Calculus 2 Workbook Solutions

Telescoping series

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MATH

CONVERGENCE OF A TELESCOPING SERIES

■ 1. Say whether the telescoping series converges or diverges.

$$\sum_{n=1}^{\infty} \left(5^n - 5^{n-1}\right)$$

Solution:

Rewrite the series.

$$\sum_{n=1}^{\infty} \left(5^n - 5^{n-1}\right)$$

$$\sum_{n=1}^{\infty} \left(5 \cdot 5^{n-1} - 5^{n-1}\right)$$

$$\sum_{n=1}^{\infty} (5-1)5^{n-1}$$

$$\sum_{n=1}^{\infty} (4)5^{n-1}$$

Matching this to

$$\sum_{n=1}^{\infty} a_1 \cdot r^{n-1}$$

tells us that $r=5$. Because the series only converges when $|r|<1$, this series diverges.

■ 2. Say whether the telescoping series converges or diverges.

$$\sum_{n=1}^{\infty}\frac{1}{n(n+2)}$$

Solution:

Using a partial fractions decomposition on the series,

$$\sum_{n=1}^{\infty}\frac{1}{n(n+2)}=\sum_{n=1}^{\infty}\frac{1}{2n}-\frac{1}{2(n+2)}$$

Find the first few terms of this rewritten series.

$$n=1 \qquad \frac{1}{2(1)}-\frac{1}{2(1+2)}=\frac{1}{2}-\frac{1}{6}$$

$$n=2 \qquad \frac{1}{2(2)}-\frac{1}{2(2+2)}=\frac{1}{4}-\frac{1}{8}$$

$$n=3 \qquad \frac{1}{2(3)}-\frac{1}{2(3+2)}=\frac{1}{6}-\frac{1}{10}$$

$$n=4 \qquad \frac{1}{2(4)}-\frac{1}{2(4+2)}=\frac{1}{8}-\frac{1}{12}$$

$$n = 5 \qquad \frac{1}{2(5)} - \frac{1}{2(5+2)} = \frac{1}{10} - \frac{1}{14}$$

If we use these terms to write out the expanded series, we get

$$\left(\frac{1}{2} - \frac{1}{6}\right) + \left(\frac{1}{4} - \frac{1}{8}\right) + \left(\frac{1}{6} - \frac{1}{10}\right)$$

$$+\left(\frac{1}{8} - \frac{1}{12}\right) + \left(\frac{1}{10} - \frac{1}{14}\right) + \ldots + \left(\frac{1}{2n} - \frac{1}{2(n+2)}\right) + \ldots$$

$$\frac{1}{2} - \frac{1}{6} + \frac{1}{4} - \frac{1}{8} + \frac{1}{6} - \frac{1}{10}$$

$$+\frac{1}{8} - \frac{1}{12} + \frac{1}{10} - \frac{1}{14} + \ldots + \frac{1}{2n} - \frac{1}{2(n+2)} + \ldots$$

$$\frac{1}{2} - \frac{1}{6} + \frac{1}{6} + \frac{1}{4} - \frac{1}{8} + \frac{1}{8}$$

$$-\frac{1}{10} + \frac{1}{10} - \frac{1}{12} - \frac{1}{14} + \ldots + \frac{1}{2n} - \frac{1}{2(n+2)} + \ldots$$

$$\frac{1}{2} + \frac{1}{4} - \frac{1}{12} - \frac{1}{14} + \ldots + \frac{1}{2n} - \frac{1}{2(n+2)} + \ldots$$

But if we were to continue with the pattern, those middle terms cancel, and we're left with only

$$\frac{1}{2} + \frac{1}{4} - \frac{1}{2((n-1)+2)} - \frac{1}{2(n+2)}$$

$$\frac{1}{2} + \frac{1}{4} - \frac{1}{2(n+1)} - \frac{1}{2(n+2)}$$

$$\frac{3}{4} - \frac{1}{2(n+1)} - \frac{1}{2(n+2)}$$

Take the limit as $n \to \infty$.

$$\lim_{n\to\infty} \frac{3}{4} - \frac{1}{2(n+1)} - \frac{1}{2(n+2)}$$

$$\frac{3}{4} - \frac{1}{2(\infty+1)} - \frac{1}{2(\infty+2)}$$

$$\frac{3}{4} - 0 - 0$$

$$\frac{3}{4}$$

The series converges.

3. Say whether the telescoping series converges or diverges.

$$\sum_{n=1}^{\infty} \frac{2}{n^2+n}$$

Solution:

Using a partial fractions decomposition on the series,

$$\sum_{n=1}^{\infty} \frac{2}{n^2+n} = \sum_{n=1}^{\infty} \frac{2}{n} - \frac{2}{n+1}$$

Find the first few terms of this rewritten series.

$n=1$ $$\frac{2}{n}-\frac{2}{n+1}=\frac{2}{1}-\frac{2}{1+1}=2-1=1$$

$n=2$ $$\frac{2}{n}-\frac{2}{n+1}=\frac{2}{2}-\frac{2}{2+1}=1-\frac{2}{3}=\frac{1}{3}$$

$n=3$ $$\frac{2}{n}-\frac{2}{n+1}=\frac{2}{3}-\frac{2}{3+1}=\frac{2}{3}-\frac{1}{2}=\frac{1}{6}$$

$n=4$ $$\frac{2}{n}-\frac{2}{n+1}=\frac{2}{4}-\frac{2}{4+1}=\frac{1}{2}-\frac{2}{5}=\frac{1}{10}$$

$n=5$ $$\frac{2}{n}-\frac{2}{n+1}=\frac{2}{5}-\frac{2}{5+1}=\frac{2}{5}-\frac{1}{3}=\frac{1}{15}$$

If we use these terms to write out the expanded series, we get

$$(2-1)+\left(1-\frac{2}{3}\right)+\left(\frac{2}{3}-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{2}{5}\right)+\left(\frac{2}{5}-\frac{1}{3}\right)+\ldots+\left(\frac{2}{n}-\frac{2}{n+1}\right)+\ldots$$

$$2-1+1-\frac{2}{3}+\frac{2}{3}-\frac{1}{2}+\frac{1}{2}-\frac{2}{5}+\frac{2}{5}-\frac{1}{3}+\ldots+\frac{2}{n}-\frac{2}{n+1}+\ldots$$

$$2-\frac{1}{3}+\ldots+\frac{2}{n}-\frac{2}{n+1}+\ldots$$

But if we were to continue with the pattern, those middle terms cancel, and we're left with only

$$2-\frac{2}{n+1}$$

Take the limit as $n\to\infty$.

$$\lim_{n\to\infty} 2 - \frac{2}{n+1} = 2 - \frac{2}{\infty + 1} = 2 - 0 = 2$$

The series converges.

4. Say whether the telescoping series converges or diverges.

$$\sum_{n=1}^{\infty} \frac{4}{n^2 + 3n + 2}$$

Solution:

Using a partial fractions decomposition on the series,

$$\sum_{n=1}^{\infty} \frac{4}{n^2 + 3n + 2} = \sum_{n=1}^{\infty} \frac{4}{n+1} - \frac{4}{n+2}$$

Find the first few terms of this rewritten series.

$$n = 1 \qquad \frac{4}{n+1} - \frac{4}{n+2} = \frac{4}{1+1} - \frac{4}{1+2} = 2 - \frac{4}{3}$$

$$n = 2 \qquad \frac{4}{n+1} - \frac{4}{n+2} = \frac{4}{2+1} - \frac{4}{2+2} = \frac{4}{3} - 1$$

$$n = 3 \qquad \frac{4}{n+1} - \frac{4}{n+2} = \frac{4}{3+1} - \frac{4}{3+2} = 1 - \frac{4}{5}$$

$$n = 4 \qquad \frac{4}{n+1} - \frac{4}{n+2} = \frac{4}{4+1} - \frac{4}{4+2} = \frac{4}{5} - \frac{2}{3}$$

$$n = 5 \qquad \frac{4}{n+1} - \frac{4}{n+2} = \frac{4}{5+1} - \frac{4}{5+2} = \frac{2}{3} - \frac{4}{7}$$

If we use these terms to write out the expanded series, we get

$$\left(2 - \frac{4}{3}\right) + \left(\frac{4}{3} - 1\right) + \left(1 - \frac{4}{5}\right)$$

$$+\left(\frac{4}{5} - \frac{2}{3}\right) + \left(\frac{2}{3} - \frac{4}{7}\right) + \dots + \left(\frac{4}{n+1} - \frac{4}{n+2}\right) + \dots$$

$$2 - \frac{4}{3} + \frac{4}{3} - 1 + 1 - \frac{4}{5} + \frac{4}{5} - \frac{2}{3} + \frac{2}{3} - \frac{4}{7} + \dots + \frac{4}{n+1} - \frac{4}{n+2} + \dots$$

$$2 - \frac{4}{7} + \dots + \frac{4}{n+1} - \frac{4}{n+2} + \dots$$

But if we were to continue with the pattern, those middle terms cancel, and we're left with only

$$2 - \frac{4}{n+2}$$

Take the limit as $n \to \infty$.

$$\lim_{n\to\infty} 2 - \frac{4}{n+2} = 2 - \frac{4}{\infty+2} = 2 - 0 = 2$$

The series converges.

■ 5. Say whether the telescoping series converges or diverges.

$$\sum_{n=1}^{\infty} \frac{5}{n+1} - \frac{5}{n+2}$$

Solution:

Find the first few terms of this rewritten series.

$$n=1 \qquad \frac{5}{n+1} - \frac{5}{n+2} = \frac{5}{1+1} - \frac{5}{1+2} = \frac{5}{2} - \frac{5}{3} = \frac{5}{6}$$

$$n=2 \qquad \frac{5}{n+1} - \frac{5}{n+2} = \frac{5}{2+1} - \frac{5}{2+2} = \frac{5}{3} - \frac{5}{4} = \frac{5}{12}$$

$$n=3 \qquad \frac{5}{n+1} - \frac{5}{n+2} = \frac{5}{3+1} - \frac{5}{3+2} = \frac{5}{4} - 1 = \frac{1}{4}$$

$$n=4 \qquad \frac{5}{n+1} - \frac{5}{n+2} = \frac{5}{4+1} - \frac{5}{4+2} = 1 - \frac{5}{6} = \frac{1}{6}$$

$$n=5 \qquad \frac{5}{n+1} - \frac{5}{n+2} = \frac{5}{5+1} - \frac{5}{5+2} = \frac{5}{6} - \frac{5}{7} = \frac{5}{42}$$

If we use these terms to write out the expanded series, we get

$$\left(\frac{5}{2} - \frac{5}{3}\right) + \left(\frac{5}{3} - \frac{5}{4}\right) + \left(\frac{5}{4} - 1\right)$$

$$+\left(1 - \frac{5}{6}\right) + \left(\frac{5}{6} - \frac{5}{7}\right) + \dots + \left(\frac{5}{n+1} - \frac{5}{n+2}\right) + \dots$$

$$\frac{5}{2} - \frac{5}{3} + \frac{5}{3} - \frac{5}{4} + \frac{5}{4} - 1 + 1 - \frac{5}{6} + \frac{5}{6} - \frac{5}{7} + \dots + \frac{5}{n+1} - \frac{5}{n+2} + \dots$$

$$\frac{5}{2}-\frac{5}{7}+\ldots+\frac{5}{n+1}-\frac{5}{n+2}+\ldots$$

But if we were to continue with the pattern, those middle terms cancel, and we're left with only

$$\frac{5}{2}-\frac{5}{n+2}$$

Take the limit as $n \to \infty$.

$$\lim_{n\to\infty}\frac{5}{2}-\frac{5}{n+2}=\frac{5}{2}-\frac{5}{\infty+1}=\frac{5}{2}-0=\frac{5}{2}$$

The series converges.

SUM OF A TELESCOPING SERIES

1. Calculate the sum of the telescoping series.

$$\sum_{n=1}^{\infty} \frac{3}{n^2+n}$$

Solution:

Use a partial fractions decomposition to rewrite the series.

$$\sum_{n=1}^{\infty} \frac{3}{n^2+n} = \sum_{n=1}^{\infty} \frac{3}{n} - \frac{3}{n+1}$$

Then the first few terms of the series are

$$n = 1 \qquad \frac{3}{n} - \frac{3}{n+1} = \frac{3}{1} - \frac{3}{1+1} = 3 - \frac{3}{2}$$

$$n = 2 \qquad \frac{3}{n} - \frac{3}{n+1} = \frac{3}{2} - \frac{3}{2+1} = \frac{3}{2} - 1$$

$$n = 3 \qquad \frac{3}{n} - \frac{3}{n+1} = \frac{3}{3} - \frac{3}{3+1} = 1 - \frac{3}{4}$$

$$n = 4 \qquad \frac{3}{n} - \frac{3}{n+1} = \frac{3}{4} - \frac{3}{4+1} = \frac{3}{4} - \frac{3}{5}$$

$$n = 5 \qquad \frac{3}{n} - \frac{3}{n+1} = \frac{3}{5} - \frac{3}{5+1} = \frac{3}{5} - \frac{1}{2}$$

If we use these terms to write out the expanded series, we get

$$\left(3-\frac{3}{2}\right)+\left(\frac{3}{2}-1\right)+\left(1-\frac{3}{4}\right)$$

$$+\left(\frac{3}{4}-\frac{3}{5}\right)+\left(\frac{3}{5}-\frac{1}{2}\right)+\ldots+\left(\frac{3}{n}-\frac{3}{n+1}\right)+\ldots$$

$$3-\frac{3}{2}+\frac{3}{2}-1+1-\frac{3}{4}+\frac{3}{4}-\frac{3}{5}+\frac{3}{5}-\frac{1}{2}+\ldots+\left(\frac{3}{n}-\frac{3}{n+1}\right)+\ldots$$

$$3-\frac{1}{2}+\ldots+\left(\frac{3}{n}-\frac{3}{n+1}\right)+\ldots$$

But if we were to continue with the pattern, those middle terms cancel, and we're left with only

$$3-\frac{3}{n+1}$$

Take the limit as $n \to \infty$.

$$\lim_{n\to\infty} 3-\frac{3}{n+1}=3-\frac{3}{\infty+1}=3-0=3$$

The sum of the series is 3.

2. Calculate the sum of the telescoping series.

$$\sum_{n=1}^{\infty}\frac{4}{n^2+3n+2}$$

Solution:

Use a partial fractions decomposition to rewrite the series.

$$\sum_{n=1}^{\infty} \frac{4}{n^2+3n+2} = \sum_{n=1}^{\infty} \frac{4}{n+1} - \frac{4}{n+2}$$

Then the first few terms of the series are

$$n=1 \qquad \frac{4}{n+1} - \frac{4}{n+2} = \frac{4}{1+1} - \frac{4}{1+2} = 2 - \frac{4}{3} = \frac{2}{3}$$

$$n=2 \qquad \frac{4}{n+1} - \frac{4}{n+2} = \frac{4}{2+1} - \frac{4}{2+2} = \frac{4}{3} - 1 = \frac{1}{3}$$

$$n=3 \qquad \frac{4}{n+1} - \frac{4}{n+2} = \frac{4}{3+1} - \frac{4}{3+2} = 1 - \frac{4}{5} = \frac{1}{5}$$

$$n=4 \qquad \frac{4}{n+1} - \frac{4}{n+2} = \frac{4}{4+1} - \frac{4}{4+2} = \frac{4}{5} - \frac{2}{3} = \frac{2}{15}$$

$$n=5 \qquad \frac{4}{n+1} - \frac{4}{n+2} = \frac{4}{5+1} - \frac{4}{5+2} = \frac{2}{3} - \frac{4}{7} = \frac{2}{21}$$

If we use these terms to write out the expanded series, we get

$$\left(2-\frac{4}{3}\right)+\left(\frac{4}{3}-1\right)+\left(1-\frac{4}{5}\right)$$

$$+\left(\frac{4}{5}-\frac{2}{3}\right)+\left(\frac{2}{3}-\frac{4}{7}\right)+\dots+\left(\frac{4}{n+1}-\frac{4}{n+2}\right)+\dots$$

$$2-\frac{4}{3}+\frac{4}{3}-1+1-\frac{4}{5}+\frac{4}{5}-\frac{2}{3}+\frac{2}{3}-\frac{4}{7}+\ldots+\frac{4}{n+1}-\frac{4}{n+2}+\ldots$$

$$2-\frac{4}{7}+\ldots+\frac{4}{n+1}-\frac{4}{n+2}+\ldots$$

But if we were to continue with the pattern, those middle terms cancel, and we're left with only

$$2-\frac{4}{n+2}$$

Take the limit as $n \to \infty$.

$$\lim_{n\to\infty} 2-\frac{4}{n+2}=2-\frac{4}{\infty+2}=2-0=2$$

The sum of the series is 2.

■ 3. Calculate the sum of the telescoping series.

$$\sum_{n=1}^{\infty}\frac{6}{n+2}-\frac{6}{n+3}$$

Solution:

Then the first few terms of the series are

$$n=1 \qquad \frac{6}{n+2}-\frac{6}{n+3}=\frac{6}{1+2}-\frac{6}{1+3}=2-\frac{3}{2}$$

$n=2$ $\qquad \dfrac{6}{n+2}-\dfrac{6}{n+3}=\dfrac{6}{2+2}-\dfrac{6}{2+3}=\dfrac{3}{2}-\dfrac{6}{5}$

$n=3$ $\qquad \dfrac{6}{n+2}-\dfrac{6}{n+3}=\dfrac{6}{3+2}-\dfrac{6}{3+3}=\dfrac{6}{5}-1$

$n=4$ $\qquad \dfrac{6}{n+2}-\dfrac{6}{n+3}=\dfrac{6}{4+2}-\dfrac{6}{4+3}=1-\dfrac{6}{7}$

$n=5$ $\qquad \dfrac{6}{n+2}-\dfrac{6}{n+3}=\dfrac{6}{5+2}-\dfrac{6}{5+3}=\dfrac{6}{7}-\dfrac{3}{4}$

If we use these terms to write out the expanded series, we get

$$\left(2-\frac{3}{2}\right)+\left(\frac{3}{2}-\frac{6}{5}\right)+\left(\frac{6}{5}-1\right)$$

$$+\left(1-\frac{6}{7}\right)+\left(\frac{6}{7}-\frac{3}{4}\right)+\ldots+\left(\frac{6}{n+2}-\frac{6}{n+3}\right)+\ldots$$

$$2-\frac{3}{2}+\frac{3}{2}-\frac{6}{5}+\frac{6}{5}-1+1-\frac{6}{7}+\frac{6}{7}-\frac{3}{4}+\ldots+\frac{6}{n+2}-\frac{6}{n+3}+\ldots$$

$$2-\frac{3}{4}+\ldots.+\frac{6}{n+2}-\frac{6}{n+3}+\ldots$$

But if we were to continue with the pattern, those middle terms cancel, and we're left with only

$$2-\frac{6}{n+3}$$

Take the limit as $n\to\infty$.

$$\lim_{n\to\infty} 2 - \frac{6}{n+3} = 2 - \frac{6}{\infty + 3} = 2 - 0 = 2$$

The sum of the series is 2.

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