# Calculus 2 Workbook Solutions

## Error bounds

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## MIDPOINT RULE ERROR BOUND

1. Calculate the area under the curve. Then use the Midpoint Rule with $n=3$ to approximate the same area. Compare the actual area to the result to determine the error of the of the Midpoint Rule approximation.

$$\int_0^6 3x^2-2x+5\ dx$$

*Solution:*

The area under the curve is

$$A=\frac{3x^3}{3}-\frac{2x^2}{2}+5x\Big|_0^6=x^3-x^2+5x\Big|_0^6$$

$$A=6^3-6^2+5(6)-\left(0^3-0^2+5(0)\right)$$

$$A=210$$

With $n=3$, the integration interval of $[0,6]$ is split into the three subintervals $[0,2]$, $[2,4]$, and $[4,6]$. The midpoints of those subintervals are $1$, $3$, and $5$. Evaluate $3x^2-2x+5$ at each of these values.

At $x=1$, $3(1)^2-2(1)+5=6$

At $x=3$, $3(3)^2-2(3)+5=26$

At $x=5$, $3(5)^2-2(5)+5=70$

Since the original interval is $[0,6]$ and $n=3$, each subinterval is $2$ units wide. So the Midpoint Rule gives

$$A_M = 6(2) + 26(2) + 70(2)$$

$$A_M = 204$$

Compared to the actual area under the curve, the Midpoint Rule gives an error of $210 - 204 = 6$.

■ 2. Calculate the area under the curve. Then use the Midpoint Rule with $n=4$ to approximate the same area. Compare the actual area to the result to determine the error of the of the Midpoint Rule approximation. Round your answer to the nearest $3$ decimal places.

$$\int_5^{13} 4\sqrt{x-2}\ dx$$

*Solution:*

The area under the curve is

$$A = \frac{8}{3}(x-2)^{\frac{3}{2}}\bigg|_5^{13}$$

$$A = \frac{8}{3}(13-2)^{\frac{3}{2}} - \frac{8}{3}(5-2)^{\frac{3}{2}}$$

$$A = 97.287661 - 13.856406 = 83.431255$$

With $n = 4$, the integration interval of $[5,13]$ is split into the four subintervals $[5,7]$, $[7,9]$, $[9,11]$, and $[11,13]$. The midpoints of those subintervals are $6$, $8$, $10$, and $12$. Evaluate $4\sqrt{x-2}$ at each of these values.

At $x = 6$, $4\sqrt{6-2} = 8$

At $x = 8$, $4\sqrt{8-2} = 9.797959$

At $x = 10$, $4\sqrt{10-2} = 11.313708$

At $x = 12$, $4\sqrt{12-2} = 12.649111$

Since the original interval is $[5,13]$ and $n = 4$, each subinterval is $2$ units wide. So the Midpoint Rule gives

$$A_M \approx 8(2) + 9.797959(2) + 11.313708\,(2) + 12.649111(2)$$

$$A_M \approx 83.521556$$

Compared to the actual area under the curve, the Midpoint Rule gives an error of $|83.431255 - 83.521556| \approx 0.090301 \approx 0.090$.

■ 3. Calculate the area under the curve. Then use the Midpoint Rule with $n = 4$ to approximate the same area. Compare the actual area to the result to determine the error of the of the Midpoint Rule approximation.

$$\int_2^{10} 4x^3 - 3x^2 + 2x - 1\ dx$$

*Solution:*

The area under the curve is

$$A = \frac{4x^4}{4} - \frac{3x^3}{3} + \frac{2x^2}{2} - x \Big|_2^{10} = x^4 - x^3 + x^2 - x \Big|_2^{10}$$

$$A = 10^4 - 10^3 + 10^2 - 10 - \left(2^4 - 2^3 + 2^2 - 2\right) = 9{,}090 - 10$$

$$A = 9{,}080$$

With $n = 4$, the integration interval of $[2,10]$ is split into the four subintervals $[2,4]$, $[4,6]$, $[6,8]$, and $[8,10]$. The midpoints of those subintervals are $3$, $5$, $7$, and $9$. Evaluate $4x^3 - 3x^2 + 2x - 1$ at each of these values.

At $x = 3$, $4(3)^3 - 3(3)^2 + 2(3) - 1 = 86$

At $x = 5$, $4(5)^3 - 3(5)^2 + 2(5) - 1 = 434$

At $x = 7$, $4(7)^3 - 3(7)^2 + 2(7) - 1 = 1{,}238$

At $x = 9$, $4(9)^3 - 3(9)^2 + 2(9) - 1 = 2{,}690$

Since the original interval is $[2,10]$ and $n = 4$, each subinterval is $2$ units wide. So the Midpoint Rule gives

$$A_M = 86(2) + 434(2) + 1{,}238(2) + 2{,}690(2)$$

$$A_M = 8{,}896$$

Compared to the actual area under the curve, the Midpoint Rule gives an error of $9{,}080 - 8{,}896 = 184$.

## TRAPEZOIDAL RULE ERROR BOUND

■ 1. Calculate the area under the curve. Then use the Trapezoidal Rule with $n = 4$ to approximate the same area. Compare the actual area to the result to determine the error of the of the Trapezoidal Rule approximation.

$$\int_1^5 6x^2 - 8x + 5\ dx$$

*Solution:*

The area under the curve is

$$A = \frac{6x^3}{3} - \frac{8x^2}{2} + 5x\Big|_1^5 = 2x^3 - 4x^2 + 5x\Big|_1^5$$

$$A = 2(5)^3 - 4(5)^2 + 5(5) - \left(2(1)^3 - 4(1)^2 + 5(1)\right) = 175 - 3$$

$$A = 172$$

With $n = 4$, the integration interval of $[1,5]$ is split into the four subintervals $[1,2]$, $[2,3]$, $[3,4]$, and $[4,5]$. Evaluate the integrand at the endpoints of each subinterval.

At $x = 1$, $6(1)^2 - 8(1) + 5 = 3$

At $x = 2$, $6(2)^2 - 8(2) + 5 = 13$

At $x = 3$, $6(3)^2 - 8(3) + 5 = 35$

At $x = 4$, $6(4)^2 - 8(4) + 5 = 69$

At $x = 5$, $6(5)^2 - 8(5) + 5 = 115$

Use these values in the Trapezoidal Rule with $\Delta x = 1$.

$$A_T = \frac{1}{2}\left[3 + 2(13) + 2(35) + 2(69) + 115\right]$$

$$A_T = 176$$

Compared to the actual area under the curve, the Trapezoidal Rule gives an error of $|172 - 176| = 4$.

■ 2. Calculate the area under the curve. Then use the Trapezoidal Rule with $n = 5$ to approximate the same area. Compare the actual area to the result to determine the error of the of the Trapezoidal Rule approximation. Round your answer to the nearest $3$ decimal places.

$$\int_2^{12} e^{-x} + 3\ dx$$

*Solution:*

The area under the curve is

$$A = -e^{-x} + 3x\Big|_2^{12}$$

$$A = -e^{-12} + 3(12) - \left(-e^{-2} + 3(2)\right)$$

$$A = 30.135329$$

With $n = 4$, the integration interval of $[2,12]$ is split into the five subintervals $[2,4]$, $[4,6]$, $[6,8]$, $[8,10]$, and $[10,12]$. Evaluate the integrand at the endpoints of each subinterval.

At $x = 2$, $e^{-2} + 3 = 3.135335$

At $x = 4$, $e^{-4} + 3 = 3.018316$

At $x = 6$, $e^{-6} + 3 = 3.002479$

At $x = 8$, $e^{-8} + 3 = 3.000335$

At $x = 10$, $e^{-10} + 3 = 3.000045$

At $x = 12$, $e^{-12} + 3 = 3.000006$

Use these values in the Trapezoidal Rule with $\Delta x = 2$.

$$A_T = \frac{2}{2}\big[3.135335 + 2(3.018316) + 2(3.002479)$$

$$+2(3.000335) + 2(3.000045) + 3.000006\big]$$

$$A_T = 30.177691$$

Compared to the actual area under the curve, the Trapezoidal Rule gives an error of $|30.135329 - 30.177691| = 0.042362$.

3. Calculate the area under the curve. Then use the Trapezoidal Rule with $n=4$ to approximate the same area. Compare the actual area to the result to determine the error of the of the Trapezoidal Rule approximation. Round your answer to the nearest three decimal places.

$$\int_0^2 4\sqrt{x}+1\ dx$$

*Solution:*

The area under the curve is

$$A=\frac{8}{3}x^{\frac{3}{2}}+x\Big|_0^2$$

$$A=\frac{8}{3}(2)^{\frac{3}{2}}+2-\left(\frac{8}{3}(0)^{\frac{3}{2}}+0\right)$$

$$A=9.542472$$

With $n=4$, the integration interval of $[0,2]$ is split into the four subintervals $[0,0.5]$, $[0.5,1]$, $[1,1.5]$, and $[1.5,2]$. Evaluate the integrand at the endpoints of each subinterval.

At $x=0$, $4\sqrt{0}+1=1$

At $x=0.5$, $4\sqrt{0.5}+1=3.828427$

At $x=1$, $4\sqrt{1}+1=5$

At $x=1.5$, $4\sqrt{1.5}+1=5.898979$

At $x=2$, $4\sqrt{2}+1=6.656854$

Use these values in the Trapezoidal Rule with $\Delta x=0.5$.

$$A_T=\frac{0.5}{2}\left[1+2(3.828427)+2(5)+2(5.898979)+6.656854\right]$$

$$A_T=9.277917$$

Compared to the actual area under the curve, the Trapezoidal Rule gives an error of $|9.542472-9.277917|=0.264826\approx 0.265$.

## SIMPSON'S RULE ERROR BOUND

1. Calculate the area under the curve. Then use Simpson's Rule with $n = 6$ to approximate the same area. Compare the actual area to the result to determine the error of the of Simpson's Rule approximation. Round your answer to the nearest three decimal places.

$$\int_{2.2}^{3.4} x^2 - x + 2\ dx$$

*Solution:*

The area under the curve is

$$A = \frac{x^3}{3} - \frac{x^2}{2} + 2x \Big|_{2.2}^{3.4}$$

$$A = \frac{(3.4)^3}{3} - \frac{(3.4)^2}{2} + 2(3.4) - \left(\frac{(2.2)^3}{3} - \frac{(2.2)^2}{2} + 2(2.2)\right)$$

$$A = 8.592$$

Since $n = 6$, the interval of integration is $[2.2,3.4]$, and $\Delta x = 0.2$,

$$x_1 = 2.2$$

$$x_2 = 2.4$$

$$x_3 = 2.6$$

$$x_4 = 2.8$$

$$x_5 = 3.0$$

$$x_6 = 3.2$$

$$x_7 = 3.4$$

Evaluate the integrand at each of these values.

At $2.2$, $(2.2)^2 - 2.2 + 2 = 4.64$

At $2.4$, $(2.4)^2 - 2.4 + 2 = 5.36$

At $2.6$, $(2.6)^2 - 2.6 + 2 = 6.16$

At $2.8$, $(2.8)^2 - 2.8 + 2 = 7.04$

At $3.0$, $(3.0)^2 - 3.0 + 2 = 8.00$

At $3.2$, $(3.2)^2 - 3.2 + 2 = 9.04$

At $3.4$, $(3.4)^2 - 3.4 + 2 = 10.16$

Use these values in Simpson's Rule with $\Delta x = 0.2$.

$$A_S \approx \frac{0.2}{3}\left[4.64 + 4(5.36) + 2(6.16) + 4(7.04) + 2(8) + 4(9.04) + 10.16\right]$$

$$A_S \approx 8.592$$

Simpson's Rule gives an error of $|8.592 - 8.592| = 0$. There is no error in this problem.

2. Calculate the area under the curve. Then use Simpson's Rule with $n = 4$ to approximate the same area. Compare the actual area to the result to determine the error of the of Simpson's Rule approximation. Round your answer to the nearest four decimal places.

$$\int_0^{1.2} e^x - 2x + 3\ dx$$

*Solution:*

The area under the curve is

$$A = e^x - \frac{2x^2}{2} + 3x \Big|_0^{1.2}$$

$$A = e^{1.2} - (1.2)^2 + 3(1.2) - \left(e^0 - (0)^2 + 3(0)\right)$$

$$A = 4.480117$$

Since $n = 4$, the interval of integration is $[0,1.2]$, and $\Delta x = 0.3$,

$$x_1 = 0$$

$$x_2 = 0.3$$

$$x_3 = 0.6$$

$x_4 = 0.9$

$x_4 = 1.2$

Evaluate the integrand at each of these values.

At $0$, $e^0 - 2(0) + 3 = 4$

At $0.3$, $e^{0.3} - 2(0.3) + 3 = 3.749859$

At $0.6$, $e^{0.6} - 2(0.6) + 3 = 3.622119$

At $0.9$, $e^{0.9} - 2(0.9) + 3 = 3.659603$

At $1.2$, $e^{1.2} - 2(1.2) + 3 = 3.920117$

Use these values in Simpson's Rule with $\Delta x = 0.3$.

$$A_S \approx \frac{0.3}{3}\left[4 + 4(3.749859) + 2(3.622119) + 4(3.659603) + 3.920117\right]$$

$$A_S \approx 4.480220$$

Simpson's Rule gives an error of $|4.480117 - 4.480220| = 0.0001033$.

■ 3. Calculate the area under the curve. Then use Simpson's Rule with $n = 4$ to approximate the same area. Compare the actual area to the result to determine the error of the of Simpson's Rule approximation. Round your answer to the nearest three decimal places.

$$\int_{-4}^{4} 2x^2 + 3x + 4 \ dx$$

*Solution:*

The area under the curve is

$$A = \frac{2x^3}{3} + \frac{3x^2}{2} + 4x\Big|_{-4}^{4}$$

$$A = \frac{2(4)^3}{3} + \frac{3(4)^2}{2} + 4(4) - \left(\frac{2(-4)^3}{3} + \frac{3(-4)^2}{2} + 4(-4)\right)$$

$$A = 117\frac{1}{3}$$

Since $n = 4$, the interval of integration is $[-4,4]$, and $\Delta x = 2$,

$$x_1 = -4$$

$$x_2 = -2$$

$$x_3 = 0$$

$$x_4 = 2$$

$$x_4 = 4$$

Evaluate the integrand at each of these values.

At $-4$, $2(-4)^2 + 3(-4) + 4 = 24$

At $-2$, $2(-2)^2 + 3(-2) + 4 = 6$

At $0$, $(0)^2 + 3(0) + 4 = 4$

At $2$, $2(2)^2 + 3(2) + 4 = 18$

At $4$, $(4)^2 + 3(4) + 4 = 4$

Use these values in Simpson's Rule with $\Delta x = 2$.

$$A_S = \frac{2}{3}\left[24 + 4(6) + 2(4) + 4(18) + 48\right]$$

$$A_S = 117\frac{1}{3}$$

Simpson's Rule gives an error of $|117.33 - 117.33| = 0$.

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