Topic: Sum of a telescoping series

Question: Calculate the sum of the telescoping series.

$$\sum_{n=1}^{\infty} \frac{3}{9n^2 - 3n - 2}$$

Answer choices:

A 3

B 2

C 1

D 0

Solution: C

The sum of a telescoping series is given by the formula

$$\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} s_n$$

We know that s_n is the series of partial sums, so we can say that the sum of the telescoping series a_n is the limit as $n \to \infty$ of its corresponding series of partial sums s_n.

Before we get to the telescoping series, we'll first decompose the rational expression into simpler fractions using partial fractions.

$$\frac{3}{9n^2 - 3n - 2} = \frac{3}{(3n-2)(3n+1)}$$

Because $(3n-2)$ and $(3n+1)$ are distinct linear factors, the decomposition will be

$$\frac{3}{(3n-2)(3n+1)} = \frac{A}{3n-2} + \frac{B}{3n+1}$$

$$\left[\frac{3}{(3n-2)(3n+1)} = \frac{A}{3n-2} + \frac{B}{3n+1}\right](3n-2)(3n+1)$$

$$3 = A(3n+1) + B(3n-2)$$

$$3 = 3An + A + 3Bn - 2B$$

$$3 = (3An + 3Bn) + (A - 2B)$$

$$3 = (3A + 3B)n + (A - 2B)$$

Equating coefficients on both sides, we get the simultaneous equations

$3A + 3B = 0$ and $A - 2B = 3$

$A = 3 + 2B$

$3(3 + 2B) + 3B = 0$

$9 + 6B + 3B = 0$

$9 + 9B = 0$

$9B = -9$

$B = -1$

$A = 3 + 2(-1)$

$A = 3 - 2$

$A = 1$

Having solved for our constants A and B, we'll plug them into our partial fractions decomposition.

$$\frac{3}{9n^2 - 3n - 2} = \frac{A}{3n - 2} + \frac{B}{3n + 1}$$

$$\frac{3}{9n^2 - 3n - 2} = \frac{1}{3n - 2} + \frac{-1}{3n + 1}$$

$$\frac{3}{9n^2 - 3n - 2} = \frac{1}{3n - 2} - \frac{1}{3n + 1}$$

Plugging the decomposition back into the summation notation, we get

$$\sum_{n=1}^{\infty} \frac{1}{3n-2} - \frac{1}{3n+1}$$

Now we'll start working on the convergence of this series by first expanding the series. Let's use $n = 1$, $n = 2$, $n = 3$ and $n = 4$.

$$n = 1 \qquad a_1 = \frac{1}{3(1)-2} - \frac{1}{3(1)+1} = \frac{1}{1} - \frac{1}{4}$$

$$n = 2 \qquad a_2 = \frac{1}{3(2)-2} - \frac{1}{3(2)+1} = \frac{1}{4} - \frac{1}{7}$$

$$n = 3 \qquad a_3 = \frac{1}{3(3)-2} - \frac{1}{3(3)+1} = \frac{1}{7} - \frac{1}{10}$$

$$n = 4 \qquad a_4 = \frac{1}{3(4)-2} - \frac{1}{3(4)+1} = \frac{1}{10} - \frac{1}{13}$$

Writing these terms into our expanded series and including the last term of the series, we get

$$\sum_{n=1}^{\infty} \frac{1}{3n-2} - \frac{1}{3n+1} = \left(\frac{1}{1} - \frac{1}{4}\right) + \left(\frac{1}{4} - \frac{1}{7}\right) + \left(\frac{1}{7} - \frac{1}{10}\right)$$

$$+ \left(\frac{1}{10} - \frac{1}{13}\right) + \dots + \left(\frac{1}{3n-2} - \frac{1}{3n+1}\right)$$

Canceling everything but the first half of the first term and the second half of the last term gives an expression for the series of partial sums.

$$s_n = \frac{1}{1} - \frac{1}{3n+1}$$

$$s_n = 1 - \frac{1}{3n+1}$$

To find the sum of the telescoping series, we'll take the limit as $n \to \infty$ of the series or partial sums s_n.

$$\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} s_n$$

$$\sum_{n=1}^{\infty} \frac{1}{3n-2} - \frac{1}{3n+1} = \lim_{n\to\infty} 1 - \frac{1}{3n+1}$$

$$\sum_{n=1}^{\infty} \frac{1}{3n-2} - \frac{1}{3n+1} = 1 - \frac{1}{3(\infty)+1}$$

$$\sum_{n=1}^{\infty} \frac{1}{3n-2} - \frac{1}{3n+1} = 1 - \frac{1}{\infty+1}$$

$$\sum_{n=1}^{\infty} \frac{1}{3n-2} - \frac{1}{3n+1} = 1 - \frac{1}{\infty}$$

$$\sum_{n=1}^{\infty} \frac{1}{3n-2} - \frac{1}{3n+1} = 1 - 0$$

$$\sum_{n=1}^{\infty} \frac{1}{3n-2} - \frac{1}{3n+1} = 1$$

The sum of the series is 1.

Topic: Sum of a telescoping series

Question: Calculate the sum of the telescoping series.

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n+3}} - \frac{1}{\sqrt{n+4}}$$

Answer choices:

A $\frac{2}{3}$

B 1

C 0

D $\frac{1}{2}$

Solution: D

The sum of a telescoping series is given by the formula

$$\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} s_n$$

We know that s_n is the series of partial sums, so we can say that the sum of the telescoping series a_n is the limit as $n \to \infty$ of its corresponding series of partial sums s_n.

We'll start by expanding the series. Let's use $n = 1$, $n = 2$, $n = 3$ and $n = 4$.

$$n = 1 \qquad a_1 = \frac{1}{\sqrt{1+3}} - \frac{1}{\sqrt{1+4}} = \frac{1}{\sqrt{4}} - \frac{1}{\sqrt{5}}$$

$$n = 2 \qquad a_2 = \frac{1}{\sqrt{2+3}} - \frac{1}{\sqrt{2+4}} = \frac{1}{\sqrt{5}} - \frac{1}{\sqrt{6}}$$

$$n = 3 \qquad a_3 = \frac{1}{\sqrt{3+3}} - \frac{1}{\sqrt{3+4}} = \frac{1}{\sqrt{6}} - \frac{1}{\sqrt{7}}$$

$$n = 4 \qquad a_4 = \frac{1}{\sqrt{4+3}} - \frac{1}{\sqrt{4+4}} = \frac{1}{\sqrt{7}} - \frac{1}{\sqrt{8}}$$

Writing these terms into our expanded series and including the last term of the series, we get

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n+3}} - \frac{1}{\sqrt{n+4}} = \left(\frac{1}{\sqrt{4}} - \frac{1}{\sqrt{5}}\right) + \left(\frac{1}{\sqrt{5}} - \frac{1}{\sqrt{6}}\right) + \left(\frac{1}{\sqrt{6}} - \frac{1}{\sqrt{7}}\right)$$

$$+\left(\frac{1}{\sqrt{7}}-\frac{1}{\sqrt{8}}\right)+\ldots+\left(\frac{1}{\sqrt{n+3}}-\frac{1}{\sqrt{n+4}}\right)$$

Canceling everything but the first half of the first term and the second half of the last term gives an expression for the series of partial sums.

$$s_n=\frac{1}{\sqrt{4}}-\frac{1}{\sqrt{n+4}}$$

$$s_n=\frac{1}{2}-\frac{1}{\sqrt{n+4}}$$

To find the sum of the telescoping series, we'll take the limit as $n \to \infty$ of the series or partial sums s_n.

$$\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} s_n$$

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n+3}}-\frac{1}{\sqrt{n+4}} = \lim_{n\to\infty} \frac{1}{2}-\frac{1}{\sqrt{n+4}}$$

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n+3}}-\frac{1}{\sqrt{n+4}} = \frac{1}{2}-\frac{1}{\sqrt{\infty+4}}$$

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n+3}}-\frac{1}{\sqrt{n+4}} = \frac{1}{2}-\frac{1}{\sqrt{\infty}}$$

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n+3}}-\frac{1}{\sqrt{n+4}} = \frac{1}{2}-\frac{1}{\infty}$$

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n+3}}-\frac{1}{\sqrt{n+4}} = \frac{1}{2}-0$$

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n+3}} - \frac{1}{\sqrt{n+4}} = \frac{1}{2}$$

The sum of the series is $1/2$.

Topic: Sum of a telescoping series

Question: Calculate the sum of the telescoping series.

$$\sum_{n=1}^{\infty} 3^{\frac{1}{n}} - 3^{\frac{1}{n+1}}$$

Answer choices:

A $\quad 2$

B $\quad 3$

C $\quad \infty$

D $\quad$ The series diverges

Solution: A

The sum of a telescoping series is given by the formula

$$\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} s_n$$

We know that s_n is the series of partial sums, so we can say that the sum of the telescoping series a_n is the limit as $n \to \infty$ of its corresponding series of partial sums s_n.

We'll start by expanding the series. Let's use $n = 1$, $n = 2$, $n = 3$ and $n = 4$.

$$n = 1 \qquad a_1 = 3^{\frac{1}{1}} - 3^{\frac{1}{1+1}} = 3 - \sqrt{3}$$

$$n = 2 \qquad a_2 = 3^{\frac{1}{2}} - 3^{\frac{1}{2+1}} = \sqrt{3} - \sqrt[3]{3}$$

$$n = 3 \qquad a_3 = 3^{\frac{1}{3}} - 3^{\frac{1}{3+1}} = \sqrt[3]{3} - \sqrt[4]{3}$$

$$n = 4 \qquad a_4 = 3^{\frac{1}{4}} - 3^{\frac{1}{4+1}} = \sqrt[4]{3} - \sqrt[5]{3}$$

Writing these terms into our expanded series and including the last term of the series, we get

$$\sum_{n=1}^{\infty} 3^{\frac{1}{n}} - 3^{\frac{1}{n+1}} = \left(3 - \sqrt{3}\right) + \left(\sqrt{3} - \sqrt[3]{3}\right) + \left(\sqrt[3]{3} - \sqrt[4]{3}\right)$$

$$+\left(\sqrt[4]{3} - \sqrt[5]{3}\right) + \dots + \left(3^{\frac{1}{n}} - 3^{\frac{1}{n+1}}\right)$$

Canceling everything but the first half of the first term and the second half of the last term gives an expression for the series of partial sums.

$$s_n = 3 - 3^{\frac{1}{n+1}}$$

To find the sum of the telescoping series, we'll take the limit as $n \to \infty$ of the series or partial sums s_n.

$$\sum_{n=1}^{\infty} a_n = \lim_{n \to \infty} s_n$$

$$\sum_{n=1}^{\infty} 3^{\frac{1}{n}} - 3^{\frac{1}{n+1}} = \lim_{n \to \infty} 3 - 3^{\frac{1}{n+1}}$$

$$\sum_{n=1}^{\infty} 3^{\frac{1}{n}} - 3^{\frac{1}{n+1}} = 3 - 3^{\frac{1}{\infty+1}}$$

$$\sum_{n=1}^{\infty} 3^{\frac{1}{n}} - 3^{\frac{1}{n+1}} = 3 - 3^0$$

$$\sum_{n=1}^{\infty} 3^{\frac{1}{n}} - 3^{\frac{1}{n+1}} = 3 - 1$$

$$\sum_{n=1}^{\infty} 3^{\frac{1}{n}} - 3^{\frac{1}{n+1}} = 2$$

The sum of the series is 2.