Increasing, decreasing, and not monotonic

Sequences are always either monotonic or not monotonic. If a sequence is monotonic, it means that it's always increasing or always decreasing. If a sequence is sometimes increasing and sometimes decreasing and therefore doesn't have a consistent direction, it means that the sequence is not monotonic. In other words, a non-monotonic sequence is increasing for parts of the sequence and decreasing for others.

The fastest way to make a **guess** about the behavior of a sequence is to calculate the first few terms of the sequence and visually determine if it's increasing, decreasing or not monotonic.

If we want to get more technical and **prove** the behavior of the sequence, we can use the following inequalities.

A sequence is increasing if $a_n \leq a_{n+1}$

A sequence is decreasing if $a_n \geq a_{n+1}$

A sequence is not monotonic if $a_n \leq a_{n+1} \geq a_{n+2}$ or $a_n \geq a_{n+1} \leq a_{n+2}$.

Example

Is the sequence increasing, decreasing or not monotonic?

$$a_n = n^3 + 9$$

We can start by determining the first few values of the sequence. Let's calculate $n=1$, $n=2$, $n=3$ and $n=4$.

When $n=1$ $\quad a_1=(1)^3+9$ $\quad$ so $\quad a_1=10$

When $n=2$ $\quad a_2=(2)^3+9$ $\quad$ so $\quad a_2=17$

When $n=3$ $\quad a_3=(3)^3+9$ $\quad$ so $\quad a_3=36$

When $n=4$ $\quad a_4=(4)^3+9$ $\quad$ so $\quad a_4=73$

The first four terms of our sequence are $\{10,\ 17,\ 36,\ 73\}$. Looking at these first few terms, we can see that the sequence is increasing. If we want to be more strict, we can prove it by showing that $a_n \le a_{n+1}$.

$$n^3+9 \le (n+1)^3+9$$

$$n^3+9 \le n^3+3n^2+3n+10$$

Looking at our inequality we can see that $a_n \le a_{n+1}$ is true for our sequence. After all, n^3+10 on its own has to be greater than or equal to n^3+9, and adding $3n^2+3n$ will definitely make it greater, so we know for sure that

$$n^3+9 \le \left(n^3+10\right)+3n^2+3n$$

If we're still unsure, we can always plug in a few values of n to confirm our conclusion.

The sequence $a_n=n^3+9$ is increasing, which means it's also monotonic.