Absolute and conditional convergence

We can calculate the convergence of a series using any of the many convergence tests, depending on the format of the series we've been given. Sometimes we're asked to say whether the series converges conditionally or converges absolutely.

A series converges conditionally if

a_n converges but $\left|a_n\right|$ diverges

In other words, $a_n \neq \left|a_n\right|$ for all possible values of n.

A series converges absolutely if

a_n and $\left|a_n\right|$ both converge

In other words, $a_n = |a_n|$ for all possible values of n.

The ratio and root tests both use absolute value bars. If we can drop the absolute value bars by the end of the problem because the expression inside the absolute value bars will always be positive anyway, then we know that the series converges absolutely.

On the other hand, if we can't drop the absolute value bars because it's possible that the expression inside them could be negative, then we know that the series converges conditionally.

Let's try an example where we use the root test to determine absolute or conditional convergence.

Example

Determine whether the series converges absolutely or conditionally.

$$\sum_{n=1}^{\infty}\left(\frac{n^3-1}{6n^3+4}\right)^n$$

We know we can use the ratio test or the root test to determine absolute convergence, and this series looks like a great candidate for the root test, since the whole thing is raised to the nth power.

To use the root test, we need to solve for the limit

$$L=\lim_{n\to\infty}\sqrt[n]{|a_n|}$$

and then evaluate the value of L.

$$L=\lim_{n\to\infty}\sqrt[n]{\left|\left(\frac{n^3-1}{6n^3+4}\right)^n\right|}$$

$$L=\lim_{n\to\infty}\left[\left|\left(\frac{n^3-1}{6n^3+4}\right)^n\right|\right]^{\frac{1}{n}}$$

$$L=\lim_{n\to\infty}\left|\frac{n^3-1}{6n^3+4}\right|^{\frac{n}{n}}$$

$$L = \lim_{n \to \infty} \left| \frac{n^3 - 1}{6n^3 + 4} \right|$$

$$L = \lim_{n \to \infty} \left| \frac{n^3 - 1}{6n^3 + 4} \left(\frac{\frac{1}{n^3}}{\frac{1}{n^3}} \right) \right|$$

$$L = \lim_{n \to \infty} \left| \frac{\frac{n^3}{n^3} - \frac{1}{n^3}}{\frac{6n^3}{n^3} + \frac{4}{n^3}} \right|$$

$$L = \lim_{n \to \infty} \left| \frac{1 - \frac{1}{n^3}}{6 + \frac{4}{n^3}} \right|$$

$$L = \left| \frac{1 - \frac{1}{\infty}}{6 + \frac{4}{\infty}} \right|$$

$$L = \left| \frac{1 - 0}{6 + 0} \right|$$

$$L = \left| \frac{1}{6} \right|$$

Since $L < 1$, and since we can drop the absolute value bars and it wouldn't change the value of L, we can say that a_n and $\left| a_n \right|$ both converge, and therefore that a_n converges absolutely.