Area under a parametric curve

Given a parametric curve where our function is defined by two equations, one for x and one for y, and both of them in terms of a parameter t,

$$x = f(t)$$

$$y = g(t)$$

we'll find the area under the curve using the integral formula

$$A = \int_{\alpha}^{\beta} y(t)x'(t)\ dt$$

where A is the area under the curve, $y(t)$ is $y = g(t)$, and $x'(t)$ is the derivative of $x = f(t)$.

Keep in mind as you're working these kinds of problems that this area formula won't give us a real-number answer. Instead, it'll give us a function that represents the area under any part of the parametric curve. In order to find a number value for the area, we'll have to use a definite integral by defining an interval for the area.

Example

Find the function that defines the area under the parametric curve.

$$x = 2\theta - \cos\theta$$

$$y = 2 + \sin\theta$$

Don't be confused by the fact that the parameter is θ instead of t. It's still a parameter value, because x and y are both defined in terms of θ.

We've already been given $y(\theta)$, but we need to find $x'(\theta)$ before we can plug into the area formula.

$$x = 2\theta - \cos\theta$$

$$x'(\theta) = 2 + \sin\theta$$

Plugging $y(\theta)$ and $x'(\theta)$ into the area formula, we get

$$A = \int (2 + \sin\theta)(2 + \sin\theta)\ d\theta$$

$$A = \int 4 + 4\sin\theta + \sin^2\theta\ d\theta$$

Using the formula

$$\sin^2\theta = \frac{1}{2}(1 - \cos 2\theta)$$

we'll make a substitution for $\sin^2\theta$.

$$A = \int 4 + 4\sin\theta + \frac{1}{2}(1 - \cos 2\theta)\ d\theta$$

$$A = \int 4 + 4\sin\theta + \frac{1}{2} - \frac{1}{2}\cos 2\theta\ d\theta$$

$$A = \int \frac{9}{2} + 4\sin\theta - \frac{1}{2}\cos 2\theta\ d\theta$$

$$A = \int \frac{9}{2}\ d\theta + \int 4 \sin \theta\ d\theta - \int \frac{1}{2} \cos 2\theta\ d\theta$$

$$A = \frac{9}{2}\theta - 4 \cos \theta - \frac{1}{4} \sin 2\theta$$