Radius and interval of convergence of a Maclaurin series

Sometimes we'll be asked for the radius and interval of convergence of a Maclaurin series. In order to find these things, we'll first have to find a power series representation for the Maclaurin series, which we can do by hand, or using a table of common Maclaurin series.

Once we have the Maclaurin series represented as a power series, we'll identify a_n and a_{n+1} and plug them into the limit formula from the ratio test in order to say where the series is convergent and give the radius of convergence.

Then we'll use the radius of convergence to find the interval of convergence, making sure to test the endpoints of the interval to verify whether or not the series converges at one or both endpoints.

Example

Find the radius and interval of convergence of the Maclaurin series of the function.

$$f(x) = \ln(1 + 2x)$$

Using a table of common Maclaurin series, we know that the power series representation of the Maclaurin series for $f(x) = \ln(1 + x)$ is

$$\ln(1 + x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} x^n$$

Since our series has a $2x$ in place of x, we'll make that substitution on both sides of the equation and get a power series representation for the given function.

$$\ln(1+2x)=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}(2x)^n$$

Now that we have a power series representation for the given function, we're able to find the radius of convergence using the ratio test.

Since the ratio test tells us that a series converges if $L<1$ when

$$L=\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|$$

we just need to find a_n and a_{n+1}, and then plug them into the equation for L. Using the power series we generated for our function, we'll say that

$$a_n=\frac{(-1)^{n+1}}{n}(2x)^n$$

$$a_{n+1}=\frac{(-1)^{n+2}}{n+1}(2x)^{n+1}$$

Plugging these into the equation for L from the ratio test, we get

$$L=\lim_{n\to\infty}\left|\frac{\frac{(-1)^{n+2}(2x)^{n+1}}{n+1}}{\frac{(-1)^{n+1}(2x)^n}{n}}\right|$$

$$L = \lim_{n\to\infty}\left|\frac{(-1)^{n+2}(2x)^{n+1}}{n+1}\left[\frac{n}{(-1)^{n+1}(2x)^n}\right]\right|$$

$$L = \lim_{n\to\infty}\left|\frac{(-1)^{n+2}}{(-1)^{n+1}}\cdot\frac{(2x)^{n+1}}{(2x)^n}\cdot\frac{n}{n+1}\right|$$

$$L = \lim_{n\to\infty}\left|\frac{(-1)^{n+2-(n+1)}(2x)^{n+1-n}n}{n+1}\right|$$

$$L = \lim_{n\to\infty}\left|\frac{(-1)^{n+2-n-1}(2x)^1 n}{n+1}\right|$$

$$L = \lim_{n\to\infty}\left|\frac{(-1)^1 2xn}{n+1}\right|$$

$$L = \lim_{n\to\infty}\left|\frac{-2xn}{n+1}\right|$$

The absolute value brackets cancel the -1, so we get

$$L = \lim_{n\to\infty}\left|\frac{2xn}{n+1}\right|$$

The limit only deals with n, not x, so we can pull $2x$ out of the limit, as long as we keep it in absolute value bars.

$$L = |2x|\lim_{n\to\infty}\left|\frac{n}{n+1}\right|$$

$$L = |2x| \lim_{n\to\infty} \left| \frac{n}{n+1} \left(\frac{\frac{1}{n}}{\frac{1}{n}} \right) \right|$$

$$L = |2x| \lim_{n\to\infty} \left| \frac{\frac{n}{n}}{\frac{n}{n} + \frac{1}{n}} \right|$$

$$L = |2x| \lim_{n\to\infty} \left| \frac{1}{1 + \frac{1}{n}} \right|$$

$$L = |2x| \left| \frac{1}{1 + \frac{1}{\infty}} \right|$$

$$L = |2x| \left| \frac{1}{1+0} \right|$$

$$L = |2x| \, |1|$$

$$L = |2x|$$

Since by the ratio test we know that the series will converge when $L < 1$, we'll set

$$|2x| < 1$$

$$|x| < \frac{1}{2}$$

With the inequality in the form $|x-a|<R$, we can say that the radius of convergence of the Maclaurin series is

$$R=\frac{1}{2}$$

To find the interval of convergence of the Maclaurin series, we'll remove the absolute value bars from the radius of convergence.

$$|x|<\frac{1}{2}$$

$$-\frac{1}{2}<x<\frac{1}{2}$$

But before we can call this the interval of convergence, we have to verify whether or not the series converges at one or both endpoints, $x=-1/2$ and $x=1/2$. To do this, we'll plug each endpoint into the original series.

For $x=-1/2$,

$$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}(2x)^n$$

$$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\left[2\left(-\frac{1}{2}\right)\right]^n$$

$$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}(-1)^n$$

$$\sum_{n=1}^{\infty}\frac{(-1)^{2n+1}}{n}$$

$$\sum_{n=1}^{\infty} (-1)^{2n+1} \frac{1}{n}$$

This is an alternating series where

$$a_n = \frac{1}{n}$$

which means we can use the alternating series test to say whether or not it converges. Remember, the alternating series test tells us that a series converges if $\lim_{n \to \infty} a_n = 0$.

$$\lim_{n \to \infty} a_n$$

$$\lim_{n \to \infty} \frac{1}{n}$$

$$\frac{1}{\infty}$$

$$0$$

Because the limit is 0, the series converges by the alternating series test, which means the Maclaurin series converges at the left endpoint of the interval, $x = -1/2$.

Now we'll test $x = 1/2$.

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} (2x)^n$$

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \left[2\left(\frac{1}{2}\right)\right]^n$$

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}(1)^n$$

Since $1^n = 1$ for all values of n, we can cancel it out.

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$$

$$\sum_{n=1}^{\infty} (-1)^{n+1}\frac{1}{n}$$

This is an alternating series where

$$a_n = \frac{1}{n}$$

This is the same series we used to find the convergence of the left endpoint of the interval, and we already know that it converges by the alternating series test. Therefore, we can say that the series also converges at the right endpoint of the interval, $x = 1/2$.

Since the series converges at both endpoints of the interval, the interval of convergence of the Maclaurin series of $f(x) = \ln(1 + 2x)$ is

$$-\frac{1}{2} \leq x \leq \frac{1}{2}$$
