Proving that the limit does not exist

Given what we now know about how the existence of the one-sided limits dictates the existence of the general limit, we can use this relationship as a test for showing whether or not the general limit exists.

Remember we said before that the general limit exists at a point $x = c$ if

1. the left-hand limit exists at $x = c$,

2. the right-hand limit exists at $x = c$, and

3. those left- and right-hand limits are equal to one another.

Therefore, the general limit **does not exist (DNE)** at $x = c$ if

1. the left-hand limit does not exist at $x = c$, and/or

2. the right-hand limit does not exist at $x = c$, and/or

3. the left- and right-hand limits both exist, but aren't equal to one another.

Let's do an example where we show algebraically that the limit does not exist.

Example

Prove that the limit does not exist.

$$\lim_{x \to 2} \frac{1}{x - 2}$$

If we try substitution, we get an undefined value, because the denominator of the fraction becomes 0.

$$\frac{1}{2-2}$$

$$\frac{1}{0}$$

Because we can't use substitution, we'll instead use values on either side of $x = 2$, very close to $x = 2$, to determine how the function is behaving as $x \to 2$.

$$f(1.9999) = \frac{1}{1.9999 - 2} = \frac{1}{-0.0001} = -10{,}000$$

$$f(2.0001) = \frac{1}{2.0001 - 2} = \frac{1}{0.0001} = 10{,}000$$

From the function's values around $x = 2$, we can tell that the function is tending toward $-\infty$ to the left of $x = 2$, and toward ∞ to the right of $x = 2$.

$$\lim_{x \to 2^-} \frac{1}{x-2} = -\infty$$

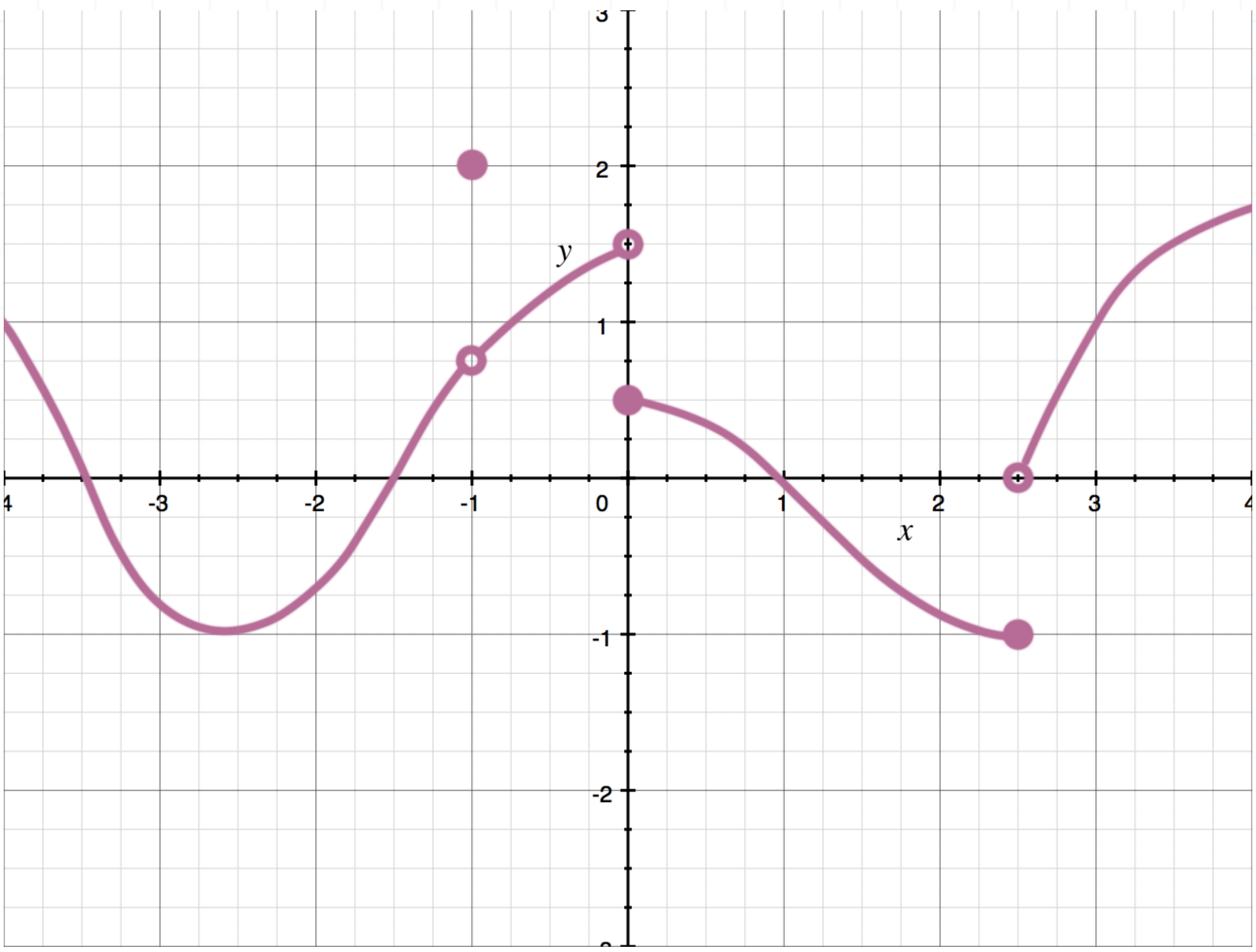
$$\lim_{x \to 2^+} \frac{1}{x-2} = \infty$$

Because the left- and right-hand limits aren't equal, we've proven that the general limit of this function does not exist at $x = 2$.

We can also determine graphically that the limit does not exist.

Example

Use the graph to determine whether or not the limit exists at $x = 0$.



At $x = 0$, the function is approaching $3/2$ from the left side. But from the right side, the function is approaching $1/2$. So if we say that the graph represents the function $f(x)$, then the one-sided limits are

$$\lim_{x \to 0^-} f(x) = \frac{3}{2}$$

$$\lim_{x \to 0^+} f(x) = \frac{1}{2}$$

Because the left- and right-hand limits aren't equal, we've proven that the general limit of this function does not exist at $x = 0$.

Infinite one-sided limits

We also want to look at what happens to the general limit when both one-sided limits are infinite.

1. If both one-sided limits are ∞, then the general limit exists and is equal to ∞.

2. If both one-sided limits are $-\infty$, then the general limit exists and is equal to $-\infty$.

3. If one of the one-sided limits is ∞ while the other one-sided limit at the same point is $-\infty$, then the general limit doesn't exist.

Let's do an example where the limit is infinite.

Example

Find the limit.

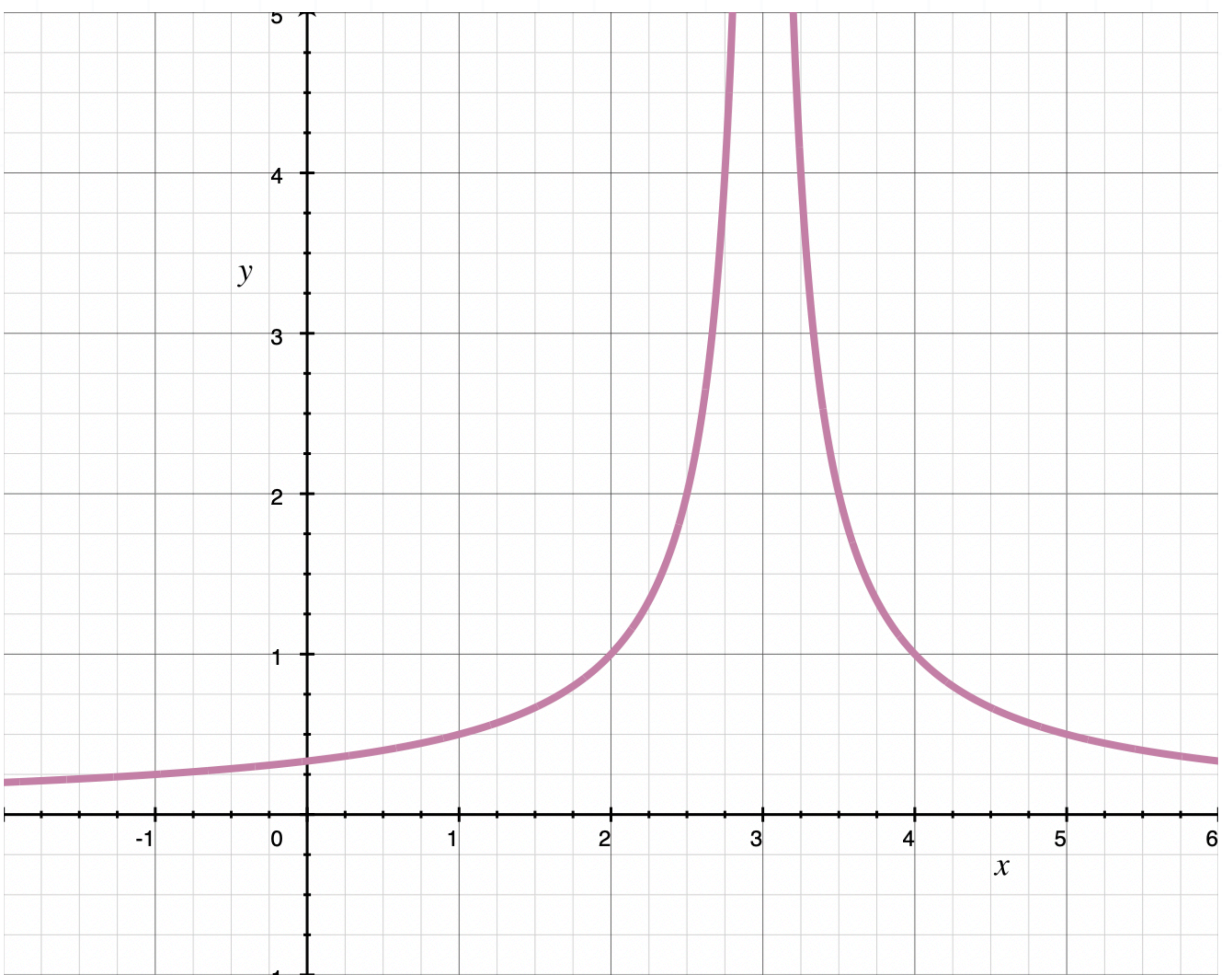
$$\lim_{x \to 3} \frac{1}{|x - 3|}$$

We can get a sense of the one-sided limits if we evaluate the function at values close to $x = 3$.

$$\lim_{x \to 3^-} \frac{1}{|x-3|} \approx \frac{1}{|2.999999 - 3|} \approx \frac{1}{0.000001} \approx 1{,}000{,}000$$

$$\lim_{x \to 3^+} \frac{1}{|x-3|} \approx \frac{1}{|3.000001 - 3|} \approx \frac{1}{0.000001} \approx 1{,}000{,}000$$

When we evaluate the function at values close to $x = 3$, we get a sense of the fact that both one-sided limits are headed toward ∞. If we use the graph to confirm this hunch,



we see that at $x = 3$, the function is approaching ∞ from the left side and ∞ from the right side.

$$\lim_{x \to 3^-} \frac{1}{|x-3|} = \infty$$

$$\lim_{x \to 3^+} \frac{1}{|x-3|} = \infty$$

Because the one-sided limits are equal, the general limit exists and is equal to that same value.

$$\lim_{x \to 3} \frac{1}{|x-3|} = \infty$$