Moments and center of mass of the system

The center of mass of a region is the single point where the system is balanced. In other words, if you could take the region into physical space and set it on a pencil point, there's one point in the region where it would balance on that point. Setting it on the pencil at any other point will make the system fall to one side or the another.

With this idea in mind, realize that a perfectly symmetrical region, like a square or a circle, has it's center or mass right in the center. Any region which is not symmetrical (asymmetrical), will have a center of mass closer to the larger part of the region.

When we're looking for the center of mass of a region, we'll first calculate the area bounded by the two curves that define the region, and by the given interval. Remember, if the problem doesn't specify an interval, you'll need to use the points of intersection of the curves by setting them equal to one another and solving for x.

The equation for this area is

[A] $$A = \int_a^b f(x) - g(x)\ dx$$

Next we'll find the moments of the system, which are values that tell us how easily the function can be rotated around the x- and y-axes. We'll calculate the moment of the system in the x direction, and the moment in the y direction using

[B] $M_x = \int_a^b \frac{1}{2}\left(\left[f(x)\right]^2 - \left[g(x)\right]^2\right)\ dx$

[C] $M_y = \int_a^b x\left[f(x) - g(x)\right]\ dx$

Once we have area and the moments of the system, we need to use both to calculate the coordinates of the center of mass using

[D] $\bar{x} = \frac{M_y}{A} = \frac{1}{A}\int_a^b x\left[f(x) - g(x)\right]\ dx$

[E] $\bar{y} = \frac{M_x}{A} = \frac{1}{A}\int_a^b \frac{1}{2}\left(\left[f(x)\right]^2 - \left[g(x)\right]^2\right)\ dx$

Example

Find the center of mass of the region bounded by the curves over the interval $[0,2]$.

$$y = x^4$$

$$y = 0$$

We'll solve for the area of the region using formula **[A]**.

$$A = \int_0^2 x^4 - 0\ dx$$

$$A = \int_0^2 x^4 \, dx$$

$$A = \frac{1}{5}x^5 \Big|_0^2$$

$$A = \frac{1}{5}(2)^5 - \frac{1}{5}(0)^5$$

$$A = \frac{32}{5}$$

Now we'll find moments of the system. Using formula [B] to find M_x, we get

$$M_x = \int_0^2 \frac{1}{2}\left[\left(x^4\right)^2 - (0)^2\right] \, dx$$

$$M_x = \int_0^2 \frac{1}{2}x^8 \, dx$$

$$M_x = \frac{1}{2}\int_0^2 x^8 \, dx$$

$$M_x = \frac{x^9}{18}\Big|_0^2$$

$$M_x = \frac{2^9}{18} - \frac{0^9}{18}$$

$$M_x = \frac{512}{18}$$

$$M_x = \frac{256}{9}$$

Using formula [C] to find M_y, we get

$$M_y = \int_0^2 x\left(x^4 - 0\right)\ dx$$

$$M_y = \int_0^2 x^5\ dx$$

$$M_y = \frac{x^6}{6}\Big|_0^2$$

$$M_y = \frac{(2)^6}{6} - \frac{(0)^6}{6}$$

$$M_y = \frac{64}{6}$$

$$M_y = \frac{32}{3}$$

With area and moments of the system, we can find the coordinates of the center or mass, $(\bar{x}, \bar{y})$.

Using formula [D] to find $\bar{x}$, we get

$$\bar{x} = \frac{M_y}{A}$$

$$\bar{x} = \frac{\frac{32}{3}}{\frac{32}{5}}$$

$$\bar{x} = \left(\frac{32}{3}\right)\left(\frac{5}{32}\right)$$

$$\bar{x} = \frac{5}{3}$$

Using formula **[E]** to find $\bar{y}$, we get

$$\bar{y} = \frac{M_x}{A}$$

$$\bar{y} = \frac{\frac{256}{9}}{\frac{32}{5}}$$

$$\bar{y} = \left(\frac{256}{9}\right)\left(\frac{5}{32}\right)$$

$$\bar{y} = \frac{40}{9}$$

The center of mass of the region bounded by $y = x^4$ and $y = 0$ on the interval $[0,2]$ is

$$(\bar{x}, \bar{y}) = \left(\frac{5}{3}, \frac{40}{9}\right)$$
