Horizontal and slant asymptotes

Now that we know how to find a function's vertical asymptotes, if it has any, let's turn our attention toward finding any horizontal and slant asymptotes for the function.

Then in the next lesson, we'll finally put this all together to sketch the function's graph.

Horizontal asymptotes

We primarily find horizontal asymptotes in the graphs of rational functions (but we do see them in other functions as well, like exponential functions).

When we're looking for horizontal asymptotes of rational functions, we only care about the highest-degree term in the numerator and the highest-degree term in the denominator.

Remember that the "degree" is the power of the term. So the degree of x^4 is four, the degree of $3x^2$ is two, and the degree of $7x$ is 1, since the x variable is raised to the first power. The degree of a constant term, like -8, is zero, because -8 can be rewritten as $-8x^0$.

Here's how we test for horizontal asymptotes:

1. If the degree of the numerator is less than the degree of the denominator, then the x-axis is a horizontal asymptote.

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2. If the degree of the numerator is equal to the degree of the denominator, then the ratio of the coefficients on these highest-degree terms is the equation of the horizontal asymptote.

3. If the degree of the numerator is greater than the degree of the denominator, there is no horizontal asymptote.

As an example, let's say we have a rational function written as

$$f(x) = \frac{x^3 + \text{lower-degree terms}}{x^2 + \text{lower-degree terms}}$$

The x^3 term is the highest-degree term in the numerator, and all the numerator's other terms have a degree lower than three. The x^2 term is the highest-degree term in the denominator, and all the denominator's other terms have a degree lower than two.

Then the degree of the numerator is 3 and the degree of the denominator is 2. Because the degree of the numerator is greater than the degree of the denominator, that function has no horizontal asymptote.

A rational functional will always have zero or one horizontal asymptote; it'll never have more than one.

Let's find any horizontal asymptote for the function we've been working with throughout this section.

Example

Find the function's horizontal asymptote, if it has one.

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$$f(x) = x + \frac{4}{x}$$

Before we can use our rule for finding horizontal asymptotes of rational functions, we need to rewrite $f(x)$ so that the function is just one fraction. We'll multiply the first term by x/x, which will give us a common denominator.

$$f(x) = x\left(\frac{x}{x}\right) + \frac{4}{x}$$

$$f(x) = \frac{x^2}{x} + \frac{4}{x}$$

With a common denominator, we can add the fractions.

$$f(x) = \frac{x^2 + 4}{x}$$

In this form, we can see that the degree of the numerator is 2, and the degree of the denominator is 1. So the degree of the numerator is greater than the degree of the denominator, which means the function doesn't have a horizontal asymptote.

Slant asymptotes

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A slant asymptote exists when the degree of the numerator is exactly one greater than the degree of the denominator. So the hypothetical function we mentioned earlier,

$$f(x) = \frac{x^3 + \text{lower-degree terms}}{x^2 + \text{lower-degree terms}}$$

would have a slant asymptote, because the degree of the numerator is exactly one greater than the degree of the denominator, $3 > 2$.

If we've determined that the function has a slant asymptote, then in order to find its equation, we divide the denominator into the numerator using polynomial long division.

Let's see whether our ongoing example problem has a slant asymptote.

Example

Show that the function has a slant asymptote, then find its equation.

$$f(x) = x + \frac{4}{x}$$

In the previous example, remember that we've already rewritten this function as

$$f(x) = \frac{x^2 + 4}{x}$$

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The degree of the numerator is exactly one greater than the degree of the denominator, $2 > 1$, so the function has a slant asymptote.

To find it equation, we divide the numerator by the denominator using polynomial long division. When we do, we get

$$f(x) = x + \frac{4}{x}$$

We get right back to the original function. That won't always happen, it's just that this particular function happened to be the composition of the quotient and remainder.

Whenever we use long division in this way to find the slant asymptote, the quotient gives the equation of the slant asymptote, and the denominator of the fraction gives the equation of the vertical asymptote.

Therefore, in this case, the slant asymptote is the line $y = x$, and the vertical asymptote is the line $x = 0$.

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