Calculus 2
Final Exam Solutions

krista king
MATH

Calculus 2 Final Exam Answer Key

1. (5 pts)	A	B	■	D	E
2. (5 pts)	A	B	C	■	E
3. (5 pts)	■	B	C	D	E
4. (5 pts)	A	■	C	D	E
5. (5 pts)	A	B	C	D	■
6. (5 pts)	■	B	C	D	E
7. (5 pts)	A	■	C	D	E
8. (5 pts)	A	B	C	D	■

9. (15 pts) 24π

10. (15 pts) $\frac{\sqrt{3}}{4}\int \tan\theta \sec\theta \ d\theta$

11. (15 pts) $V_y = \frac{80\pi}{3}$

12. (15 pts) $R = 1$

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1. C. Use integration by parts with $u = 2x$ and $dv = \sin x\ dx$. Then differentiate u and integrate dv to get $du = 2\ dx$ and $v = -\cos x$.

Plug these values into the integration by parts formula.

$$\int u\ dv = uv - \int v\ du$$

$$\int 2x \sin x\ dx = 2x(-\cos x) - \int (-\cos x)(2\ dx)$$

$$\int 2x \sin x\ dx = -2x \cos x + 2 \int \cos x\ dx$$

Evaluate the remaining integral on the right side to get

$$-2x \cos x + 2 \sin x + C$$

$$2 \sin x - 2x \cos x + C$$

2. D. The four equal subintervals in the interval $[2,10]$ are $[2,4]$, $[4,6]$, $[6,8]$ and $[8,10]$. Each subinterval is 2 units wide, and the midpoints of the four subintervals are 3, 5, 7, and 9. The function's "height" at each midpoint is

$$f(3) = 2(3)^3 - 7(3)^2 + 2(3) - 6 = -9$$

$$f(5) = 2(5)^3 - 7(5)^2 + 2(5) - 6 = 79$$

$$f(7) = 2(7)^3 - 7(7)^2 + 2(7) - 6 = 351$$

$$f(9) = 2(9)^3 - 7(9)^2 + 2(9) - 6 = 903$$

Then the area of each rectangle is given by the product of its width, 2, and its height, which comes from the values we just calculated.

$$A_3 = -9 \times 2 = -18$$

$$A_5 = 79 \times 2 = 158$$

$$A_7 = 351 \times 2 = 702$$

$$A_9 = 903 \times 2 = 1{,}806$$

Add these areas together to get the final approximation.

$$A = -18 + 158 + 702 + 1{,}806 = 2{,}648$$

3. A. For the given spring where $F(x) = 100$ lbs and $x = 0.5$ ft, we'll solve for k by substituting these values into Hooke's Law.

$$F(x) = kx$$

$$k = \frac{F(x)}{x}$$

$$k = \frac{100}{0.5}$$

$$k = 200$$

The spring constant is 200 lbs/ft. Then the work done to stretch the spring 3 foot beyond its natural length is

$$W = \int_a^b F(x)\ dx$$

$$W = \int_0^3 200x\ dx$$

$$W = \frac{200}{2}x^2\Big|_0^3$$

$$W = 100x^2\Big|_0^3$$

$$W = 100(3)^2 - 100(0)^2$$

$$W = 900$$

So the work done in stretching the spring 3 foot beyond its natural length is 900 ft-lbs.

4. B. We'll match each term in the sequence with its corresponding n value.

$n = 1$	$n = 2$	$n = 3$	$n = 4$
$\frac{1}{3}$	$-\frac{4}{4}$	$\frac{9}{5}$	$-\frac{16}{6}$

Now we can start examining the sequence. We notice that the signs of the terms are alternating, such that the even terms are negative and the odd terms are positive, which means that the general term will include $(-1)^{n-1}$ or $(-1)^{n+1}$.

Turning to the numerator, we can see that each numerator is the square of its n-value.

$$n = 1 \qquad 1 = 1^2 = n^2$$

$$n = 2 \qquad 4 = 2^2 = n^2$$

$$n = 3 \qquad 9 = 3^3 = n^2$$

$$n = 4 \qquad 16 = 4^2 = n^2$$

This tells us that the numerator of the general term will be n^2.

Now let's look at the denominator of each term in the sequence. Taking the denominator only, we see that

$a_1 = 3$ when $n = 1$

$a_2 = 4$ when $n = 2$

$a_3 = 5$ when $n = 3$

$a_4 = 6$ when $n = 4$

The denominator of each term is always 2 higher than the corresponding value of n, so the denominator of the general term will be $n + 2$.

Putting this all together gives the general term as

$$a_n = (-1)^{n-1}\frac{n^2}{n+2} \quad \text{or} \quad a_n = (-1)^{n+1}\frac{n^2}{n+2}$$

5. E. To find the distance between two polar coordinates, we'll use the formula

$$D = \sqrt{r_1^2 + r_2^2 - 2r_1r_2\cos(\theta_1 - \theta_2)}$$

where (r_1, θ_1) and (r_2, θ_2) are the given polar points. It doesn't matter which point we use for (r_1, θ_1) and (r_2, θ_2), but it's easier to make θ_1 the larger of the two θ values, since we subtract θ_2 from θ_1.

We'll set

$$(r_1, \theta_1) = \left(1, \frac{3\pi}{4}\right)$$

$$(r_2, \theta_2) = \left(4, \frac{\pi}{2}\right)$$

Plugging these points into the distance formula, we get

$$D = \sqrt{1^2 + 4^2 - 2(1)(4)\cos\left(\frac{3\pi}{4} - \frac{\pi}{2}\right)}$$

$$D = \sqrt{1 + 16 - 8\cos\left(\frac{3\pi}{4} - \frac{2\pi}{4}\right)}$$

$$D=\sqrt{17-8\cos\left(\frac{\pi}{4}\right)}$$

$$D=\sqrt{17-8\left(\frac{\sqrt{2}}{2}\right)}$$

$$D=\sqrt{17-4\sqrt{2}}$$

6. A. We need to get the series into standard form for a geometric series to make sure the series is geometric. Since the index starts at $n=0$, standard form is

$$\sum_{n=0}^{\infty} ar^n$$

so we'll rewrite the series as

$$\sum_{n=0}^{\infty} \frac{4^{n-2}}{3^{2n}}$$

$$\sum_{n=0}^{\infty} \frac{4^n 4^{-2}}{9^n}$$

$$\sum_{n=0}^{\infty} 4^{-2}\left(\frac{4^n}{9^n}\right)$$

$$\sum_{n=0}^{\infty} \frac{1}{16} \left(\frac{4}{9} \right)^n$$

Comparing this to the standard form, we'll say that

$$a = \frac{1}{16}$$

and

$$r = \frac{4}{9}$$

Since

$$\left| \frac{4}{9} \right| = \frac{4}{9} < 1$$

the series converges by the geometric series test for convergence, which means we can find the sum of the series as

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}$$

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{\frac{1}{16}}{1 - \frac{4}{9}}$$

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{\frac{1}{16}}{\frac{9}{9} - \frac{4}{9}}$$

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{\frac{1}{16}}{\frac{5}{9}}$$

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{1}{16} \cdot \frac{9}{5}$$

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{9}{80}$$

The sum of the series is $9/80$.

7. B. Since the curves we're given are expressed for x in terms of y, we can say these are "left and right curves." To find their points of intersection, we'll set the curves equal to each other.

$$(y-1)^2 + 1 = 6 - y^2$$

$$y^2 - 2y + 1 + 1 = 6 - y^2$$

$$y^2 - 2y + 2 = 6 - y^2$$

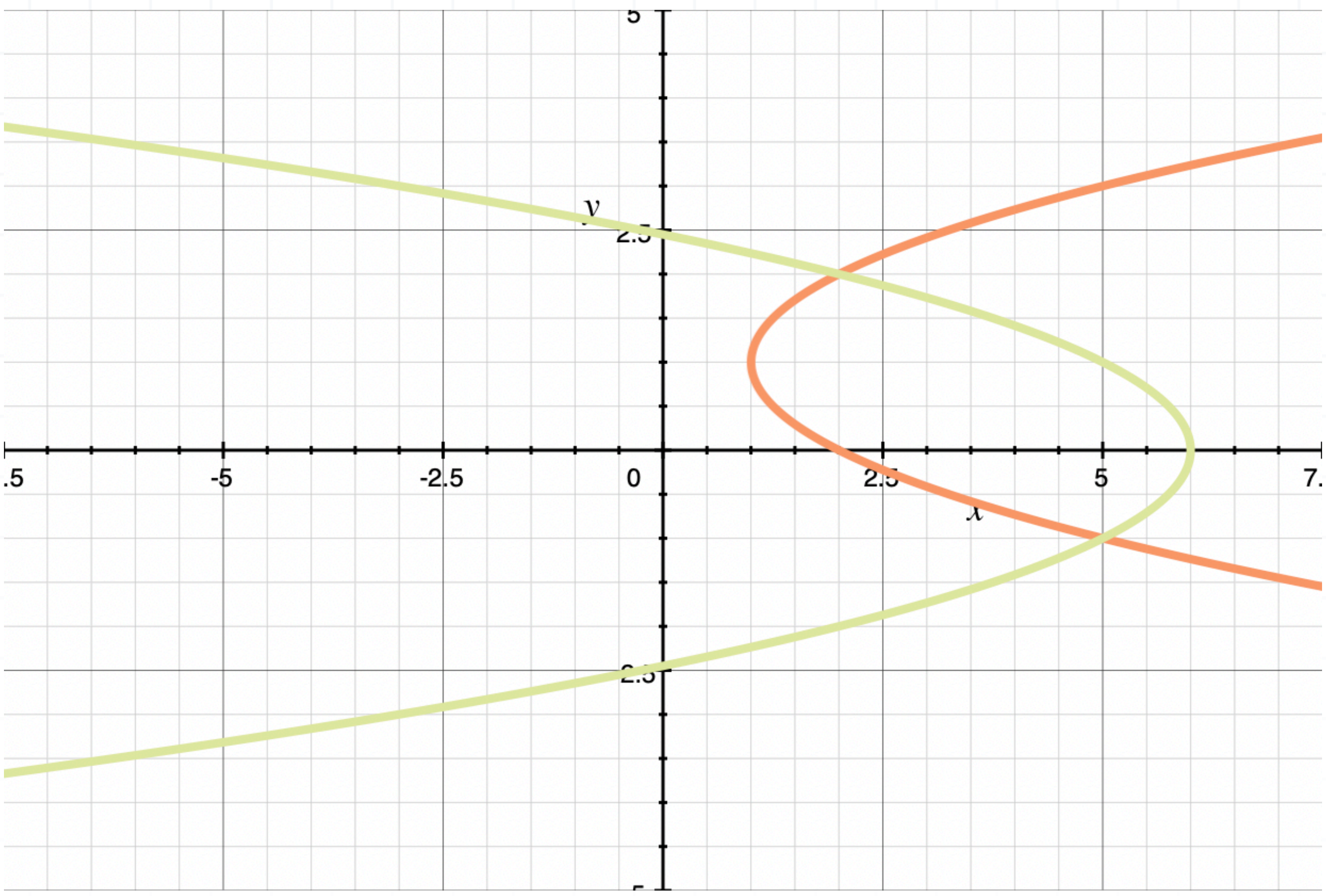
$$2y^2 - 2y - 4 = 0$$

$$y^2 - y - 2 = 0$$

$$(y+1)(y-2) = 0$$

$y = -1$ and $y = 2$

It's helpful to sketch the graphs to see which curve is the upper curve and which curve is the lower curve.



These look like left-right curves, with $y^2 = 6 - x$ being the "right" curve and $x = (y-1)^2 + 1$ being the "left" curve. We'll rewrite these equations as functions.

$$f(y) = 6 - y^2$$

$$g(y) = (y-1)^2 + 1$$

Now we can plug these functions and the interval we found earlier into the formula for the area between left and right curves.

$$\int_d^c f(y) - g(y)\ dy$$

$$\int_{-1}^{2} 6 - y^2 - ((y-1)^2 + 1)\ dy$$

$$\int_{-1}^{2} 6 - y^2 - (y-1)^2 - 1\ dy$$

$$\int_{-1}^{2} 5 - y^2 - (y^2 - 2y + 1)\ dy$$

$$\int_{-1}^{2} 5 - y^2 - y^2 + 2y - 1\ dy$$

$$\int_{-1}^{2} 4 - 2y^2 + 2y\ dy$$

Integrate, then evaluate over the interval.

$$\frac{-2y^3}{3} + \frac{2y^2}{2} + 4y\Big|_{-1}^{2}$$

$$-\frac{2y^3}{3} + y^2 + 4y\Big|_{-1}^{2}$$

$$\left[\frac{-2(2)^3}{3} + 2^2 + 4(2)\right] - \left[\frac{-2(-1)^3}{3} + (-1)^2 + 4(-1)\right]$$

$$\left(\frac{-16}{3} + 4 + 8\right) - \left(\frac{2}{3} + 1 - 4\right)$$

$$-\frac{16}{3} + 12 - \frac{2}{3} + 3$$

$$-\frac{18}{3}+15$$

$$-6+15$$

$$9$$

8. E. Find the derivatives dx/dt and dy/dt.

$$x=2e^{6t}-3t+6$$

$$\frac{dx}{dt}=12e^{6t}-3$$

and

$$y=4e^{3t}-1$$

$$\frac{dy}{dt}=12e^{3t}$$

Then the arc length of the parametric curve over the interval $2 \le t \le 5$ is given by

$$L=\int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\ dt$$

$$L=\int_2^5 \sqrt{(12e^{6t}-3)^2+(12e^{3t})^2}\ dt$$

$$L=\int_2^5 \sqrt{144e^{12t}-72e^{6t}+9+144e^{6t}}\ dt$$

$$L=\int_{2}^{5}\sqrt{144e^{12t}+72e^{6t}+9}\ dt$$

$$L=\int_{2}^{5}\sqrt{(12e^{6t}+3)^2}\ dt$$

$$L=\int_{2}^{5}12e^{6t}+3\ dt$$

Integrate, then evaluate over the interval.

$$L=2e^{6t}+3t\Big|_{2}^{5}$$

$$L=2e^{6(5)}+3(5)-(2e^{6(2)}+3(2))$$

$$L=2e^{30}+15-2e^{12}-6$$

$$L=2e^{30}-2e^{12}+9$$

9. Because our curve is defined in the form $y=f(x)$ and our limits of integration are defined as $x=-2$ and $x=1$, the formula we use to find the surface area of revolution is

$$A=\int_{a}^{b}2\pi y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\ dx$$

The derivative of $y=\sqrt{16-x^2}$ is

$$\frac{dy}{dx} = \frac{-x}{\sqrt{16-x^2}}$$

Plugging the derivative and limits of integration into the surface area of revolution formula gives

$$A = \int_{-2}^{1} 2\pi\sqrt{16-x^2}\sqrt{1+\left(\frac{-x}{\sqrt{16-x^2}}\right)^2}\ dx$$

$$A = 2\pi\int_{-2}^{1} \sqrt{16-x^2}\sqrt{1+\frac{x^2}{16-x^2}}\ dx$$

Find a common denominator and combine fractions.

$$A = 2\pi\int_{-2}^{1} \sqrt{16-x^2}\sqrt{\frac{16-x^2}{16-x^2}+\frac{x^2}{16-x^2}}\ dx$$

$$A = 2\pi\int_{-2}^{1} \sqrt{16-x^2}\sqrt{\frac{16}{16-x^2}}\ dx$$

$$A = 2\pi\int_{-2}^{1} \sqrt{16-x^2}\cdot\frac{\sqrt{16}}{\sqrt{16-x^2}}\ dx$$

$$A = 8\pi\int_{-2}^{1}\ dx$$

$$A = 8\pi x\Big|_{-2}^{1}$$

$$A = 8\pi(1) - 8\pi(-2)$$

$$A = 8\pi + 16\pi$$

$$A = 24\pi$$

10. We want to set up the integral for trigonometric substitution. In

$$\int \frac{4x}{\sqrt{3+16x^2}}\ dx$$

we see the expression $a^2 + u^2$, where $a^2 = 3$ and $u^2 = 16x^2$.

$$a^2 = 3$$

$$a = \sqrt{3}$$

and

$$u^2 = 16x^2$$

$$u = 4x$$

This format requires us to use the tangent substitution $u = a\tan\theta$.

$$u = a\tan\theta$$

$$4x = \sqrt{3}\tan\theta$$

$$\tan\theta = \frac{4x}{\sqrt{3}}$$

$$\theta = \arctan\left(\frac{4x}{\sqrt{3}}\right)$$

We also get

$$4x = \sqrt{3}\tan\theta$$

$$x = \frac{\sqrt{3}}{4}\tan\theta$$

$$dx = \frac{\sqrt{3}}{4}\sec^2\theta\ d\theta$$

In a right triangle, we know the side opposite of θ will have length $4x$, the side adjacent to θ will have length $\sqrt{3}$, and therefore that they hypotenuse of the triangle will have length $\sqrt{3+16x^2}$.

Then the trigonometric substitution will be

$$\int \frac{4x}{\sqrt{3+16x^2}}\ dx$$

$$\int \frac{\sqrt{3}\tan\theta}{\sqrt{3+16\left(\frac{\sqrt{3}}{4}\tan\theta\right)^2}}\left(\frac{\sqrt{3}}{4}\sec^2\theta\ d\theta\right)$$

$$\int \frac{\frac{3}{4}\tan\theta\sec^2\theta}{\sqrt{3+16\left(\frac{3}{16}\tan^2\theta\right)}}\ d\theta$$

$$\frac{3}{4}\int \frac{\tan\theta\sec^2\theta}{\sqrt{3+3\tan^2\theta}}\ d\theta$$

$$\frac{3}{4}\int \frac{\tan\theta\sec^2\theta}{\sqrt{3(1+\tan^2\theta)}}\ d\theta$$

Use the Pythagorean identity $1+\tan^2\theta=\sec^2\theta$ to make a substitution.

$$\frac{3}{4}\int \frac{\tan\theta\sec^2\theta}{\sqrt{3\sec^2\theta}}\ d\theta$$

$$\frac{3}{4}\int \frac{\tan\theta\sec^2\theta}{\sqrt{3}\sec\theta}\ d\theta$$

$$\frac{\sqrt{3}}{4}\int \tan\theta\sec\theta\ d\theta$$

11. Since we're rotating around the y-axis, we'll use the formula

$$V_y=\int_{\alpha}^{\beta}\pi x^2\left(\frac{dy}{dt}\right)\ dt$$

The problem gave the interval $-\pi/2\le t\le\pi/2$, so $\alpha=-\pi/2$ and $\beta=\pi/2$. Now we need to find dy/dt so that we can plug it into the volume formula.

$$y=5\sin t$$

$$\frac{dy}{dt} = 5\cos t$$

Plugging everything into the volume formula, we get

$$V_y = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \pi(2\cos t)^2(5\cos t)\ dt$$

$$V_y = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \pi(4\cos^2 t)(5\cos t)\ dt$$

$$V_y = 20\pi \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^3 t\ dt$$

$$V_y = 20\pi \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos t\cos^2 t\ dt$$

Make a substitution using the Pythagorean identity $\cos^2 t = 1 - \sin^2 t$.

$$V_y = 20\pi \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos t(1 - \sin^2 t)\ dt$$

$$V_y = 20\pi \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos t - \cos t\sin^2 t\ dt$$

$$V_y = 20\pi\sin t\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} - 20\pi \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos t\sin^2 t\ dt$$

Use a u-substitution with $u = \sin^2 t$ and $du = 2\sin t\cos t\ dt$.

$$V_y = 20\pi \sin t \Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} - 20\pi \int_{t=-\frac{\pi}{2}}^{t=\frac{\pi}{2}} u \cos t \left(\frac{du}{2 \sin t \cos t} \right)$$

$$V_y = 20\pi \sin t \Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} - 20\pi \int_{t=-\frac{\pi}{2}}^{t=\frac{\pi}{2}} u \left(\frac{du}{2 \sin t} \right)$$

$$V_y = 20\pi \sin t \Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} - 20\pi \int_{t=-\frac{\pi}{2}}^{t=\frac{\pi}{2}} u \left(\frac{du}{2\sqrt{u}} \right)$$

$$V_y = 20\pi \sin t \Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} - 10\pi \int_{t=-\frac{\pi}{2}}^{t=\frac{\pi}{2}} \sqrt{u} \ du$$

Integrate with respect to u, then back-substitute for u.

$$V_y = 20\pi \sin t \Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} - 10\pi \left(\frac{2}{3} u^{\frac{3}{2}} \right) \Big|_{t=-\frac{\pi}{2}}^{t=\frac{\pi}{2}}$$

$$V_y = 20\pi \sin t \Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} - \frac{20\pi}{3} u^{\frac{3}{2}} \Big|_{t=-\frac{\pi}{2}}^{t=\frac{\pi}{2}}$$

$$V_y = 20\pi \sin t \Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} - \frac{20\pi}{3} (\sin^2 t)^{\frac{3}{2}} \Big|_{t=-\frac{\pi}{2}}^{t=\frac{\pi}{2}}$$

$$V_y = 20\pi \sin t \Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} - \frac{20\pi}{3} \sin^3 t \Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$$

$$V_y = 20\pi \sin t - \frac{20\pi}{3} \sin^3 t \Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$$

Evaluate over the interval.

$$V_y = 20\pi \sin\left(\frac{\pi}{2}\right) - \frac{20\pi}{3}\sin^3\left(\frac{\pi}{2}\right) - \left(20\pi \sin\left(-\frac{\pi}{2}\right) - \frac{20\pi}{3}\sin^3\left(-\frac{\pi}{2}\right)\right)$$

$$V_y = 20\pi(1) - \frac{20\pi}{3}(1)^3 - \left(20\pi(-1) - \frac{20\pi}{3}(-1)^3\right)$$

$$V_y = 20\pi - \frac{20\pi}{3} - \left(-20\pi + \frac{20\pi}{3}\right)$$

$$V_y = 20\pi - \frac{20\pi}{3} + 20\pi - \frac{20\pi}{3}$$

$$V_y = 40\pi - \frac{40\pi}{3}$$

$$V_y = \frac{120\pi}{3} - \frac{40\pi}{3}$$

$$V_y = \frac{80\pi}{3}$$

12. To get a power series representation of the series, we need to remember that the formula for the Taylor polynomial of f at a is

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \ldots + \frac{f^n(a)}{n!}(x-a)^n$$

Which means that the power series representation of the function will be the last term,

$$\sum_{n=1}^{\infty} \frac{f^n(a)}{n!}(x-a)^n$$

We can rewrite the given series as

$$5 + \frac{1}{2}(x-5) + \frac{1}{3}(x-5)^2 + \frac{1}{4}(x-5)^3$$

$$5(x-5)^0 + \frac{1}{2}(x-5)^1 + \frac{1}{3}(x-5)^2 + \frac{1}{4}(x-5)^3$$

$$5(x-5)^0 + \frac{1}{1+1}(x-5)^1 + \frac{1}{2+1}(x-5)^2 + \frac{1}{3+1}(x-5)^3$$

The first term doesn't follow the same pattern as the other terms, so we can pull it out in front of the sum and represent the series as

$$5 + \sum_{n=1}^{\infty} \frac{1}{n+1}(x-5)^n$$

To find the radius of convergence of this series, we'll first identify a_n and a_{n+1}.

$$a_n = \frac{1}{n+1}(x-5)^n$$

$$a_{n+1} = \frac{1}{n+2}(x-5)^{n+1}$$

Now we can use the ratio test to find the radius of convergence. The ratio test tells us that a series converges if $L < 1$, when

$$L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$$

Plugging the values we found for a_n and a_{n+1} into this formula for L, we get

$$L = \lim_{n\to\infty} \left| \frac{\frac{1}{n+2}(x-5)^{n+1}}{\frac{1}{n+1}(x-5)^n} \right|$$

$$L = \lim_{n\to\infty} \left| \frac{\frac{1}{n+2}(x-5)^{n+1-n}}{\frac{1}{n+1}} \right|$$

$$L = \lim_{n\to\infty} \left| \frac{\frac{1}{n+2}}{\frac{1}{n+1}}(x-5) \right|$$

$$L = \lim_{n\to\infty} \left| \frac{n+1}{n+2}(x-5) \right|$$

The limit only effects n, not x, so we can pull $(x-5)$ out in front of the limit, as long as we keep it inside absolute value bars.

$$L = |x-5| \lim_{n\to\infty} \left| \frac{n+1}{n+2} \right|$$

$$L = |x-5| \lim_{n\to\infty} \left| \frac{n+1}{n+2} \left(\frac{\frac{1}{n}}{\frac{1}{n}} \right) \right|$$

$$L = |x-5| \lim_{n\to\infty} \left| \frac{1+\frac{1}{n}}{1+\frac{2}{n}} \right|$$

$$L = |x-5| \left| \frac{1+0}{1+0} \right|$$

$$L = |x-5|$$

Now we can set $L < 1$ to find the radius of convergence.

$$|x-5| < 1$$

Comparing this to $|x-a| < R$, where R is the radius of convergence, we can say that the radius of convergence of the series is $R = 1$.

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