Topic: Comparison theorem

Question: Use the comparison theorem to say whether the integral converges or diverges.

$$\int_{2}^{\infty} \frac{\cos^2 x}{x^2}\, dx$$

Answer choices:

A Converges

B Diverges

C Comparison theorem does not apply

D Cannot compare the integral

Solution: A

The comparison test is used to determine convergence of divergence of an improper integral that cannot be evaluated directly. In this test, we find another improper integral, and compare the two integrals to each other.

If we want to confirm that an integral converges, we find another integral that contains an integrand that always has values greater than or equal to the integrand in our integral, in the interval of integration, and converges. Using the comparison theorem, we could say that our integral also converges.

Conversely, if we want to confirm that an integral diverges, we find another integral that contains an integrand that always has values less than or equal to the integrand in our integral, in the interval of integration, and diverges. Using the comparison theorem, we could say that our integral also diverges.

If we compare the given integral

$$\int_2^\infty \frac{\cos^2 x}{x^2}\ dx$$

to

$$\int_2^\infty \frac{1}{x^2}\ dx$$

and prove that the given integrand is always less than or equal to the comparison function, and then prove that the comparison function converges, then we'll have proven that the given integral converges.

First, let's confirm that

$$\frac{\cos^2 x}{x^2} \le \frac{1}{x^2}$$

on $[2,\infty)$. We can say

$$\cos^2 x \le 1$$

We know that the value of $\cos x$ is always between -1 and 1, so when we square $\cos x$, the value will always be between 0 and 1. So we can confirm that the inequality is true. We could have also confirmed this by graphing both functions.

Now we'll test the convergence of

$$\int_2^\infty \frac{1}{x^2}\ dx$$

Convert the improper integral to a limit and convert the integrand so we can integrate using the power rule.

$$\lim_{b\to\infty} \int_2^b x^{-2}\ dx$$

$$\lim_{b\to\infty} \frac{x^{-1}}{-1}\Bigg|_2^b$$

$$-\lim_{b\to\infty} \frac{1}{x}\Bigg|_2^b$$

Evaluate over the interval.

$$-\lim_{b\to\infty}\left(\frac{1}{b}-\frac{1}{2}\right)$$

$$-\left(0-\frac{1}{2}\right)$$

$$\frac{1}{2}$$

We have just proven that the comparison integral converges to $1/2$. Therefore, by the comparison theorem, the given integral also converges.

Topic: Comparison theorem

Question: Use the comparison theorem to say whether the integral converges or diverges.

$$\int_1^\infty \frac{1+2\sin^2(2x)}{\sqrt{x}}\,dx$$

Answer choices:

A Converges

B Diverges

C Comparison theorem does not apply

D Cannot compare the integral

Solution: B

The comparison test is used to determine convergence of divergence of an improper integral that cannot be evaluated directly. In this test, we find another improper integral, and compare the two integrals to each other.

If we want to confirm that an integral converges, we find another integral that contains an integrand that always has values greater than or equal to the integrand in our integral, in the interval of integration, and converges. Using the comparison theorem, we could say that our integral also converges.

Conversely, if we want to confirm that an integral diverges, we find another integral that contains an integrand that always has values less than or equal to the integrand in our integral, in the interval of integration, and diverges. Using the comparison theorem, we could say that our integral also diverges.

If we compare the given integral

$$\int_1^\infty \frac{1+2\sin^2(2x)}{\sqrt{x}}\ dx$$

to

$$\int_1^\infty \frac{1}{\sqrt{x}}\ dx$$

and prove that the given integrand is always greater than or equal to the comparison function, and then prove that the comparison function diverges, then we'll have proven that the given integral diverges.

First, let's confirm that

$$\frac{1+2\sin^2(2x)}{\sqrt{x}} \geq \frac{1}{\sqrt{x}}$$

on $[1,\infty)$. We can say

$$1+2\sin^2(2x) \geq 1$$

$$2\sin^2(2x) \geq 0$$

$$\sin^2(2x) \geq 0$$

We know that the value of $\sin x$ is always between -1 and 1, so when we square $\sin x$, the value will always be between 0 and 1. So we can confirm that the inequality is true. We could have also confirmed this by graphing both functions.

Now we'll test the convergence of

$$\int_1^\infty \frac{1}{\sqrt{x}}\ dx$$

Convert the improper integral to a limit and convert the integrand so we can integrate using the power rule.

$$\lim_{b\to\infty} \int_1^b x^{-\frac{1}{2}}\ dx$$

$$\lim_{b\to\infty} \frac{x^{\frac{1}{2}}}{\frac{1}{2}}\Bigg|_1^b$$

$$2 \lim_{b \to \infty} \sqrt{x} \Big|_1^b$$

Evaluate over the interval.

$$2 \lim_{b \to \infty} \left(\sqrt{b} - \sqrt{1} \right)$$

$$\lim_{b \to \infty} \left(2\sqrt{b} - 2 \right)$$

$$\infty$$

We have just proven that the comparison integral diverges. Therefore, by the comparison theorem, the given integral also diverges.

Topic: Comparison theorem

Question: Use the comparison theorem to say whether the integral converges or diverges.

$$\int_{2}^{\infty} \frac{2 - e^{-x}}{x} \, dx$$

when compared to $\displaystyle\int_{2}^{\infty} \frac{1}{2x} \, dx$

Answer choices:

A Converges

B Diverges

C Comparison theorem does not apply

D Cannot compare the integral

Solution: B

The comparison test is used to determine convergence of divergence of an improper integral that cannot be evaluated directly. In this test, we find another improper integral, and compare the two integrals to each other.

If we want to confirm that an integral converges, we find another integral that contains an integrand that always has values greater than or equal to the integrand in our integral, in the interval of integration, and converges. Using the comparison theorem, we could say that our integral also converges.

Conversely, if we want to confirm that an integral diverges, we find another integral that contains an integrand that always has values less than or equal to the integrand in our integral, in the interval of integration, and diverges. Using the comparison theorem, we could say that our integral also diverges.

If we compare the given integral

$$\int_2^\infty \frac{2-e^{-x}}{x}\ dx$$

to

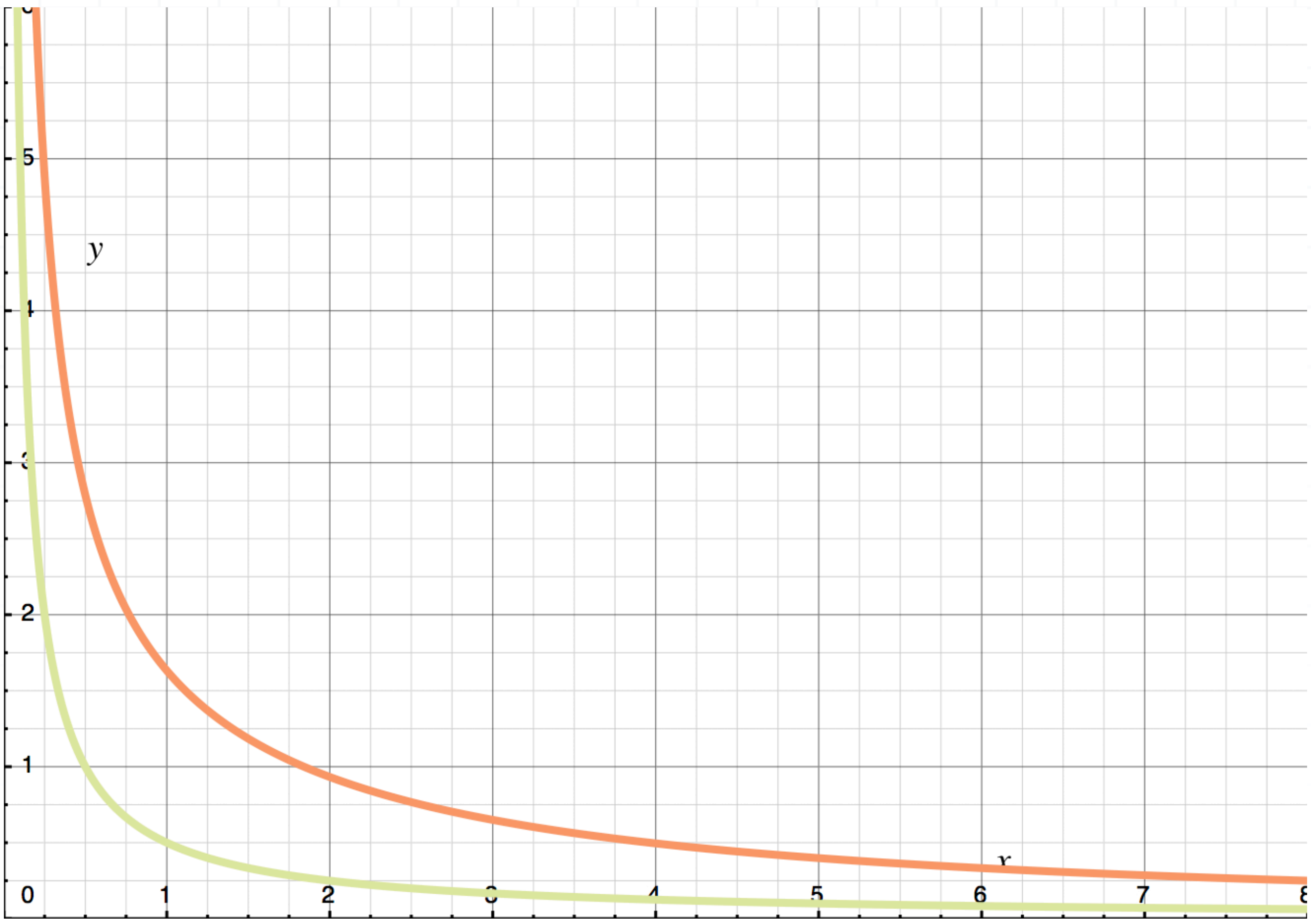
$$\int_2^\infty \frac{1}{2x}\ dx$$

and prove that the given integrand is always greater than or equal to the comparison function, and then prove that the comparison function diverges, then we'll have proven that the given integral also diverges.

First, let's confirm that

$$\frac{2-e^{-x}}{x} \geq \frac{1}{2x}$$

on $[2,\infty)$. We can do that graphically. The graph of both functions is below.



From the graphs, we can confirm that the inequality is true.

Now we'll test the convergence of

$$\int_{2}^{\infty} \frac{1}{2x}\ dx$$

Convert the improper integral to a limit, then integrate.

$$\frac{1}{2}\lim_{b\to\infty}\int_{2}^{b}\frac{1}{x}\ dx$$

$$\frac{1}{2}\lim_{b\to\infty}\ln|x|\Big|_2^b$$

Evaluate over the interval.

$$\frac{1}{2}\lim_{b\to\infty}\left(\ln|b|-\ln|2|\right)$$

$$\frac{1}{2}\left(\infty-\ln 2\right)$$

$$\infty$$

We have just proven that the comparison integral diverges. Therefore, by the comparison theorem, the given integral also diverges.