Integration by parts two times

What happens if you apply integration by parts and the integral you're left with still isn't easy to solve?

The first thing you should do in this situation is make sure that you assigned u and dv correctly. Try assigning u and dv to opposite components of your original integral and see if you end up with a better answer.

If you still get an integral you can't evaluate, maybe you need to use u-substitution on the left-over integral after you've already used integration by parts. Or, maybe integration by parts wasn't the right integration technique to use in the first place. Check to see if u-substitution works better on your original integral.

If all else fails, the trick might be to use integration by parts a second or third time. In other words, you might need to apply integration by parts several times in a row to get to an integral you can easily solve.

With practice, you'll start to realize that integrals like these ones often require multiple applications of integration by parts:

Power $\times$ Exponential

$$\int x^n e^x \ dx$$

Set $u = x^n$, apply IBP n times to reduce x^n to 1

Power $\times$ Trigonometric

$$\int x^n \sin x \, dx$$ Set $u = x^n$, apply IBP n times to reduce x^n to 1

$$\int x^n \cos x \, dx$$ Set $u = x^n$, apply IBP n times to reduce x^n to 1

Exponential $\times$ Trigonometric

$$\int e^x \sin x \, dx$$ Set $u = \sin x$, apply IBP twice to get back to $\sin x$, combine with the left-hand side

$$\int e^x \cos x \, dx$$ Set $u = \cos x$, apply IBP twice to get back to $\cos x$, combine with the left-hand side

While these aren't the only functions that force you to apply integration by parts multiple times, they are by far the most common, so it's helpful to remember them if you can.

Here's an example of a power function with a trigonometric function.

Example

Use integration by parts to evaluate the integral.

$$\int x^3 \cos x \, dx$$

First, we'll assign u and dv, then differentiate u to get du and integrate dv to get v.

$$u = x^3 \qquad \text{differentiate} \qquad du = 3x^2\ dx$$

$$dv = \cos x\ dx \qquad \text{integrate} \qquad v = \sin x$$

Plugging all four components into the formula gives

$$(x^3)(\sin x) - \int (\sin x)(3x^2\ dx)$$

$$x^3 \sin x - \int 3x^2 \sin x\ dx$$

What remains inside the integral is not easy to evaluate. Since u-substitution won't get us anywhere, we try integration by parts again, using our most recent integral.

$$u = 3x^2 \qquad \text{differentiate} \qquad du = 6x\ dx$$

$$dv = \sin x\ dx \qquad \text{integrate} \qquad v = -\cos x$$

Plugging in again to our last integral gives

$$x^3 \sin x - \left[(3x^2)(-\cos x) - \int (-\cos x)(6x\ dx)\right]$$

$$x^3 \sin x - \left(-3x^2 \cos x + \int 6x \cos x\ dx\right)$$

$$x^3 \sin x + 3x^2 \cos x - \int 6x \cos x\ dx$$

We still don't have an easy integral, so we use integration by parts one more time.

$u = 6x$ differentiate $du = 6\ dx$

$dv = \cos x\ dx$ integrate $v = \sin x$

Using the integration by parts formula to again transform the integral, we get:

$$x^3 \sin x + 3x^2 \cos x - \left[(6x)(\sin x) - \int (\sin x)(6\ dx) \right]$$

$$x^3 \sin x + 3x^2 \cos x - 6x \sin x + \int 6 \sin x\ dx$$

We finally have something we can easily integrate, so the answer is

$$x^3 \sin x + 3x^2 \cos x - 6x \sin x - 6 \cos x + C$$

Sometimes after applying integration by parts twice, you end up with an integral that is the same as your original problem. When that happens, instead of feeling like you're right back where you started, realize that you can add the integral on the right side of your equation to the integral on the left. You're just combining like terms like you did in algebra, except, instead of combining something simple like x^2 and $3x^2$, you're combining equal integrals.

Let's look at an example of an exponential function with a trigonometric function.

Example

Use integration by parts to evaluate the integral.

$$\int e^x \cos x \ dx$$

First, we'll assign u and dv, then differentiate u to get du and integrate dv to get v.

$u = \cos x$ differentiate $du = -\sin x \ dx$

$dv = e^x \ dx$ integrate $v = e^x$

Plugging all four components into the integration by parts formula gives

$$(\cos x)(e^x) - \int (e^x)(-\sin x \ dx)$$

$$e^x \cos x + \int e^x \sin x \ dx$$

We use integration by parts again to simplify the remaining integral.

$u = \sin x$ differentiate $du = \cos x \ dx$

$dv = e^x \ dx$ integrate $v = e^x$

Plugging in again, we get

$$e^x \cos x + \left[(\sin x)(e^x) - \int (e^x)(\cos x \ dx)\right]$$

$$\int e^x \cos x \ dx = e^x \cos x + e^x \sin x - \int e^x \cos x \ dx$$

See how our new integral on the right is the same as the original integral on the left? It seems like we're right back to the beginning of our problem and that all hope is lost. Instead, we can combine like terms and add the integral on the right to the left side of our equation.

$$2\int e^x \cos x \ dx = e^x \cos x + e^x \sin x$$

We'll just simplify the right-hand side and then divide both sides of the equation by 2 to solve for the original integral and get our final answer.

$$2\int e^x \cos x \ dx = e^x(\cos x + \sin x)$$

$$\int e^x \cos x \ dx = \frac{e^x(\cos x + \sin x)}{2}$$