Probability density functions

Probability density refers to the probability that a continuous random variable X will exist within a set of conditions. It follows that using the probability density equations will tell us the likelihood of an X existing in the interval $[a, b]$.

A probability density function $f(x)$ must meet these conditions:

1. $f(x) \geq 0$ for all values of x

2. $\int_{-\infty}^{\infty} f(x)\ dx = 1$

The equation for probability density is

$$P(a \leq X \leq b) = \int_{a}^{b} f(x)\ dx$$

where $P(a \leq X \leq b)$ is the probability that X exists in $[a, b]$.

Example

Show that $f(x)$ is a probability density function and find $P(1 \leq X \leq 4)$.

$$f(x) = \left(\frac{x^3}{5{,}000}\right)(10 - x)$$

for $0 \leq x \leq 10$ and $f(x) = 0$ for all other values of x

The first thing we need to do is show that $f(x)$ is a probability density function. We can see that the interval $0 \leq x \leq 10$ is positive. For all other possibilities we know that $f(x) = 0$. This means we've satisfied the first criteria for a probability density equation. Now we need to verify that

$$\int_{-\infty}^{\infty} f(x)\ dx = 1$$

We can set the interval to $[0,10]$ since it's only in this interval that the equation doesn't equal 0.

$$\int_{-\infty}^{\infty} f(x)\ dx = \int_{0}^{10} \left(\frac{x^3}{5,000}\right)(10 - x)\ dx$$

$$\int_{-\infty}^{\infty} f(x)\ dx = \int_{0}^{10} \frac{x^3}{500} - \frac{x^4}{5,000}\ dx$$

$$\int_{-\infty}^{\infty} f(x)\ dx = \int_{0}^{10} \frac{x^3}{500}\ dx + \int_{0}^{10} -\frac{x^4}{5,000}\ dx$$

$$\int_{-\infty}^{\infty} f(x)\ dx = \frac{x^4}{2,000} - \frac{x^5}{25,000}\Bigg|_{0}^{10}$$

$$\int_{-\infty}^{\infty} f(x)\ dx = \left[\frac{(10)^4}{2,000} - \frac{(10)^5}{25,000}\right] - \left[\frac{(0)^4}{2,000} - \frac{(0)^5}{25,000}\right]$$

$$\int_{-\infty}^{\infty} f(x)\ dx = 1$$

The equation has met both of the criteria, so we've verified that it's a probability density function.

In order to solve for $P(1 \le X \le 4)$, we'll identify the interval $[1,4]$ and plug it into the probability density equation.

$$P(a \le X \le b) = \int_a^b f(x)\ dx$$

$$P(1 \le X \le 4) = \int_1^4 \left(\frac{x^3}{5{,}000}\right)(10 - x)\ dx$$

$$P(1 \le X \le 4) = \int_1^4 \frac{x^3}{500} - \frac{x^4}{5{,}000}\ dx$$

$$P(1 \le X \le 4) = \int_1^4 \frac{x^3}{500}\ dx + \int_0^{10} -\frac{x^4}{5{,}000}\ dx$$

$$P(1 \le X \le 4) = \frac{x^4}{2{,}000} - \frac{x^5}{25{,}000}\Bigg|_1^4$$

$$P(1 \le X \le 4) = \left[\frac{(4)^4}{2{,}000} - \frac{(4)^5}{25{,}000}\right] - \left[\frac{(1)^4}{2{,}000} - \frac{(1)^5}{25{,}000}\right]$$

$$P(1 \le X \le 4) = 0.0866$$

The answer tell us that the probability of X existing between 1 and 4 is about $8.66\,\%$.
