## Surface area of revolution of a polar curve

We can find the surface area of the object created when we rotate a polar curve around either the x-axis or the y-axis using the formulas

$$S_{x} = \int_{\alpha}^{\beta} 2\pi y \sqrt{r^{2} + \left(\frac{dr}{d\theta}\right)^{2}} \ d\theta$$

around the x-axis

$$S_{y} = \int_{\alpha}^{\beta} 2\pi x \sqrt{r^{2} + \left(\frac{dr}{d\theta}\right)^{2}} \ d\theta$$

around the y-axis

where

r is the equation of the polar curve

 $\frac{dr}{d\theta}$  is the derivative of the polar curve

 $[\alpha, \beta]$  is the interval

We can solve for x and y as needed using the conversion formulas

$$x = r\cos\theta$$

$$y = r \sin \theta$$

## Example

Find the surface area of revolution of the polar curve over the given interval.

$$r = 5\cos\theta$$

$$0 \le \theta \le \pi$$

around the y-axis

Before we can plug into the formula, we need to find x and  $dr/d\theta$ .

Since  $x = r \cos \theta$ , we get

$$x = 5\cos\theta\cos\theta$$

$$x = 5\cos^2\theta$$

To find  $dr/d\theta$ , we'll take the derivative of the given polar equation.

$$r = 5\cos\theta$$

$$\frac{dr}{d\theta} = -5\sin\theta$$

We'll plug everything into the formula for the surface area of revolution about the y-axis.

$$S_{y} = \int_{\alpha}^{\beta} 2\pi x \sqrt{r^{2} + \left(\frac{dr}{d\theta}\right)^{2}} \ d\theta$$

$$S_{y} = \int_{0}^{\pi} 2\pi \left( 5\cos^{2}\theta \right) \sqrt{\left( 5\cos\theta \right)^{2} + \left( -5\sin\theta \right)^{2}} \ d\theta$$

$$S_y = 10\pi \int_0^{\pi} \cos^2 \theta \sqrt{25 \cos^2 \theta + 25 \sin^2 \theta} \ d\theta$$



$$S_y = 10\pi \int_0^{\pi} \cos^2 \theta \sqrt{25 \left(\cos^2 \theta + \sin^2 \theta\right)} \ d\theta$$

Since  $\cos^2 \theta + \sin^2 \theta = 1$ ,

$$S_y = 50\pi \int_0^\pi \cos^2 \theta \sqrt{1} \ d\theta$$

$$S_y = 50\pi \int_0^\pi \cos^2 \theta \ d\theta$$

Since 
$$\cos^2 \theta = \frac{1}{2} \left[ 1 + \cos(2\theta) \right]$$
,

$$S_{y} = 50\pi \int_{0}^{\pi} \frac{1}{2} \left[ 1 + \cos(2\theta) \right] d\theta$$

$$S_y = 50\pi \int_0^{\pi} \frac{1}{2} + \frac{1}{2}\cos(2\theta) \ d\theta$$

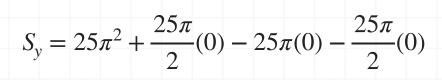
$$S_y = 25\pi \int_0^{\pi} 1 + \cos(2\theta) \ d\theta$$

$$S_{y} = 25\pi \left[ \theta + \frac{1}{2} \sin(2\theta) \right] \Big|_{0}^{\pi}$$

$$S_y = 25\pi\theta + \frac{25\pi}{2}\sin(2\theta)\bigg|_0^{\pi}$$

$$S_y = 25\pi(\pi) + \frac{25\pi}{2}\sin(2\pi) - \left[25\pi(0) + \frac{25\pi}{2}\sin(2(0))\right]$$





$$S_y = 25\pi^2$$

