Topic: Integration by parts three times

Question: Use integration by parts to evaluate the integral.

$$\int 2x^3e^x\ dx$$

Answer choices:

A $2e^x\left(x^3+3x^2-6x+6\right)+C$

B $e^x\left(x^3-3x^2+6x-6\right)+C$

C $2e^x\left(x^3-3x^2+6x-6\right)+C$

D $e^x\left(x^3+3x^2-6x+6\right)+C$

Solution: C

Sometimes integration by parts is the correct tool to use to evaluate the integral, but using it only once doesn't simplify the integral enough, and we have to use it a second, or even a third time. However many times we use it, we always use the same integration by parts formula,

$$\int u\ dv = uv - \int v\ du$$

We need to identify a value for u and a value for dv in our original integral, and then take the derivative of u to get du, and take the integral of dv to get v. Let's do that for this integral.

$$u = 2x^3$$

$$du = 6x^2\ dx$$

and

$$dv = e^x\ dx$$

$$v = e^x$$

Plugging these values into the right side of the integration by parts formula, we get

$$\int 2x^3e^x\ dx = 2x^3e^x - \int e^x(6x^2\ dx)$$

$$\int 2x^3e^x\ dx = 2x^3e^x - \int 6x^2e^x\ dx$$

$$\int 2x^3e^x\ dx = 2x^3e^x - 6\int x^2e^x\ dx$$

Since we haven't managed to simplify our integral to the point where we can evaluate it, we'll have to try integration by parts a second time.

$$u_2 = x^2$$

$$du_2 = 2x\ dx$$

and

$$dv_2 = e^x\ dx$$

$$v_2 = e^x$$

Plugging these values into the right side of the integration by parts formula to replace just the remaining integral, we get

$$\int 2x^3e^x\ dx = 2x^3e^x - 6\left[x^2e^x - \int e^x(2x\ dx)\right]$$

$$\int 2x^3e^x\ dx = 2x^3e^x - 6\left[x^2e^x - 2\int xe^x\ dx\right]$$

$$\int 2x^3e^x\ dx = 2x^3e^x - 6x^2e^x + 12\int xe^x\ dx$$

Since we haven't managed to simplify our integral to the point where we can evaluate it, we'll have to try integration by parts a third time. We can sense that we're getting closer, because we started with an x^3 value in our original integral, after integration by parts it became x^2, then x after the second application. So we suspect that if we apply integration by parts one more time, we'll go from x to 1, and this term will drop out completely, leaving an integral we can actually evaluate.

$$u_3 = x$$

$$du_3 = 1\ dx$$

and

$$dv_3 = e^x\ dx$$

$$v_3 = e^x$$

Plugging these values into the right side of the integration by parts formula to replace just the remaining integral, we get

$$\int 2x^3e^x\ dx = 2x^3e^x - 6x^2e^x + 12\left[xe^x - \int e^x(1\ dx)\right]$$

$$\int 2x^3e^x\ dx = 2x^3e^x - 6x^2e^x + 12xe^x - 12\int e^x\ dx$$

$$\int 2x^3e^x\ dx = 2x^3e^x - 6x^2e^x + 12xe^x - 12e^x + C$$

$$\int 2x^3e^x\ dx = 2e^x\left(x^3 - 3x^2 + 6x - 6\right) + C$$

Topic: Integration by parts three times

Question: Use integration by parts to evaluate the integral.

$$\int 3x^3e^{-x}\ dx$$

Answer choices:

A $\quad -e^{-x}\left(x^3+3x^2+6x+6\right)+C$

B $\quad -e^{-x}\left(3x^3+9x^2+18x+18\right)+C$

C $\quad e^{-x}\left(3x^3+9x^2+18x+18\right)+C$

D $\quad -e^{-x}\left(-3x^3-9x^2-18x-18\right)+C$

Solution: B

Sometimes integration by parts is the correct tool to use to evaluate the integral, but using it only once doesn't simplify the integral enough, and we have to use it a second, or even a third time. However many times we use it, we always use the same integration by parts formula,

$$\int u\ dv = uv - \int v\ du$$

We need to identify a value for u and a value for dv in our original integral, and then take the derivative of u to get du, and take the integral of dv to get v. Let's do that for this integral.

$$u = 3x^3$$

$$du = 9x^2\ dx$$

and

$$dv = e^{-x}\ dx$$

$$v = -e^{-x}$$

Plugging these values into the right side of the integration by parts formula, we get

$$\int 3x^3e^{-x}\ dx = -3x^3e^{-x} - \int (-e^{-x})\left(9x^2\ dx\right)$$

$$\int 3x^3e^{-x}\ dx = -3x^3e^{-x} + 9\int x^2e^{-x}\ dx$$

Since we haven't managed to simplify our integral to the point where we can evaluate it, we'll have to try integration by parts a second time.

$$u_2 = x^2$$

$$du_2 = 2x\ dx$$

and

$$dv_2 = e^{-x}\ dx$$

$$v_2 = -e^{-x}$$

Plugging these values into the right side of the integration by parts formula to replace just the remaining integral, we get

$$\int 3x^3e^{-x}\ dx = -3x^3e^{-x} + 9\left[-x^2e^{-x} - \int (-e^{-x})(2x\ dx)\right]$$

$$\int 3x^3e^{-x}\ dx = -3x^3e^{-x} + 9\left[-x^2e^{-x} + 2\int xe^{-x}\ dx\right]$$

$$\int 3x^3e^{-x}\ dx = -3x^3e^{-x} - 9x^2e^{-x} + 18\int xe^{-x}\ dx$$

Since we haven't managed to simplify our integral to the point where we can evaluate it, we'll have to try integration by parts a third time. We can sense that we're getting closer, because we started with an x^3 value in our original integral, after integration by parts it became x^2, then x after the second application. So we suspect that if we apply integration by parts one more time, we'll go from x to 1, and this term will drop out completely, leaving an integral we can actually evaluate.

$u_3 = x$

$du_3 = 1 \ dx$

and

$dv_3 = e^{-x} \ dx$

$v_3 = -e^{-x}$

Plugging these values into the right side of the integration by parts formula to replace just the remaining integral, we get

$$\int 3x^3e^{-x} \ dx = -3x^3e^{-x} - 9x^2e^{-x} + 18\left[-xe^{-x} - \int (-e^{-x})(1 \ dx)\right]$$

$$\int 3x^3e^{-x} \ dx = -3x^3e^{-x} - 9x^2e^{-x} - 18xe^{-x} + 18\int e^{-x} \ dx$$

$$\int 3x^3e^{-x} \ dx = -3x^3e^{-x} - 9x^2e^{-x} - 18xe^{-x} - 18e^{-x} + C$$

$$\int 3x^3e^{-x} \ dx = -e^{-x}\left(3x^3 + 9x^2 + 18x + 18\right) + C$$

Topic: Integration by parts three times

Question: Evaluate the integral using integration by parts.

$$\int 2x^3e^{2x}\ dx$$

Answer choices:

A $\quad e^{2x}\left(x^3-\frac{3}{2}x^2+\frac{3}{2}x-\frac{3}{4}\right)+C$

B $\quad e^{2x}\left(x^3-\frac{3}{2}x^2+\frac{3}{2}x-\frac{3}{4}\right)$

C $\quad x^3e^{2x}-\frac{3}{2}x^2e^{2x}\frac{3}{2}xe^{2x}-\frac{3}{4}e^{2x}$

D $\quad x^3e^{2x}-\frac{3}{2}x^2e^{2x}\frac{3}{2}xe^{2x}-\frac{3}{4}e^{2x}+1$

Solution: A

Integration by parts is a method of evaluating an integral that cannot be evaluated using normal integration techniques, by using integration by substitution, or by using integration formulas. The general formula for integration by parts is

$$\int u\ dv = uv - \int v\ du$$

In this formula, we separate the integrand into two parts; one part is called u and the other part is called dv. In making these two parts, we must use all of the integrand.

Although there is sometimes flexibility in choosing u, we can generally use the following sequence of choices to select the best part of the integrand to be u. This method involves the acronym LIPET, where we select the first u in the sequence of the list below. The letters mean

L Logarithmic expression

I Inverse trigonometric expression

P Polynomial expression

E Exponential expression

T Trigonometric function expression

In this problem, the integrand is $2x^3e^{2x}$ where we have a polynomial exponential expression and an exponential expression. In the LIPET sequence, polynomial comes before exponential, so u is the polynomial expression. Let's identify the parts we need to integrate. Additionally, since this problem will take more than one integration by parts, we will use subscripts with the u, v, du, and dv.

$$u_1 = 2x^3$$

$$du_1 = 6x^2\ dx$$

$$dv_1 = e^{2x}\ dx$$

$$v_1 = \frac{1}{2}e^{2x}$$

We're now ready to integrate by parts using the general formula.

$$\int u_1\ dv_1 = u_1v_1 - \int v_1\ du_1$$

$$\int 2x^3e^{2x}\ dx = \left(2x^3\right)\left(\frac{1}{2}e^{2x}\right) - \int \left(\frac{1}{2}e^{2x}\right)\left(6x^2\right)\ dx$$

$$\int 2x^3e^{2x}\ dx = x^3e^{2x} - \int 3x^2e^{2x}\ dx$$

Next, let's evaluate the new integral, which will require integration by parts a second time.

$$\int 3x^2e^{2x}\ dx$$

Using the LIPET sequence again, we again have a polynomial expression and an exponential expression. The exponential expression will be the u and the trigonometric function will be the dv.

$$\int u_2 \, dv_2 = u_2 v_2 - \int v_2 \, du_2$$

$$u_2 = 3x^2$$

$$du_2 = 6x \ dx$$

$$dv_2 = e^{2x} \ dx$$

$$v_2 = \frac{1}{2}e^{2x}$$

$$\int 3x^2 e^{2x} \ dx = (3x^2)\left(\frac{1}{2}e^{2x}\right) - \int \left(\frac{1}{2}e^{2x}\right)(6x) \ dx$$

$$\int 3x^2 e^{2x} \ dx = \frac{3}{2}x^2 e^{2x} - \int 3xe^{2x} \ dx$$

Now, we'll rewrite the equation using the original integral.

$$\int 2x^3 e^{2x} \ dx = x^3 e^{2x} - \left(\frac{3}{2}x^2 e^{2x} - \int 3xe^{2x} \ dx\right)$$

$$\int 2x^3 e^{2x} \ dx = x^3 e^{2x} - \frac{3}{2}x^2 e^{2x} + \int 3xe^{2x} \ dx$$

Next, let's evaluate the new integral, which will require integration by parts a third time.

$$\int 3xe^{2x} \ dx$$

Using the LIPET sequence again, we again have a polynomial expression and an exponential expression. The exponential expression will be the u and the trigonometric function will be the dv.

$$\int u_3\ dv_3 = u_3v_3 - \int v_3\ du_3$$

$$u_3 = 3x$$

$$du_3 = 3\ dx$$

$$dv_3 = e^{2x}\ dx$$

$$v_3 = \frac{1}{2}e^{2x}$$

$$\int 3xe^{2x}\ dx = (3x)\left(\frac{1}{2}e^{2x}\right) - \int \left(\frac{1}{2}e^{2x}\right)(3)\ dx$$

$$\int 3xe^{2x}\ dx = \frac{3}{2}xe^{2x} - \int \frac{3}{2}e^{2x}\ dx$$

$$\int \frac{3}{2}e^{2x}\ dx = \frac{3}{2}\left(\frac{e^{2x}}{2}\right) + C = \frac{3}{4}e^{2x} + C$$

Since we will have an indefinite integral equal a term, we will add an arbitrary constant C to accommodate the possibility of a constant term in the answer.

Now, we will rewrite the equation, again using the original integral. Each term comes from the three integration by parts processes.

$$\int 2x^3e^{2x}\ dx = x^3e^{2x} - \frac{3}{2}x^2e^{2x} + \frac{3}{2}xe^{2x} - \frac{3}{4}e^{2x} + C$$

Although we have an acceptable final answer, we can factor out the greatest common factor from the terms. The arbitrary constant does not contain the GCF.

$$\int 2x^3e^{2x}\ dx = e^{2x}\left(x^3 - \frac{3}{2}x^2 + \frac{3}{2}x - \frac{3}{4}\right) + C$$