Topic: Interval of convergence

Question: Find the interval of convergence of the series.

$$\sum_{n=1}^{\infty} \frac{(x-6)^n}{8^n}$$

Answer choices:

A $2 \leq x \leq 14$

B $2 < x < 14$

C $-2 \leq x \leq 14$

D $-2 < x < 14$

Solution: D

We can use the ratio test for convergence to find the interval of convergence of a series. The ratio test tells us that, for a series a_n, if

$$L=\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|$$

then the series converges absolutely if $L<1$. Therefore, we'll find the value of L for the given series, plug it into $L<1$, and then solve for the variable.

In order to get L, we'll need a_n and a_{n+1}.

$$a_n=\frac{(x-6)^n}{8^n}$$

$$a_{n+1}=\frac{(x-6)^{n+1}}{8^{n+1}}$$

Plugging these into the formula for L, we get

$$L=\lim_{n\to\infty}\left|\frac{\frac{(x-6)^{n+1}}{8^{n+1}}}{\frac{(x-6)^n}{8^n}}\right|$$

$$L=\lim_{n\to\infty}\left|\frac{(x-6)^{n+1}}{8^{n+1}}\cdot\frac{8^n}{(x-6)^n}\right|$$

Pairing similar numerators and denominators together, we get

$$L=\lim_{n\to\infty}\left|\frac{(x-6)^{n+1}}{(x-6)^n}\cdot\frac{8^n}{8^{n+1}}\right|$$

$$L = \lim_{n \to \infty} \left| (x-6)^{n+1-n} \cdot 8^{n-(n+1)} \right|$$

$$L = \lim_{n \to \infty} \left| (x-6)^{1} \cdot 8^{n-n-1} \right|$$

$$L = \lim_{n \to \infty} \left| (x-6) \cdot 8^{-1} \right|$$

$$L = \lim_{n \to \infty} \left| \frac{x-6}{8} \right|$$

Since there are no ns remaining, and the limit only effects n, we can eliminate the limit.

$$L = \left| \frac{x-6}{8} \right|$$

The ratio test tells us that the series converges when $L < 1$. Plugging L into this inequality, we get

$$\left| \frac{x-6}{8} \right| < 1$$

$$-1 < \frac{x-6}{8} < 1$$

$$-8 < x-6 < 8$$

$$-2 < x < 14$$

We always have to check both endpoints of the interval before we can give a final answer for the interval of convergence. For this particular series, we can use the nth-term test to check the convergence of the endpoints. By the nth-term test,

if $\lim_{n\to\infty} a_n = 0$, the test is inconclusive

if $\lim_{n\to\infty} a_n \neq 0$, the series diverges

We'll check the endpoints.

At $x = -2$:

$$\lim_{n\to\infty} \frac{(-2-6)^n}{8^n}$$

$$\lim_{n\to\infty} \frac{(-8)^n}{8^n}$$

$$\lim_{n\to\infty} \frac{(-1)^n 8^n}{8^n}$$

$$\lim_{n\to\infty} (-1)^n$$

No limit

At $x = 14$:

$$\lim_{n\to\infty} \frac{(14-6)^n}{8^n}$$

$$\lim_{n\to\infty} \frac{8^n}{8^n}$$

$$\lim_{n\to\infty} 1$$

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Since both endpoints returned a non-zero value, by the nth-term test the series diverges at both endpoints. Therefore, the interval of convergence is the open interval $-2 < n < 14$.

Topic: Interval of convergence

Question: Find the interval of convergence of the series.

$$\sum_{n=0}^{\infty} \frac{5x^{2n+1}}{n!}$$

Answer choices:

A $-1 < x < 1$

B $-1 \leq x \leq 1$

C $-\infty < x < \infty$

D $0 < x < \infty$

Solution: C

We can use the ratio test for convergence to find the interval of convergence of a series. The ratio test tells us that, for a series a_n, if

$$L=\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|$$

then the series converges absolutely if $L<1$. Therefore, we'll find the value of L for the given series, plug it into $L<1$, and then solve for the variable.

In order to get L, we'll need a_n and a_{n+1}.

$$a_n=\frac{5x^{2n+1}}{n!}$$

$$a_{n+1}=\frac{5x^{2(n+1)+1}}{(n+1)!}=\frac{5x^{2n+2+1}}{(n+1)!}=\frac{5x^{2n+3}}{(n+1)!}$$

Plugging these into the formula for L, we get

$$L=\lim_{n\to\infty}\left|\frac{\frac{5x^{2n+3}}{(n+1)!}}{\frac{5x^{2n+1}}{n!}}\right|$$

$$L=\lim_{n\to\infty}\left|\frac{5x^{2n+3}}{(n+1)!}\cdot\frac{n!}{5x^{2n+1}}\right|$$

Pairing similar numerators and denominators together, we get

$$L = \lim_{n \to \infty} \left| \frac{5x^{2n+3}}{5x^{2n+1}} \cdot \frac{n!}{(n+1)!} \right|$$

Expanding the factorials so that we can get an idea of what we can cancel, and then canceling terms, we get

$$L = \lim_{n \to \infty} \left| x^{2n+3-(2n+1)} \cdot \frac{n(n-1)(n-2)(n-3)\ldots}{(n+1)(n+1-1)(n+1-2)(n+1-3)(n+1-4)\ldots} \right|$$

$$L = \lim_{n \to \infty} \left| x^{2n+3-2n-1} \cdot \frac{n(n-1)(n-2)(n-3)\ldots}{(n+1)(n)(n-1)(n-2)(n-3)\ldots} \right|$$

$$L = \lim_{n \to \infty} \left| x^2 \cdot \frac{1}{n+1} \right|$$

Since the limit only effects n, we can pull x out in front of the limit, as long as we keep it inside absolute value brackets.

$$L = \left| x^2 \right| \lim_{n \to \infty} \left| \frac{1}{n+1} \right|$$

$$L = \left| x^2 \right| \left| \frac{1}{\infty + 1} \right|$$

$$L = \left| x^2 \right| \left| \frac{1}{\infty} \right|$$

$$L = \left| x^2 \right| \left| 0 \right|$$

$$L = 0$$

Since the limit is 0 and $0 < 1$ is always true regardless of the value of n, the series converges for all values of n. Therefore, we don't need to check any endpoints, and the interval of convergence is $-\infty < n < \infty$.

Topic: Interval of convergence

Question: Find the interval of convergence of the series.

$$\sum_{n=1}^{\infty} \frac{(x-3)^n}{n^n}$$

Answer choices:

A $0 < x < 3$

B $0 < x < \infty$

C $-\infty < x < \infty$

D $-3 < x < 3$

Solution: C

We can use the root test for convergence to find the interval of convergence of a series. The root test tells us that, for a series a_n, if

$$L = \lim_{n \to \infty} \sqrt[n]{\left| a_n \right|}$$

then the series converges absolutely if $L < 1$. Therefore, we'll find the value of L for the given series, plug it into $L < 1$, and then solve for the variable.

Plugging a_n into the formula for L, we get

$$L = \lim_{n \to \infty} \sqrt[n]{\left| \frac{(x-3)^n}{n^n} \right|}$$

$$L = \lim_{n \to \infty} \sqrt[n]{\left| \left(\frac{x-3}{n} \right)^n \right|}$$

$$L = \lim_{n \to \infty} \left| \left(\frac{x-3}{n} \right)^n \right|^{\frac{1}{n}}$$

$$L = \lim_{n \to \infty} \left| \left(\frac{x-3}{n} \right)^{n \cdot \frac{1}{n}} \right|$$

$$L = \lim_{n \to \infty} \left| \frac{x-3}{n} \right|$$

Since the limit only effects n, we can pull x out in front of the limit, as long as we keep it inside absolute value brackets.

$$L = \left| x - 3 \right| \lim_{n \to \infty} \left| \frac{1}{n} \right|$$

$$L = \left| x - 3 \right| \left| \frac{1}{\infty} \right|$$

$$L = \left| x - 3 \right| \left| 0 \right|$$

$$L = 0$$

Since the limit is 0 and $0 < 1$ is always true regardless of the value of n, the series converges for all values of n. Therefore, we don't need to check any endpoints, and the interval of convergence is $-\infty < n < \infty$.