Calculus 2
Final Exam Solutions

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MATH

Calculus 2 Final Exam Answer Key

1. (5 pts)	A	■	C	D	E
2. (5 pts)	■	B	C	D	E
3. (5 pts)	■	B	C	D	E
4. (5 pts)	A	B	■	D	E
5. (5 pts)	A	B	C	■	E
6. (5 pts)	A	B	C	■	E
7. (5 pts)	A	B	C	D	■
8. (5 pts)	A	■	C	D	E

9. (15 pts) 4

10. (15 pts) $V = (17/15)\pi$

11. (15 pts) 2π

12. (15 pts) Interval of $[1/6, 1/2]$; Radius of $1/6$

Calculus 2 Final Exam Solutions

1. B. Find the antiderivative of each term, and remember to add C at the end to account for the constant.

$$\int x^3 + \frac{x}{4} + \frac{2}{x} + 5\ dx$$

$$\frac{x^4}{4} + \frac{x^2}{2 \cdot 4} + 2 \ln x + 5x + C$$

$$\frac{x^4}{4} + \frac{x^2}{8} + 2 \ln x + 5x + C$$

2. A. Simpson's rule is

$$S_n = \frac{\Delta x}{3} \left[f(x_0) + 4f(x_1) + 2f(x_2) + \ldots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n) \right]$$

From the given integral,

$$\int_2^3 3x^2\ dx$$

we can see that the interval is $[2,3]$, so we'll use that to find Δx.

$$\Delta x = \frac{3-2}{4} = \frac{1}{4}$$

Now we can identify the x-values we'll use for x_0, x_1, x_2, etc.

$$x_0 = 2$$

$$x_1 = 2 + \frac{1}{4} = \frac{9}{4}$$

$$x_2 = \frac{9}{4} + \frac{1}{4} = \frac{10}{4} = \frac{5}{2}$$

$$x_3 = \frac{10}{4} + \frac{1}{4} = \frac{11}{4}$$

$$x_4 = 3$$

Plugging everything into the Simpson's rule formula, we get

$$S_n = \frac{\frac{1}{4}}{3}\left[3 \cdot 2^2 + 4 \cdot 3 \cdot \left(\frac{9}{4}\right)^2 + 2 \cdot 3 \cdot \left(\frac{5}{2}\right)^2 + 4 \cdot 3 \cdot \left(\frac{11}{4}\right)^2 + 3 \cdot 3^2\right]$$

$$S_n = \frac{1}{4}\left[2^2 + 4 \cdot \left(\frac{9}{4}\right)^2 + 2 \cdot \left(\frac{5}{2}\right)^2 + 4 \cdot \left(\frac{11}{4}\right)^2 + 3^2\right]$$

$$S_n = \frac{1}{4}\left(4 + 4 \cdot \frac{81}{16} + 2 \cdot \frac{25}{4} + 4 \cdot \frac{121}{16} + 9\right)$$

$$S_n = \frac{1}{4}\left(4 + \frac{81}{4} + \frac{25}{2} + \frac{121}{4} + 9\right)$$

$$S_n = 1 + \frac{81}{16} + \frac{25}{8} + \frac{121}{16} + \frac{9}{4}$$

$$S_n = \frac{16}{16} + \frac{81}{16} + \frac{50}{16} + \frac{121}{16} + \frac{36}{16}$$

$$S_n = \frac{304}{16}$$

$$S_n = 19$$

3. A. Solve using u-substitution. Set u equal to the inside function $3x+4$.

$$u = 3x + 4$$

$$\frac{du}{dx} = 3, \text{ so } du = 3\ dx, \text{ so } dx = \frac{du}{3}$$

Make substitutions.

$$\int 3(3x+4)^4\ dx$$

$$\int 3u^4\ \frac{du}{3}$$

$$\int u^4\ du$$

$$\frac{u^5}{5} + C$$

Back-substitute.

$$\frac{1}{5}(3x+4)^5 + C$$

4. C. Solve using integration by parts. Let $u=\sin x$ and $dv=e^x\ dx$, which means also that $du=\cos x\ dx$ and $v=e^x$.

$$\int_0^\pi e^x\sin x\ dx=e^x\sin x\Big|_0^\pi-\int_0^\pi e^x\cos x\ dx$$

Use integration by parts again, with $u=\cos x$, $dv=e^x\ dx$, $du=-\sin x\ dx$, and $v=e^x$.

$$\int_0^\pi e^x\sin x\ dx=e^x\sin x\Big|_0^\pi-\left[e^x\cos x\Big|_0^\pi+\int_0^\pi e^x\sin x\ dx\right]$$

$$\int_0^\pi e^x\sin x\ dx=e^x\sin x\Big|_0^\pi-e^x\cos x\Big|_0^\pi-\int_0^\pi e^x\sin x\ dx$$

Add $\int_0^\pi e^x\sin x\ dx$ to both sides.

$$2\int_0^\pi e^x\sin x\ dx=e^x\sin x\Big|_0^\pi-e^x\cos x\Big|_0^\pi$$

Divide both sides by 2 and factor out the e^x on the right side.

$$\int_0^\pi e^x\sin x\ dx=\frac{1}{2}e^x(\sin x-\cos x)\Big|_0^\pi$$

So the value of the integral will be

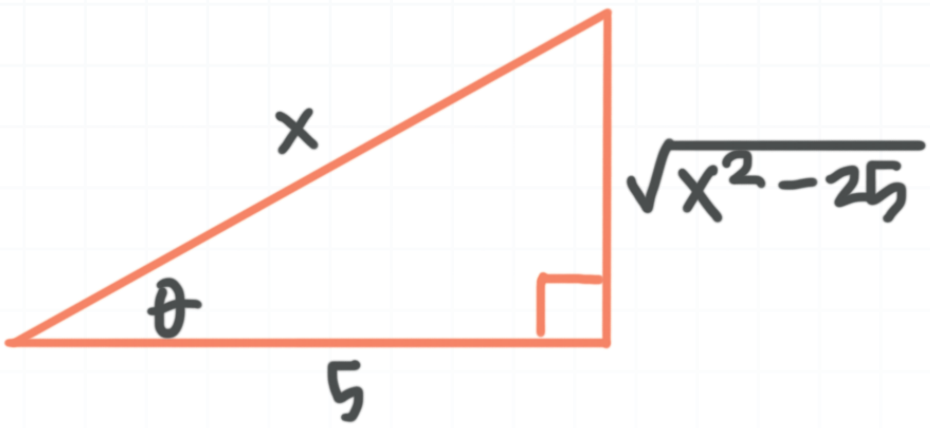
$$\frac{1}{2}e^\pi(\sin\pi-\cos\pi)-\frac{1}{2}e^0(\sin 0-\cos 0)$$

$$\frac{1}{2}e^\pi(0-(-1))-\frac{1}{2}(0-1)$$

$$\frac{1}{2}e^{\pi} + \frac{1}{2}$$

$$\frac{1}{2}(1 + e^{\pi})$$

5. D. Draw a reference triangle.



Then we can say

$$x = 5\sec\theta$$

$$dx = 5\sec\theta\tan\theta\ d\theta$$

Substitute into the integral.

$$\int \frac{1}{x^2\sqrt{x^2-25}}\ dx$$

$$\int \frac{5\sec\theta\tan\theta}{(5\sec\theta)^2\sqrt{(5\sec\theta)^2-25}}\ d\theta$$

$$\int \frac{5\sec\theta\tan\theta}{25\sec^2\theta\sqrt{25\sec^2\theta-25}}\ d\theta$$

$$\int \frac{\tan\theta}{5\sec\theta\sqrt{25(\sec^2\theta - 1)}}\ d\theta$$

Use the trig identity $\sec^2\theta - 1 = \tan^2\theta$ to simplify.

$$\int \frac{\tan\theta}{5\sec\theta\sqrt{25(\tan^2\theta)}}\ d\theta$$

$$\int \frac{\tan\theta}{(5\sec\theta)(5\tan\theta)}\ d\theta$$

$$\int \frac{\tan\theta}{25\sec\theta\tan\theta}\ d\theta$$

$$\int \frac{1}{25\sec\theta}\ d\theta$$

Knowing the reciprocal identity $\cos\theta = 1/\sec\theta$, we get

$$\frac{1}{25}\int \cos\theta\ d\theta$$

$$\frac{1}{25}\sin\theta + C$$

Use the reference triangle to find $\sin\theta$ (sine=opposite/hypotenuse), and simplify.

$$\frac{1}{25}\cdot\frac{\sqrt{x^2-25}}{x} + C$$

$$\frac{\sqrt{x^2-25}}{25x} + C$$

6. D. Use the formula for arc length:

$$L=\int_a^b \sqrt{1+[f'(x)]^2}\ dx$$

We know from the given interval that $[a,b]=[1,4]$. Find the derivative so that we can plug in for $f'(x)$.

$$y=\frac{2}{3}(x-1)^{\frac{3}{2}}$$

$$y'=(x-1)^{\frac{1}{2}}$$

Plug everything into the arc length formula.

$$L=\int_1^4 \sqrt{1+\left((x-1)^{\frac{1}{2}}\right)^2}\ dx$$

$$L=\int_1^4 \sqrt{1+x-1}\ dx$$

$$L=\int_1^4 \sqrt{x}\ dx$$

Integrate and evaluate over the interval.

$$L=\frac{2}{3}x^{\frac{3}{2}}\Big|_1^4$$

$$L=\frac{2}{3}(4)^{\frac{3}{2}}-\frac{2}{3}(1)^{\frac{3}{2}}$$

$$L=\frac{2}{3}(8)-\frac{2}{3}$$

$$L=\frac{16}{3}-\frac{2}{3}$$

$$L=\frac{14}{3}$$

7. E. To find the total work required, we have to find the work required to lift the load and add that to the work required to lift the rope. Remember that $W=Fd$, where W is work, F is force (weight), and d is distance. The work required to lift the load is

$$W_L=300\text{ lbs}\cdot 600\text{ ft}$$

$$W_L=180{,}000\text{ ft-lbs}$$

The work required to lift the rope is

$$W_R=\int_0^{600}3x\ dx$$

$$W_R=\frac{3}{2}x^2\Big|_0^{600}$$

$$W_R=\frac{3}{2}(600)^2-\frac{3}{2}(0)^2$$

$$W_R=\frac{3}{2}(360{,}000)$$

$$W_R = 540{,}000 \text{ ft-lbs}$$

The total work required is therefore

$$W = 540{,}000 + 180{,}000$$

$$W = 720{,}000 \text{ ft-lbs}$$

8. B. Use the formula for area under a parametric curve.

$$A = \int_{\alpha}^{\beta} y(t)x'(t)\ dt$$

The interval is given by $[\alpha, \beta] = [0,3]$. We can also find $y(t)$ and $x'(t)$.

$$y(t) = g(t) = t - 3$$

$$x'(t) = f'(t) = 6t$$

Substitute into the integral formula.

$$A = \int_{0}^{3} (t-3)(6t)\ dt$$

$$A = \int_{0}^{3} 6t^2 - 18t\ dt$$

$$A = 2t^3 - 9t^2 \Big|_0^3$$

$$A = 2(3)^3 - 9(3)^2 - (2(0)^3 - 9(0)^2)$$

$$A = 2(27) - 9(9)$$

$$A = 54 - 81$$

$$A = -27$$

9. Since both of the equations are functions, these are upper and lower curves. Find the points of intersection by setting the curves equal to each other.

$$3x^2 + x - 2 = x + 1$$

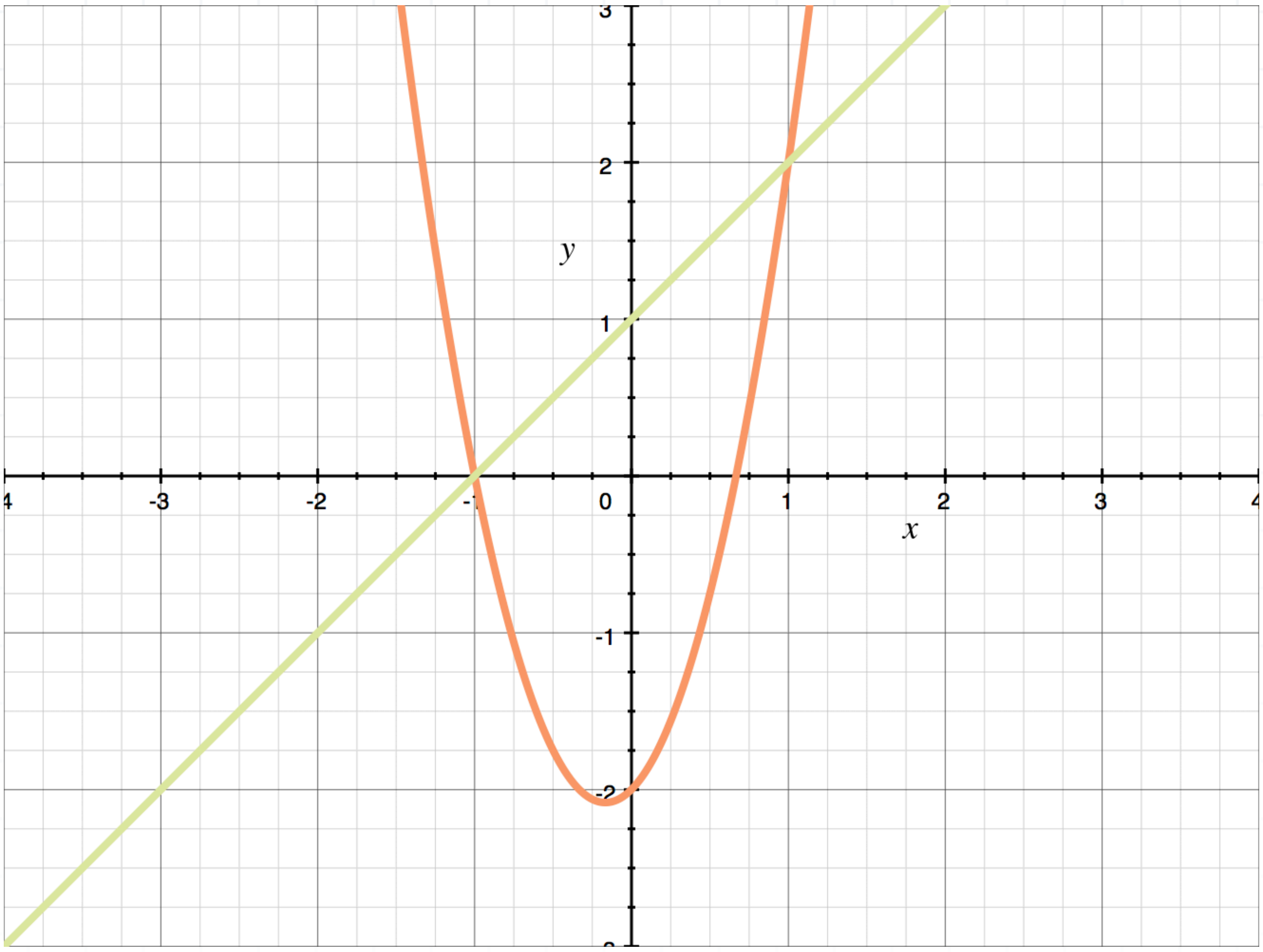
$$3x^2 - 3 = 0$$

$$x^2 - 1 = 0$$

$$(x + 1)(x - 1) = 0$$

$$x = -1, 1$$

It's helpful to sketch the graphs to understand which curve is the upper curve. You could also plug in x-values to the equations to see which y-values are greater.



From the graph, we see that $y = x + 1$ is the upper curve, so the integral to find the area between the two curves is

$$\int_{-1}^{1} (x+1) - (3x^2 + x - 2)\ dx$$

$$\int_{-1}^{1} (x+1) - (3x^2 + x - 2)\ dx$$

$$\int_{-1}^{1} x + 1 - 3x^2 - x + 2\ dx$$

$$\int_{-1}^{1} -3x^2 + 3\ dx$$

$$-\frac{3}{3}x^3+3x\Big|_{-1}^{1}$$

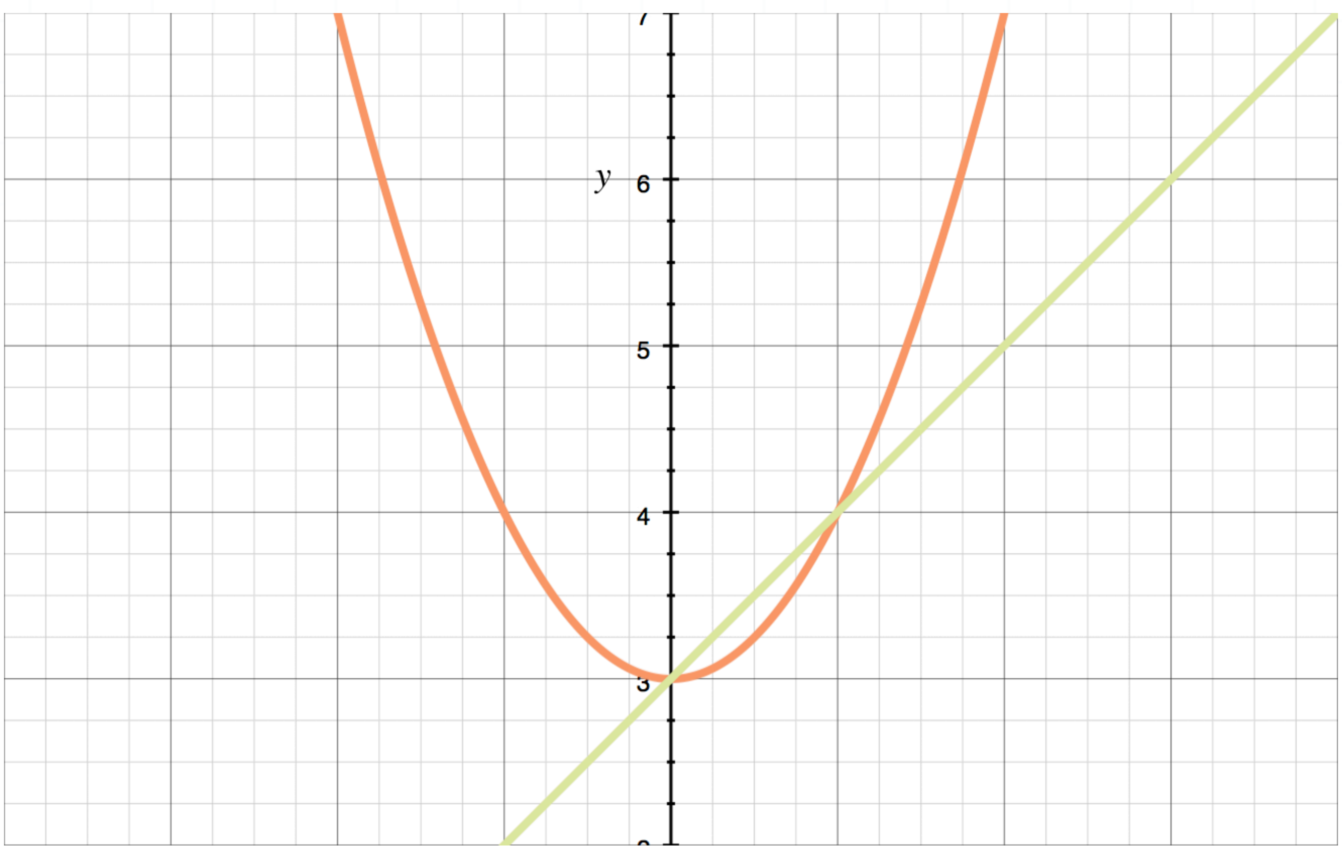
$$-(1)^3+3(1)-(-(-1)^3+3(-1))$$

$$-1+3-(1-3)$$

$$2-(-2)$$

$$4$$

10. It's helpful to sketch the graph of the curves.



Find the intersection points to find the interval $[a,b]$.

$$x^2+3=x+3$$

$$x^2-x+3-3=x-x+3-3$$

$$x^2 - x = 0$$

$$x(x-1) = 0$$

$$x = 0,\ 1$$

The interval $[a, b]$ is $[0,1]$, $f(x) = x + 3$ because the line is above the parabola in the interval $[0,1]$, and $g(x) = x^2 + 3$.

$$V = \pi \int_a^b [f(x)]^2 - [g(x)]^2\ dx$$

$$V = \pi \int_0^1 (x+3)^2 - (x^2+3)^2\ dx$$

$$V = \pi \int_0^1 x^2 + 6x + 9 - (x^4 + 6x^2 + 9)\ dx$$

$$V = \pi \int_0^1 -x^4 - 5x^2 + 6x\ dx$$

$$V = \pi \left(-\frac{x^5}{5} - \frac{5}{3}x^3 + 3x^2 \right) \Bigg|_0^1$$

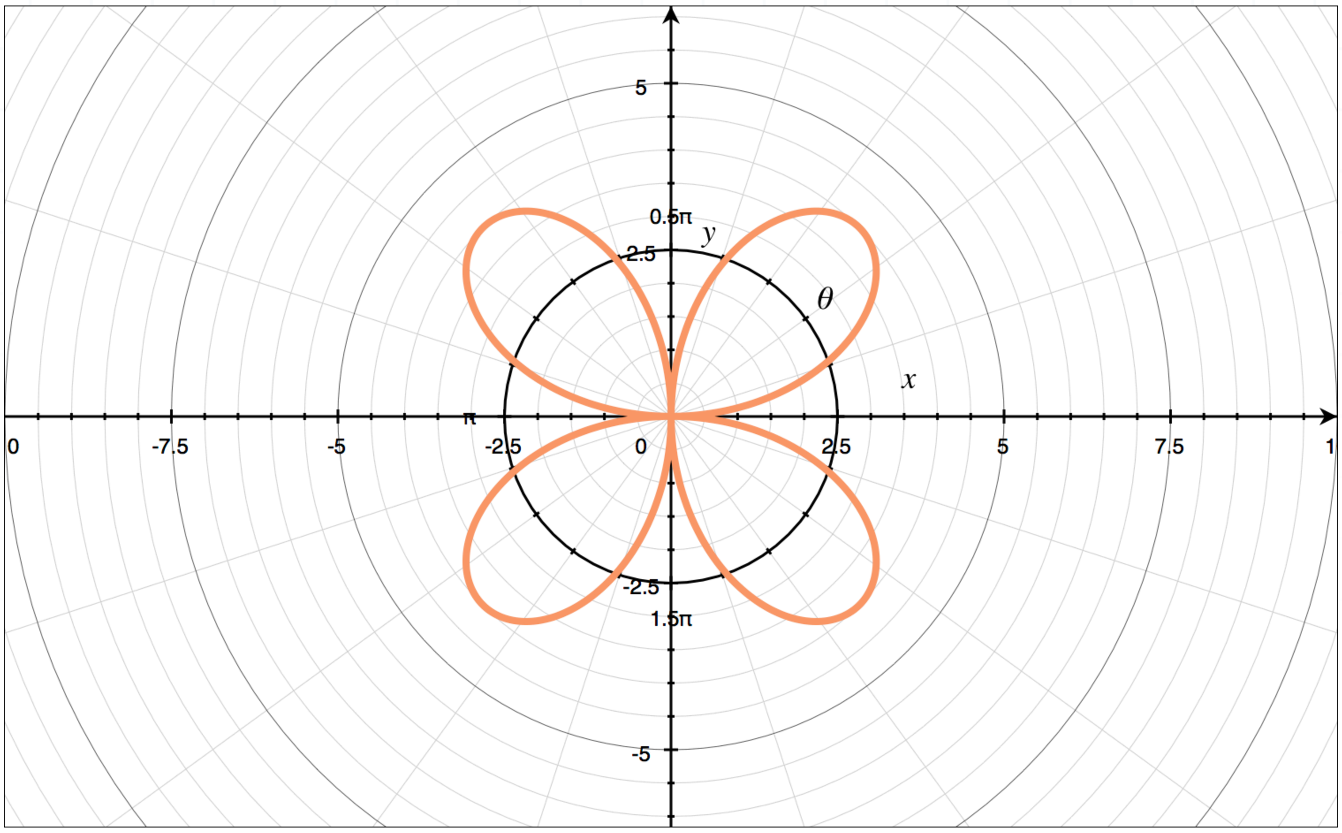
$$V = \pi \left[-\frac{1^5}{5} - \frac{5}{3}(1)^3 + 3(1)^2 - \left(-\frac{0^5}{5} - \frac{5}{3}(0)^3 + 3(0)^2 \right) \right]$$

$$V = \pi \left(-\frac{1}{5} - \frac{5}{3} + 3 - 0 \right)$$

$$V=\pi\left(-\frac{3}{15}-\frac{25}{15}+\frac{45}{15}\right)$$

$$V=\frac{17}{15}\pi$$

11. It's helpful to sketch the graph of the polar curve.



Notice that each loop starts and ends at the origin, so we need to find θ when $r=0$.

$$0=4\sin 2\theta$$

$$0=\sin 2\theta$$

$$\sin^{-1}0=2\theta$$

$$2\theta=0,\ \pi,\ 2\pi$$

$$\theta = 0, \frac{\pi}{2}, \pi$$

Since we're only finding the area of one loop, we'll evaluate the area integral from 0 to $\pi/2$.

$$A = \int_a^b \frac{1}{2} r^2 \ d\theta$$

$$\int_0^{\frac{\pi}{2}} \frac{1}{2}(4 \sin 2\theta)^2 \ d\theta$$

$$\int_0^{\frac{\pi}{2}} \frac{1}{2} \cdot 16 \sin^2(2\theta) \ d\theta$$

$$8 \int_0^{\frac{\pi}{2}} \sin^2(2\theta) \ d\theta$$

Remember the double-angle identity $\cos 2\theta = 1 - 2 \sin^2 \theta$ and solve for $\sin^2 \theta$. Substitute

$$\sin^2 \theta = \frac{1 - \cos 2\theta}{2}$$

into the integral

$$8 \int_0^{\frac{\pi}{2}} \frac{1 - \cos 2(2\theta)}{2} \ d\theta$$

$$8 \int_0^{\frac{\pi}{2}} \frac{1}{2} - \frac{\cos 4\theta}{2} \ d\theta$$

$$4\int_0^{\frac{\pi}{2}} 1-\cos 4\theta\ d\theta$$

$$4\left(\theta-\frac{\sin 4\theta}{4}\right)\Bigg|_0^{\frac{\pi}{2}}$$

$$4\left[\frac{\pi}{2}-\frac{\sin 2\pi}{4}-\left(0-\frac{\sin 0}{4}\right)\right]$$

$$4\left(\frac{\pi}{2}-0-0+0\right)$$

$$2\pi$$

12. Use the ratio test to see if the series converges

$$\lim_{n\to\infty}\frac{a_{n+1}}{a_n}$$

$$\lim_{n\to\infty}\frac{2^{n+1}(3x-1)^{n+1}}{((n+1)+1)^2}\cdot\frac{(n+1)^2}{2^n(3x-1)^n}$$

$$\lim_{n\to\infty}2(3x-1)\left(\frac{n+1}{n+2}\right)^2$$

The value

$$\left(\frac{n+1}{n+2}\right)^2$$

converges to 1, so the ratio between consecutive terms is $2(3x-1)$. A series converges if and only if $|\text{ratio}| < 1$.

$$|2(3x-1)| < 1$$

$$-1 < 6x - 2 < 1$$

$$1 < 6x < 3$$

$$\frac{1}{6} < x < \frac{1}{2}$$

Now we need to check each endpoint. Plugging in $x = 1/6$ gives

$$\sum_{n=0}^{\infty} \frac{\left(6\left(\frac{1}{6}-2\right)\right)^n}{(n+1)^2} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)^2}$$

The left endpoint converges by the alternating series test. Plugging in $x = 1/2$ gives

$$\sum_{n=0}^{\infty} \frac{\left(6\left(\frac{1}{2}-2\right)\right)^n}{(n+1)^2} = \sum_{n=0}^{\infty} \frac{(1)^n}{(n+1)^2} = \sum_{n=0}^{\infty} \frac{1}{(n+1)^2}$$

The right endpoint is converges by the comparison test with the p-series $1/n^2$ with $p > 1$.

Therefore, the interval of convergence is $[1/6,1/2]$. The radius is half of the distance between the endpoints

$$\frac{1}{2}\left(\frac{1}{2}-\frac{1}{6}\right) = \frac{1}{2}\left(\frac{1}{3}\right) = \frac{1}{6}$$

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