Topic: Dividing the area between curves into equal parts

Question: The line $y = k$ divides the area bounded by the curves into two equal parts. Find k.

$$f(x) = x^2$$

$$g(x) = 25$$

Answer choices:

A $\quad k = \sqrt{\frac{25\sqrt[3]{2}}{2}}$

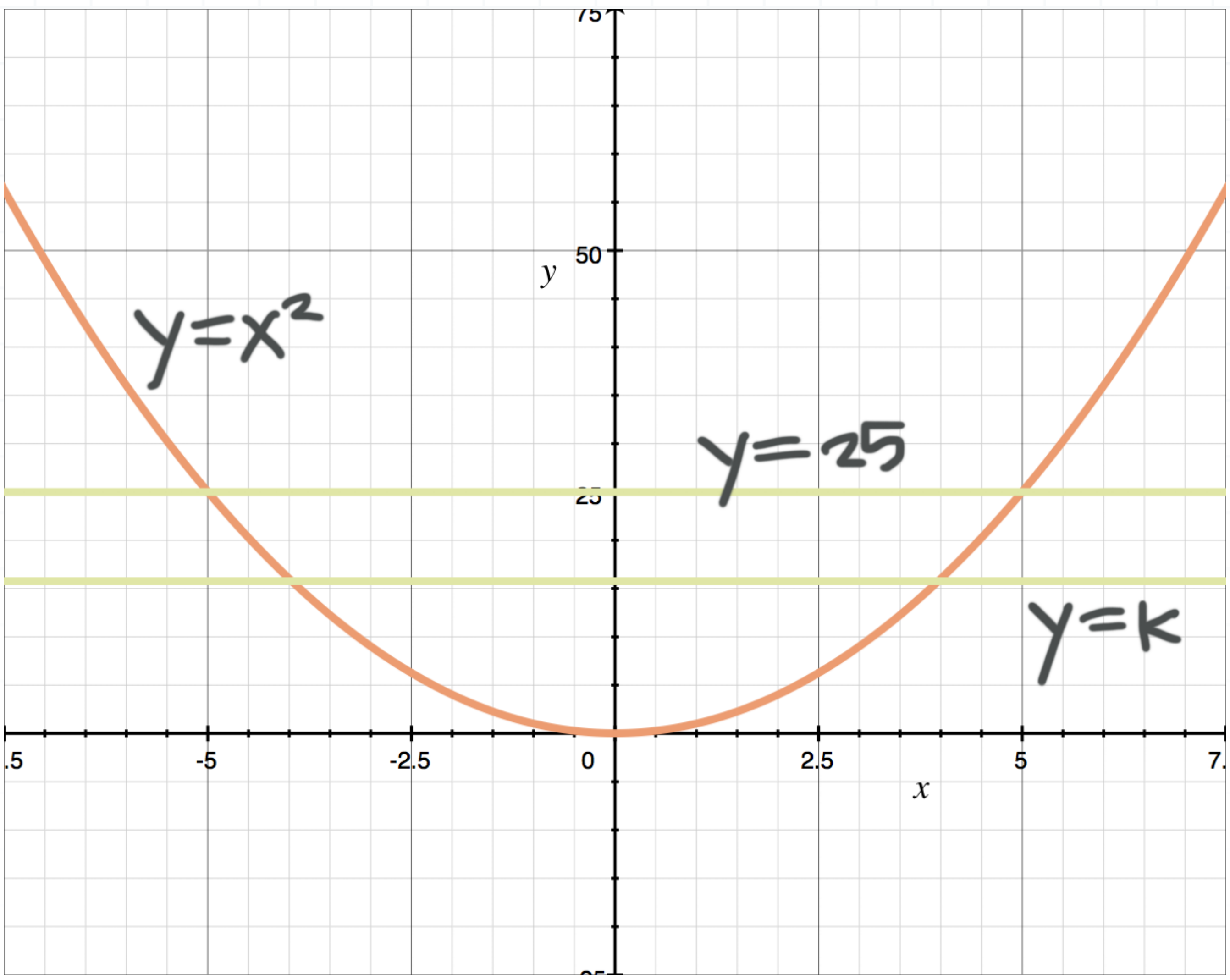
B $\quad k = -\frac{25\sqrt[3]{2}}{2}$

C $\quad k = -\sqrt{\frac{25\sqrt[3]{2}}{2}}$

D $\quad k = \frac{25\sqrt[3]{2}}{2}$

Solution: D

The graph of $f(x) = 25 - x^2$ with a line $y = k$ is



To answer the question, first, we will find the entire area of the bounded region. Notice from the graph that the interval on the x-axis for the bounded region is $[-5,5]$. Let's find the area of this region by integrating the upper function minus the lower function in that interval. You can see that $y = 25$ is the upper function and $y = x^2$ is the lower function.

$$\int_{-5}^{5} 25 - x^2 \, dx$$

$$25x - \frac{1}{3}x^3 \Big|_{-5}^{5}$$

$$25(5)-\frac{1}{3}(5)^3-\left(25(-5)-\frac{1}{3}(-5)^3\right)$$

$$125-\frac{125}{3}-\left(-125+\frac{125}{3}\right)$$

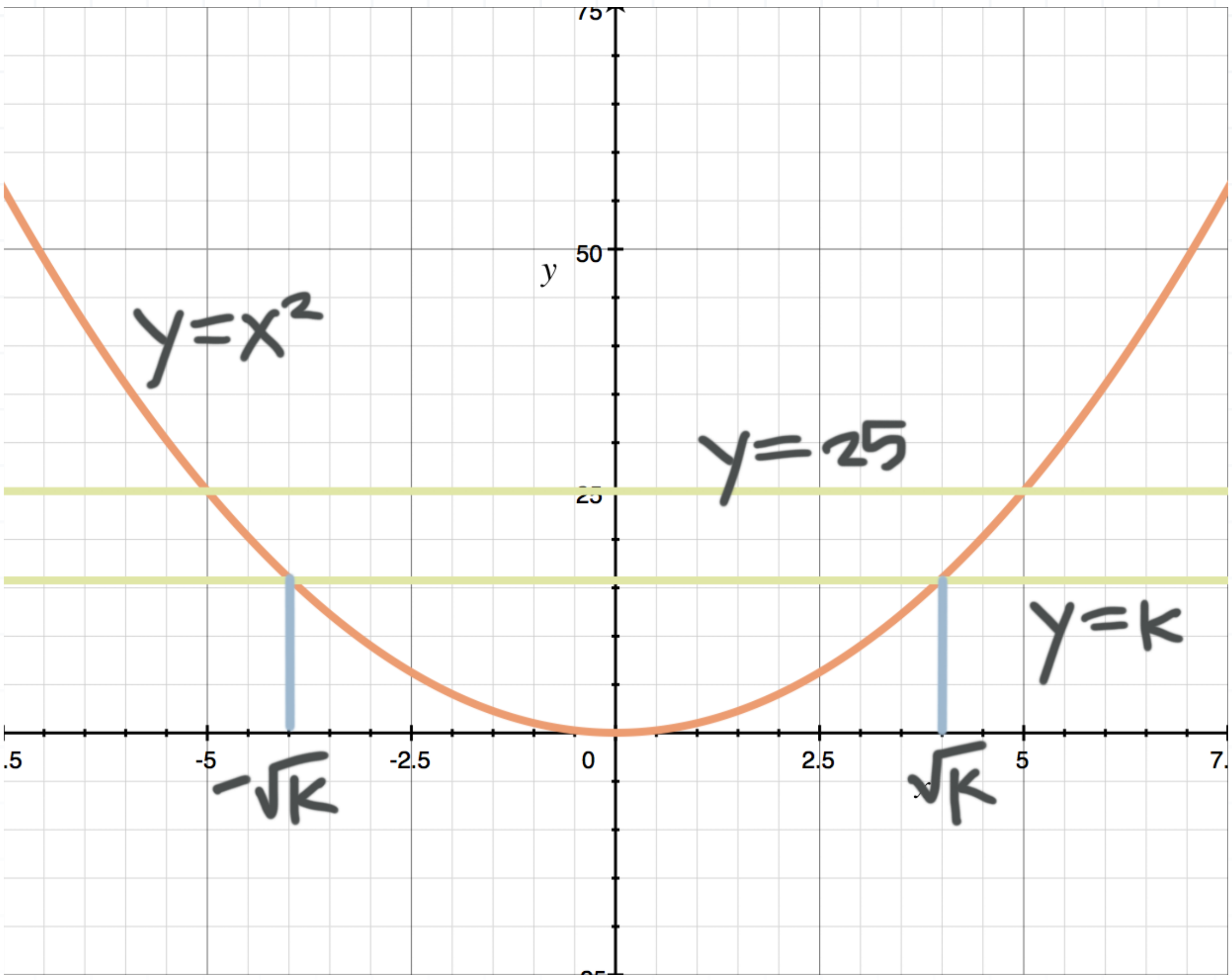
$$125-\frac{125}{3}+125-\frac{125}{3}$$

$$\frac{500}{3}$$

This is the area of the entire bounded region.

If we will find the value of k in the equation $y=k$ that divides the bounded region's area into two equal parts, we know that the two equal parts will have an area equal to $1/2$ of $500/3$, which is $250/3$.

The region that represents one-half of the bounded region is the portion of the original region that is below the line $y=k$ as shown in the graph below.



Notice from the graph that since the functions are $y = x^2$ and $y = k$, the points of intersection are $\left(-\sqrt{k}, k\right)$ and $\left(\sqrt{k}, k\right)$. Therefore, the interval of the integration will be $\left[-\sqrt{k}, \sqrt{k}\right]$.

Now, let's prepare an integral. Let's find the area of this region by integrating the upper function minus the lower function in that interval. You can see that $y = k$ is the upper function and $y = x^2$ is the lower function.

$$A = \int_{-\sqrt{k}}^{\sqrt{k}} k - x^2 \, dx$$

$$A = kx - \frac{1}{3}x^3 \Big|_{-\sqrt{k}}^{\sqrt{k}}$$

$$A = k\sqrt{k} - \frac{1}{3}\left(\sqrt{k}\right)^3 - \left(k\left(-\sqrt{k}\right) - \frac{1}{3}\left(-\sqrt{k}\right)^3\right)$$

$$A = k^{\frac{3}{2}} - \frac{1}{3}k^{\frac{3}{2}} - \left(-k^{\frac{3}{2}} + \frac{1}{3}k^{\frac{3}{2}}\right)$$

$$A = k^{\frac{3}{2}} - \frac{1}{3}k^{\frac{3}{2}} + k^{\frac{3}{2}} - \frac{1}{3}k^{\frac{3}{2}}$$

$$A = \frac{4}{3}k^{\frac{3}{2}}$$

Now recall that earlier we said that the area of this region had to be equal to $1/2$ of the area of the original bounded region. The area of the original bounded region was $500/3$. Which means the area A we found above must be $250/3$ square units.

$$\frac{4}{3}k^{\frac{3}{2}} = \frac{250}{3}$$

$$4k^{\frac{3}{2}} = 250$$

$$16k^3 = 250^2$$

$$k^3 = \frac{250^2}{16}$$

$$k = \sqrt[3]{\frac{250^2}{16}}$$

$$k = \sqrt[3]{\frac{25 \cdot 25 \cdot 10 \cdot 10}{16}}$$

$$k = \sqrt[3]{\frac{25 \cdot 25 \cdot 5 \cdot 5}{4}}$$

$$k = \frac{25}{\sqrt[3]{4}}$$

Rationalize the denominator.

$$k = \frac{25\sqrt[3]{16}}{4}$$

$$k = \frac{25\sqrt[3]{8}\sqrt[3]{2}}{4}$$

$$k = \frac{25\sqrt[3]{2}}{2}$$

Topic: Dividing the area between curves into equal parts

Question: The line $x = a$ divides the area bounded by the curves into two equal parts. Find a.

$$x = y^2$$

$$x = 4$$

Answer choices:

A $a = 2\sqrt{2}$

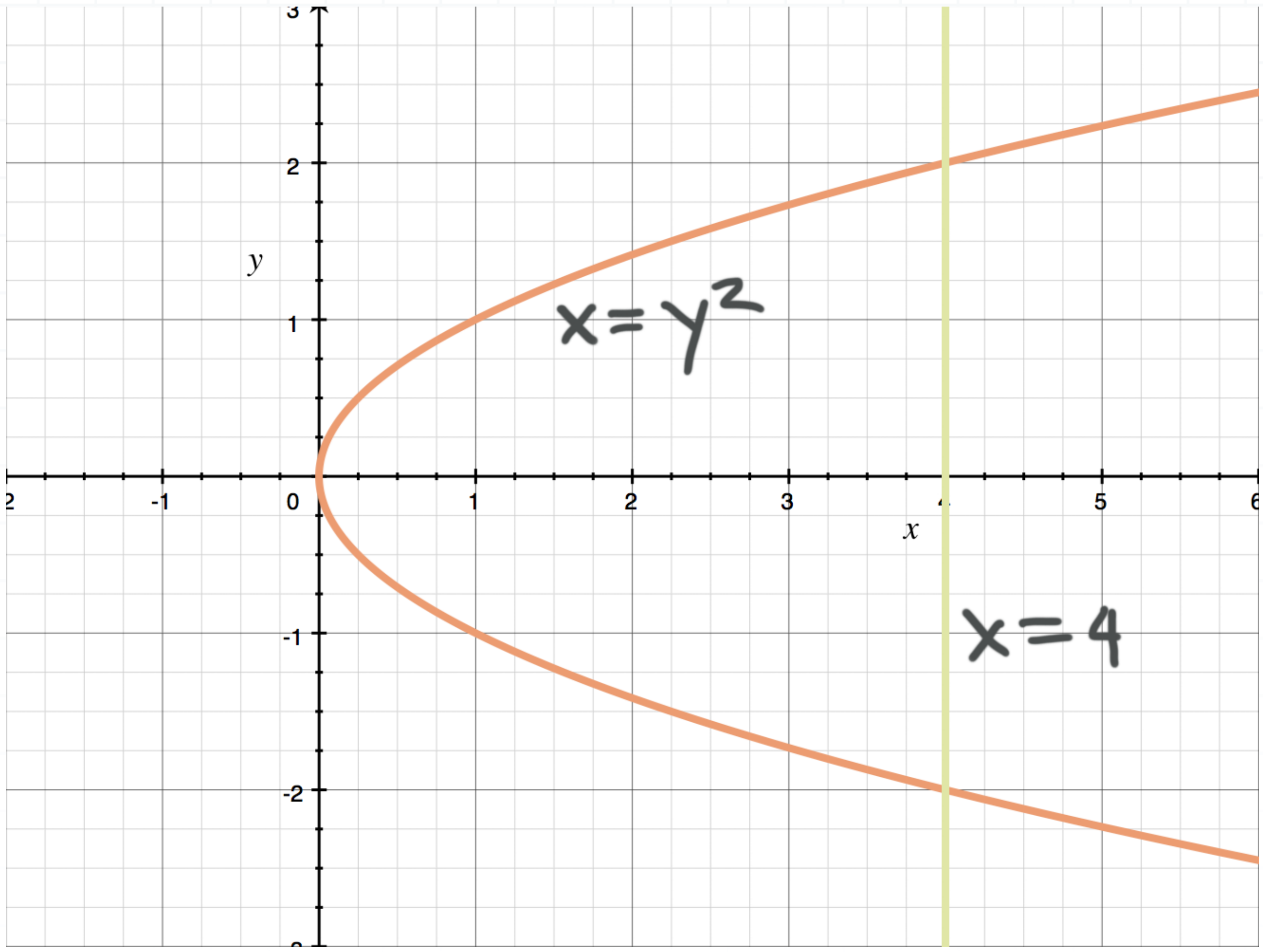
B $a = 2\sqrt[3]{2}$

C $a = 3\sqrt[3]{2}$

D $a = 3\sqrt{2}$

Solution: B

The graph of $x = y^2$ with a line $x = 4$ is



To answer the question, first, we will find the entire area of the bounded region. Since the equations in the question are not functions, and x is expressed in terms of y, we will integrate with respect to y.

Notice, from the graph that the interval on the y-axis for the bounded region is $[-2,2]$. Let's find the area of this region by integrating the right curve minus the left curve in that interval. You can see that $x = 4$ is the right curve and $x = y^2$ is the left curve.

$$\int_{-2}^{2} 4 - y^2 \, dy$$

Integrate, then evaluate over the interval.

$$4y - \frac{1}{3}y^3 \Big|_{-2}^{2}$$

$$4(2) - \frac{1}{3}(2)^3 - \left(4(-2) - \frac{1}{3}(-2)^3\right)$$

$$8 - \frac{8}{3} + 8 - \frac{8}{3}$$

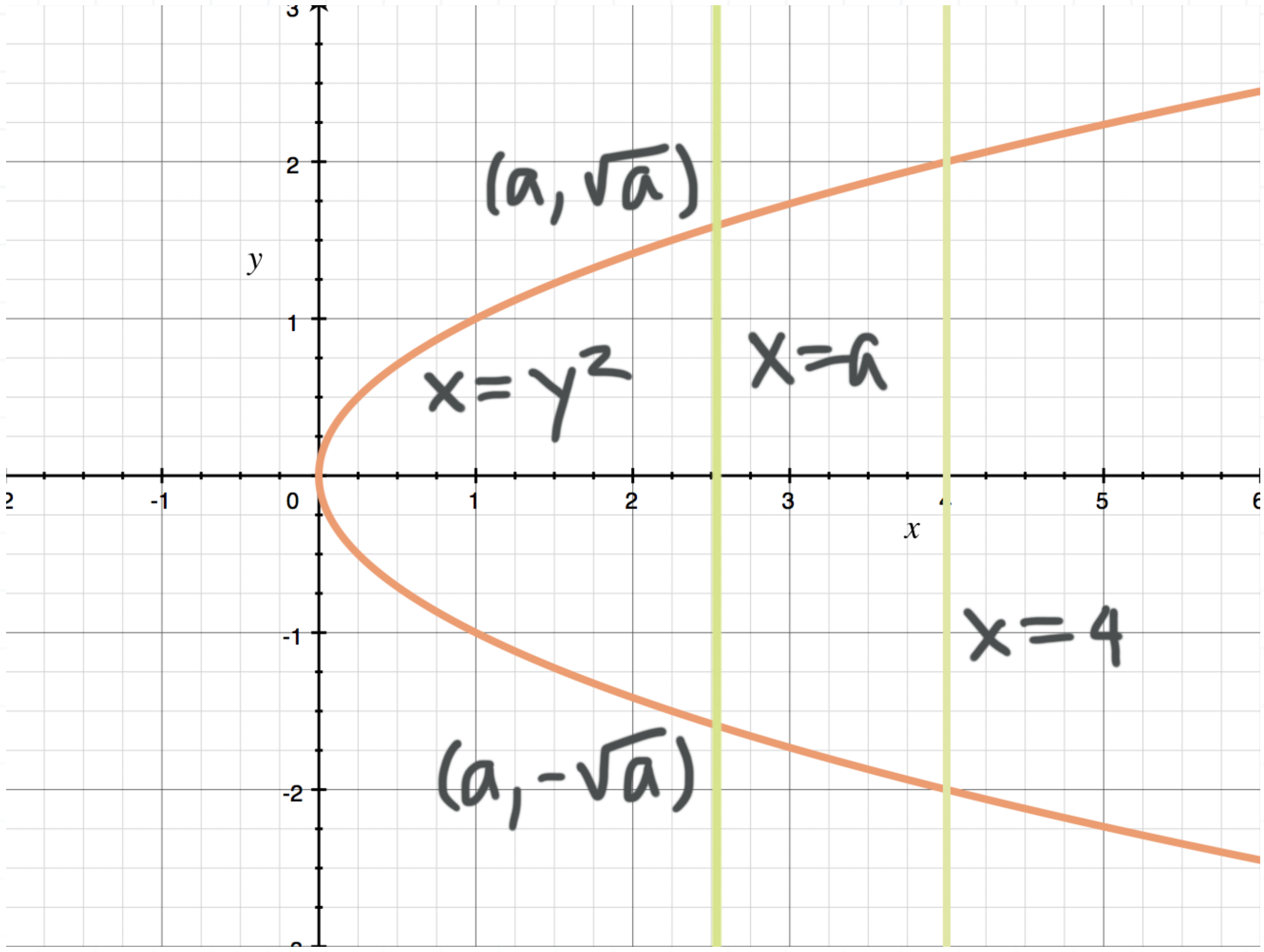
$$16 - \frac{16}{3}$$

$$\frac{32}{3}$$

This is the area of the entire bounded region.

Now, if we will find the value of a in the equation that divides the bounded region's area into two equal parts, we know that the two equal parts will have an area equal to $1/2$ of $32/3$, which is $16/3$.

The region that represents $1/2$ of the bounded region is the portion of the original region that is left of the line as shown in the graph below.



Notice from the graph that since the curves are $x=y^2$ and $x=a$, the points of intersection are $\left(a,-\sqrt{a}\right)$ and $\left(a,\sqrt{a}\right)$. Therefore, the interval of the integration will be $\left[-\sqrt{a},\sqrt{a}\right]$.

Now, let's prepare an integral. Let's find the area of this region by integrating the right curve minus the left curve in that interval, again, with respect to y. You can see that $x=a$ is to the right of $x=y^2$.

$$A=\int_{-\sqrt{a}}^{\sqrt{a}} a-y^2\ dy$$

Integrate, then evaluate over the interval.

$$A=ay-\frac{1}{3}y^3\Big|_{-\sqrt{a}}^{\sqrt{a}}$$

$$A = a\sqrt{a} - \frac{1}{3}\left(\sqrt{a}\right)^3 - \left(a\left(-\sqrt{a}\right) - \frac{1}{3}\left(-\sqrt{a}\right)^3\right)$$

$$A = a^{\frac{3}{2}} - \frac{1}{3}a^{\frac{3}{2}} + a^{\frac{3}{2}} - \frac{1}{3}a^{\frac{3}{2}}$$

$$A = \frac{4}{3}a^{\frac{3}{2}}$$

Now, recall that earlier we said that the area of this region has to be equal to $1/2$ of the area of the original bounded region. We stated that the area of this region is $16/3$ square units. Thus we will make the expression above equal to $16/3$.

$$\frac{4}{3}a^{\frac{3}{2}} = \frac{16}{3}$$

$$4a^{\frac{3}{2}} = 16$$

$$a^{\frac{3}{2}} = 4$$

$$a^3 = 16$$

$$a = \sqrt[3]{16}$$

$$a = \sqrt[3]{8 \cdot 2}$$

$$a = \sqrt[3]{8}\sqrt[3]{2}$$

$$a = 2\sqrt[3]{2}$$

Topic: Dividing the area between curves into equal parts

Question: The line $y = a$ divides the area bounded by the curves into two equal parts. Find a.

$$f(x) = -x^2 + 4$$

$$g(x) = -\frac{1}{4}x^2 + 1$$

Answer choices:

A $a = -1.9199$

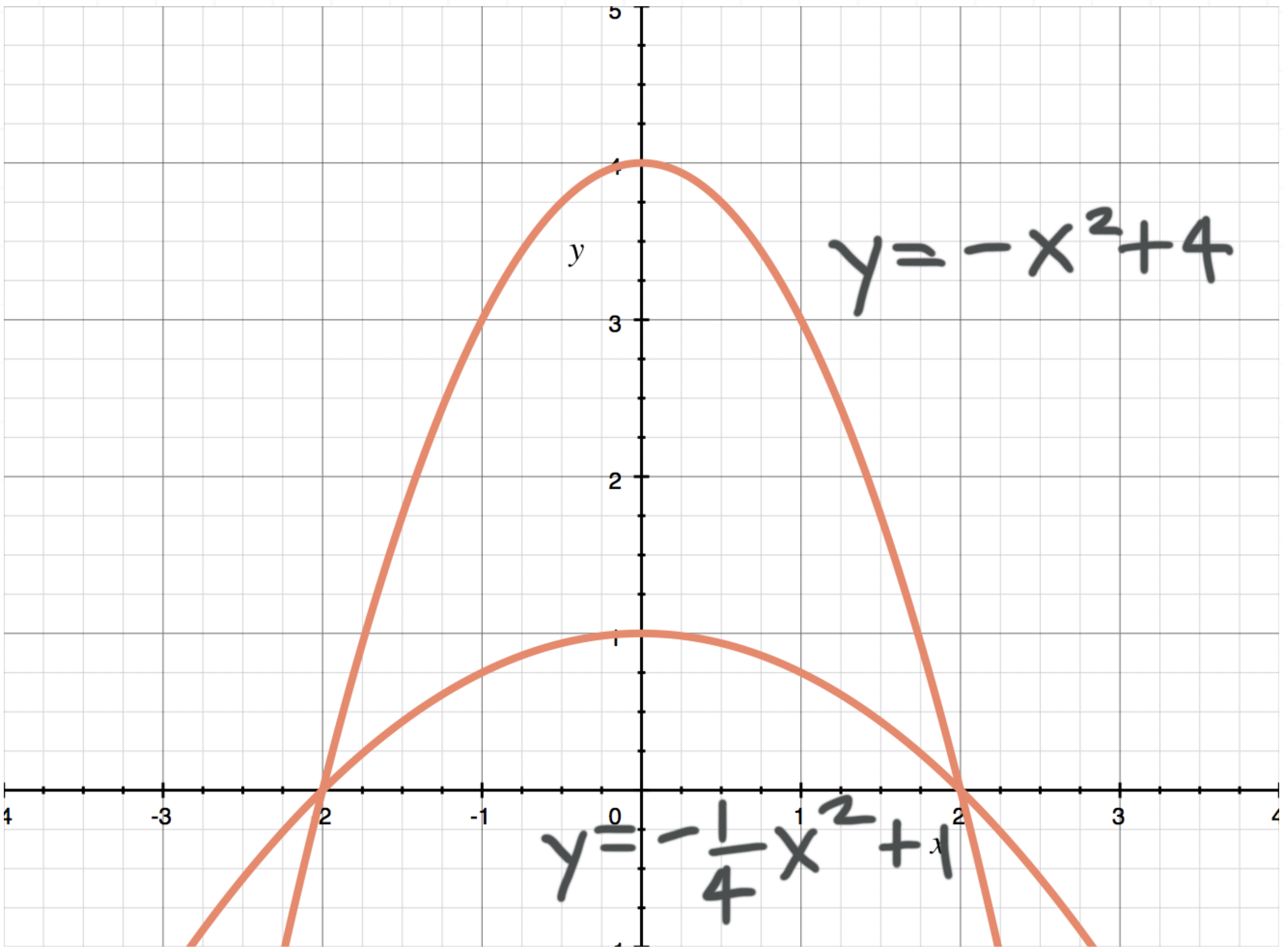
B $a = 1.4423$

C $a = 1.9199$

D $a = -1.4423$

Solution: C

The graphs of $f(x) = -x^2 + 4$ and $g(x) = -\frac{1}{4}x^2 + 1$ are shown below.



To answer the question, first, we will find the entire area of the bounded region. Notice from the graph that the interval on the x-axis for the bounded region is $[-2,2]$. Let's find the area of this region by integrating the upper function minus the lower function in that interval. You can see that $y = -x^2 + 4$ is the upper function and $y = -(1/4)x^2 + 1$ is the lower function.

Let's write the integral first.

$$\int_{-2}^{2} -x^2 + 4 - \left(-\frac{1}{4}x^2 + 1\right) \ dx$$

$$\int_{-2}^{2} -x^2 + 4 + \frac{1}{4}x^2 - 1 \ dx$$

$$\int_{-2}^{2} 3 - \frac{3}{4}x^2 \ dx$$

Integrate, then evaluate over the interval.

$$3x - \frac{1}{4}x^3 \Big|_{-2}^{2}$$

$$3(2) - \frac{1}{4}(2)^3 - \left(3(-2) - \frac{1}{4}(-2)^3\right)$$

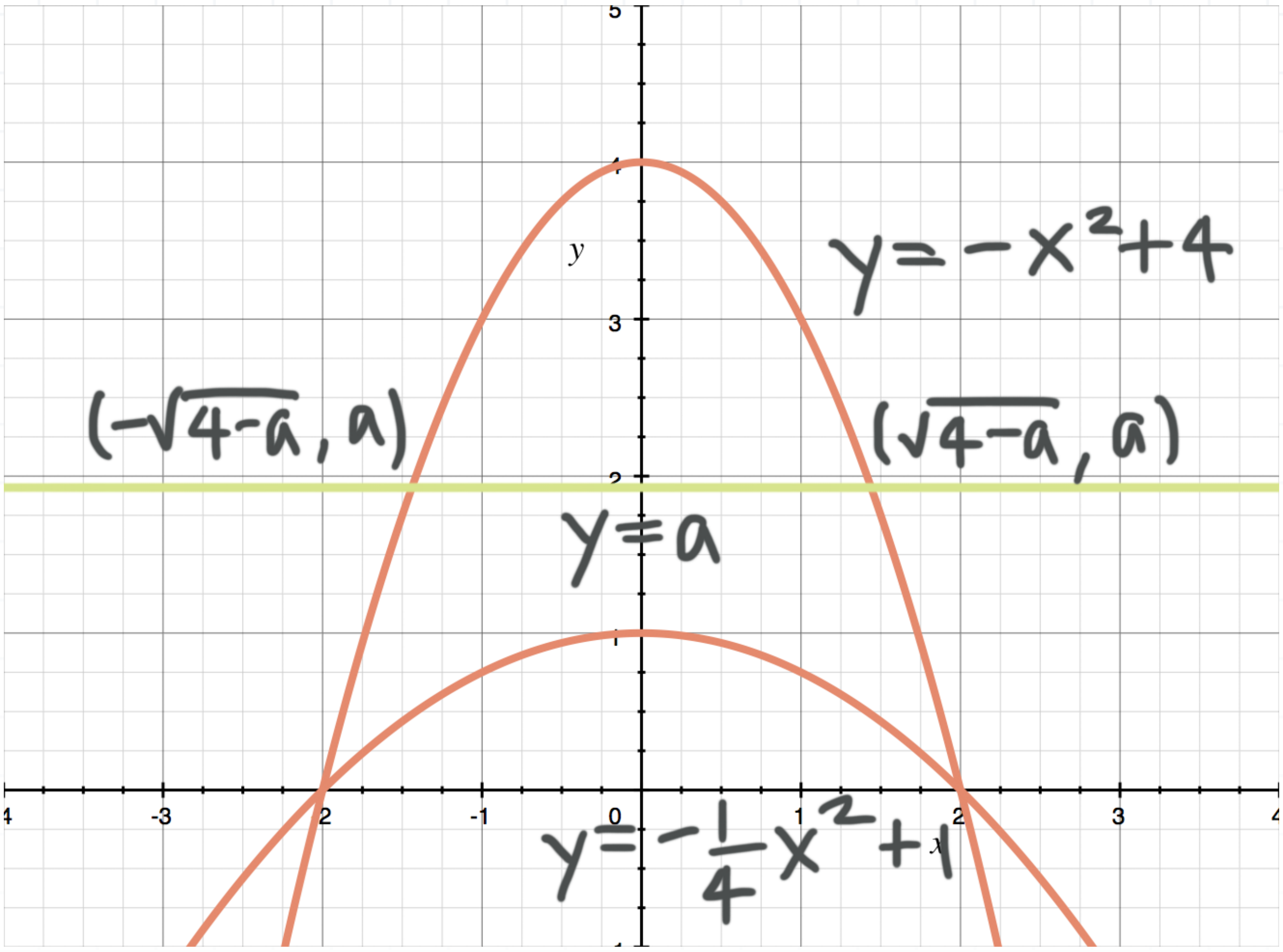
$$6 - 2 - (-6 + 2)$$

$$8$$

The area of the entire bounded region is 8 square units.

Now, if we will find the value of a in the equation $y = a$ that divides the bounded region's area into two equal parts, we know that the two equal parts will have an area equal to $1/2$ of 8, which is 4.

The region that represents $1/2$ of the bounded region is the portion of the original region that is below the curve $y = -x^2 + 4$ and above the line $y = a$ as shown in the graph below.



Notice from the graph that the functions $y = -x^2 + 4$ and $y = a$ intersect at the points where the value of y is equal to a. To find the x-values in the points of intersection, set the function equal to a, and solve for x.

$$-x^2 + 4 = a$$

$$-x^2 = a - 4$$

$$x^2 = 4 - a$$

$$x = \pm\sqrt{4 - a}$$

Therefore, the points of intersection are $\left(-\sqrt{4-a}, a\right)$ and $\left(\sqrt{4-a}, a\right)$. Thus, the interval of the integration will be $\left[-\sqrt{4-a}, \sqrt{4-a}\right]$.

Now, let's prepare an integral. Let's find the area of this region by integrating the upper function minus the lower function in that interval. You can see that $y=-x^2+4$ is the upper function and $y=a$ is the lower function.

$$A=\int_{-\sqrt{4-a}}^{\sqrt{4-a}}-x^2+4-a\ dx$$

Integrate, then evaluate over the interval.

$$A=-\frac{1}{3}x^3+4x-ax\Big|_{-\sqrt{4-a}}^{\sqrt{4-a}}$$

$$A=-\frac{1}{3}\left(\sqrt{4-a}\right)^3+4\sqrt{4-a}-a\sqrt{4-a}-\left(-\frac{1}{3}\left(-\sqrt{4-a}\right)^3+4\left(-\sqrt{4-a}\right)-a\left(-\sqrt{4-a}\right)\right)$$

$$A=-\frac{1}{3}(4-a)^{\frac{3}{2}}+4\sqrt{4-a}-a\sqrt{4-a}-\left(\frac{1}{3}(4-a)^{\frac{3}{2}}-4\sqrt{4-a}+a\sqrt{4-a}\right)$$

$$A=-\frac{1}{3}(4-a)^{\frac{3}{2}}+4\sqrt{4-a}-a\sqrt{4-a}-\frac{1}{3}(4-a)^{\frac{3}{2}}+4\sqrt{4-a}-a\sqrt{4-a}$$

$$A=-\frac{2}{3}(4-a)^{\frac{3}{2}}+8\sqrt{4-a}-2a\sqrt{4-a}$$

$$A=-\frac{2}{3}(4-a)^{\frac{3}{2}}+(8-2a)\sqrt{4-a}$$

$$A=-\frac{2}{3}(4-a)^{\frac{3}{2}}+2(4-a)\sqrt{4-a}$$

$$A=-\frac{2}{3}(4-a)^{\frac{3}{2}}+2(4-a)^{\frac{3}{2}}$$

$$A = \frac{4}{3}(4-a)^{\frac{3}{2}}$$

Now, recall that earlier we said that the area of this region had to be equal to $1/2$ of the area of the original bounded region. The area of the original bounded region was 8, which means that the area of this region is 4 square units.

$$\frac{4}{3}(4-a)^{\frac{3}{2}} = 4$$

$$(4-a)^{\frac{3}{2}} = 3$$

$$(4-a)^3 = 9$$

$$4 - a = \sqrt[3]{9}$$

$$-a = -4 + \sqrt[3]{9}$$

$$a = 4 - \sqrt[3]{9}$$

$$a = 1.9199$$