Topic: Arc length of a parametric curve

Question: Find the length of the parametric curve on the given interval.

$$x = 2 - t$$

$$y = 3 + 4t$$

$$-2 \leq t \leq 3$$

Answer choices:

A $\sqrt{17}$

B $5\sqrt{15}$

C $\sqrt{15}$

D $5\sqrt{17}$

Solution: D

The length of a curve over an interval $a \le t \le b$ is given by

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\ dt$$

We'll calculate the derivatives of x and y so that we can plug them into the formula.

$$x = 2 - t$$

$$\frac{dx}{dt} = -1$$

and

$$y = 3 + 4t$$

$$\frac{dy}{dt} = 4$$

Plugging these into our formula, we get

$$L = \int_{-2}^{3} \sqrt{(-1)^2 + (4)^2}\ dt$$

$$L = \int_{-2}^{3} \sqrt{17}\ dt$$

$$L = \sqrt{17}t\Big|_{-2}^{3}$$

$$L = \sqrt{17}(3) - \sqrt{17}(-2)$$

$$L = 3\sqrt{17} + 2\sqrt{17}$$

$$L = 5\sqrt{17}$$

Topic: Arc length of a parametric curve

Question: Find the length of the parametric curve on the given interval.

$$x = 3e^{3t} - 4t$$

$$y = 8e^{\frac{3t}{2}}$$

$$1 \le t \le 2$$

Answer choices:

A $3e^6$

B $6e^6$

C $6e^6 - 4$

D $3e^6 - 3e^3 + 4$

Solution: D

The length of a curve over an interval $a \leq t \leq b$ is given by

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\ dt$$

Let's find the derivatives of our equations so that we can plug them into our arc length formula.

$$x = 3e^{3t} - 4t$$

$$\frac{dx}{dt} = 9e^{3t} - 4$$

and

$$y = 8e^{\frac{3t}{2}}$$

$$\frac{dy}{dt} = 12e^{\frac{3t}{2}}$$

Plugging these into our formula, we get

$$L = \int_1^2 \sqrt{\left(9e^{3t} - 4\right)^2 + \left(12e^{\frac{3t}{2}}\right)^2}\ dt$$

$$L = \int_1^2 \sqrt{81e^{6t} - 72e^{3t} + 16 + 144e^{3t}}\ dt$$

$$L = \int_1^2 \sqrt{81e^{6t} + 72e^{3t} + 16}\ dt$$

$$L = \int_1^2 \sqrt{\left(9e^{3t} + 4\right)^2}\ dt$$

$$L=\int_1^2 9e^{3t}+4\ dt$$

$$L=3e^{3t}+4t\Big|_1^2$$

$$L=3e^{3(2)}+4(2)-\left[3e^{3(1)}+4(1)\right]$$

$$L=3e^6+8-3e^3-4$$

$$L=3e^6-3e^3+4$$

Topic: Arc length of a parametric curve

Question: A parametric curve is defined by $x = ae^t \cos t$ and $y = be^t \sin t$, where a and b are positive real numbers. For which values of a and b between $t = 0$ and $t = \pi/2$, is this the length of the curve?

$$L = 2\sqrt{2}\left(e^{\frac{\pi}{2}} - 1\right)$$

Answer choices:

A	$a = 2$	and	$b = 2$
B	$a = 2$	and	$b = 3$
C	$a = 3$	and	$b = 2$
D	$a = 3$	and	$b = 3$

Solution: A

Choose $a = 2$ and $b = 2$. Then the given functions

$$x = ae^t \cos t$$

$$y = be^t \sin t$$

take on the following forms:

$$x = 2e^t \cos t$$

$$y = 2e^t \sin t$$

Differentiate both functions, and square the results.

$$\frac{dx}{dt} = 2e^t \cos t - 2e^t \sin t$$

$$\left(\frac{dx}{dt}\right)^2 = \left(2e^t \cos t - 2e^t \sin t\right)^2$$

$$\left(\frac{dx}{dt}\right)^2 = 4e^{2t}\left(\cos t - \sin t\right)^2$$

$$\left(\frac{dx}{dt}\right)^2 = 4e^{2t}\left(\cos^2 t + \sin^2 t - 2 \sin t \cos t\right)$$

$$\left(\frac{dx}{dt}\right)^2 = 4e^{2t}\left(1 - 2 \sin t \cos t\right)$$

and

$$\frac{dy}{dt} = 2e^t \sin t + 2e^t \cos t$$

$$\left(\frac{dy}{dt}\right)^2 = 4e^{2t}\left(\sin t + \cos t\right)^2$$

$$\left(\frac{dy}{dt}\right)^2 = 4e^{2t}\left(\sin^2 t + \cos^2 t + 2\sin t \cos t\right)$$

$$\left(\frac{dy}{dt}\right)^2 = 4e^{2t}\left(1 + 2\sin t \cos t\right)$$

Plug into the arc length formula. Remember that we were given the limits of integration $[0,\pi/2]$ in the problem.

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\ dt$$

$$L = \int_0^{\frac{\pi}{2}} \sqrt{4e^{2t}\left(1 - 2\sin t \cos t\right) + 4e^{2t}\left(1 + 2\sin t \cos t\right)}\ dt$$

$$L = \int_0^{\frac{\pi}{2}} \sqrt{4e^{2t}\left(1 - 2\sin t \cos t + 1 + 2\sin t \cos\right)}\ dt$$

$$L = \int_0^{\frac{\pi}{2}} \sqrt{8e^{2t}}\ dt$$

$$L = 2\sqrt{2}\int_0^{\frac{\pi}{2}} e^t\ dt$$

Integrate.

$$L = 2\sqrt{2}e^t \Big|_0^{\frac{\pi}{2}}$$

$$L = 2\sqrt{2}\left(e^{\frac{\pi}{2}} - 1\right)$$

Because this matches the arc length we were given, we know that $a = 2$ and $b = 2$ are the values we needed to choose. So answer choice A is correct.