Theorem of Pappus

The Theorem of Pappus tells us that the volume of a three-dimensional solid object that's created by rotating a two-dimensional shape around an axis is given by

$$V = Ad$$

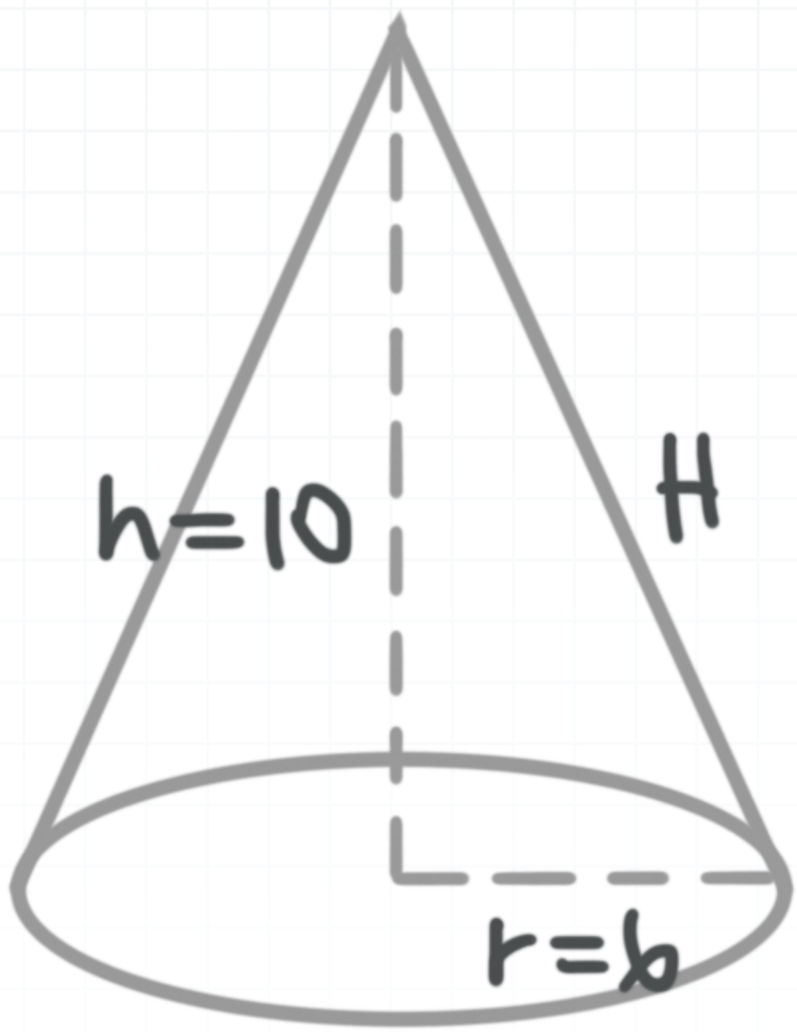
where V is the volume of the three-dimensional object, A is the area of the two-dimensional figure being revolved, and d is the distance traveled by the centroid of the two-dimensional figure.

Example

Use the Theorem of Pappus to find the volume of a right circular cone with radius $r = 6$ and height $h = 10$.

The Theorem of Pappus defines volume as $V = Ad$. Before we can solve for volume we need to find the area of the triangle we're revolving. Our shape, the right circular cone, can be described as a triangle rotated around an axis. The formula for area of a triangle is

$$A = \frac{1}{2}bh$$



The base of the triangle will be the radius $r = b = 6$, and the height of the triangle will be $h = 10$.

$$A = \frac{1}{2}(6)(10)$$

$$A = 30$$

Next, we need to solve for distance, d. Distance will involve the relationship of the triangle's centroid and the rotation it experiences. In other words, $d = 2\pi\bar{x}$ where $\bar{x}$ is the x-coordinate of the centroid and 2π refers to the fact that the object is being rotated around an axis. The equation for $\bar{x}$ is

$$\bar{x} = \frac{1}{A}\int_a^b xf(x)\ dx$$

Looking at this equation we realize we're still missing $f(x)$, which is the third side of the triangle, H. If we position the center of the base of the cone at the origin $(0,0)$, then the right edge of the base of the cone sits at $(6,0)$, and the point at the top of the cone sits at $(0,10)$.

Therefore, the equation that models the hypotenuse H is the equation of the line passing through $(0,10)$ and $(6,0)$. The slope of that line is

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{10 - 0}{0 - 6} = \frac{10}{-6} = -\frac{5}{3}$$

Then the equation of the line modeling the hypotenuse is

$$y - y_1 = m(x - x_1)$$

$$y - 0 = -\frac{5}{3}(x - 6)$$

$$y = -\frac{5}{3}x + 10$$

Now we can solve for $\bar{x}$.

$$\bar{x} = \frac{1}{30}\int_0^6 x\left(-\frac{5}{3}x + 10\right)\ dx$$

$$\bar{x} = -\frac{1}{30}\int_0^6 \frac{5}{3}x^2 - 10x\ dx$$

$$\bar{x} = -\frac{1}{30}\left(\frac{5}{9}x^3 - 5x^2\right)\Big|_0^6$$

$$\bar{x} = \frac{1}{6}x^2 - \frac{1}{54}x^3 \Big|_0^6$$

$$\bar{x} = \frac{1}{6}(6)^2 - \frac{1}{54}(6)^3 - \left(\frac{1}{6}(0)^2 - \frac{1}{54}(0)^3\right)$$

$$\bar{x} = 6 - 4$$

$$\bar{x} = 2$$

Now we can solve for distance $d = 2\pi\bar{x}$.

$$d = 2\pi(2)$$

$$d = 4\pi$$

Finally, we can solve for volume using $V = Ad$.

$$V = 30(4\pi)$$

$$V = 120\pi$$