Calculus 1 Workbook Solutions

Definition of the derivative

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DEFINITION OF THE DERIVATIVE

■ 1. Use the definition of the derivative to find the derivative of $f(x)=2x^2+2x-12$ at $(4,28)$.

Solution:

At $(4,28)$, the derivative will be

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$f'(4)=\lim_{h\to 0}\frac{f(4+h)-f(4)}{h}$$

$$f'(4)=\lim_{h\to 0}\frac{(2(4+h)^2+2(4+h)-12)-(2(4)^2+2(4)-12)}{h}$$

$$f'(4)=\lim_{h\to 0}\frac{(2(16+8h+h^2)+8+2h-12)-(32+8-12)}{h}$$

$$f'(4)=\lim_{h\to 0}\frac{32+16h+2h^2+8+2h-12-32-8+12}{h}$$

$$f'(4)=\lim_{h\to 0}\frac{18h+2h^2}{h}$$

$$f'(4)=\lim_{h\to 0}(18+2h)$$

Evaluate the limit to find the derivative at $(4,28)$.

$$f'(4) = 18 + 2(0)$$

$$f'(4) = 18$$

■ 2. Use the definition of the derivative to find the derivative of $g(x) = 3x^3 - 4x + 7$ at $(-2, -9)$.

Solution:

At $(-2, -9)$, the derivative will be

$$g'(x) = \lim_{h \to 0} \frac{g(x+h) - g(x)}{h}$$

$$g'(-2) = \lim_{h \to 0} \frac{g(-2+h) - g(-2)}{h}$$

$$g'(-2) = \lim_{x \to 0} \frac{(3(-2+h)^3 - 4(-2+h) + 7) - (3(-2)^3 - 4(-2) + 7)}{h}$$

$$g'(-2) = \lim_{x \to 0} \frac{(3(-8 + 4h + 8h - 4h^2 - 2h^2 + h^3) + 8 - 4h + 7) - (-24 + 8 + 7)}{h}$$

$$g'(-2) = \lim_{x \to 0} \frac{-24 + 12h + 24h - 12h^2 - 6h^2 + 3h^3 + 8 - 4h + 7 + 24 - 8 - 7}{h}$$

$$g'(-2) = \lim_{x \to 0} \frac{3h^3 - 18h^2 + 32h}{h}$$

$$g'(-2) = \lim_{x \to 0} (3h^2 - 18h + 32)$$

Evaluate the limit to find the derivative at $(-2, -9)$.

$$g'(-2) = 3(0)^2 - 18(0) + 32$$

$$g'(-2) = 32$$

■ 3. Use the definition of the derivative to find the derivative at $(-1, -1)$.

$$f(x) = \frac{x}{x+2}$$

Solution:

At $(-1, -1)$, the derivative will be

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$f'(-1) = \lim_{h \to 0} \frac{f(-1+h) - f(-1)}{h}$$

$$f'(-1) = \lim_{h \to 0} \frac{\frac{-1+h}{-1+h+2} - \frac{-1}{-1+2}}{h}$$

$$f'(-1) = \lim_{h \to 0} \frac{\frac{-1+h}{h+1} - \frac{-1}{1}}{h}$$

$$f'(-1) = \lim_{h \to 0} \frac{\frac{-1+h}{h+1} + 1}{h}$$

$$f'(-1) = \lim_{h \to 0} \frac{\frac{-1+h+h+1}{h+1}}{h}$$

$$f'(-1) = \lim_{h \to 0} \frac{2h}{h(h+1)}$$

$$f'(-1) = \lim_{h \to 0} \frac{2}{h+1}$$

Evaluate the limit to find the derivative at $(-1, -1)$.

$$f'(-1) = \frac{2}{0+1}$$

$$f'(-1) = 2$$

■ 4. Use the definition of the derivative to find the derivative of $f(x) = \sqrt{5x-4}$ at $x = 4$.

Solution:

At $x = 4$, the derivative will be

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$f'(4) = \lim_{h \to 0} \frac{f(4+h) - f(4)}{h}$$

$$f'(4) = \lim_{h \to 0} \frac{\sqrt{5(4+h)-4} - \sqrt{5(4)-4}}{h}$$

$$f'(4)=\lim_{h\to 0}\frac{\sqrt{20+5h-4}-\sqrt{20-4}}{h}$$

$$f'(4)=\lim_{h\to 0}\frac{\sqrt{16+5h}-\sqrt{16}}{h}$$

$$f'(4)=\lim_{h\to 0}\frac{\sqrt{16+5h}-4}{h}$$

Use conjugate method.

$$f'(4)=\lim_{h\to 0}\frac{\sqrt{16+5h}-4}{h}\cdot\frac{\sqrt{16+5h}+4}{\sqrt{16+5h}+4}$$

$$f'(4)=\lim_{h\to 0}\frac{16+5h-16}{h(\sqrt{16+5h}+4)}$$

$$f'(4)=\lim_{h\to 0}\frac{5h}{h(\sqrt{16+5h}+4)}$$

$$f'(4)=\lim_{h\to 0}\frac{5}{\sqrt{16+5h}+4}$$

Evaluate the limit to find the derivative at $x=4$.

$$f'(4)=\frac{5}{\sqrt{16+5(0)}+4}$$

$$f'(4)=\frac{5}{4+4}$$

$$f'(4) = \frac{5}{8}$$

■ 5. Use the definition of the derivative to find the derivative of $g(x) = \cos(x-1)$ at $x = \pi/2$.

Solution:

At $x = \pi/2$, the derivative will be

$$g'(x) = \lim_{h \to 0} \frac{g(x+h) - g(x)}{h}$$

$$g'(x) = \lim_{h \to 0} \frac{\cos(x+h-1) - \cos(x-1)}{h}$$

$$g'(x) = \lim_{h \to 0} \frac{\cos(x-1+h) - \cos(x-1)}{h}$$

Apply the sum-identity for cosine.

$$g'(x) = \lim_{h \to 0} \frac{\cos(x-1)\cos h - \sin(x-1)\sin h - \cos(x-1)}{h}$$

$$g'(x) = \lim_{h \to 0} \frac{\cos(x-1)(\cos h - 1) - \sin(x-1)\sin h}{h}$$

$$g'(x) = \lim_{h \to 0} \left(\frac{\cos(x-1)(\cos h - 1)}{h} - \frac{\sin(x-1)\sin h}{h} \right)$$

Pull the expressions with x out in front of the limits, which only apply to h.

$$g'(x) = \cos(x-1)\lim_{h\to 0}\frac{\cos h - 1}{h} - \sin(x-1)\lim_{h\to 0}\frac{\sin h}{h}$$

Applying formulas for the limits of these common trigonometric functions, we get

$$g'(x) = \cos(x-1)(0) - \sin(x-1)(1)$$

$$g'(x) = -\sin(x-1)$$

Evaluate the limit to find the derivative at $x = \pi/2$.

$$g'\left(\frac{\pi}{2}\right) = -\sin\left(\frac{\pi}{2} - 1\right)$$

If we choose, we could apply the cofunction identity for cosine,

$$\cos\theta = \sin\left(\frac{\pi}{2} - \theta\right)$$

Using that identity allows us to simplify the answer to

$$g'\left(\frac{\pi}{2}\right) = -\cos(1)$$

■ 6. Use the definition of the derivative to find the derivative of $g(x) = |x|$ at $x = 0$.

Solution:

At $x = 0$, the derivative will be

$$g'(x) = \lim_{h \to 0} \frac{g(x+h) - g(x)}{h}$$

$$g'(0) = \lim_{h \to 0} \frac{g(0+h) - g(0)}{h}$$

$$g'(0) = \lim_{h \to 0} \frac{|h| - |0|}{h}$$

$$g'(0) = \lim_{h \to 0} \frac{|h|}{h}$$

We know that

$$|h| = \begin{cases} -h & h < 0 \\ h & h \geq 0 \end{cases}$$

$$\lim_{h \to 0^-} \frac{|h|}{h} = \lim_{h \to 0^-} \frac{-h}{h} = -1$$

$$\lim_{h \to 0^+} \frac{|h|}{h} = \lim_{h \to 0^+} \frac{h}{h} = 1$$

Since the left- and right-hand limits aren't equal, this limit doesn't exist, which means the function isn't differentiable at $x = 0$, and we can't find its derivative there.

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