Calculus 2 Workbook Solutions

Polar curves

krista king MATH

POLAR COORDINATES

1. Convert the rectangular point $(2, -2)$ to a polar point.

Solution:

Use $x^2 + y^2 = r^2$ to find r.

$$2^2 + (-2)^2 = r^2$$

$$4 + 4 = r^2$$

$$8 = r^2$$

$$r = \sqrt{8}$$

$$r = 2\sqrt{2}$$

Use $\theta = \tan^{-1}(y/x)$ to find θ.

$$\theta = \tan^{-1}\left(\frac{-2}{2}\right)$$

$$\theta = \tan^{-1}(-1)$$

$$\theta = \frac{3\pi}{4}, \frac{7\pi}{4}$$

Since the point $(2, -2)$ is in quadrant IV, $\theta = 7\pi/4$. Therefore, the polar point is

$$\left(2\sqrt{2},\frac{7\pi}{4}\right)$$

■ 2. Convert the polar point $(3,\pi/4)$ to a rectangular point.

Solution:

Use $x=r\cos\theta$ and $y=r\sin\theta$ to find the rectangular point.

$$x=r\cos\theta$$

$$x=3\cos\left(\frac{\pi}{4}\right)$$

$$x=3\left(\frac{\sqrt{2}}{2}\right)$$

$$x=\frac{3\sqrt{2}}{2}$$

and

$$y=r\sin\theta$$

$$y=3\sin\left(\frac{\pi}{4}\right)$$

$$y=3\left(\frac{\sqrt{2}}{2}\right)$$

$$y = \frac{3\sqrt{2}}{2}$$

Therefore, the rectangular point is

$$\left(\frac{3\sqrt{2}}{2}, \frac{3\sqrt{2}}{2}\right)$$

■ 3. Convert the rectangular point $\left(-5\sqrt{3},5\right)$ to a polar point.

Solution:

Use $x^2 + y^2 = r^2$ to find r.

$$\left(-5\sqrt{3}\right)^2 + (5)^2 = r^2$$

$$75 + 25 = r^2$$

$$100 = r^2$$

$$r = \sqrt{100}$$

$$r = 10$$

Use $\theta = \tan^{-1}(y/x)$ to find θ.

$$\theta = \tan^{-1}\left(\frac{5}{-5\sqrt{3}}\right)$$

$$\theta = \tan^{-1}\left(-\frac{1}{\sqrt{3}}\right)$$

$$\theta = \frac{5\pi}{6}, \frac{11\pi}{6}$$

Since the point $\left(-5\sqrt{3},5\right)$ is in quadrant II, $\theta = 5\pi/6$. Therefore, the polar point is

$$\left(10,\frac{5\pi}{6}\right)$$

4. Convert the polar point $(8,11\pi/6)$ to a rectangular point.

Solution:

Use $x = r\cos\theta$ and $y = r\sin\theta$ to find the rectangular point.

$$x = r\cos\theta$$

$$x = 8\cos\left(\frac{11\pi}{6}\right)$$

$$x = 8\left(\frac{\sqrt{3}}{2}\right)$$

$$x = 4\sqrt{3}$$

and

$$y = r\sin\theta$$

$$y = 8\sin\left(\frac{11\pi}{6}\right)$$

$$y = 8\left(-\frac{1}{2}\right)$$

$$y = -4$$

Therefore, the rectangular point is

$$\left(4\sqrt{3}, -4\right)$$

CONVERTING RECTANGULAR EQUATIONS

1. Convert the rectangular equation to an equivalent polar equation.

$$4x^2 + 4y^2 = 64$$

Solution:

When converting from rectangular to polar, use $x = r\cos\theta$ and $y = r\sin\theta$. Simplify the given equation and then substitute.

$$4x^2 + 4y^2 = 64$$

$$x^2 + y^2 = 16$$

$$(r\cos\theta)^2 + (r\sin\theta)^2 = 16$$

$$r^2\cos^2\theta + r^2\sin^2\theta = 16$$

$$r^2\left(\cos^2\theta + \sin^2\theta\right) = 16$$

$$r^2(1) = 16$$

$$r^2 = 16$$

$$r = 4$$

This is the equivalent polar equation.

2. Convert the rectangular equation to an equivalent polar equation.

$$\frac{x^2}{9}+\frac{y^2}{4}=1$$

Solution:

When converting from rectangular to polar, use $x=r\cos\theta$ and $y=r\sin\theta$. Eliminate the denominators and then substitute.

$$\frac{x^2}{9}+\frac{y^2}{4}=1$$

$$4x^2+9y^2=36$$

$$4(r\cos\theta)^2+9(r\sin\theta)^2=36$$

$$4r^2\cos^2\theta+9r^2\sin^2\theta=36$$

$$r^2\left(4\cos^2\theta+9\sin^2\theta\right)=36$$

$$r^2\left[4\cos^2\theta+4\sin^2\theta+5\sin^2\theta\right]=36$$

$$r^2\left[4\left(\cos^2\theta+\sin^2\theta\right)+5\sin^2\theta\right]=36$$

$$r^2(4(1)+5\sin^2\theta)=36$$

$$r^2(4+5\sin^2\theta)=36$$

$$r^2=\frac{36}{4+5\sin^2\theta}$$

$$r = \frac{6}{\sqrt{4 + 5\sin^2\theta}}$$

This is the equivalent polar equation.

■ 3. Convert the rectangular equation to an equivalent polar equation.

$$(x-2)^2 + (y+2)^2 = 8$$

Solution:

When converting from rectangular to polar, use $x = r\cos\theta$ and $y = r\sin\theta$. Square the binomials and then substitute.

$$(x-2)^2 + (y+2)^2 = 8$$

$$x^2 - 4x + 4 + y^2 + 4y + 4 = 8$$

$$x^2 + y^2 - 4x + 4y = 0$$

$$r^2 - 4r\cos x + 4r\sin x = 0$$

$$r^2 = 4r\cos x - 4r\sin x$$

$$r = 4\cos x - 4\sin x$$

This is the equivalent polar equation.

4. Convert the rectangular equation to an equivalent polar equation.

$$\frac{x^2}{9} - \frac{y^2}{8} = 1$$

Solution:

When converting from rectangular to polar, use $x = r\cos\theta$ and $y = r\sin\theta$. Eliminate the denominators and then substitute.

$$\frac{x^2}{9} - \frac{y^2}{8} = 1$$

$$8x^2 - 9y^2 = 72$$

$$8(r\cos\theta)^2 - 9(r\sin\theta)^2 = 72$$

$$8r^2\cos^2\theta - 9r^2\sin^2\theta = 72$$

$$r^2\left(8\cos^2\theta - 9\sin^2\theta\right) = 72$$

$$r^2\left(8\cos^2\theta - 8\sin^2\theta - \sin^2\theta\right) = 72$$

$$8r^2\left(\cos^2\theta - \sin^2\theta\right) - r^2\sin^2\theta = 72$$

Use the identity $\cos(2\theta) = \cos^2\theta - \sin^2\theta$ to make a substitution.

$$8r^2\cos(2\theta) - r^2\sin^2\theta = 72$$

$$r^2\left(8\cos(2\theta) - \sin^2\theta\right) = 72$$

$$r^2 = \frac{72}{8\cos(2\theta) - \sin^2\theta}$$

$$r = \frac{6\sqrt{2}}{\sqrt{8\cos(2\theta) - \sin^2\theta}}$$

This is the equivalent polar equation.

CONVERTING POLAR EQUATIONS

■ 1. Convert the polar equation to an equivalent rectangular equation.

$$r = 4\cos\theta + 4\sin\theta$$

Solution:

When converting from polar to rectangular, use $x = r\cos\theta$ and $y = r\sin\theta$. For this problem, rewrite those as

$$\cos\theta = \frac{x}{r}$$

$$\sin\theta = \frac{y}{r}$$

Substitute these values.

$$r = 4\cos\theta + 4\sin\theta$$

$$r = \frac{4x}{r} + \frac{4y}{r}$$

$$r^2 = 4x + 4y$$

Replace r^2 with $x^2 + y^2$.

$$x^2 + y^2 = 4x + 4y$$

$$x^2 - 4x + y^2 - 4y = 0$$

$$x^2 - 4x + 4 + y^2 - 4y + 4 = 4 + 4$$

$$\left(x^2 - 4x + 4\right) + \left(y^2 - 4y + 4\right) = 8$$

$$(x - 2)^2 + (y - 2)^2 = 8$$

This is the equivalent rectangular equation.

■ 2. Convert the polar equation to an equivalent rectangular equation.

$$r = 12\cos\theta - 12\sin\theta$$

Solution:

When converting from polar to rectangular, use $x = r\cos\theta$ and $y = r\sin\theta$. For this problem, rewrite those as

$$\cos\theta = \frac{x}{r}$$

$$\sin\theta = \frac{y}{r}$$

Substitute these values.

$$r = 12\cos\theta - 12\sin\theta$$

$$r = \frac{12x}{r} - \frac{12y}{r}$$

$$r^2 = 12x - 12y$$

$$x^2 + y^2 = 12x - 12y$$

$$x^2 - 12x + y^2 + 12y = 0$$

$$x^2 - 12x + 36 + y^2 + 12y + 36 = 36 + 36$$

$$\left(x^2 - 12x + 36\right) + \left(y^2 + 12y + 36\right) = 72$$

$$(x - 6)^2 + (y + 6)^2 = 72$$

This is the equivalent rectangular equation.

3. Convert the polar equation to an equivalent rectangular equation.

$$r = 3\sin\left(\theta + \frac{\pi}{4}\right)$$

Solution:

Use the identity $\sin(a + b) = \sin a \cos b + \cos a \sin b$ to rewrite the polar equation.

$$r = 3\sin\left(\theta + \frac{\pi}{4}\right)$$

$$r = 3\left(\sin\theta\cos\left(\frac{\pi}{4}\right) + \cos\theta\sin\left(\frac{\pi}{4}\right)\right)$$

$$r = 3\left(\frac{\sqrt{2}}{2}\sin\theta + \frac{\sqrt{2}}{2}\cos\theta\right)$$

$$r = \frac{3\sqrt{2}}{2}\sin\theta + \frac{3\sqrt{2}}{2}\cos\theta$$

When converting from polar to rectangular, use $x = r\cos\theta$ and $y = r\sin\theta$. For this problem, rewrite those as

$$\cos\theta = \frac{x}{r}$$

$$\sin\theta = \frac{y}{r}$$

Substitute these values.

$$r = \frac{3\sqrt{2}}{2}\frac{y}{r} + \frac{3\sqrt{2}}{2}\frac{x}{r}$$

$$r^2 = \frac{3\sqrt{2}}{2}y + \frac{3\sqrt{2}}{2}x$$

$$x^2 + y^2 = \frac{3\sqrt{2}}{2}y + \frac{3\sqrt{2}}{2}x$$

$$x^2 - \frac{3\sqrt{2}}{2}x + y^2 - \frac{3\sqrt{2}}{2}y = 0$$

Complete the square with respect to both variables.

$$x^2 - \frac{3\sqrt{2}}{2}x + \frac{9}{8} + y^2 - \frac{3\sqrt{2}}{2}y + \frac{9}{8} = \frac{9}{8} + \frac{9}{8}$$

$$\left(x^2-\frac{3\sqrt{2}}{2}x+\frac{9}{8}\right)+\left(y^2-\frac{3\sqrt{2}}{2}y+\frac{9}{8}\right)=\frac{9}{4}$$

$$\left(x-\frac{3\sqrt{2}}{4}\right)^2+\left(y-\frac{3\sqrt{2}}{4}\right)^2=\frac{9}{4}$$

This is the equivalent rectangular equation.

■ 4. Convert the polar equation to an equivalent rectangular equation.

$$r=6\cos\theta-10\sin\theta$$

Solution:

When converting from polar to rectangular, use $x=r\cos\theta$ and $y=r\sin\theta$. For this problem, rewrite those as

$$\cos\theta=\frac{x}{r}$$

$$\sin\theta=\frac{y}{r}$$

Substitute these values.

$$r=6\cos\theta-10\sin\theta$$

$$r=\frac{6x}{r}-\frac{10y}{r}$$

$$r^2 = 6x - 10y$$

$$x^2 + y^2 = 6x - 10y$$

$$x^2 - 6x + y^2 + 10y = 0$$

$$x^2 - 6x + 9 + y^2 + 10y + 25 = 9 + 25$$

$$\left(x^2 - 6x + 9\right) + \left(y^2 + 10y + 25\right) = 34$$

$$(x - 3)^2 + (y + 5)^2 = 34$$

This is the equivalent rectangular equation.

■ 5. Convert the polar equation to an equivalent rectangular equation.

$$r = 12 \sin \theta$$

Solution:

When converting from polar to rectangular, use $x = r \cos \theta$ and $y = r \sin \theta$. For this problem, rewrite those as

$$\cos \theta = \frac{x}{r}$$

$$\sin \theta = \frac{y}{r}$$

Substitute these values.

$$r = 12\sin\theta$$

$$r = \frac{12y}{r}$$

$$r^2 = 12y$$

$$x^2 + y^2 = 12y$$

$$x^2 + y^2 - 12y = 0$$

$$x^2 + y^2 - 12y + 36 = 36$$

$$x^2 + \left(y^2 - 12y + 36\right) = 36$$

$$x^2 + (y - 6)^2 = 36$$

This is the equivalent rectangular equation.

DISTANCE BETWEEN POLAR POINTS

1. Calculate the distance between the polar coordinate points.

$$\left(2,\frac{\pi}{3}\right) \text{ and } \left(2,\frac{11\pi}{6}\right)$$

Solution:

Find the distance between two polar coordinate points with the formula

$$D=\sqrt{{r_1}^2+{r_2}^2-2r_1r_2\cos\left(\theta_1-\theta_2\right)}$$

Plugging the points into this distance formula, we get

$$D=\sqrt{2^2+2^2-2(2)(2)\cos\left(\frac{11\pi}{6}-\frac{\pi}{3}\right)}$$

$$D=\sqrt{4+4-8\cos\left(\frac{3\pi}{2}\right)}$$

$$D=\sqrt{8-8(0)}$$

$$D=\sqrt{8}$$

$$D=2\sqrt{2}$$

This is the distance between the polar points.

2. Calculate the distance between the polar coordinate points.

$$\left(4, \frac{7\pi}{12}\right) \text{ and } \left(2, \frac{\pi}{12}\right)$$

Solution:

Find the distance between two polar coordinate points with the formula

$$D = \sqrt{{r_1}^2 + {r_2}^2 - 2r_1r_2\cos\left(\theta_1 - \theta_2\right)}$$

Plugging the points into this distance formula, we get

$$D = \sqrt{4^2 + 2^2 - 2(4)(2)\cos\left(\frac{7\pi}{12} - \frac{\pi}{12}\right)}$$

$$D = \sqrt{16 + 4 - 16\cos\left(\frac{\pi}{2}\right)}$$

$$D = \sqrt{20 - 16(0)}$$

$$D = \sqrt{20}$$

$$D = 2\sqrt{5}$$

This is the distance between the polar points.

3. Calculate the distance between the polar coordinate points.

$\left(4,\frac{\pi}{4}\right)$ and $\left(9,\frac{3\pi}{4}\right)$

Solution:

Find the distance between two polar coordinate points with the formula

$$D=\sqrt{{r_1}^2+{r_2}^2-2r_1r_2\cos\left(\theta_1-\theta_2\right)}$$

Plugging the points into this distance formula, we get

$$D=\sqrt{4^2+9^2-2(4)(9)\cos\left(\frac{3\pi}{4}-\frac{\pi}{4}\right)}$$

$$D=\sqrt{16+81-72\cos\left(\frac{\pi}{2}\right)}$$

$$D=\sqrt{97-72(0)}$$

$$D=\sqrt{97}$$

This is the distance between the polar points.

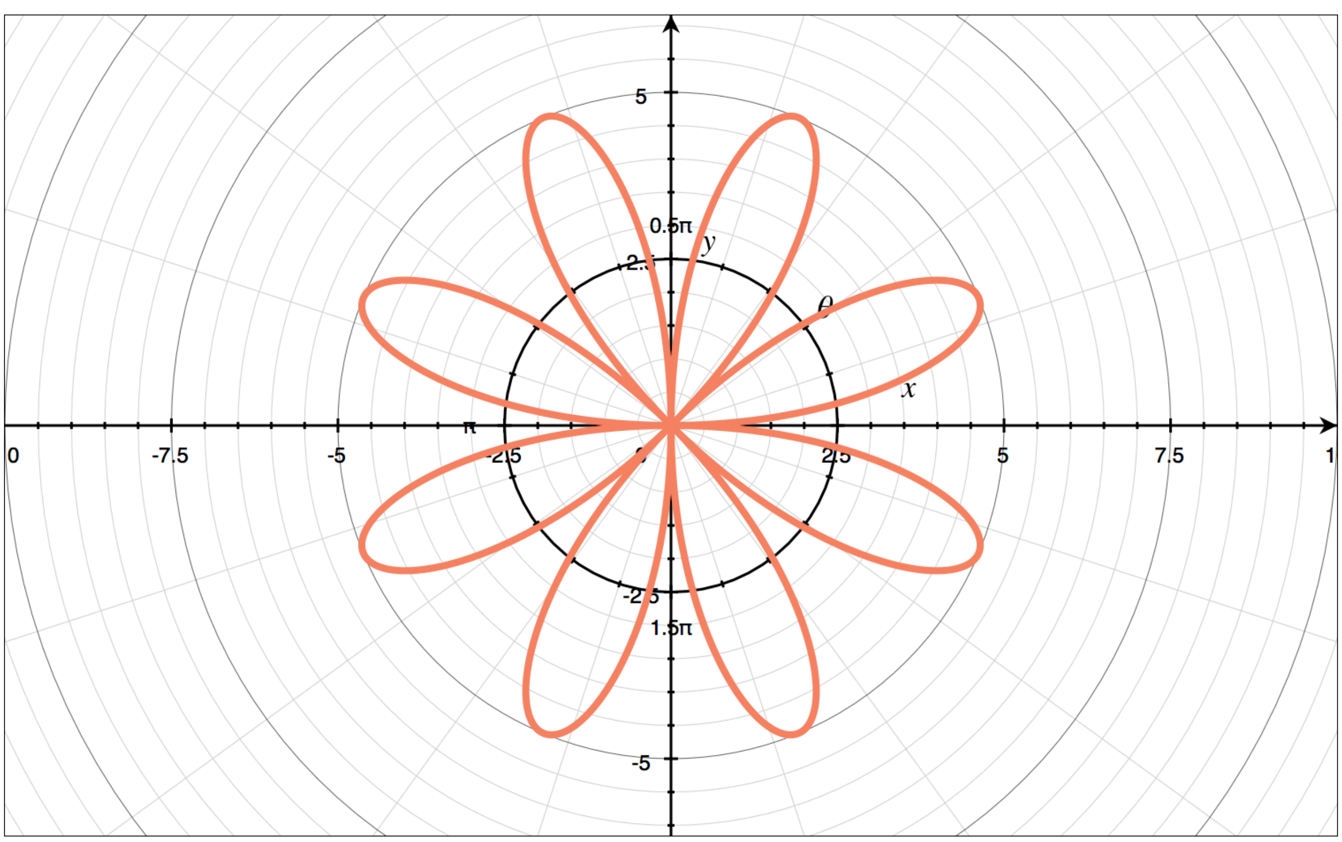
SKETCHING POLAR CURVES

■ 1. Graph the polar curve. How many petals does the curve have, and what is the length of each petal?

$$r = 5\sin(4\theta)$$

Solution:

The polar equation represents a rose. The length of the petals of a curve in the form $r = a\sin(b\theta)$ is a units. The number of petals depends on the value of b. If b is an odd number, then the graph has b petals. If b is an even number, then the graph has $2b$ petals. In this question, $a = 5$, $b = 4$. Therefore, the graph has 8 petals and the length of each petal is 5 units. The graph of the given polar equation is



W W W . K R I S T A K I N G M A T H . C O M