**Topic**: Sketching the area between curves

**Question**: Sketch the region(s) enclosed by the $x$-axis and the curve. Determine the best way to find total area of the regions, then calculate the total area.

$$f(x) = x^3 - 3x^2 + 2x$$

**Answer choices**:

A $\displaystyle \int_0^1 x^3 - 3x^2 + 2x \ dx + \int_1^2 x^3 - 3x^2 + 2x \ dx = \frac{1}{2}$

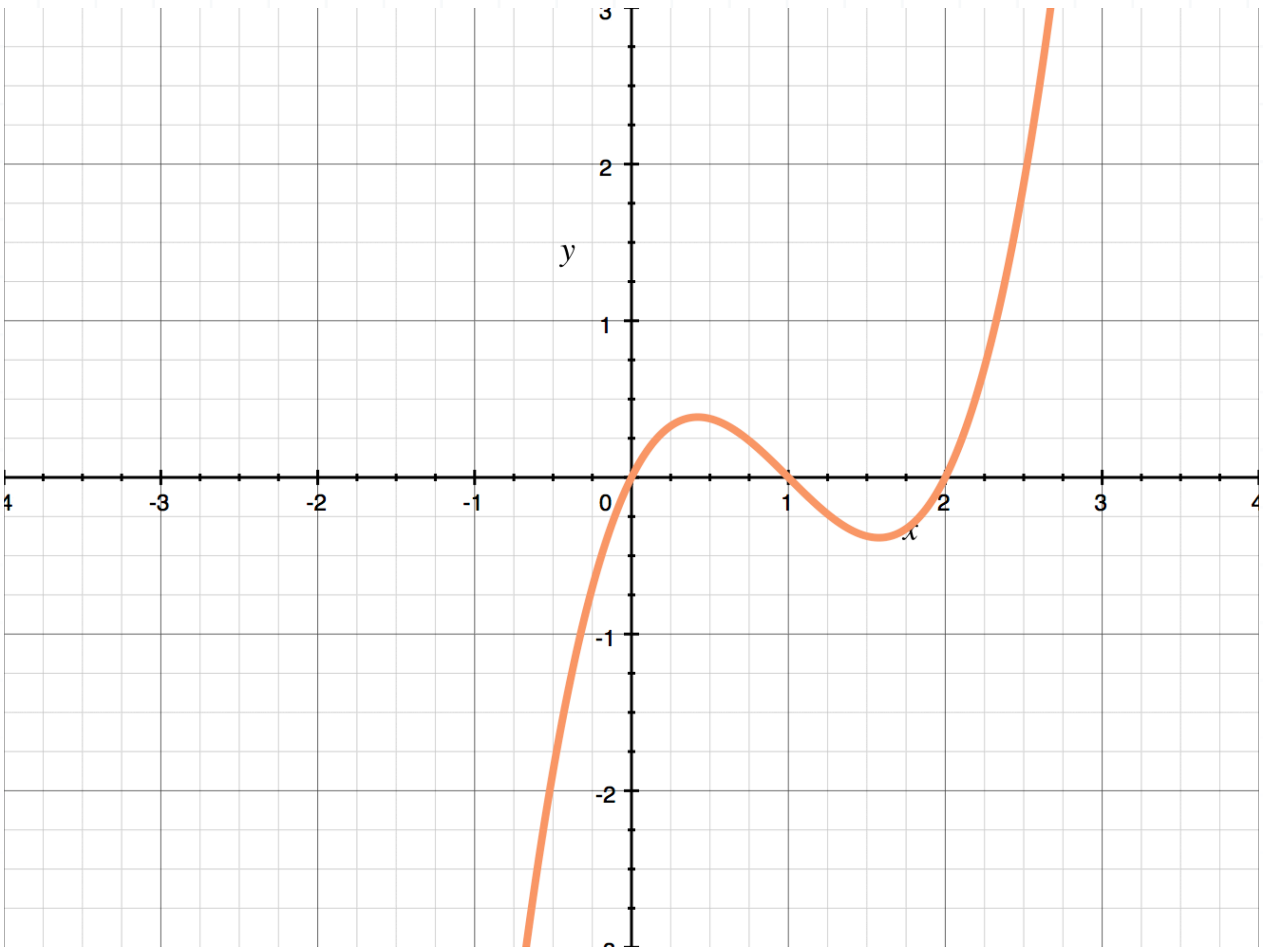
B $\displaystyle \int_0^1 x^3 - 3x^2 + 2x \ dx + \left| \int_1^2 x^3 - 3x^2 + 2x \ dx \right| = \frac{1}{2}$

C $\displaystyle \int_0^2 x^3 - 3x^2 + 2x \ dx = 0$

D $\displaystyle \int_0^1 x^3 - 3x^2 + 2x \ dx + \int_1^2 x^3 - 3x^2 + 2x \ dx = 0$

**Solution**: B

The graph of $f(x) = x^3 - 3x^2 + 2x$ is the graph of a cubic function. A sketch of $f(x)$ is




Notice that the graph of $f(x)$ is above the $x$-axis in the interval $[0,1]$ and below the $x$-axis in the interval $[1,2]$.

We integrate a function to find the area between a curve and the x-axis. However, the integral of $f(x)$ over the entire interval $[0,2]$ would give us the net area, and not the total area.

Additionally, integrating a function on an interval where the function is above the $x$-axis gives the area between the curve and the $x$-axis, but integrating a function on an interval where the function is below the $x$-axis gives a negative value of the area between the curve and the $x$-axis.

Therefore, if we want the total area between $f(x)$ and the $x$-axis, we will have to integrate the absolute value of $f(x)$ on the interval in which $f(x)$ is below the $x$-axis.

To find the total area, we will integrate $f(x)$ on the interval $[0,1]$ and then integrate $|f(x)|$ on the interval $[1,2]$, and then add the results of the integration.

$$A=\int_0^1 f(x)\ dx+\int_1^2 \left|f(x)\right|\ dx$$

$$A=\int_0^1 x^3-3x^2+2x\ dx+\left|\int_1^2 x^3-3x^2+2x\ dx\right|$$

Integrate using the power rule, then evaluate over the interval.

$$A=\frac{1}{4}x^4-x^3+x^2\Big|_0^1+\left|\frac{1}{4}x^4-x^3+x^2\Big|_1^2\right|$$

$$A=\frac{1}{4}(1)^4-(1)^3+(1)^2-\left(\frac{1}{4}(0)^4-(0)^3+(0)^2\right)+\left|\frac{1}{4}(2)^4-(2)^3+(2)^2-\left(\frac{1}{4}(1)^4-(1)^3+(1)^2\right)\right|$$

$$A=\frac{1}{4}-1+1+\left|\frac{1}{4}(16)-8+4-\left(\frac{1}{4}-1+1\right)\right|$$

$$A=\frac{1}{4}-1+1+\left|4-8+4-\frac{1}{4}+1-1\right|$$

$$A=\frac{1}{4}+\left|-\frac{1}{4}\right|$$

$$A = \frac{1}{2}$$

**Topic**: Sketching the area between curves

**Question**: Sketch the region(s) enclosed by the curves. Determine the best integration method to find total area of the regions. Then, calculate the total area.

$$f(x) = x(x-3)$$

$$x = 0$$

$$x = 5$$

**Answer choices**:

A $\int_0^3 x^2 - 3x\ dx + \int_3^5 x^2 - 3x\ dx = \frac{25}{6}$

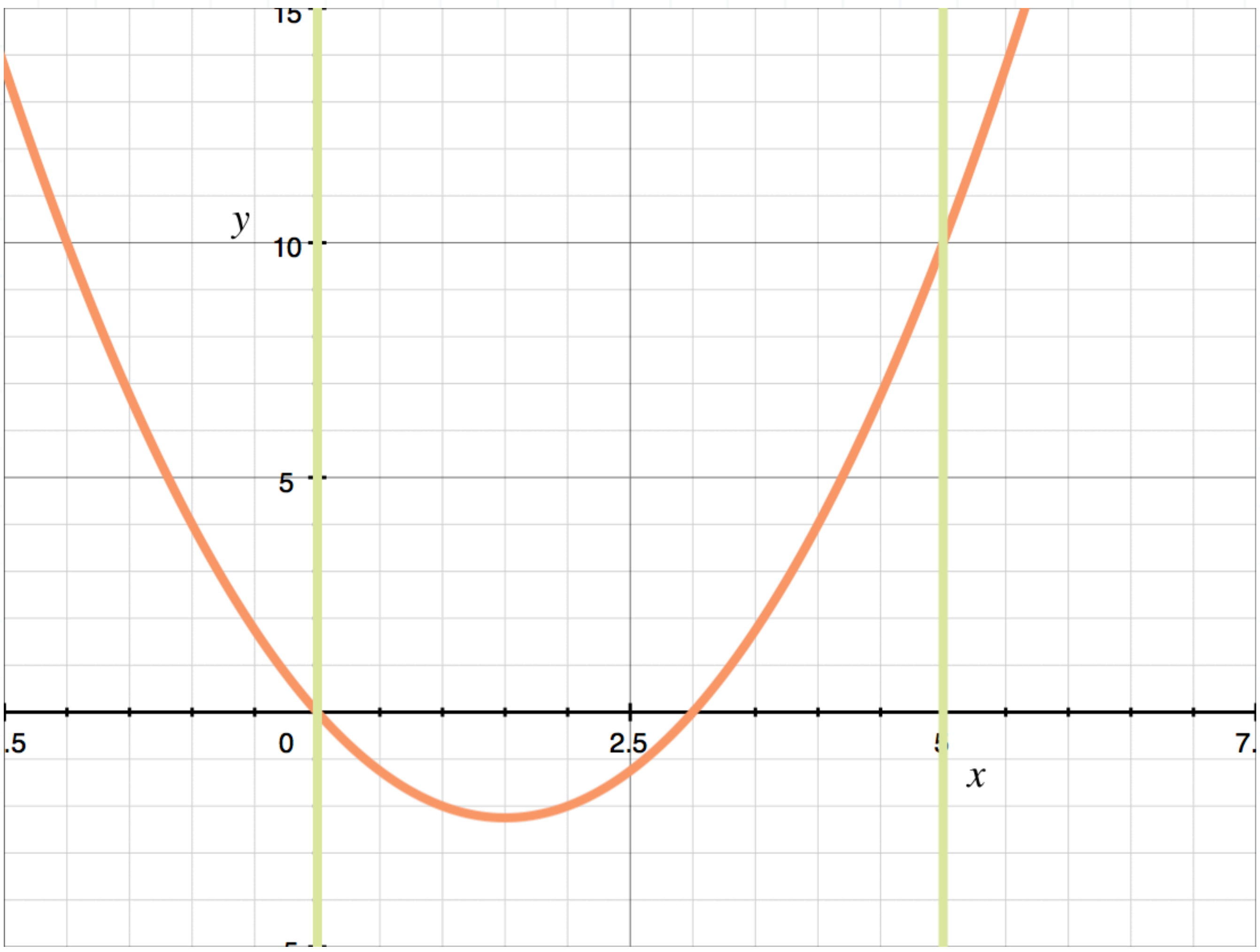
B $\int_0^3 x^2 - 3x\ dx + \left| \int_3^5 x^2 - 3x\ dx \right| = \frac{25}{6}$

C $\int_0^3 x^2 - 3x\ dx + \int_3^5 x^2 - 3x\ dx = \frac{79}{6}$

D $\left| \int_0^3 x^2 - 3x\ dx \right| + \int_3^5 x^2 - 3x\ dx = \frac{79}{6}$

**Solution**: D

The graph of $f(x) = x^2 - 3x$ is the graph of a quadratic function. A sketch of $f(x)$ between the lines $x = 0$ and $x = 5$ is




Notice that the graph of $f(x)$ is below the $x$-axis on the interval $[0,3]$ and above the $x$-axis on the interval $[3,5]$.

We integrate a function to find the area between a curve and the $x$-axis. However, the integral of $f(x)$ over the entire interval $[0,5]$ would give us the net area, and not the total area.

Additionally, integrating a function on an interval where the function is above the $x$-axis gives the area between the curve and the $x$-axis, but integrating a function on an interval where the function is below the $x$-axis gives a negative value of the area between the curve and the $x$-axis.

Therefore, if we want the total area between $f(x)$ and the $x$-axis between the lines $x=0$ and $x=5$, we will have to integrate the absolute value of $f(x)$ on the interval in which $f(x)$ is below the $x$-axis.

To find the total area, we will integrate $|f(x)|$ on the interval $[0,3]$ and then integrate $f(x)$ on the interval $[3,5]$, and then add the results of the integration.

$$\left|\int_0^3 f(x)\ dx\right|+\int_3^5 f(x)\ dx$$

$$\left|\int_0^3 x(x-3)\ dx\right|+\int_3^5 x(x-3)\ dx$$

$$\left|\int_0^3 x^2-3x\ dx\right|+\int_3^5 x^2-3x\ dx$$

Integrate.

$$\left|\frac{1}{3}x^3-\frac{3}{2}x^2\Big|_0^3\right|+\frac{1}{3}x^3-\frac{3}{2}x^2\Big|_3^5$$

Evaluate over each interval.

$$\left|\frac{1}{3}(3)^3-\frac{3}{2}(3)^2-\left(\frac{1}{3}(0)^3-\frac{3}{2}(0)^2\right)\right|+\frac{1}{3}(5)^3-\frac{3}{2}(5)^2-\left(\frac{1}{3}(3)^3-\frac{3}{2}(3)^2\right)$$

$$\left|9-\frac{27}{2}\right|+\frac{125}{3}-\frac{75}{2}-\left(9-\frac{27}{2}\right)$$

$$\left|\frac{18}{2}-\frac{27}{2}\right|+\frac{125}{3}-\frac{75}{2}-9+\frac{27}{2}$$

$$\frac{125}{3}-\frac{39}{2}-9$$

$$\frac{250}{6}-\frac{117}{6}-\frac{54}{6}$$

$$\frac{79}{6}$$

**Topic**: Sketching the area between curves

**Question**: Sketch the region enclosed by the curves. Determine the best integration method to find total area of the region. Then, calculate the total area.

$$x = y^2$$

$$x = y + 2$$

**Answer choices**:

A $\int_{-1}^{2} y + 2 - y^2 \, dy = \frac{9}{2}$

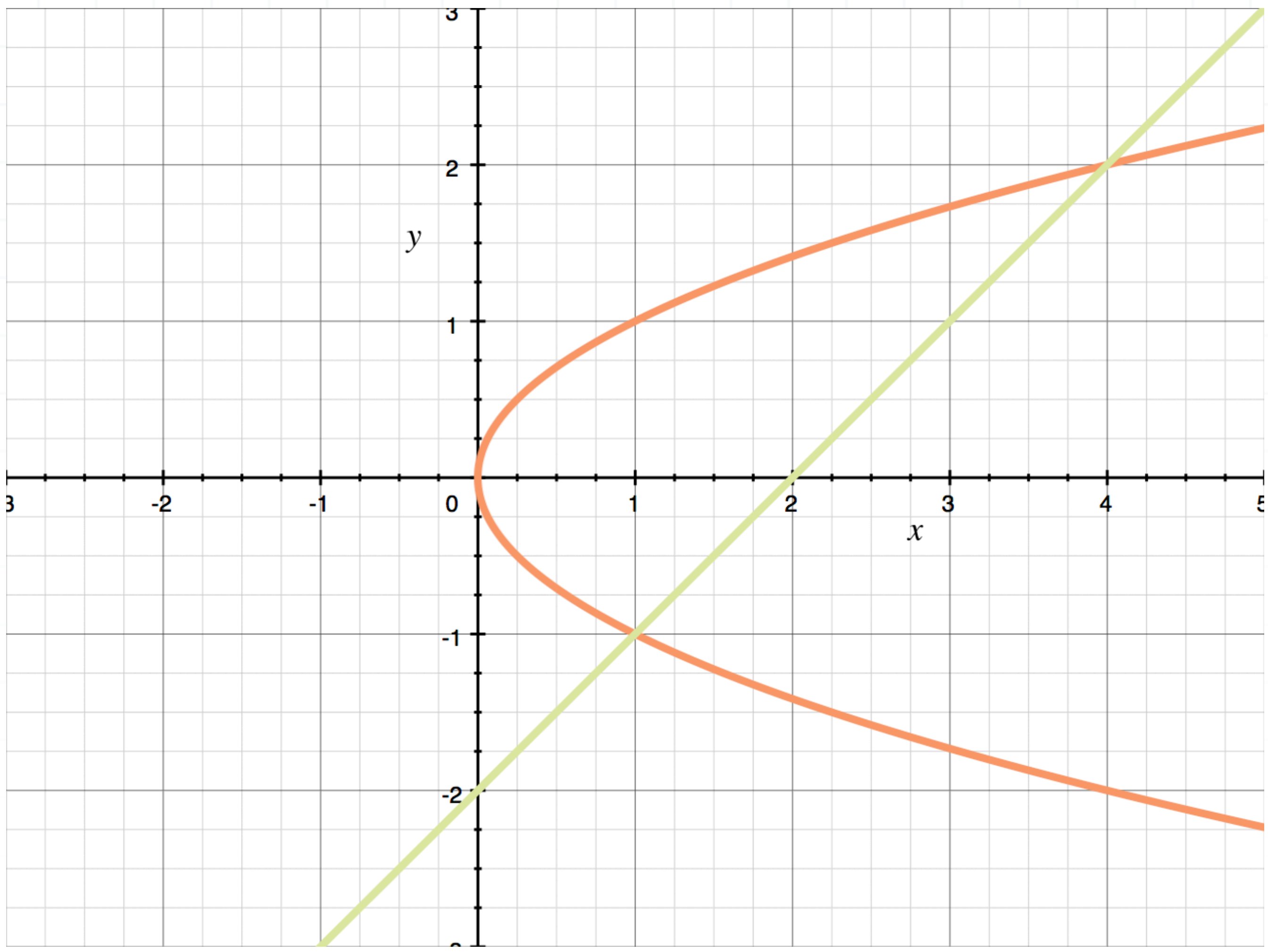
B $\int_{-1}^{2} y^2 - y - 2 \, dy = \frac{9}{2}$

C $\int_{0}^{5} x^2 - x - 2 \, dx = 19\frac{1}{6}$

D $\int_{1}^{5} x^2 - x - 2 \, dx = 21\frac{1}{3}$

**Solution**: A

The graphs of $x = y^2$ and $x = y + 2$ is the graph of a quadratic curve and a linear function. A sketch is




Notice that the graph of the parabola opens toward the right and is not a function because it fails the vertical line test.

We usually integrate to find the area between the two curves with respect to $x$. If we do that in this problem, we will need more than one integral because one curve is not a function, and the curves intersect more than once. However, if we integrate with respect to $y$, we can find the area enclosed by the two curves with a single integral.

Also note, from the graph above, that the two curves intersect at the points $(1, -1)$ and $(4,2)$ which gives us our integration limits, and since we are integrating with respect to $y$, the integration limits are the $y$-values in the points of intersection. Thus, the limits of integration will be from $-1$ to $2$.

When integrating with respect to $x$ to find the area between two functions, we subtract the lower function from the higher function in the integrand. When we integrate with respect to $y$ to find the area between two curves, we subtract the left curve from the right curve.

To find the area, we will use this integral

$$\int_{-1}^{2} y+2-y^2\ dy$$

Integrate.

$$\frac{1}{2}y^2+2y-\frac{1}{3}y^3\Big|_{-1}^{2}$$

Evaluate over the interval.

$$\frac{1}{2}(2)^2+2(2)-\frac{1}{3}(2)^3-\left(\frac{1}{2}(-1)^2+2(-1)-\frac{1}{3}(-1)^3\right)$$

$$2+4-\frac{8}{3}-\left(\frac{1}{2}-2+\frac{1}{3}\right)$$

$$2+4-\frac{8}{3}-\frac{1}{2}+2-\frac{1}{3}$$

$$5-\frac{1}{2}$$

$$\frac{9}{2}$$