Trigonometric substitution

Trigonometric substitution is another tool we can use to help solve integrals that are too complex for simpler strategies.

You should check to see whether u-substitution, integration by parts, or partial fractions can be used to evaluate the integral before you try trigonometric substitution, because they're often easier and faster. If you want to use trigonometric substitution, the integral must contain one of the values below.

If the integral contains $\sqrt{b^2x^2 - a^2}$ use the substitution $x = \frac{a}{b}\sec\theta$

If the integral contains $\sqrt{a^2 - b^2x^2}$ use the substitution $x = \frac{a}{b}\sin\theta$

If the integral contains $\sqrt{a^2 + b^2x^2}$ use the substitution $x = \frac{a}{b}\tan\theta$

Some integrals can be solved with trigonometric substitution but don't necessarily contain one of the values above. When this is the case, we may have to complete the square or use a simple u-substitution before proceeding with the trigonometric substitution.

Example

Evaluate the integral.

$$\int \frac{x}{\sqrt{2x^2 - 4x - 7}}\ dx$$

Since our function doesn't already contain the format we need for a trigonometric substitution, but the value inside the square root is a quadratic function, we'll complete the square to see if we can get the quadratic function into the right format.

To save some space, let's just work with the value underneath the radical, then we'll plug it back into the integral.

$$2x^2 - 4x - 7$$

$$2\left(x^2 - 2x - \frac{7}{2}\right)$$

$$2\left(x^2 - 2x + 1 - 1 - \frac{7}{2}\right)$$

$$2\left[\left(x^2 - 2x + 1\right) - 1 - \frac{7}{2}\right]$$

$$2\left[(x-1)^2 - \frac{9}{2}\right]$$

$$2(x-1)^2 - 9$$

Plugging this back into the integral, we get

$$\int \frac{x}{\sqrt{2(x-1)^2 - 9}}\ dx$$

Now we have the format we need for the secant substitution,

$$x = \frac{a}{b}\sec\theta$$

If we say that

$$x = x - 1$$

$$a = 3$$

$$b = \sqrt{2}$$

then the substitutions are

$$x - 1 = \frac{3}{\sqrt{2}}\sec\theta$$

$$x = 1 + \frac{3}{\sqrt{2}}\sec\theta$$

$$dx = \frac{3}{\sqrt{2}}\sec\theta\tan\theta\ d\theta$$

$$\sec\theta = \frac{\sqrt{2}(x-1)}{3}$$

Plugging into the integral for x and dx, we get

$$\int \frac{x}{\sqrt{2(x-1)^2 - 9}}\ dx$$

$$\int \frac{1+\frac{3}{\sqrt{2}}\sec\theta}{\sqrt{2\left(1+\frac{3}{\sqrt{2}}\sec\theta-1\right)^2-9}}\left(\frac{3}{\sqrt{2}}\sec\theta\tan\theta\ d\theta\right)$$

We'll focus on simplifying the denominator.

$$\int \frac{1+\frac{3}{\sqrt{2}}\sec\theta}{\sqrt{2\left(\frac{3}{\sqrt{2}}\sec\theta\right)^2-9}}\left(\frac{3}{\sqrt{2}}\sec\theta\tan\theta\right)\ d\theta$$

$$\int \frac{1+\frac{3}{\sqrt{2}}\sec\theta}{\sqrt{2\left(\frac{9}{2}\sec^2\theta\right)-9}}\left(\frac{3}{\sqrt{2}}\sec\theta\tan\theta\right)\ d\theta$$

$$\int \frac{1+\frac{3}{\sqrt{2}}\sec\theta}{\sqrt{9\sec^2\theta-9}}\left(\frac{3}{\sqrt{2}}\sec\theta\tan\theta\right)\ d\theta$$

$$\int \frac{1+\frac{3}{\sqrt{2}}\sec\theta}{\sqrt{9\left(\sec^2\theta-1\right)}}\left(\frac{3}{\sqrt{2}}\sec\theta\tan\theta\right)\ d\theta$$

$$\int \frac{1+\frac{3}{\sqrt{2}}\sec\theta}{3\sqrt{\sec^2\theta-1}}\left(\frac{3}{\sqrt{2}}\sec\theta\tan\theta\right)\ d\theta$$

Remembering the trigonometric identity $\tan^2\theta=\sec^2\theta-1$, we get

$$\int \frac{1+\frac{3}{\sqrt{2}}\sec\theta}{3\sqrt{\tan^2\theta}}\left(\frac{3}{\sqrt{2}}\sec\theta\tan\theta\right)\ d\theta$$

$$\int \frac{1+\frac{3}{\sqrt{2}}\sec\theta}{3\tan\theta}\left(\frac{3}{\sqrt{2}}\sec\theta\tan\theta\right)\ d\theta$$

Now we can cancel $3\tan\theta$.

$$\int \left(1+\frac{3}{\sqrt{2}}\sec\theta\right)\left(\frac{1}{\sqrt{2}}\sec\theta\right)\ d\theta$$

We'll find a common denominator so that we can make the whole function one fraction.

$$\int \left(\frac{\sqrt{2}}{\sqrt{2}}+\frac{3\sec\theta}{\sqrt{2}}\right)\left(\frac{\sec\theta}{\sqrt{2}}\right)\ d\theta$$

$$\int \left(\frac{\sqrt{2}+3\sec\theta}{\sqrt{2}}\right)\left(\frac{\sec\theta}{\sqrt{2}}\right)\ d\theta$$

Multiplying these functions together, we get

$$\int \frac{\sqrt{2}\sec\theta+3\sec^2\theta}{2}\ d\theta$$

$$\frac{1}{2}\int \sqrt{2}\sec\theta+3\sec^2\theta\ d\theta$$

We'll break this into two integrals.

$$\frac{1}{2}\int \sqrt{2}\sec\theta\ d\theta + \frac{1}{2}\int 3\sec^2\theta\ d\theta$$

$$\frac{\sqrt{2}}{2}\int \sec\theta\ d\theta + \frac{3}{2}\int \sec^2\theta\ d\theta$$

Now we can integrate.

$$\frac{\sqrt{2}}{2}\ln\left|\sec\theta + \tan\theta\right| + \frac{3}{2}\tan\theta + C$$

To finish the problem, we need to put the answer back in terms of x instead of θ. Remember that we solved for

$$\sec\theta = \frac{\sqrt{2}(x-1)}{3}$$

when we were setting up the substitution. Plugging that into what we have so far, we get

$$\frac{\sqrt{2}}{2}\ln\left|\frac{\sqrt{2}(x-1)}{3} + \tan\theta\right| + \frac{3}{2}\tan\theta + C$$

Now we'll take a break from this function in order to find $\tan\theta$. Once we've found it, we'll come back to this function to plug it in. In order to find $\tan\theta$, we'll remember that

$$\cos\theta = \frac{1}{\sec\theta}$$

$$\cos\theta = \frac{1}{\frac{\sqrt{2}(x-1)}{3}}$$

$$\cos\theta = \frac{3}{\sqrt{2}(x-1)}$$

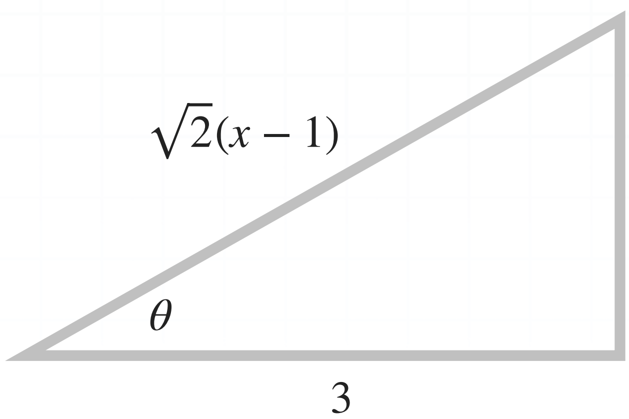
We know that

$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}}$$

so

$$\text{adjacent} = 3$$

$$\text{hypotenuse} = \sqrt{2}(x-1)$$



Using the pythagorean theorem,

$$a^2 + b^2 = c^2$$

we'll plug in the adjacent side and the hypotenuse to solve for the opposite side.

$$(3)^2 + b^2 = \left[\sqrt{2}(x-1)\right]^2$$

$$9 + b^2 = 2(x-1)^2$$

$$9 + b^2 = 2(x-1)(x-1)$$

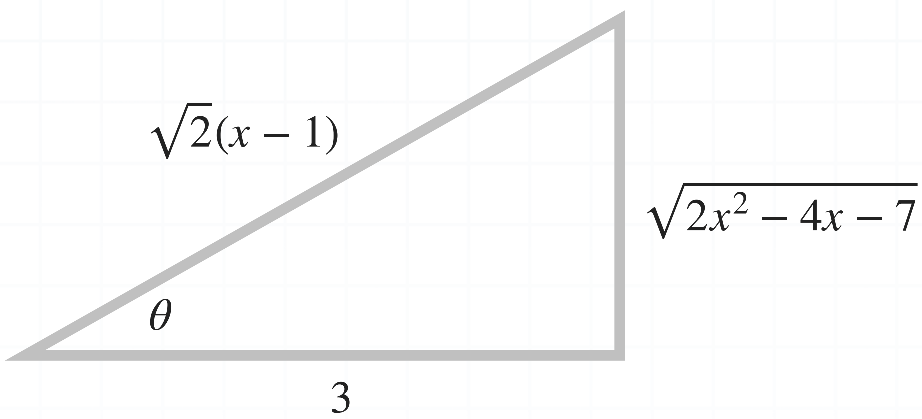
$$9+b^2=2\left(x^2-2x+1\right)$$

$$9+b^2=2x^2-4x+2$$

$$b^2=2x^2-4x-7$$

$$b=\sqrt{2x^2-4x-7}$$

Now our triangle is



Knowing that

$$\tan\theta=\frac{\text{opposite}}{\text{adjacent}}$$

we can say that

$$\tan\theta=\frac{\sqrt{2x^2-4x-7}}{3}$$

Now we can plug this value for $\tan\theta$ into the function we've been working with.

$$\frac{\sqrt{2}}{2}\ln\left|\frac{\sqrt{2}(x-1)}{3}+\tan\theta\right|+\frac{3}{2}\tan\theta+C$$

$$\frac{\sqrt{2}}{2}\ln\left|\frac{\sqrt{2}(x-1)}{3}+\frac{\sqrt{2x^2-4x-7}}{3}\right|+\frac{3}{2}\left(\frac{\sqrt{2x^2-4x-7}}{3}\right)+C$$

$$\frac{\sqrt{2}}{2}\ln\left|\frac{\sqrt{2}(x-1)+\sqrt{2x^2-4x-7}}{3}\right|+\frac{\sqrt{2x^2-4x-7}}{2}+C$$

$$\frac{1}{2}\left[\sqrt{2}\ln\left|\frac{\sqrt{2}(x-1)+\sqrt{2x^2-4x-7}}{3}\right|+\sqrt{2x^2-4x-7}\right]+C$$