Calculus 2 Workbook Solutions

Area between curves

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MATH

AREA BETWEEN UPPER AND LOWER CURVES

1. Find the area, in square units, between the two curves. Round your answer to two decimal places.

$$f(x) = -2x^2 + 7$$

$$g(x) = -x + 3$$

Solution:

Find the intersection points of the curves.

$$-2x^2 + 7 = -x + 3$$

$$2x^2 - x - 4 = 0$$

Use the quadratic formula.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(2)(-4)}}{2(2)} = \frac{1 \pm \sqrt{1 + 32}}{4} = \frac{1 \pm \sqrt{33}}{4}$$

Between these two points, $f(x) > g(x)$. Therefore, the area between the curves is

$$A=\int_{\frac{1-\sqrt{33}}{4}}^{\frac{1+\sqrt{33}}{4}}-2x^2+7-(-x+3)\ dx$$

$$A=\int_{\frac{1-\sqrt{33}}{4}}^{\frac{1+\sqrt{33}}{4}}-2x^2+7+x-3\ dx$$

$$A=\int_{\frac{1-\sqrt{33}}{4}}^{\frac{1+\sqrt{33}}{4}}-2x^2+x+4\ dx$$

Integrate and evaluate over the interval.

$$A=-\frac{2}{3}x^3+\frac{1}{2}x^2+4x\Big|_{\frac{1-\sqrt{33}}{4}}^{\frac{1+\sqrt{33}}{4}}$$

$$A=-\frac{2}{3}\left(\frac{1+\sqrt{33}}{4}\right)^3+\frac{1}{2}\left(\frac{1+\sqrt{33}}{4}\right)^2+4\left(\frac{1+\sqrt{33}}{4}\right)$$

$$-\left[-\frac{2}{3}\left(\frac{1-\sqrt{33}}{4}\right)^3+\frac{1}{2}\left(\frac{1-\sqrt{33}}{4}\right)^2+4\left(\frac{1-\sqrt{33}}{4}\right)\right]$$

$$A=-\frac{2}{3}\left(\frac{1+\sqrt{33}}{4}\right)^3+\frac{1}{2}\left(\frac{1+\sqrt{33}}{4}\right)^2+4\left(\frac{1+\sqrt{33}}{4}\right)$$

$$+\frac{2}{3}\left(\frac{1-\sqrt{33}}{4}\right)^3-\frac{1}{2}\left(\frac{1-\sqrt{33}}{4}\right)^2-4\left(\frac{1-\sqrt{33}}{4}\right)$$

$$A\approx 7.90$$

2. Find the area, in square units, between the two curves.

$$f(x) = -3x^2 + 9x$$

$$g(x) = 3x^2 - 9x$$

Solution:

Find the intersection points of the curves.

$$-3x^2 + 9x = 3x^2 - 9x$$

$$6x^2 - 18x = 0$$

$$x^2 - 3x = 0$$

$$x(x-3) = 0$$

$$x = 0, 3$$

Between these two points, $f(x) > g(x)$. Therefore, the area between the curves is

$$A = \int_0^3 f(x) - g(x)\ dx$$

$$A = \int_0^3 -3x^2 + 9x - \left(3x^2 - 9x\right)\ dx$$

$$A = \int_0^3 -3x^2 + 9x - 3x^2 + 9x \ dx$$

$$A = \int_0^3 -6x^2 + 18x \ dx$$

Integrate and evaluate over the interval.

$$A = -\frac{6}{3}x^3 + \frac{18}{2}x^2 \Big|_0^3$$

$$A = -2x^3 + 9x^2 \Big|_0^3$$

$$A = 9x^2 - 2x^3 \Big|_0^3$$

$$A = 9(3)^2 - 2(3)^3 - \left(9(0)^2 - 2(0)^3\right)$$

$$A = 9(9) - 2(27)$$

$$A = 81 - 54$$

$$A = 27$$

AREA BETWEEN LEFT AND RIGHT CURVES

1. Find the area, in square units, between the two curves. Round your answer to two decimal places.

$$f(y) = 2y^2 + 12y + 15$$

$$g(y) = -2y^2 - 12y - 15$$

Solution:

Find the intersection points of the curves.

$$2y^2 + 12y + 15 = -2y^2 - 12y - 15$$

$$4y^2 + 24y + 30 = 0$$

$$2y^2 + 12y + 15 = 0$$

Use the quadratic formula.

$$y = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$y = \frac{-12 \pm \sqrt{12^2 - 4(2)(15)}}{2(2)} = \frac{-12 \pm \sqrt{144 - 120}}{4} = \frac{-12 \pm \sqrt{24}}{4}$$

$$= \frac{-12 \pm 2\sqrt{6}}{4} = \frac{-6 \pm \sqrt{6}}{2}$$

Between these two points, $g(y) > f(y)$. Therefore, the area between the curves is

$$A = \int_{\frac{-6-\sqrt{6}}{2}}^{\frac{-6+\sqrt{6}}{2}} -2y^2 - 12y - 15 - \left(2y^2 + 12y + 15\right)\ dy$$

$$A = \int_{\frac{-6-\sqrt{6}}{2}}^{\frac{-6+\sqrt{6}}{2}} -2y^2 - 12y - 15 - 2y^2 - 12y - 15\ dy$$

$$A = \int_{\frac{-6-\sqrt{6}}{2}}^{\frac{-6+\sqrt{6}}{2}} -4y^2 - 24y - 30\ dy$$

Integrate and evaluate over the interval.

$$A = -\frac{4}{3}y^3 - \frac{24}{2}y^2 - 30y\Big|_{\frac{-6-\sqrt{6}}{2}}^{\frac{-6+\sqrt{6}}{2}}$$

$$A = -\frac{4}{3}y^3 - 12y^2 - 30y\Big|_{\frac{-6-\sqrt{6}}{2}}^{\frac{-6+\sqrt{6}}{2}}$$

$$A = -\frac{4}{3}\left(\frac{-6+\sqrt{6}}{2}\right)^3 - 12\left(\frac{-6+\sqrt{6}}{2}\right)^2 - 30\left(\frac{-6+\sqrt{6}}{2}\right)$$

$$-\left[-\frac{4}{3}\left(\frac{-6-\sqrt{6}}{2}\right)^3 - 12\left(\frac{-6-\sqrt{6}}{2}\right)^2 - 30\left(\frac{-6-\sqrt{6}}{2}\right)\right]$$

$$A = -\frac{4}{3}\left(\frac{-6+\sqrt{6}}{2}\right)^3 - 12\left(\frac{-6+\sqrt{6}}{2}\right)^2 - 30\left(\frac{-6+\sqrt{6}}{2}\right)$$

$$+\frac{4}{3}\left(\frac{-6-\sqrt{6}}{2}\right)^3 + 12\left(\frac{-6-\sqrt{6}}{2}\right)^2 + 30\left(\frac{-6-\sqrt{6}}{2}\right)$$

$$A \approx 9.80$$

■ 2. Find the area, in square units, between the two curves, and between $y = -2$ and $y = -5$.

$$f(y) = 2y^2 + 12y + 19$$

$$g(y) = -\frac{y^2}{2} - 4y - 10$$

Solution:

Find the intersection points of the curves.

$$2y^2 + 12y + 19 = -\frac{y^2}{2} - 4y - 10$$

$$4y^2 + 24y + 38 = -y^2 - 8y - 20$$

$$5y^2 + 32y + 58 = 0$$

Use the quadratic formula.

$$y=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

$$y=\frac{-32\pm\sqrt{32^2-4(5)(58)}}{2(5)}=\frac{-32\pm\sqrt{1{,}024-1{,}160}}{10}=\frac{-32\pm\sqrt{-136}}{10}$$

Because we can't take the square root of a negative number, this means that the curves do not intersect. Which means only $y=-2$ and $y=-5$ provide the limits of integration. $f(y)$ is to the right of $g(y)$, so

$$A=\int_{-5}^{-2}2y^2+12y+19-\left(-\frac{y^2}{2}-4y-10\right)\ dy$$

$$A=\int_{-5}^{-2}2y^2+12y+19+\frac{y^2}{2}+4y+10\ dy$$

$$A=\int_{-5}^{-2}\frac{5}{2}y^2+16y+29\ dy$$

Integrate and evaluate over the interval.

$$A=\frac{5y^3}{6}+\frac{16y^2}{2}+29y\Big|_{-5}^{-2}$$

$$A=\frac{5y^3}{6}+8y^2+29y\Big|_{-5}^{-2}$$

$$A=\frac{5(-2)^3}{6}+8(-2)^2+29(-2)-\left(\frac{5(-5)^3}{6}+8(-5)^2+29(-5)\right)$$

$$A=\frac{5(-8)}{6}+8(4)-58-\left(\frac{5(-125)}{6}+8(25)-145\right)$$

$$A=-\frac{40}{6}+32-58-\left(-\frac{625}{6}+200-145\right)$$

$$A=-\frac{40}{6}+32-58+\frac{625}{6}-200+145$$

$$A=\frac{585}{6}-81$$

$$A=16.5$$

■ 3. Find the area, in square units, between the two curves.

$$f(y)=-y^3+6y$$

$$g(y)=-y^2$$

Solution:

Find the intersection points of the curves.

$$-y^3+6y=-y^2$$

$$y^3-y^2-6y=0$$

$$y(y^2-y-6)=0$$

$$y(y-3)(y+2)=0$$

$$y=-2,\ 0,\ 3$$

Between $y=-2$ and $y=0$, $g(y)$ is to the right of $f(y)$. And between $y=0$ and $y=3$, $f(y)$ is to the right of $g(y)$. Therefore, the area between the curves is

$$A=\int_{-2}^{0}-y^2-(-y^3+6y)\ dy+\int_{0}^{3}-y^3+6y-(-y^2)\ dy$$

$$A=\int_{-2}^{0}-y^2+y^3-6y\ dy+\int_{0}^{3}-y^3+6y+y^2\ dy$$

$$A=\int_{-2}^{0}y^3-y^2-6y\ dy+\int_{0}^{3}-y^3+y^2+6y\ dy$$

Integrate and evaluate over the interval.

$$A=\left(\frac{1}{4}y^4-\frac{1}{3}y^3-3y^2\right)\Bigg|_{-2}^{0}+\left(-\frac{1}{4}y^4+\frac{1}{3}y^3+\frac{6}{2}y^2\right)\Bigg|_{0}^{3}$$

$$A=\frac{1}{4}(0)^4-\frac{1}{3}(0)^3-3(0)^2-\left(\frac{1}{4}(-2)^4-\frac{1}{3}(-2)^3-3(-2)^2\right)$$

$$+\left[-\frac{1}{4}(3)^4+\frac{1}{3}(3)^3+\frac{6}{2}(3)^2-\left(-\frac{1}{4}(0)^4+\frac{1}{3}(0)^3+\frac{6}{2}(0)^2\right)\right]$$

$$A=-\frac{1}{4}(-2)^4+\frac{1}{3}(-2)^3+3(-2)^2-\frac{1}{4}(3)^4+\frac{1}{3}(3)^3+\frac{6}{2}(3)^2$$

$$A=-\frac{1}{4}(16)+\frac{1}{3}(-8)+3(4)-\frac{1}{4}(81)+\frac{1}{3}(27)+\frac{6}{2}(9)$$

$$A=-4-\frac{8}{3}+12-\frac{81}{4}+9+27$$

$$A=-\frac{8}{3}-\frac{81}{4}+44$$

$$A=-\frac{32}{12}-\frac{243}{12}+\frac{528}{12}$$

$$A=\frac{253}{12}$$

4. Find the area, in square units, between the two curves.

$$f(y)=\frac{y^2}{2}-3y-\frac{1}{2}$$

$$g(y)=3$$

Solution:

Find the intersection points of the curves.

$$\frac{y^2}{2}-3y-\frac{1}{2}=3$$

$$y^2-6y-1=6$$

$$y^2-6y-7=0$$

$$(y-7)(y+1)=0$$

$$y=-1,\ 7$$

Between these two points, $g(y)$ is to the right of $f(y)$. Therefore, the area between the curves is

$$A=\int_{-1}^{7}3-\left(\frac{y^2}{2}-3y-\frac{1}{2}\right)\ dy$$

$$A=\int_{-1}^{7}3-\frac{y^2}{2}+3y+\frac{1}{2}\ dy$$

$$A=\int_{-1}^{7}-\frac{y^2}{2}+3y+\frac{7}{2}\ dy$$

Integrate and evaluate over the interval.

$$A=-\frac{y^3}{6}+\frac{3}{2}y^2+\frac{7}{2}y\Big|_{-1}^{7}$$

$$A=-\frac{(7)^3}{6}+\frac{3}{2}(7)^2+\frac{7}{2}(7)-\left(-\frac{(-1)^3}{6}+\frac{3}{2}(-1)^2+\frac{7}{2}(-1)\right)$$

$$A=-\frac{343}{6}+\frac{147}{2}+\frac{49}{2}-\frac{1}{6}-\frac{3}{2}+\frac{7}{2}$$

$$A=-\frac{344}{6}+\frac{200}{2}$$

$$A=\frac{300}{3}-\frac{172}{3}$$

$$A=\frac{128}{3}$$

5. Find the area, in square units, between the two curves, and between $y=0$ and $y=4$.

$$f(y)=2y^2-8y+9$$

$$g(y)=\frac{y^2}{2}-2y-1$$

Solution:

Find the intersection points of the curves.

$$2y^2-8y+9=\frac{y^2}{2}-2y-1$$

$$4y^2-16y+18=y^2-4y-2$$

$$3y^2-12y+20=0$$

Use the quadratic formula.

$$y=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

$$y=\frac{-(-12)\pm\sqrt{(-12)^2-4(3)(20)}}{2(3)}=\frac{12\pm\sqrt{144-240}}{6}=\frac{12\pm\sqrt{-96}}{6}$$

Because we can't take the square root of a negative number, this means that the curves do not intersect. Which means only $y=0$ and $y=4$ provide the limits of integration. $f(y)$ is to the right of $g(y)$, so

$$A = \int_0^4 2y^2 - 8y + 9 - \left(\frac{y^2}{2} - 2y - 1\right)\ dy$$

$$A = \int_0^4 2y^2 - 8y + 9 - \frac{y^2}{2} + 2y + 1\ dy$$

$$A = \int_0^4 \frac{3}{2}y^2 - 6y + 10\ dy$$

Integrate and evaluate over the interval.

$$A = \frac{3}{2(3)}y^3 - \frac{6}{2}y^2 + 10y\Big|_0^4$$

$$A = \frac{1}{2}y^3 - 3y^2 + 10y\Big|_0^4$$

$$A = \frac{1}{2}(4)^3 - 3(4)^2 + 10(4) - \left(\frac{1}{2}(0)^3 - 3(0)^2 + 10(0)\right)$$

$$A = \frac{1}{2}(64) - 3(16) + 40$$

$$A = 32 - 48 + 40$$

$$A = 24$$

SKETCHING THE AREA BETWEEN CURVES

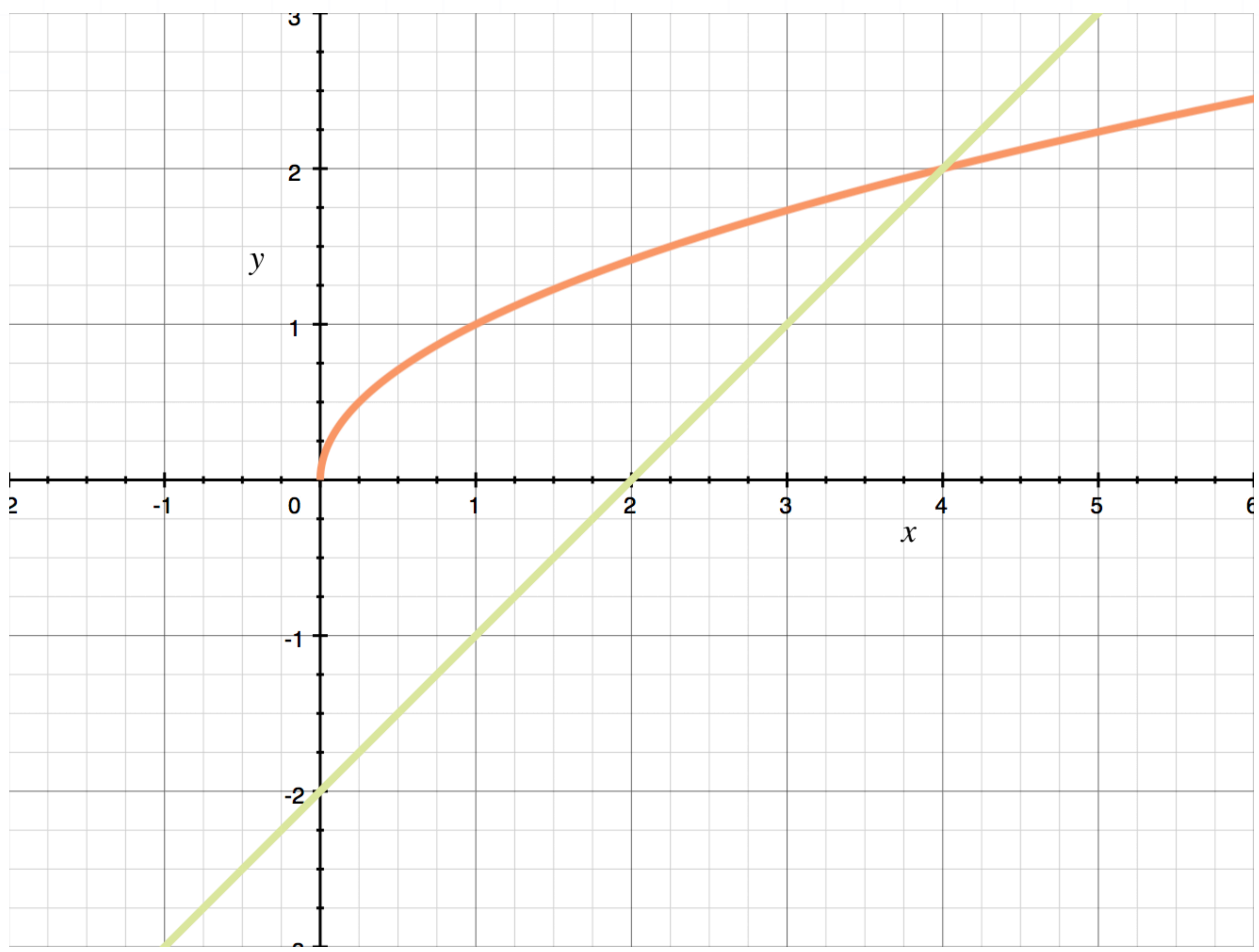
■ 1. Find the area of the region in the first quadrant that's enclosed by the graphs of the curves.

$$y = \sqrt{x}$$

$$y = x - 2$$

Solution:

The graph of the region is



Since these are more left-right curves, we should integrate with respect to y, which means we need to solve both equations for x.

$y=\sqrt{x}$ becomes $x=y^2$

$y=x-2$ becomes $x=y+2$

Find the intersection points of the curves.

$$y^2=y+2$$

$$y^2-y-2=0$$

$$(y-2)(y+1)=0$$

$$y=-1,\ 2$$

So

$$x=y+2$$

$$x=-1+2$$

$$x=1$$

and

$$x=y+2$$

$$x=2+2$$

$$x=4$$

The curves intersect at $(1,-1)$ and $(4,2)$, but only $(4,2)$ is in the first quadrant. With respect to y, that means the region is bounded below by $y=0$ and bounded above by $y=2$.

So the area enclosed by the curves in the first quadrant is

$$A=\int_0^2 (y+2)-y^2\ dy$$

Integrate and evaluate over the interval.

$$A=\frac{y^2}{2}+2y-\frac{y^3}{3}\Big|_0^2$$

$$A=\left(\frac{2^2}{2}+2(2)-\frac{2^3}{3}\right)-\left(\frac{0^2}{2}+2(0)-\frac{0^3}{3}\right)$$

$$A=\left(2+4-\frac{8}{3}\right)-(0+0-0)$$

$$A=\frac{10}{3}$$

2. Find the area of the region that's enclosed by the graphs of the curves.

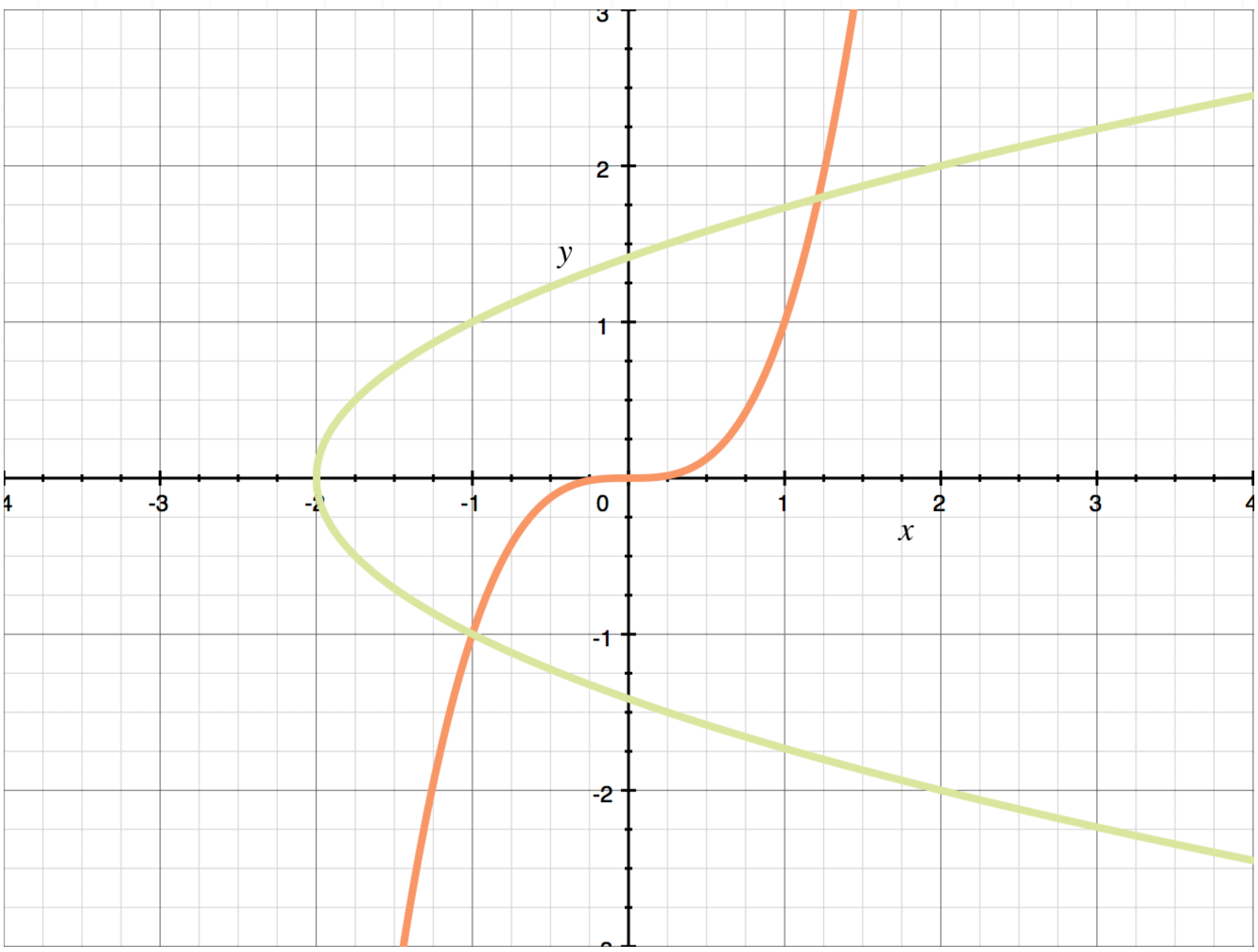
$$y=x^3$$

$$y=\sqrt{x+2}$$

$$y = -\sqrt{x+2}$$

Solution:

The graph of the region is



Since these are more left-right curves, we should integrate with respect to y, which means we need to solve both equations for x.

$y = x^3$ becomes $x = y^{\frac{1}{3}}$

$y = \sqrt{x+2}$ and $y = -\sqrt{x+2}$ both become $x = y^2 - 2$

Find the intersection points of the curves.

$$y^{\frac{1}{3}} = y^2 - 2$$

$$y = (y^2 - 2)^3$$

$$y = (y^4 - 4y^2 + 4)(y^2 - 2)$$

$$y = y^6 - 6y^4 + 12y^2 - 8$$

The roots of this polynomial are $y = -1$ and $y \approx 1.79$

So

$$x = y^2 - 2$$

$$x = (-1)^2 - 2$$

$$x = -1$$

and

$$x = y^2 - 2$$

$$x = 1.79^2 - 2$$

$$x \approx 1.20$$

The curves intersect at $(-1, -1)$ and $(1.20, 1.79)$. With respect to y, that means the region is bounded below by $y = -1$ and bounded above by $y = 1.79$.

So the area enclosed by the curves in the first quadrant is

$$A = \int_{-1}^{1.79} y^{\frac{1}{3}} - (y^2 - 2)\ dy$$

Integrate and evaluate over the interval.

$$A=\frac{3}{4}y^{\frac{4}{3}}-\frac{1}{3}y^3+2y\Big|_{-1}^{1.79}$$

$$A=\frac{3}{4}(1.79)^{\frac{4}{3}}-\frac{1}{3}(1.79)^3+2(1.79)-\left(\frac{3}{4}(-1)^{\frac{4}{3}}-\frac{1}{3}(-1)^3+2(-1)\right)$$

$$A=\frac{3}{4}(1.79)^{\frac{4}{3}}-\frac{1}{3}(1.79)^3+2(1.79)-\frac{3}{4}-\frac{1}{3}+2$$

$$A\approx 4.215$$

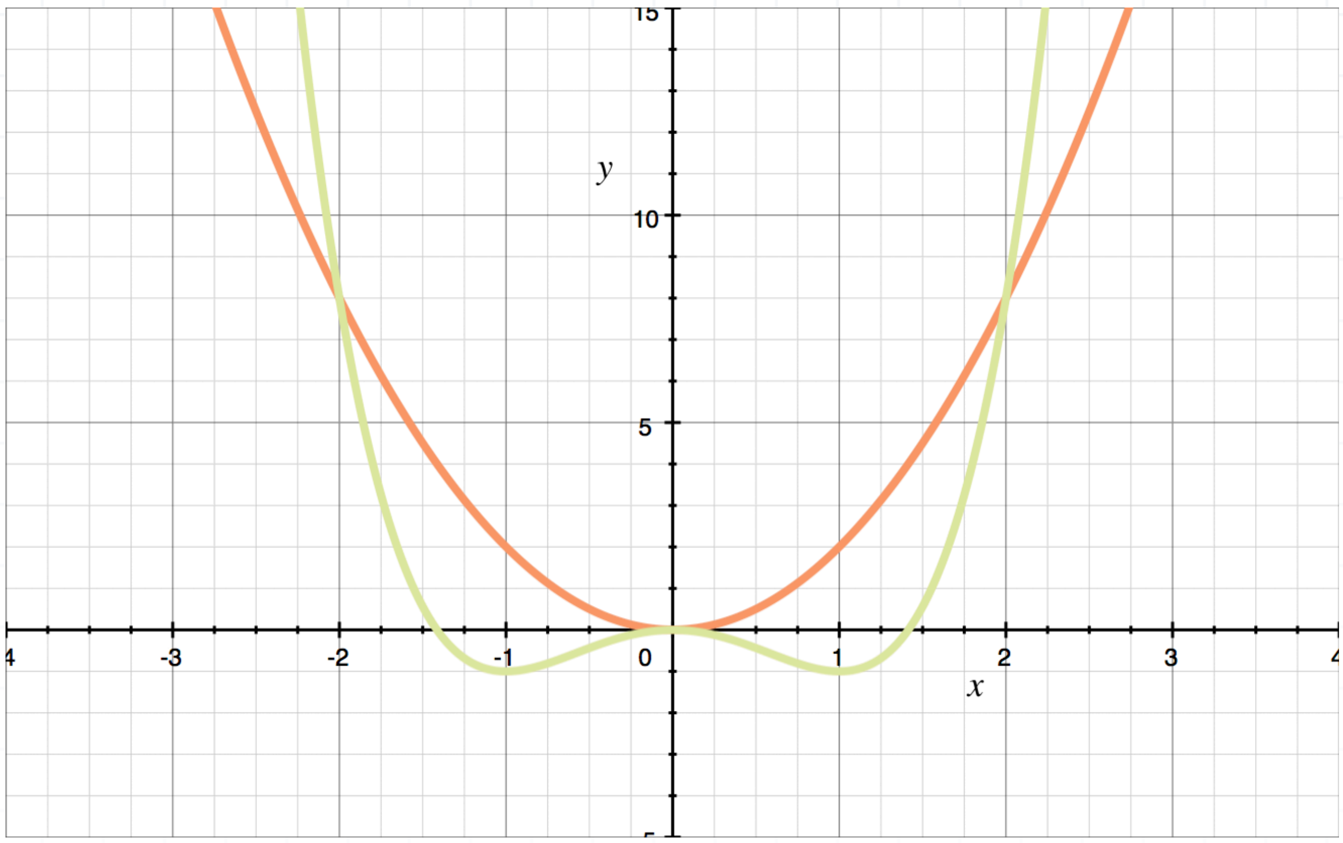
■ 3. Find the area of the region that's enclosed by the graphs of the curves.

$$y=2x^2$$

$$y=x^4-2x^2$$

Solution:

The graph of the region is



The region is symmetric about the y-axis, so the best way to calculate the area is by integrating half the region with respect to x and then doubling the answer.

Find the intersection points of the curves.

$$2x^2 = x^4 - 2x^2$$

$$x^4 - 4x^2 = 0$$

$$x^2(x^2 - 4) = 0$$

$$x^2(x + 2)(x - 2) = 0$$

$$x = -2, 0, 2$$

So

$$y = 2x^2$$

$$y = 2(-2)^2$$

$$y = 8$$

and

$$y = 2x^2$$

$$y = 2(0)^2$$

$$y = 0$$

and

$$y = 2x^2$$

$$y = 2(-2)^2$$

$$y = 8$$

The curves intersect at $(-2,8)$, $(0,0)$, and $(2,8)$. With respect to x, that means we'll integrate from $x = -2$ to $x = 0$, and then double the result.

So the area enclosed by the curves in the first quadrant is

$$A = 2\int_0^2 2x^2 - (x^4 - 2x^2)\ dx$$

$$A = 2\int_0^2 2x^2 - x^4 + 2x^2\ dx$$

$$A = 2\int_0^2 4x^2 - x^4\ dx$$

$$A=\int_0^2 8x^2-2x^4\ dx$$

Integrate and evaluate over the interval.

$$A=\frac{8}{3}x^3-\frac{2}{5}x^5\Big|_0^2$$

$$A=\frac{8}{3}(2)^3-\frac{2}{5}(2)^5-\left(\frac{8}{3}(0)^3-\frac{2}{5}(0)^5\right)$$

$$A=\frac{8}{3}(8)-\frac{2}{5}(32)$$

$$A=\frac{64}{3}-\frac{64}{5}$$

$$A=\frac{320}{15}-\frac{192}{15}$$

$$A=\frac{128}{15}$$

DIVIDING THE AREA BETWEEN CURVES INTO EQUAL PARTS

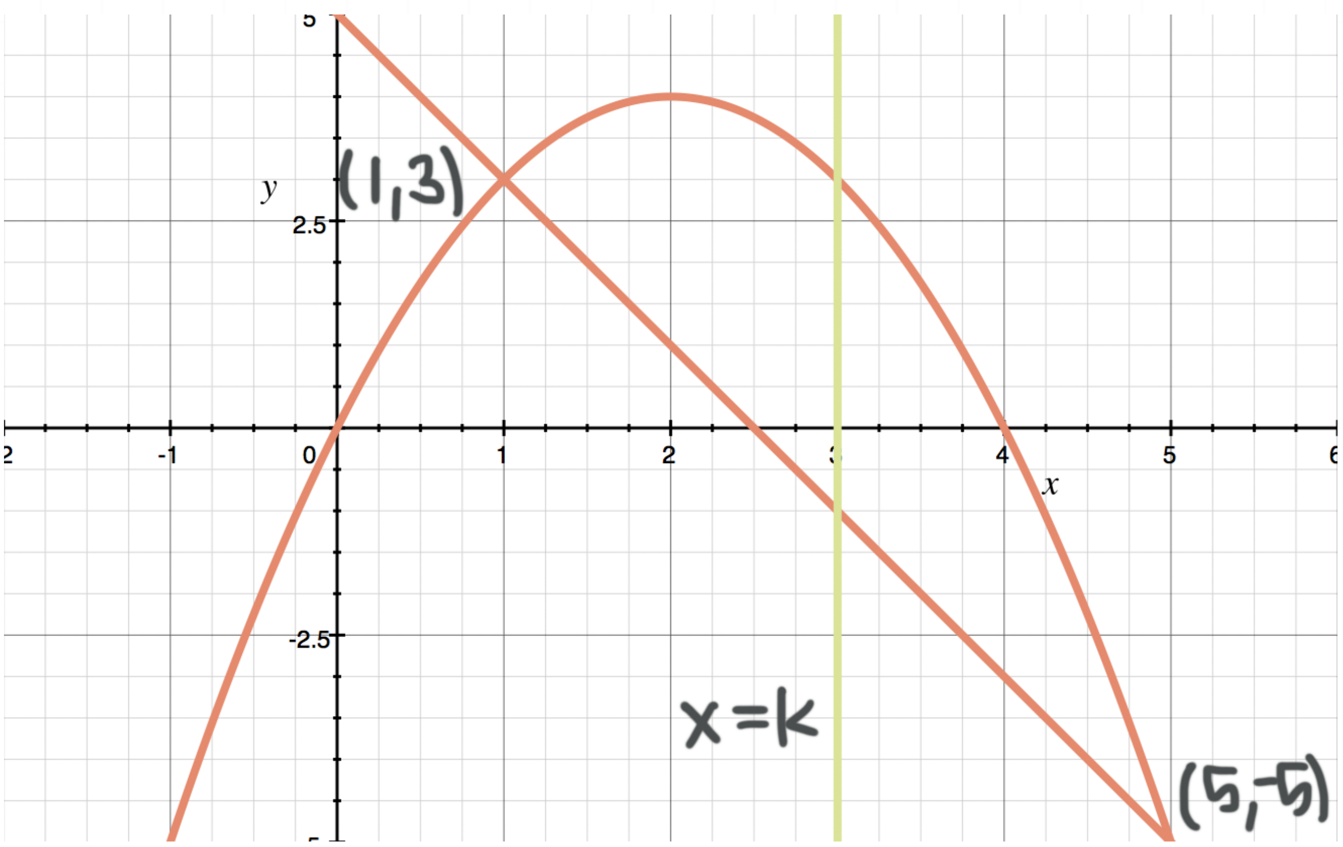
■ 1. The line $x = k$ divides the area bounded by the curves into two equal parts. Find k.

$$f(x) = 4x - x^2$$

$$g(x) = 5 - 2x$$

Solution:

The graph of the area, with the line $x = k$, bounded by the two functions is:



Find the intersection points of the curves.

$$4x - x^2 = 5 - 2x$$

$$x^2 - 6x + 5 = 0$$

$$(x - 5)(x - 1) = 0$$

$$x = 1,\ 5$$

So

$$y = 5 - 2x$$

$$y = 5 - 2(1)$$

$$y = 3$$

and

$$y = 5 - 2x$$

$$y = 5 - 2(5)$$

$$y = -5$$

The curves intersect at $(1,3)$ and $(5,-5)$. With respect to x, that means the region is bounded below by $x = 1$ and bounded above by $x = 5$.

$$\int_1^5 (4x - x^2) - (5 - 2x)\ dx$$

$$\int_1^5 4x - x^2 - 5 + 2x\ dx$$

$$\int_1^5 -x^2 + 6x - 5\ dx$$

Integrate and evaluate over the interval.

$$-\frac{1}{3}x^3+3x^2-5x\Big|_1^5$$

$$-\frac{1}{3}(5)^3+3(5)^2-5(5)-\left(-\frac{1}{3}(1)^3+3(1)^2-5(1)\right)$$

$$-\frac{125}{3}+75-25+\frac{1}{3}-3+5$$

$$-\frac{124}{3}+52$$

$$-\frac{124}{3}+\frac{156}{3}$$

$$\frac{32}{3}$$

Half of this area is $16/3$, which means we can set up an integral on $[1,k]$ that's equal to $16/3$.

$$\int_1^k (4x-x^2)-(5-2x)\ dx=\frac{16}{3}$$

$$\int_1^k 4x-x^2-5+2x\ dx=\frac{16}{3}$$

$$\int_1^k -x^2+6x-5\ dx=\frac{16}{3}$$

$$-\frac{1}{3}x^3 + 3x^2 - 5x\Big|_1^k = \frac{16}{3}$$

$$-\frac{1}{3}k^3 + 3k^2 - 5k - \left(-\frac{1}{3}(1)^3 + 3(1)^2 - 5(1)\right) = \frac{16}{3}$$

$$-\frac{1}{3}k^3 + 3k^2 - 5k + \frac{1}{3} - 3 + 5 = \frac{16}{3}$$

$$-k^3 + 9k^2 - 15k + 1 - 9 + 15 = 16$$

$$-k^3 + 9k^2 - 15k = 9$$

The roots of the polynomial are

$$k = 3$$

$$k = 3 \pm 2\sqrt{3}$$

but $x = 3 - 2\sqrt{3}$ and $x = 3 + 2\sqrt{3}$ are both outside the interval of the bounded region. Which means $x = 3$ must be the line that divides the area of the region in half.

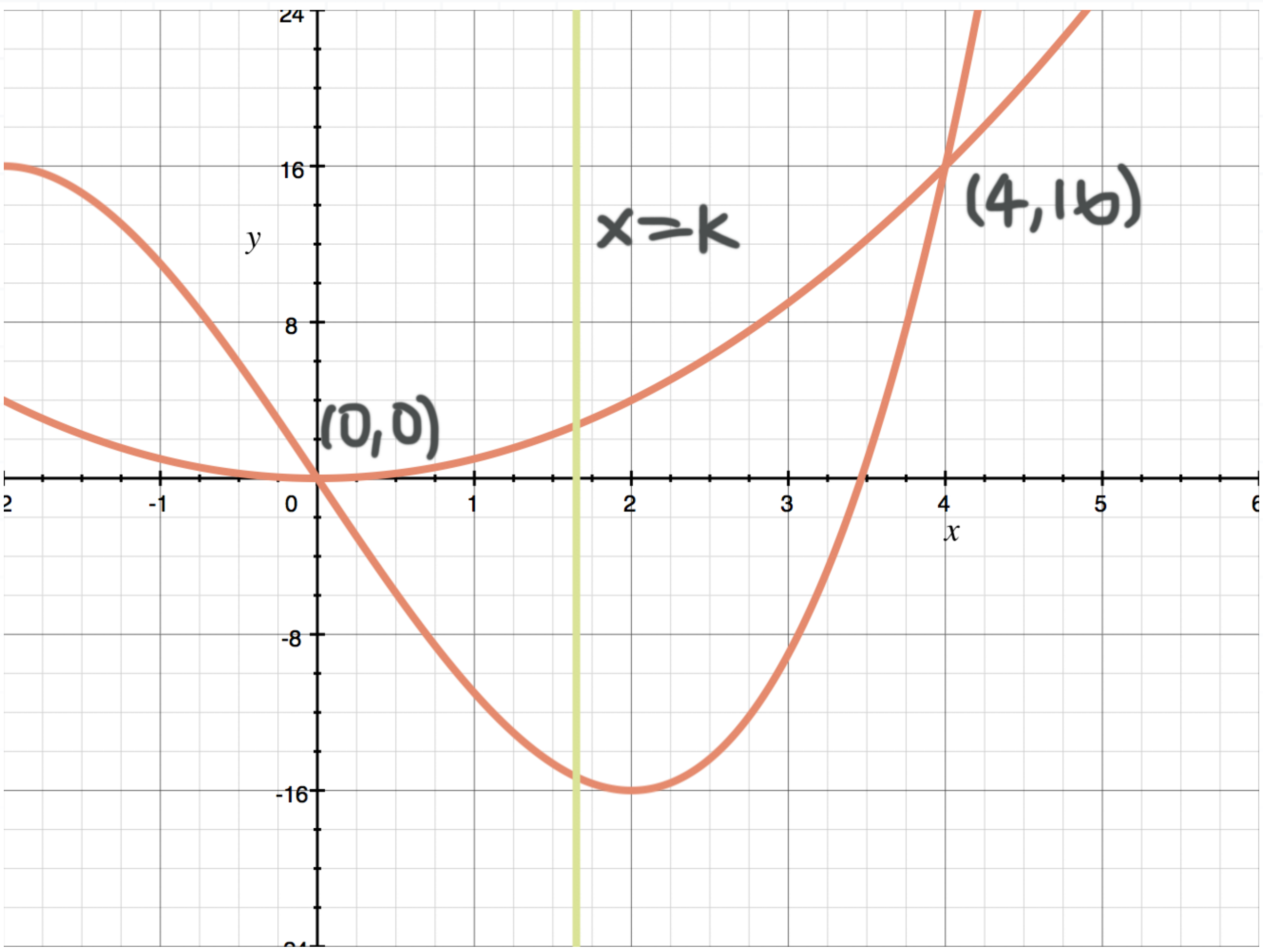
2. The line $x = k$ divides the area bounded by the curves into two equal parts, for $x > 0$. Find k. Round your answer to the nearest three decimal places.

$$f(x) = x^3 - 12x$$

$$g(x) = x^2$$

Solution:

The graph of the area, with the line $x = k$, bounded by the two functions is:



Find the intersection points of the curves.

$$x^3 - 12x = x^2$$

$$x^3 - x^2 - 12x = 0$$

$$x(x^2 - x - 12) = 0$$

$$x(x - 4)(x + 3) = 0$$

$$x = -3,\ 0,\ 4$$

So

$$y = x^2$$

$$y = (-3)^2$$

$$y = 9$$

and

$$y = x^2$$

$$y = 0^2$$

$$y = 0$$

and

$$y = x^2$$

$$y = 4^2$$

$$y = 16$$

The two curves intersect at $(-3,9)$, $(0,0)$ and $(4,16)$. But we're only interested in $x > 0$, so we can ignore $(-3,9)$. So the area of the enclosed region is:

$$\int_0^4 (x^2) - (x^3 - 12x)\ dx$$

$$\int_0^4 x^2 - x^3 + 12x\ dx$$

$$\frac{1}{3}x^3-\frac{1}{4}x^4+6x^2\Big|_0^4$$

$$\frac{1}{3}(4)^3-\frac{1}{4}(4)^4+6(4)^2-\left(\frac{1}{3}(0)^3-\frac{1}{4}(0)^4+6(0)^2\right)$$

$$\frac{1}{3}(64)-\frac{1}{4}(256)+6(16)$$

$$\frac{64}{3}-64+96$$

$$\frac{64}{3}+32$$

$$\frac{64}{3}+\frac{96}{3}$$

$$\frac{160}{3}$$

Half of this area is $80/3$, which means we can set up an integral on $[0,k]$ that's equal to $80/3$.

$$\int_0^k (x^2)-(x^3-12x)\ dx=\frac{80}{3}$$

$$\int_0^k x^2-x^3+12x\ dx=\frac{80}{3}$$

$$\frac{1}{3}x^3-\frac{1}{4}x^4+6x^2\Big|_0^k=\frac{80}{3}$$

$$\frac{1}{3}k^3-\frac{1}{4}k^4+6k^2-\left(\frac{1}{3}(0)^3-\frac{1}{4}(0)^4+6(0)^2\right)=\frac{80}{3}$$

$$\frac{1}{3}k^3-\frac{1}{4}k^4+6k^2=\frac{80}{3}$$

$$(12)\frac{1}{3}k^3-(12)\frac{1}{4}k^4+(12)6k^2=(12)\frac{80}{3}$$

$$4k^3-3k^4+72k^2=320$$

$$3k^4-4k^3-72k^2+320=0$$

The roots of the polynomial are

$$k\approx 2.20$$

$$k\approx 5.19$$

but only $k\approx 2.20$ is inside the interval $x=[0,4]$. Which means $x\approx 2.20$ must be the line that divides the area of the region in half.

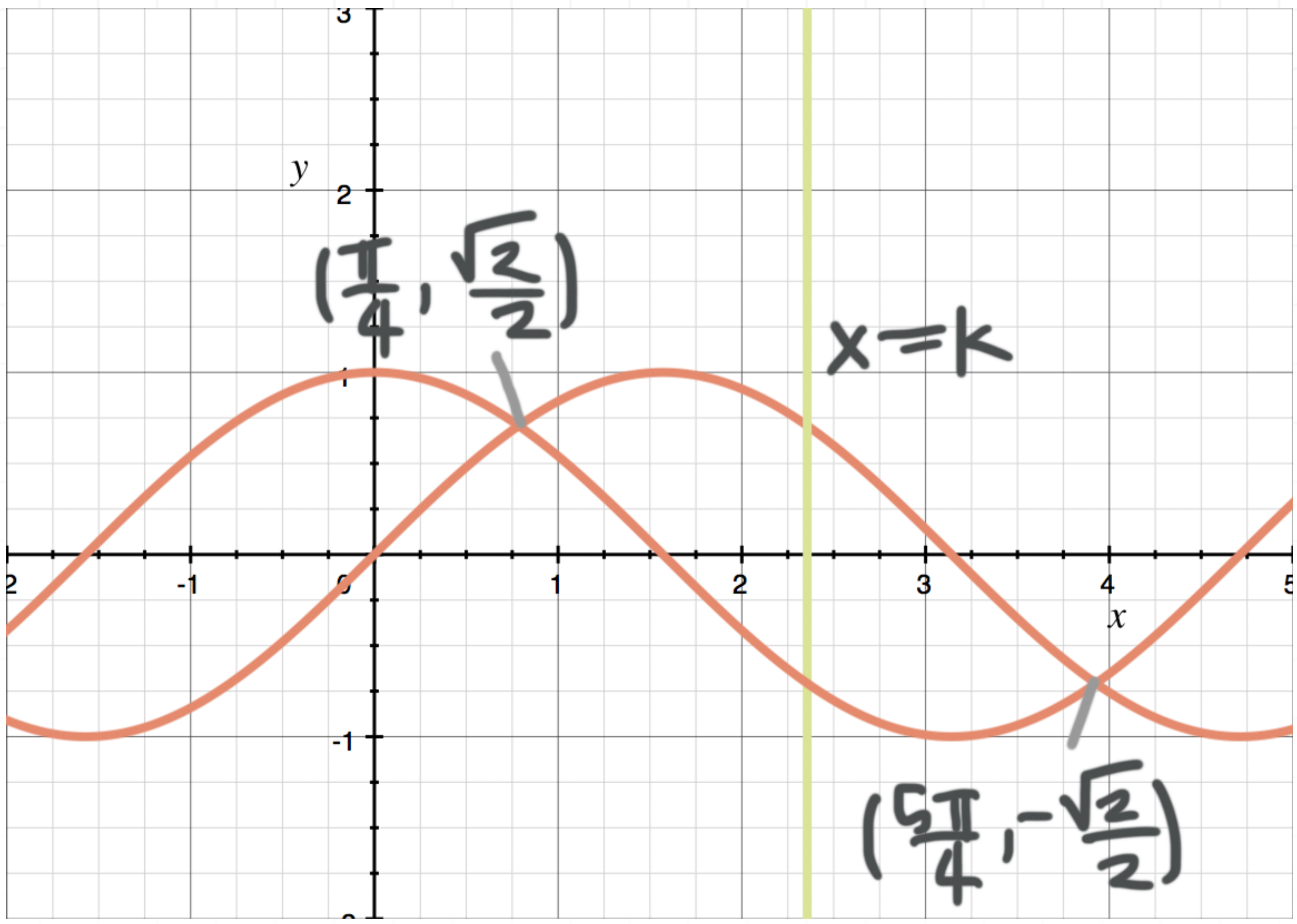
■ 3. The line $x=k$ divides the area bounded by the curves on $\pi/4\le x\le 5\pi/4$ into two equal parts. Find k.

$$f(x)=\sin x$$

$$g(x)=\cos x$$

Solution:

The graph of the area, with the line $x = k$, bounded by the two functions is:



Find the intersection points of the curves.

$$\sin x = \cos x$$

$$x = \frac{\pi}{4}, \frac{5\pi}{4}$$

So

$$y = \sin x$$

$$y = \sin \frac{\pi}{4}$$

$$y = \frac{\sqrt{2}}{2}$$

and

$$y = \sin x$$

$$y = \sin \frac{5\pi}{4}$$

$$y = -\frac{\sqrt{2}}{2}$$

The two curves intersect at $(\pi/4, \sqrt{2}/2)$ and $(5\pi/4, -\sqrt{2}/2)$. So the area of the enclosed region is:

$$\int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} \sin x - \cos x \ dx$$

$$-\cos x - \sin x \Big|_{\frac{\pi}{4}}^{\frac{5\pi}{4}}$$

$$-\cos \frac{5\pi}{4} - \sin \frac{5\pi}{4} - \left(-\cos \frac{\pi}{4} - \sin \frac{\pi}{4} \right)$$

$$-\cos \frac{5\pi}{4} - \sin \frac{5\pi}{4} + \cos \frac{\pi}{4} + \sin \frac{\pi}{4}$$

$$-\left(-\frac{\sqrt{2}}{2} \right) - \left(-\frac{\sqrt{2}}{2} \right) + \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}$$

$$\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}$$

$$\frac{4\sqrt{2}}{2}$$

$$2\sqrt{2}$$

Half of this area is $\sqrt{2}$, which means we can set up an integral on $[\pi/4,k]$ that's equal to $\sqrt{2}$.

$$\int_{\frac{\pi}{4}}^{k} \sin x - \cos x \ dx = \sqrt{2}$$

$$-\cos x - \sin x \Big|_{\frac{\pi}{4}}^{k} = \sqrt{2}$$

$$-\cos k - \sin k - \left(-\cos\frac{\pi}{4} - \sin\frac{\pi}{4}\right) = \sqrt{2}$$

$$-\cos k - \sin k + \cos\frac{\pi}{4} + \sin\frac{\pi}{4} = \sqrt{2}$$

$$-\cos k - \sin k + \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} = \sqrt{2}$$

$$-\cos k - \sin k + \sqrt{2} = \sqrt{2}$$

$$-\cos k - \sin k = 0$$

$$-\cos k = \sin k$$

$$k = \frac{3\pi}{4}$$

Which means $x = 3\pi/4$ must be the line that divides the area of the region in half.

W W W . K R I S T A K I N G M A T H . C O M