Topic: Area bounded by one loop of a polar curve

Question: Find the area bounded by one loop of the polar curve.

$$r = 5\cos 3\theta$$

Answer choices:

A $\frac{25}{12}$

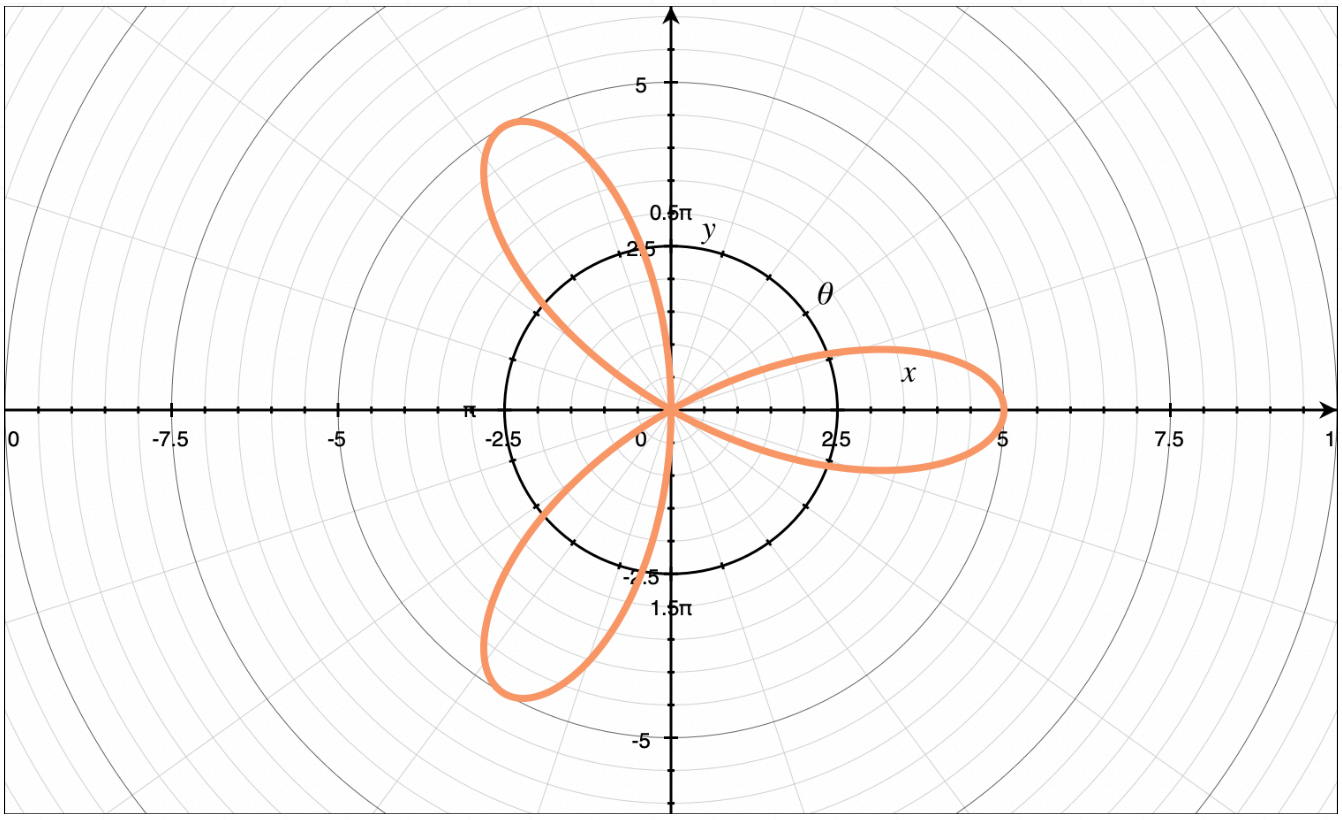
B $\frac{25\pi}{2}$

C $\frac{25\pi}{12}$

D π

Solution: C

The graph of the polar curve looks like this:



The graph of the polar curve has three loops and the best loop to consider is the one that lies on the positive x-axis. The loop is symmetric about the x-axis, so we'll consider only the top half for integration, and then we'll double that area. So our area formula is

$$A = 2\left(\frac{1}{2}\int_{\alpha}^{\beta} r^2 \ d\theta\right) = \int_{\alpha}^{\beta} r^2 \ d\theta$$

To get the limits of integration, we begin substituting values for θ and solve for r.

When $\theta = 0$, $r = 5$. When $\theta = \pi/6$, the polar curve loops back to the origin, so $r = 0$. Therefore, the limits of integration are $\alpha = 0$ and $\beta = \pi/6$.

$$A=\int_0^{\frac{\pi}{6}}(5\cos 3\theta)^2\ d\theta$$

$$A=25\int_0^{\frac{\pi}{6}}\cos^2 3\theta\ d\theta$$

Using the power reduction formula

$$\cos^2\theta=\frac{1}{2}+\frac{1}{2}\cos 2\theta$$

we get

$$A=25\int_0^{\frac{\pi}{6}}\frac{1}{2}+\frac{1}{2}\cos 6\theta\ d\theta$$

$$A=25\left(\frac{1}{2}\theta+\frac{1}{12}\sin 6\theta\right)\Bigg|_0^{\frac{\pi}{6}}$$

$$A=25\left[\left(\frac{1}{2}\left(\frac{\pi}{6}\right)+\frac{1}{12}\sin 6\left(\frac{\pi}{6}\right)\right)-\left(\frac{1}{2}(0)+\frac{1}{12}\sin 6(0)\right)\right]$$

$$A=25\left[\left(\frac{\pi}{12}+\frac{1}{12}(0)\right)-\left(0+\frac{1}{12}(0)\right)\right]$$

$$A=\frac{25\pi}{12}$$

Topic: Area bounded by one loop of a polar curve

Question: Find the area bounded by one loop of the polar curve.

$$r = \sin(2\theta)$$

Answer choices:

A $\frac{\pi}{16}$

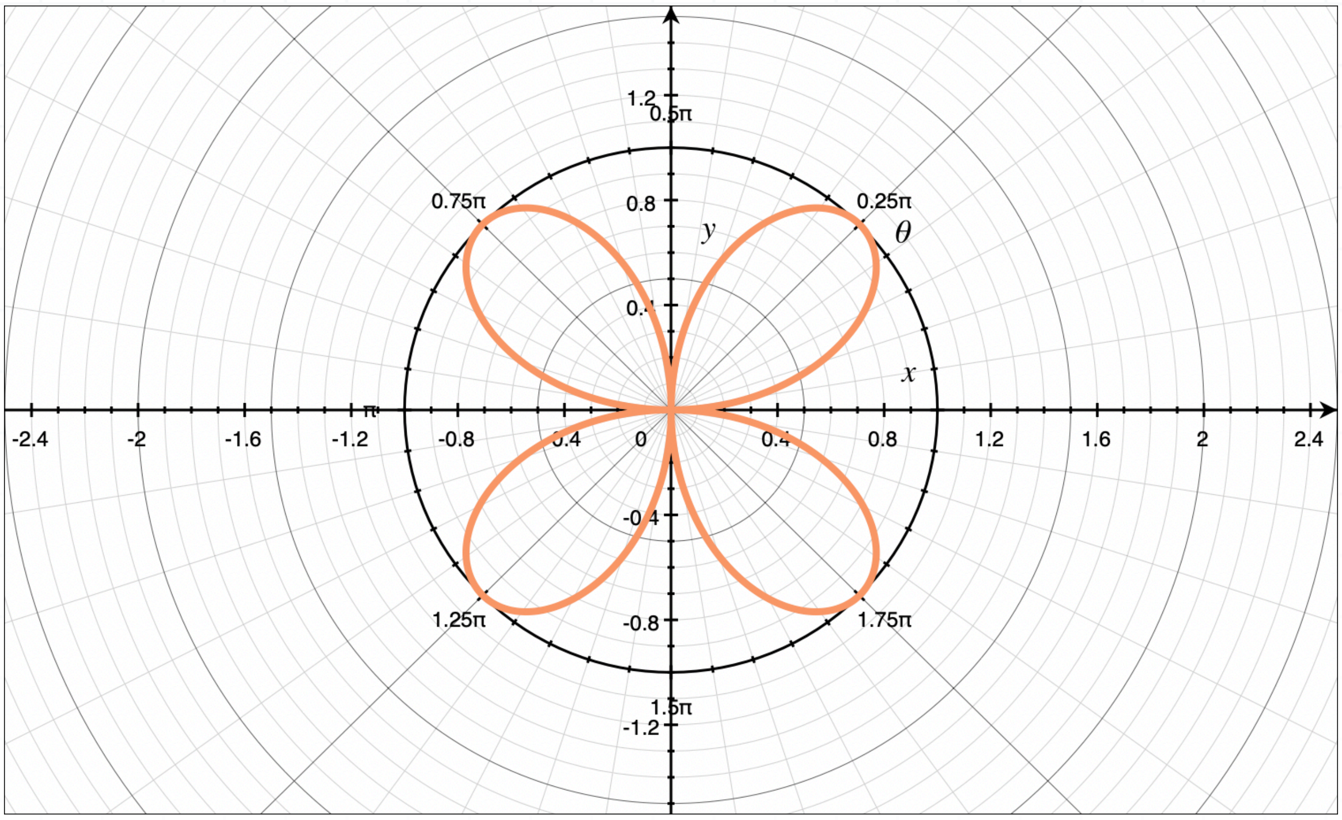
B $\frac{\pi}{2}$

C $\frac{\pi}{8}$

D $\frac{\pi}{4}$

Solution: C

The graph of the polar curve looks like this:



The graph of the polar curve has four loops and the best loop to consider is the one that lies in the first quadrant. The loop is symmetric about the line $\theta = \pi/4$, so we'll consider only the bottom half of that loop for integration (the part that lies below the line $\theta = \pi/4$), and then we'll double that area. So our area formula is

$$A = 2\left(\frac{1}{2}\int_{\alpha}^{\beta} r^2 \ d\theta\right) = \int_{\alpha}^{\beta} r^2 \ d\theta$$

To get the limits of integration, we begin substituting values for θ and solve for r.

When $\theta = 0$, $r = 0$. When $\theta = \pi/4$, the polar curve loops out to the tip of the first petal, so $r = 1$. Therefore, the limits of integration are $\alpha = 0$ and $\beta = \pi/4$.

$$A = \int_0^{\frac{\pi}{4}} (\sin 2\theta)^2 \ d\theta$$

$$A = \int_0^{\frac{\pi}{4}} \sin^2 2\theta \ d\theta$$

Using the power reduction formula

$$\sin^2 \theta = \frac{1}{2}(1 - \cos(2\theta))$$

we get

$$A = \int_0^{\frac{\pi}{4}} \frac{1}{2}(1 - \cos(2(2\theta))) \ d\theta$$

$$A = \frac{1}{2}\int_0^{\frac{\pi}{4}} 1 - \cos(4\theta) \ d\theta$$

We'll integrate, then evaluate over the interval.

$$A = \frac{1}{2}\left[\theta - \frac{1}{4}\sin(4\theta)\right]\Bigg|_0^{\frac{\pi}{4}}$$

$$A = \frac{1}{2}\left[\frac{\pi}{4} - \frac{1}{4}\sin\left(4 \cdot \frac{\pi}{4}\right)\right] - \frac{1}{2}\left[0 - \frac{1}{4}\sin(4 \cdot 0)\right]$$

$$A=\frac{1}{2}\left[\frac{\pi}{4}-\frac{1}{4}(0)\right]-\frac{1}{2}\left[0-\frac{1}{4}(0)\right]$$

$$A=\frac{1}{2}\left(\frac{\pi}{4}\right)$$

$$A=\frac{\pi}{8}$$

Topic: Area bounded by one loop of a polar curve

Question: Find the area bounded by one loop of the polar curve.

$$r = \cos(3\theta)$$

Answer choices:

A $\frac{\pi}{12}$

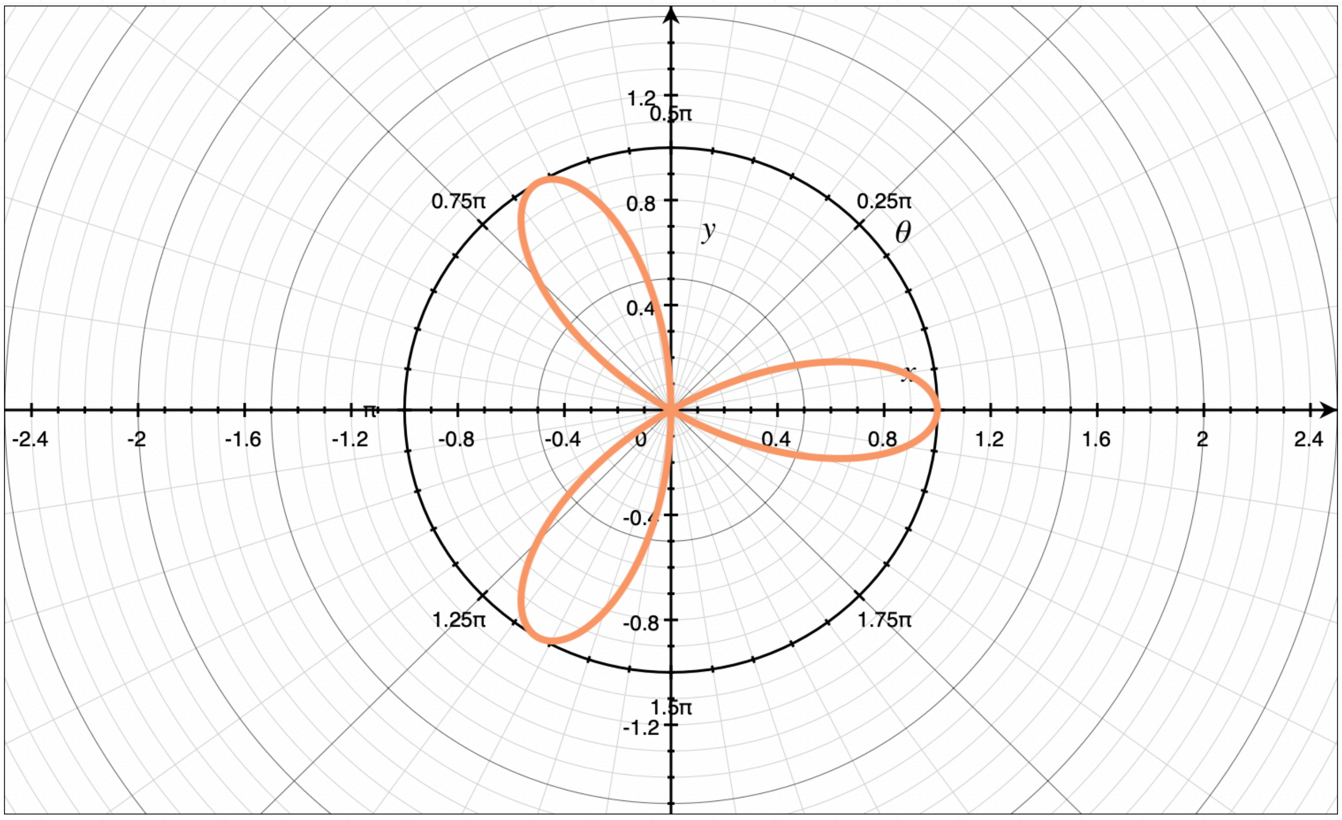
B $\frac{\pi}{10}$

C $\frac{\pi}{8}$

D $\frac{\pi}{14}$

Solution: A

The graph of the polar curve looks like this:



The graph of the polar curve has three loops and the best loop to consider is the one that straddles the positive side of the x-axis. The loop is symmetric about the axis, so we'll consider only the top half of that loop for integration (the part that lies above the line $\theta = 0$), and then we'll double that area. So our area formula is

$$A = 2\left(\frac{1}{2}\int_{\alpha}^{\beta} r^2\ d\theta\right) = \int_{\alpha}^{\beta} r^2\ d\theta$$

To get the limits of integration, we begin substituting values for θ and solve for r.

When $\theta = 0$, $r = 1$. When $\theta = \pi/6$, the polar curve loops out to the tip of the first petal, so $r = 0$. Therefore, the limits of integration are $\alpha = 0$ and $\beta = \pi/6$.

$$A = \int_0^{\frac{\pi}{6}} (\cos(3\theta))^2 \ d\theta$$

$$A = \int_0^{\frac{\pi}{6}} \cos^2(3\theta) \ d\theta$$

Using the power reduction formula

$$\cos^2\theta = \frac{1}{2} + \frac{1}{2}\cos 2\theta$$

we get

$$A = \int_0^{\frac{\pi}{6}} \frac{1}{2} + \frac{1}{2}\cos(3(2\theta)) \ d\theta$$

$$A = \frac{1}{2}\int_0^{\frac{\pi}{6}} 1 + \cos(6\theta) \ d\theta$$

We'll integrate, then evaluate over the interval.

$$A = \frac{1}{2}\left[\theta + \frac{1}{6}\sin(6\theta)\right]\Bigg|_0^{\frac{\pi}{6}}$$

$$A = \frac{1}{2}\left[\frac{\pi}{6} + \frac{1}{6}\sin\left(6 \cdot \frac{\pi}{6}\right)\right] - \frac{1}{2}\left[0 + \frac{1}{6}\sin(6(0))\right]$$

$$A=\frac{1}{2}\left[\frac{\pi}{6}+\frac{1}{6}(0)\right]-\frac{1}{2}\left[0+\frac{1}{6}(0)\right]$$

$$A=\frac{1}{2}\left(\frac{\pi}{6}\right)$$

$$A=\frac{\pi}{12}$$