Topic: Maclaurin series to estimate a definite integral

Question: Use a Maclaurin series to estimate the definite integral.

$$\int_0^2 xe^x \, dx$$

Answer choices:

A $\quad 8.0$

B $\quad 1.0$

C $\quad 0.8$

D $\quad 2.0$

Solution: A

When we're asked to use a Maclaurin series to estimate a definite integral, it means we're supposed to find a power series representation for the function we've been asked to integrate, and then integrate that power series instead of the original function.

To find the power series representation of the given function, we'll start with the known Maclaurin series

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

and then manipulate it until it matches the given series. To get it to match the given series, we'll multiply both sides by x.

$$xe^x = x\sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$xe^x = \sum_{n=0}^{\infty} \frac{x^{n+1}}{n!}$$

Now we can integrate the power series instead of the original function.

$$\int_0^2 xe^x \, dx = \int_0^2 \sum_{n=0}^{\infty} \frac{x^{n+1}}{n!} \, dx$$

Since we're integrating with respect to x, we can remove from the integral on the right any term that doesn't involve x.

$$\int_0^2 xe^x\ dx = \sum_{n=0}^{\infty} \frac{1}{n!} \int_0^2 x^{n+1}\ dx$$

$$\int_0^2 xe^x\ dx = \sum_{n=0}^{\infty} \frac{1}{n!} \cdot \frac{x^{n+2}}{n+2} \Big|_0^2$$

$$\int_0^2 xe^x\ dx = \sum_{n=0}^{\infty} \frac{x^{n+2}}{n!(n+2)} \Big|_0^2$$

Now we'll expand the power series through its first eight terms. That should be enough to make sure that our answer is stable to one decimal place, which is the number of decimal places given in the answer choices.

$$\int_0^2 xe^x\ dx = \frac{x^{0+2}}{0!(0+2)} + \frac{x^{1+2}}{1!(1+2)} + \frac{x^{2+2}}{2!(2+2)} + \frac{x^{3+2}}{3!(3+2)} + \frac{x^{4+2}}{4!(4+2)}$$

$$+ \frac{x^{5+2}}{5!(5+2)} + \frac{x^{6+2}}{6!(6+2)} + \frac{x^{7+2}}{7!(7+2)} + \frac{x^{8+2}}{8!(8+2)} + \dots \Big|_0^2$$

$$\int_0^2 xe^x\ dx = \frac{x^2}{1(2)} + \frac{x^3}{1(3)} + \frac{x^4}{2(4)} + \frac{x^5}{6(5)} + \frac{x^6}{24(6)}$$

$$+ \frac{x^7}{120(7)} + \frac{x^8}{720(8)} + \frac{x^9}{5{,}040(9)} + \frac{x^{10}}{40{,}320(10)} \Big|_0^2$$

$$\int_0^2 xe^x\ dx = \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{8} + \frac{x^5}{30} + \frac{x^6}{144} + \frac{x^7}{840} + \frac{x^8}{5{,}760} + \frac{x^9}{45{,}360} + \frac{x^{10}}{403{,}200} \Big|_0^2$$

Now we can evaluate the expansion over the interval. Since plugging in $x=0$ will give 0 for every term, we only need to evaluate at $x=2$.

$$\int_0^2 xe^x\ dx = \frac{2^2}{2}+\frac{2^3}{3}+\frac{2^4}{8}+\frac{2^5}{30}+\frac{2^6}{144}+\frac{2^7}{840}+\frac{2^8}{5{,}760}+\frac{2^9}{45{,}360}+\frac{2^{10}}{403{,}200}$$

$$\int_0^2 xe^x\ dx = \frac{4}{2}+\frac{8}{3}+\frac{16}{8}+\frac{32}{30}+\frac{64}{144}+\frac{128}{840}+\frac{256}{5{,}760}+\frac{512}{45{,}360}+\frac{1{,}024}{403{,}200}$$

$$\int_0^2 xe^x\ dx \approx 2.00000+2.66667+2.00000+1.06667+0.44444$$

$$+0.15238+0.04444+0.01128+0.00254$$

Now we need to add the terms together until we get an answer that's stable to one decimal place.

$$2.00000+2.66667=4.66667$$

$$4.66667+2.00000=6.66667$$

$$6.66667+1.06667=7.73334$$

$$7.73334+0.04444=7.77778$$

$$7.77778+0.15238=7.93016$$

$$7.93016+0.04444=7.97460$$

$$7.97460+0.01128=7.98588$$

$$7.98588+0.00254=7.98842$$

Since we got a 9 in the tenths place four times in a row, we know the answer will be stable to this value. We'll round the last answer to the tenths place and get 8.0 as an approximation of the definite integral.

Topic: Maclaurin series to estimate a definite integral

Question: Use a Maclaurin series to estimate the definite integral.

$$\int_0^{\frac{1}{4}} \frac{2x}{1-3x}\,dx$$

Answer choices:

A 1.1

B 1.0

C 0.1

D 2.0

Solution: C

When we're asked to use a Maclaurin series to estimate a definite integral, it means we're supposed to find a power series representation for the function we've been asked to integrate, and then integrate that power series instead of the original function.

To find the power series representation of the given function, we'll start with the known Maclaurin series

$$\frac{1}{1-x}=\sum_{n=0}^{\infty}x^n$$

and then manipulate it until it matches the given series. To get it to match the given series, we'll substitute $3x$ for x and then multiply by $2x$.

$$\frac{1}{1-3x}=\sum_{n=0}^{\infty}(3x)^n$$

$$\frac{2x}{1-3x}=2x\sum_{n=0}^{\infty}(3x)^n$$

$$\frac{2x}{1-3x}=\sum_{n=0}^{\infty}2x3^nx^n$$

$$\frac{2x}{1-3x}=\sum_{n=0}^{\infty}2(3^n)x^{n+1}$$

Now we can integrate the power series instead of the original function.

$$\int_0^{\frac{1}{4}} \frac{2x}{1-3x}\ dx = \int_0^{\frac{1}{4}} \sum_{n=0}^{\infty} 2(3^n)x^{n+1}\ dx$$

Since we're integrating with respect to x, we can remove from the integral on the right any term that doesn't involve x.

$$\int_0^{\frac{1}{4}} \frac{2x}{1-3x}\ dx = \sum_{n=0}^{\infty} 2(3^n) \int_0^{\frac{1}{4}} x^{n+1}\ dx$$

$$\int_0^{\frac{1}{4}} \frac{2x}{1-3x}\ dx = \sum_{n=0}^{\infty} 2(3^n) \cdot \frac{x^{n+2}}{n+2} \Bigg|_0^{\frac{1}{4}}$$

$$\int_0^{\frac{1}{4}} \frac{2x}{1-3x}\ dx = \sum_{n=0}^{\infty} \frac{2(3^n)x^{n+2}}{n+2} \Bigg|_0^{\frac{1}{4}}$$

Now we'll expand the power series through its first eight terms. That should be enough to make sure that our answer is stable to one decimal place, which is the number of decimal places given in the answer choices.

$$\int_0^{\frac{1}{4}} \frac{2x}{1-3x}\ dx = \frac{2(3^0)x^{0+2}}{0+2} + \frac{2(3^1)x^{1+2}}{1+2} + \frac{2(3^2)x^{2+2}}{2+2} + \frac{2(3^3)x^{3+2}}{3+2} + \frac{2(3^4)x^{4+2}}{4+2}$$

$$+ \frac{2(3^5)x^{5+2}}{5+2} + \frac{2(3^6)x^{6+2}}{6+2} + \frac{2(3^7)x^{7+2}}{7+2} + \frac{2(3^8)x^{8+2}}{8+2} \Bigg|_0^{\frac{1}{4}}$$

$$\int_0^{\frac{1}{4}} \frac{2x}{1-3x}\ dx = \frac{2(1)x^2}{2} + \frac{2(3)x^3}{3} + \frac{2(9)x^4}{4} + \frac{2(27)x^5}{5} + \frac{2(81)x^6}{6}$$

$$+\frac{2(243)x^7}{7}+\frac{2(729)x^8}{8}+\frac{2(2,187)x^9}{9}+\frac{2(6,561)x^{10}}{10}\Bigg|_0^{\frac{1}{4}}$$

$$\int_0^{\frac{1}{4}}\frac{2x}{1-3x}\ dx=\frac{2x^2}{2}+\frac{6x^3}{3}+\frac{18x^4}{4}+\frac{54x^5}{5}+\frac{162x^6}{6}$$

$$+\frac{486x^7}{7}+\frac{1,458x^8}{8}+\frac{4,374x^9}{9}+\frac{13,122x^{10}}{10}\Bigg|_0^{\frac{1}{4}}$$

Now we can evaluate the expansion over the interval. Since plugging in $x=0$ will give 0 for every term, we only need to evaluate at $x=1/4$.

$$\int_0^{\frac{1}{4}}\frac{2x}{1-3x}\ dx=\frac{2\left(\frac{1}{4}\right)^2}{2}+\frac{6\left(\frac{1}{4}\right)^3}{3}+\frac{18\left(\frac{1}{4}\right)^4}{4}+\frac{54\left(\frac{1}{4}\right)^5}{5}+\frac{162\left(\frac{1}{4}\right)^6}{6}$$

$$+\frac{486\left(\frac{1}{4}\right)^7}{7}+\frac{1,458\left(\frac{1}{4}\right)^8}{8}+\frac{4,374\left(\frac{1}{4}\right)^9}{9}+\frac{13,122\left(\frac{1}{4}\right)^{10}}{10}$$

$$\int_0^{\frac{1}{4}}\frac{2x}{1-3x}\ dx=\frac{2\left(\frac{1}{16}\right)}{2}+\frac{6\left(\frac{1}{64}\right)}{3}+\frac{18\left(\frac{1}{256}\right)}{4}+\frac{54\left(\frac{1}{1,024}\right)}{5}+\frac{162\left(\frac{1}{4,096}\right)}{6}$$

$$+\frac{486\left(\frac{1}{16,384}\right)}{7}+\frac{1,458\left(\frac{1}{65,536}\right)}{8}+\frac{4,374\left(\frac{1}{262,144}\right)}{9}+\frac{13,122\left(\frac{1}{1,048,576}\right)}{10}$$

$$\int_0^{\frac{1}{4}}\frac{2x}{1-3x}\ dx=\frac{1}{16}+\frac{1}{32}+\frac{9}{512}+\frac{27}{2,560}+\frac{27}{4,096}$$

$$+\frac{243}{57,344}+\frac{729}{262,144}+\frac{243}{131,072}+\frac{6,561}{5,242,880}$$

$$\int_0^{\frac{1}{4}} \frac{2x}{1-3x}\ dx = 0.06250 + 0.03125 + 0.01758 + 0.01055 + 0.00659$$

$$+0.00424 + 0.00278 + 0.00185 + 0.00125$$

Now we need to add the terms together until we get an answer that's stable to one decimal place.

$$0.06250 + 0.03125 = 0.09375$$

$$0.09375 + 0.01758 = 0.11133$$

$$0.11133 + 0.01055 = 0.12188$$

$$0.12188 + 0.00659 = 0.12847$$

$$0.12847 + 0.00424 = 0.13271$$

$$0.13271 + 0.00278 = 0.13549$$

$$0.13549 + 0.00185 = 0.13734$$

$$0.13734 + 0.00125 = 0.13859$$

Since we got a 1 in the tenths place four times in a row, we know the answer will be stable to this value. We'll round the last answer to the tenths place and get 0.1 as an approximation of the definite integral.

Topic: Maclaurin series to estimate a definite integral

Question: Use a Maclaurin series to estimate the definite integral.

$$\int_0^1 \frac{\cos x}{2}\ dx$$

Answer choices:

A 0.1

B 0.4

C -0.2

D 0.2

Solution: B

When we're asked to use a Maclaurin series to estimate a definite integral, it means we're supposed to find a power series representation for the function we've been asked to integrate, and then integrate that power series instead of the original function.

To find the power series representation of the given function, we'll start with the known Maclaurin series

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

and then manipulate it until it matches the given series. To get it to match the given series, we'll multiply by $1/2$.

$$\frac{1}{2}\cos x = \frac{1}{2}\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

$$\frac{\cos x}{2} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2(2n)!}$$

Now we can integrate the power series instead of the original function.

$$\int_0^1 \frac{\cos x}{2}\ dx = \int_0^1 \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2(2n)!}\ dx$$

Since we're integrating with respect to x, we can remove from the integral on the right any term that doesn't involve x.

$$\int_0^1 \frac{\cos x}{2}\ dx = \sum_{n=0}^{\infty} \frac{(-1)^n}{2(2n)!} \int_0^1 x^{2n}\ dx$$

$$\int_0^1 \frac{\cos x}{2}\ dx = \sum_{n=0}^{\infty} \frac{(-1)^n}{2(2n)!} \cdot \frac{x^{2n+1}}{2n+1} \Bigg|_0^1$$

$$\int_0^1 \frac{\cos x}{2}\ dx = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2(2n+1)(2n)!} \Bigg|_0^1$$

Now we'll expand the power series through its first five terms. That should be enough to make sure that our answer is stable to one decimal place, which is the number of decimal places given in the answer choices.

$$\int_0^1 \frac{\cos x}{2}\ dx = \frac{(-1)^0 x^{2(0)+1}}{2(2(0)+1)(2(0))!} + \frac{(-1)^1 x^{2(1)+1}}{2(2(1)+1)(2(1))!} + \frac{(-1)^2 x^{2(2)+1}}{2(2(2)+1)(2(2))!}$$

$$+ \frac{(-1)^3 x^{2(3)+1}}{2(2(3)+1)(2(3))!} + \frac{(-1)^4 x^{2(4)+1}}{2(2(4)+1)(2(4))!} + \frac{(-1)^5 x^{2(5)+1}}{2(2(5)+1)(2(5))!} \Bigg|_0^1$$

$$\int_0^1 \frac{\cos x}{2}\ dx = \frac{x^1}{2(1)0!} + \frac{-x^3}{2(3)2!} + \frac{x^5}{2(5)4!} + \frac{-x^7}{2(7)6!} + \frac{x^9}{2(9)8!} + \frac{-x^{11}}{2(11)10!} \Bigg|_0^1$$

$$\int_0^1 \frac{\cos x}{2}\ dx = \frac{x^1}{(2)0!} - \frac{x^3}{(6)2!} + \frac{x^5}{(10)4!} - \frac{x^7}{(14)6!} + \frac{x^9}{(18)8!} - \frac{x^{11}}{(22)10!} \Bigg|_0^1$$

Now we can evaluate the expansion over the interval. Since plugging in $x = 0$ will give 0 for every term, we only need to evaluate at $x = 1$.

$$\int_0^1 \frac{\cos x}{2}\ dx = \frac{1^1}{(2)0!} - \frac{1^3}{(6)2!} + \frac{1^5}{(10)4!} - \frac{1^7}{(14)6!} + \frac{1^9}{(18)8!} - \frac{1^{11}}{(22)10!}$$

$$\int_0^1 \frac{\cos x}{2}\ dx = \frac{1}{2} - \frac{1}{12} + \frac{1}{240} - \frac{1}{10{,}080} + \frac{1}{933{,}120} - \frac{1}{102{,}643{,}200}$$

$$\int_0^1 \frac{\cos x}{2}\ dx = 0.50000 - 0.08333 + 0.00417 - 0.00010 + 0.00000 - 0.00000$$

Now we need to add the terms together until we get an answer that's stable to one decimal place.

$$0.50000 - 0.08333 = 0.41667$$

$$0.41667 + 0.00417 = 0.42084$$

$$0.42084 - 0.00010 = 0.42074$$

$$0.42074 + 0.00000 = 0.42074$$

$$0.42074 - 0.00000 = 0.42074$$

Since we got a 4 in the tenths place five times in a row, we know the answer will be stable to this value. We'll round the last answer to the tenths place and get 0.4 as an approximation of the definite integral.