Topic: Income stream, compounded continuously, future value

Question: Money is invested at a rate of $\$12{,}000$ annually and the bank pays $14.5\,\%$ interest, compounded continuously. How many years will it take for the investment to reach half a million dollars?

Answer choices:

A 13.5

B 14.5

C 15.5

D 16.5

Solution: A

Plugging the values we know into the future value formula for a continuous income stream, we get

$$FV=\int_0^N S(t)e^{r(N-t)}\ dt$$

$$500{,}000=\int_0^N 12{,}000e^{0.145(N-t)}\ dt$$

$$500{,}000=12{,}000\int_0^N e^{0.145N-0.145t}\ dt$$

$$500{,}000=12{,}000\int_0^N e^{0.145N}e^{-0.145t}\ dt$$

$$500{,}000=12{,}000e^{0.145N}\int_0^N e^{-0.145t}\ dt$$

Integrate, then evaluate over the interval.

$$500{,}000=12{,}000e^{0.145N}\left(\frac{1}{-0.145}e^{-0.145t}\right)\Bigg|_0^N$$

$$\frac{500{,}000(-0.145)}{12{,}000e^{0.145N}}=e^{-0.145t}\Big|_0^N$$

$$\frac{500{,}000(-0.145)}{12{,}000e^{0.145N}}=e^{-0.145N}-e^{-0.145(0)}$$

$$\frac{500{,}000(-0.145)}{12{,}000e^{0.145N}} = e^{-0.145N} - 1$$

Multiply through by $e^{0.145N}$ to collect the N variables on one side of the equation.

$$\frac{500{,}000(-0.145)}{12{,}000} = e^{-0.145N}e^{0.145N} - 1e^{0.145N}$$

$$\frac{500{,}000(-0.145)}{12{,}000} = 1 - e^{0.145N}$$

$$e^{0.145N} = 1 - \frac{500{,}000(-0.145)}{12{,}000}$$

$$e^{0.145N} = 1 + \frac{125(0.145)}{3}$$

Take the natural log of both sides to solve for N.

$$\ln(e^{0.145N}) = \ln\left(1 + \frac{125(0.145)}{3}\right)$$

$$0.145N = \ln\left(1 + \frac{125(0.145)}{3}\right)$$

$$N = \frac{1}{0.145}\ln\left(1 + \frac{125(0.145)}{3}\right)$$

$$N \approx 13.5$$

Topic: Income stream, compounded continuously, future value

Question: Money is invested at $\$12{,}000$ annually and the bank pays $4.5\,\%$ interest, compounded continuously. What is the balance in the account after five years?

Answer choices:

A $\$67{,}286$

B $\$65{,}286$

C $\$65{,}000$

D $\$60{,}286$

Solution: A

Plugging the values we've been given into the future value formula for a continuous income stream, we get

$$FV = \int_0^N S(t)e^{r(N-t)}\ dt$$

$$FV = \int_0^5 12{,}000e^{0.045(5-t)}\ dt$$

$$FV = 12{,}000\int_0^5 e^{0.225}e^{-0.045t}\ dt$$

$$FV = 12{,}000e^{0.225}\int_0^5 e^{-0.045t}\ dt$$

Integrate, then evaluate over the interval.

$$FV = 12{,}000e^{0.225}\left(\frac{1}{-0.045}e^{-0.045t}\right)\Bigg|_0^5$$

$$FV = -\frac{12{,}000e^{0.225}}{0.045}(e^{-0.045(5)} - e^{-0.045(0)})$$

$$FV = -\frac{12{,}000e^{0.225}}{0.045}(e^{-0.225} - 1)$$

$$FV \approx \$67{,}286$$

Topic: Income stream, compounded continuously, future value

Question: Find the present and future value of an income stream given by $S(t) = \$1{,}600e^{0.02t}$, if the bank pays $4\,\%$ interest, compounded continuously, for 10 years.

Answer choices:

A $PV = \$12{,}501.45$ and $FV = \$21{,}001.76$

B $PV = \$14{,}501.54$ and $FV = \$21{,}633.76$

C $PV = \$16{,}323.65$ and $FV = \$23{,}633.26$

D $PV = \$14{,}105.54$ and $FV = \$19{,}366.75$

Solution: B

Plugging the values we've been given into the present value formula for a continuous income stream, we get

$$PV = \int_0^N S(t)e^{-rt}\ dt$$

$$PV = \int_0^{10} 1{,}600e^{0.02t}e^{-0.04t}\ dt$$

$$PV = 1{,}600\int_0^{10} e^{0.02t-0.04t}\ dt$$

$$PV = 1{,}600\int_0^{10} e^{-0.02t}\ dt$$

Integrate, then evaluate over the interval.

$$PV = \frac{1{,}600}{-0.02}e^{-0.02t}\Big|_0^{10}$$

$$PV = -80{,}000(e^{-0.02(10)} - e^{-0.02(0)})$$

$$PV = -80{,}000(e^{-0.2} - 1)$$

$$PV \approx \$14{,}501.54$$

We'll rewrite the formula for the future value of a continuous income stream,

$$FV = \int_0^N S(t)e^{r(N-t)}\ dt$$

$$FV = \int_0^N S(t)e^{rN}e^{-rt}\ dt$$

$$FV = e^{rN}\int_0^N S(t)e^{-rt}\ dt$$

such that the remaining integral is now equivalent to the present value of the income stream.

$$FV = e^{rN}PV$$

$$FV = e^{(0.04)(10)}(14{,}501.54)$$

$$FV \approx \$21{,}633.76$$