Topic: Vertical and horizontal tangent lines to the polar curve

Question: Which are the points at which the curve has horizontal tangent lines?

$$r = 6(1 - \cos\theta)$$

Answer choices:

A $\left(1,\frac{2\pi}{3}\right)$ and $\left(1,\frac{4\pi}{3}\right)$ and $(0,0)$

B $\left(5,\frac{3\pi}{2}\right)$ and $\left(5,\frac{5\pi}{3}\right)$ and $(0,0)$

C $\left(9,\frac{2\pi}{3}\right)$ and $\left(9,\frac{4\pi}{3}\right)$ and $(0,0)$

D $\left(9,\frac{3\pi}{2}\right)$ and $\left(9,\frac{5\pi}{3}\right)$ and $(0,0)$

Solution: C

We'll use the conversion equations $x = r\cos\theta$ and $y = r\sin\theta$, and plug in the given polar equation, $r = 6(1 - \cos\theta)$.

$$x = r\cos\theta$$

$$x = 6(1 - \cos\theta)\cos\theta$$

$$x = 6\cos\theta - 6\cos^2\theta$$

and

$$y = r\sin\theta$$

$$y = 6(1 - \cos\theta)\sin\theta$$

$$y = 6\sin\theta - 6\sin\theta\cos\theta$$

Take the derivative of each of these.

$$\frac{dx}{d\theta} = -6\sin\theta + 12\sin\theta\cos\theta$$

$$\frac{dy}{d\theta} = 6\cos\theta + 6\sin^2\theta - 6\cos^2\theta$$

Now we'll get the derivative dy/dx.

$$\frac{dy}{dx} = \frac{6\cos\theta + 6\sin^2\theta - 6\cos^2\theta}{-6\sin\theta + 12\sin\theta\cos\theta}$$

Horizontal tangent lines exist where this derivative is equal to 0. But because the derivative is a fraction, it can only be 0 where the numerator is 0. Therefore

$$0 = 6\cos\theta + 6\sin^2\theta - 6\cos^2\theta$$

$$0 = \cos\theta + \sin^2\theta - \cos^2\theta$$

$$0 = \cos\theta + 1 - \cos^2\theta - \cos^2\theta$$

$$2\cos^2\theta - \cos\theta - 1 = 0$$

$$(2\cos\theta + 1)(\cos\theta - 1) = 0$$

$$2\cos\theta + 1 = 0 \quad \text{or} \quad \cos\theta - 1 = 0$$

$$\cos\theta = -\frac{1}{2} \quad \text{or} \quad \cos\theta = 1$$

$$\theta = \frac{2\pi}{3}, \frac{4\pi}{3} \quad \text{or} \quad \theta = 0$$

If we plug these values for θ back into the original polar curve, we get

$$r = 6\left(1 - \cos\frac{2\pi}{3}\right)$$

$$r = 6\left(1 + \frac{1}{2}\right)$$

$$r = 9$$

and

$$r=6\left(1-\cos\frac{4\pi}{3}\right)$$

$$r=6\left(1+\frac{1}{2}\right)$$

$$r=9$$

and

$$r=6(1-\cos 0)$$

$$r=6(1-1)$$

$$r=0$$

Thus, the horizontal tangents pass through

$$\left(9,\frac{2\pi}{3}\right)$$

$$\left(9,\frac{4\pi}{3}\right)$$

$$(0,0)$$

Topic: Vertical and horizontal tangent lines to the polar curve

Question: Where are the horizontal and vertical tangent lines to the polar curve?

$$r = 4\cos\theta$$

Answer choices:

A	Vertical tangents at	$(4,0)$	and	$\left(0,\frac{\pi}{2}\right)$
	Horizontal tangents at	$\left(2\sqrt{2},\frac{\pi}{4}\right)$	and	$\left(-2\sqrt{2},\frac{3\pi}{4}\right)$
B	Vertical tangents at	$(4,0)$	and	$\left(0,\frac{3\pi}{2}\right)$
	Horizontal tangents at	$\left(\frac{4}{\sqrt{2}},\frac{\pi}{4}\right)$	and	$\left(2\sqrt{2},\frac{\pi}{4}\right)$
C	Vertical tangents at	$(4,0)$	and	$\left(0,\frac{\pi}{2}\right)$
	Horizontal tangents at	$\left(\frac{4}{\sqrt{2}},\frac{3\pi}{4}\right)$	and	$\left(-\frac{4}{\sqrt{2}},\frac{3\pi}{4}\right)$

D Vertical tangents at $\left(2\sqrt{2},\frac{\pi}{4}\right)$ and $\left(0,\frac{5\pi}{2}\right)$

Horizontal tangents at $\left(\frac{4}{\sqrt{2}},\frac{\pi}{4}\right)$ and $\left(-\frac{4}{\sqrt{2}},\frac{3\pi}{4}\right)$

Solution: A

The function $r = 4\cos\theta$ can be described by

$$x = r\cos\theta$$

$$x = (4\cos\theta)\cos\theta$$

$$x = 4\cos^2\theta$$

and

$$y = r\sin\theta$$

$$y = (4\cos\theta)\sin\theta$$

$$y = 4\sin\theta\cos\theta$$

Take the derivative of each of these.

$$\frac{dx}{d\theta} = -8\sin\theta\cos\theta$$

$$\frac{dy}{d\theta} = 4\cos^2\theta - 4\sin^2\theta$$

Horizontal tangent lines exist where $dy/d\theta = 0$, and vertical tangent lines exist where $dx/d\theta = 0$. Therefore, set the derivatives equal to 0 and solve for θ.

Vertical tangent lines at:

$$-8\sin\theta\cos\theta = 0$$

$$2\sin\theta\cos\theta = 0$$

$$\sin 2\theta = 0$$

$\theta = 0$ or $\theta = \dfrac{\pi}{2}$

and

Horizontal tangent lines at:

$$4\cos^2 - 4\sin^2\theta = 0$$

$$\cos^2 - \sin^2\theta = 0$$

$$\cos 2\theta = 0$$

$\theta = \dfrac{\pi}{4}$ or $\theta = \dfrac{3\pi}{4}$

Plug each of these into the original function.

$$r = 4\cos 0$$

$$r = 4(1)$$

$$r = 4$$

and

$$r = 4\cos\frac{\pi}{2}$$

$$r = 4(0)$$

$$r = 0$$

and

$$r = 4\cos\frac{\pi}{4}$$

$$r = 4\left(\frac{\sqrt{2}}{2}\right)$$

$$r = 2\sqrt{2}$$

and

$$r = 4\cos\frac{3\pi}{4}$$

$$r = 4\left(-\frac{\sqrt{2}}{2}\right)$$

$$r = -2\sqrt{2}$$

Therefore, there are

vertical tangents at $(4,0)$ and $\left(0,\frac{\pi}{2}\right)$

horizontal tangents at $\left(2\sqrt{2},\frac{\pi}{4}\right)$ and $\left(-2\sqrt{2},\frac{3\pi}{4}\right)$

Topic: Vertical and horizontal tangent lines to the polar curve

Question: Which function has vertical tangent lines that pass through these points?

$$\left(3,\frac{\pi}{6}\right) \text{ and } \left(3,\frac{5\pi}{6}\right)$$

Answer choices:

A $\quad r = 3(1 - \sin\theta)$

B $\quad r = -2(1 + \sin\theta)$

C $\quad r = 3(1 + \sin\theta)$

D $\quad r = 2(1 + \sin\theta)$

Solution: D

Starting with the function from answer choice D, $r=2(1+\sin\theta)$, we'll use the conversion formulas

$$x=r\cos\theta$$

$$y=r\sin\theta$$

and plug in the given polar curve.

$$x=2(1+\sin\theta)\cos\theta$$

$$x=2\cos\theta+2\sin\theta\cos\theta$$

and

$$y=2(1+\sin\theta)\sin\theta$$

$$y=2\sin^2\theta+2\sin\theta$$

Take the derivative of each of these.

$$\frac{dy}{d\theta}=4\sin\theta\cos\theta+2\cos\theta$$

$$\frac{dx}{d\theta}=-2\sin\theta+2\cos^2\theta-2\sin^2\theta$$

Now find the derivative dy/dx.

$$\frac{dy}{dx}=\frac{4\sin\theta\cos\theta+2\cos\theta}{-2\sin\theta+2\cos^2\theta-2\sin^2\theta}$$

Vertical tangent lines will exist where this derivative is undefined, which means we'll find vertical tangent lines wherever the denominator is equal to 0.

$$-2\sin\theta + 2\cos^2\theta - 2\sin^2\theta = 0$$

$$-\sin\theta + \cos^2\theta - \sin^2\theta = 0$$

$$-\sin\theta + 1 - \sin^2\theta - \sin^2\theta = 0$$

$$-2\sin^2\theta - \sin\theta + 1 = 0$$

$$2\sin^2\theta + \sin\theta - 1 = 0$$

$$(2\sin\theta - 1)(\sin\theta + 1) = 0$$

$$2\sin\theta - 1 = 0 \qquad \text{or} \qquad \sin\theta + 1 = 0$$

$$\sin\theta = \frac{1}{2} \qquad \text{or} \qquad \sin\theta = -1$$

$$\theta = \frac{\pi}{6}, \frac{5\pi}{6} \qquad \text{or} \qquad \theta = \frac{3\pi}{2}$$

The angle $3\pi/2$ doesn't exist in the given points, so we can ignore this value. Plug the other θ values into the original function.

$$r = 2(1 + \sin\theta)$$

$$r = 2\left(1 + \sin\frac{\pi}{6}\right)$$

$$r = 3$$

and

$$r = 2(1 + \sin\theta)$$

$$r = 2\left(1 + \sin\frac{5\pi}{6}\right)$$

$$r = 3$$

Therefore, the vertical tangents of answer choice D pass through

$$\left(3, \frac{\pi}{6}\right) \text{ and } \left(3, \frac{5\pi}{6}\right)$$

Because these are the points we were given, we know that answer choice D is correct.