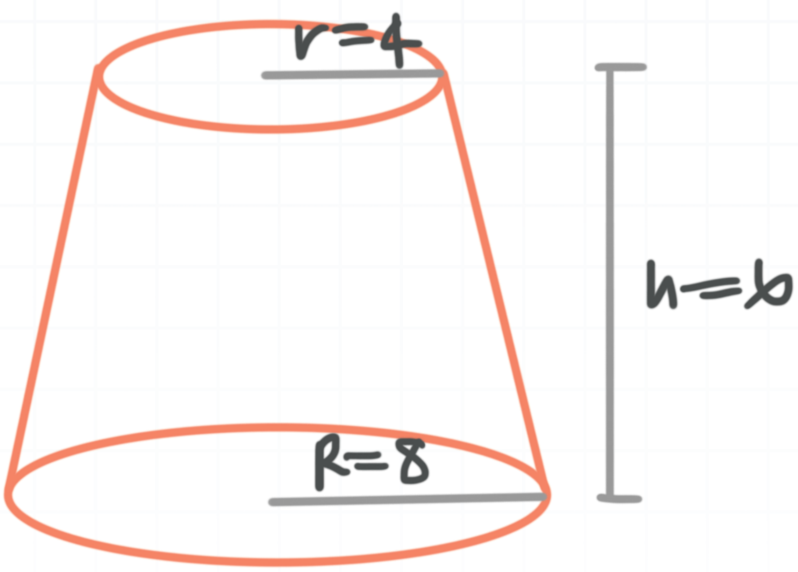
Topic: Disks, volume of a frustum

Question: Use disks to find the volume of the frustum of a right circular cone with height $h=12$ inches, a lower base radius $R=8$ inches, and an upper radius of $r=4$ inches.



Answer choices:

A $\quad V=224$ cubic inches

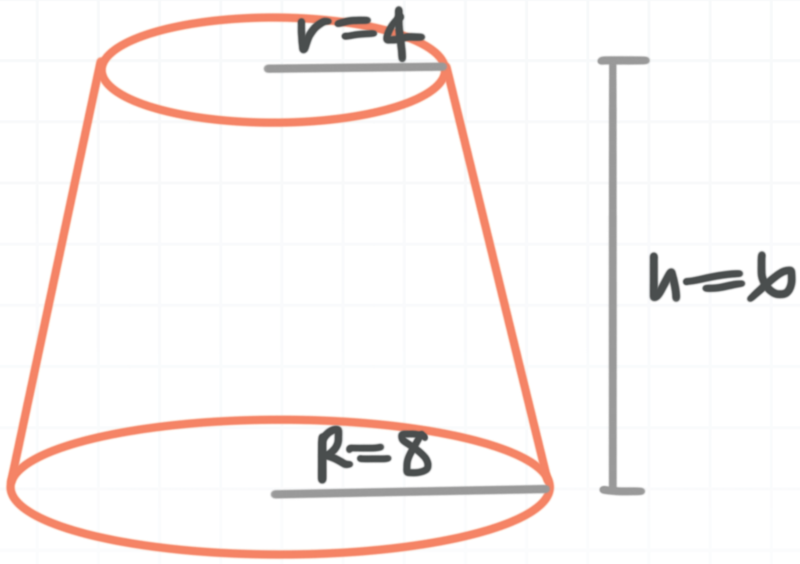
B $\quad V=224\pi$ cubic inches

C $\quad V=45\pi$ cubic inches

D $\quad V=54\pi$ cubic inches

Solution: B

The frustum in this problem looks like this



We will use the y-axis as the center-height of the cone. From the problem, the lower base radius is $R = 8$. This means that the edge of the cone is at the point $(8,0)$. The height of the cone is $h = 12$, so the vertex of the cone is at the point $(0,12)$. Therefore, we can determine the equation of the line that is formed by the lateral surface of the cone.

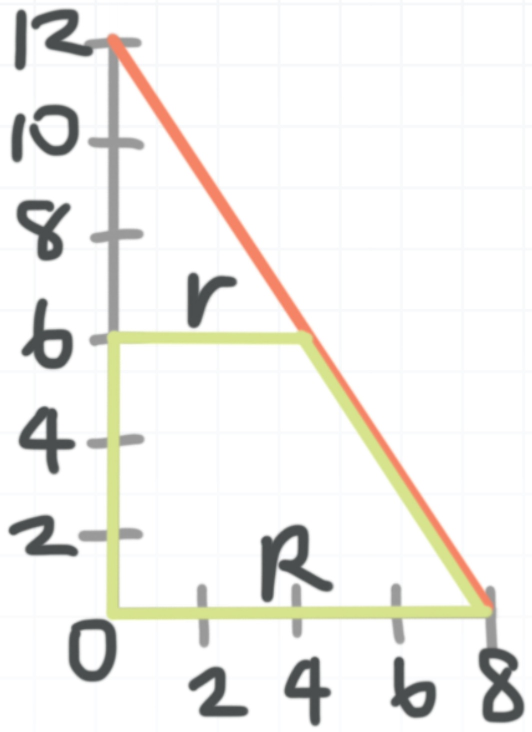
We will use the slope-intercept form to determine the equation. First, the slope is

$$\frac{0-12}{8-0} = -\frac{12}{8} = -\frac{3}{2}$$

The vertex of the cone is on the y-axis at the point $(0,12)$. Thus, the equation of the line that forms the lateral surface of the cone, as a rotation is

$$y = -\frac{3}{2}x + 12$$

The frustum is formed by rotating the region shown below about the y-axis.



To find its volume, we'll realize that the axis of rotation is vertical. Because the slices we'll use to approximate volume must always be perpendicular to the axis of rotation, that means the slices must be horizontal. Which means we'll represent the width of each infinitely thin slice as dy. Which means we'll be integrating with respect to y. Therefore, the limits of integration will be given as $y=[0,6]$, and we can write the volume integral as

$$V=\int_c^d \pi\left[f(y)\right]^2\ dy$$

$$V=\int_0^6 \pi\left[f(y)\right]^2\ dy$$

The outer radius $f(y)$ is given by

$$y=-\frac{3}{2}x+12$$

but we need to rearrange the equation so that it's solved for x in terms of y.

$$y-12=-\frac{3}{2}x$$

$$2y-24=-3x$$

$$x=8-\frac{2}{3}y$$

The volume of the frustum is then

$$V=\int_0^6 \pi\left(8-\frac{2}{3}y\right)^2\ dy$$

$$V=\int_0^6 \pi\left(64-\frac{32}{3}y+\frac{4}{9}y^2\right)\ dy$$

$$V=\int_0^6 64\pi-\frac{32}{3}\pi y+\frac{4}{9}\pi y^2\ dy$$

Integrate, then evaluate over the interval.

$$V=64\pi y-\frac{16}{3}\pi y^2+\frac{4}{27}\pi y^3\Big|_0^6$$

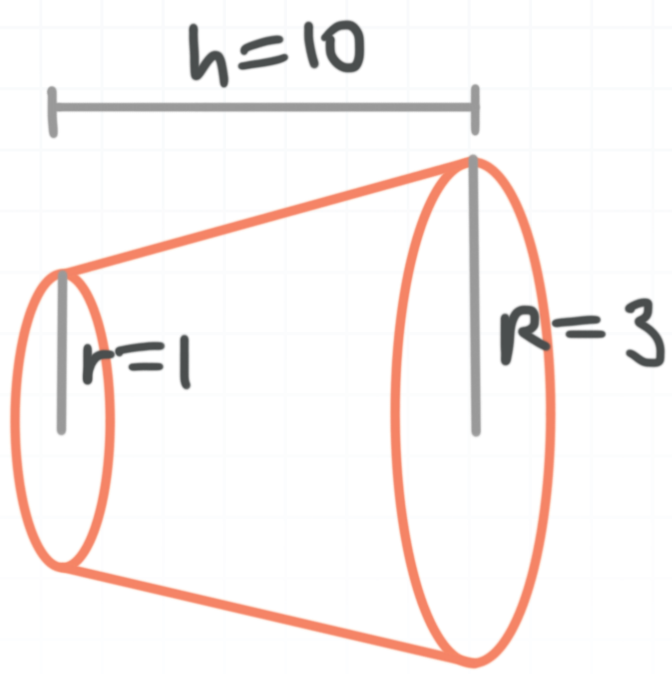
$$V=64\pi(6)-\frac{16}{3}\pi(6)^2+\frac{4}{27}\pi(6)^3-\left(64\pi(0)-\frac{16}{3}\pi(0)^2+\frac{4}{27}\pi(0)^3\right)$$

$$V=384\pi-192\pi+32\pi$$

$$V=224\pi$$

Topic: Disks, volume of a frustum

Question: Use disks to find the volume of the frustum of a right circular cone with height $h = 15$ cm, a lower base radius $R = 3$ cm, and an upper radius of $r = 1$ cm.



Answer choices:

A $\quad V = \dfrac{130}{3}\pi$ cubic cm

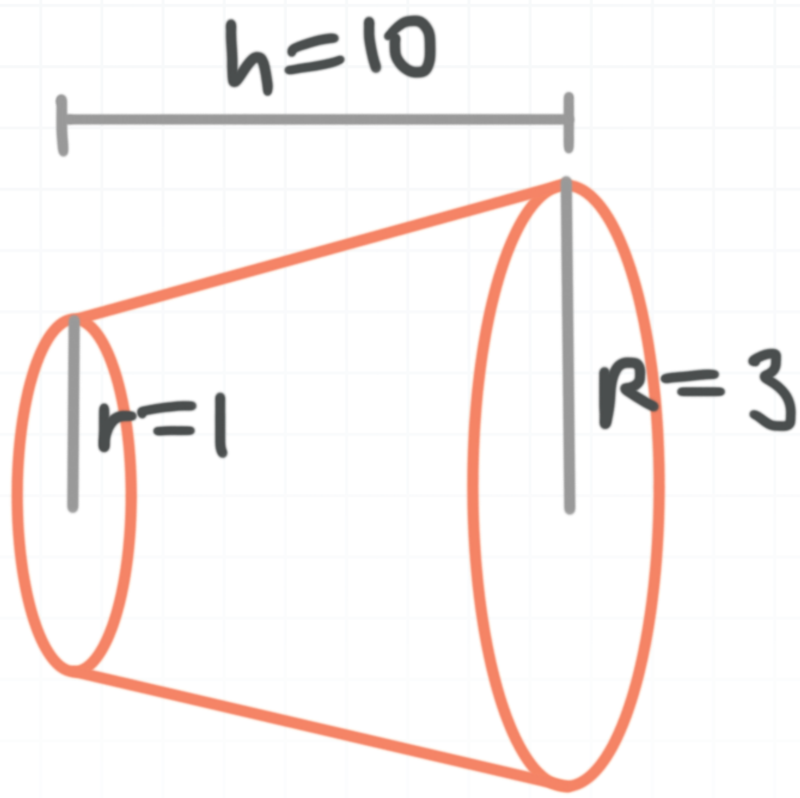
B $\quad V = 130\pi$ cubic cm

C $\quad V = \dfrac{130}{3}$ cubic cm

D $\quad V = \dfrac{130}{3}\pi$ square cm

Solution: A

The frustum in this problem looks like this



We will use the x-axis as the center-height of the cone. From the problem, the lower base radius is $R = 3$. This means that the edge of the cone is at the point $(15,3)$. The height of the cone is $h = 15$, so the vertex of the cone is at the point $(0,0)$. Therefore, we can determine the equation of the line that is formed by the lateral surface of the cone.

We will use the slope-intercept form to determine the equation. First, the slope is

$$\frac{3-0}{15-0} = \frac{3}{15} = \frac{1}{5}$$

The vertex of the cone is on the x-axis at the point $(0,0)$. Thus, the equation of the line that forms the lateral surface of the cone, as a rotation is

$$y = \frac{1}{5}x$$

The frustum is formed by rotating the region shown below about the x-axis.

To find its volume, we'll realize that the axis of rotation is horizontal. Because the slices we'll use to approximate volume must always be perpendicular to the axis of rotation, that means the slices must be vertical. Which means we'll represent the width of each infinitely thin slice as dx. Which means we'll be integrating with respect to x. Therefore, the limits of integration will be given as $x = [5,15]$, and we can write the volume integral as

$$V = \int_a^b \pi \left[f(x)\right]^2 \ dx$$

$$V = \int_5^{15} \pi \left[f(x)\right]^2 \ dx$$

The outer radius $f(x)$ is given by

$$y = \frac{1}{5}x$$

The volume of the frustum is then

$$V = \int_{5}^{15} \pi \left(\frac{1}{5}x \right)^2 \ dx$$

$$V = \int_{5}^{15} \frac{1}{25} \pi x^2 \ dx$$

Integrate, then evaluate over the interval.

$$V = \frac{1}{75} \pi x^3 \Big|_{5}^{15}$$

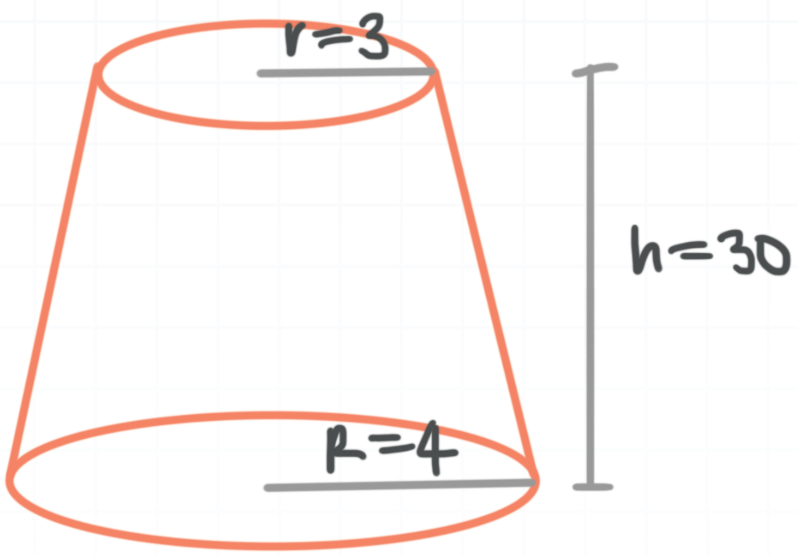
$$V = \frac{1}{75} \pi (15)^3 - \left(\frac{1}{75} \pi (5)^3 \right)$$

$$V = \frac{3{,}375}{75} \pi - \frac{125}{75} \pi$$

$$V = \frac{130}{3} \pi$$

Topic: Disks, volume of a frustum

Question: Use disks to find the volume of the frustum of a right circular cone with height $h=120$ mm, a lower base radius $R=4$ mm, and an upper radius of $r=3$ mm.



Answer choices:

A $V=37\pi$ cubic mm

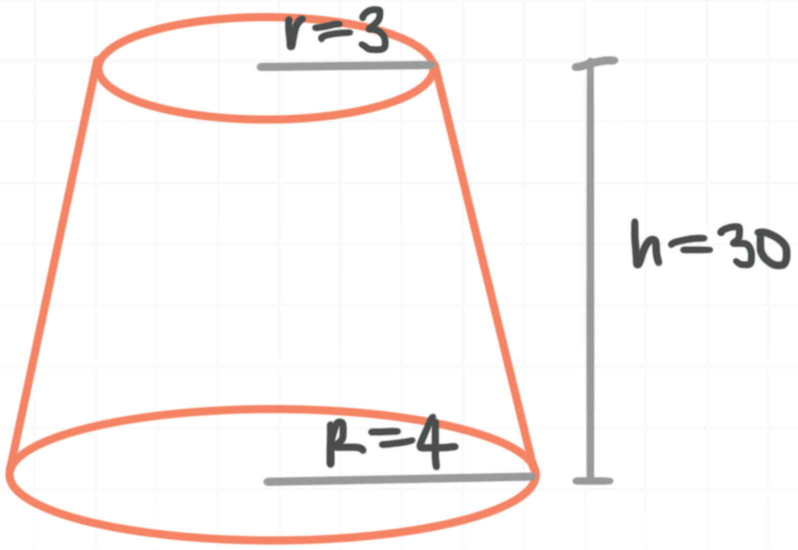
B $V=370\pi$ square mm

C $V=370$ cubic mm

D $V=370\pi$ cubic mm

Solution: D

The frustum in this problem looks like this



We will use the y-axis as the center-height of the cone. From the problem, the lower base radius is $R = 4$. This means that the edge of the cone is at the point $(4,0)$. The height of the cone is $h = 120$, so the vertex of the cone is at the point $(0,120)$. Therefore, we can determine the equation of the line that is formed by the lateral surface of the cone.

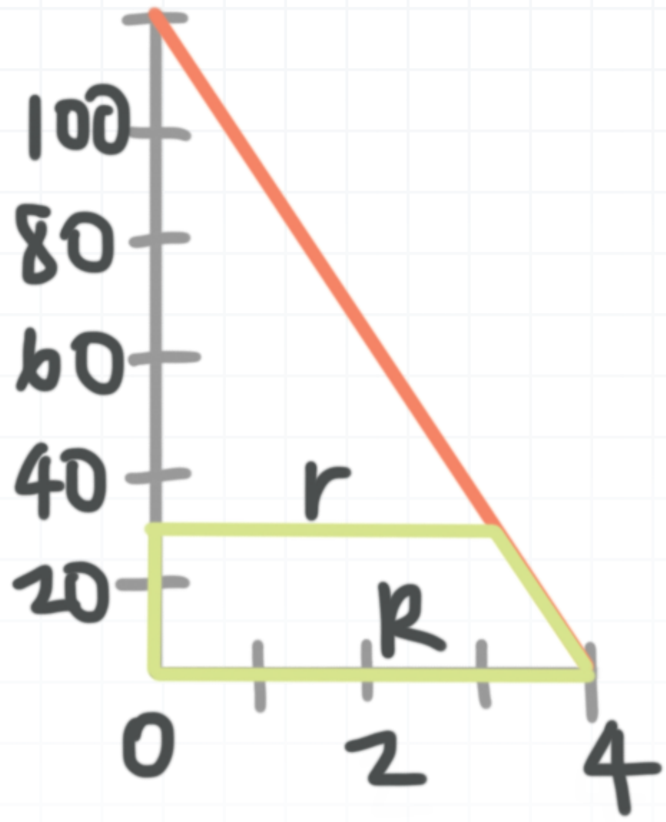
We will use the slope-intercept form to determine the equation. First, the slope is

$$\frac{0-120}{4-0} = -\frac{120}{4} = -30$$

The vertex of the cone is on the y-axis at the point $(0,120)$. Thus, the equation of the line that forms the lateral surface of the cone, as a rotation is

$$y = -30x + 120$$

The frustum is formed by rotating the region shown below about the y-axis.



To find its volume, we'll realize that the axis of rotation is vertical. Because the slices we'll use to approximate volume must always be perpendicular to the axis of rotation, that means the slices must be horizontal. Which means we'll represent the width of each infinitely thin slice as dy. Which means we'll be integrating with respect to y. Therefore, the limits of integration will be given as $y=[0,30]$, and we can write the volume integral as

$$V=\int_c^d \pi\left[f(y)\right]^2\ dy$$

$$V=\int_0^{30} \pi\left[f(y)\right]^2\ dy$$

The outer radius $f(y)$ needs to come from

$$y=-30x+120$$

So we need to solve this for x in terms of y.

$$y - 120 = -30x$$

$$x = -\frac{1}{30}y + 4$$

The volume of the frustum is then

$$V = \int_0^{30} \pi \left(-\frac{1}{30}y + 4\right)^2 \ dy$$

$$V = \int_0^{30} \pi \left(\frac{1}{900}y^2 - \frac{4}{15}y + 16\right) \ dy$$

$$V = \int_0^{30} \frac{1}{900}\pi y^2 - \frac{4}{15}\pi y + 16\pi \ dy$$

Integrate, then evaluate over the interval.

$$V = \frac{1}{2{,}700}\pi y^3 - \frac{2}{15}\pi y^2 + 16\pi y\Big|_0^{30}$$

$$V = \frac{1}{2{,}700}\pi(30)^3 - \frac{2}{15}\pi(30)^2 + 16\pi(30) - \left(\frac{1}{2{,}700}\pi(0)^3 - \frac{2}{15}\pi(0)^2 + 16\pi(0)\right)$$

$$V = 10\pi - 120\pi + 480\pi$$

$$V = 370\pi$$