Topic: Riemann sums, left endpoints

Question: Use a lower sum (inscribed rectangles) to find the area under the curve.

$$y = \sqrt{x} + 1$$

on the interval $[0,3]$

with $n = 6$

Answer choices:

A 3.75

B 6.48

C 6.83

D 5.96

Solution: D

The Riemann sum is a tool we can use to approximate the area under a function over a set interval $a \le x \le b$.

We'll divide the area into rectangles and then sum the areas of all of the rectangles in order to get an approximation of area. The greater the number of rectangles, the more accurate the approximation will be. Of course, if we use an infinite number of rectangles, taking the limit as $n \to \infty$ of the sum of the area of each rectangle, then we'd be taking the integral and calculating exact area.

When we approximate area with Riemann sums we consider the area above the x-axis to be positive, and the area below the x-axis to be negative. If our final result is positive, it tells us that there's more area above the x-axis than below it. On the other hand, if our final result is negative, it means that there's more area below the x-axis than above it.

The Riemann sum formula is

$$R_n = \sum_{i=1}^{n} f(x_i)\Delta x$$

where $\Delta x = (b - a)/n$ and Δx is the width of each rectangle, and where n is the number of rectangles we're using to approximate area. If we expand the Riemann sum, we get the formula

$$R_n = \Delta x \left[f(x_1) + f(x_2) + \dots + f(x_n)\right]$$

Our plan is to solve for Δx, divide the interval into even segments that are each Δx wide, and then use an endpoint of each segment as the values of

x_n. When we're using a Riemann sum to approximate area, we can choose the left endpoints, right endpoints, or midpoints of our rectangles.

Plugging the interval and the value of n we've been given into the formula for Δx, we get

$$\Delta x = \frac{b-a}{n}$$

$$\Delta x = \frac{3-0}{6}$$

$$\Delta x = \frac{1}{2}$$

Next, we need to figure out whether to use left endpoints, right endpoints, or a combination in order to get the lower sum, so we take the derivative of the function to see where it's increasing and decreasing.

$$y' = \frac{1}{2}x^{-\frac{1}{2}}$$

Since this derivative function is always positive, the original function is increasing on the interval $[0,3]$, and so we use the left endpoints to find the lower sum.

$$c_i = a + (i-1)\Delta x$$

$$c_1 = 0 \qquad c_2 = \frac{1}{2} \qquad c_3 = 1 \qquad c_4 = \frac{3}{2} \qquad c_5 = 2 \qquad c_6 = \frac{5}{2}$$

Finally, we plug everything into the lower sum formula.

$$s(n) = \sum_{i=1}^{n} f(c_i)\Delta x$$

$$s(6) = \sum_{i=1}^{6} f(c_i)\Delta x$$

$$s(6) = \left(\sqrt{0}+1\right)\left(\frac{1}{2}\right) + \left(\sqrt{\frac{1}{2}}+1\right)\left(\frac{1}{2}\right) + \left(\sqrt{1}+1\right)\left(\frac{1}{2}\right)$$

$$+\left(\sqrt{\frac{3}{2}}+1\right)\left(\frac{1}{2}\right) + \left(\sqrt{2}+1\right)\left(\frac{1}{2}\right) + \left(\sqrt{\frac{5}{2}}+1\right)\left(\frac{1}{2}\right)$$

$$s(6) \approx 5.96$$

Topic: Riemann sums, left endpoints

Question: Approximate the area under the curve using a left rectangular approximation method, and five equal subintervals.

$$f(x) = \frac{x^2 + 6x + 5}{x + 4}$$

on the interval $[0,5]$

Answer choices:

A $\quad \dfrac{20{,}171}{840}$

B $\quad \dfrac{4{,}857}{280}$

C $\quad \dfrac{4{,}507}{280}$

D $\quad \dfrac{19{,}121}{840}$

Solution: B

The term rectangular approximation method means we will approximate the area under the curve using rectangles. We calculate the area of each rectangle by multiplying the height of the rectangle (the function value) times the width of the rectangle (the length of the subinterval).

Because we are using a left rectangular approximation method, we will find the height of the rectangle by calculating the function value at the left endpoint of each subinterval.

The five equal subintervals in the interval $[0,5]$ are $[0,1]$, $[1,2]$, $[2,3]$, $[3,4]$, and $[4,5]$. Each subinterval is 1 unit wide. We will calculate the function values at 0, 1, 2, 3, and 4.

$$f(0) = \frac{(0)^2 + 6(0) + 5}{0 + 4} = \frac{5}{4}$$

$$f(1) = \frac{(1)^2 + 6(1) + 5}{1 + 4} = \frac{1 + 6 + 5}{5} = \frac{12}{5}$$

$$f(2) = \frac{(2)^2 + 6(2) + 5}{2 + 4} = \frac{4 + 12 + 5}{6} = \frac{21}{6} = \frac{7}{2}$$

$$f(3) = \frac{(3)^2 + 6(3) + 5}{3 + 4} = \frac{9 + 18 + 5}{7} = \frac{32}{7}$$

$$f(4) = \frac{(4)^2 + 6(4) + 5}{4 + 4} = \frac{16 + 24 + 5}{8} = \frac{45}{8}$$

Now that we know the height of each rectangle at the left endpoints of the subintervals, we will add the areas together to get the final approximation.

Since the widths of the subintervals are all one unit, we do not have to multiply the heights by the width in this question.

$$\frac{5}{4}+\frac{12}{5}+\frac{7}{2}+\frac{32}{7}+\frac{45}{8}$$

$$\frac{350}{280}+\frac{672}{280}+\frac{980}{280}+\frac{1{,}280}{280}+\frac{1{,}575}{280}$$

$$\frac{4{,}857}{280}$$

Topic: Riemann sums, left endpoints

Question: Approximate the area under the curve using a left rectangular approximation method, and three equal subintervals.

$$g(x) = -\frac{1}{2}x^3 + 5x^2 - 3x - 8$$

on the interval $[2,8]$

Answer choices:

A $\frac{349}{2}$

B 76

C 106

D 152

Solution: D

The term rectangular approximation method means we will approximate the area under the curve using rectangles. We calculate the area of each rectangle by multiplying the height of the rectangle (the function value) times the width of the rectangle (the length of the subinterval).

Because we are using a left rectangular approximation method, we will find the height of the rectangle by calculating the function value at the left endpoint of each subinterval.

The three equal subintervals in the interval $[2,8]$ are $[2,4]$, $[4,6]$, and $[6,8]$. Each subinterval is 2 units wide. We will calculate the function values at 2, 4, and 6, and then multiply each value by 2 to find the area of the rectangles.

$$g(2) = -\frac{1}{2}(2)^3 + 5(2)^2 - 3(2) - 8 = -4 + 20 - 6 - 8 = 2$$

Area: $2 \times 2 = 4$

$$g(4) = -\frac{1}{2}(4)^3 + 5(4)^2 - 3(4) - 8 = -32 + 80 - 12 - 8 = 28$$

Area: $2 \times 28 = 56$

$$g(6) = -\frac{1}{2}(6)^3 + 5(6)^2 - 3(6) - 8 = -108 + 180 - 18 - 8 = 46$$

Area: $2 \times 46 = 92$

Now we know the area of each rectangle at the left endpoints of the subintervals, we will add the areas together to get the final approximation.

$$4 + 56 + 92 = 152$$