Calculus 2 Workbook Solutions

Trigonometric substitution

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MATH

TRIGONOMETRIC SUBSTITUTION WITH SECANT

1. Set up and simplify the integral for trig substitution, but don't integrate.

$$\int \frac{3}{\sqrt{9x^2+6x}}\ dx$$

Solution:

Rewrite the integrand.

$$\int \frac{3}{\sqrt{9\left(x^2+\frac{2}{3}x\right)}}\ dx$$

$$\int \frac{1}{\sqrt{x^2+\frac{2}{3}x}}\ dx$$

$$\int \frac{1}{\sqrt{\left(x^2+\frac{2}{3}x+\frac{1}{9}\right)-\frac{1}{9}}}\ dx$$

$$\int \frac{1}{\sqrt{\left(x+\frac{1}{3}\right)^2-\frac{1}{9}}}\ dx$$

Set up the trig substitution.

$$u^2 - a^2 = \left(x + \frac{1}{3}\right)^2 - \frac{1}{9}$$

$u = x + \frac{1}{3}$ and $a = \frac{1}{3}$

$x + \frac{1}{3} = \frac{1}{3}\sec\theta$ so $x = -\frac{1}{3} + \frac{1}{3}\sec\theta$

$$dx = \frac{1}{3}\sec\theta\tan\theta\ d\theta$$

Substitute.

$$\int \frac{1}{\sqrt{\left(-\frac{1}{3} + \frac{1}{3}\sec\theta + \frac{1}{3}\right)^2 - \frac{1}{9}}} \cdot \frac{1}{3}\sec\theta\tan\theta\ d\theta$$

$$\frac{1}{3}\int \frac{\sec\theta\tan\theta}{\sqrt{\left(\frac{1}{3}\sec\theta\right)^2 - \frac{1}{9}}}\ d\theta$$

$$\frac{1}{3}\int \frac{\sec\theta\tan\theta}{\sqrt{\frac{1}{9}\sec^2\theta - \frac{1}{9}}}\ d\theta$$

$$\frac{1}{3}\int \frac{\sec\theta\tan\theta}{\sqrt{\frac{1}{9}(\sec^2\theta - 1)}}\ d\theta$$

Simplify with the trig identity $\sec^2\theta - 1 = \tan^2\theta$.

$$\frac{1}{3}\int \frac{\sec\theta\tan\theta}{\sqrt{\frac{1}{9}\tan^2\theta}}\ d\theta$$

$$\frac{1}{3}\int \frac{\sec\theta\tan\theta}{\frac{1}{3}\tan\theta}\ d\theta$$

$$\int \sec\theta\ d\theta$$

■ 2. Set up and simplify the integral for trig substitution, but don't integrate.

$$\int \frac{5}{\sqrt{4x^2+4x}}\ dx$$

Solution:

Rewrite the integrand.

$$\int \frac{5}{\sqrt{4(x^2+x)}}\ dx$$

$$\frac{5}{2}\int \frac{1}{\sqrt{x^2+x}}\ dx$$

$$\frac{5}{2}\int \frac{1}{\sqrt{x^2+x+\frac{1}{4}-\frac{1}{4}}}\ dx$$

$$\frac{5}{2}\int \frac{1}{\sqrt{\left(x+\frac{1}{2}\right)^2-\frac{1}{4}}}\ dx$$

Set up the trig substitution.

$$u^2-a^2=\left(x+\frac{1}{2}\right)^2-\frac{1}{4}$$

$u=x+\dfrac{1}{2}$ and $a=\dfrac{1}{2}$

$x+\dfrac{1}{2}=\dfrac{1}{2}\sec\theta$ so $x=-\dfrac{1}{2}+\dfrac{1}{2}\sec\theta$

$$dx=\frac{1}{2}\sec\theta\tan\theta\ d\theta$$

Substitute.

$$\frac{5}{2}\int \frac{1}{\sqrt{\left(-\frac{1}{2}+\frac{1}{2}\sec\theta+\frac{1}{2}\right)^2-\frac{1}{4}}}\cdot\frac{1}{2}\sec\theta\tan\theta\ d\theta$$

$$\frac{5}{4}\int \frac{\sec\theta\tan\theta}{\sqrt{\left(\frac{1}{2}\sec\theta\right)^2-\frac{1}{4}}}\ d\theta$$

$$\frac{5}{4}\int \frac{\sec\theta\tan\theta}{\sqrt{\frac{1}{4}\sec^2\theta-\frac{1}{4}}}\ d\theta$$

$$\frac{5}{4}\int \frac{\sec\theta\tan\theta}{\sqrt{\frac{1}{4}(\sec^2\theta-1)}}\ d\theta$$

Simplify with the trig identity $\sec^2\theta - 1 = \tan^2\theta$.

$$\frac{5}{4}\int \frac{\sec\theta\tan\theta}{\sqrt{\frac{1}{4}\tan^2\theta}}\ d\theta$$

$$\frac{5}{4}\int \frac{\sec\theta\tan\theta}{\frac{1}{2}\tan\theta}\ d\theta$$

$$\frac{5}{2}\int \sec\theta\ d\theta$$

3. Set up and simplify the integral for trig substitution, but don't integrate.

$$\int \frac{dx}{x^2\sqrt{x^2-9}}$$

Solution:

Set up the trig substitution.

$$u^2 - a^2 = x^2 - 9$$

$u = x$ and $a = 3$

$$x = 3\sec\theta$$

$$dx = 3\sec\theta\tan\theta\ d\theta$$

Substitute.

$$\int \frac{3\sec\theta\tan\theta\ d\theta}{(3\sec\theta)^2\sqrt{(3\sec\theta)^2 - 9}}$$

$$\int \frac{3\sec\theta\tan\theta\ d\theta}{9\sec^2\theta\sqrt{9\sec^2\theta - 9}}$$

$$\int \frac{\tan\theta}{3\sec\theta\sqrt{9(\sec^2\theta - 1)}}\ d\theta$$

Simplify with the trig identity $\sec^2\theta - 1 = \tan^2\theta$.

$$\int \frac{\tan\theta}{3\sec\theta\sqrt{9\tan^2\theta}}\ d\theta$$

$$\int \frac{\tan\theta}{3\sec\theta(3\tan\theta)}\ d\theta$$

$$\frac{1}{9}\int \frac{1}{\sec\theta}\ d\theta$$

$$\frac{1}{9}\int \cos\theta\ d\theta$$

■ 4. Set up and simplify the integral for trig substitution, but don't integrate.

$$\int \frac{4\ dx}{x^2\sqrt{x^2-25}}$$

Solution:

Set up the trig substitution.

$$u^2 - a^2 = x^2 - 25$$

$u = x$ and $a = 5$

$$x = 5\sec\theta$$

$$dx = 5\sec\theta\tan\theta\ d\theta$$

Substitute.

$$\int \frac{4(5\sec\theta\tan\theta\ d\theta)}{(5\sec\theta)^2\sqrt{(5\sec\theta)^2-25}}$$

$$\int \frac{20\sec\theta\tan\theta\ d\theta}{25\sec^2\theta\sqrt{25\sec^2\theta-25}}$$

$$\int \frac{4\tan\theta}{5\sec\theta\sqrt{25(\sec^2\theta-1)}}\ d\theta$$

Simplify with the trig identity $\sec^2\theta - 1 = \tan^2\theta$.

$$\int \frac{4\tan\theta}{5\sec\theta\sqrt{25\tan^2\theta}}\ d\theta$$

$$\int \frac{4\tan\theta}{5\sec\theta(5\tan\theta)}\ d\theta$$

$$\frac{4}{25}\int \frac{1}{\sec\theta}\ d\theta$$

$$\frac{4}{25}\int \cos\theta\ d\theta$$

TRIGONOMETRIC SUBSTITUTION WITH SINE

1. Set up and simplify the integral for trig substitution, but don't integrate.

$$\int \frac{3x}{\sqrt{64-49x^2}}\ dx$$

Solution:

Set up the trig substitution.

$$a^2 - u^2 = 64 - 49x^2$$

$u = 7x$ and $a = 8$

$7x = 8\sin\theta$ so $x = \frac{8}{7}\sin\theta$

$$dx = \frac{8}{7}\cos\theta\ d\theta$$

Substitute.

$$\int \frac{3 \cdot \frac{8}{7}\sin\theta}{\sqrt{64-49\left(\frac{8}{7}\sin\theta\right)^2}} \cdot \frac{8}{7}\cos\theta\ d\theta$$

$$\frac{192}{49}\int \frac{\sin\theta\cos\theta}{\sqrt{64-49\left(\frac{64}{49}\sin^2\theta\right)}}\ d\theta$$

$$\frac{192}{49}\int \frac{\sin\theta\cos\theta}{\sqrt{64-64\sin^2\theta}}\ d\theta$$

$$\frac{192}{49}\int \frac{\sin\theta\cos\theta}{\sqrt{64(1-\sin^2\theta)}}\ d\theta$$

Simplify with the trig identity $1-\sin^2\theta=\cos^2\theta$.

$$\frac{192}{49}\int \frac{\sin\theta\cos\theta}{\sqrt{64\cos^2\theta}}\ d\theta$$

$$\frac{192}{49}\int \frac{\sin\theta\cos\theta}{8\cos\theta}\ d\theta$$

$$\frac{24}{49}\int \sin\theta\ d\theta$$

■ 2. Set up and simplify the integral for trig substitution, but don't integrate.

$$\int \frac{2x}{\sqrt{121-144x^2}}\ dx$$

Solution:

Set up the trig substitution.

$$a^2 - u^2 = 121 - 144x^2$$

$u = 12x$ and $a = 11$

$12x = 11\sin\theta$ so $x = \frac{11}{12}\sin\theta$

$$dx = \frac{11}{12}\cos\theta\ d\theta$$

Substitute.

$$\int \frac{2 \cdot \frac{11}{12}\sin\theta}{\sqrt{121 - 144\left(\frac{11}{12}\sin\theta\right)^2}} \cdot \frac{11}{12}\cos\theta\ d\theta$$

$$\frac{121}{72}\int \frac{\sin\theta\cos\theta}{\sqrt{121 - 144\left(\frac{121}{144}\sin^2\theta\right)}}\ d\theta$$

$$\frac{121}{72}\int \frac{\sin\theta\cos\theta}{\sqrt{121 - 121\sin^2\theta}}\ d\theta$$

$$\frac{121}{72}\int \frac{\sin\theta\cos\theta}{\sqrt{121(1 - \sin^2\theta)}}\ d\theta$$

Simplify with the trig identity $1 - \sin^2\theta = \cos^2\theta$.

$$\frac{121}{72}\int \frac{\sin\theta\cos\theta}{11\cos\theta}\ d\theta$$

$$\frac{11}{72}\int \sin\theta\ d\theta$$

■ 3. Set up and simplify the integral for trig substitution, but don't integrate.

$$\int \frac{6x}{\sqrt{81-36x^2}}\ dx$$

Solution:

Rewrite the integrand.

$$\int \frac{6x}{\sqrt{9(9-4x^2)}}\ dx$$

$$\int \frac{6x}{3\sqrt{9-4x^2}}\ dx$$

$$\int \frac{2x}{\sqrt{9-4x^2}}\ dx$$

Set up the trig substitution.

$$a^2-u^2=9-4x^2$$

$u=2x$ and $a=3$

$$2x = 3\sin\theta \text{ so } x = \frac{3}{2}\sin\theta$$

$$dx = \frac{3}{2}\cos\theta\ d\theta$$

Substitute.

$$\int \frac{2 \cdot \frac{3}{2}\sin\theta}{\sqrt{9 - 4\left(\frac{3}{2}\sin\theta\right)^2}} \cdot \frac{3}{2}\cos\theta\ d\theta$$

$$\frac{9}{2}\int \frac{\sin\theta\cos\theta}{\sqrt{9 - 4\left(\frac{9}{4}\sin^2\theta\right)}}\ d\theta$$

$$\frac{9}{2}\int \frac{\sin\theta\cos\theta}{\sqrt{9 - 9\sin^2\theta}}\ d\theta$$

$$\frac{9}{2}\int \frac{\sin\theta\cos\theta}{\sqrt{9(1 - \sin^2\theta)}}\ d\theta$$

$$\frac{3}{2}\int \frac{\sin\theta\cos\theta}{\sqrt{1 - \sin^2\theta}}\ d\theta$$

Simplify with the trig identity $1 - \sin^2\theta = \cos^2\theta$.

$$\frac{3}{2}\int \frac{\sin\theta\cos\theta}{\sqrt{\cos^2\theta}}\ d\theta$$

$$\frac{3}{2}\int \frac{\sin\theta\cos\theta}{\cos\theta}\ d\theta$$

$$\frac{3}{2}\int \sin\theta\ d\theta$$

■ 4. Set up and simplify the integral for trig substitution, but don't integrate.

$$\int \frac{35x}{\sqrt{25-100x^2}}\ dx$$

Solution:

Rewrite the integrand.

$$\int \frac{35x}{\sqrt{25(1-4x^2)}}\ dx$$

$$\int \frac{35x}{5\sqrt{1-4x^2}}\ dx$$

$$\int \frac{7x}{\sqrt{1-4x^2}}\ dx$$

Set up the trig substitution.

$$a^2-u^2=1-4x^2$$

$u=2x$ and $a=1$

$$2x = \sin\theta \text{ so } x = \frac{1}{2}\sin\theta$$

$$dx = \frac{1}{2}\cos\theta \ d\theta$$

Substitute.

$$\int \frac{7 \cdot \frac{1}{2}\sin\theta}{\sqrt{1 - 4\left(\frac{1}{2}\sin\theta\right)^2}} \cdot \frac{1}{2}\cos\theta \ d\theta$$

$$\frac{7}{4}\int \frac{\sin\theta\cos\theta}{\sqrt{1 - 4\left(\frac{1}{4}\sin^2\theta\right)}} \ d\theta$$

$$\frac{7}{4}\int \frac{\sin\theta\cos\theta}{\sqrt{1 - \sin^2\theta}} \ d\theta$$

Simplify with the trig identity $1 - \sin^2\theta = \cos^2\theta$.

$$\frac{7}{4}\int \frac{\sin\theta\cos\theta}{\sqrt{\cos^2\theta}} \ d\theta$$

$$\frac{7}{4}\int \frac{\sin\theta\cos\theta}{\cos\theta} \ d\theta$$

$$\frac{7}{4}\int \sin\theta \ d\theta$$

TRIGONOMETRIC SUBSTITUTION WITH TANGENT

■ 1. Set up and simplify the integral for trig substitution, but don't integrate.

$$\int \sqrt{36x^2 + 25}\ dx$$

Solution:

Set up the trig substitution.

$$u^2 + a^2 = 36x^2 + 25$$

$u = 6x$ and $a = 5$

$6x = 5 \tan \theta$ so $x = \frac{5}{6} \tan \theta$

$$dx = \frac{5}{6} \sec^2 \theta\ d\theta$$

Substitute.

$$\int \sqrt{36 \left(\frac{5}{6} \tan \theta \right)^2 + 25} \cdot \frac{5}{6} \sec^2 \theta\ d\theta$$

$$\frac{5}{6} \int \sec^2 \theta \sqrt{36 \left(\frac{25}{36} \tan^2 \theta \right) + 25}\ d\theta$$

$$\frac{5}{6}\int \sec^2\theta\sqrt{25\tan^2\theta+25}\ d\theta$$

$$\frac{5}{6}\int \sec^2\theta\sqrt{25(\tan^2\theta+1)}\ d\theta$$

$$\frac{25}{6}\int \sec^2\theta\sqrt{\tan^2\theta+1}\ d\theta$$

Simplify with the trig identity $\tan^2\theta+1=\sec^2\theta$.

$$\frac{25}{6}\int \sec^2\theta\sqrt{\sec^2\theta}\ d\theta$$

$$\frac{25}{6}\int \sec^2\theta\sec\theta\ d\theta$$

$$\frac{25}{6}\int \sec^3\theta\ d\theta$$

■ 2. Set up and simplify the integral for trig substitution, but don't integrate.

$$\int\sqrt{4x^2+81}\ dx$$

Solution:

Set up the trig substitution.

$$u^2+a^2=4x^2+81$$

$u = 2x$ and $a = 9$

$2x = 9\tan\theta$ so $x = \frac{9}{2}\tan\theta$

$dx = \frac{9}{2}\sec^2\theta\ d\theta$

Substitute.

$$\int \sqrt{4\left(\frac{9}{2}\tan\theta\right)^2 + 81} \cdot \frac{9}{2}\sec^2\theta\ d\theta$$

$$\frac{9}{2}\int \sec^2\theta\sqrt{4\left(\frac{81}{4}\tan^2\theta\right) + 81}\ d\theta$$

$$\frac{9}{2}\int \sec^2\theta\sqrt{81\tan^2\theta + 81}\ d\theta$$

$$\frac{9}{2}\int \sec^2\theta\sqrt{81(\tan^2\theta + 1)}\ d\theta$$

$$\frac{81}{2}\int \sec^2\theta\sqrt{\tan^2\theta + 1}\ d\theta$$

Simplify with the trig identity $\tan^2\theta + 1 = \sec^2\theta$.

$$\frac{81}{2}\int \sec^2\theta\sqrt{\sec^2\theta}\ d\theta$$

$$\frac{81}{2}\int \sec^2\theta\sec\theta\ d\theta$$

$$\frac{81}{2}\int \sec^3\theta\ d\theta$$

■ 3. Set up and simplify the integral for trig substitution, but don't integrate.

$$\int \frac{7}{\sqrt{x^2+4x+8}}\ dx$$

Solution:

Rewrite the integrand by completing the square.

$$\int \frac{7}{\sqrt{(x^2+4x+4)+4}}\ dx$$

$$\int \frac{7}{\sqrt{(x+2)^2+4}}\ dx$$

Set up the trig substitution.

$u^2+a^2=(x+2)^2+4$

$u=x+2$ and $a=2$

$x+2=2\tan\theta$ so $x=-2+2\tan\theta$

$dx=2\sec^2\theta\ d\theta$

Substitute.

$$\int \frac{7}{\sqrt{(-2+2\tan\theta+2)^2+4}} \cdot 2\sec^2\theta \ d\theta$$

$$14\int \frac{\sec^2\theta}{\sqrt{(2\tan\theta)^2+4}} \ d\theta$$

$$14\int \frac{\sec^2\theta}{\sqrt{4\tan^2\theta+4}} \ d\theta$$

$$14\int \frac{\sec^2\theta}{\sqrt{4(\tan^2\theta+1)}} \ d\theta$$

$$7\int \frac{\sec^2\theta}{\sqrt{\tan^2\theta+1}} \ d\theta$$

Simplify with the trig identity $\tan^2\theta + 1 = \sec^2\theta$.

$$7\int \frac{\sec^2\theta}{\sqrt{\sec^2\theta}} \ d\theta$$

$$7\int \frac{\sec^2\theta}{\sec\theta} \ d\theta$$

$$7\int \sec\theta \ d\theta$$

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