**Topic**: Area between polar curves

**Question**: Find the area between the polar curves.

$$r = 3 - 3\cos\theta$$

$$r = 3$$

**Answer choices**:

A $45$

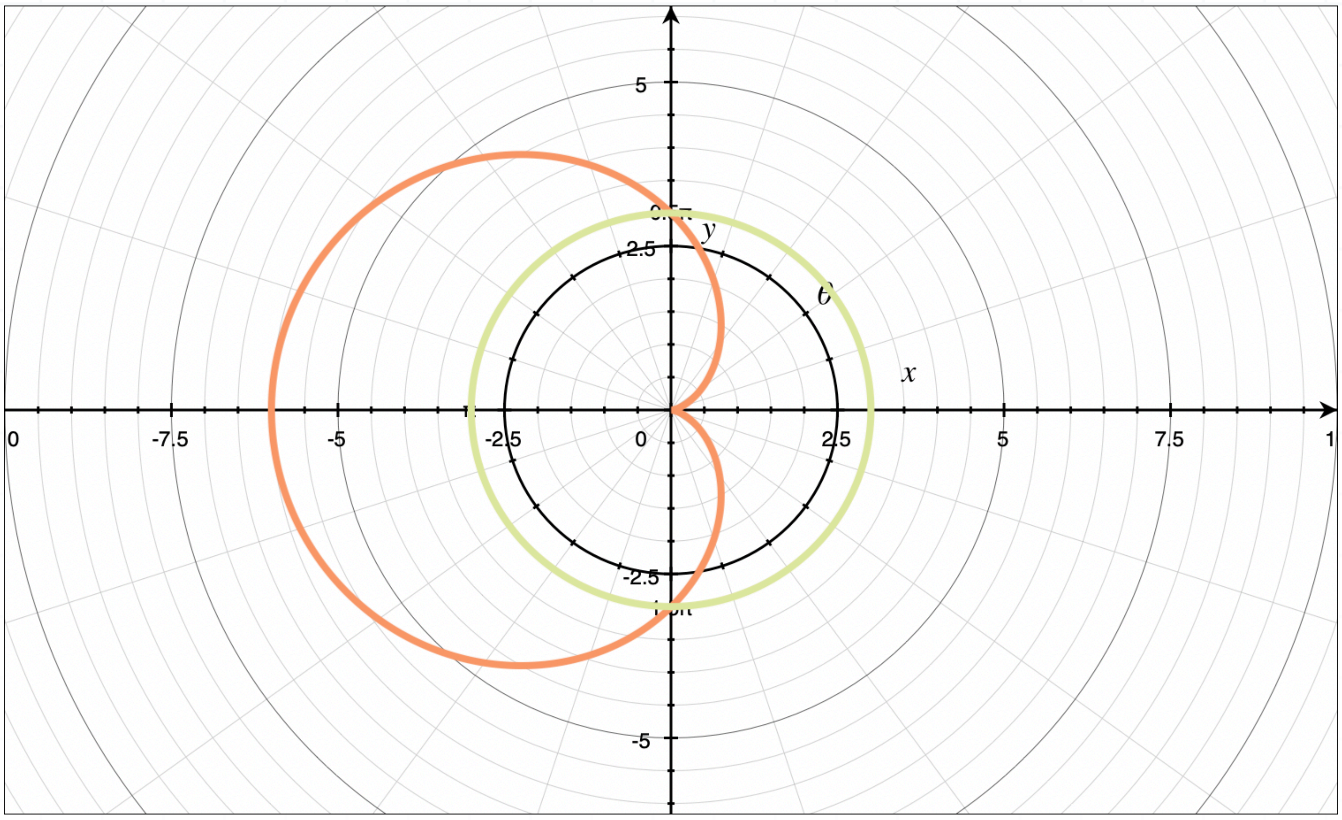
B $36$

C $18$

D $9$

**Solution**: B

The area between the polar curves looks like this:



We can split the area between the curves into two parts:

1. The area outside the cardioid but inside the circle in the first and fourth quadrants, and

2. The area outside the circle but inside the cardioid in the second and third quadrants.

The curves intersect at $(3,\pi/2)$ and $(3,3\pi/2)$. We can also write $(3,3\pi/2)$ as $(3,-\pi/2)$. So our area formula will be

$$A=\frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} r_{\text{circle}}^2-r_{\text{cardioid}}^2\ d\theta+\frac{1}{2}\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} r_{\text{cardioid}}^2-r_{\text{circle}}^2\ d\theta$$

$$A=\frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}3^2-(3-3\cos\theta)^2\ d\theta+\frac{1}{2}\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}}(3-3\cos\theta)^2-3^2\ d\theta$$

$$A=\frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}9-(9-18\cos\theta+9\cos^2\theta)\ d\theta+\frac{1}{2}\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}}(9-18\cos\theta+9\cos^2\theta)-9\ d\theta$$

$$A=\frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}9-9+18\cos\theta-9\cos^2\theta\ d\theta+\frac{1}{2}\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}}9-18\cos\theta+9\cos^2\theta-9\ d\theta$$

$$A=\frac{9}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}2\cos\theta-\cos^2\theta\ d\theta+\frac{9}{2}\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}}\cos^2\theta-2\cos\theta\ d\theta$$

Using the power reduction formula

$$\cos^2\theta=\frac{1}{2}+\frac{1}{2}\cos 2\theta$$

we get

$$A=\frac{9}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}2\cos\theta-\left(\frac{1}{2}+\frac{1}{2}\cos 2\theta\right)\ d\theta+\frac{9}{2}\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}}\frac{1}{2}+\frac{1}{2}\cos 2\theta-2\cos\theta\ d\theta$$

$$A=\frac{9}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}2\cos\theta-\frac{1}{2}-\frac{1}{2}\cos 2\theta\ d\theta+\frac{9}{2}\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}}\frac{1}{2}+\frac{1}{2}\cos 2\theta-2\cos\theta\ d\theta$$

Integrate term by term.

$$A=\frac{9}{2}\left(2\sin\theta-\frac{1}{2}\theta-\frac{1}{4}\sin 2\theta\right)\Bigg|_{-\frac{\pi}{2}}^{\frac{\pi}{2}}+\frac{9}{2}\left(\frac{1}{2}\theta+\frac{1}{4}\sin 2\theta-2\sin\theta\right)\Bigg|_{\frac{\pi}{2}}^{\frac{3\pi}{2}}$$

$$A=\frac{9}{2}\left[2\sin\frac{\pi}{2}-\frac{1}{2}\left(\frac{\pi}{2}\right)-\frac{1}{4}\sin\left(2\cdot\frac{\pi}{2}\right)\right]-\frac{9}{2}\left[2\sin\left(-\frac{\pi}{2}\right)-\frac{1}{2}\left(-\frac{\pi}{2}\right)-\frac{1}{4}\sin\left(2\cdot-\frac{\pi}{2}\right)\right]$$

$$+\frac{9}{2}\left[\frac{1}{2}\left(\frac{3\pi}{2}\right)+\frac{1}{4}\sin\left(2\cdot\frac{3\pi}{2}\right)-2\sin\frac{3\pi}{2}\right]-\frac{9}{2}\left[\frac{1}{2}\left(\frac{\pi}{2}\right)+\frac{1}{4}\sin\left(2\cdot\frac{\pi}{2}\right)-2\sin\frac{\pi}{2}\right]$$

$$A=\frac{9}{2}\left[2(1)-\frac{\pi}{4}-\frac{1}{4}\sin\pi\right]-\frac{9}{2}\left[2(-1)+\frac{\pi}{4}-\frac{1}{4}\sin(-\pi)\right]$$

$$+\frac{9}{2}\left[\frac{3\pi}{4}+\frac{1}{4}\sin(3\pi)-2(-1)\right]-\frac{9}{2}\left[\frac{\pi}{4}+\frac{1}{4}\sin\pi-2(1)\right]$$

$$A=\frac{9}{2}\left[2-\frac{\pi}{4}-\frac{1}{4}(0)\right]-\frac{9}{2}\left[-2+\frac{\pi}{4}-\frac{1}{4}(0)\right]$$

$$+\frac{9}{2}\left[\frac{3\pi}{4}+\frac{1}{4}(0)+2\right]-\frac{9}{2}\left[\frac{\pi}{4}+\frac{1}{4}(0)-2\right]$$

$$A=9-\frac{9\pi}{8}+9-\frac{9\pi}{8}+\frac{27\pi}{8}+9-\frac{9\pi}{8}+9$$

$$A=36-\frac{27\pi}{8}+\frac{27\pi}{8}$$

$$A=36$$

**Topic**: Area between polar curves

**Question**: Find the area between the polar curves.

$$r = \frac{1}{2} + \cos\theta$$

$$r = 1$$

**Answer choices**:

A $\dfrac{-2\pi - 27\sqrt{3}}{24}$

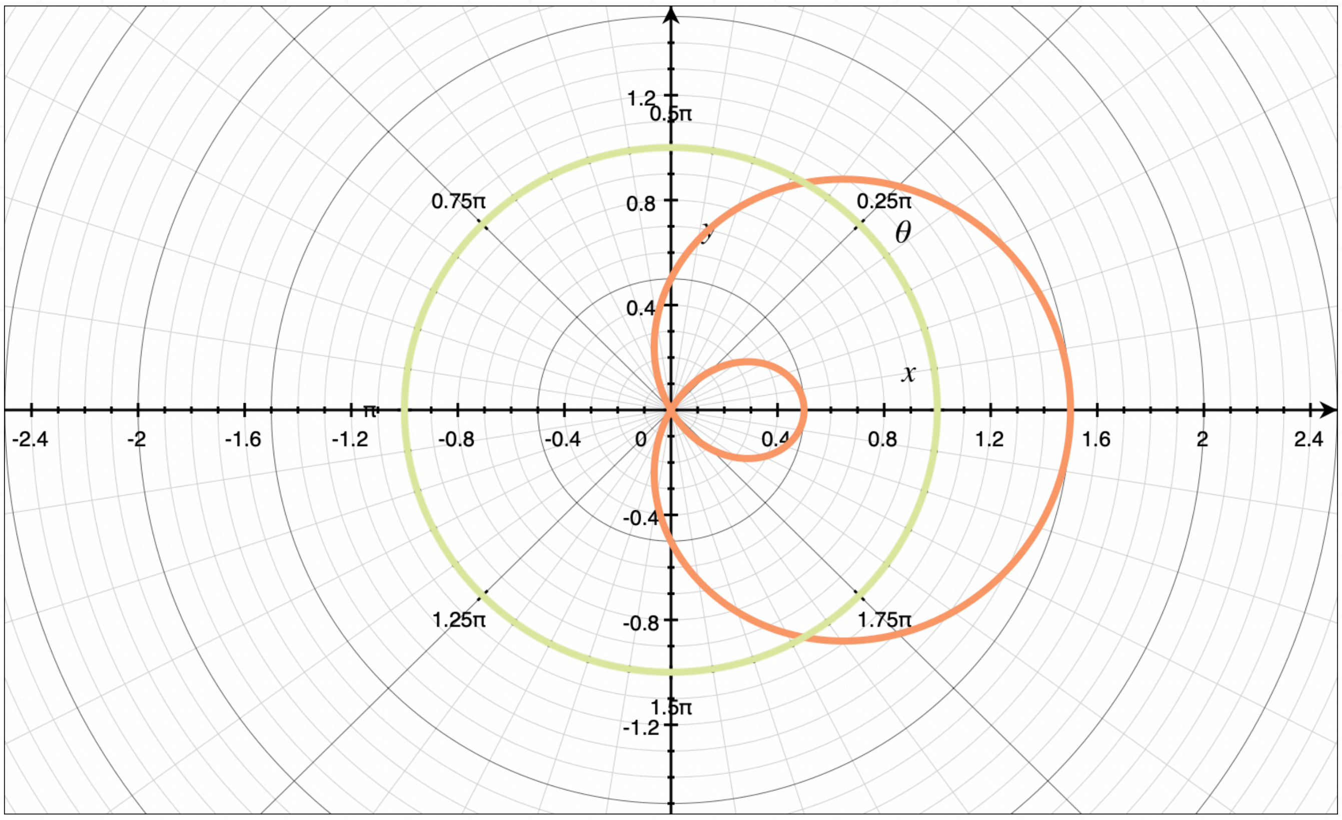
B $\dfrac{-2 + 27\sqrt{3}}{24}$

C $\dfrac{\pi - 15\sqrt{3}}{12}$

D $\dfrac{\pi + 15\sqrt{3}}{12}$

**Solution**: D

The area between the polar curves looks like this:




We can split the area between the curves into two parts:

1. The area outside the circle but inside the cardioid in the first and fourth quadrants, and

2. The area outside the cardioid but inside the circle (on the left).

The curves intersect at $(1,\pi/3)$ and $(1,5\pi/3)$. We can also write $(1,5\pi/3)$ as $(1,-\pi/3)$. So our area formula will be

$$A=\frac{1}{2}\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} r^2_{\text{cardioid}}-r^2_{\text{circle}}\ d\theta+\frac{1}{2}\int_{\frac{\pi}{3}}^{\frac{5\pi}{3}} r^2_{\text{circle}}-r^2_{\text{cardioid}}\ d\theta$$

$$A=\frac{1}{2}\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \left(\frac{1}{2}+\cos\theta\right)^2-(1)^2\ d\theta+\frac{1}{2}\int_{\frac{\pi}{3}}^{\frac{5\pi}{3}} (1)^2-\left(\frac{1}{2}+\cos\theta\right)^2\ d\theta$$

$$A=\frac{1}{2}\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}}\frac{1}{4}+\cos\theta+\cos^2\theta-1\ d\theta+\frac{1}{2}\int_{\frac{\pi}{3}}^{\frac{5\pi}{3}}1-\left(\frac{1}{4}+\cos\theta+\cos^2\theta\right)\ d\theta$$

$$A=\frac{1}{2}\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}}\cos^2\theta+\cos\theta-\frac{3}{4}\ d\theta+\frac{1}{2}\int_{\frac{\pi}{3}}^{\frac{5\pi}{3}}\frac{3}{4}-\cos\theta-\cos^2\theta\ d\theta$$

Using the power reduction formula

$$\cos^2\theta=\frac{1}{2}+\frac{1}{2}\cos 2\theta$$

we get

$$A=\frac{1}{2}\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}}\frac{1}{2}+\frac{1}{2}\cos 2\theta+\cos\theta-\frac{3}{4}\ d\theta+\frac{1}{2}\int_{\frac{\pi}{3}}^{\frac{5\pi}{3}}\frac{3}{4}-\cos\theta-\left(\frac{1}{2}+\frac{1}{2}\cos 2\theta\right)\ d\theta$$

$$A=\frac{1}{2}\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}}\frac{1}{2}\cos 2\theta+\cos\theta-\frac{1}{4}\ d\theta+\frac{1}{2}\int_{\frac{\pi}{3}}^{\frac{5\pi}{3}}\frac{1}{4}-\cos\theta-\frac{1}{2}\cos 2\theta\ d\theta$$

Integrate term by term.

$$A=\frac{1}{2}\left(\frac{1}{4}\sin 2\theta+\sin\theta-\frac{1}{4}\theta\right)\Bigg|_{-\frac{\pi}{3}}^{\frac{\pi}{3}}+\frac{1}{2}\left(\frac{1}{4}\theta-\sin\theta-\frac{1}{4}\sin 2\theta\right)\Bigg|_{\frac{\pi}{3}}^{\frac{5\pi}{3}}$$

$$A=\frac{1}{8}(\sin 2\theta+4\sin\theta-\theta)\Bigg|_{-\frac{\pi}{3}}^{\frac{\pi}{3}}+\frac{1}{8}(\theta-4\sin\theta-\sin 2\theta)\Bigg|_{\frac{\pi}{3}}^{\frac{5\pi}{3}}$$

$$A=\frac{1}{8}\left(\sin\frac{2\pi}{3}+4\sin\frac{\pi}{3}-\frac{\pi}{3}\right)-\frac{1}{8}\left(\sin\left(-\frac{2\pi}{3}\right)+4\sin\left(-\frac{\pi}{3}\right)+\frac{\pi}{3}\right)$$

$$+\frac{1}{8}\left(\frac{5\pi}{3}-4\sin\frac{5\pi}{3}-\sin\frac{10\pi}{3}\right)-\frac{1}{8}\left(\frac{\pi}{3}-4\sin\frac{\pi}{3}-\sin\frac{2\pi}{3}\right)$$

$$A=\frac{1}{8}\left(\frac{\sqrt{3}}{2}+4\frac{\sqrt{3}}{2}-\frac{\pi}{3}\right)-\frac{1}{8}\left[-\frac{\sqrt{3}}{2}+4\left(-\frac{\sqrt{3}}{2}\right)+\frac{\pi}{3}\right]$$

$$+\frac{1}{8}\left[\frac{5\pi}{3}-4\left(-\frac{\sqrt{3}}{2}\right)-\left(-\frac{\sqrt{3}}{2}\right)\right]-\frac{1}{8}\left(\frac{\pi}{3}-4\frac{\sqrt{3}}{2}-\frac{\sqrt{3}}{2}\right)$$

$$A=\frac{1}{8}\left(\frac{5\sqrt{3}}{2}-\frac{\pi}{3}\right)-\frac{1}{8}\left(-\frac{5\sqrt{3}}{2}+\frac{\pi}{3}\right)+\frac{1}{8}\left(\frac{5\pi}{3}+\frac{5\sqrt{3}}{2}\right)-\frac{1}{8}\left(\frac{\pi}{3}-\frac{5\sqrt{3}}{2}\right)$$

$$A=\frac{5\sqrt{3}}{16}-\frac{\pi}{24}+\frac{5\sqrt{3}}{16}-\frac{\pi}{24}+\frac{5\pi}{24}+\frac{5\sqrt{3}}{16}-\frac{\pi}{24}+\frac{5\sqrt{3}}{16}$$

$$A=\frac{5\sqrt{3}}{4}+\frac{\pi}{12}$$

$$A=\frac{\pi+15\sqrt{3}}{12}$$

**Topic**: Area between polar curves

**Question**: Find the area between the polar curves.

$$r = -2\cos\theta$$

$$r = 1$$

**Answer choices**:

A $\quad -\dfrac{4\pi + 3\sqrt{3}}{6}$

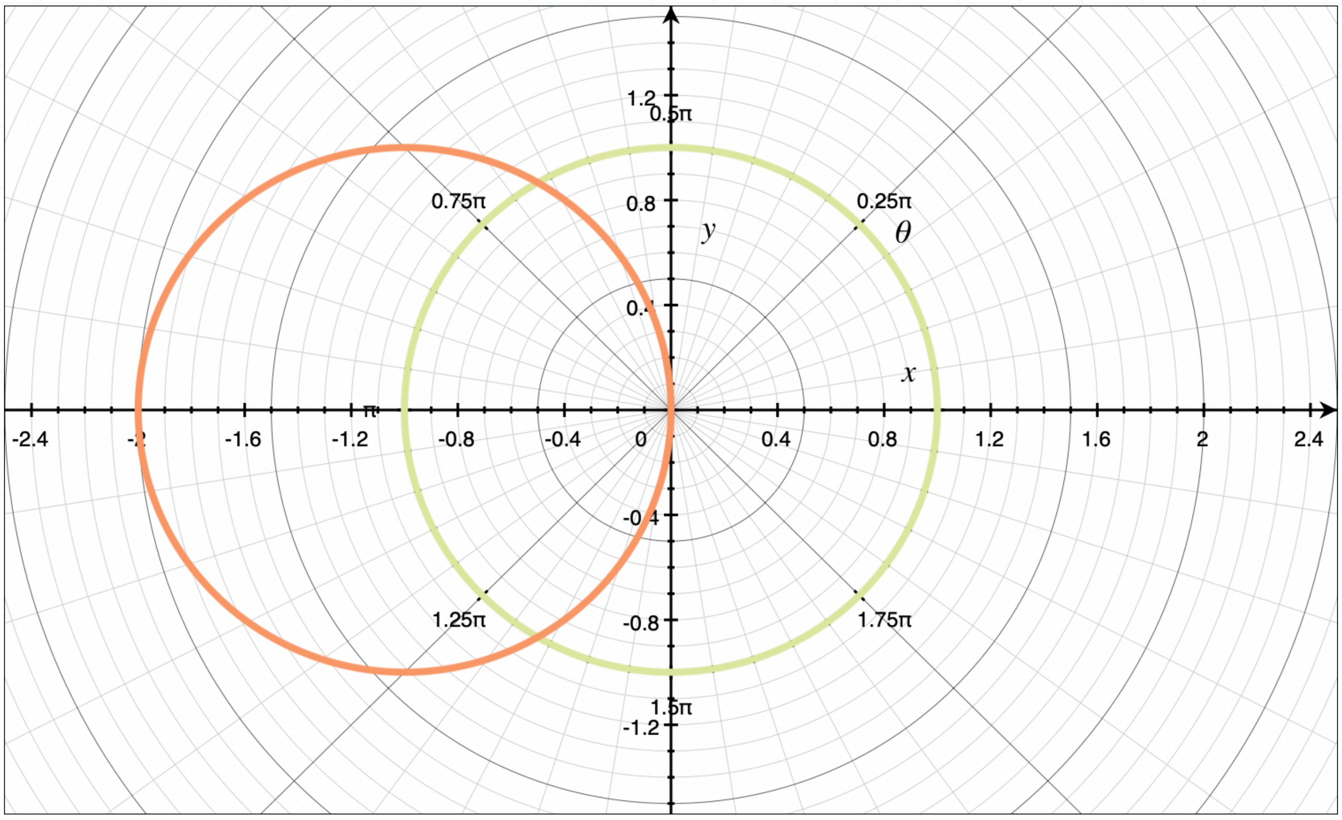
B $\quad \dfrac{4\pi + 3\sqrt{3}}{6}$

C $\quad \dfrac{-\pi + 3\sqrt{3}}{3}$

D $\quad \dfrac{\pi - 3\sqrt{3}}{3}$

**Solution**: C

The area between the polar curves looks like this:




We can split the area between the curves into two parts:

1. The area outside $r=-2\cos\theta$ but inside the circle (on the right), and

2. The area outside the circle but inside $r=-2\cos\theta$ (on the left).

The curves intersect at $(1,2\pi/3)$ and $(1,4\pi/3)$. We can also write $(1,4\pi/3)$ as $(1,-2\pi/3)$. So our area formula will be

$$A=\frac{1}{2}\int_{-\frac{2\pi}{3}}^{\frac{2\pi}{3}}(1)^2-(-2\cos\theta)^2\ d\theta+\frac{1}{2}\int_{\frac{2\pi}{3}}^{\frac{4\pi}{3}}(-2\cos\theta)^2-(1)^2\ d\theta$$

$$A=\frac{1}{2}\int_{-\frac{2\pi}{3}}^{\frac{2\pi}{3}}1-4\cos^2\theta\ d\theta+\frac{1}{2}\int_{\frac{2\pi}{3}}^{\frac{4\pi}{3}}4\cos^2\theta-1\ d\theta$$

Using the power reduction formula

$$\cos^2\theta = \frac{1}{2} + \frac{1}{2}\cos 2\theta$$

we get

$$A = \frac{1}{2}\int_{-\frac{2\pi}{3}}^{\frac{2\pi}{3}} 1 - 4\left(\frac{1}{2} + \frac{1}{2}\cos 2\theta\right)\ d\theta + \frac{1}{2}\int_{\frac{2\pi}{3}}^{\frac{4\pi}{3}} 4\left(\frac{1}{2} + \frac{1}{2}\cos 2\theta\right) - 1\ d\theta$$

$$A = \frac{1}{2}\int_{-\frac{2\pi}{3}}^{\frac{2\pi}{3}} 1 - 2 - 2\cos 2\theta\ d\theta + \frac{1}{2}\int_{\frac{2\pi}{3}}^{\frac{4\pi}{3}} 2 + 2\cos 2\theta - 1\ d\theta$$

$$A = -\frac{1}{2}\int_{-\frac{2\pi}{3}}^{\frac{2\pi}{3}} 1 + 2\cos 2\theta\ d\theta + \frac{1}{2}\int_{\frac{2\pi}{3}}^{\frac{4\pi}{3}} 1 + 2\cos 2\theta\ d\theta$$

Integrate term by term.

$$A = -\frac{1}{2}(\theta + \sin 2\theta)\Big|_{-\frac{2\pi}{3}}^{\frac{2\pi}{3}} + \frac{1}{2}(\theta + \sin 2\theta)\Big|_{\frac{2\pi}{3}}^{\frac{4\pi}{3}}$$

$$A = -\frac{1}{2}\left(\frac{2\pi}{3} + \sin\left(2\cdot\frac{2\pi}{3}\right)\right) + \frac{1}{2}\left(-\frac{2\pi}{3} + \sin\left(2\cdot -\frac{2\pi}{3}\right)\right)$$

$$+\frac{1}{2}\left(\frac{4\pi}{3} + \sin\left(2\cdot\frac{4\pi}{3}\right)\right) - \frac{1}{2}\left(\frac{2\pi}{3} + \sin\left(2\cdot\frac{2\pi}{3}\right)\right)$$

$$A = -\frac{1}{2}\left(\frac{2\pi}{3} + \sin\frac{4\pi}{3}\right) + \frac{1}{2}\left(-\frac{2\pi}{3} + \sin\left(-\frac{4\pi}{3}\right)\right)$$

$$+\frac{1}{2}\left(\frac{4\pi}{3}+\sin\frac{8\pi}{3}\right)-\frac{1}{2}\left(\frac{2\pi}{3}+\sin\frac{4\pi}{3}\right)$$

$$A=-\frac{1}{2}\left(\frac{2\pi}{3}-\frac{\sqrt{3}}{2}\right)+\frac{1}{2}\left(-\frac{2\pi}{3}+\frac{\sqrt{3}}{2}\right)+\frac{1}{2}\left(\frac{4\pi}{3}+\frac{\sqrt{3}}{2}\right)-\frac{1}{2}\left(\frac{2\pi}{3}-\frac{\sqrt{3}}{2}\right)$$

$$A=-\frac{\pi}{3}+\frac{\sqrt{3}}{4}-\frac{\pi}{3}+\frac{\sqrt{3}}{4}+\frac{2\pi}{3}+\frac{\sqrt{3}}{4}-\frac{\pi}{3}+\frac{\sqrt{3}}{4}$$

$$A=-\frac{\pi}{3}+\frac{4\sqrt{3}}{4}$$

$$A=\frac{-\pi+3\sqrt{3}}{3}$$