Topic: tan^m sec^n, odd m

Question: Evaluate the trigonometric integral.

$$\int \tan^5 x \sec x \ dx$$

Answer choices:

A $\quad \frac{1}{5}\tan^5 x - \frac{2}{3}\tan^3 x + \tan x + C$

B $\quad \frac{1}{5}\tan^5 x + \frac{2}{3}\tan^3 x + \tan x + C$

C $\quad \frac{1}{5}\sec^5 x - \frac{2}{3}\sec^3 x + \sec x + C$

D $\quad \frac{1}{5}\sec^5 x + \frac{2}{3}\sec^3 x + \sec x + C$

Solution: C

In the specific case where our function is the product of

an **odd** number of **tangent** factors and

an **even or odd** number of **secant** factors,

our plan is to

1. save one $\sec x \tan x$ factor and use the identity $\tan^2 x = \sec^2 x - 1$ to write the other cosine factors in terms of secant, then

2. use u-substitution with $u = \sec x$.

We'll separate a single tangent to make $\sec x \tan x$, and then replace the remaining tangent factors using the identity.

$$\int \tan^5 x \sec x \ dx$$

$$\int \tan^4 x \tan x \sec x \ dx$$

$$\int \left(\tan^2 x\right)^2 \tan x \sec x \ dx$$

$$\int \left(\sec^2 x - 1\right)^2 \tan x \sec x \ dx$$

Using u-substitution with $u = \sec x$, we get

$$u = \sec x$$

$$du = \sec x \tan x \ dx$$

Substitute into the integral.

$$\int \left(u^2 - 1\right)^2 \ du$$

$$\int u^4 - 2u^2 + 1 \ du$$

$$\frac{1}{5}u^5 - \frac{2}{3}u^3 + u + C$$

Back-substituting for u, we get

$$\frac{1}{5}\sec^5 x - \frac{2}{3}\sec^3 x + \sec x + C$$

Topic: tan^m sec^n, odd m

Question: Evaluate the trigonometric integral.

$$\int \tan^3 x \sec x \ dx$$

Answer choices:

A $\quad \frac{1}{3}\tan^3 x + \tan x + C$

B $\quad \frac{1}{3}\tan^3 x - \tan x + C$

C $\quad \frac{1}{3}\sec^3 x + \sec x + C$

D $\quad \frac{1}{3}\sec^3 x - \sec x + C$

Solution: D

In the specific case where our function is the product of

an **odd** number of **tangent** factors and

an **even or odd** number of **secant** factors,

our plan is to

1. save one $\sec x \tan x$ factor and use the identity $\tan^2 x = \sec^2 x - 1$ to write the other cosine factors in terms of secant, then

2. use u-substitution with $u = \sec x$.

We'll separate a single tangent to make $\sec x \tan x$, and then replace the remaining tangent factors using the identity.

$$\int \tan^3 x \sec x \ dx$$

$$\int \tan^2 x \tan x \sec x \ dx$$

$$\int \left(\sec^2 x - 1\right) \tan x \sec x \ dx$$

Using u-substitution with $u = \sec x$, we get

$$u = \sec x$$

$$du = \sec x \tan x \ dx$$

Substitute into the integral.

$$\int u^2 - 1 \ du$$

$$\frac{1}{3}u^3 - u + C$$

Back-substituting for u, we get

$$\frac{1}{3}\sec^3 x - \sec x + C$$

Topic: tan^m sec^n, odd m

Question: Evaluate the trigonometric integral.

$$\int_{\frac{2\pi}{3}}^{\pi} \tan^5 x \sec x \ dx$$

Answer choices:

A $\frac{15}{38}$

B $\frac{38}{15}$

C $\frac{5}{18}$

D $\frac{18}{5}$

Solution: B

In the specific case where our function is the product of

an **odd** number of **tangent** factors and

an **even or odd** number of **secant** factors,

our plan is to

1. save one $\sec x \tan x$ factor and use the identity $\tan^2 x = \sec^2 x - 1$ to write the other cosine factors in terms of secant, then

2. use u-substitution with $u = \sec x$.

We'll separate a single tangent to make $\sec x \tan x$, and then replace the remaining tangent factors using the identity.

$$\int_{\frac{2\pi}{3}}^{\pi} \tan^5 x \sec x \ dx$$

$$\int_{\frac{2\pi}{3}}^{\pi} \tan^4 x \tan x \sec x \ dx$$

$$\int_{\frac{2\pi}{3}}^{\pi} \left(\tan^2 x\right)^2 \tan x \sec x \ dx$$

$$\int_{\frac{2\pi}{3}}^{\pi} \left(\sec^2 x - 1\right)^2 \tan x \sec x \ dx$$

Using u-substitution with $u = \sec x$, we get

$$u = \sec x$$

$$du = \sec x \tan x \ dx$$

Because we're dealing with a definite integral, we have to either change the limits of integration when we make our substitution, or we have to indicate that the limits of integration are in terms of x until we back-substitute. Substitute into the integral.

$$\int_{x=\frac{2\pi}{3}}^{x=\pi} \left(u^2 - 1\right)^2 \ du$$

$$\int_{x=\frac{2\pi}{3}}^{x=\pi} u^4 - 2u^2 + 1 \ du$$

$$\frac{1}{5}u^5 - \frac{2}{3}u^3 + u \Big|_{x=\frac{2\pi}{3}}^{x=\pi}$$

Back-substituting for u, we get

$$\frac{1}{5}\sec^5 x - \frac{2}{3}\sec^3 x + \sec x \Big|_{\frac{2\pi}{3}}^{\pi}$$

$$\frac{1}{5}\sec^5 \pi - \frac{2}{3}\sec^3 \pi + \sec \pi - \left(\frac{1}{5}\sec^5 \frac{2\pi}{3} - \frac{2}{3}\sec^3 \frac{2\pi}{3} + \sec \frac{2\pi}{3}\right)$$

$$\frac{1}{5\cos^5 \pi} - \frac{2}{3\cos^3 \pi} + \frac{1}{\cos \pi} - \left(\frac{1}{5\cos^5 \frac{2\pi}{3}} - \frac{2}{3\cos^3 \frac{2\pi}{3}} + \frac{1}{\cos \frac{2\pi}{3}}\right)$$

$$\frac{1}{5(-1)^5}-\frac{2}{3(-1)^3}+\frac{1}{(-1)}-\left[\frac{1}{5\left(-\frac{1}{2}\right)^5}-\frac{2}{3\left(-\frac{1}{2}\right)^3}+\frac{1}{\left(-\frac{1}{2}\right)}\right]$$

$$-\frac{1}{5}+\frac{2}{3}-1-\left(-\frac{1}{\frac{5}{32}}+\frac{2}{\frac{3}{8}}-2\right)$$

$$-\frac{1}{5}+\frac{2}{3}-1+\frac{32}{5}-\frac{16}{3}+2$$

$$\frac{31}{5}-\frac{14}{3}+1$$

$$\frac{93}{15}-\frac{70}{15}+\frac{15}{15}$$

$$\frac{38}{15}$$