Topic: Arc length of x=g(y)

Question: Find the arc length of the curve over the given interval.

$$6xy = y^4 + 3$$

on the interval $[1,2]$

Answer choices:

A $\frac{11}{8}$

B $\frac{17}{16}$

C $\frac{11}{12}$

D $\frac{17}{12}$

Solution: D

The formula for arc length for a curve defined as $x = g(y)$ and with limits of integration given as $y = c$ and $y = d$ is

$$L = \int_c^d \sqrt{1 + [g'(y)]^2}\ dy$$

We already know that $c = 1$ and $d = 2$. The only thing we need for our formula is $g'(y)$, which we'll find by taking the derivative of our original function and solving for x'.

$$x = \frac{y^4 + 3}{6y}$$

$$x = \frac{y^4}{6y} + \frac{3}{6y}$$

$$x = \frac{y^3}{6} + \frac{1}{2y}$$

$$x' = \frac{y^2}{2} - \frac{1}{2y^2}$$

Plugging all of these values into the formula, we get

$$L = \int_1^2 \sqrt{1 + \left(\frac{y^2}{2} - \frac{1}{2y^2}\right)^2}\ dy$$

Find a common denominator and combine fractions.

$$L=\int_1^2\sqrt{1+\left(\frac{y^4}{2y^2}-\frac{1}{2y^2}\right)^2}\ dy$$

$$L=\int_1^2\sqrt{1+\left(\frac{y^4-1}{2y^2}\right)^2}\ dy$$

$$L=\int_1^2\sqrt{1+\frac{\left(y^4-1\right)^2}{4y^4}}\ dy$$

Find a common denominator and combine fractions.

$$L=\int_1^2\sqrt{\frac{4y^4}{4y^4}+\frac{\left(y^4-1\right)^2}{4y^4}}\ dy$$

$$L=\int_1^2\sqrt{\frac{4y^4+\left(y^4-1\right)^2}{4y^4}}\ dy$$

Take the square root of the numerator and denominator separately.

$$L=\frac{1}{2}\int_1^2\frac{1}{y^2}\sqrt{4y^4+\left(y^4-1\right)^2}\ dy$$

$$L=\frac{1}{2}\int_1^2\frac{1}{y^2}\sqrt{4y^4+y^8-2y^4+1}\ dy$$

$$L=\frac{1}{2}\int_1^2\frac{1}{y^2}\sqrt{y^8+2y^4+1}\ dy$$

$$L=\frac{1}{2}\int_1^2\frac{1}{y^2}\sqrt{\left(y^4+1\right)^2}\ dy$$

$$L=\frac{1}{2}\int_1^2\frac{1}{y^2}\cdot\left(y^4+1\right)\ dy$$

$$L=\frac{1}{2}\int_1^2\frac{y^4+1}{y^2}\ dy$$

$$L=\frac{1}{2}\int_1^2\frac{y^4}{y^2}+\frac{1}{y^2}\ dy$$

$$L=\frac{1}{2}\int_1^2 y^2+y^{-2}\ dy$$

Take the integral, then evaluate over the given interval.

$$L=\frac{1}{2}\left(\frac{1}{3}y^3-y^{-1}\right)\Bigg|_1^2$$

$$L=\frac{1}{2}\left(\frac{y^3}{3}-\frac{1}{y}\right)\Bigg|_1^2$$

$$L=\frac{1}{2}\left[\left(\frac{(2)^3}{3}-\frac{1}{(2)}\right)-\left(\frac{(1)^3}{3}-\frac{1}{(1)}\right)\right]$$

$$L=\frac{1}{2}\left[\left(\frac{8}{3}-\frac{1}{2}\right)-\left(\frac{1}{3}-1\right)\right]$$

$$L=\frac{1}{2}\left(\frac{8}{3}-\frac{1}{2}-\frac{1}{3}+1\right)$$

$$L=\frac{1}{2}\left(\frac{7}{3}-\frac{1}{2}+1\right)$$

$$L=\frac{1}{2}\left(\frac{14}{6}-\frac{3}{6}+\frac{6}{6}\right)$$

$$L=\frac{17}{12}$$

Topic: Arc length of x=g(y)

Question: Find the arc length of the curve over the given interval.

$$3x = 2\left(y^2+1\right)^{\frac{3}{2}}$$

on the interval $[0,3]$

Answer choices:

A 21

B 22

C 23

D 24

Solution: A

First, rewrite the given curve as

$$x = \frac{2}{3}\left(y^2+1\right)^{\frac{3}{2}}$$

Differentiating with respect to y gives us

$$\frac{dx}{dy} = \frac{2}{3} \cdot \frac{3}{2}\left(y^2+1\right)^{\frac{1}{2}} \cdot 2y$$

$$\frac{dx}{dy} = 2y\left(y^2+1\right)^{\frac{1}{2}}$$

The applicable formula for arc length is

$$L = \int_c^d \sqrt{1+\left(\frac{dx}{dy}\right)^2}\ dy$$

Putting limits of integration and the derivative into the formula gives

$$L = \int_0^3 \sqrt{1+\left[2y\left(y^2+1\right)^{\frac{1}{2}}\right]^2}\ dy$$

$$L = \int_0^3 \sqrt{1+4y^2\left(y^2+1\right)}\ dy$$

$$L = \int_0^3 \sqrt{1+4y^2+4y^4}\ dy$$

$$L = \int_0^3 \sqrt{\left(1+2y^2\right)^2}\ dy$$

$$L=\int_0^3 1+2y^2\ dy$$

Take the integral, then evaluate over the given interval.

$$L=y+\frac{2}{3}y^3\Big|_0^3$$

$$L=3+\frac{2}{3}(3)^3-\left[0+\frac{2}{3}(0)^3\right]$$

$$L=21$$

Topic: Arc length of x=g(y)

Question: Find the arc length of the function on the interval.

$$x = \frac{y^3}{3} + \frac{1}{4y}$$

on the interval $y = [1,3]$

Answer choices:

A $\frac{33}{2}$

B $\frac{53}{6}$

C 17

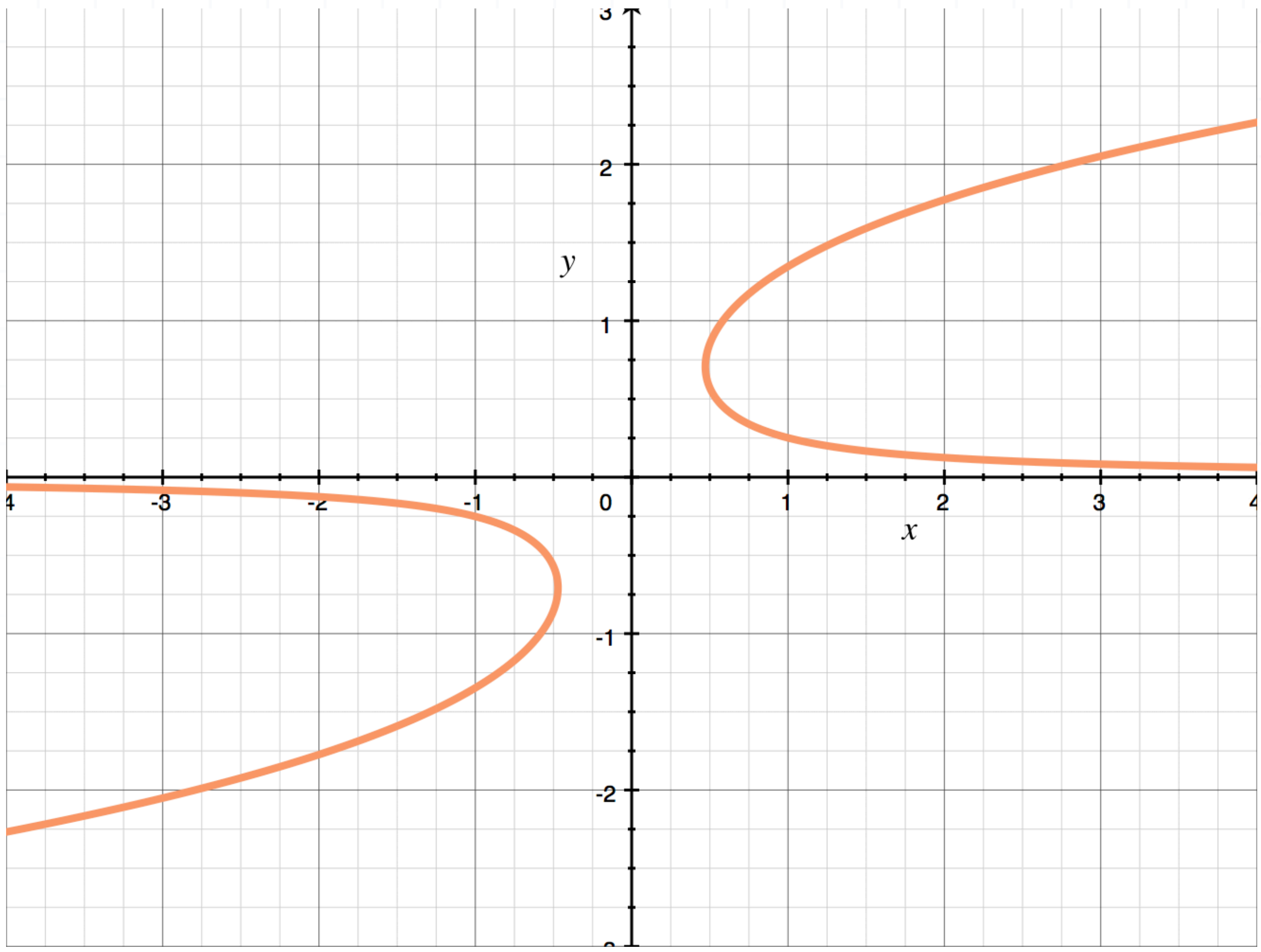
D $\frac{58}{6}$

Solution: B

The graph of

$$x = \frac{y^3}{3} + \frac{1}{4y}$$

is shown below.



We can see that the curve is not a function, so it cannot be integrated with respect to x. We can also see that x is a smooth curve of y on the interval $y = [1,3]$. Therefore, the arc length can be given by

$$L = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\ dy$$

We'll find dx/dy.

$$x=\frac{y^3}{3}+\frac{1}{4y}$$

$$x=\frac{1}{3}y^3+\frac{1}{4}y^{-1}$$

$$\frac{dx}{dy}=\left(\frac{1}{3}\right)\left(3y^2\right)+\left(\frac{1}{4}\right)(-1)\left(y^{-2}\right)$$

$$\frac{dx}{dy}=y^2-\frac{1}{4y^2}$$

Now plug everything into the integral formula.

$$L=\int_c^d\sqrt{1+\left(\frac{dx}{dy}\right)^2}\ dy$$

$$L=\int_1^3\sqrt{1+\left(y^2-\frac{1}{4y^2}\right)^2}\ dy$$

$$L=\int_1^3\sqrt{1+y^4-\frac{1}{2}+\frac{1}{16y^4}}\ dy$$

$$L=\int_1^3\sqrt{\frac{1}{2}+y^4+\frac{1}{16y^4}}\ dy$$

The expression inside the radical becomes a perfect square.

$$L = \int_1^3 \sqrt{\left(y^2 + \frac{1}{4y^2}\right)^2}\ dy$$

$$L = \int_1^3 y^2 + \frac{1}{4y^2}\ dy$$

Integrate, then evaluate over the interval.

$$L = \frac{1}{3}y^3 - \frac{1}{4y}\Big|_1^3$$

$$L = \frac{1}{3}(3)^3 - \frac{1}{4(3)} - \left(\frac{1}{3}(1)^3 - \frac{1}{4(1)}\right)$$

$$L = 9 - \frac{1}{12} - \left(\frac{1}{3} - \frac{1}{4}\right)$$

$$L = \frac{108}{12} - \frac{1}{12} - \frac{4}{12} + \frac{3}{12}$$

$$L = \frac{106}{12}$$

$$L = \frac{53}{6}$$