Topic: sin^m cos^n, m and n even

Question: Evaluate the trigonometric integral.

$$\int \sin^2\theta\cos^2\theta\ d\theta$$

Answer choices:

A $\quad \frac{1}{8}\theta + \frac{1}{32}\sin 4\theta + C$

B $\quad \frac{1}{8}\theta - \frac{1}{32}\sin 4\theta + C$

C $\quad \frac{1}{8}\theta - \frac{1}{32}\cos 4\theta + C$

D $\quad \frac{1}{8}\theta + \frac{1}{32}\cos 4\theta + C$

Solution: B

In the specific case where our function is the product of

an **even** number of **sine** factors and

an **even** number of **cosine** factors,

our plan is to

1. create sets of $\sin x \cos x$ and replace each of them with

    a. $\sin x \cos x = \frac{1}{2} \sin 2x$,

2. then use the half-angle formulas to make substitutions,

    a. $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$

    b. $\cos^2 x = \frac{1}{2}(1 + \cos 2x)$

3. remembering that we may need to use the identity

    a. $\cos a \cos b = \frac{1}{2}\left[\cos(a - b) + \cos(a + b)\right]$

We'll create sets of $\sin x \cos x$ and then use the $\sin x \cos x$ identity to make a substitution.

$$\int \sin^2 \theta \cos^2 \theta \ d\theta$$

$$\int \left(\sin\theta\cos\theta\right)^2\ d\theta$$

$$\int \left(\frac{1}{2}\sin 2\theta\right)^2\ d\theta$$

$$\int \frac{1}{4}\sin^2 2\theta\ d\theta$$

$$\frac{1}{4}\int \sin^2 2\theta\ d\theta$$

Now we'll use the $\sin^2 x$ identity to make a second substitution.

$$\frac{1}{4}\int \frac{1}{2}\left[1-\cos 2(2\theta)\right]\ d\theta$$

$$\frac{1}{8}\int 1-\cos 4\theta\ d\theta$$

$$\frac{1}{8}\left(\theta-\frac{1}{4}\sin 4\theta\right)+C$$

$$\frac{1}{8}\theta-\frac{1}{32}\sin 4\theta+C$$

Topic: sin^m cos^n, m and n even

Question: Evaluate the trigonometric integral.

$$\int \sin^6 x \cos^4 x \ dx$$

Answer choices:

A $\quad \frac{1}{256}\left(3x - \frac{1}{2}\sin 2x - \frac{1}{2}\cos 8x \sin 2x - \sin 4x + \frac{1}{8}\sin 8x + \frac{1}{5}\sin 10x\right) + C$

B $\quad -\frac{1}{256}\left(3x + \sin^2 4x \sin 2x - \frac{5}{2}\sin 2x - \sin 4x + \frac{1}{8}\sin 8x - \frac{1}{10}\sin 10x\right) + C$

C $\quad \frac{1}{256}\left(3x + \sin^2 4x \sin 2x - \frac{5}{2}\sin 2x - \sin 4x\right) + C$

D $\quad -\frac{1}{256}\left(3x + \sin^2 4x \sin 2x - \frac{5}{2}\sin 2x - \sin 4x\right) + C$

Solution: A

In the specific case where our function is the product of

an **even** number of **sine** factors and

an **even** number of **cosine** factors,

our plan is to

1. create sets of $\sin x \cos x$ and replace each of them with

   a. $\sin x \cos x = \frac{1}{2}\sin 2x$,

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   a. $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$

   b. $\cos^2 x = \frac{1}{2}(1 + \cos 2x)$

3. remembering that we may need to use the identity

   a. $\cos a \cos b = \frac{1}{2}\left[\cos(a - b) + \cos(a + b)\right]$

We'll create sets of $\sin x \cos x$ and then use the $\sin x \cos x$ identity to make a substitution.

$$\int \sin^6 x \cos^4 x \ dx$$

$$\int \sin^2 x \left(\sin x \cos x\right)^4 \ dx$$

$$\int \sin^2 x \left(\frac{1}{2}\sin 2x\right)^4 \ dx$$

$$\frac{1}{16}\int \sin^2 x \sin^4 2x \ dx$$

$$\frac{1}{16}\int \sin^2 x \left(\sin^2 2x\right)^2 \ dx$$

Now we'll use the $\sin^2 x$ identity to make a second substitution.

$$\frac{1}{16}\int \frac{1}{2}(1-\cos 2x)\left[\frac{1}{2}(1-\cos 2(2x))\right]^2 \ dx$$

$$\frac{1}{32}\int (1-\cos 2x)\left(\frac{1}{2}-\frac{1}{2}\cos 4x\right)^2 \ dx$$

$$\frac{1}{32}\int (1-\cos 2x)\left(\frac{1}{4}-\frac{1}{2}\cos 4x+\frac{1}{4}\cos^2 4x\right) \ dx$$

$$\frac{1}{32}\int \frac{1}{4}-\frac{1}{2}\cos 4x+\frac{1}{4}\cos^2 4x$$

$$-\frac{1}{4}\cos 2x+\frac{1}{2}\cos 2x\cos 4x-\frac{1}{4}\cos^2 4x\cos 2x \ dx$$

$$\frac{1}{32}\left(\frac{1}{4}x-\frac{1}{8}\sin 4x-\frac{1}{8}\sin 2x\right)$$

$$+\frac{1}{32}\int \frac{1}{4}\cos^2 4x+\frac{1}{2}\cos 2x\cos 4x-\frac{1}{4}\cos^2 4x\cos 2x \ dx$$

Using the identity

$$\cos a \cos b = \frac{1}{2}\left[\cos(a-b)+\cos(a+b)\right]$$

we'll simplify the integral.

$$\frac{1}{32}\left(\frac{1}{4}x-\frac{1}{8}\sin 4x-\frac{1}{8}\sin 2x\right)$$

$$+\frac{1}{32}\int \frac{1}{4}\cos^2 4x+\frac{1}{2}\left[\frac{1}{2}\left(\cos(4x-2x)+\cos(4x+2x)\right)\right]$$

$$-\frac{1}{4}\cos^2 4x\cos 2x\ dx$$

$$\frac{1}{32}\left(\frac{1}{4}x-\frac{1}{8}\sin 4x-\frac{1}{8}\sin 2x\right)$$

$$+\frac{1}{32}\int \frac{1}{4}\cos^2 4x+\frac{1}{4}\cos 2x+\frac{1}{4}\cos 6x-\frac{1}{4}\cos^2 4x\cos 2x\ dx$$

$$\frac{1}{32}\left(\frac{1}{4}x-\frac{1}{8}\sin 4x-\frac{1}{8}\sin 2x+\frac{1}{8}\sin 2x\right)$$

$$+\frac{1}{32}\int \frac{1}{4}\cos^2 4x+\frac{1}{4}\cos 6x-\frac{1}{4}\cos^2 4x\cos 2x\ dx$$

$$\frac{1}{32}\left(\frac{1}{4}x-\frac{1}{8}\sin 4x-\frac{1}{8}\sin 2x+\frac{1}{8}\sin 2x+\frac{1}{24}\sin 6x\right)$$

$$+\frac{1}{32}\int \frac{1}{4}\cos^2 4x-\frac{1}{4}\cos^2 4x\cos 2x\ dx$$

$$\frac{1}{32}\left(\frac{1}{4}x-\frac{1}{8}\sin 4x+\frac{1}{24}\sin 6x\right)$$

$$+\frac{1}{32}\int \frac{1}{4}\cos^2 4x-\frac{1}{4}\cos^2 4x\cos 2x\ dx$$

$$\frac{1}{128}\left(x-\frac{1}{2}\sin 4x+\frac{1}{6}\sin 6x\right)+\frac{1}{128}\int \cos^2 4x-\cos^2 4x\cos 2x\ dx$$

$$\frac{1}{128}\left(x-\frac{1}{2}\sin 4x+\frac{1}{6}\sin 6x\right)+\frac{1}{128}\int \cos^2 4x\ dx-\frac{1}{128}\int \cos^2 4x\cos 2x\ dx$$

Use the identity

$$\cos^2 x=\frac{1}{2}+\frac{1}{2}\cos(2x)$$

to rewrite the first integral.

$$\frac{1}{128}\left(x-\frac{1}{2}\sin 4x+\frac{1}{6}\sin 6x\right)+\frac{1}{128}\int \frac{1}{2}+\frac{1}{2}\cos(2(4x))\ dx-\frac{1}{128}\int \cos^2 4x\cos 2x\ dx$$

$$\frac{1}{128}\left(x-\frac{1}{2}\sin 4x+\frac{1}{6}\sin 6x\right)+\frac{1}{256}\int 1+\cos 8x\ dx-\frac{1}{128}\int \cos^2 4x\cos 2x\ dx$$

$$\frac{1}{128}\left(x-\frac{1}{2}\sin 4x+\frac{1}{6}\sin 6x\right)+\frac{1}{256}\left(x+\frac{1}{8}\sin 8x\right)-\frac{1}{128}\int \cos^2 4x\cos 2x\ dx$$

Use the identity $\cos^2 x=1-\sin^2 x$ to rewrite the second integral.

$$\frac{1}{128}\left(x-\frac{1}{2}\sin 4x+\frac{1}{6}\sin 6x\right)+\frac{1}{256}\left(x+\frac{1}{8}\sin 8x\right)-\frac{1}{128}\int (1-\sin^2 4x)\cos 2x\ dx$$

$$\frac{1}{128}\left(x-\frac{1}{2}\sin 4x+\frac{1}{6}\sin 6x\right)+\frac{1}{256}\left(x+\frac{1}{8}\sin 8x\right)$$

$$-\frac{1}{128}\int \cos 2x-\sin^2 4x\cos 2x\ dx$$

$$\frac{1}{128}\left(x-\frac{1}{2}\sin 4x+\frac{1}{6}\sin 6x\right)+\frac{1}{256}\left(x+\frac{1}{8}\sin 8x\right)$$

$$-\frac{1}{128}\left(\frac{1}{2}\sin 2x\right)-\frac{1}{128}\int -\sin^2 4x\cos 2x\ dx$$

$$\frac{1}{128}\left(x-\frac{1}{2}\sin 4x+\frac{1}{6}\sin 6x\right)+\frac{1}{256}\left(x+\frac{1}{8}\sin 8x\right)$$

$$-\frac{1}{256}\sin 2x+\frac{1}{128}\int \sin^2 4x\cos 2x\ dx$$

Now use integration by parts with

$$u=\sin^2 4x$$

$$du=8\sin 4x\cos 4x\ dx$$

$$dv=\cos 2x\ dx$$

$$v=\frac{1}{2}\sin 2x$$

Focusing on just the remaining integral, we can say

$$\int \sin^2 4x\cos 2x\ dx=uv-\int v\ du$$

$$\int \sin^2 4x \cos 2x \ dx = (\sin^2 4x)\left(\frac{1}{2}\sin 2x\right) - \int \frac{1}{2}\sin 2x(8\sin 4x\cos 4x \ dx)$$

$$\int \sin^2 4x \cos 2x \ dx = \frac{1}{2}\sin^2 4x \sin 2x - 4\int \sin 2x \sin 4x \cos 4x \ dx$$

Use the identity

$$\sin\alpha\sin\beta = \frac{1}{2}(\cos(\alpha-\beta) - \cos(\alpha+\beta))$$

to rewrite the integral.

$$\int \sin^2 4x \cos 2x \ dx = \frac{1}{2}\sin^2 4x \sin 2x - 4\int \cos 4x\left[\frac{1}{2}(\cos(4x-2x) - \cos(4x+2x))\right] \ dx$$

$$\int \sin^2 4x \cos 2x \ dx = \frac{1}{2}\sin^2 4x \sin 2x - 2\int \cos 4x(\cos 2x - \cos 6x) \ dx$$

$$\int \sin^2 4x \cos 2x \ dx = \frac{1}{2}\sin^2 4x \sin 2x - 2\int \cos 2x\cos 4x - \cos 4x \cos 6x \ dx$$

$$\int \sin^2 4x \cos 2x \ dx = \frac{1}{2}\sin^2 4x \sin 2x - 2\int \cos 2x\cos 4x \ dx + 2\int \cos 4x \cos 6x \ dx$$

Use the identity

$$\cos\alpha\cos\beta = \frac{1}{2}(\cos(\alpha-\beta) + \cos(\alpha+\beta))$$

to rewrite both integrals.

$$\int \sin^2 4x \cos 2x \ dx = \frac{1}{2}\sin^2 4x \sin 2x$$

$$-2\int\frac{1}{2}(\cos(4x-2x)+\cos(4x+2x))\ dx+2\int\frac{1}{2}(\cos(6x-4x)+\cos(6x+4x))\ dx$$

$$\int\sin^2 4x\cos 2x\ dx=\frac{1}{2}\sin^2 4x\sin 2x$$

$$-\int\cos 2x+\cos 6x\ dx+\int\cos 2x+\cos 10x\ dx$$

$$\int\sin^2 4x\cos 2x\ dx=\frac{1}{2}\sin^2 4x\sin 2x$$

$$-\int\cos 2x\ dx-\int\cos 6x\ dx+\int\cos 2x\ dx+\int\cos 10x\ dx$$

$$\int\sin^2 4x\cos 2x\ dx=\frac{1}{2}\sin^2 4x\sin 2x-\frac{1}{2}\sin 2x-\frac{1}{6}\sin 6x+\frac{1}{2}\sin 2x+\frac{1}{10}\sin 10x$$

$$\int\sin^2 4x\cos 2x\ dx=\frac{1}{2}\sin^2 4x\sin 2x-\frac{1}{6}\sin 6x+\frac{1}{10}\sin 10x$$

Now we can plug this value back in for just the integral in

$$\frac{1}{128}\left(x-\frac{1}{2}\sin 4x+\frac{1}{6}\sin 6x\right)+\frac{1}{256}\left(x+\frac{1}{8}\sin 8x\right)$$

$$-\frac{1}{256}\sin 2x+\frac{1}{128}\int\sin^2 4x\cos 2x\ dx$$

We get

$$\frac{1}{128}\left(x-\frac{1}{2}\sin 4x+\frac{1}{6}\sin 6x\right)+\frac{1}{256}\left(x+\frac{1}{8}\sin 8x\right)$$

$$-\frac{1}{256}\sin 2x+\frac{1}{128}\left[\frac{1}{2}\sin^2 4x\sin 2x-\frac{1}{6}\sin 6x+\frac{1}{10}\sin 10x\right]$$

$$\frac{1}{128}x-\frac{1}{256}\sin 4x+\frac{1}{768}\sin 6x+\frac{1}{256}x+\frac{1}{2{,}048}\sin 8x$$

$$-\frac{1}{256}\sin 2x+\frac{1}{256}\sin^2 4x\sin 2x-\frac{1}{768}\sin 6x+\frac{1}{1{,}280}\sin 10x$$

$$\frac{3}{256}x-\frac{1}{256}\sin 2x-\frac{1}{256}\sin 4x+\frac{1}{2{,}048}\sin 8x+\frac{1}{1{,}280}\sin 10x+\frac{1}{256}\sin^2 4x\sin 2x$$

$$\frac{1}{256}\left(3x+\sin^2 4x\sin 2x-\sin 2x-\sin 4x+\frac{1}{8}\sin 8x+\frac{1}{5}\sin 10x\right)+C$$

To reduce the degree of the $\sin^2$ term, we could use the identity

$$\sin^2 x=\frac{1}{2}(1-\cos 2x)$$

to say

$$\sin^2 4x=\frac{1}{2}(1-\cos(2\cdot 4x))$$

$$\sin^2 4x=\frac{1}{2}-\frac{1}{2}\cos 8x$$

Then the expression becomes

$$\frac{1}{256}\left[3x+\left(\frac{1}{2}-\frac{1}{2}\cos 8x\right)\sin 2x-\sin 2x-\sin 4x+\frac{1}{8}\sin 8x+\frac{1}{5}\sin 10x\right]+C$$

$$\frac{1}{256}\left(3x+\frac{1}{2}\sin 2x-\frac{1}{2}\cos 8x\sin 2x-\sin 2x-\sin 4x+\frac{1}{8}\sin 8x+\frac{1}{5}\sin 10x\right)+C$$

$$\frac{1}{256}\left(3x-\frac{1}{2}\sin 2x-\frac{1}{2}\cos 8x\sin 2x-\sin 4x+\frac{1}{8}\sin 8x+\frac{1}{5}\sin 10x\right)+C$$

Topic: sin^m cos^n, m and n even

Question: Evaluate the trigonometric integral.

$$\int_0^{\frac{\pi}{2}} \sin^2\theta\cos^4\theta\ d\theta$$

Answer choices:

A $-\frac{\pi}{8}$

B $-\frac{\pi}{32}$

C $\frac{\pi}{8}$

D $\frac{\pi}{32}$

Solution: D

In the specific case where our function is the product of

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an **even** number of **cosine** factors,

our plan is to

1. create sets of $\sin x \cos x$ and replace each of them with

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2. then use the half-angle formulas to make substitutions,

    a. $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$

    b. $\cos^2 x = \frac{1}{2}(1 + \cos 2x)$

3. remembering that we may need to use the identity

    a. $\cos a \cos b = \frac{1}{2}\left[\cos(a - b) + \cos(a + b)\right]$

We'll create sets of $\sin x \cos x$ and then use the $\sin x \cos x$ identity to make a substitution.

$$\int_0^{\frac{\pi}{2}} \sin^2 \theta \cos^4 \theta \ d\theta$$

$$\int_0^{\frac{\pi}{2}} (\sin\theta\cos\theta)^2\cos^2\theta \ d\theta$$

$$\int_0^{\frac{\pi}{2}} \left(\frac{1}{2}\sin 2\theta\right)^2\cos^2\theta \ d\theta$$

$$\frac{1}{4}\int_0^{\frac{\pi}{2}} \sin^2 2\theta\cos^2\theta \ d\theta$$

Now we'll use the $\sin^2 x$ identity to make a second substitution.

$$\frac{1}{4}\int_0^{\frac{\pi}{2}} \frac{1}{2}\left[1-\cos 2(2\theta)\right]\cos^2\theta \ d\theta$$

$$\frac{1}{4}\int_0^{\frac{\pi}{2}} \left(\frac{1}{2}-\frac{1}{2}\cos 4\theta\right)\cos^2\theta \ d\theta$$

Now we'll use the $\cos^2 x$ identity to make a third substitution.

$$\frac{1}{4}\int_0^{\frac{\pi}{2}} \left(\frac{1}{2}-\frac{1}{2}\cos 4\theta\right)\frac{1}{2}(1+\cos 2\theta) \ d\theta$$

$$\frac{1}{4}\int_0^{\frac{\pi}{2}} \left(\frac{1}{2}-\frac{1}{2}\cos 4\theta\right)\left(\frac{1}{2}+\frac{1}{2}\cos 2\theta\right) \ d\theta$$

$$\frac{1}{4}\int_0^{\frac{\pi}{2}} \frac{1}{4}+\frac{1}{4}\cos 2\theta-\frac{1}{4}\cos 4\theta-\frac{1}{4}\cos 4\theta\cos 2\theta \ d\theta$$

$$\frac{1}{16}\int_0^{\frac{\pi}{2}} 1+\cos 2\theta-\cos 4\theta-\cos 4\theta\cos 2\theta \ d\theta$$

Using the identity

$$\cos a \cos b = \frac{1}{2}\left[\cos(a-b)+\cos(a+b)\right]$$

we'll simplify the integral.

$$\frac{1}{16}\int_0^{\frac{\pi}{2}} 1+\cos 2\theta-\cos 4\theta-\left[\frac{1}{2}\left(\cos(4\theta-2\theta)+\cos(4\theta+2\theta)\right)\right]\ d\theta$$

$$\frac{1}{16}\int_0^{\frac{\pi}{2}} 1+\cos 2\theta-\cos 4\theta-\frac{1}{2}\cos 2\theta-\frac{1}{2}\cos 6\theta\ d\theta$$

$$\frac{1}{16}\left(\theta+\frac{1}{2}\sin 2\theta-\frac{1}{4}\sin 4\theta-\frac{1}{4}\sin 2\theta-\frac{1}{12}\sin 6\theta\right)\Bigg|_0^{\frac{\pi}{2}}$$

$$\frac{1}{16}\left[\frac{\pi}{2}+\frac{1}{2}\sin\left(2\cdot\frac{\pi}{2}\right)-\frac{1}{4}\sin\left(4\cdot\frac{\pi}{2}\right)-\frac{1}{4}\sin\left(2\cdot\frac{\pi}{2}\right)-\frac{1}{12}\sin\left(6\cdot\frac{\pi}{2}\right)\right]$$

$$-\frac{1}{16}\left[0+\frac{1}{2}\sin 2(0)-\frac{1}{4}\sin 4(0)-\frac{1}{4}\sin 2(0)-\frac{1}{12}\sin 6(0)\right]$$

$$\frac{1}{16}\left(\frac{\pi}{2}+\frac{1}{2}\sin\pi-\frac{1}{4}\sin 2\pi-\frac{1}{4}\sin\pi-\frac{1}{12}\sin 3\pi\right)$$

$$-\frac{1}{16}\left(0+\frac{1}{2}\sin 0-\frac{1}{4}\sin 0-\frac{1}{4}\sin 0-\frac{1}{12}\sin 0\right)$$

$$\frac{1}{16}\left[\frac{\pi}{2}+\frac{1}{2}(0)-\frac{1}{4}(0)-\frac{1}{4}(0)-\frac{1}{12}(0)\right]-\frac{1}{16}\left[0+\frac{1}{2}(0)-\frac{1}{4}(0)-\frac{1}{4}(0)-\frac{1}{12}(0)\right]$$

$$\frac{\pi}{32}$$