Calculus 2 Formulas

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Integrals

Midpoint rule

$$\int_a^b f(x)\,dx \approx \sum_{i=1}^{n} f\left(\bar{x}_i\right)\Delta x = \Delta x\left[f\left(\bar{x}_1\right) + \ldots + f\left(\bar{x}_n\right)\right]$$

where $\left(x_{i-1}, x_i\right)$ and

$$\Delta x = \frac{b-a}{n}$$ and

$$\bar{x}_i = \frac{1}{2}(x_{i-1} + x_i)$$

Trapezoidal rule

$$\int_a^b f(x)\,dx \approx T_n = \frac{\Delta x}{2}\left[f(x_0) + 2f(x_1) + 2f(x_2) + \ldots + 2f(x_{n-1}) + f(x_n)\right]$$

where $\Delta x = \dfrac{b-a}{n}$ and

$$x_i = a + i\Delta x$$

Midpoint and trapezoidal error bounds

E_T and E_M are the errors in the trapezoidal and midpoint rules

$$|E_T| \leq \frac{K(b-a)^3}{12n^2} \qquad \text{and} \qquad |E_M| \leq \frac{K(b-a)^3}{24n^2}$$

where $\quad |f''(x)| \leq K$

for $\quad a \leq x \leq b$

Simpson's rule

$$\int_a^b f(x)\ dx \approx S_n = \frac{\Delta x}{3}\left[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + \dots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)\right]$$

where $\quad n$ is even $\quad$ and

$$\Delta x = \frac{b-a}{n}$$

Simpson's error bounds

E_S is the error in Simpson's rule

$$|E_S| \leq \frac{K(b-a)^5}{180n^4}$$

where $\quad \left|f^{(4)}(x)\right| \leq K$

for $\quad a \leq x \leq b$

Symmetric functions

Suppose f is continuous on $[-a, a]$.

If f is **even** $\quad [f(-x) = f(x)]$, then $\quad \int_{-a}^{a} f(x)\ dx = 2\int_{0}^{a} f(x)\ dx$

If f is **odd** $\quad [f(-x) = -f(x)]$, then $\quad \int_{-a}^{a} f(x)\ dx = 0$

Limit process for area under the curve

$$\int_{a}^{b} f(x)\ dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$$

where $\quad \Delta x = \frac{b-a}{n}$

and $\quad x_i = a + i\Delta x$

Summation formulas for the limit process

$\sum_{i=1}^{n} k = kn \quad$ where k is any non-zero constant

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2} = \frac{n^2+n}{2}$$

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} = \frac{2n^3+3n^2+n}{6}$$

$$\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4} = \frac{n^4+2n^3+n^2}{4}$$

$$\sum_{i=1}^{n} i^4 = \frac{n(2n+1)(n+1)(3n^2+3n-1)}{30} = \frac{6n^5+15n^4+10n^3+6n^2-n}{30}$$

Net change theorem

$$\int_a^b F'(x)\ dx = F(b) - F(a)$$

Fundamental theorem of calculus

Suppose f is continuous on $[a, b]$.

Part 1

Given integral	How to solve it
$f(x) = \int_a^x f(t)\ dt$	Plug x in for t.
$f(x) = \int_x^a f(t)\ dt$	Reverse limits of integration and multiply by -1, then plug x in for t.

$f(x) = \int_a^{g(x)} f(t)\ dt$

Plug $g(x)$ in for t, then multiply by dg/dx.

$f(x) = \int_{g(x)}^{a} f(t)\ dt$

Reverse limits of integration and multiply by -1, then plug $g(x)$ in for t and multiply by dg/dx.

$f(x) = \int_{g(x)}^{h(x)} f(t)\ dt$

Split the limits of integration as $\int_{g(x)}^{0} f(t)\ dt + \int_{0}^{h(x)} f(t)\ dt$. Reverse limits of integration on $\int_{g(x)}^{0} f(t)\ dt$ and multiply by -1, then plug $g(x)$ and $h(x)$ in for t, multiplying by dg/dx and dh/dx respectively.

Part 2

$$\int_a^b f(x)\ dx = F(b) - F(a)$$

where F is any antiderivative of f, that is, a function such that $F' = f$

Integration by parts

$$\int u\ dv = uv - \int v\ du$$

Properties of integrals

$$\int_a^b c\ dx = c(b-a)$$

$$\int_a^b f(x) + g(x)\ dx = \int_a^b f(x)\ dx + \int_a^b g(x)\ dx$$

$$\int_a^b cf(x)\ dx = c\int_a^b f(x)\ dx$$

$$\int_a^b f(x) - g(x)\ dx = \int_a^b f(x)\ dx - \int_a^b g(x)\ dx$$

Common indefinite integrals

$$\int k\ dx = kx + C$$

$$\int x^n\ dx = \frac{x^{n+1}}{n+1} + C$$ with $n \neq -1$

$$\int \frac{1}{x}\ dx = \ln|x| + C$$

$$\int e^x\ dx = e^x + C$$

$$\int a^x\ dx = \frac{a^x}{\ln a} + C$$

Integrals of trig functions

$$\int \sin x \ dx = -\cos x + C$$

$$\int \csc x \ dx = \ln \left| \csc x - \cot x \right| + C$$

or $$\int \csc x \ dx = \ln \left(\sin \frac{x}{2} \right) - \ln \left(\cos \frac{x}{2} \right) + C$$

$$\int \cos x \ dx = \sin x + C$$

$$\int \sec x \ dx = \ln \left| \sec x + \tan x \right| + C$$

or

$$\int \sec x \ dx = \ln \left(\sin \frac{x}{2} + \cos \frac{x}{2} \right) - \ln \left(\cos \frac{x}{2} - \sin \frac{x}{2} \right) + C$$

$$\int \tan x \ dx = -\ln \cos x + C$$

$$\int \cot x \ dx = \ln \sin x + C$$

Other common trig integrals

$$\int \sec^2 x \ dx = \tan x + C$$

$$\int \csc^2 x \ dx = -\cot x + C$$

$$\int \sec x \tan x \ dx = \sec x + C$$

$$\int \csc x \cot x \ dx = -\csc x + C$$

$$\int \frac{1}{x^2 + 1} \ dx = \tan^{-1} x + C$$

$$\int \frac{1}{\sqrt{1 - x^2}} \ dx = \sin^{-1} x + C$$

$$\int \sinh x \ dx = \cosh x + C$$

$$\int \cosh x \ dx = \sinh x + C$$

Rewriting inverse hyperbolic functions

$$\sinh^{-1} x = \ln(x + \sqrt{x^2 + 1})$$

$$\cosh^{-1} x = \ln(x \pm \sqrt{x^2 - 1}) = \pm \ln(x + \sqrt{x^2 - 1})$$

$$\tanh^{-1} x = \frac{1}{2} \ln \left(\frac{1 + x}{1 - x} \right)$$

$$\text{sech}^{-1} x = \ln \left(\frac{1 + \sqrt{1 - x^2}}{x} \right)$$

$$\text{csch}^{-1} x = \ln \left(\frac{1}{x} + \frac{\sqrt{1 - x^2}}{|x|} \right)$$

$$\coth^{-1} x = \frac{1}{2} \ln \left(\frac{x + 1}{x - 1} \right)$$

Integrals of inverse hyperbolic trig functions

$$\int \sinh^{-1} x \ dx = x \sinh^{-1} x - \sqrt{x^2 + 1} + C$$

$$\int \cosh^{-1} x \ dx = x \cosh^{-1} x - \sqrt{x - 1}\sqrt{x + 1} + C$$

$$\int \tanh^{-1} x \ dx = \frac{1}{2} \log(1 - x^2) + x \tanh^{-1} x + C$$

$$\int \coth^{-1} x \ dx = \frac{1}{2}\log(1-x^2) + x\coth^{-1} x + C$$

Integrals resulting in inverse hyperbolic trig functions

$$\int \frac{1}{\sqrt{x^2+1}} \ dx = \sinh^{-1} x$$

$$\int \frac{1}{\sqrt{x-1}\sqrt{x+1}} \ dx = \cosh^{-1} x$$

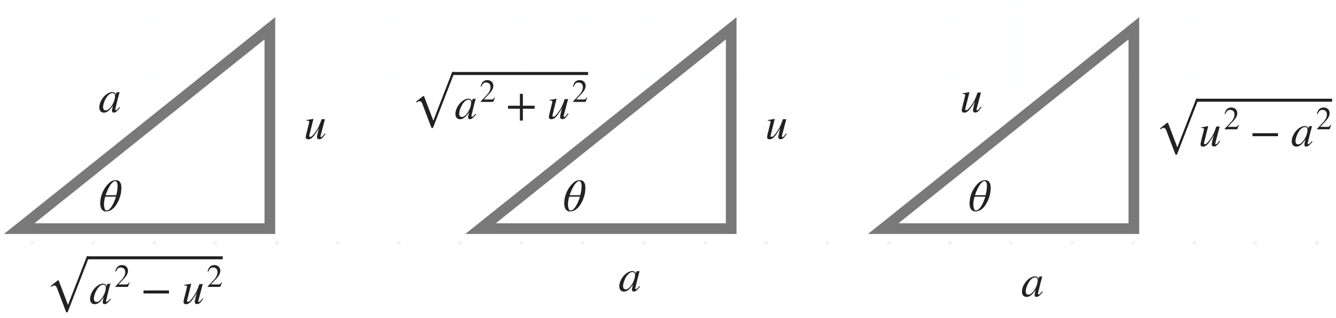
$$\int \frac{1}{1-x^2} \ dx = \tanh^{-1} x$$

$$\int \frac{1}{1-x^2} \ dx = \coth^{-1} x$$

Trig substitution setup

	sin	tan	sec
the integral includes	$\sqrt{a^2-u^2}$	$\sqrt{a^2+u^2}$	$\sqrt{u^2-a^2}$
so substitute	$u=a\sin\theta$	$u=a\tan\theta$	$u=a\sec\theta$
and use the identity	$1-\sin^2\theta=\cos^2\theta$	$1+\tan^2\theta=\sec^2\theta$	$\sec^2\theta-1=\tan^2\theta$
solve for the trig	$\sin\theta=\dfrac{u}{a}$	$\tan\theta=\dfrac{u}{a}$	$\sec\theta=\dfrac{u}{a}$
and for du	$du=a\cos\theta\ d\theta$	$du=a\sec^2\theta\ d\theta$	$du=a\sec\theta\tan\theta\ d\theta$
and for θ	$\theta=\arcsin\dfrac{u}{a}$	$\theta=\arctan\dfrac{u}{a}$	$\theta=\text{arcsec}\dfrac{u}{a}$

reference triangle



Trig substitution simplification

	$\arcsin x$	$\arccos x$	$\arctan x$	$\text{arccsc}x$	$\text{arcsec}x$	$\text{arccot}x$
sin of	x	$\sqrt{1-x^2}$	$\frac{x}{\sqrt{x^2+1}}$	$\frac{1}{x}$	$\sqrt{1-\frac{1}{x^2}}$	$-\frac{1}{x\sqrt{\frac{1}{x^2}+1}}$
cos of	$\sqrt{1-x^2}$	x	$\frac{1}{\sqrt{x^2+1}}$	$\sqrt{1-\frac{1}{x^2}}$	$\frac{1}{x}$	$\frac{1}{\sqrt{\frac{1}{x^2}+1}}$
tan of	$\frac{x}{\sqrt{1-x^2}}$	$\frac{\sqrt{1-x^2}}{x}$	x	$\frac{1}{x\sqrt{1-\frac{1}{x^2}}}$	$x\sqrt{1-\frac{1}{x^2}}$	$\frac{1}{x}$
csc of	$\frac{1}{x}$	$\frac{1}{\sqrt{1-x^2}}$	$\frac{\sqrt{x^2+1}}{x}$	x	$\frac{1}{\sqrt{1-\frac{1}{x^2}}}$	$x\sqrt{\frac{1}{x^2}+1}$
sec of	$\frac{1}{\sqrt{1-x^2}}$	$\frac{1}{x}$	$\sqrt{x^2+1}$	$\frac{1}{\sqrt{1-\frac{1}{x^2}}}$	x	$\sqrt{\frac{1}{x^2}+1}$
cot of	$\frac{\sqrt{1-x^2}}{x}$	$\frac{x}{\sqrt{1-x^2}}$	$\frac{1}{x}$	$x\sqrt{1-\frac{1}{x^2}}$	$\frac{1}{x\sqrt{1-\frac{1}{x^2}}}$	x

Applications of Integrals

Average value

$$f_{ave} = \frac{1}{b-a}\int_a^b f(x)\ dx$$

Area between curves

$$A = \int_a^b |f(x) - g(x)|\ dx$$

Arc length

$$L = \int_a^b \sqrt{1 + \left[f'(x)\right]^2}\ dx = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\ dx$$ for a function $y = f(x)$

on the interval $a \le x \le b$

$$L = \int_c^d \sqrt{1 + \left[g'(y)\right]^2}\ dy = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\ dy$$ for a function $x = g(y)$

on the interval $c \le y \le d$

Surface area of revolution

The surface area of revolution is

$$S = \int_a^b 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\ dx$$

for a function $y = f(x)$

rotated about the x-axis

on the interval $a \le x \le b$

$$S = \int_a^b 2\pi x \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\ dx$$

for a function $y = f(x)$

rotated about the y-axis

on the interval $a \le x \le b$

$$S = \int_c^d 2\pi y \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\ dy$$

for a function $x = g(y)$

rotated about the x-axis

on the interval $c \le y \le d$

$$S = \int_c^d 2\pi x \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\ dy$$

for a function $x = g(y)$

rotated about the y-axis

on the interval $c \le y \le d$

Volume of revolution

Axis	Disks	Washers	Shells
	$\int$ area width	$\int$ area width	$\int$ circumference height width

Axis of revolution: HORIZONTAL

Axis	Disks	Washers	Shells
x-axis	$\int_a^b \pi \left[f(x)\right]^2 \ dx$	$\int_a^b \pi \left[f(x)\right]^2 - \pi \left[g(x)\right]^2 \ dx$	$\int_c^d 2\pi y \left[f(y) - g(y)\right] \ dy$
$y = -k$		$\int_a^b \pi \left[k + f(x)\right]^2 - \pi \left[k + g(x)\right]^2 \ dx$	$\int_c^d 2\pi(y + k)\left[f(y) - g(y)\right] \ dy$
$y = k$		$\int_a^b \pi \left[k - f(x)\right]^2 - \pi \left[k - g(x)\right]^2 \ dx$	$\int_c^d 2\pi(k - y)\left[f(y) - g(y)\right] \ dy$

Axis of revolution: VERTICAL

Axis	Disks	Washers	Shells
y-axis	$\int_c^d \pi \left[f(y)\right]^2 \ dy$	$\int_c^d \pi \left[f(y)\right]^2 - \pi \left[g(y)\right]^2 \ dy$	$\int_a^b 2\pi x \left[f(x) - g(x)\right] \ dx$
$x = -k$		$\int_c^d \pi \left[k + f(y)\right]^2 - \pi \left[k + g(y)\right]^2 \ dy$	$\int_a^b 2\pi(x + k)\left[f(x) - g(x)\right] \ dx$
$x = k$		$\int_c^d \pi \left[k - f(y)\right]^2 - \pi \left[k - g(y)\right]^2 \ dy$	$\int_a^b 2\pi(k - x)\left[f(x) - g(x)\right] \ dx$

Mean value theorem for integrals

If f is continuous on $[a, b]$, then c exists in $[a, b]$ such that

$$f(c) = f_{ave} = \frac{1}{b-a}\int_a^b f(x)\ dx, \quad \text{that is,} \quad \int_a^b f(x)\ dx = f(c)(b-a)$$

Moments of the region

The moment of the region

about the y-axis is $$M_y = \lim_{n\to\infty}\sum_{i=1}^{n}\rho\bar{x}_i f(\bar{x}_i)\Delta x = \rho\int_a^b xf(x)\ dx$$

about the x-axis is $$M_x = \lim_{n\to\infty}\sum_{i=1}^{n}\rho\cdot\frac{1}{2}\left[f(\bar{x}_i)\right]^2\Delta x = \rho\int_a^b \frac{1}{2}\left[f(x)\right]^2\ dx$$

Center of mass of the region

The center of mass is located at $(\bar{x}, \bar{y})$ where

$$\bar{x} = \frac{M_y}{m} = \frac{1}{A}\int_a^b xf(x)\ dx \quad \text{and} \quad \bar{y} = \frac{M_x}{m} = \frac{1}{A}\int_a^b \frac{1}{2}\left[f(x)\right]^2\ dx$$

Center of mass of the region bounded by two curves

The center of mass is located at $(\bar{x}, \bar{y})$ where

$$\bar{x} = \frac{1}{A}\int_a^b x\left[f(x) - g(x)\right]\ dx \qquad \text{and} \qquad \bar{y} = \frac{1}{A}\int_a^b \frac{1}{2}\left\{\left[f(x)\right]^2 - \left[g(x)\right]^2\right\}\ dx$$

Polar & Parametric

Parametric

Derivatives

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} \qquad \text{where} \qquad \frac{dx}{dt} \neq 0$$

Area

$$A = \int_a^b y\ dx = \int_\alpha^\beta g(t)f'(t)\ dt$$

Surface area of revolution

The surface area of a parametric curve rotated

about the x-axis is $$S_x = \int_a^b 2\pi y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\ dt$$

about the y-axis is $$S_y = \int_a^b 2\pi x \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\ dt$$

Arc length

$$L = \int_{\alpha}^{\beta} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\ dt$$

Volume of revolution

The volume created by rotating a parametric curve

about the x-axis is $$V_x = \int_a^b \pi y^2 \left[x'(t)\right]\ dt$$

about the y-axis is $$V_y = \int_a^b \pi x^2 \left[y'(t)\right]\ dt$$

Polar

Conversion between cartesian and polar coordinates

$$x = r\cos\theta \qquad y = r\sin\theta \qquad x^2 + y^2 = r^2$$

Distance between two points

The distance between two polar coordinate points (r_1, θ_1) and (r_2, θ_2) is

$$D = \sqrt{\left(r_2 \cos \theta_2 - r_1 \cos \theta_1\right)^2 + \left(r_2 \sin \theta_2 - r_1 \sin \theta_1\right)^2}$$

Area

The area enclosed by a polar curve is

$$A = \int_a^b \frac{1}{2} \left[f(\theta)\right]^2 \ d\theta = \int_a^b \frac{1}{2} r^2 \ d\theta$$

Area between curves

The area between two polar curves is

$$A = \frac{1}{2} \int_\alpha^\beta (r_{\text{outer}})^2 - (r_{\text{inner}})^2 \ d\theta$$

Arc length

$$L = \int_a^b \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} \ d\theta$$

Surface area of revolution

The surface area of revolution of a polar parametric curve rotated about the x-axis is $$S_x = \int_\alpha^\beta 2\pi y \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\ d\theta$$

about the y-axis is $$S_y = \int_\alpha^\beta 2\pi x \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\ d\theta$$

Sequences & Series

Limit of a sequence

The limit of a sequence $\{a_n\}$ is L

$$\lim_{n\to\infty} a_n = L \qquad \text{or} \qquad a_n \to L \text{ as } n \to \infty$$

if we can make the terms of a_n closer and closer to L as we we make n larger and larger. If

$$\lim_{n\to\infty} a_n$$

exists, the sequence converges (is convergent). Otherwise it diverges (is divergent).

Precise definition of the limit of a sequence

The limit of a sequence $\{a_n\}$ is L

$$\lim_{n\to\infty} a_n = L \qquad \text{or} \qquad a_n \to L \text{ as } n \to \infty$$

if for every $\epsilon > 0$ there is a corresponding integer N such that

$$\text{if} \qquad n > N \qquad \text{then} \qquad |a_n - L| < \epsilon$$

Limit laws for sequences

If $\{a_n\}$ and $\{b_n\}$ are convergent sequences and c is a constant, then

$$\lim_{n\to\infty} (a_n + b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n$$

$$\lim_{n\to\infty} (a_n - b_n) = \lim_{n\to\infty} a_n - \lim_{n\to\infty} b_n$$

$$\lim_{n\to\infty} ca_n = c \lim_{n\to\infty} a_n$$

$$\lim_{n\to\infty} c = c$$

$$\lim_{n\to\infty} (a_n b_n) = \lim_{n\to\infty} a_n \cdot \lim_{n\to\infty} b_n$$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n}$$ if $\lim_{n\to\infty} b_n \neq 0$

$$\lim_{n\to\infty} a_n^p = \left[\lim_{n\to\infty} a_n\right]^p$$ if $p > 0$ and $a_n > 0$

Squeeze theorem for sequences

If $a_n \leq b_n \leq c_n$ for $n \geq n_0$ and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$ then $\lim_{n\to\infty} b_n = L$.

Absolute value of a sequence

If $\lim_{n\to\infty} |a_n| = 0$, then $\lim_{n\to\infty} a_n = 0$.

Convergence of a sequence r^n

The sequence $\{r^n\}$ is convergent if $-1 < r \leq 1$ and divergent for all other values of r.

$$\lim_{n\to\infty} r^n \quad = 0 \quad \text{if} \quad -1 < r < 1$$

$$= 1 \quad \text{if} \quad r = 1$$

Increasing, decreasing, and monotonic sequences

A sequence $\{a_n\}$ is

increasing if $a_n < a_{n+1}$ for all $n \geq 1$, ($a_1 < a_2 < a_3 < \dots$)

decreasing if $a_n > a_{n+1}$ for all $n \geq 1$, ($a_1 > a_2 > a_3 > \dots$)

monotonic if it's either increasing or decreasing

Bounded sequences

A sequence $\{a_n\}$ is

bounded above if there's a number M such that $a_n \leq M$ for all $n \geq 1$

bounded below if there's a number m such that $m \leq a_n$ for all $n \geq 1$

a bounded sequence if it's bounded above and below

Monotonic sequence theorem

Every bounded, monotonic sequence is convergent.

Partial sum of the series

Given a series $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \dots$, let s_n denote its nth partial sum.

$$s_n = \sum_{i=1}^{n} a_i = a_1 + a_2 + \ldots + a_n$$

If the sequence $\{s_n\}$ converges and $\lim_{n \to \infty} s_n = s$ exists as a real number, then the series $\sum a_n$ converges and we can say

$$a_1 + a_2 + \ldots + a_n + \ldots = s \qquad \text{or} \qquad \sum_{n=1}^{\infty} a_n = s$$

The number s is the sum of the series. If the sequence $\{s_n\}$ diverges, then the series diverges.

Convergence and sum of a geometric series

The geometric series

$$\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + \dots$$

converges if $|r| < 1$, otherwise it diverges (if $|r| \geq 1$). The sum of the convergent series is

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r} \qquad |r| < 1$$

Convergence of a_n

If the series $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n\to\infty} a_n = 0$.

Test for divergence

If $\lim_{n\to\infty} a_n$ doesn't exist or if $\lim_{n\to\infty} a_n \neq 0$, then the series $\sum_{n=1}^{\infty} a_n$ diverges.

Laws of convergent series

If the series a_n and b_n both converge, then so do these (where c is a constant):

$$\sum_{n=1}^{\infty} ca_n = c\sum_{n=1}^{\infty} a_n$$

$$\sum_{n=1}^{\infty} (a_n + b_n) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n$$

$$\sum_{n=1}^{\infty} (a_n - b_n) = \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{\infty} b_n$$

Integral test for convergence

Suppose f is a continuous, positive, decreasing function on $[1,\infty)$, and let $a_n = f(n)$.

If $\int_1^{\infty} f(x)\ dx$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.

If $\int_1^{\infty} f(x)\ dx$ diverges, then $\sum_{n=1}^{\infty} a_n$ diverges.

Remainder estimate for the integral test

Suppose $f(k) = a_k$, where f is a continuous, positive, decreasing function for $x \geq n$ and $\sum a_n$ converges. If $R_n = s - s_n$, then

$$\int_{n+1}^{\infty} f(x)\ dx \leq R_n \leq \int_{n}^{\infty} f(x)\ dx$$

p-Series test for convergence

The p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$

converges if $p > 1$

diverges if $p \leq 1$

Comparison test for convergence

Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms.

If $\sum b_n$ converges and $a_n \leq b_n$ for all n, then $\sum a_n$ converges.

If $\sum b_n$ diverges and $a_n \geq b_n$ for all n, then $\sum a_n$ diverges.

Limit comparison test for convergence

Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms.

If $\lim_{n \to \infty} \frac{a_n}{b_n} = c$

where c is a finite number and

where $0 < c < \infty$

then either both series converge or both diverge.

Alternating series test for convergence

If the alternating series

$$\sum_{n=1}^{\infty}(-1)^{n-1}b_n = b_1 - b_2 + b_3 - b_4 + b_5 - b_6 + \dots \qquad b_n > 0$$

satisfies

$$b_{n+1} \leq b_n \qquad \text{for all } n$$

$$\lim_{n\to\infty} b_n = 0$$

then the series converges.

Alternating series estimation theorem

If $s = \sum (-1)^{n-1}b_n$ is the sum of an alternating series that satisfies

$$b_{n+1} \leq b_n$$

$$\lim_{n\to\infty} b_n = 0$$

then $|R_n| = |s - s_n| \leq b_{n+1}$.

Absolute convergence

A series $\sum a_n$ is absolutely convergent if the series of absolute values $\sum |a_n|$ is convergent.

If a series $\sum a_n$ is absolutely convergent, then it's convergent.

Conditional convergence

A series $\sum a_n$ is conditionally convergent if it's convergent but not absolutely convergent.

Ratio test for convergence

If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L < 1$, then the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent.

If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L > 1$ or $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \infty$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 1$, the ratio test is inconclusive about the convergence of $\sum_{n=1}^{\infty} a_n$, which means we'll have to use a different convergence test to determine convergence.

Root test for convergence

If $\lim_{n\to\infty}\sqrt[n]{|a_n|} = L < 1$, then the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent.

If $\lim_{n\to\infty}\sqrt[n]{|a_n|} = L > 1$ or $\lim_{n\to\infty}\sqrt[n]{|a_n|} = \infty$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = 1$, the root test is inconclusive about the convergence of $\sum_{n=1}^{\infty} a_n$, which means we'll have to use a different convergence test to determine convergence.

Convergence of power series

Given a power series $\sum_{n=0}^{\infty} c_n(x-a)^n$,

the series converges only when $x = a$ **or**

the series converges for all x **or**

there is a positive number R such that the series converges if $|x-a| < R$ and diverges if $|x-a| > R$

Differentiation and integration of power series

If the power series $\sum c_n(x-a)^n$ has a radius of convergence $R > 0$, then the function f defined by

$$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + \dots = \sum_{n=0}^{\infty} c_n(x-a)^n$$

is differentiable (and therefore continuous) on the interval $(a-R, a+R)$ and

$$f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + \ldots = \sum_{n=1}^{\infty} nc_n(x-a)^{n-1}$$

$$\int f(x)\ dx = C + c_0(x-a) + c_1\frac{(x-a)^2}{2} + c_2\frac{(x-a)^3}{3} + \ldots = C + \sum_{n=0}^{\infty} c_n\frac{(x-a)^{n+1}}{n+1}$$

Note: The radii of convergence of these two power series is R.

Power series representation (expansion)

If f has a power series representation (expansion) at a, that is, if

$$f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n \qquad |x-a| < R$$

then its coefficients are given by

$$c_n = \frac{f^{(n)}(a)}{n!}$$

and the power series has the form

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty}\frac{f^{(n)}(a)}{n!}(x-a)^n \\ &= f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \ldots \end{aligned}$$

which is the taylor series of the function f at a (or about a or centered at a).

Taylor series

The taylor series of a function f at a (or about a or centered at a) is

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

$$= f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \dots$$

Remainder of the taylor series

If $f(x) = T_n(x) + R_n(x)$ where T_n is the nth-degree taylor polynomial of f at a and

$$\lim_{n \to \infty} R_n(x) = 0$$

for $|x-a| < R$, then f is equal to the sum of its taylor series on the interval $|x-a| < R$.

Taylor's inequality

If

$$|f^{(n+1)}(x)| \le M \text{ for } |x-a| \le d$$

then the remainder $R_n(x)$ of the taylor series satisfies the inequality

$$|R_n(x)| \le \frac{M}{(n+1)!}|x-a|^{n+1} \qquad \text{for} \qquad |x-a| \le d$$

Maclaurin series

The maclaurin series is a specific instance of the Taylor series where $a = 0$. In other words, it's just the Taylor series of a function f at 0 (or about 0 or centered at 0).

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}(x-0)^n$$

$$= \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n$$

$$= f(0) + \frac{f'(0)}{1!}(x-0) + \frac{f''(0)}{2!}(x-0)^2 + \frac{f'''(0)}{3!}(x-0)^3 + \dots$$

$$= f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots$$

Common maclaurin series and their radii of convergence

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \dots \qquad R = 1$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots \qquad R = \infty$$

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots \qquad R = \infty$$

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots \qquad R = \infty$$

$$\tan x = \sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n+2}(2^{2n+2}-1)B_{2n+2}}{(2n+2)!} x^{2n+1} = x + \frac{x^3}{3} + \frac{2x^5}{15} + \frac{17x^7}{315} + \frac{62x^9}{2{,}835} + \dots$$

$$\sin^{-1} x = \sum_{n=0}^{\infty} \frac{\Gamma\left(n+\frac{1}{2}\right)}{\sqrt{\pi}(2n+1)n!} x^{2n+1} = x + \frac{x^3}{6} + \frac{3x^5}{40} + \frac{5x^7}{112} + \frac{35x^9}{1{,}152} - \dots$$

$$\tan^{-1} x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \qquad R = 1$$

$$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots \qquad R = 1$$

$$(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n = 1 + kx + \frac{k(k-1)}{2!}x^2 + \frac{k(k-1)(k-2)}{3!}x^3 + \dots \qquad R = 1$$

Exponential series

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \qquad \text{for all } x$$

$$e = \sum_{n=0}^{\infty} \frac{1}{n!} = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \dots$$

Binomial series

If k is any real number and $|x| < 1$, then

$$(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n = 1 + kx + \frac{k(k-1)}{2!}x^2 + \frac{k(k-1)(k-2)}{3!}x^3 + \ldots$$

Recognizing types of series

Use this visual guide to train yourself to quickly recognize different types of series. Sometimes the best way to recognize the type of series is to write out the first few terms of the series and then match it to one of the types of series below.

Geometric series

$$2.3171717... = 2.3 + \frac{17}{10^3} + \frac{17}{10^5} + \frac{17}{10^7} + \ldots = 2.3 + \frac{17}{10^3}\left(1 + \frac{1}{10^2} + \frac{1}{10^4} + \ldots\right)$$

$$5 - \frac{10}{3} + \frac{20}{9} - \frac{40}{27} + \ldots = 5\left[1 - \frac{2}{3} + \left(\frac{2}{3}\right)^2 - \left(\frac{2}{3}\right)^3 + \ldots\right]$$

$$4 + 3 + \frac{9}{4} + \frac{27}{16} + \ldots = 4\left[1 + \frac{3}{4} + \left(\frac{3}{4}\right)^2 + \left(\frac{3}{4}\right)^3 + \ldots\right]$$

$$3 - 4 + \frac{16}{3} - \frac{64}{9} + \ldots = 3\left[1 - \frac{4}{3} + \left(\frac{4}{3}\right)^2 - \left(\frac{4}{3}\right)^3 + \ldots\right]$$

$$10 - 2 + 0.4 - 0.08 + \ldots = 10\left[1 - 0.2 + (0.2)^2 - (0.2)^3 + \ldots\right]$$

$$2+0.5+0.125+0.03125+=2\left[1+0.25+(0.25)^2+(0.25)^3+\dots\right]$$

p-series

$$\sum_{n=1}^{\infty}\frac{1}{n^3}=\frac{1}{1^3}+\frac{1}{2^3}+\frac{1}{3^3}+\frac{1}{4^3}+\dots$$

$$\sum_{n=1}^{\infty}\frac{1}{n^{\frac{1}{3}}}=\sum_{n=1}^{\infty}\frac{1}{\sqrt[3]{n}}=1+\frac{1}{\sqrt[3]{2}}+\frac{1}{\sqrt[3]{3}}+\frac{1}{\sqrt[3]{4}}+\dots$$

Telescoping series

$$\sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{n+1}\right)=\left(1-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\dots+\left(\frac{1}{n}-\frac{1}{n+1}\right)$$

Alternating series

$(-1)^n$	$(-1)^{n-1}$	$(-1)^{n+1}$
Will start with –	Will start with +	Will start with +
$\sum_{n=1}^{\infty}(-1)^n\frac{\sqrt{n}}{2n+3}$	$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{2n+1}$	$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n^2}{n^3+4}$
$\sum_{n=1}^{\infty}(-1)^n\frac{n}{\sqrt{n^3+2}}$	$\sum_{n=1}^{\infty}(-1)^{n-1}e^{\frac{2}{n}}$	$\sum_{n=1}^{\infty}(-1)^{n+1}ne^{-n}$

$$\sum_{n=1}^{\infty} (-1)^n \frac{3n-1}{2n+1}$$

$$\sum_{n=1}^{\infty} (-1)^{n-1} \arctan n$$

$$\sum_{n=1}^{\infty} (-1)^n e^{-n}$$

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\ln(n+4)}$$