Topic: Integration by parts two times

Question: Use integration by parts to evaluate the integral.

$$\int e^{2x}\sin(4x)\ dx$$

Answer choices:

A $\quad \dfrac{e^{2x}\sin(4x) - 2e^{2x}\cos(4x)}{10} + C$

B $\quad \dfrac{e^{2x}\sin(4x) - e^{2x}\cos(4x)}{10} + C$

C $\quad \dfrac{e^{2x}\sin(4x) - e^{2x}\cos(4x)}{5} + C$

D $\quad \dfrac{e^{2x}\sin(4x) - 2e^{2x}\cos(4x)}{5} + C$

Solution: A

Sometimes integration by parts is the correct tool to use to evaluate the integral, but using it only once doesn't simplify the integral enough, and we have to use it a second, or even a third time. However many times we use it, we always use the same integration by parts formula,

$$\int u\ dv = uv - \int v\ du$$

We need to identify a value for u and a value for dv in our original integral, and then take the derivative of u to get du, and take then integral of dv to get v. Let's do that for this integral.

$$u = \sin(4x)$$

$$du = 4\cos(4x)\ dx$$

and

$$dv = e^{2x}\ dx$$

$$v = \frac{e^{2x}}{2}$$

Plugging these values into the right side of the integration by parts formula, we get

$$\int e^{2x}\sin(4x)\ dx = [\sin(4x)]\left(\frac{e^{2x}}{2}\right) - \int\left(\frac{e^{2x}}{2}\right)[4\cos(4x)\ dx]$$

$$\int e^{2x}\sin(4x)\ dx = \frac{e^{2x}\sin(4x)}{2} - 2\int e^{2x}\cos(4x)\ dx$$

Since we haven't managed to simplify our integral to the point where we can evaluate it, we'll have to try integration by parts a second time.

$$u_2 = \cos(4x)$$

$$du_2 = -4\sin(4x)\ dx$$

and

$$dv_2 = e^{2x}\ dx$$

$$v_2 = \frac{e^{2x}}{2}$$

Plugging these values into the right side of the integration by parts formula to replace just the remaining integral, we get

$$\int e^{2x}\sin(4x)\ dx = \frac{e^{2x}\sin(4x)}{2} - 2\left[\cos(4x)\left(\frac{e^{2x}}{2}\right) - \int\left(\frac{e^{2x}}{2}\right)\left[-4\sin(4x)\ dx\right]\right]$$

$$\int e^{2x}\sin(4x)\ dx = \frac{e^{2x}\sin(4x)}{2} - 2\left[\frac{e^{2x}\cos(4x)}{2} + 2\int e^{2x}\sin(4x)\ dx\right]$$

$$\int e^{2x}\sin(4x)\ dx = \frac{e^{2x}\sin(4x)}{2} - e^{2x}\cos(4x) - 4\int e^{2x}\sin(4x)\ dx$$

We've simplified the right-hand side as much as we can, and our remaining integral is the same as our original integral, and the same as the integral on the left-hand side, which means we can add the integral from the right side to the left side, and then solve for our original integral.

$$\int e^{2x}\sin(4x)\ dx + 4\int e^{2x}\sin(4x)\ dx = \frac{e^{2x}\sin(4x)}{2} - e^{2x}\cos(4x) + C$$

$$5\int e^{2x}\sin(4x)\ dx = \frac{e^{2x}\sin(4x)}{2} - e^{2x}\cos(4x) + C$$

$$\int e^{2x}\sin(4x)\ dx = \frac{e^{2x}\sin(4x)}{10} - \frac{e^{2x}\cos(4x)}{5} + C$$

$$\int e^{2x}\sin(4x)\ dx = \frac{e^{2x}\sin(4x) - 2e^{2x}\cos(4x)}{10} + C$$

Topic: Integration by parts two times

Question: Use integration by parts to evaluate the integral.

$$\int x^2e^x\ dx$$

Answer choices:

A $\quad e^x\left(x^2-2x+2\right)+5$

B $\quad e^x\left(x^2-2x+2\right)+C$

C $\quad x^2e^x-2xe^x+2e^x$

D $\quad e^x\left(x^2-2x+2\right)$

Solution: B

The question asks us to evaluate

$$\int x^2e^x\ dx$$

using integration by parts.

Integration by parts is a method of evaluating an integral that cannot be evaluated using normal integration techniques, by using integration by substitution, or by using integration formulas.

The general formula for integration by parts is

$$\int u\ dv = uv - \int v\ du$$

In this formula, we separate the integrand into two parts; one part is called u and the other part is called dv. In making these two parts, we must use all of the integrand.

Although there is sometimes flexibility in choosing u, we can generally use the following sequence of choices to select the best part of the integrand to be u. This method involves the acronym LIPET, where we select the first u in the sequence of the list below. The letters mean

L Logarithmic expression

I Inverse trigonometric expression

P Polynomial expression

E Exponential expression

T Trigonometric function expression

In this problem, the integrand is x^2e^x where we have a polynomial exponential expression and an exponential expression. In the LIPET sequence, polynomial comes before exponential, so u is the polynomial expression. Let's identify the parts we need to integrate. Additionally, since this problem will take more than one integration by parts, we will use subscripts with the u, v, du and dv.

$$u_1 = x^2$$

$$du_1 = 2x\ dx$$

$$dv_1 = e^x\ dx$$

$$v_1 = e^x$$

We are now ready to integrate by parts using the general formula.

$$\int u_1\ dv_1 = u_1v_1 - \int v_1\ du_1$$

$$\int x^2e^x\ dx = x^2e^x - \int 2xe^x\ dx$$

Next, let's evaluate the new integral, which will require integration by parts a second time.

$$\int 2xe^x\ dx$$

Using the LIPET sequence again, we again have an exponential expression and a trigonometric expression. The exponential expression will be the u and the trigonometric function will be the dv.

$$\int u_2\ dv_2 = u_2v_2 - \int v_2\ du_2$$

$$u_2 = 2x$$

$$du_2 = 2\ dx$$

$$dv_2 = e^x\ dx$$

$$v_2 = e^x$$

$$\int 2e^x\ dx = 2xe^x - \int 2e^x\ dx$$

Now, we will rewrite the equation using the original integral.

$$\int x^2e^x\ dx = x^2e^x - \left(2xe^x - \int 2e^x\ dx\right)$$

Next, we'll distribute the negative and evaluate the integral on the right side of the equation.

$$\int x^2e^x\ dx = x^2e^x - 2xe^x + \int 2e^x\ dx$$

Since we will have an indefinite integral equal the sum of terms, we will add an arbitrary constant C to accommodate the possibility of a constant term in the answer.

$$\int x^2e^x\,dx = x^2e^x - 2xe^x + 2e^x + C$$

Although we have an acceptable final answer, we can factor out the greatest common factor from the terms. The arbitrary constant does not contain the GCF.

$$\int x^2e^x\,dx = e^x\left(x^2 - 2x + 2\right) + C$$

Topic: Integration by parts two times

Question: Use integration by parts to evaluate the integral.

$$\int 2e^{5x}\sin(2x)\ dx$$

Answer choices:

A $\quad -e^{5x}\cos(2x)+\frac{5}{2}e^{5x}\sin(2x)+C$

B $\quad -e^{5x}\cos(2x)+\frac{5}{2}e^{5x}\sin(2x)$

C $\quad \frac{4}{29}\left(-e^{5x}\cos(2x)+\frac{5}{2}e^{5x}\sin(2x)\right)$

D $\quad \frac{4}{29}\left(-e^{5x}\cos(2x)+\frac{5}{2}e^{5x}\sin(2x)\right)+C$

Solution: D

The question asks us to evaluate

$$\int 2e^{5x}\sin(2x)\ dx$$

using integration by parts.

Integration by parts is a method of evaluating an integral that cannot be evaluated using normal integration techniques, by using integration by substitution, or by using integration formulas.

The general formula for integration by parts is

$$\int u\ dv = uv - \int v\ du$$

In this formula, we separate the integrand into two parts; one part is called u and the other part is called dv. In making these two parts, we must use all of the integrand.

Although there is sometimes flexibility in choosing u, we can generally use the following sequence of choices to select the best part of the integrand to be u. This method involves the acronym LIPET, where we select the first u in the sequence of the list below. The letters mean

L Logarithmic expression

I Inverse trigonometric expression

P Polynomial expression

E Exponential expression

T Trigonometric function expression

In this problem, the integrand is $2e^{5x}\sin(2x)$ where we have an exponential expression and a trigonometric function. In the LIPET sequence, exponential comes before trigonometric function, so u is the exponential expression. Let's identify the parts we need to integrate. Additionally, since this problem will take more than one integration by parts, we will use subscripts with the u, v, du and dv.

$$u_1 = 2e^{5x}$$

$$du_1 = 10e^{5x}\ dx$$

$$dv_1 = \sin(2x)\ dx$$

$$v_1 = -\frac{1}{2}\cos(2x)$$

We're now ready to integrate by parts using the general formula.

$$\int u_1\ dv_1 = u_1v_1 - \int v_1\ du_1$$

$$\int 2e^{5x}\sin(2x)\ dx = \left(2e^{5x}\right)\left(-\frac{1}{2}\cos(2x)\right) - \int -\frac{1}{2}\cos(2x)\left(10e^{5x}\right)\ dx$$

$$\int 2e^{5x}\sin(2x)\ dx = -e^{5x}\cos(2x) + 5\int e^{5x}\cos(2x)\ dx$$

Next, let's evaluate the new integral, which will require integration by parts a second time.

$$5\int e^{5x}\cos(2x)\ dx$$

Using the LIPET sequence again, we have an exponential expression and a trigonometric expression. The exponential expression will be the u and the trigonometric function will be the dv.

$$\int u_2\ dv_2 = u_2v_2 - \int v_2\ du_2$$

$$u_2 = e^{5x}$$

$$du_2 = 5e^{5x}\ dx$$

$$dv_2 = \cos(2x)\ dx$$

$$v_2 = \frac{1}{2}\sin(2x)$$

$$5\int e^{5x}\cos(2x)\ dx = 5\left[\frac{1}{2}e^{5x}\sin(2x) - \int \frac{1}{2}\left(5e^{5x}\right)\sin(2x)\ dx\right]$$

$$5\int e^{5x}\cos(2x)\ dx = \frac{5}{2}e^{5x}\sin(2x) - \frac{25}{2}\int e^{5x}\sin(2x)\ dx$$

Now, we'll rewrite the equation using the original integral.

$$\int 2e^{5x}\sin(2x)\ dx = -e^{5x}\cos(2x) + \frac{5}{2}e^{5x}\sin(2x) - \frac{25}{2}\int e^{5x}\sin(2x)\ dx$$

Notice the similarity between the original integral and the new integral in the above equation. We'll multiply the new integral by $1/2$ on the outside and by 2 on the inside to make the two integrals the same.

$$\int 2e^{5x}\sin(2x)\ dx = -e^{5x}\cos(2x) + \frac{5}{2}e^{5x}\sin(2x) - \frac{25}{4}\int 2e^{5x}\sin(2x)\ dx$$

Next, we will combine like terms by adding

$$\frac{25}{4}\int 2e^{5x}\sin(2x)\ dx$$

to both sides of the equation.

$$\int 2e^{5x}\sin(2x)\ dx + \frac{25}{4}\int 2e^{5x}\sin(2x)\ dx$$

$$= -e^{5x}\cos(2x) + \frac{5}{2}e^{5x}\sin(2x) - \frac{25}{4}\int 2e^{5x}\sin(2x)\ dx + \frac{25}{4}\int 2e^{5x}\sin(2x)\ dx$$

The integrals on the left side of the equation can be combined. The integrals on the right side of the equation cancel each other. Additionally, since we have an equation that shows an indefinite integral equals the sum of expressions, we will add a constant "C" to accommodate a possible constant term in the final answer.

Our last step is to multiply both sides of the equation by the reciprocal of $29/4$.

$$\left(\frac{4}{29}\right)\left(\frac{29}{4}\right)\int 2e^{5x}\sin(2x)\ dx = \frac{4}{29}\left[-e^{5x}\cos(2x) + \frac{5}{2}e^{5x}\sin(2x)\right] + C$$

It is not necessary to multiply the "C" by this fraction because it is an arbitrary constant. The final answer is

$$\int 2e^{5x}\sin(2x)\ dx = \frac{4}{29}\left[-e^{5x}\cos(2x) + \frac{5}{2}e^{5x}\sin(2x)\right] + C$$