Calculus 2 Workbook Solutions

Integration by parts

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INTEGRATION BY PARTS

1. Use integration by parts to evaluate the integral.

$$\int 9x \sin x \ dx$$

Solution:

Pick

$u = 9x$ differentiating $du = 9 \ dx$

$dv = \sin x \ dx$ integrating $v = -\cos x$

Plug into the integration by parts formula.

$$\int 9x \sin x \ dx = (9x)(-\cos x) - \int (-\cos x)(9 \ dx)$$

$$\int 9x \sin x \ dx = -9x \cos x + 9 \int \cos x \ dx$$

$$\int 9x \sin x \ dx = -9x \cos x + 9 \sin x + C$$

2. Use integration by parts to evaluate the integral.

$$\int 5xe^x \ dx$$

Solution:

Pick

$u = 5x$ differentiating $du = 5 \ dx$

$dv = e^x \ dx$ integrating $v = e^x$

Plug into the integration by parts formula.

$$\int 5xe^x \ dx = (5x)(e^x) - \int (e^x)(5 \ dx)$$

$$\int 5xe^x \ dx = 5xe^x - 5 \int e^x \ dx$$

$$\int 5xe^x \ dx = 5xe^x - 5e^x + C$$

You could leave the answer this way, or factor it as

$$\int 5xe^x \ dx = 5e^x(x - 1) + C$$

■ 3. Use integration by parts to evaluate the integral.

$$\int 7x \ln x \ dx$$

Solution:

Pick

$$u = \ln x \qquad \text{differentiating} \qquad du = \frac{1}{x}\ dx$$

$$dv = 7x\ dx \qquad \text{integrating} \qquad v = \frac{7}{2}x^2$$

Plug into the integration by parts formula.

$$\int 7x \ln x\ dx = (\ln x)\left(\frac{7}{2}x^2\right) - \int \left(\frac{7}{2}x^2\right)\left(\frac{1}{x}\ dx\right)$$

$$\int 7x \ln x\ dx = \frac{7}{2}x^2 \ln x - \frac{7}{2}\int x\ dx$$

$$\int 7x \ln x\ dx = \frac{7}{2}x^2 \ln x - \frac{7}{2}\left(\frac{1}{2}x^2\right) + C$$

$$\int 7x \ln x\ dx = \frac{7}{2}x^2 \ln x - \frac{7}{4}x^2 + C$$

You could leave the answer this way, or factor it as

$$\int 7x \ln x\ dx = \frac{7}{2}x^2\left(\ln x - \frac{1}{2}\right) + C$$

■ 4. Use integration by parts to evaluate the integral.

$$\int 2x \cos x \ dx$$

Solution:

Pick

$u = 2x$ differentiating $du = 2 \ dx$

$dv = \cos x \ dx$ integrating $v = \sin x$

Plug into the integration by parts formula.

$$\int 2x \cos x \ dx = (2x)(\sin x) - \int (\sin x)(2 \ dx)$$

$$\int 2x \cos x \ dx = 2x \sin x - 2 \int \sin x \ dx$$

$$\int 2x \cos x \ dx = 2x \sin x - 2(-\cos x) + C$$

$$\int 2x \cos x \ dx = 2x \sin x + 2 \cos x + C$$

■ 5. Use integration by parts to evaluate the integral.

$$\int 3\sqrt{x} \ln x \ dx$$

Solution:

Pick

$$u = \ln x \qquad \text{differentiating} \qquad du = \frac{1}{x}\ dx$$

$$dv = 3\sqrt{x}\ dx \qquad \text{integrating} \qquad v = 3\left(\frac{2}{3}x^{\frac{3}{2}}\right) = 2x^{\frac{3}{2}}$$

Plug into the integration by parts formula.

$$\int 3\sqrt{x}\ln x\ dx = (\ln x)(2x^{\frac{3}{2}}) - \int (2x^{\frac{3}{2}})\left(\frac{1}{x}\ dx\right)$$

$$\int 3\sqrt{x}\ln x\ dx = 2x^{\frac{3}{2}}\ln x - 2\int \frac{x^{\frac{3}{2}}}{x}\ dx$$

$$\int 3\sqrt{x}\ln x\ dx = 2x^{\frac{3}{2}}\ln x - 2\int x^{\frac{1}{2}}\ dx$$

$$\int 3\sqrt{x}\ln x\ dx = 2x^{\frac{3}{2}}\ln x - 2\left(\frac{2}{3}x^{\frac{3}{2}}\right) + C$$

$$\int 3\sqrt{x}\ln x\ dx = 2x^{\frac{3}{2}}\ln x - \frac{4}{3}x^{\frac{3}{2}} + C$$

You could leave the answer this way, or factor it as

$$\int 3\sqrt{x}\ln x\ dx = 2x^{\frac{3}{2}}\left(\ln x - \frac{2}{3}\right) + C$$

INTEGRATION BY PARTS TWO TIMES

1. Apply integration by parts two times to evaluate the integral.

$$\int 3x^2e^x \ dx$$

Solution:

Pick

$u = 3x^2$ differentiating $du = 6x \ dx$

$dv = e^x \ dx$ integrating $v = e^x$

Plug into the integration by parts formula.

$$\int 3x^2e^x \ dx = (3x^2)(e^x) - \int (e^x)(6x \ dx)$$

$$\int 3x^2e^x \ dx = 3x^2e^x - 6\int xe^x \ dx$$

Apply integration by parts again to replace the integral on the right side.
Pick

$u = x$ differentiating $du = 1 \ dx$

$dv = e^x \ dx$ integrating $v = e^x$

Plug into the integration by parts formula.

$$\int xe^x\ dx = (x)(e^x) - \int (e^x)(1\ dx)$$

$$\int xe^x\ dx = xe^x - \int e^x\ dx$$

The integral on the right is now simple enough to evaluate directly.

$$\int xe^x\ dx = xe^x - e^x + C$$

Take the right side of this equation, and plug it into the equation from earlier.

$$\int 3x^2e^x\ dx = 3x^2e^x - 6\int xe^x\ dx$$

$$\int 3x^2e^x\ dx = 3x^2e^x - 6\left(xe^x - e^x + C\right)$$

$$\int 3x^2e^x\ dx = 3x^2e^x - 6xe^x + 6e^x - 6C$$

If C is a constant, then $-6C$ is also a constant, so we can simplify.

$$\int 3x^2e^x\ dx = 3x^2e^x - 6xe^x + 6e^x + C$$

You could leave the answer this way, or factor it as

$$\int 3x^2e^x\ dx = 3e^x(x^2 - 2x + 2) + C$$

2. Use integration by parts to evaluate the integral.

$$\int e^{3x}\cos(5x)\ dx$$

Solution:

First, break down the given integral into suitable expressions for u and dv as follows:

$$u = \cos(5x)$$

$$dv = e^{3x}\ dx$$

Differentiating u and integrating dv, we get

$$du = -5\sin(5x)\ dx$$

$$v = \int e^{3x}\ dx = \frac{1}{3}e^{3x}$$

Plug the values into the formula for integration by parts.

$$\int u\ dv = uv - \int v\ du$$

$$\int e^{3x}\cos(5x)\ dx = \frac{1}{3}e^{3x}\cos(5x) - \int \frac{1}{3}e^{3x}\left[-5\sin(5x)\right]\ dx$$

$$\int e^{3x}\cos(5x)\ dx = \frac{1}{3}e^{3x}\cos(5x) + \frac{5}{3}\int e^{3x}\sin(5x)\ dx$$

Notice that the resulting integral on the right side of the equal sign is still not readily integrable. We again use integration by parts and define a new set of u and dv.

$$u = \sin(5x)$$

$$dv = e^{3x}\ dx$$

and

$$du = 5\cos(5x)\ dx$$

$$v = \int e^{3x}\ dx = \frac{1}{3}e^{3x}$$

Replacing the integral on the right with the integration by parts formula and the new values we found, we get

$$\int e^{3x}\cos(5x)\ dx = \frac{1}{3}e^{3x}\cos(5x) + \frac{5}{3}\left[uv - \int v\ du\right]$$

$$\int e^{3x}\cos(5x)\ dx = \frac{1}{3}e^{3x}\cos(5x) + \frac{5}{3}\left[(\sin(5x))\left(\frac{1}{3}e^{3x}\right) - \int \frac{1}{3}e^{3x}(5\cos(5x)\ dx)\right]$$

$$\int e^{3x}\cos(5x)\ dx = \frac{1}{3}e^{3x}\cos(5x) + \frac{5}{3}\left[\frac{1}{3}e^{3x}\sin(5x) - \frac{5}{3}\int e^{3x}\cos(5x)\ dx\right]$$

$$\int e^{3x}\cos(5x)\ dx = \frac{1}{3}e^{3x}\cos(5x) + \frac{5}{9}e^{3x}\sin(5x) - \frac{25}{9}\int e^{3x}\cos(5x)\ dx$$

Notice that the resulting integral on the right side of the equal sign is exactly the same as the given integral. So we can use a little algebra and move it to the left-hand side to combine it with the given integral.

$$\int e^{3x}\cos(5x)\ dx+\frac{25}{9}\int e^{3x}\cos(5x)\ dx=\frac{1}{3}e^{3x}\cos(5x)+\frac{5}{9}e^{3x}\sin(5x)+C$$

$$\frac{9}{9}\int e^{3x}\cos(5x)\ dx+\frac{25}{9}\int e^{3x}\cos(5x)\ dx=\frac{1}{3}e^{3x}\cos(5x)+\frac{5}{9}e^{3x}\sin(5x)+C$$

$$\frac{34}{9}\int e^{3x}\cos(5x)\ dx=\frac{1}{3}e^{3x}\cos(5x)+\frac{5}{9}e^{3x}\sin(5x)+C$$

Now multiply both sides by $9/34$ to solve for the given integral, keeping in mind that the $9/34$ gets absorbed into the constant C.

$$\int e^{3x}\cos(5x)\ dx=\frac{9}{34}\left[\frac{1}{3}e^{3x}\cos(5x)+\frac{5}{9}e^{3x}\sin(5x)+C\right]$$

$$\int e^{3x}\cos(5x)\ dx=\frac{3}{34}e^{3x}\cos(5x)+\frac{5}{34}e^{3x}\sin(5x)+C$$

$$\int e^{3x}\cos(5x)\ dx=\frac{1}{34}e^{3x}\left[3\cos(5x)+5\sin(5x)\right]+C$$

INTEGRATION BY PARTS THREE TIMES

1. Apply integration by parts three times to evaluate the integral.

$$\int 7x^3e^x\ dx$$

Solution:

Pick

$u = 7x^3$ differentiating $du = 21x^2\ dx$

$dv = e^x\ dx$ integrating $v = e^x$

Plug into the integration by parts formula.

$$\int 7x^3e^x\ dx = (7x^3)(e^x) - \int (e^x)(21x^2\ dx)$$

$$\int 7x^3e^x\ dx = 7x^3e^x - 21\int x^2e^x\ dx$$

Apply integration by parts again to replace the integral on the right side. Pick

$u = x^2$ differentiating $du = 2x\ dx$

$dv = e^x\ dx$ integrating $v = e^x$

Plug into the integration by parts formula.

$$\int x^2e^x\ dx=(x^2)(e^x)-\int(e^x)(2x\ dx)$$

$$\int x^2e^x\ dx=x^2e^x-2\int xe^x\ dx$$

Apply integration by parts again to replace the integral on the right side. Pick

$u=x$	differentiating	$du=1\ dx$
$dv=e^x\ dx$	integrating	$v=e^x$

Plug into the integration by parts formula.

$$\int xe^x\ dx=(x)(e^x)-\int(e^x)(1\ dx)$$

$$\int xe^x\ dx=xe^x-\int e^x\ dx$$

The integral on the right is now simple enough to evaluate directly.

$$\int xe^x\ dx=xe^x-e^x+C$$

Take the right side of this equation, and plug it into the equation from earlier.

$$\int x^2e^x\ dx=x^2e^x-2\int xe^x\ dx$$

$$\int x^2e^x\ dx=x^2e^x-2\left(xe^x-e^x+C\right)$$

$$\int x^2e^x\ dx = x^2e^x - 2xe^x + 2e^x - 2C$$

If C is a constant, then $-2C$ is also a constant, so we can simplify.

$$\int x^2e^x\ dx = x^2e^x - 2xe^x + 2e^x + C$$

Take the right side of this equation, and plug it into the equation from earlier.

$$\int 7x^3e^x\ dx = 7x^3e^x - 21\int x^2e^x\ dx$$

$$\int 7x^3e^x\ dx = 7x^3e^x - 21\left(x^2e^x - 2xe^x + 2e^x + C\right)$$

$$\int 7x^3e^x\ dx = 7x^3e^x - 21x^2e^x + 42xe^x - 42e^x - 21C$$

If C is a constant, then $-21C$ is also a constant, so we can simplify.

$$\int 7x^3e^x\ dx = 7x^3e^x - 21x^2e^x + 42xe^x - 42e^x + C$$

You could leave the answer this way, or factor it as

$$\int 7x^3e^x\ dx = 7e^x\left(x^3 - 3x^2 + 6x - 6\right) + C$$

■ 2. Apply integration by parts three times to evaluate the integral.

$$\int \left(2x^3 + x^2\right) e^x \ dx$$

Solution:

Pick

$u = 2x^3 + x^2$ differentiating $du = 6x^2 + 2x \ dx$

$dv = e^x \ dx$ integrating $v = e^x$

Plug into the integration by parts formula.

$$\int \left(2x^3 + x^2\right) e^x \ dx = (2x^3 + x^2)(e^x) - \int (e^x)(6x^2 + 2x \ dx)$$

$$\int \left(2x^3 + x^2\right) e^x \ dx = 2x^3e^x + x^2e^x - \int (6x^2 + 2x)(e^x) \ dx$$

$$\int \left(2x^3 + x^2\right) e^x \ dx = 2x^3e^x + x^2e^x - 2\int (3x^2 + x)(e^x) \ dx$$

Apply integration by parts again to replace the integral on the right side. Pick

$u = 3x^2 + x$ differentiating $du = 6x + 1 \ dx$

$dv = e^x \ dx$ integrating $v = e^x$

Plug into the integration by parts formula.

$$\int (3x^2 + x)(e^x) \ dx = (3x^2 + x)(e^x) - \int (e^x)(6x + 1 \ dx)$$

$$\int (3x^2 + x)(e^x)\ dx = 3x^2e^x + xe^x - \int (6x+1)(e^x)\ dx$$

Apply integration by parts again to replace the integral on the right side. Pick

$u = 6x + 1$	differentiating	$du = 6\ dx$
$dv = e^x\ dx$	integrating	$v = e^x$

Plug into the integration by parts formula.

$$\int (6x+1)(e^x)\ dx = (6x+1)(e^x) - \int (e^x)(6\ dx)$$

$$\int (6x+1)(e^x)\ dx = 6xe^x + e^x - 6\int e^x\ dx$$

The integral on the right is now simple enough to evaluate directly.

$$\int (6x+1)(e^x)\ dx = 6xe^x + e^x - 6e^x + C$$

Take the right side of this equation, and plug it into the equation from earlier.

$$\int (3x^2 + x)(e^x)\ dx = 3x^2e^x + xe^x - \int (6x+1)(e^x)\ dx$$

$$\int (3x^2 + x)(e^x)\ dx = 3x^2e^x + xe^x - \left(6xe^x + e^x - 6e^x + C\right)$$

$$\int (3x^2 + x)(e^x)\ dx = 3x^2e^x + xe^x - 6xe^x - e^x + 6e^x - C$$

$$\int (3x^2+x)(e^x)\ dx = 3x^2e^x - 5xe^x + 5e^x - C$$

If C is a constant, then $-C$ is also a constant, so we can simplify.

$$\int (3x^2+x)(e^x)\ dx = 3x^2e^x - 5xe^x + 5e^x + C$$

Take the right side of this equation, and plug it into the equation from earlier.

$$\int \left(2x^3+x^2\right) e^x\ dx = 2x^3e^x + x^2e^x - 2\int (3x^2+x)(e^x)\ dx$$

$$\int \left(2x^3+x^2\right) e^x\ dx = 2x^3e^x + x^2e^x - 2\left(3x^2e^x - 5xe^x + 5e^x + C\right)$$

$$\int \left(2x^3+x^2\right) e^x\ dx = 2x^3e^x + x^2e^x - 6x^2e^x + 10xe^x - 10e^x - 2C$$

$$\int \left(2x^3+x^2\right) e^x\ dx = 2x^3e^x - 5x^2e^x + 10xe^x - 10e^x - 2C$$

If C is a constant, then $-2C$ is also a constant, so we can simplify.

$$\int \left(2x^3+x^2\right) e^x\ dx = 2x^3e^x - 5x^2e^x + 10xe^x - 10e^x + C$$

You could leave the answer this way, or factor it as

$$\int \left(2x^3+x^2\right) e^x\ dx = e^x\left(2x^3 - 5x^2 + 10x - 10\right) + C$$

3. Use integration by parts three times to evaluate the integral.

$$\int (\ln x)^3 \ dx$$

Solution:

Pick

$u = (\ln x)^3$ differentiating $du = 3(\ln x)^2\left(\frac{1}{x}\right) \ dx$

$dv = dx$ integrating $v = x$

Plug into the integration by parts formula.

$$\int (\ln x)^3 \ dx = ((\ln x)^3)(x) - \int (x)\left(3(\ln x)^2\left(\frac{1}{x}\right) \ dx\right)$$

$$\int (\ln x)^3 \ dx = x(\ln x)^3 - 3\int (\ln x)^2 \ dx$$

Apply integration by parts again to replace the integral on the right side.
Pick

$u = (\ln x)^2$ differentiating $du = 2(\ln x)\left(\frac{1}{x}\right) \ dx$

$dv = dx$ integrating $v = x$

Plug into the integration by parts formula.

$$\int (\ln x)^2 \ dx = ((\ln x)^2)(x) - \int (x)\left(2(\ln x)\left(\frac{1}{x}\right) \ dx \right)$$

$$\int (\ln x)^2 \ dx = x(\ln x)^2 - 2\int \ln x \ dx$$

Apply integration by parts again to replace the integral on the right side. Pick

$u = \ln x$ differentiating $du = \frac{1}{x} \ dx$

$dv = dx$ integrating $v = x$

Plug into the integration by parts formula.

$$\int \ln x \ dx = (\ln x)(x) - \int (x)\left(\frac{1}{x} \ dx\right)$$

$$\int \ln x \ dx = x \ln x - \int \ dx$$

The integral on the right is now simple enough to evaluate directly.

$$\int \ln x \ dx = x \ln x - x + C$$

Take the right side of this equation, and plug it into the equation from earlier.

$$\int (\ln x)^2 \ dx = x(\ln x)^2 - 2\int \ln x \ dx$$

$$\int (\ln x)^2 \ dx = x(\ln x)^2 - 2\left(x \ln x - x + C\right)$$

$$\int (\ln x)^2 \ dx = x(\ln x)^2 - 2x \ln x + 2x - 2C$$

If C is a constant, then $-2C$ is also a constant, so we can simplify.

$$\int (\ln x)^2 \ dx = x(\ln x)^2 - 2x \ln x + 2x + C$$

Take the right side of this equation, and plug it into the equation from earlier.

$$\int (\ln x)^3 \ dx = x(\ln x)^3 - 3 \int (\ln x)^2 \ dx$$

$$\int (\ln x)^3 \ dx = x(\ln x)^3 - 3\left(x(\ln x)^2 - 2x \ln x + 2x + C\right)$$

$$\int (\ln x)^3 \ dx = x(\ln x)^3 - 3x(\ln x)^2 + 6x \ln x - 6x - 3C$$

If C is a constant, then $-3C$ is also a constant, so we can simplify.

$$\int (\ln x)^3 \ dx = x(\ln x)^3 - 3x(\ln x)^2 + 6x \ln x - 6x + C$$

You could leave the answer this way, or factor it as

$$\int (\ln x)^3 \ dx = x\left[(\ln x)^3 - 3(\ln x)^2 + 6 \ln x - 6\right] + C$$

INTEGRATION BY PARTS WITH U-SUBSTITUTION

■ 1. Use integration by parts and substitution to evaluate the integral.

$$\int \tan^{-1} x \ dx$$

Solution:

Use integration by parts first. Pick

$u = \tan^{-1} x$ differentiating $du = \frac{1}{x^2+1} \ dx$

$dv = dx$ integrating $v = x$

Plug into the integration by parts formula.

$$\int \tan^{-1} x \ dx = (\tan^{-1} x)(x) - \int (x)\left(\frac{1}{x^2+1} \ dx\right)$$

$$\int \tan^{-1} x \ dx = x \tan^{-1} x - \int \frac{x}{x^2+1} \ dx$$

Use substitution to evaluate the integral that remains. Let

$k = x^2 + 1$

$dk = 2x \ dx$ so $dx = \frac{dk}{2x}$

Substitute into the integral on the right.

$$\int \tan^{-1} x \ dx = x \tan^{-1} x - \int \frac{x}{x^2+1} \ dx$$

$$\int \tan^{-1} x \ dx = x \tan^{-1} x - \int \frac{x}{k}\left(\frac{dk}{2x}\right)$$

$$\int \tan^{-1} x \ dx = x \tan^{-1} x - \frac{1}{2}\int \frac{1}{k} \ dk$$

Integrate, then back-substitute.

$$\int \tan^{-1} x \ dx = x \tan^{-1} x - \frac{1}{2} \ln|k| + C$$

$$\int \tan^{-1} x \ dx = x \tan^{-1} x - \frac{1}{2} \ln|x^2+1| + C$$

$$\int \tan^{-1} x \ dx = x \tan^{-1} x - \frac{1}{2} \ln(x^2+1) + C$$

■ 2. Use integration by parts and substitution to evaluate the integral.

$$\int 7x\cos(9x) \ dx$$

Solution:

Use integration by parts first. Pick

$u = 7x$ differentiating $du = 7\ dx$

$dv = \cos(9x)\ dx$ integrating $v = \frac{1}{9}\sin(9x)$

Plug into the integration by parts formula.

$$\int 7x\cos(9x)\ dx = (7x)\left(\frac{1}{9}\sin(9x)\right) - \int\left(\frac{1}{9}\sin(9x)\right)(7\ dx)$$

$$\int 7x\cos(9x)\ dx = \frac{7}{9}x\sin(9x) - \frac{7}{9}\int \sin(9x)\ dx$$

Use substitution to evaluate the integral that remains. Let

$k = 9x$

$dk = 9\ dx$ so $dx = \frac{dk}{9}$

Substitute into the integral on the right.

$$\int 7x\cos(9x)\ dx = \frac{7}{9}x\sin(9x) - \frac{7}{9}\int \sin k\left(\frac{dk}{9}\right)$$

$$\int 7x\cos(9x)\ dx = \frac{7}{9}x\sin(9x) - \frac{7}{81}\int \sin k\ dk$$

Integrate, then back-substitute.

$$\int 7x\cos(9x)\ dx = \frac{7}{9}x\sin(9x) - \frac{7}{81}(-\cos k) + C$$

$$\int 7x\cos(9x)\ dx = \frac{7}{9}x\sin(9x) + \frac{7}{81}\cos k + C$$

$$\int 7x\cos(9x)\ dx = \frac{7}{9}x\sin(9x) + \frac{7}{81}\cos(9x) + C$$

■ 3. Use integration by parts and substitution to evaluate the integral.

$$\int \ln(3x+5)\ dx$$

Solution:

Use substitution first. Let

$$k = 3x + 5$$

$$dk = 3\ dx \text{ so } dx = \frac{dk}{3}$$

Substitute into the integral.

$$\int \ln(3x+5)\ dx$$

$$\int \ln k\left(\frac{dk}{3}\right)$$

$$\frac{1}{3}\int \ln k\ dk$$

Now use integration by parts. Pick

$u = \ln k$ differentiating $du = \frac{1}{k}\ dk$

$dv = dk$ integrating $v = k$

Plug into the integration by parts formula.

$$\int \ln k\ dk = (\ln k)(k) - \int (k)\left(\frac{1}{k}\ dk\right)$$

$$\int \ln k\ dk = k\ln k - \int\ dk$$

Integrate.

$$\int \ln k\ dk = k\ln k - k + C$$

Plug the value from the right side of this equation into the equation from earlier.

$$\frac{1}{3}\int \ln k\ dk$$

$$\frac{1}{3}(k\ln k - k + C)$$

Now back substitute.

$$\frac{1}{3}((3x+5)\ln(3x+5) - (3x+5) + C)$$

$$\frac{1}{3}\left[(3x+5)\ln(3x+5) - (3x+5)\right] + \frac{1}{3}C$$

If C is a constant, then $(1/3)C$ is also a constant, so we can simplify.

$$\frac{1}{3}\left[(3x+5)\ln(3x+5)-(3x+5)\right]+C$$

PROVE THE REDUCTION FORMULA

1. Use integration by parts, and $n = 8$, to prove the reduction formula for the integral.

$$\int x^n \sin x \ dx = -x^n \cos x + n \int x^{n-1} \cos x \ dx$$

Solution:

If $n = 8$, then

$$\int x^n \sin x \ dx = \int x^8 \sin x \ dx$$

If we're going to apply integration by parts to the integral on the right side of the equation, then pick

$u = x^8$	differentiating	$du = 8x^7 \ dx$
$dv = \sin x \ dx$	integrating	$v = -\cos x$

Plugging these values into the integration by parts formula gives

$$\int x^8 \sin x \ dx = (x^8)(-\cos x) - \int (-\cos x)(8x^7 \ dx)$$

$$\int x^8 \sin x \ dx = -x^8 \cos x + 8 \int x^7 \cos x \ dx$$

$$\int x^8 \sin x \ dx = -x^8 \cos x + 8 \int x^{8-1} \cos x \ dx$$

The format of this equation now matches the format of the reduction formula.

■ 2. Use integration by parts, and $n = 11$, to prove the reduction formula for the integral.

$$\int x^n \cos x \ dx = x^n \sin x - n \int x^{n-1} \sin x \ dx$$

Solution:

If $n = 11$, then

$$\int x^n \cos x \ dx = \int x^{11} \cos x \ dx$$

If we're going to apply integration by parts to the integral on the right side of the equation, then pick

$u = x^{11}$	differentiating	$du = 11x^{10} \ dx$
$dv = \cos x \ dx$	integrating	$v = \sin x$

Plugging these values into the integration by parts formula gives

$$\int x^{11} \cos x \ dx = (x^{11})(\sin x) - \int (\sin x)(11x^{10} \ dx)$$

$$\int x^{11} \cos x \, dx = x^{11} \sin x - 11 \int x^{10} \sin x \, dx$$

$$\int x^{11} \cos x \, dx = x^{11} \sin x - 11 \int x^{11-1} \sin x \, dx$$

The format of this equation now matches the format of the reduction formula.

■ 3. Use integration by parts, $a = 5$, and $n = 9$, to prove the reduction formula for the integral.

$$\int x^n a^x \, dx = \frac{x^n a^x}{\ln a} - \frac{n}{\ln a} \int x^{n-1} a^x \, dx$$

Solution:

If $a = 5$, and $n = 9$, then

$$\int x^n a^x \, dx = \int x^9 5^x \, dx$$

If we're going to apply integration by parts to the integral on the right side of the equation, then pick

$u = x^9$	differentiating	$du = 9x^8 \, dx$
$dv = 5^x \, dx$	integrating	$v = \dfrac{5^x}{\ln 5}$

Plugging these values into the integration by parts formula gives

$$\int x^9 5^x \ dx = (x^9)\left(\frac{5^x}{\ln 5}\right) - \int \left(\frac{5^x}{\ln 5}\right)(9x^8 \ dx)$$

$$\int x^9 5^x \ dx = \frac{x^9 5^x}{\ln 5} - \frac{9}{\ln 5}\int x^8 5^x \ dx$$

$$\int x^9 5^x \ dx = \frac{x^9 5^x}{\ln 5} - \frac{9}{\ln 5}\int x^{9-1} 5^x \ dx$$

The format of this equation now matches the format of the reduction formula.

TABULAR INTEGRATION

■ 1. Use tabular integration to evaluate the integral.

$$\int \left(5x^2 + 4x - 3\right) e^{2x}\ dx$$

Solution:

Let $f(x) = 5x^2 + 4x - 3$ and $g(x) = e^{2x}$.

Derivatives of $f(x)$	**Antiderivatives of $g(x)$**
$5x^2 + 4x - 3$	e^{2x}
$10x + 4$	$\frac{1}{2}e^{2x}$
10	$\frac{1}{4}e^{2x}$
0	$\frac{1}{8}e^{2x}$

Evaluate the integral by multiplying the entry in the first line, first column, by the entry in the second line, second column, beginning with a positive product. Then continue to pattern going down the table using opposite signs. The value of the integral will be

$$(5x^2+4x-3)\left(\frac{e^{2x}}{2}\right)-(10x+4)\left(\frac{e^{2x}}{4}\right)+10\left(\frac{e^{2x}}{8}\right)+C$$

Factor.

$$\frac{e^{2x}}{2}\left[(5x^2+4x-3)-(10x+4)\left(\frac{1}{2}\right)+10\left(\frac{1}{4}\right)\right]+C$$

$$\frac{e^{2x}}{2}\left(5x^2+4x-3-5x-2+\frac{5}{2}\right)+C$$

$$\frac{e^{2x}}{2}\left(5x^2-x-\frac{5}{2}\right)+C$$

■ 2. Use tabular integration to evaluate the integral.

$$\int x^3\cos(3x)\ dx$$

Solution:

Let $f(x)=x^3$ and $g(x)=\cos x$.

Derivatives of $f(x)$	**Antiderivatives of $g(x)$**
x^3	$\cos(3x)$
$3x^2$	$\frac{1}{3}\sin(3x)$

$6x$	$-\frac{1}{9}\cos(3x)$
6	$-\frac{1}{27}\sin(3x)$
0	$\frac{1}{81}\cos(3x)$

Evaluate the integral by multiplying the entry in the first line, first column, by the entry in the second line, second column, beginning with a positive product. Then continue to pattern going down the table using opposite signs. The value of the integral will be

$$\frac{x^3\sin 3x}{3}+\frac{3x^2\cos 3x}{9}-\frac{6x\sin 3x}{27}-\frac{6\cos 3x}{81}+C$$

$$\frac{x^3\sin 3x}{3}+\frac{x^2\cos 3x}{3}-\frac{2x\sin 3x}{9}-\frac{2\cos 3x}{27}+C$$

3. Use tabular integration to evaluate the integral.

$$\int \frac{x^4e^x}{6}\ dx$$

Solution:

Let $f(x)=x^4$ and $g(x)=e^x/6$.

Derivatives of $f(x)$ **Antiderivatives of $g(x)$**

x^4	$\frac{1}{6}e^x$
$4x^3$	$\frac{1}{6}e^x$
$12x^2$	$\frac{1}{6}e^x$
$24x$	$\frac{1}{6}e^x$
24	$\frac{1}{6}e^x$
0	$\frac{1}{6}e^x$

Evaluate the integral by multiplying the entry in the first line, first column, by the entry in the second line, second column, beginning with a positive product. Then continue to pattern going down the table using opposite signs. The value of the integral will be

$$x^4 \cdot \frac{e^x}{6} - 4x^3 \cdot \frac{e^x}{6} + 12x^2 \cdot \frac{e^x}{6} - 24x \cdot \frac{e^x}{6} + 24 \cdot \frac{e^x}{6} + C$$

$$\frac{e^x}{6}\left(x^4 - 4x^3 + 12x^2 - 24x + 24\right) + C$$

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