Work done to lift a mass or weight

To calculate the work done when we lift a weight or mass vertically some distance, we'll use the formula

$$W = \int_a^b F(x)\ dx$$

where W is the work done, $F(x)$ is the force equation, and $[a, b]$ is the starting and ending height of the weight or mass.

Oftentimes problems like these will have us use a rope or cable to lift an object up some vertical height. In a problem like this, we'll need to determine the combined force required to lift the rope and the object. The formula for force is

$$F = mg$$

where F is force, m is the mass of the object, and g is the gravitational constant 9.8 m/s^2.

If we're given a weight instead of a mass, we can say that the force required to lift the weight is the same as the weight itself, because gravity has already been factored in when we have the "weight" of the object. Otherwise, if we have a mass, we have to multiply by the gravitational constant in order to get the force required to lift it.

Example

Movers are trying to bring a piano into a third floor apartment. Since the front door is too small, they're going to hoist the piano up the side of the building and in through a window. The steel cable they're using has a mass of 3 kg/m and the piano has a mass of 550 kg. Find the work done to lift the piano to the third floor window, 7 m above the ground.

In order to find the work required to lift the piano, we'll find the work required to lift just the piano, then we'll find the work required to lift the cable, and then we'll add the two together to find the total work required.

Remember that work is just the integral of force over the interval $[a, b]$,

$$W = \int_a^b F(x)\ dx$$

which means that we need to find the force for the piano and the force for the cable. If we start with the piano, we know the mass of the piano is 550 kg, so the force required to lift it is

$$F_p = m_p \cdot g$$

$$F_p = 550 \cdot 9.8$$

$$F_p = 5{,}390$$

Now we can find the work required to lift the piano from 0 m at ground level to 7 m at third floor level.

$$W = \int_a^b F(x)\ dx$$

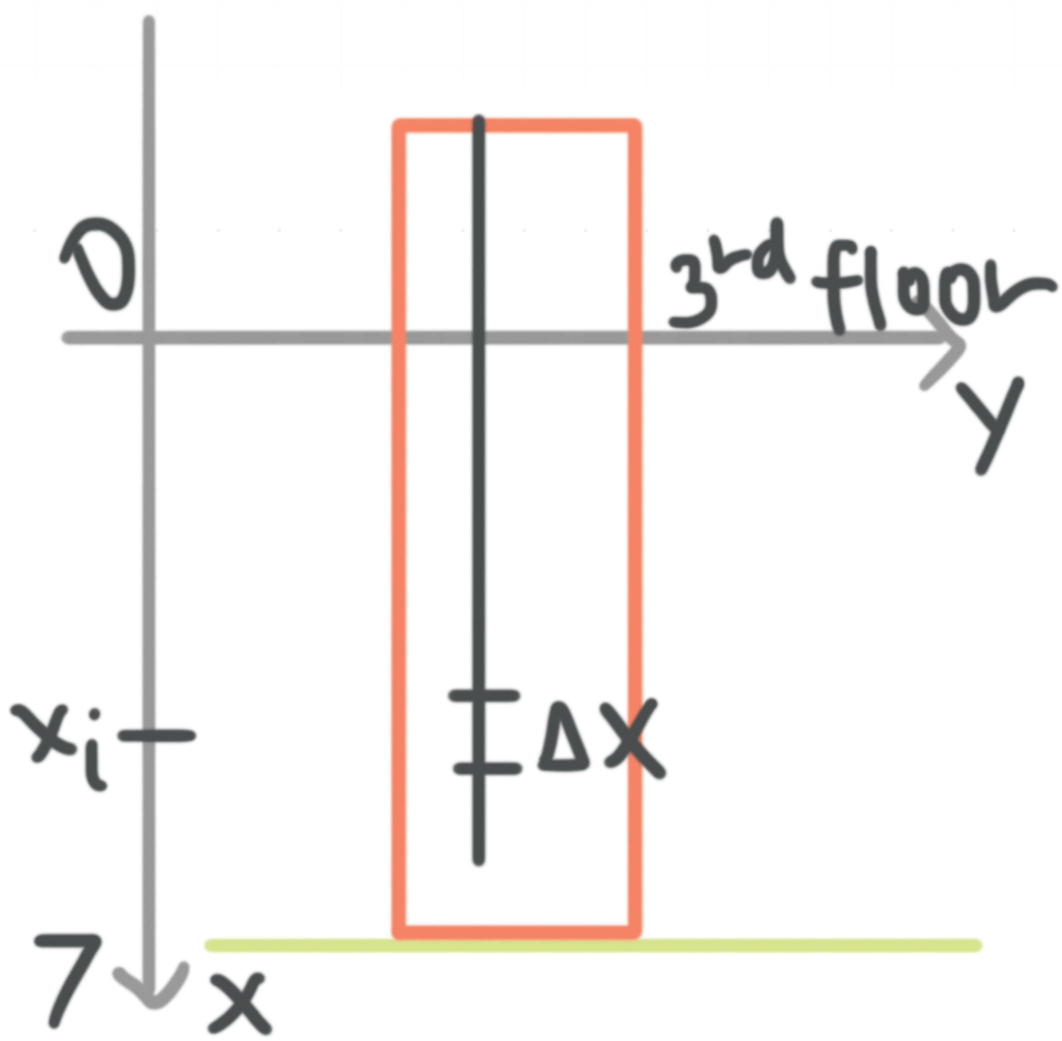
$$W_p = \int_0^7 5{,}390\ dx$$

$$W_p = 5{,}390x \Big|_0^7$$

$$W_p = 5{,}390(7) - 5{,}390(0)$$

$$W_p = 37{,}730$$

Next, we'll find the force for the cable F_c. If we put the apartment building into a coordinate system, with the third floor of the building at the origin and the x-axis extending out toward the base of the building, then we get



We divide the cable into small parts, each with a length of Δx. If we try to model the force for just a single section of Δx, we can say that we have to move that section a distance of x_i to get it to the third floor window, 7 m above the ground. Which means the force exerted on one slice of the cable is

$$F_i = m_i g$$

$$F_i = 3 \cdot 9.8$$

$$F_i = 29.4$$

Now we can find the work required to lift just the same slice of cable.

$$W_i = F_i d$$

$$W_i = 29.4 \cdot x_i$$

$$W_i = 29.4x_i$$

If we want to find the work required to lift the entire cable, and not just a single slice, we'll use an infinite number of slices n and sum the work required to lift every slice.

$$W_c = \lim_{n \to \infty} \sum_{i=1}^{n} 29.4x_i \Delta x$$

Taking the infinite sum is the same as taking the integral, so we'll convert this to integral notation, changing x_i to x and Δx to dx when we do.

$$W_c = \int 29.4x \ dx$$

Since the section of cable at the third floor has to be pulled up 0 m to get to the third floor, and the section of cable at the ground has to be pulled up 7 m to get to the third floor, the integral becomes

$$W_c = \int_0^7 29.4x\ dx$$

$$W_c = \frac{29.4}{2}x^2 \Big|_0^7$$

$$W_c = 14.7x^2 \Big|_0^7$$

$$W_c = 14.7(7)^2 - 14.7(0)^2$$

$$W_c = 14.7(49)$$

$$W_c = 720.3$$

To find the total work required, we'll add together the work required to lift both the piano and the cable.

$$W = W_p + W_c$$

$$W = 37{,}730 + 720.3$$

$$W = 38{,}450.3 \text{ J}$$