Topic: Area between upper and lower curves

Question: Find the area between the curves.

$$y = \sec^2 x$$

$$y = 1$$

and

$$x = 0$$

$$x = \frac{\pi}{4}$$

Answer choices:

A $\frac{\pi}{4} - 1$

B $1 - \frac{\pi}{4}$

C 1

D $\sqrt{2} - \frac{\pi}{4}$

Solution: B

In order to calculate the area between two curves, we need to follow these steps:

1. Decide whether the curves are
    a. upper and lower curves, or
    b. left and right curves.
2. Find points of intersection.
3. Determine which curve has the larger value between each point of intersection.
4. Plug everything into the appropriate formula.

Since the curves we're given are both expressed for y in terms of x, it means these are upper and lower curves. If we're using these curves, then the lines $x = 0$ and $x = \pi/4$ define our interval. Remember though, just because we're given the interval doesn't mean we can skip step 2. We still need to make sure that there are no points of intersection inside the given interval.

To check for points of intersection, we'll set the curves equal to each other.

$$\sec^2 x = 1$$

$$\frac{1}{\cos^2 x} = 1$$

$$1 = \cos^2 x$$

$$\sqrt{1} = \sqrt{\cos^2 x}$$

$$1 = \cos x$$

$$x = 0$$

Since this point of intersection is the same as the left endpoint of our interval, and there are no other points of intersection inside the interval, we know that one curve is always above the other curve throughout the entire interval. Our next step is to determine which curve has a larger y-value on the x-interval $[0,\pi/4]$.

We can do this by picking an x-value within the interval and plugging it into both functions. Whichever curve returns a larger value we'll call $f(x)$, and whichever curve returns a lower value we'll call $g(x)$.

Plugging $x = \pi/6$ into both functions, we get

$$y = 1$$

and

$$y = \sec^2 x$$

$$y = \sec^2 \frac{\pi}{6}$$

$$y = \frac{1}{\cos^2 \frac{\pi}{6}}$$

$$y = \frac{1}{\left(\cos \frac{\pi}{6}\right)^2}$$

$$y = \frac{1}{\left(\frac{\sqrt{3}}{2}\right)^2}$$

$$y = \frac{1}{\frac{3}{4}}$$

$$y = \frac{4}{3}$$

Since $y = \sec^2 x$ gives a larger value, we'll say

$$f(x) = \sec^2 x$$

and

$$g(x) = 1$$

Now we can plug these functions and the interval we found earlier into the formula for area between upper and lower curves.

$$\int_0^{\frac{\pi}{4}} \sec^2 x - 1 \ dx$$

$$\tan x - x \Big|_0^{\frac{\pi}{4}}$$

$$\tan\frac{\pi}{4} - \frac{\pi}{4} - (\tan 0 - 0)$$

$$\frac{\sin\frac{\pi}{4}}{\cos\frac{\pi}{4}} - \frac{\pi}{4} - \frac{\sin 0}{\cos 0}$$

$$\frac{\frac{\sqrt{2}}{2}}{\frac{\sqrt{2}}{2}} - \frac{\pi}{4} - \frac{0}{1}$$

$$1 - \frac{\pi}{4}$$

Topic: Area between upper and lower curves

Question: Find the area between the curves.

$$y = x^3 + 2x^2 + 1$$

$$y = x + 3$$

Answer choices:

A $\frac{37}{12}$

B $-\frac{37}{12}$

C $-\frac{9}{4}$

D $\frac{9}{4}$

Solution: A

In order to calculate the area between two curves, we need to follow these steps:

1. Decide whether the curves are

    a. upper and lower curves, or

    b. left and right curves.

2. Find points of intersection.

3. Determine which curve has the larger value between each point of intersection.

4. Plug everything into the appropriate formula.

Since the curves we're given are both expressed for y in terms of x, it means these are upper and lower curves.

To find points of intersection, we'll set the curves equal to each other.

$$x^3 + 2x^2 + 1 = x + 3$$

$$x^3 + 2x^2 - x - 2 = 0$$

$$x^2(x + 2) - x - 2 = 0$$

$$x^2(x + 2) - (x + 2) = 0$$

$$(x + 2)\left(x^2 - 1\right) = 0$$

$$(x + 2)(x + 1)(x - 1) = 0$$

$$x = -2, \ -1, \ 1$$

These left-most and right-most points define the endpoints of our interval in terms of x. Because there's a third point of intersection inside the interval $[-2,1]$, we know that this is a point where the curves cross each other, which means our next step is to determine which curve has a larger y-value on the x-interval $[-2,-1]$. We know that they'll have the opposite orientation on the x-interval $[-1,1]$.

We can do this by picking an x-value within the interval and plugging it into both functions. Whichever curve returns a larger value we'll call $f(x)$, and whichever curve returns a lower value we'll call $g(x)$.

Plugging $x = -3/2$ into both functions, we get

$$y = x^3 + 2x^2 + 1$$

$$y = \left(-\frac{3}{2}\right)^3 + 2\left(-\frac{3}{2}\right)^2 + 1$$

$$y = -\frac{27}{8} + \frac{18}{4} + 1$$

$$y = -\frac{27}{8} + \frac{36}{8} + \frac{8}{8}$$

$$y = \frac{17}{8}$$

and

$$y = x + 3$$

$$y=-\frac{3}{2}+3$$

$$y=-\frac{3}{2}+\frac{6}{2}$$

$$y=\frac{3}{2}$$

Since $y=x^3+2x^2+1$ gives a larger value, we'll say

$$f(x)=x^3+2x^2+1$$

and

$$g(x)=x+3$$

Now we can plug these functions and the interval we found earlier into the formula for area between upper and lower curves. Remember, since the curves cross each other at $x=-1$, we have to use two separate integrals.

$$\int_a^b f(x)-g(x)\ dx+\int_b^c g(x)-f(x)\ dx$$

$$\int_{-2}^{-1}\left(x^3+2x^2+1\right)-(x+3)\ dx+\int_{-1}^{1}(x+3)-\left(x^3+2x^2+1\right)\ dx$$

$$\int_{-2}^{-1}x^3+2x^2-x-2\ dx+\int_{-1}^{1}-x^3-2x^2+x+2\ dx$$

$$\left(\frac{1}{4}x^4+\frac{2}{3}x^3-\frac{1}{2}x^2-2x\right)\Bigg|_{-2}^{-1}+\left(-\frac{1}{4}x^4-\frac{2}{3}x^3+\frac{1}{2}x^2+2x\right)\Bigg|_{-1}^{1}$$

$$\left(\frac{x^4}{4}+\frac{2x^3}{3}-\frac{x^2}{2}-2x\right)\Bigg|_{-2}^{-1}+\left(-\frac{x^4}{4}-\frac{2x^3}{3}+\frac{x^2}{2}+2x\right)\Bigg|_{-1}^{1}$$

$$\frac{(-1)^4}{4}+\frac{2(-1)^3}{3}-\frac{(-1)^2}{2}-2(-1)-\left[\frac{(-2)^4}{4}+\frac{2(-2)^3}{3}-\frac{(-2)^2}{2}-2(-2)\right]$$

$$+\left[-\frac{(1)^4}{4}-\frac{2(1)^3}{3}+\frac{(1)^2}{2}+2(1)\right]-\left[-\frac{(-1)^4}{4}-\frac{2(-1)^3}{3}+\frac{(-1)^2}{2}+2(-1)\right]$$

$$\frac{1}{4}-\frac{2}{3}-\frac{1}{2}+2-\left(\frac{16}{4}-\frac{16}{3}-\frac{4}{2}+4\right)+\left(-\frac{1}{4}-\frac{2}{3}+\frac{1}{2}+2\right)-\left(-\frac{1}{4}+\frac{2}{3}+\frac{1}{2}-2\right)$$

$$\frac{1}{4}-\frac{2}{3}-\frac{1}{2}+2-\frac{16}{4}+\frac{16}{3}+\frac{4}{2}-4-\frac{1}{4}-\frac{2}{3}+\frac{1}{2}+2+\frac{1}{4}-\frac{2}{3}-\frac{1}{2}+2$$

$$\frac{1}{4}-\frac{16}{4}-\frac{1}{4}+\frac{1}{4}-\frac{2}{3}+\frac{16}{3}-\frac{2}{3}-\frac{2}{3}-\frac{1}{2}+\frac{4}{2}+\frac{1}{2}-\frac{1}{2}+2-4+2+2$$

$$-\frac{15}{4}+\frac{10}{3}+\frac{3}{2}+2$$

$$-\frac{45}{12}+\frac{40}{12}+\frac{18}{12}+\frac{24}{12}$$

$$\frac{37}{12}$$

Topic: Area between upper and lower curves

Question: Determine the area of the region enclosed by the curves.

$$f(x) = 1 + 4x - x^2$$

$$g(x) = 6 - 2x$$

Answer choices:

A $\frac{33}{2}$

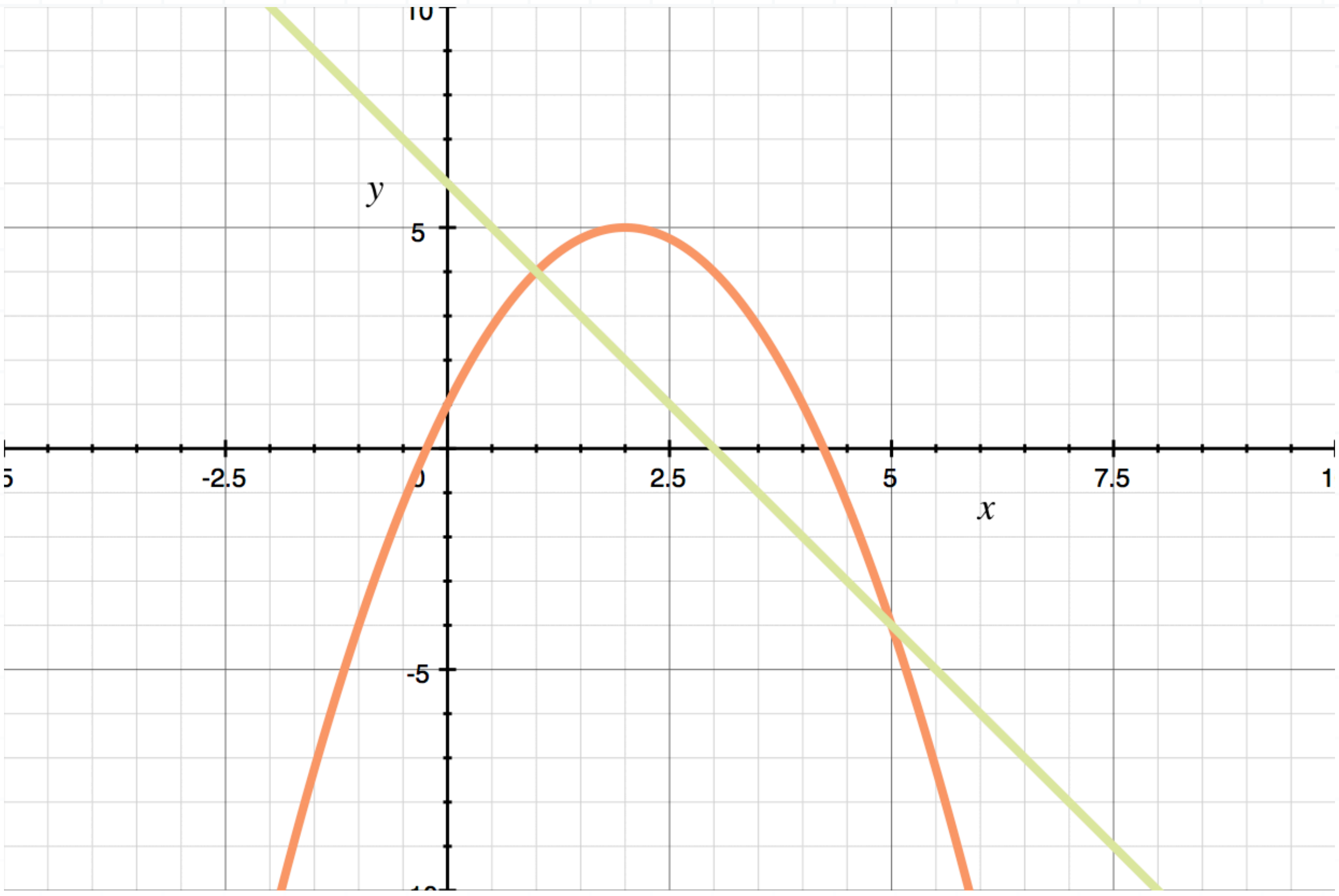
B $\frac{32}{3}$

C $-\frac{32}{3}$

D $-\frac{33}{2}$

Solution: B

The graph of the two functions is



The graph shows that the quadratic function $f(x) = 1 + 4x - x^2$ is higher than the linear function $g(x) = 6 - 2x$. The graph also shows that the two graphs intersect at the points $(1, 4)$ and $(5, -4)$. The x-values in these points give the integration limits. The integral to find the area between the two curves is

$$A = \int_1^5 \left(1 + 4x - x^2\right) - \left(6 - 2x\right)\ dx$$

$$A = \int_1^5 1 + 4x - x^2 - 6 + 2x\ dx$$

$$A = \int_1^5 -x^2 + 6x - 5\ dx$$

Integrate, then evaluate over the interval.

$$A = -\frac{1}{3}x^3 + 3x^2 - 5x \Big|_1^5$$

$$A = -\frac{1}{3}(5)^3 + 3(5)^2 - 5(5) - \left(-\frac{1}{3}(1)^3 + 3(1)^2 - 5(1)\right)$$

$$A = -\frac{125}{3} + 75 - 25 - \left(-\frac{1}{3} + 3 - 5\right)$$

$$A = -\frac{125}{3} + 75 - 25 + \frac{1}{3} - 3 + 5$$

$$A = -\frac{124}{3} + 52$$

$$A = \frac{32}{3}$$