**Topic**: Convergence of a telescoping series

**Question**: Say whether or not the telescoping series converges.

$$\sum_{n=1}^{\infty} 2^n - 2^{n+1}$$

**Answer choices**:

A The series diverges $s_n = 2 - 2^{n-1}$

B The series converges $s_n = 2 - 2^{n+1}$

C The series converges $s_n = 2 - 2^{n-1}$

D The series diverges $s_n = 2 - 2^{n+1}$

**Solution**: D

To see whether or not a telescoping series converges or diverges, we'll find the $n$th partial sum of the series $s_n$, and then take the limit as $n \to \infty$ of $s_n$, or

$$s = \lim_{n\to\infty} s_n$$

If we get a real-number value for $s$, then $s_n$ converges, and therefore so does $a_n$.

We'll expand the telescoping series by calculating the first few terms, making sure to also include the last term of the series, then simplify the sum by canceling all of the terms in the middle. The remaining series will be the series of partial sums $s_n$.

In order to show that the series is telescoping, we'll need to start by expanding the series. Let's use $n = 1$, $n = 2$, $n = 3$ and $n = 4$.

| | | |
|---|---|---|
| $n = 1$ | $2^1 - 2^2$ | $2 - 4$ |
| $n = 2$ | $2^2 - 2^3$ | $4 - 8$ |
| $n = 3$ | $2^3 - 2^4$ | $8 - 16$ |
| $n = 4$ | $2^4 - 2^5$ | $16 - 32$ |

Writing these terms into our expanded series and including the last term of the series, we get

$$\sum_{n=1}^{\infty} 2^n - 2^{n+1} = (2 - 4) + (4 - 8) + \left(8 - 16\right) + \left(16 - 32\right) + \ldots + \left(2^n - 2^{n+1}\right)$$

When we look at our expanded series, we see that the second half of the first term will cancel with the first half of the second term, that the second half of the second term will cancel with the first half of the third term, and so on, so we can say that the series is telescoping.

Canceling everything but the first half of the first term and the second half of the last term gives an expression for the series of partial sums.

$$s_n = 2 - 2^{n+1}$$

Remember that we're looking for a real-number value for $s$, where

$$s = \lim_{n \to \infty} s_n$$

so we'll plug $s_n$ into this equation and get

$$s = \lim_{n \to \infty} 2 - 2^{n+1}$$

If we find the limit as $n \to \infty$, the exponent in this equation will become infinitely large. Raising $2$ to an infinitely large exponent will make the term $2^{n+1}$ infinitely large. Subtracting an infinitely large value from $2$ will give us an infinitely negative value, so we can say

$$s = -\infty$$

Because the value of $s$ is not a real number, the sum of the series does not exist, $s_n$ diverges, and therefore $a_n$ also diverges.

**Topic**: Convergence of a telescoping series

**Question**: Say whether or not the telescoping series converges.

$$\sum_{n=1}^{\infty} \frac{1}{\ln n} - \frac{1}{\ln(n+1)}$$

**Answer choices**:

A The series diverges $\quad s_n = \frac{1}{\ln 1} + \frac{1}{\ln(n+1)}$

B The series converges $\quad s_n = \frac{1}{\ln 1} + \frac{1}{\ln(n+1)}$

C The series converges $\quad s_n = \frac{1}{\ln 1} - \frac{1}{\ln(n+1)}$

D The series diverges $\quad s_n = \frac{1}{\ln 1} - \frac{1}{\ln(n+1)}$

**Solution**: D

To see whether or not a telescoping series converges or diverges, we'll find the $n$th partial sum of the series $s_n$, and then take the limit as $n \to \infty$ of $s_n$, or

$$s = \lim_{n \to \infty} s_n$$

If we get a real-number value for $s$, then $s_n$ converges, and therefore so does $a_n$.

We'll expand the telescoping series by calculating the first few terms, making sure to also include the last term of the series, then simplify the sum by canceling all of the terms in the middle. The remaining series will be the series of partial sums $s_n$.

In order to show that the series is telescoping, we'll need to start by expanding the series. Let's use $n = 1$, $n = 2$, $n = 3$ and $n = 4$.

$$n = 1 \qquad \frac{1}{\ln 1} - \frac{1}{\ln(1+1)} \qquad \frac{1}{\ln 1} - \frac{1}{\ln 2}$$

$$n = 2 \qquad \frac{1}{\ln 2} - \frac{1}{\ln(2+1)} \qquad \frac{1}{\ln 2} - \frac{1}{\ln 3}$$

$$n = 3 \qquad \frac{1}{\ln 3} - \frac{1}{\ln(3+1)} \qquad \frac{1}{\ln 3} - \frac{1}{\ln 4}$$

$$n = 4 \qquad \frac{1}{\ln 4} - \frac{1}{\ln(4+1)} \qquad \frac{1}{\ln 4} - \frac{1}{\ln 5}$$

Writing these terms into our expanded series and including the last term of the series, we get

$$\sum_{n=1}^{\infty} \frac{1}{\ln n} - \frac{1}{\ln(n+1)} = \left(\frac{1}{\ln 1} - \frac{1}{\ln 2}\right) + \left(\frac{1}{\ln 2} - \frac{1}{\ln 3}\right) + \left(\frac{1}{\ln 3} - \frac{1}{\ln 4}\right)$$

$$+\left(\frac{1}{\ln 4} - \frac{1}{\ln 5}\right) + \dots + \left(\frac{1}{\ln n} - \frac{1}{\ln(n+1)}\right)$$

When we look at our expanded series, we see that the second half of the first term will cancel with the first half of the second term, that the second half of the second term will cancel with the first half of the third term, and so on, so we can say that the series is telescoping.

Canceling everything but the first half of the first term and the second half of the last term gives an expression for the series of partial sums.

$$s_n = \frac{1}{\ln 1} - \frac{1}{\ln(n+1)}$$

Remember that we're looking for a real-number value for $s$, where

$$s = \lim_{n \to \infty} s_n$$

so we'll plug $s_n$ into this equation and get

$$s = \lim_{n \to \infty} \frac{1}{\ln 1} - \frac{1}{\ln(n+1)}$$

If we find the limit as $n \to \infty$, then $n + 1$ is an infinitely large value, and $\ln(n+1)$ will be infinitely large. When the denominator of a fraction becomes infinitely large, the fraction itself tends toward $0$, so

$$s = \lim_{n \to \infty} \frac{1}{\ln 1} - 0$$

$$s = \lim_{n \to \infty} \frac{1}{\ln 1}$$

There is no $n$ value remaining in the expression, and the limit only effects $n$, which means we can remove it.

$$s = \frac{1}{\ln 1}$$

Because $\ln 1 = 0$, we'll get a $0$ in the denominator of this fraction, which makes the fraction undefined. Therefore, the value of $s$ is undefined, so the sum of the series does not have a real-number value.

Because the value of $s$ is not a real number, the sum of the series does not exist, $s_n$ diverges, and therefore $a_n$ also diverges.

**Topic**: Convergence of a telescoping series

**Question**: Say whether or not the telescoping series converges.

$$\sum_{n=1}^{\infty} \frac{4}{n} - \frac{4}{n+1}$$

**Answer choices**:

A The series diverges $s_n = 4 + \frac{4}{n+1}$

B The series converges $s_n = 4 + \frac{4}{n+1}$

C The series diverges $s_n = 4 - \frac{4}{n+1}$

D The series converges $s_n = 4 - \frac{4}{n+1}$

**Solution**: D

To see whether or not a telescoping series converges or diverges, we'll find the $n$th partial sum of the series $s_n$, and then take the limit as $n \to \infty$ of $s_n$, or

$$s = \lim_{n \to \infty} s_n$$

If we get a real-number value for $s$, then $s_n$ converges, and therefore so does $a_n$.

We'll expand the telescoping series by calculating the first few terms, making sure to also include the last term of the series, then simplify the sum by canceling all of the terms in the middle. The remaining series will be the series of partial sums $s_n$.

In order to show that the series is telescoping, we'll need to start by expanding the series. Let's use $n = 1$, $n = 2$, $n = 3$ and $n = 4$.

$$n = 1 \qquad \frac{4}{1} - \frac{4}{1+1} \qquad \frac{4}{1} - \frac{4}{2}$$

$$n = 2 \qquad \frac{4}{2} - \frac{4}{2+1} \qquad \frac{4}{2} - \frac{4}{3}$$

$$n = 3 \qquad \frac{4}{3} - \frac{4}{3+1} \qquad \frac{4}{3} - \frac{4}{4}$$

$$n = 4 \qquad \frac{4}{4} - \frac{4}{4+1} \qquad \frac{4}{4} - \frac{4}{5}$$

Writing these terms into our expanded series and including the last term of the series, we get

$$\sum_{n=1}^{\infty} \frac{4}{n} - \frac{4}{n+1} = \left(\frac{4}{1} - \frac{4}{2}\right) + \left(\frac{4}{2} - \frac{4}{3}\right) + \left(\frac{4}{3} - \frac{4}{4}\right) + \left(\frac{4}{4} - \frac{4}{5}\right) + \ldots + \left(\frac{4}{n} - \frac{4}{n+1}\right)$$

When we look at our expanded series, we see that the second half of the first term will cancel with the first half of the second term, that the second half of the second term will cancel with the first half of the third term, and so on, so we can say that the series is telescoping.

Canceling everything but the first half of the first term and the second half of the last term gives an expression for the series of partial sums.

$$s_n = \frac{4}{1} - \frac{4}{n+1}$$

$$s_n = 4 - \frac{4}{n+1}$$

Remember that we're looking for a real-number value for $s$, where

$$s = \lim_{n\to\infty} s_n$$

so we'll plug $s_n$ into this equation and get

$$s = \lim_{n\to\infty} 4 - \frac{4}{n+1}$$

$$s = \lim_{n\to\infty} 4 - \frac{\frac{4}{n}}{\frac{n}{n} + \frac{1}{n}}$$

$$s = \lim_{n\to\infty} 4 - \frac{\frac{4}{n}}{1 + \frac{1}{n}}$$

If we find the limit as $n \to \infty$, then we get

$$s = 4 - \frac{0}{1+0}$$

$$s = 4 - 0$$

$$s = 4$$

Because the value of $s$ is a real number, the sum of the series is $s = 4$, $s_n$ converges, and therefore $a_n$ also converges.