Topic: Integration by parts

Question: Use integration by parts to evaluate the integral.

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} x \csc^2 x \ dx$$

Answer choices:

A $\frac{\sqrt{3}}{3}\pi + \ln 2$

B $\frac{\sqrt{3}}{6}\pi - \ln 2$

C $\frac{\sqrt{3}}{6}\pi + \ln 2$

D $\frac{\sqrt{3}}{3}\pi - \ln 2$

Solution: C

To use integration by parts, first identify suitable expressions for u and dv such as

$$u = x$$

$$dv = \csc^2 x \ dx$$

Take the differential of u and the integral of dv.

$$du = dx$$

$$v = \int \csc^2 x \ dx = -\cot x$$

Plug the values into the formula for integration by parts.

$$\int u \ dv = uv - \int v \ du$$

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} x \csc^2 x \ dx = -x \cot x \Big|_{\frac{\pi}{6}}^{\frac{\pi}{2}} - \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} -\cot x \ dx$$

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} x \csc^2 x \ dx = -x \cot x \Big|_{\frac{\pi}{6}}^{\frac{\pi}{2}} + \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \cot x \ dx$$

We know that $\int \cot x \ dx = \ln(\sin x) + C$, so we apply this formula and get

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} x \csc^2 x \ dx$$

$$-x\cot x+\ln(\sin x)\Big|_{\frac{\pi}{6}}^{\frac{\pi}{2}}$$

$$\left[-\frac{\pi}{2}\cot\frac{\pi}{2}+\ln\left(\sin\frac{\pi}{2}\right)\right]-\left[-\frac{\pi}{6}\cot\frac{\pi}{6}+\ln\left(\sin\frac{\pi}{6}\right)\right]$$

$$-\frac{\pi}{2}\cot\frac{\pi}{2}+\ln\left(\sin\frac{\pi}{2}\right)+\frac{\pi}{6}\cot\frac{\pi}{6}-\ln\left(\sin\frac{\pi}{6}\right)$$

$$-\frac{\pi}{2}\cdot\frac{\cos\frac{\pi}{2}}{\sin\frac{\pi}{2}}+\ln 1+\frac{\pi}{6}\cdot\frac{\cos\frac{\pi}{6}}{\sin\frac{\pi}{6}}-\ln\frac{1}{2}$$

$$-\frac{\pi}{2}\cdot\frac{0}{1}+\ln 1+\frac{\pi}{6}\cdot\frac{\sqrt{3}/2}{1/2}-\ln\frac{1}{2}$$

$$0+0+\frac{\pi}{6}\cdot\frac{\sqrt{3}}{2}\cdot\frac{2}{1}-\ln\frac{1}{2}$$

$$\pi\frac{\sqrt{3}}{6}-\ln\frac{1}{2}$$

$$\pi\frac{\sqrt{3}}{6}-\ln 2^{-1}$$

$$\pi\frac{\sqrt{3}}{6}-(-1)\ln 2$$

$$\frac{\sqrt{3}}{6}\pi+\ln 2$$

Topic: Integration by parts

Question: Use integration by parts to evaluate the integral.

$$\int xe^x \, dx$$

Answer choices:

A $e^x(x+1)+C$

B $e^x(x-1)+C$

C xe^x+C

D $e^{2x}+C$

Solution: B

To use integration by parts, first identify suitable expressions for u and dv such as

$$u = x$$

$$dv = e^x\ dx$$

Take the differential of u and the integral of dv.

$$du = dx$$

$$v = e^x$$

Plug the values into the formula for integration by parts.

$$\int u\ dv = uv - \int v\ du$$

$$\int xe^x\ dx = xe^x - \int e^x\ dx$$

Now we can integrate, and the value of the integral is

$$xe^x - e^x + C$$

$$e^x(x - 1) + C$$

Topic: Integration by parts

Question: Use integration by parts to evaluate the integral.

$$\int x \cos x \ dx$$

Answer choices:

A $\quad x \sin x + \cos x$

B $\quad x \sin x + \cos x + C$

C $\quad x \cos x - \sin x + C$

D $\quad x \sin x - \cos x + C$

Solution: B

The question asks us to evaluate

$$\int x \cos x \ dx$$

using integration by parts.

Integration by parts is a method of evaluating an integral that cannot be evaluated using normal integration techniques, by using integration by substitution, or by using integration formulas.

The general formula for integration by parts is

$$\int u \ dv = uv - \int v \ du$$

In this formula, we separate the integrand into two parts; one part is called u and the other part is called dv. In making these two parts, we must use all of the integrand.

Although there is sometimes flexibility in choosing u, we can generally use the following sequence of choices to select the best part of the integrand to be u. This method involves the acronym LIPET, where we select the first u in the sequence of the list below. The letters mean

L Logarithmic expression

I Inverse trigonometric expression

P Polynomial expression

E Exponential expression

T Trigonometric function expression

In this problem, the integrand is $x \cos x$ where we have a polynomial and a trigonometric function. In the LIPET sequence, polynomial comes before trigonometric function, so u is the polynomial. Let's identify the parts we need to integrate.

$$u = x$$

$$du = dx$$

$$dv = \cos x \ dx$$

$$v = \sin x$$

We are now ready to integrate by parts using the general formula.

$$\int u \ dv = uv - \int v \ du$$

$$\int x \cos x \ dx = x \sin x - \int \sin x \ dx$$

Next, let's evaluate the new integral.

$$x \sin x - \int \sin x \ dx = x \sin x - (-\cos x) + C$$

We added a "C" to accommodate the possibility of a constant in the function. The last step is to simplify the result of the integration.

$$\int x\cos x\ dx = x\sin x + \cos x + C$$