Topic: Comparison test

Question: Use the comparison test to say whether or not the series converges.

$$\sum_{n=1}^{\infty}\left(\frac{n}{2n+6}\right)^{n}$$

Answer choices:

A $\quad a_n$ converges

B $\quad a_n$ diverges

C $\quad b_n$ converges but you can't verify $0 \leq b_n \leq a_n$ so the test is inconclusive

D $\quad b_n$ diverges but you can't verify $a_n \geq b_n \geq 0$ so the test is inconclusive

Solution: A

The comparison test for convergence lets us determine the convergence or divergence of the given series a_n by *comparing* it to a similar, but simpler comparison series b_n.

We're usually trying to find a comparison series that's a geometric or p-series, since it's very easy to determine the convergence of a geometric or p-series.

We can use the comparison test to show that

the original series a_n is **diverging** if

the original series a_n is greater than or equal to the comparison series b_n and both series are positive, $a_n \geq b_n \geq 0$, and

the comparison series b_n is diverging

Note: If $a_n < b_n$, the test is inconclusive

the original series is **converging** if

the original series a_n is less than or equal to the comparison series b_n and both series are positive, $0 \leq a_n \leq b_n$, and

the comparison series b_n is converging

Note: If $b_n < a_n$, the test is inconclusive

Before we can use the comparison test with the series a_n that we're given in this problem, we need to create a similar, but simpler comparison series b_n.

We'll use the numerator from a_n for b_n, since the numerator is already pretty simple. In the denominator, the $2n$ carries a lot more weight and will affect the series more than the 6, so we'll use only the $2n$ in the denominator of the comparison series.

$$b_n = \left(\frac{n}{2n}\right)^n$$

$$b_n = \left(\frac{1}{2}\right)^n$$

The comparison series is a geometric series. The geometric series test for convergence says that

if $|r| < 1$ then the series converges

if $|r| \geq 1$ then the series diverges

when we're pulling r from the expanded form of the geometric series

$$\sum_{n=1}^{\infty} ar^{n-1} = a\left\{1 + r + r^2 + r^3 + \dots\right\}$$

Expanding b_n until it matches this expanded form of a geometric series, we get

$$b_n = \left(\frac{1}{2}\right)^1 + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3 + \left(\frac{1}{2}\right)^4 + \left(\frac{1}{2}\right)^5 + \dots$$

$$b_n = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \dots$$

$$b_n = \frac{1}{2}\left(1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \dots\right)$$

We'll pull r from the term immediately following the 1 inside the parentheses, so $r = 1/2$. Applying the geometric series test, we see that

$$\left|\frac{1}{2}\right| = \frac{1}{2} < 1$$

which means that the comparison series converges.

Knowing that the comparison series converges, we need to show that

$$0 \le a_n \le b_n$$

in order to prove that the original series a_n is also converging. If we can't verify this inequality, then the comparison test will be inconclusive. To verify the inequality, we'll compare a few points from a_n and b_n. Let's use $n = 1$, $n = 2$ and $n = 3$.

	a_n	b_n
$n = 1$	$a_1 = \left(\frac{1}{2(1)+6}\right)^1 = \frac{1}{8}$	$b_1 = \left(\frac{1}{2}\right)^1 = \frac{1}{2}$
$n = 2$	$a_2 = \left(\frac{2}{2(2)+6}\right)^2 = \frac{1}{25}$	$b_2 = \left(\frac{1}{2}\right)^2 = \frac{1}{4}$
$n = 3$	$a_3 = \left(\frac{3}{2(3)+6}\right)^3 = \frac{1}{64}$	$b_3 = \left(\frac{1}{2}\right)^3 = \frac{1}{8}$

Looking at just these few terms, we can see that $0 \leq a_n \leq b_n$ for all n, which means we can conclude that a_n converges.

Topic: Comparison test

Question: Use the comparison test to say whether or not the series converges.

$$\sum_{n=1}^{\infty} \left(\frac{4}{n} - \frac{4}{n^2} \right)$$

Answer choices:

A a_n converges

B a_n diverges

C b_n converges but you can't verify $0 \leq b_n \leq a_n$ so the test is inconclusive

D b_n diverges but you can't verify $a_n \geq b_n \geq 0$ so the test is inconclusive

Solution: D

The comparison test for convergence lets us determine the convergence or divergence of the given series a_n by *comparing* it to a similar, but simpler comparison series b_n.

We're usually trying to find a comparison series that's a geometric or p-series, since it's very easy to determine the convergence of a geometric or p-series.

We can use the comparison test to show that

the original series a_n is **diverging** if

the original series a_n is greater than or equal to the comparison series b_n and both series are positive, $a_n \geq b_n \geq 0$, and

the comparison series b_n is diverging

Note: If $a_n < b_n$, the test is inconclusive

the original series is **converging** if

the original series a_n is less than or equal to the comparison series b_n and both series are positive, $0 \leq a_n \leq b_n$, and

the comparison series b_n is converging

Note: If $b_n < a_n$, the test is inconclusive

Before we can use the comparison test with the series a_n that we're given in this problem, we need to create a similar, but simpler comparison series

b_n. Let's combine the given series into one fraction before creating the comparison series.

$$\sum_{n=1}^{\infty}\left(\frac{4}{n}-\frac{4}{n^2}\right)=\sum_{n=1}^{\infty}\left[\frac{4}{n}\left(\frac{n}{n}\right)-\frac{4}{n^2}\right]$$

$$\sum_{n=1}^{\infty}\left(\frac{4}{n}-\frac{4}{n^2}\right)=\sum_{n=1}^{\infty}\left(\frac{4n}{n^2}-\frac{4}{n^2}\right)$$

$$\sum_{n=1}^{\infty}\left(\frac{4}{n}-\frac{4}{n^2}\right)=\sum_{n=1}^{\infty}\frac{4n-4}{n^2}$$

We'll use the denominator from a_n for b_n, since the denominator is already pretty simple. In the numerator, the $4n$ carries a lot more weight and will affect the series more than the -4, so we'll use only the $4n$ in the numerator of the comparison series.

$$b_n=\frac{4n}{n^2}$$

$$b_n=\frac{4}{n}$$

$$b_n=4\left(\frac{1}{n}\right)$$

$$b_n=4\left(\frac{1}{n^1}\right)$$

The comparison series is a p-series. Since the p-series test tells us that the series will converge when $p > 1$

diverge when $p \le 1$

we can say that $1 \le 1$ and therefore that b_n diverges.

Knowing that the comparison series diverges, we need to show that

$$a_n \ge b_n \ge 0$$

in order to prove that the original series a_n is also diverging. If we can't verify this inequality, then the comparison test will be inconclusive. To verify the inequality, we'll compare a few points from a_n and b_n. Let's use $n = 1$, $n = 2$ and $n = 3$.

	a_n	b_n
$n = 1$	$a_1 = \frac{4(1) - 4}{1^2} = 0$	$b_1 = \frac{4}{1} = 4$
$n = 2$	$a_2 = \frac{4(2) - 4}{2^2} = 1$	$b_2 = \frac{4}{2} = 2$
$n = 3$	$a_3 = \frac{4(3) - 4}{3^2} = \frac{8}{9}$	$b_3 = \frac{4}{3}$

Looking at just these few terms, we can see that $0 \le a_n \le b_n$. This is the opposite of what we were looking for, $a_n \ge b_n \ge 0$, which means the test is inconclusive.

Topic: Comparison test

Question: Use the comparison test to say whether or not the series converges.

$$\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^2 + 1}$$

Answer choices:

A a_n converges

B a_n diverges

C b_n converges but you can't verify $0 \leq b_n \leq a_n$ so the test is inconclusive

D b_n diverges but you can't verify $a_n \geq b_n \geq 0$ so the test is inconclusive

Solution: A

The comparison test for convergence lets us determine the convergence or divergence of the given series a_n by *comparing* it to a similar, but simpler comparison series b_n.

We're usually trying to find a comparison series that's a geometric or p-series, since it's very easy to determine the convergence of a geometric or p-series.

We can use the comparison test to show that

the original series a_n is **diverging** if

the original series a_n is greater than or equal to the comparison series b_n and both series are positive, $a_n \geq b_n \geq 0$, and

the comparison series b_n is diverging

Note: If $a_n < b_n$, the test is inconclusive

the original series is **converging** if

the original series a_n is less than or equal to the comparison series b_n and both series are positive, $0 \leq a_n \leq b_n$, and

the comparison series b_n is converging

Note: If $b_n < a_n$, the test is inconclusive

Before we can use the comparison test with the series a_n that we're given in this problem, we need to create a similar, but simpler comparison series b_n.

We'll use the numerator from a_n for b_n, since the numerator is already pretty simple. In the denominator, the n^2 carries a lot more weight and will affect the series more than the 1, so we'll use only the n^2 in the denominator of the comparison series.

$$b_n = \frac{\sqrt{n}}{n^2}$$

$$b_n = \frac{n^{\frac{1}{2}}}{n^2}$$

$$b_n = \frac{1}{n^{2-\frac{1}{2}}}$$

$$b_n = \frac{1}{n^{\frac{3}{2}}}$$

The comparison series is a p-series. Since the p-series test tells us that the series will

converge when $p > 1$

diverge when $p \leq 1$

we can say that $3/2 > 1$ and therefore that b_n converges.

Knowing that the comparison series converges, we need to show that

$$0 \leq a_n \leq b_n$$

in order to prove that the original series a_n is also converging. If we can't verify this inequality, then the comparison test will be inconclusive. To verify the inequality, we'll compare a few points from a_n and b_n. Since we've got a square root in a_n, let's use squares, like $n=1$, $n=4$ and $n=9$.

	a_n	b_n
$n=1$	$a_1=\frac{\sqrt{1}}{1^2+1}=\frac{1}{2}$	$b_1=\frac{1}{1^{\frac{3}{2}}}=1$
$n=4$	$a_4=\frac{\sqrt{4}}{4^2+1}=\frac{2}{17}$	$b_4=\frac{1}{4^{\frac{3}{2}}}=\frac{1}{8}$
$n=9$	$a_9=\frac{\sqrt{9}}{9^2+1}=\frac{3}{82}$	$b_9=\frac{1}{9^{\frac{3}{2}}}=\frac{1}{27}$

Looking at just these few terms, we can see that $0 \le a_n \le b_n$ for all n, which means we can conclude that a_n converges.