# Volume of revolution, cylindrical shells

We can use integrals to find the volume of the three-dimensional object created by rotating a function around either the $x$-axis (or some other horizontal axis with the equation $y = b$) or around the $y$-axis (or some other vertical axis with the equation $x = a$).

We can do this using the disk method, the washer method, or using cylindrical shells. The cylindrical shell method rotates the function in a perpendicular fashion. This means that $y = f(x)$ is rotated around the $y$-axis and $x = g(y)$ is rotated around the $x$-axis.

The cylindrical shells method formulas we use to find volume of rotation are different depending on the form of the function and the axis of rotation.

1. If the function is in the form $y = f(x)$ and we're rotating around the $y$-axis over the interval $[a, b]$, the formula for the volume of the solid is

$$V = \int_a^b 2\pi x f(x)\ dx$$

2. If the function is in the form $x = g(y)$ and we're rotating around the $x$-axis over the interval $[c, d]$, the formula for the volume of the solid is

$$V = \int_c^d 2\pi y f(y)\ dy$$

The table below will help guide you through how to solve a volume problem when you're using cylindrical shells to find the volume. Start in the first row of the table, and determine the line of rotation or revolution. The problem will usually tell you the line of rotation. If you're asked to rotate about the $y$-axis or some line defined for $x$ in terms of $y$, then stay in the first column of the table. If you're asked to rotate about the $x$-axis or some line defined for $y$ in terms of $x$, then stay in the second column of the table.

The best way to figure out whether you need to use cylindrical shells instead of either disks or washers is to graph the functions and the axis of rotation and draw a picture of the rotated volume.

| Axis | Disks | Washers | Shells |
|---|---|---|---|
| | $\int$ area width | $\int$ area width | $\int$ circumference height width |
| | | **Axis of revolution: HORIZONTAL** | |
| $x$-axis | $\int_a^b \pi \left[f(x)\right]^2\ dx$ | $\int_a^b \pi \left[f(x)\right]^2 - \pi \left[g(x)\right]^2\ dx$ | $\int_c^d 2\pi y \left[f(y) - g(y)\right]\ dy$ |
| $y = -k$ | | $\int_a^b \pi \left[k + f(x)\right]^2 - \pi \left[k + g(x)\right]^2\ dx$ | $\int_c^d 2\pi(y + k)\left[f(y) - g(y)\right]\ dy$ |
| $y = k$ | | $\int_a^b \pi \left[k - g(x)\right]^2 - \pi \left[k - f(x)\right]^2\ dx$ | $\int_c^d 2\pi(k - y)\left[f(y) - g(y)\right]\ dy$ |
| | | **Axis of revolution: VERTICAL** | |
| $y$-axis | $\int_c^d \pi \left[f(y)\right]^2\ dy$ | $\int_c^d \pi \left[f(y)\right]^2 - \pi \left[g(y)\right]^2\ dy$ | $\int_a^b 2\pi x \left[f(x) - g(x)\right]\ dx$ |
| $x = -k$ | | $\int_c^d \pi \left[k + f(y)\right]^2 - \pi \left[k + g(y)\right]^2\ dy$ | $\int_a^b 2\pi(x + k)\left[f(x) - g(x)\right]\ dx$ |
| $x = k$ | | $\int_c^d \pi \left[k - g(y)\right]^2 - \pi \left[k - f(y)\right]^2\ dy$ | $\int_a^b 2\pi(k - x)\left[f(x) - g(x)\right]\ dx$ |

## Example

Find the volume of the solid created by rotating the curve about the $y$-axis over the interval $[1,2]$.

$$y = x^3$$

Looking at the function, we can see that it's in the form $y = f(x)$ and revolved around the $y$-axis, which means the volume is given by

$$V = \int_a^b 2\pi x f(x)\ dx$$

$$V = \int_1^2 2\pi x \left(x^3\right)\ dx$$

$$V = \int_1^2 2\pi x^4\ dx$$

$$V = 2\pi \int_1^2 x^4\ dx$$

Integrating, we get

$$V = 2\pi \left(\frac{x^5}{5}\right)\Bigg|_1^2$$

$$V = \frac{2\pi x^5}{5}\Bigg|_1^2$$

Now we can evaluate over the interval.

$$V = \frac{2\pi(2)^5}{5} - \left[\frac{2\pi(1)^5}{5}\right]$$

$$V = \frac{62}{5}\pi$$

This is the volume of the solid object created by rotating $y = x^3$ about the $y$-axis over the interval $[1,2]$.

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Let's try another example where the curve is defined for $x$ in terms of $y$.

### Example

Find the volume of the solid created by rotating the curve about the $x$-axis over the interval $y = 2$ to $y = 4$.

$$x = 3y^4$$

Looking at the function, we can see that it's in the form $x = g(y)$ and revolved around the $x$-axis, which means the volume is given by

$$V = \int_a^b 2\pi y f(y)\ dy$$

$$V = \int_2^4 2\pi y \left(3y^4\right)\ dy$$

$$V = \int_2^4 6\pi y^5\ dy$$

$$V = 6\pi \int_2^4 y^5\ dy$$

Integrating, we get

$$V = 6\pi \left( \frac{y^6}{6} \right) \Bigg|_2^4$$

$$V = \pi y^6 \Big|_2^4$$

Now we can evaluate over the interval.

$$V = \pi(4)^6 - \pi(2)^6$$

$$V = 4{,}032\pi$$

This is the volume of rotation for the solid object created by rotating $x = 3y^4$ about the $x$-axis over the interval $y = 2$ to $y = 4$.