Surface area of revolution of a parametric curve, horizontal axis

The surface area of the solid created by revolving a parametric curve around the x-axis is given by

$$S_x = \int_a^b 2\pi y \sqrt{\left[f'(t)\right]^2 + \left[g'(t)\right]^2}\ dt$$

where the curve is defined over the interval $[a, b]$,

where $f'(t)$ is the derivative of the curve $f(t)$

where $g'(t)$ is the derivative of the curve $g(t)$

Example

Find the surface area of revolution of the solid created when the parametric curve is rotated around the given axis over the given interval.

$$x = \cos^3 t$$

$$y = \sin^3 t$$

for $0 \le t \le \frac{\pi}{2}$, rotated around the x-axis

We'll call the parametric equations

$$f(t) = \cos^3 t$$

$$g(t) = \sin^3 t$$

The limits of integration are defined in the problem, but we need to find both derivatives before we can plug into the formula.

$$f'(t) = -3\cos^2 t \sin t$$

$$g'(t) = 3\sin^2 t \cos t$$

Now we'll plug into the formula for the surface area of revolution.

$$S_x = \int_0^{\frac{\pi}{2}} 2\pi(\sin^3 t)\sqrt{\left(-3\cos^2 t \sin t\right)^2 + \left(3\sin^2 t \cos t\right)^2}\ dt$$

$$S_x = \int_0^{\frac{\pi}{2}} 2\pi \sin^3 t\sqrt{9\cos^4 t \sin^2 t + 9\sin^4 t \cos^2 t}\ dt$$

$$S_x = \int_0^{\frac{\pi}{2}} 2\pi \sin^3 t\sqrt{9\sin^2 t \cos^2 t\left(\cos^2 t + \sin^2 t\right)}\ dt$$

Since $\sin^2 t + \cos^2 t = 1$, we get

$$S_x = \int_0^{\frac{\pi}{2}} 2\pi \sin^3 t\sqrt{9\sin^2 t \cos^2 t\,(1)}\ dt$$

$$S_x = \int_0^{\frac{\pi}{2}} 2\pi \sin^3 t\sqrt{9\sin^2 t \cos^2 t}\ dt$$

$$S_x = \int_0^{\frac{\pi}{2}} 2\pi \sin^3 t\left(3\sin t \cos t\right)\ dt$$

$$S_x = 6\pi \int_0^{\frac{\pi}{2}} \sin^4 t \cos t \ dt$$

We'll use u-substitution, letting

$$u = \sin t$$

$$du = \cos t \ dt$$

We'll make the substitution.

$$S_x = 6\pi \int_{x=0}^{x=\frac{\pi}{2}} u^4 \ du$$

$$S_x = \frac{6\pi}{5} u^5 \Big|_{x=0}^{x=\frac{\pi}{2}}$$

Back-substituting for u, we get

$$S_x = \frac{6\pi}{5} \sin^5 t \Big|_0^{\frac{\pi}{2}}$$

$$S_x = \left(\frac{6\pi}{5} \sin^5 \frac{\pi}{2}\right) - \left(\frac{6\pi}{5} \sin^5 0\right)$$

$$S_x = \frac{6\pi}{5}(1)^5 - \frac{6\pi}{5}(0)^5$$

$$S_x = \frac{6\pi}{5}$$