Topic: Integration by parts with u-substitution

Question: Use u-substitution and then integration by parts to evaluate the integral.

$$\int_{\sqrt[3]{\frac{\pi}{2}}}^{\sqrt[3]{\pi}} 3\theta^5 \sin\left(\theta^3\right)\ d\theta$$

Answer choices:

A $\pi + 1$

B $\pi - 1$

C $\sqrt[3]{\pi} - 1$

D $\sqrt[3]{\pi} + 1$

Solution: B

First, we'll use u-substitution to simplify the integral. The integrand includes a trigonometric function of another function. Typically, the function that represents the angle will be removed/changed using the u-substitution process. Since we're going to use integration by parts, we'll use x as the new variable after the substitution. Therefore, we'll make the following substitutions:

$$x = \theta^3$$

$$dx = 3\theta^2\ d\theta$$

$$d\theta = \frac{dx}{3\theta^2}$$

Since the integration limits are in terms of θ, we'll also change the integration limits to match our substitution. If we don't do this now, we'll have to do it later.

Lower limit:

$$\theta = \sqrt[3]{\frac{\pi}{2}}$$

$$\theta^3 = \left(\sqrt[3]{\frac{\pi}{2}}\right)^3$$

$$x = \left(\sqrt[3]{\frac{\pi}{2}}\right)^3$$

$$x = \frac{\pi}{2}$$

Upper limit:

$$\theta = \sqrt[3]{\pi}$$

$$\theta^3 = \left(\sqrt[3]{\pi}\right)^3$$

$$x = \left(\sqrt[3]{\pi}\right)^3$$

$$x = \pi$$

Now, let's rewrite the integral in terms of x instead of θ.

$$\int_{\sqrt[3]{\frac{\pi}{2}}}^{\sqrt[3]{\pi}} 3\theta^5 \sin\left(\theta^3\right)\ d\theta$$

$$\int_{\frac{\pi}{2}}^{\pi} 3\theta^5 \sin x \frac{dx}{3\theta^2}$$

$$\int_{\frac{\pi}{2}}^{\pi} \theta^3 \sin x\ dx$$

$$\int_{\frac{\pi}{2}}^{\pi} x \sin x\ dx$$

We are now prepared to integrate by parts. Integration by parts is a method of evaluating an integral that cannot be evaluated using normal integration techniques, by using integration by substitution, or by using integration formulas. The general formula for integration by parts is

$$\int u \; dv = uv - \int v \; du$$

In this formula, we separate the integrand into two parts; one part is called u and the other part is called dv. In making these two parts, we must use all of the integrand.

Although there is sometimes flexibility in choosing u, we can generally use the following sequence of choices to select the best part of the integrand to be u. This method involves the acronym LIPET, where we select the first u in the sequence of the list below. The letters mean

L	Logarithmic expression
I	Inverse trigonometric expression
P	Polynomial expression
E	Exponential expression
T	Trigonometric function expression

In this problem, the integrand is $x \sin x$ where we have a polynomial and a trigonometric function. In the LIPET sequence, polynomial comes before trigonometric function, so u is the polynomial. Let's identify the parts we need to integrate.

$$u = x$$

$$du = dx$$

$$dv = \sin x \; dx$$

$$v = -\cos x$$

We're now ready to integrate by parts using the general formula.

$$\int u\ dv = uv - \int v\ du$$

$$\int x \sin x\ dx = -x \cos x - \int -\cos x\ dx$$

$$\int x \sin x\ dx = -x \cos x + \int \cos x\ dx$$

$$\int x \sin x\ dx = -x \cos x + \sin x$$

We didn't add a C to accommodate the possibility of a constant in the function because this problem involves a definite integral. Now we'll evaluate the result of the integration using the integration limits.

$$\int_{\frac{\pi}{2}}^{\pi} x \sin x\ dx = \left(-x \cos x + \sin x\right)\Big|_{\frac{\pi}{2}}^{\pi}$$

$$\int_{\frac{\pi}{2}}^{\pi} x \sin x\ dx = \left(-\pi \cos \pi + \sin \pi\right) - \left(-\frac{\pi}{2} \cos \frac{\pi}{2} + \sin \frac{\pi}{2}\right)$$

$$\int_{\frac{\pi}{2}}^{\pi} x \sin x\ dx = \left[-\pi(-1) + (0)\right] - \left[-\frac{\pi}{2}(0) + (1)\right]$$

$$\int_{\frac{\pi}{2}}^{\pi} x \sin x\ dx = \pi - 1$$

Therefore, $\pi - 1$ is the value of the integral.

Topic: Integration by parts with u-substitution

Question: Use u-substitution and then integration by parts to evaluate the integral.

$$\int_{\sqrt{\frac{3\pi}{2}}}^{\sqrt{\frac{5\pi}{2}}} 4\theta^3 \cos\left(\theta^2\right)\ d\theta$$

Answer choices:

A $8\pi - 1$

B $8\pi + 1$

C $8\pi + 2$

D 8π

Solution: D

First, we'll use u-substitution to simplify the integral. The integrand includes a trigonometric function of another function. Typically, the function that represents the angle will be removed/changed using the u-substitution process. Since we're going to use integration by parts, we'll use x as the new variable after the substitution. Therefore, we'll make the following substitutions:

$$x = \theta^2$$

$$dx = 2\theta\ d\theta$$

$$d\theta = \frac{dx}{2\theta}$$

Since the integration limits are in terms of θ, we'll also change the integration limits to match our substitution. If we don't do this now, we'll have to do it later.

Lower limit:

$$\theta = \sqrt{\frac{3\pi}{2}}$$

$$\theta^2 = \left(\sqrt{\frac{3\pi}{2}}\right)^2$$

$$x = \left(\sqrt{\frac{3\pi}{2}}\right)^2$$

$$x = \frac{3\pi}{2}$$

Upper limit:

$$\theta = \sqrt{\frac{5\pi}{2}}$$

$$\theta^2 = \left(\sqrt{\frac{5\pi}{2}}\right)^2$$

$$x = \left(\sqrt{\frac{5\pi}{2}}\right)^2$$

$$x = \frac{5\pi}{2}$$

Now, let's rewrite the integral in terms of x instead of θ.

$$\int_{\sqrt{\frac{3\pi}{2}}}^{\sqrt{\frac{5\pi}{2}}} 4\theta^3 \cos\left(\theta^2\right)\ d\theta$$

$$\int_{\frac{3\pi}{2}}^{\frac{5\pi}{2}} 4\theta^3 \cos x \frac{dx}{2\theta}$$

$$\int_{\frac{3\pi}{2}}^{\frac{5\pi}{2}} 2\theta^2 \cos x\ dx$$

$$\int_{\frac{3\pi}{2}}^{\frac{5\pi}{2}} 2x \cos x\ dx$$

$$2\int_{\frac{3\pi}{2}}^{\frac{5\pi}{2}} x\cos x\ dx$$

We are now prepared to integrate by parts. Integration by parts is a method of evaluating an integral that cannot be evaluated using normal integration techniques, by using integration by substitution, or by using integration formulas. The general formula for integration by parts is

$$\int u\ dv = uv - \int v\ du$$

In this formula, we separate the integrand into two parts; one part is called u and the other part is called dv. In making these two parts, we must use all of the integrand.

Although there is sometimes flexibility in choosing u, we can generally use the following sequence of choices to select the best part of the integrand to be u. This method involves the acronym LIPET, where we select the first u in the sequence of the list below. The letters mean

L	Logarithmic expression
I	Inverse trigonometric expression
P	Polynomial expression
E	Exponential expression
T	Trigonometric function expression

In this problem, the integrand is $x\cos x$ where we have a polynomial and a trigonometric function. In the LIPET sequence, polynomial comes before trigonometric function, so u is the polynomial. Let's identify the parts we need to integrate.

$$u = x$$

$$du = dx$$

$$dv = \cos x \ dx$$

$$v = \sin x$$

We're now ready to integrate by parts using the general formula.

$$\int u \ dv = uv - \int v \ du$$

$$2\int x \cos x \ dx = 2\left[x \sin x - \int \sin x \ dx\right]$$

$$2\int x \cos x \ dx = 2\left[x \sin x - (-\cos x)\right]$$

$$2\int x \cos x \ dx = 2x \sin x + 2\cos x$$

We didn't add a C to accommodate the possibility of a constant in the function because this problem involves a definite integral. Now we'll evaluate the result of the integration using the integration limits.

$$2\int_{\frac{3\pi}{2}}^{\frac{5\pi}{2}} x \cos x \ dx = \left(2x \sin x + 2\cos x\right)\Big|_{\frac{3\pi}{2}}^{\frac{5\pi}{2}}$$

$$2\int_{\frac{3\pi}{2}}^{\frac{5\pi}{2}} x\cos x\ dx = \left[2\left(\frac{5\pi}{2}\right)\sin\frac{5\pi}{2} + 2\cos\frac{5\pi}{2}\right] - \left[2\left(\frac{3\pi}{2}\right)\sin\frac{3\pi}{2} + 2\cos\frac{3\pi}{2}\right]$$

$$2\int_{\frac{3\pi}{2}}^{\frac{5\pi}{2}} x\cos x\ dx = \left[5\pi\sin\frac{5\pi}{2} + 2\cos\frac{5\pi}{2}\right] - \left[3\pi\sin\frac{3\pi}{2} + 2\cos\frac{3\pi}{2}\right]$$

$$2\int_{\frac{3\pi}{2}}^{\frac{5\pi}{2}} x\cos x\ dx = [5\pi(1) + 2(0)] - [3\pi(-1) + 2(0)]$$

$$2\int_{\frac{3\pi}{2}}^{\frac{5\pi}{2}} x\cos x\ dx = 5\pi + 3\pi$$

$$2\int_{\frac{3\pi}{2}}^{\frac{5\pi}{2}} x\cos x\ dx = 8\pi$$

Therefore, 8π is the value of the integral.

Topic: Integration by parts with u-substitution

Question: Use u-substitution and then integration by parts to evaluate the integral.

$$\int_0^{\sqrt[4]{\pi}} 6\theta^7 \sin\left(\theta^4\right)\ d\theta$$

Answer choices:

A $\quad \dfrac{5\pi}{2}$

B $\quad \dfrac{\pi}{2}$

C $\quad \dfrac{3}{2}\pi$

D $\quad \dfrac{7}{2}(\pi - 1)$

Solution: C

First, we'll use u-substitution to simplify the integral. The integrand includes a trigonometric function of another function. Typically, the function that represents the angle will be removed/changed using the u-substitution process. Since we're going to use integration by parts, we'll use x as the new variable after the substitution. Therefore, we'll make the following substitutions:

$$x = \theta^4$$

$$dx = 4\theta^3\ d\theta$$

$$d\theta = \frac{dx}{4\theta^3}$$

Since the integration limits are in terms of θ, we'll also change the integration limits to match our substitution. If we don't do this now, we'll have to do it later.

Lower limit:

$$\theta = 0$$

$$\theta^4 = 0^4$$

$$x = 0^4$$

$$x = 0$$

Upper limit:

$$\theta = \sqrt[4]{\pi}$$

$$\theta^4 = \left(\sqrt[4]{\pi}\right)^4$$

$$x = \left(\sqrt[4]{\pi}\right)^4$$

$$x = \pi$$

Now, let's rewrite the integral in terms of x instead of θ.

$$\int_0^{\sqrt[4]{\pi}} 6\theta^7 \sin\left(\theta^4\right)\ d\theta$$

$$\int_0^{\pi} 6\theta^7 \sin x \frac{dx}{4\theta^3}$$

$$\int_0^{\pi} 3\theta^4 \sin x \frac{dx}{2}$$

$$\int_0^{\pi} \frac{3}{2}\theta^4 \sin x\ dx$$

$$\frac{3}{2}\int_0^{\pi} x \sin x\ dx$$

We are now prepared to integrate by parts. Integration by parts is a method of evaluating an integral that cannot be evaluated using normal integration techniques, by using integration by substitution, or by using integration formulas. The general formula for integration by parts is

$$\int u\ dv = uv - \int v\ du$$

In this formula, we separate the integrand into two parts; one part is called u and the other part is called dv. In making these two parts, we must use all of the integrand.

Although there is sometimes flexibility in choosing u, we can generally use the following sequence of choices to select the best part of the integrand to be u. This method involves the acronym LIPET, where we select the first u in the sequence of the list below. The letters mean

L Logarithmic expression

I Inverse trigonometric expression

P Polynomial expression

E Exponential expression

T Trigonometric function expression

In this problem, the integrand is $6\theta^7 \sin\left(\theta^4\right)$ where we have a polynomial and a trigonometric function. In the LIPET sequence, polynomial comes before trigonometric function, so u is the polynomial. Let's identify the parts we need to integrate.

$$u = x$$

$$du = dx$$

$$dv = \sin x \ dx$$

$$v = -\cos x$$

We're now ready to integrate by parts using the general formula.

$$\int u\ dv = uv - \int v\ du$$

$$\frac{3}{2}\int x\sin x\ dx = \frac{3}{2}\left(-x\cos x - \int -\cos x\ dx\right)$$

$$\frac{3}{2}\int x\sin x\ dx = \frac{3}{2}\left(-x\cos x + \int \cos x\ dx\right)$$

$$\frac{3}{2}\int x\sin x\ dx = \frac{3}{2}\left(-x\cos x + \sin x\right)$$

We didn't add a C to accommodate the possibility of a constant in the function because this problem involves a definite integral. Now we'll evaluate the result of the integration using the integration limits.

$$\frac{3}{2}\int_0^\pi x\sin x\ dx = \frac{3}{2}\left(-x\cos x + \sin x\right)\Big|_0^\pi$$

$$\frac{3}{2}\int_0^\pi x\sin x\ dx = \frac{3}{2}\left[-\pi\cos\pi + \sin\pi\right] - \frac{3}{2}\left[-(0)\cos(0) + \sin(0)\right]$$

$$\frac{3}{2}\int_0^\pi x\sin x\ dx = \frac{3}{2}\left[-\pi(-1) + 0\right] - \frac{3}{2}\left[-(0)(1) + 0\right]$$

$$\frac{3}{2}\int_0^\pi x\sin x\ dx = \frac{3}{2}\pi - 0$$

$$\frac{3}{2}\int_0^\pi x\sin x\ dx = \frac{3}{2}\pi$$

Therefore,

$$\frac{3}{2}\pi$$

is the value of the integral.