

Power series multiplication

Previously we learned how to create a power series representation for a function by modifying a similar, known series to match the function.

Sometimes the function we're given is the product of two other functions, like

$$f(x) = \frac{\cos x}{1 - x}$$

This function is the product of $g(x) = \cos x$ and $h(x) = 1/(1 - x)$.

$$f(x) = \frac{1}{1 - x} \cdot \cos x$$

If we already know the power series representations of $g(x) = \cos x$ and $h(x) = 1/(1 - x)$, we can multiply the expanded power series together to find a power series representation of $f(x)$, since $f(x)$ is the product of $g(x)$ and $h(x)$.

In other words, since we already know from a table to standard Maclaurin series that

$$\cos x \approx 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \frac{1}{720}x^6 + \dots$$

$$\frac{1}{1 - x} \approx 1 + x + x^2 + x^3 + \dots$$

we can say that



$$f(x) \approx \left(1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \frac{1}{720}x^6 + \dots\right) (1 + x + x^2 + x^3 + \dots)$$

Keep in mind also that multiplying power series together is just like multiplying two simple polynomials together. We use the distributive property from algebra, and we multiply the first term in the first series by all of the terms in the second series, then we multiply the second term in the first series by all of the terms in the second series, and so on.

$$\begin{aligned} f(x) &\approx 1(1 + x + x^2 + x^3 + \dots) - \frac{1}{2}x^2(1 + x + x^2 + x^3 + \dots) \\ &\quad + \frac{1}{24}x^4(1 + x + x^2 + x^3 + \dots) - \frac{1}{720}x^6(1 + x + x^2 + x^3 + \dots) + \dots \end{aligned}$$

$$\begin{aligned} f(x) &\approx 1 + x + x^2 + x^3 - \frac{1}{2}x^2 - \frac{1}{2}x^3 - \frac{1}{2}x^4 - \frac{1}{2}x^5 \\ &\quad + \frac{1}{24}x^4 + \frac{1}{24}x^5 + \frac{1}{24}x^6 + \frac{1}{24}x^7 - \frac{1}{720}x^6 - \frac{1}{720}x^7 - \frac{1}{720}x^8 - \frac{1}{720}x^9 \end{aligned}$$

We'll collect like-terms, and get

$$f(x) \approx 1 + x + x^2 - \frac{1}{2}x^2 + x^3 - \frac{1}{2}x^3 - \frac{1}{2}x^4 + \frac{1}{24}x^4 - \frac{1}{2}x^5 + \frac{1}{24}x^5$$

$$+ \frac{1}{24}x^6 - \frac{1}{720}x^6 + \frac{1}{24}x^7 - \frac{1}{720}x^7 - \frac{1}{720}x^8 - \frac{1}{720}x^9$$

$$f(x) \approx 1 + x + \frac{1}{2}x^2 + \frac{1}{2}x^3 - \frac{11}{24}x^4 - \frac{11}{24}x^5$$

$$+ \frac{1}{24}x^6 - \frac{1}{720}x^6 + \frac{1}{24}x^7 - \frac{1}{720}x^7 - \frac{1}{720}x^8 - \frac{1}{720}x^9$$



Remember to list the terms in the series in ascending order of degree, with any constant first, followed by x , followed by x^2 , x^3 , x^4 , etc.

For most power series multiplication problems, we'll be asked to find a specific number of non-zero terms in the expanded power series representation of $f(x)$. With this in mind, we can actually stop multiplying once we have the number of non-zero terms we've been asked for. In the above example, if we were asked for the first five non-zero terms, we could have stopped multiplying once we had all of our x^4 terms. We would have collected like terms for all fourth- or lesser-degree terms, and given an answer of

$$f(x) \approx 1 + x + \frac{1}{2}x^2 + \frac{1}{2}x^3 - \frac{11}{24}x^4$$

Example

Use power series multiplication to find the first four non-zero terms of the Maclaurin series of the given function.

$$y = \sin(2x)e^{3x}$$

We know that the expanded versions of the Maclaurin series for $\sin x$ and e^x are

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} = x - \frac{1}{6}x^3 + \frac{1}{120}x^5 - \frac{1}{5,040}x^7 + \dots$$



$$e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \dots$$

Since we have $\sin(2x)$ instead of $\sin x$ and since we have e^{3x} instead of e^x , we'll need to modify both series.

We'll start with the $\sin x$ series, letting $x = 2x$, and the expanded series will be

$$\sin(2x) = 2x - \frac{1}{6}(2x)^3 + \frac{1}{120}(2x)^5 - \frac{1}{5,040}(2x)^7 + \dots$$

$$\sin(2x) = 2x - \frac{4}{3}x^3 + \frac{4}{15}x^5 - \frac{8}{315}x^7 + \dots$$

Now we'll modify the e^x series, letting $x = 3x$, and the expanded series will be

$$e^{3x} = 1 + 3x + \frac{1}{2}(3x)^2 + \frac{1}{6}(3x)^3 + \dots$$

$$e^{3x} = 1 + 3x + \frac{9}{2}x^2 + \frac{9}{2}x^3 + \dots$$

We'll use the distributive property from algebra to multiply the series together, as if we're multiplying two simple polynomials.

$$\sin(2x)e^{3x} = \left(2x - \frac{4}{3}x^3 + \frac{4}{15}x^5 - \frac{8}{315}x^7 + \dots \right) \left(1 + 3x + \frac{9}{2}x^2 + \frac{9}{2}x^3 + \dots \right)$$

$$\sin(2x)e^{3x} = 2x + 6x^2 + 9x^3 + 9x^4$$

$$- \frac{4}{3}x^3 - 4x^4 - 6x^5 - 6x^6$$



$$+\frac{4}{15}x^5 + \frac{4}{5}x^6 + \frac{6}{5}x^7 + \frac{6}{5}x^8$$

$$-\frac{8}{315}x^7 - \frac{24}{315}x^8 - \frac{72}{630}x^9 - \frac{72}{630}x^{10} + \dots$$

Cancel everything that isn't an x , x^2 , x^3 , or x^4 term.

$$\sin(2x)e^{3x} = 2x + 6x^2 + 9x^3 + 9x^4 - \frac{4}{3}x^3 - 4x^4$$

$$\sin(2x)e^{3x} = 2x + 6x^2 + \frac{27}{3}x^3 - \frac{4}{3}x^3 + 5x^4$$

$$\sin(2x)e^{3x} = 2x + 6x^2 + \frac{23}{3}x^3 + 5x^4$$

Since this is an approximation, the answer will be

$$\sin(2x)e^{3x} = 2x + 6x^2 + \frac{23}{3}x^3 + 5x^4$$

