Topic: Applied optimization

Question: A rancher plans to build a fenced rectangular area adjacent to an existing stone wall. He wants the fence to enclose $160{,}000$ square meters for his horses, but he's low on fencing. Which dimensions require the least amount of fencing?

Answer choices:

A 400×400

B 283×565

C $4{,}000 \times 8{,}000$

D $100 \times 1{,}600$

Solution: B

In this problem, we want to minimize the perimeter of a field and we know that we will require the fence to enclose $160{,}000$ ft square meters for the horses.

Since one side of the enclosure doesn't need any fencing, the amount of fencing needed to enclose the area is

$$P = 2x + y$$

where x is the length of a side adjacent to the wall and y is the length of the side opposite the wall. We can also write an equation for area in terms of x and y, and then solve it for y.

$$A = xy$$

$$160{,}000 = xy$$

$$y = \frac{160{,}000}{x}$$

Substitute this value into the perimeter equation.

$$P = 2x + \frac{160{,}000}{x}$$

Take the derivative,

$$P' = 2 - \frac{160{,}000}{x^2}$$

then set it equal to 0 to find critical points.

$$2-\frac{160,000}{x^2}=0$$

$$2=\frac{160,000}{x^2}$$

$$2x^2=160,000$$

$$x^2=80,000$$

$$x=\sqrt{80,000}$$

$$x\approx 283$$

Use the first derivative test with test values of 280 and 290 to confirm that $x \approx 283$ represents a minimum.

$$P'(280)=2-\frac{160,000}{280^2}$$

$$P'(280)\approx -0.04$$

and

$$P'(290)=2-\frac{160,000}{290^2}$$

$$P'(290)\approx 0.10$$

Because we get a negative value to the left of the critical point and a positive value to the right, $x \approx 283$ represents a minimum. The associated value for y is

$$y=\frac{160,000}{283}$$

$$y \approx 565$$

Topic: Applied optimization

Question: With $1{,}000$ m of new fencing material, you need to enclose a rectangular yard and maximize its area. What dimensions should you use?

Answer choices:

A 250×250

B 250×750

C 500×500

D 125×375

Solution: A

If we use l for length and w for width, we can say the perimeter of the yard is

$$P = 2l + 2w$$

$$1{,}000 = 2l + 2w$$

and that the area is

$$A = lw$$

We want to maximize area, so we'll solve the perimeter equation for width.

$$2w = 1{,}000 - 2l$$

$$w = 500 - l$$

Substitute this value into the area equation.

$$A = l(500 - l)$$

$$A = 500l - l^2$$

Take the derivative, then set it equal to 0 to find critical points.

$$A' = 500 - 2l$$

$$500 - 2l = 0$$

$$2l = 500$$

$$l = 250$$

Use the first derivative test with test values of $l = 240$ and $l = 260$ to make sure that $l = 250$ represents a maximum.

$$A'(240) = 500 - 2(240)$$

$$A'(240) = 500 - 480$$

$$A'(240) = 20$$

and

$$A'(260) = 500 - 2(260)$$

$$A'(260) = 500 - 520$$

$$A'(260) = -20$$

Since we got a positive value to the left of the critical point and a negative value to the right of it, we know $l = 250$ represents a maximum. The corresponding width is

$$w = 500 - 250$$

$$w = 250$$

Topic: Applied optimization

Question: You want to construct a box with a square bottom and you only have 36 m^2 of material. Assuming you use all of the material, what is the maximum volume of the box?

Answer choices:

A $12\sqrt{6}$ m^3

B 36 m^3

C 18 m^3

D $6\sqrt{6}$ m^3

Solution: D

The volume of a box is always given by $V = lwh$, but since we've been told that the box has a square base, we know that $l = w$, so we can simplify the volume equation to $V = l^2h$.

The surface area of a box is always given by $A = 2lw + 2lh + 2wh$. But since $l = w$, we can simplify this as

$$A = 2ll + 2lh + 2lh$$

$$A = 2l^2 + 4lh$$

We know that total surface area is 36, so

$$36 = 2l^2 + 4lh$$

$$18 = l^2 + 2lh$$

Solve this area equation for height h.

$$2lh = 18 - l^2$$

$$h = \frac{18 - l^2}{2l}$$

Substitute this value into the volume equation.

$$V = l^2h$$

$$V = l^2\left(\frac{18 - l^2}{2l}\right)$$

$$V = \frac{18l - l^3}{2}$$

$$V = 9l - \frac{1}{2}l^3$$

Take the derivative,

$$V' = 9 - \frac{3}{2}l^2$$

then set it equal to 0 to find critical points.

$$9 - \frac{3}{2}l^2 = 0$$

$$\frac{3}{2}l^2 = 9$$

$$l^2 = 6$$

$$l = \pm\sqrt{6}$$

It's nonsensical to have a negative length, so the only critical point is $l = \sqrt{6}$. Use the first derivative test with test values of $l = 2$ and $l = 3$ to confirm that the critical point represents a maximum.

$$V'(2) = 9 - \frac{3}{2}(2)^2$$

$$V'(2) = 9 - 6$$

$$V'(2) = 3$$

and

$$V'(3) = 9 - \frac{3}{2}(3)^2$$

$$V'(3) = \frac{18}{2} - \frac{27}{2}$$

$$V'(3) = -\frac{9}{2}$$

Since we get a positive value to the left of the critical point and a negative value to the right of it, the function has a maximum at the critical point.

We found the length l associated with the critical point, but we were asked for the maximum volume, so now we just need to find the volume that corresponds with this length, which we'll do by plugging $l = \sqrt{6}$ into $V = 9l - (1/2)l^3$.

$$V = 9\sqrt{6} - \frac{1}{2}(\sqrt{6})^3$$

$$V = 9\sqrt{6} - \frac{6}{2}\sqrt{6}$$

$$V = 9\sqrt{6} - 3\sqrt{6}$$

$$V = 6\sqrt{6}$$

The maximum volume of the box is $6\sqrt{6}$ m^3.