Topic: Trigonometric substitution with secant

Question: Use trigonometric substitution to evaluate the integral.

$$\int \frac{dx}{\sqrt{x^2 - 2x}}$$

Answer choices:

A $\quad \ln \left| x - 1 + \sqrt{x^2 - 2x} \right| + C$

B $\quad \ln \left| \sqrt{x^2 - 2x} \right| + C$

C $\quad \ln \left(x^2 - 2x \right) + C$

D $\quad \ln \left(x^2 + 2x \right) + C$

Solution: A

First, complete the square to rewrite the integral as

$$\int \frac{dx}{\sqrt{x^2 - 2x}}$$

$$\int \frac{dx}{\sqrt{\left(x^2 - 2x + 1\right) - 1}}$$

$$\int \frac{dx}{\sqrt{(x-1)^2 - 1^2}}$$

We can now use trigonometric substitution to evaluate the integral. Recognizing that

$$u^2 - a^2 = (x-1)^2 - 1^2$$

we get

$$u = x - 1$$

$$a = 1$$

Knowing that

$$u = a \sec \theta$$

is the substitution we use for $u^2 - a^2$, we get

$$x - 1 = 1 \sec \theta$$

$$x-1=\sec\theta$$

$$x=1+\sec\theta$$

$$dx=\sec\theta\tan\theta\ d\theta$$

$$\theta=\sec^{-1}(x-1)$$

Plugging these into the integral we get

$$\int\frac{\sec\theta\tan\theta\ d\theta}{\sqrt{(1+\sec\theta-1)^2-1^2}}$$

$$\int\frac{\sec\theta\tan\theta}{\sqrt{\sec^2\theta-1}}\ d\theta$$

We know that $\tan^2 x=\sec^2 x-1$, so we'll make a substitution to simplify the integral.

$$\int\frac{\sec\theta\tan\theta}{\sqrt{\tan^2\theta}}\ d\theta$$

$$\int\frac{\sec\theta\tan\theta}{\tan\theta}\ d\theta$$

$$\int\sec\theta\ d\theta$$

The formula for the integral of $\sec x$ is

$$\int\sec x\ dx=\ln|\sec x+\tan x|+C$$

Using the formula, the integral becomes

$$\ln|\sec\theta+\tan\theta|+C$$

Since $\theta=\sec^{-1}(x-1)$, we get

$$\ln\left|\sec\left[\sec^{-1}(x-1)\right]+\tan\left[\sec^{-1}(x-1)\right]\right|+C$$

$$\ln\left|x-1+(x-1)\sqrt{1-\frac{1}{(x-1)^2}}\right|+C$$

$$\ln\left|x-1+(x-1)\sqrt{\frac{(x-1)^2}{(x-1)^2}-\frac{1}{(x-1)^2}}\right|+C$$

$$\ln\left|x-1+(x-1)\sqrt{\frac{(x-1)^2-1}{(x-1)^2}}\right|+C$$

$$\ln\left|x-1+(x-1)\sqrt{\frac{x^2-2x+1-1}{(x-1)^2}}\right|+C$$

$$\ln\left|x-1+(x-1)\sqrt{\frac{x^2-2x}{(x-1)^2}}\right|+C$$

$$\ln\left|x-1+(x-1)\frac{\sqrt{x^2-2x}}{\sqrt{(x-1)^2}}\right|+C$$

$$\ln\left|x-1+(x-1)\frac{\sqrt{x^2-2x}}{x-1}\right|+C$$

$$\ln\left|x-1+\sqrt{x^2-2x}\right|+C$$

Topic: Trigonometric substitution with secant

Question: Use trigonometric substitution to evaluate the integral.

$$\int_5^8 \frac{dx}{x^2\sqrt{x^2-16}}$$

Answer choices:

A $\quad \dfrac{5\sqrt{3}-6}{160}$

B $\quad \dfrac{\sqrt{3}}{8}-\dfrac{3}{20}$

C $\quad \dfrac{5\sqrt{3}-6}{80}$

D $\quad \dfrac{5\sqrt{3}}{16}-\dfrac{3}{80}$

Solution: A

Recognizing that we have

$$u^2 - a^2 = x^2 - 4^2$$

in the integral, we get

$$u = x$$

$$a = 4$$

Knowing that

$$u = a \sec \theta$$

is the substitution we use for $u^2 - a^2$, we get

$$x = 4 \sec \theta$$

$$\frac{x}{4} = \sec \theta$$

$$dx = 4 \sec \theta \tan \theta \ d\theta$$

$$\theta = \sec^{-1} \frac{x}{4}$$

Plugging these into the integral we get

$$\int_5^8 \frac{dx}{x^2\sqrt{x^2 - 16}}$$

$$\int_5^8 \frac{4 \sec \theta \tan \theta \ d\theta}{(4 \sec \theta)^2\sqrt{(4 \sec \theta)^2 - 16}}$$

$$\int_5^8 \frac{4\sec\theta\tan\theta}{16\sec^2\theta\sqrt{16\sec^2\theta - 16}}\ d\theta$$

$$\int_5^8 \frac{\tan\theta}{4\sec\theta\sqrt{16\left(\sec^2\theta - 1\right)}}\ d\theta$$

$$\frac{1}{16}\int_5^8 \frac{\tan\theta}{\sec\theta\sqrt{\sec^2\theta - 1}}\ d\theta$$

We know that $\tan^2 x = \sec^2 x - 1$, so we'll make a substitution to simplify the integral.

$$\frac{1}{16}\int_5^8 \frac{\tan\theta}{\sec\theta\sqrt{\tan^2\theta}}\ d\theta$$

$$\frac{1}{16}\int_5^8 \frac{\tan\theta}{\sec\theta\tan\theta}\ d\theta$$

$$\frac{1}{16}\int_5^8 \frac{1}{\sec\theta}\ d\theta$$

$$\frac{1}{16}\int_5^8 \cos\theta\ d\theta$$

The integral of $\cos x$ is

$$\int \cos x\ dx = \sin x + C$$

so the integral becomes

$$\frac{1}{16}\sin\theta\Big|_5^8$$

Back-substituting for x before we evaluate over the interval, we get

$$\frac{1}{16}\sin\left(\sec^{-1}\frac{x}{4}\right)\Big|_5^8$$

$$\frac{1}{16}\sqrt{1-\frac{1}{\left(\frac{x}{4}\right)^2}}\Bigg|_5^8$$

$$\frac{1}{16}\sqrt{1-\frac{16}{x^2}}\Bigg|_5^8$$

$$\frac{1}{16}\left(\sqrt{1-\frac{16}{8^2}}-\sqrt{1-\frac{16}{5^2}}\right)$$

$$\frac{1}{16}\left(\sqrt{\frac{3}{4}}-\sqrt{\frac{9}{25}}\right)$$

$$\frac{1}{16}\left(\frac{\sqrt{3}}{2}-\frac{3}{5}\right)$$

$$\frac{1}{16}\left(\frac{5\sqrt{3}}{10}-\frac{6}{10}\right)$$

$$\frac{5\sqrt{3}-6}{160}$$

Topic: Trigonometric substitution with secant

Question: Use trigonometric substitution to evaluate the integral.

$$\int \frac{6}{\left(9x^2 - 16\right)^{\frac{3}{2}}} \, dx$$

Answer choices:

A $\quad \dfrac{3x}{8\sqrt{9x^2 - 16}} + C$

B $\quad -\dfrac{3x}{8\sqrt{9x^2 - 16}} + C$

C $\quad -\dfrac{3x}{\sqrt{9x^2 - 16}} + C$

D $\quad -\dfrac{x}{8\sqrt{9x^2 - 16}} + C$

Solution: B

Recognizing that we have

$$u^2 - a^2 = 9x^2 - 4^2$$

in the integral, we get

$$u = 3x$$

$$a = 4$$

Knowing that

$$u = a\sec\theta$$

is the substitution we use for $u^2 - a^2$, we get

$$3x = 4\sec\theta$$

$$\frac{3x}{4} = \sec\theta$$

$$x = \frac{4}{3}\sec\theta$$

$$dx = \frac{4}{3}\sec\theta\tan\theta\ d\theta$$

$$\theta = \sec^{-1}\frac{3x}{4}$$

Plugging these into the integral we get

$$\int \frac{6}{(9x^2-16)^{\frac{3}{2}}}\ dx$$

$$\int \frac{6}{\left[9\left(\frac{4}{3}\sec\theta\right)^2-16\right]^{\frac{3}{2}}}\left(\frac{4}{3}\sec\theta\tan\theta\ d\theta\right)$$

$$\int \frac{8\sec\theta\tan\theta}{\left[9\left(\frac{4}{3}\sec\theta\right)^2-16\right]^{\frac{3}{2}}}\ d\theta$$

$$\int \frac{8\sec\theta\tan\theta}{\left[9\left(\frac{16}{9}\sec^2\theta\right)-16\right]^{\frac{3}{2}}}\ d\theta$$

$$\int \frac{8\sec\theta\tan\theta}{(16\sec^2\theta-16)^{\frac{3}{2}}}\ d\theta$$

$$\int \frac{8\sec\theta\tan\theta}{\left[16\left(\sec^2\theta-1\right)\right]^{\frac{3}{2}}}\ d\theta$$

We know that $\tan^2 x = \sec^2 x - 1$, so we'll make a substitution to simplify the integral.

$$\int \frac{8\sec\theta\tan\theta}{(16\tan^2\theta)^{\frac{3}{2}}}\ d\theta$$

$$\int \frac{8\sec\theta\tan\theta}{\left(\sqrt{16\tan^2\theta}\right)^3}\ d\theta$$

$$\int \frac{8\sec\theta\tan\theta}{(4\tan\theta)^3}\ d\theta$$

$$\int \frac{8\sec\theta\tan\theta}{64\tan^3\theta}\ d\theta$$

$$\int \frac{\sec\theta}{8\tan^2\theta}\ d\theta$$

Make substitutions into the integral.

$$\int \frac{\frac{1}{\cos\theta}}{8\frac{\sin^2\theta}{\cos^2\theta}}\ d\theta$$

$$\frac{1}{8}\int \frac{1}{\cos\theta}\cdot\frac{\cos^2\theta}{\sin^2\theta}\ d\theta$$

$$\frac{1}{8}\int \frac{\cos\theta}{\sin^2\theta}\ d\theta$$

$$\frac{1}{8}\int \frac{\cos\theta}{\sin\theta}\cdot\frac{1}{\sin\theta}\ d\theta$$

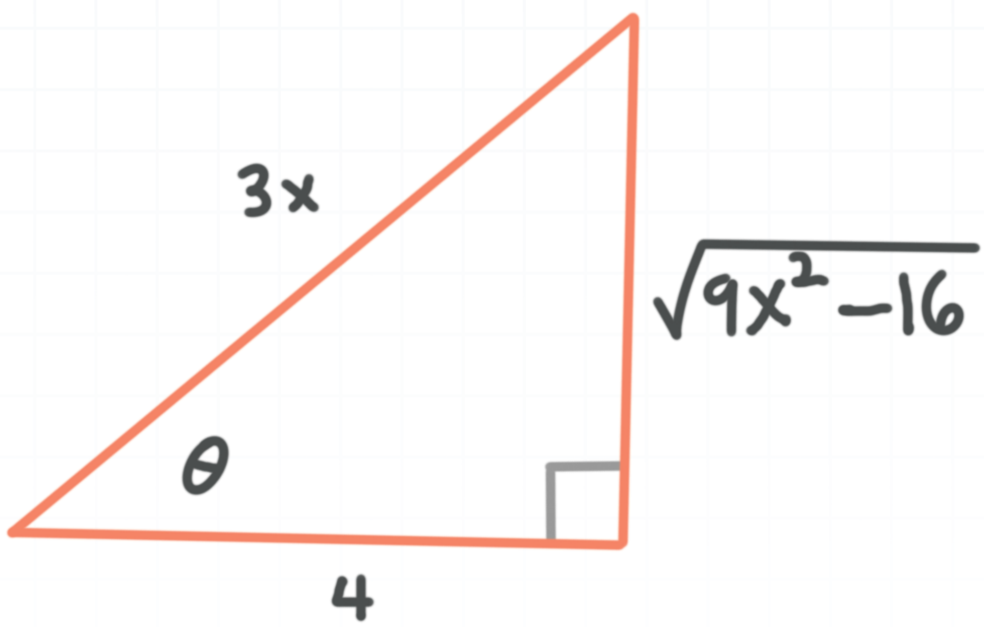
$$\frac{1}{8}\int \cot\theta\csc\theta\ d\theta$$

$$-\frac{1}{8}\csc\theta + C$$

We have successfully integrated this problem using trigonometric substitution with secant. However, to finish with an appropriate answer, we'll now put the problem back in terms of x.

$$-\frac{1}{8\sin\theta}+C$$

The reference triangle is



Because

$$\sin\theta=\frac{\text{opposite}}{\text{hypotenuse}}$$

$$\sin\theta=\frac{\sqrt{9x^2-16}}{3x}$$

Therefore,

$$-\frac{1}{8\frac{\sqrt{9x^2-16}}{3x}}+C$$

$$-\frac{3x}{8\sqrt{9x^2-16}}+C$$