Inverse trigonometric derivatives

Now that we've covered the derivatives of the six basic trig functions, we want to look at the derivatives of the six inverse trig functions.

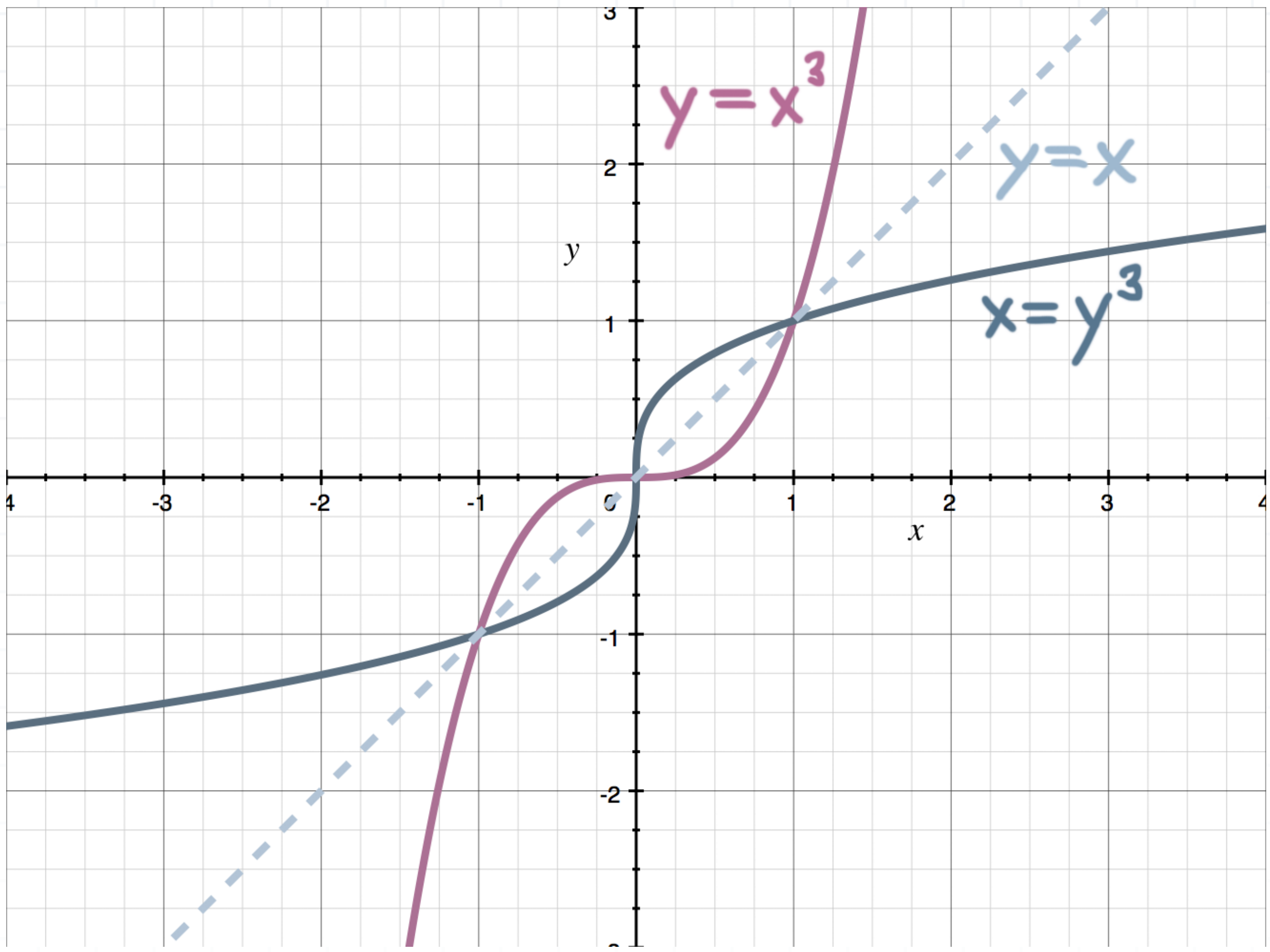
Before we get to the derivatives though, let's first define the six inverse trig functions, themselves. And before we get to the inverse trig functions, let's remind ourselves about inverse functions in general.

Inverse functions

Inverse functions are usually covered in Algebra, but let's do a quick refresher here.

To find a function's inverse, all we need to do is change the places of the x and y variables in the function's equation. For instance, given the function $y=x^3$, its inverse is $x=y^3$. Or given the function $y=3x^2-6x+2$, its inverse is $x=3y^2-6y+2$. We can see how we're just swapping out x variables for y variables and vice versa in order to get the inverse.

A function and its inverse will always be reflections of each other over the line $y=x$. As an example, if we graph $y=x^3$ and $x=y^3$ on the same set of axes, we can see that they are a perfect reflection of each other across $y=x$.



We can use the same "flip x and y" method to find the inverse of a trig function. The inverse of $y = \sin x$ is therefore $x = \sin y$. To solve this equation for y, we take inverse sine of both sides, which will cancel the $\sin$ and get y by itself.

$$x = \sin y$$

$$\sin^{-1}(x) = \sin^{-1}(\sin y)$$

$$\sin^{-1} x = y$$

$$y = \sin^{-1} x$$

So we can see that $y = \sin x$ and $y = \sin^{-1} x$ are inverses of one another. We can use the notation $\sin^{-1}$ to indicate "inverse sine," or we can use $\arcsin$ instead. Both indicate the inverse of the $\sin$ function and can be used interchangeably. If we're going to use $\sin^{-1}$ notation, we just need to remember that the -1 is **not** an exponent. Remember that the negative exponent rule tells us that

$$x^{-1}=\frac{1}{x}$$

but inverse trigonometric functions don't not follow this rule. So

$$\sin^{-1}x\neq\frac{1}{\sin x}$$

Because we have to be so careful with the $\sin^{-1}$ notation to remember that the negative exponent rule doesn't apply, many people prefer to use the $\arcsin$ notation instead of the $\sin^{-1}$ notation.

Inverse trig derivatives

Below is a table for the derivatives of the inverse trig functions.

Inverse trig function	**Derivative**
$y=\sin^{-1}x$	$y'=\frac{1}{\sqrt{1-x^2}}$
$y=\cos^{-1}x$	$y'=-\frac{1}{\sqrt{1-x^2}}$
$y=\tan^{-1}x$	$y'=\frac{1}{1+x^2}$

$y = \sec^{-1} x$ $\quad$ $y' = \dfrac{1}{|x|\sqrt{x^2 - 1}}$

$y = \csc^{-1} x$ $\quad$ $y' = -\dfrac{1}{|x|\sqrt{x^2 - 1}}$

$y = \cot^{-1} x$ $\quad$ $y' = -\dfrac{1}{1 + x^2}$

Just like with trig functions, when we differentiate an inverse trig function, we always have to apply chain rule and multiply by the derivative of the argument.

Because all the formulas above are for trig functions with an argument of x. Because the argument is x, and the derivative of x is 1, applying chain rule just means that we multiply by 1, which doesn't affect the value of the derivative.

But if the argument of the inverse trig function is anything other than x, then applying chain rule means we'll be multiplying by something other than 1, which means that applying chain rule will affect the value of the derivative.

Here's a table of formulas for the inverse trig functions when the argument is something other than x.

Inverse trig function	**Derivative**
$y = \sin^{-1}[g(x)]$	$y' = \dfrac{g'(x)}{\sqrt{1 - [g(x)]^2}}$

$$y = \cos^{-1}[g(x)] \qquad y' = -\frac{g'(x)}{\sqrt{1 - [g(x)]^2}}$$

$$y = \tan^{-1}[g(x)] \qquad y' = \frac{g'(x)}{1 + [g(x)]^2}$$

$$y = \csc^{-1}[g(x)] \qquad y' = -\frac{g'(x)}{|g(x)|\sqrt{[g(x)]^2 - 1}}$$

$$y = \sec^{-1}[g(x)] \qquad y' = \frac{g'(x)}{|g(x)|\sqrt{[g(x)]^2 - 1}}$$

$$y = \cot^{-1}[g(x)] \qquad y' = -\frac{g'(x)}{1 + [g(x)]^2}$$

With these formulas in hand, let's try an example where we find the derivative of an inverse trig function.

Example

Find the derivative of the inverse trig function.

$$y = 7\tan^{-1}(4x^3)$$

Apply the formula for the derivative of inverse tangent with $g(x) = 4x^3$.

$$y' = \frac{g'(x)}{1 + [g(x)]^2}$$

$$y' = 7\left(\frac{12x^2}{1 + (4x^3)^2}\right)$$

$$y' = \frac{84x^2}{1 + 16x^6}$$

It's common to see inverse trigonometric functions mixed into more elaborate functions, so let's try an example with some other things going on.

Example

Find the derivative of the function.

$$y = 2\sec^{-1}(x^3) - 54x^7$$

We differentiate one term at a time, which means we can differentiate $-54x^7$ using power rule. We only have to worry about the formula for the derivative of inverse secant with $g(x) = x^3$,

$$y' = \frac{g'(x)}{|g(x)|\sqrt{[g(x)]^2 - 1}}$$

when we differentiate the first term.

$$y' = 2\left(\frac{3x^2}{|x^3|\sqrt{(x^3)^2 - 1}}\right) - 378x^6$$

$$y' = \frac{6x^2}{|x^3|\sqrt{x^6 - 1}} - 378x^6$$

Let's try one more example.

Example

Find the derivative of the function.

$$y = 2x^6 - x^3 \cos^{-1}(2x) + 8\sin(3x^5)$$

We'll need to use product rule for the second term, $-x^3\cos^{-1}(2x)$. Taking the derivative term by term, using the formula for the derivative of inverse cosine with $g(x) = 2x$,

$$y' = -\frac{g'(x)}{\sqrt{1 - [g(x)]^2}}$$

and the formula from the previous lesson for the derivative of sine,

$$y' = \cos x$$

we get

$$y' = 12x^5 - \left[(3x^2)(\cos^{-1}(2x)) + (x^3)\left(-\frac{2}{\sqrt{1 - (2x)^2}}\right)\right] + 8\cos(3x^5)(15x^4)$$

$$y' = 12x^5 - \left(3x^2\cos^{-1}(2x) - \frac{2x^3}{\sqrt{1-4x^2}}\right) + 120x^4\cos(3x^5)$$

$$y' = 12x^5 - 3x^2\cos^{-1}(2x) + \frac{2x^3}{\sqrt{1-4x^2}} + 120x^4\cos(3x^5)$$