Error or remainder of a series

Imagine that you need to find the sum of a series, but you don't have a formula that you can use to do it. Instead, you have to manually add all of the series' terms together, one at a time. Of course you could never do this, because the series has an infinite number of terms, and you'd be adding forever.

But what if you knew that the sum of just the first five terms of the series was only $.00001$ less than the sum of the entire series? If that were the case, maybe you could just use the first five terms, and say that it was a *good enough estimate* of the total sum, since it's only $.00001$ different and it saves you from manually adding infinitely more terms to the sum.

If you use the estimate, then you want to be able to report next to your answer that the value you found is only $.00001$ off of the total sum. This $.00001$ value is called the remainder, or error, of the series, and it tells you how close your estimate is to the real sum.

To find the remainder of the series, we'll need to

1. Estimate the total sum by calculating a partial sum for the series.

2. Use the comparison test to say whether the series converges or diverges.

3. Use the integral test to solve for the remainder.

Example

Use the first six terms to estimate the remainder of the series.

$$\sum_{n=1}^{\infty} \frac{n}{2n^4+3}$$

The first thing we need to do is to find the sum of the first six terms s_6 of our original series a_n.

$n=1$ $\quad a_1=\frac{(1)}{2(1)^4+3}$ $\quad a_1=\frac{1}{5}$

$n=2$ $\quad a_2=\frac{(2)}{2(2)^4+3}$ $\quad a_2=\frac{2}{35}$

$n=3$ $\quad a_3=\frac{(3)}{2(3)^4+3}$ $\quad a_3=\frac{1}{54}$

$n=4$ $\quad a_4=\frac{(4)}{2(4)^4+3}$ $\quad a_4=\frac{4}{515}$

$n=5$ $\quad a_5=\frac{(5)}{2(5)^4+3}$ $\quad a_5=\frac{5}{1,253}$

$n=6$ $\quad a_6=\frac{(6)}{2(6)^4+3}$ $\quad a_6=\frac{6}{2,595}$

The sum of the first six terms of the series a_n is

$$s_6=\frac{1}{5}+\frac{2}{35}+\frac{1}{54}+\frac{4}{515}+\frac{5}{1,253}+\frac{6}{2,595}$$

$$s_6=0.2000+0.0571+0.0185+0.0078+0.0040+0.0023$$

$$s_6 = 0.2897$$

Since we've rounded our decimals, we'll say

$$s_6 \approx 0.2897$$

Next, we need to use the comparison test to figure out whether a_n converges or diverges. We will need to create a similar but simpler comparison series b_n. We can use the same numerator in b_n as the numerator from a_n, since it's already pretty simple. For the denominator, we can use n^4, since it's the element of the denominator that has the most impact on the series. The comparison series b_n will be

$$b_n = \frac{n}{n^4}$$

$$b_n = \frac{1}{n^3}$$

The comparison series b_n is a p-series where $p = 3$. The p-series test tells us that the series

will converge when $p > 1$

will diverge when $p \le 1$

Since $p = 3$, we know that b_n converges.

To use the comparison test to show that a_n also converges, we have to show that $0 \le a_n \le b_n$. We'll find some of the first few values of the comparison series b_n and compare them to a_n. Let's use $n = 1, 2, 3$.

$n = 1$ $\qquad b_1 = \frac{1}{(1)^3}$ $\qquad b_1 = 1$

$n = 2$ $\qquad b_2 = \frac{1}{(2)^3}$ $\qquad b_2 = \frac{1}{8}$

$n = 3$ $\qquad b_3 = \frac{1}{(3)^3}$ $\qquad b_3 = \frac{1}{27}$

Looking at these three terms and their corresponding terms from a_n, we can see that $0 \le a_n \le b_n$, which means that a_n converges.

Now that we know that the series converges, we'll use the integral test to find the remainder of the series a_n after the first six terms, R_6. We'll call the remainder of the comparison series b_n after the first six terms, T_6. Since we know that $0 \le a_n \le b_n$, and that a_n and b_n converge, we can say that $R_6 \le T_6$, which will be less than the total area under b_n.

$$R_6 \le T_6 \le \int_6^\infty b_n \ dx = \int_6^\infty f(x) \ dx$$

$$R_6 \le T_6 \le \int_6^\infty b_n \ dx = \int_6^\infty \frac{1}{x^3} \ dx$$

$$R_6 \le T_6 \le \int_6^\infty b_n \ dx = \int_6^\infty x^{-3} \ dx$$

$$R_6 \le \frac{x^{-2}}{-2} \Bigg|_6^\infty$$

$$R_6 \le \lim_{a \to \infty} \frac{x^{-2}}{-2} \Bigg|_6^a$$

$$R_6 \le \lim_{a\to\infty} -\frac{1}{2x^2}\Bigg|_6^a$$

$$R_6 \le \lim_{a\to\infty} -\frac{1}{2a^2} - \left(-\frac{1}{2(6)^2}\right)$$

$$R_6 \le \lim_{a\to\infty} \frac{1}{2(6)^2} - \frac{1}{2a^2}$$

$$R_6 \le \lim_{a\to\infty} \frac{1}{72} - \frac{1}{2a^2}$$

$$R_6 \le \frac{1}{72} - \frac{1}{2\infty^2}$$

$$R_6 \le \frac{1}{72} - 0$$

$$R_6 \le \frac{1}{72}$$

$$R_6 \le 0.0139$$

The sixth partial sum of the series a_n is $s_6 \approx 0.2897$, with error $R_6 \le 0.0139$.
