Estimating definite integrals

We can use power series to estimate definite integrals in the same way we used them to estimate indefinite integrals. The only difference is that we'll evaluate over the given interval once we find a power series that represents the original integral.

To evaluate over the interval, we'll expand the power series through its first few terms, and then evaluate each term separately over the interval.

Oftentimes we'll be asked to use a power series to approximate the definite integral to a certain number of decimal places. If this is the case, we need to make sure we keep more decimals than we're asked for when we evaluate over the interval. That way, we'll be able to give an accurate answer to the requested number of decimal places when we sum all of our decimal values together.

Example

Use power series to estimate the definite integral to five decimal places.

$$\int_0^{0.2} 4x\arctan(2x)\ dx$$

Since this integral includes an $\arctan$ function, we'll use the standard power series

$$\arctan x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$$

and manipulate it to get the given definite integral.

We can start by replacing x with $2x$ inside the $\arctan$ function in order to match the given function.

$$\arctan(2x) = \sum_{n=0}^{\infty} \frac{(-1)^n (2x)^{2n+1}}{2n+1}$$

Next, we'll multiply both sides by $4x$ in order to make the power series match the function.

$$4x \arctan(2x) = 4x \sum_{n=0}^{\infty} \frac{(-1)^n (2x)^{2n+1}}{2n+1}$$

$$4x \arctan(2x) = 4x^1 \sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n+1} x^{2n+1}}{2n+1}$$

$$4x \arctan(2x) = \sum_{n=0}^{\infty} \frac{4(-1)^n 2^{2n+1} x^{2n+2}}{2n+1}$$

Since the problem asks us to find the integral of the left-hand side of this equation, we'll integrate both sides.

$$\int 4x \arctan(2x) \ dx = \int \sum_{n=0}^{\infty} \frac{4(-1)^n 2^{2n+1} x^{2n+2}}{2n+1} \ dx$$

Because we want to solve for what we have on the left-hand side, we only need to integrate the right side. We're integrating with respect to x, so we can treat the n's as constants.

$$\int 4x\arctan(2x)\ dx = \sum_{n=0}^{\infty}\frac{4(-1)^n 2^{2n+1}x^{2n+3}}{(2n+1)(2n+3)}$$

Adding in the interval from the original question, we get

$$\int_0^{0.2} 4x\arctan(2x)\ dx = \left[\sum_{n=0}^{\infty}\frac{4(-1)^n 2^{2n+1}x^{2n+3}}{(2n+1)(2n+3)}\right]\Bigg|_0^{0.2}$$

In order to evaluate over the interval, we'll expand the power series through the first few terms. Remember, we need to approximate the final answer to five decimal places, which means we'll have to calculate results beyond five decimals until we get to a point where the first five decimal places aren't changing.

$$\int_0^{0.2} 4x\arctan(2x)\ dx = \frac{8x^3}{3} - \frac{32x^5}{15} + \frac{128x^7}{35} - \frac{512x^9}{63} + \frac{2{,}048x^{11}}{99} + \dots\Bigg|_0^{0.2}$$

Evaluating each term separately over the interval, we get

$$\int_0^{0.2} 4x\arctan(2x)\ dx = \left(\frac{8(0.2)^3}{3} - \frac{8(0)^3}{3}\right) - \left(\frac{32(0.2)^5}{15} - \frac{32(0)^5}{15}\right)$$

$$+\left(\frac{128(0.2)^7}{35} - \frac{128(0)^7}{35}\right) - \left(\frac{512(0.2)^9}{63} - \frac{512(0)^9}{63}\right)$$

$$+\left(\frac{2{,}048(0.2)^{11}}{99}-\frac{2{,}048(0)^{11}}{99}\right)+\ldots$$

$$\int_0^{0.2} 4x \arctan(2x)\ dx \approx 0.0213333-0.0006827+0.0000468-0.0000041+0.0000004+\ldots$$

Let's start adding the terms together.

$$n_0+n_1=0.0206506$$

$$n_0+n_1+n_2=0.0206974$$

$$n_0+n_1+n_2+n_3=0.0206933$$

$$n_0+n_1+n_2+n_3+n_4=0.0206937$$

Remember, we only need the first five decimal places. When we analyze our results, we can see that $n_0+n_1+n_2+n_3=0.0206933$ is as far as we need to add in order to get five stable decimal points. Therefore, rounding the answer to five decimal places gives

$$n_0+n_1+n_2+n_3 \approx 0.02069$$