representation. Whenever possible though, we want the power on the x variable to be n instead of $n-1$. To accomplish this, we'll do an index shift and substitute $n+1$ in for n.

$$\frac{1}{(4-x)^2} = \sum_{n+1=0}^{\infty} (n+1)(x-3)^{n+1-1}$$

$$\frac{1}{(4-x)^2} = \sum_{n=-1}^{\infty} (n+1)(x-3)^n$$

Since plugging in $n = -1$ yields a 0 term, we can shift the index back to $n = 0$ without effecting the value of the series.

$$\frac{1}{(4-x)^2} = \sum_{n=0}^{\infty} (n+1)(x-3)^n$$

This is the power series representation of the function.