Calculus 1 Workbook Solutions

Linear approximation

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LINEAR APPROXIMATION

1. Find the linear approximation of $f(x) = x^3 - 4x^2 + 2x - 3$ at $x = 3$ and use it to approximate $f(3.02)$.

Solution:

The linear approximation formula at $x = a$ is $L(x) = f(a) + f'(a)(x - a)$. In this problem, $a = 3$, so the linear approximation is $L(x) = f(3) + f'(3)(x - 3)$. Find the pieces that we need for the formula.

$$f(3) = 3^3 - 4(3)^2 + 2(3) - 3 = -6$$

$$f'(x) = 3x^2 - 8x + 2$$

$$f'(3) = 3(3)^2 - 8(3) + 2 = 5$$

Plugging these pieces into the linear approximation gives

$$L(x) = -6 + 5(x - 3)$$

$$L(x) = -6 + 5x - 15$$

$$L(x) = 5x - 21$$

Use this equation to approximate $f(3.02)$.

$$f(3.02) = 5(3.02) - 21 = -5.9$$

2. Find the linear approximation of $g(x)=\sqrt{8x-15}$ at $x=8$ and use it to approximate $f(8.05)$.

Solution:

The linear approximation formula at $x=a$ is $L(x)=g(a)+g'(a)(x-a)$. In this problem, $a=8$, so the linear approximation is $L(x)=g(8)+g'(8)(x-8)$. Find the pieces that we need for the formula.

$$g(8)=\sqrt{8(8)-15}=\sqrt{49}=7$$

$$g'(x)=\frac{8}{2\sqrt{8x-15}}=\frac{4}{\sqrt{8x-15}}$$

$$g'(8)=\frac{4}{\sqrt{8(8)-15}}=\frac{4}{\sqrt{49}}=\frac{4}{7}$$

Plugging these pieces into the linear approximation gives

$$L(x)=7+\frac{4}{7}(x-8)$$

$$L(x)=7+\frac{4}{7}x-\frac{32}{7}$$

$$L(x)=\frac{4}{7}x+\frac{17}{7}$$

Use this equation to approximate $g(8.05)$.

$$g(8.05)=\frac{4}{7}(8.05)+\frac{17}{7}=\frac{246}{35}\approx 7.029$$

■ 3. Find the linear approximation of $h(x) = 2e^{x-4} + 6$ at $x = 5$ and use it to approximate $h(5.1)$.

Solution:

The linear approximation formula at $x = a$ is $L(x) = h(a) + h'(a)(x - a)$. In this problem, $a = 5$, so the linear approximation is $L(x) = h(5) + h'(5)(x - 5)$. Find the pieces that we need for the formula.

$$h(5) = 2e^{5-4} + 6 = 2e + 6$$

$$h'(x) = 2e^{x-4}$$

$$h'(5) = 2e^{5-4} = 2e$$

Plugging these pieces into the linear approximation gives

$$L(x) = 2e + 6 + 2e(x - 5)$$

$$L(x) = 2e + 6 + 2ex - 10e$$

$$L(x) = 2ex - 8e + 6$$

Use this equation to approximate $h(5.1)$.

$$g(5.1) = 2e(5.1) - 8e + 6 = 10.2e - 8e + 6 = 2.2e + 6 \approx 11.98$$

■ 4. Find the linear approximation of $f(x) = \ln(2x - 7)$ at $x = 4$ and use it to approximate $f(3.8)$.

Solution:

The linear approximation formula at $x = a$ is $L(x) = f(a) + f'(a)(x - a)$. In this problem, $a = 4$, so the linear approximation is $L(x) = f(4) + f'(4)(x - 4)$. Find the pieces that we need for the formula.

$$f(4) = \ln(2(4) - 7) = \ln(8 - 7) = 0$$

$$f'(x) = \frac{2}{2x - 7}$$

$$f'(4) = \frac{2}{2(4) - 7} = \frac{2}{1} = 2$$

Plugging these pieces into the linear approximation gives

$$L(x) = 0 + 2(x - 4)$$

$$L(x) = 2x - 8$$

Use this equation to approximate $f(3.8)$.

$$f(3.8) = 2(3.8) - 8 = -0.4$$

■ 5. Use linear approximation to estimate $f(3.1)$.

$$f(x) = \sin(3x)$$

Solution:

The first thing we want to realize is that finding $f(3.1)$ gets pretty messy. If we substitute $x = 3.1$ into the function, we get $\sin 9.3$. That's not necessarily an easy value to find. However, $\sin 9.3$ is pretty close to $\sin 9.42 = \sin(3\pi)$, which is a very easy value to find.

Therefore, instead of trying to find $f(3.1)$, let's use a linear approximation equation and $a = \pi$ to get an approximation for $f(3.1)$.

The linear approximation formula at $x = a$ is $L(x) = f(a) + f'(a)(x - a)$. In this problem, $a = \pi$, so the linear approximation is $L(x) = f(\pi) + f'(\pi)(x - \pi)$. Find the pieces that we need for the formula.

$$f(\pi) = \sin(3\pi) = 0$$

$$f'(x) = 3\cos(3x)$$

$$f'(\pi) = 3\cos(3\pi) = -3$$

Plugging these pieces into the linear approximation gives

$$L(x) = 0 - 3(x - \pi)$$

$$L(x) = -3x + 3\pi$$

Use this equation to approximate $f(3.1)$.

$$f(3.1) = -3(3.1) + 3\pi = 0.125$$

■ 6. Use linear approximation to estimate $f(6.1)$.

$$f(x) = e^{\cos x}$$

Solution:

The first thing we want to realize is that finding $f(6.1)$ gets pretty messy. If we substitute $x = 6.1$ into the function, we get $e^{\cos 6.1}$. That's not necessarily an easy value to find. However, $e^{\cos 6.1}$ is pretty close to $e^{\cos 6.28} = e^{\cos(2\pi)}$, which is a very easy value to find.

Therefore, instead of trying to find $f(6.1)$, let's use a linear approximation equation and $a = 2\pi$ to get an approximation for $f(6.1)$.

The linear approximation formula at $x = a$ is $L(x) = f(a) + f'(a)(x - a)$. In this problem, $a = 2\pi$, so the linear approximation is $L(x) = f(2\pi) + f'(2\pi)(x - 2\pi)$. Find the pieces that we need for the formula.

$$f(2\pi) = e^{\cos 2\pi} = e^1 = e$$

$$f'(x) = -\sin x e^{\cos x}$$

$$f'(2\pi) = -\sin(2\pi)e^{\cos(2\pi)} = -0 \cdot e = 0$$

Plugging these pieces into the linear approximation gives

$$L(x) = e + 0(x - 2\pi)$$

$$L(x) = e$$

Use this equation to approximate $f(6.1)$.

$$f(6.1) = e = 2.72$$

ESTIMATING A ROOT

■ 1. Use linear approximation to estimate $\sqrt[5]{34}$.

Solution:

Let $f(x) = \sqrt[5]{x}$ and $a = 32$. The linear approximation would be given by

$$L(x) = f(a) + f'(a)(x - a)$$

$$L(x) = f(32) + f'(32)(x - 32)$$

Find the pieces needed for the formula.

$$f(32) = \sqrt[5]{32} = 2$$

$$f(x) = \sqrt[5]{x} = x^{\frac{1}{5}}$$

$$f'(x) = \frac{1}{5}(x)^{-\frac{4}{5}} = \frac{1}{5 \cdot \sqrt[5]{x^4}}$$

$$f'(32) = \frac{1}{5 \cdot \sqrt[5]{32^4}} = \frac{1}{80}$$

Then the linear approximation is

$$L(x) = 2 + \frac{1}{80}(x - 32)$$

$$L(x) = \frac{1}{80}x + \frac{8}{5}$$

Use this approximation to estimate $\sqrt[5]{34}$.

$$f(34) = \frac{1}{80}(34) + \frac{8}{5} = \frac{81}{40}$$

■ 2. Use linear approximation to estimate $\sqrt[8]{260}$.

Solution:

Let $f(x) = \sqrt[8]{x}$ and $a = 256$. The linear approximation would be given by

$$L(x) = f(a) + f'(a)(x - a)$$

$$L(x) = f(256) + f'(256)(x - 256)$$

Find the pieces needed for the formula.

$$f(256) = \sqrt[8]{256} = 2$$

$$f(x) = \sqrt[8]{x} = x^{\frac{1}{8}}$$

$$f'(x) = \frac{1}{8}(x)^{-\frac{7}{8}} = \frac{1}{8\sqrt[8]{x^7}}$$

$$f'(256) = \frac{1}{8\sqrt[8]{256^7}} = \frac{1}{1,024}$$

Then the linear approximation is

$$L(x) = 2 + \frac{1}{1{,}024}(x - 256)$$

$$L(x) = \frac{1}{1{,}024}x + \frac{7}{4}$$

Use this approximation to estimate $\sqrt[8]{260}$.

$$f(260) = \frac{1}{1{,}024}(260) + \frac{7}{4} = \frac{513}{256} \approx 2.0039$$

■ 3. Use linear approximation to estimate $\sqrt[4]{85}$.

Solution:

Let $f(x) = \sqrt[4]{x}$ and $a = 81$. The linear approximation would be given by

$$L(x) = f(a) + f'(a)(x - a)$$

$$L(x) = f(81) + f'(81)(x - 81)$$

Find the pieces needed for the formula.

$$f(81) = \sqrt[4]{81} = 3$$

$$f(x) = \sqrt[4]{x} = x^{\frac{1}{4}}$$

$$f'(x) = \frac{1}{4}x^{-\frac{3}{4}} = \frac{1}{4\sqrt[4]{x^3}}$$

$$f'(81) = \frac{1}{4\sqrt[4]{81^3}} = \frac{1}{108}$$

Then the linear approximation is

$$L(x) = 3 + \frac{1}{108}(x - 81)$$

$$L(x) = \frac{1}{108}x + \frac{9}{4}$$

Use this approximation to estimate $\sqrt[4]{85}$.

$$f(85) = \frac{1}{108}(85) + \frac{9}{4} = \frac{82}{27} \approx 3.037$$

■ 4. Use linear approximation to estimate $\sqrt[4]{615}$.

Solution:

Let $f(x) = \sqrt[4]{x}$ and $a = 625$. The linear approximation would be given by

$$L(x) = f(a) + f'(a)(x - a)$$

$$L(x) = f(625) + f'(625)(x - 625)$$

Find the pieces needed for the formula.

$$f(625) = \sqrt[4]{625} = 5$$

$$f(x) = \sqrt[4]{x} = x^{\frac{1}{4}}$$

$$f'(x) = \frac{1}{4}x^{-\frac{3}{4}} = \frac{1}{4\sqrt[4]{x^3}}$$

$$f'(625) = \frac{1}{4\sqrt[4]{625^3}} = \frac{1}{500}$$

Then the linear approximation is

$$L(x) = 5 + \frac{1}{500}(x - 625)$$

$$L(x) = \frac{1}{500}x + \frac{15}{4}$$

Use this approximation to estimate $\sqrt[4]{615}$.

$$f(615) = \frac{1}{500}(615) + \frac{15}{4} = \frac{249}{50} \approx 4.98$$

■ 5. Use linear approximation to estimate $\sqrt{95}$.

Solution:

Let $f(x) = \sqrt{x}$ and $a = 100$. The linear approximation would be given by

$$L(x) = f(a) + f'(a)(x - a)$$

$$L(x) = f(100) + f'(100)(x - 100)$$

Find the pieces needed for the formula.

$$f(100) = \sqrt{100} = 10$$

$$f(x) = \sqrt{x} = x^{\frac{1}{2}}$$

$$f'(x) = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$$

$$f'(100) = \frac{1}{2\sqrt{100}} = \frac{1}{20}$$

Then the linear approximation is

$$L(x) = 10 + \frac{1}{20}(x - 100)$$

$$L(x) = \frac{1}{20}x + 5$$

Use this approximation to estimate $\sqrt{95}$.

$$f(95) = \frac{1}{20}(95) + 5 = \frac{39}{4} \approx 9.75$$

■ 6. Use linear approximation to estimate $\sqrt[3]{700}$.

Solution:

Let $f(x) = \sqrt[3]{x}$ and $a = 729$. The linear approximation would be given by

$$L(x) = f(a) + f'(a)(x - a)$$

$$L(x) = f(729) + f'(729)(x - 729)$$

Find the pieces needed for the formula.

$$f\,(729) = \sqrt[3]{729} = 9.$$

$$f(x) = \sqrt[3]{x} = x^{\frac{1}{3}}$$

$$f'(x) = \frac{1}{3}x^{-\frac{2}{3}} = \frac{1}{3\sqrt[3]{x^2}}$$

$$f'(729) = \frac{1}{3\sqrt[3]{729^2}} = \frac{1}{243}$$

Then the linear approximation is

$$L(x) = 9 + \frac{1}{243}(x - 729)$$

$$L(x) = \frac{1}{243}x + 6$$

Use this approximation to estimate $\sqrt[3]{700}$.

$$f(700) = \frac{1}{243}(700) + 6 = \frac{2{,}158}{243} \approx 8.88$$

ABSOLUTE, RELATIVE, AND PERCENTAGE ERROR

1. Use a linear approximation to estimate the value of $e^{0.002}$, then find the absolute error of the estimate.

Solution:

We need to realize here that $e^{0.002}$ is a difficult value to find. But it's very close to e^0, which we already know is 1. So instead of thinking specifically about $e^{0.002}$, let's think about e^x, and therefore use the function $f(x) = e^x$. We'll differentiate it,

$$f'(x) = e^x$$

then evaluate the derivative at $x = 0$.

$$f'(0) = e^0$$

$$f'(0) = 1$$

So the linear approximation intersects $f(x) = e^x$ at the point of tangency $(0,1)$, and has a slope of $m = 1$. Substitute these values into the linear approximation equation.

$$L(x) = f(a) + f'(a)(x - a)$$

$$L(x) = 1 + 1(x - 0)$$

$$L(x) = 1 + x$$

Then the linear approximation at $x = 0.002$ is

$$L(0.002) = 1 + 0.002$$

$$L(0.002) = 1.002$$

In comparison, the actual value of $e^{0.002}$ is

$$f(x) = e^x$$

$$f(0.002) = e^{0.002}$$

$$f(0.002) \approx 1.002002$$

Therefore, the absolute error of the approximation is

$$E_A(a) = |f(a) - L(a)|$$

$$E_A(0.002) = |f(0.002) - L(0.002)|$$

$$E_A(0.002) \approx |1.002002 - 1.002|$$

$$E_A(0.002) \approx |0.000002|$$

$$E_A(0.002) = 0.000002$$

■ 2. Use linear approximation to estimate $f(2.15)$, then find the relative error of the estimate.

$$f(x) = 4xe^{3x-6}$$

Solution:

Instead of trying to find $f(2.15)$, let's use a linear approximation equation and $a=2$ to get an approximation for $f(2.15)$.

$$f(x)=4xe^{3x-6}$$

$$f(2)=4(2)e^{3(2)-6}$$

$$f(2)=8$$

Differentiate the function,

$$f'(x)=4e^{3x-6}+4xe^{3x-6}(3)$$

$$f'(x)=4e^{3x-6}+12xe^{3x-6}$$

$$f'(x)=4e^{3x-6}(1+3x)$$

then evaluate the derivative at $x=2$.

$$f'(2)=4e^{3(2)-6}(1+3(2))$$

$$f'(2)=4e^{0}(7)$$

$$f'(2)=28$$

So the linear approximation intersects $f(x)=4xe^{3x-6}$ at the point of tangency $(2,8)$, and has a slope of $m=28$. Substitute these values into the linear approximation equation.

$$L(x)=f(a)+f'(a)(x-a)$$

$$L(x)=8+28(x-2)$$

$$L(x) = 8 + 28x - 56$$

$$L(x) = 28x - 48$$

Then the linear approximation at $x = 2.15$ is

$$L(2.15) = 28(2.15) - 48$$

$$L(2.15) = 60.2 - 48$$

$$L(2.15) = 12.2$$

In comparison, the actual value of $f(2.15)$ is

$$f(2.15) = 4(2.15)e^{(3(2.15)-6)}$$

$$f(2.15) \approx 13.4875$$

Therefore, the absolute error of the approximation is

$$E_A(a) = |f(a) - L(a)|$$

$$E_A(2.15) = |f(2.15) - L(2.15)|$$

$$E_A(2.15) \approx |13.4875 - 12.2|$$

$$E_A(2.15) \approx |1.2875|$$

$$E_A(2.15) \approx 1.2875$$

and the relative error is

$$E_R(a) = \frac{E_A(a)}{f(a)}$$

$$E_R(2.15) = \frac{E_A(2.15)}{f(2.15)}$$

$$E_R(2.15) \approx \frac{1.2875}{13.4875}$$

$$E_R(2.15) \approx 0.095459$$

■ 3. Use linear approximation to estimate $f(1.2)$, then find the percentage error of the estimate.

$$f(x) = \sqrt[3]{x+1}$$

Solution:

Instead of trying to find $f(1.2)$, let's use a linear approximation equation and $a = 0$ to get an approximation for $f(1.2)$.

$$f(x) = \sqrt[3]{x+1}$$

$$f(0) = \sqrt[3]{0+1}$$

$$f(0) = 1$$

Differentiate the function,

$$f'(x) = \frac{1}{3}(x+1)^{-\frac{2}{3}}$$

$$f'(x) = \frac{1}{3(x+1)^{\frac{2}{3}}}$$

then evaluate the derivative at $x = 0$.

$$f'(0) = \frac{1}{3(0+1)^{\frac{2}{3}}}$$

$$f'(0) = \frac{1}{3(1)}$$

$$f'(0) = \frac{1}{3}$$

So the linear approximation intersects $f(x) = \sqrt[3]{x+1}$ at the point of tangency $(0,1)$, and has a slope of $m = 1/3$. Substitute these values into the linear approximation equation.

$$L(x) = f(a) + f'(a)(x - a)$$

$$L(x) = 1 + \frac{1}{3}(x - 0)$$

$$L(x) = 1 + \frac{1}{3}x$$

Then the linear approximation at $x = 1.2$ is

$$L(1.2) = 1 + \frac{1}{3}(1.2)$$

$$L(1.2) = 1 + 0.4$$

$$L(1.2) = 1.4$$

In comparison, the actual value of $f(1.2)$ is

$$f(x) = \sqrt[3]{x+1}$$

$$f(1.2) = \sqrt[3]{1.2+1}$$

$$f(1.2) \approx 1.30059$$

Therefore, the absolute error of the approximation is

$$E_A(a) = |f(a) - L(a)|$$

$$E_A(1.2) = |f(1.2) - L(1.2)|$$

$$E_A(1.2) \approx |1.30059 - 1.4|$$

$$E_A(1.2) \approx |-0.09941|$$

$$E_A(1.2) \approx 0.09941$$

The relative error is

$$E_R(a) = \frac{E_A(a)}{f(a)}$$

$$E_R(1.2) = \frac{E_A(1.2)}{f(1.2)}$$

$$E_R(1.2) \approx \frac{0.09941}{1.30059}$$

$$E_R(1.2) \approx 0.076435$$

The percentage error is

$$E_P(a) = 100\% \cdot E_R(a)$$

$$E_P(1.2) = 100\% \cdot E_R(1.2)$$

$$E_P(1.2) \approx 100\% \cdot 0.076435$$

$$E_P(1.2) \approx 7.6435\,\%$$

■ 4. Use a linear approximation to estimate the value of $\sqrt[3]{30}$, then find the relative error of the estimate.

Solution:

We need to realize here that $\sqrt[3]{30}$ is a difficult value to find. But it's very close to $\sqrt[3]{27}$, which we already know is 3. So instead of thinking specifically about $\sqrt[3]{30}$, let's think about $\sqrt[3]{x}$, and therefore use the function $f(x) = \sqrt[3]{x}$.

Differentiate the function,

$$f'(x) = \frac{1}{3}x^{-\frac{2}{3}}$$

$$f'(x) = \frac{1}{3x^{\frac{2}{3}}}$$

$$f'(x) = \frac{1}{3\sqrt[3]{x^2}}$$

then evaluate the derivative at $x = 27$.

$$f'(27) = \frac{1}{3\sqrt[3]{27^2}}$$

$$f'(27) = \frac{1}{3(9)}$$

$$f'(27) = \frac{1}{27}$$

So the linear approximation intersects $f(x) = \sqrt[3]{x}$ at the point of tangency $(27,3)$, and has a slope of $m = 1/27$. Substitute these values into the linear approximation equation.

$$L(x) = f(a) + f'(a)(x - a)$$

$$L(x) = 3 + \frac{1}{27}(x - 27)$$

$$L(x) = 3 + \frac{1}{27}x - 1$$

$$L(x) = \frac{1}{27}x + 2$$

Then the linear approximation at $x = 30$ is

$$L(30) = \frac{1}{27}(30) + 2$$

$$L(30) = \frac{30}{27} + 2$$

$$L(30) = \frac{30}{27} + \frac{54}{27}$$

$$L(30) = \frac{84}{27}$$

$$L(30) \approx 3.111111$$

In comparison, the actual value of $\sqrt[3]{30}$ is

$$f(x) = \sqrt[3]{x}$$

$$f(30) = \sqrt[3]{30}$$

$$f(30) \approx 3.107233$$

Therefore, the absolute error of the approximation is

$$E_A(a) = |f(a) - L(a)|$$

$$E_A(30) = |f(30) - L(30)|$$

$$E_A(30) \approx |3.107233 - 3.111111|$$

$$E_A(30) \approx |-0.003878|$$

$$E_A(30) \approx 0.003878$$

The relative error is

$$E_R(a) = \frac{E_A(a)}{f(a)}$$

$$E_R(30) = \frac{E_A(30)}{f(30)}$$

$$E_R(30) \approx \frac{0.003878}{3.107233}$$

$$E_R(30) \approx 0.001248$$

5. Find the absolute, relative, and percentage error of the approximation 2.7 to the value of e.

Solution:

The absolute error of the approximation is

$$E_A(a) = |f(a) - L(a)|$$

$$E_A(a) = |e - 2.7|$$

$$E_A(a) = |2.718282 - 2.7|$$

$$E_A(a) = |0.018282|$$

$$E_A(a) = 0.018282$$

The relative error is

$$E_R(a) = \frac{E_A(a)}{f(a)}$$

$$E_R(a) \approx \frac{0.018282}{2.718282}$$

$$E_R(a) \approx 0.0067256$$

The percentage error is

$$E_P(a) = 100\% \cdot E_R(a)$$

$$E_P(a) \approx 100\% \cdot 0.0067256$$

$$E_P(a) \approx 0.67256\,\%$$

■ 6. Use a linear approximation to estimate the value of $\sin(93^\circ)$, then find the absolute error of the estimate.

Solution:

We need to realize here that $\sin(93^\circ)$ is a difficult value to find. But it's very close to $\sin(90^\circ)$, which we already know is 1. So instead of thinking specifically about $\sin(93^\circ)$, let's think about $\sin x$, and therefore use the function $f(x) = \sin x$.

$$f(x) = \sin x$$

$$f(90^\circ) = \sin 90^\circ$$

$$f(90^\circ) = 1$$

Differentiate the function,

$$f'(x) = \cos x$$

then evaluate the derivative at $x = 90^\circ$.

$$f'(90^\circ) = \cos 90^\circ$$

$$f'(90^\circ) = 0$$

So the linear approximation intersects $f(x) = \sin x$ at the point of tangency $(90^\circ, 1)$, and has a slope of $m = 0$. Substitute these values into the linear approximation equation.

$$L(x) = f(a) + f'(a)(x - a)$$

$$L(x) = 1 + 0(x - 90^\circ)$$

$$L(x) = 1$$

Then the linear approximation at $x = 93^\circ$ is

$$L(93^\circ) = 1$$

In comparison, the actual value of $\sin(93^\circ)$ is

$$f(x) = \sin x$$

$$f(93^\circ) = \sin 93^\circ$$

$$f(93^\circ) \approx 0.998629$$

Therefore, the absolute error of the approximation is

$$E_A(a) = |f(a) - L(a)|$$

$$E_A(93^\circ) = |f(93^\circ) - L(93^\circ)|$$

$$E_A(93^\circ) \approx |0.998629 - 1|$$

$$E_A(93^\circ) \approx |-0.001371|$$

$$E_A(93^\circ) \approx 0.001371$$

W W W . K R I S T A K I N G M A T H . C O M