Surface area of revolution

We can use integrals to find the surface area of the three-dimensional figure that's created when we take a function and rotate it around an axis and over a certain interval.

The formulas we use to find surface area of revolution are different depending on the form of the original function and the axis of rotation.

1. When the function is in the form $y = f(x)$ and you're rotating around the y-axis, the interval is $a \le x \le b$ and the formula is

$$S = \int_a^b 2\pi x \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\ dx$$

2. When the function is in the form $y = f(x)$ and you're rotating around the x-axis, the interval is $a \le x \le b$ and the formula is

$$S = \int_a^b 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\ dx$$

3. When the function is in the form $x = g(y)$ and you're rotating around the y-axis, the interval is $c \le y \le d$ and the formula is

$$S = \int_c^d 2\pi x \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\ dy$$

4. When the function is in the form $x = g(y)$ and you're rotating around the x-axis, the interval is $c \le y \le d$ and the formula is

$$S=\int_c^d 2\pi y\sqrt{1+\left(\frac{dx}{dy}\right)^2}\ dy$$

Example

Find the area of the surface generated by rotating the function about the given axis over the given interval.

$$y=x^3$$

about the x-axis

$$0\le x\le 3$$

Since the equation is in the form $y=f(x)$, and we're rotating around the x-axis, we'll use the formula

$$S=\int_a^b 2\pi y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\ dx$$

We'll calculate dy/dx and then substitute it back into the equation.

$$\frac{dy}{dx}=3x^2$$

$$S=\int_0^3 2\pi x^3\sqrt{1+\left(3x^2\right)^2}\ dx$$

$$S=\int_0^3 2\pi x^3\sqrt{1+9x^4}\ dx$$

Using u-substitution and setting $u=1+9x^4$ and $du=36x^3\ dx$, we calculate

$$x=\left(\frac{u-1}{9}\right)^{\frac{1}{4}}$$

$$dx=\frac{1}{36x^3}\ du$$

$$dx=\frac{1}{36\left[\left(\frac{u-1}{9}\right)^{\frac{1}{4}}\right]^3}\ du$$

Plugging these values back into the integral, we get

$$S=\int_0^3 2\pi\left[\left(\frac{u-1}{9}\right)^{\frac{1}{4}}\right]^3\sqrt{u}\frac{1}{36\left[\left(\frac{u-1}{9}\right)^{\frac{1}{4}}\right]^3}\ du$$

$$S=\int_0^3 2\pi\left(\frac{u-1}{9}\right)^{\frac{3}{4}}\sqrt{u}\frac{1}{36\left(\frac{u-1}{9}\right)^{\frac{3}{4}}}\ du$$

$$S=\frac{\pi}{18}\int_0^3\left(\frac{u-1}{9}\right)^{\frac{3}{4}}\sqrt{u}\frac{1}{\left(\frac{u-1}{9}\right)^{\frac{3}{4}}}\ du$$

$$S=\frac{\pi}{18}\int_0^3\sqrt{u}\ du$$

Integrate.

$$S=\left(\frac{\pi}{18}\right)\left(\frac{2}{3}u^{\frac{3}{2}}\right)\Bigg|_0^3$$

$$S=\frac{\pi}{27}u^{\frac{3}{2}}\Bigg|_0^3$$

We'll plug back in for u, remembering that $u=1+9x^4$, and then evaluate over the interval.

$$S=\frac{\pi}{27}\left(1+9x^4\right)^{\frac{3}{2}}\Bigg|_0^3$$

$$S=\frac{\pi}{27}\left[1+9(3)^4\right]^{\frac{3}{2}}-\left[\frac{\pi}{27}\left(1+9(0)^4\right)^{\frac{3}{2}}\right]$$

$S=2{,}294.8$ square units

The surface area obtained by rotating $y=x^3$ around the x-axis over the interval $0\le x\le 3$ is $S=2{,}294.8$.