Volume of revolution, disk method

We can use integrals to find the volume of the three-dimensional object created by rotating a function around either the x-axis (or some other horizontal axis with the equation $y = b$) or around the y-axis (or some other vertical axis with the equation $x = a$).

We can do this using the disk method, the washer method, or using cylindrical shells. To use the disk method, the volume generated by rotating the function has to be a solid volume with no holes in the middle. It'll be bounded by the function you're rotating and the axis of rotation.

The disk method formulas we use to find volume of rotation are different depending on the form of the function and the axis of rotation.

1. If the function is in the form $y = f(x)$ and we're rotating around the x-axis over the interval $[a, b]$, the formula for the volume of the solid is

$$V = \int_a^b \pi \left[f(x)\right]^2 dx$$

2. If the function is in the form $x = g(y)$ and we're rotating around the y-axis over the interval $[c, d]$, the formula for the volume of the solid is

$$V = \int_c^d \pi \left[f(y)\right]^2 dy$$

The table below will help guide you through how to solve a volume problem when you're using disks or washers to find the volume. Start in the first row of the table, and determine the line of rotation or revolution. The problem will usually tell you the line of rotation. If you're asked to rotate about the x-axis or some line defined for y in terms of x, then stay in the first column of the table. If you're asked to rotate about the y-axis or some line defined for x in terms of y, then stay in the second column of the table.

The best way to figure out whether you need to use disks or washers is to graph the functions and the axis of rotation and draw a picture of the rotated volume.

Axis	Disks	Washers	Shells
	$\int$ area width	$\int$ area width	$\int$ circumference height width

Axis of revolution: HORIZONTAL

x-axis	$\int_a^b \pi\left[f(x)\right]^2\ dx$	$\int_a^b \pi\left[f(x)\right]^2 - \pi\left[g(x)\right]^2\ dx$	$\int_c^d 2\pi y\left[f(y) - g(y)\right]\ dy$
$y = -k$		$\int_a^b \pi\left[k + f(x)\right]^2 - \pi\left[k + g(x)\right]^2\ dx$	$\int_c^d 2\pi(y + k)\left[f(y) - g(y)\right]\ dy$
$y = k$		$\int_a^b \pi\left[k - g(x)\right]^2 - \pi\left[k - f(x)\right]^2\ dx$	$\int_c^d 2\pi(k - y)\left[f(y) - g(y)\right]\ dy$

Axis of revolution: VERTICAL

y-axis	$\int_c^d \pi\left[f(y)\right]^2\ dy$	$\int_c^d \pi\left[f(y)\right]^2 - \pi\left[g(y)\right]^2\ dy$	$\int_a^b 2\pi x\left[f(x) - g(x)\right]\ dx$
$x = -k$		$\int_c^d \pi\left[k + f(y)\right]^2 - \pi\left[k + g(y)\right]^2\ dy$	$\int_a^b 2\pi(x + k)\left[f(x) - g(x)\right]\ dx$
$x = k$		$\int_c^d \pi\left[k - g(y)\right]^2 - \pi\left[k - f(y)\right]^2\ dy$	$\int_a^b 2\pi(k - x)\left[f(x) - g(x)\right]\ dx$

Example

Find the volume of the solid created by rotating the function about the x-axis over the interval $[1,3]$.

$$y = 2x^3$$

Since the function is in the form $y = f(x)$, we'll use the formula

$$V = \int_a^b \pi \left[f(x)\right]^2 \ dx$$

and plug in what we've been given.

$$V = \int_1^3 \pi \left(2x^3\right)^2 dx$$

$$V = \int_1^3 4\pi x^6 \ dx$$

$$V = 4\pi \int_1^3 x^6 \ dx$$

We'll integrate,

$$V = 4\pi \left(\frac{x^7}{7}\right)\Bigg|_1^3$$

$$V = \frac{4\pi x^7}{7}\Bigg|_1^3$$

and then evaluate over the interval.

$$V = \frac{4\pi(3)^7}{7} - \frac{4\pi(1)^7}{7}$$

$$V = \frac{8{,}744\pi}{7}$$

$$V \approx 3{,}924.30$$

The volume of the solid object created by rotating $y = 2x^3$ about the x-axis over the interval $[1,3]$ is $V \approx 3{,}924.30$.

Let's do another example, this time where we rotate around the y-axis.

Example

Find the volume of the solid created by rotating the function about the y-axis over the interval $y = 3$ to $y = 5$.

$$x = 3y^{-2}$$

Since the function is in the form $x = f(y)$, we'll use the formula

$$V = \int_c^d \pi \left[f(y)\right]^2 \ dy$$

and plug in what we've been given.

$$V = \int_3^5 \pi \left(3y^{-2}\right)^2 \ dy$$

$$V = \int_3^5 9\pi y^{-4} \ dy$$

$$V = 9\pi \int_3^5 y^{-4}\ dy$$

We'll integrate,

$$V = 9\pi \left(\frac{y^{-3}}{-3} \right) \Bigg|_3^5$$

$$V = -3\pi y^{-3} \Big|_3^5$$

and then evaluate over the interval.

$$V = -3\pi(5)^{-3} - \left[-3\pi(3)^{-3} \right]$$

$$V = -\frac{3\pi}{5^3} + \frac{3\pi}{3^3}$$

$$V = \frac{\pi}{3^2} - \frac{3\pi}{5^3}$$

$$V = \frac{\pi}{9} - \frac{3\pi}{125}$$

$$V = \frac{125\pi}{1,125} - \frac{27\pi}{1,125}$$

$$V = \frac{98\pi}{1,125}$$

$$V \approx 0.27$$

The volume of the solid object created by rotating $x = 3y^{-2}$ about the y-axis over the interval $[3,5]$ is $V \approx 0.27$.