Topic: Limit process to find area on [a,b]

Question: Use the limit process to find the area of the region between the function and the x-axis over the given interval.

$$f(x) = 4 - x^2$$

on the interval $[1,2]$

Answer choices:

A $\frac{5}{3}$

B $\frac{3}{5}$

C $\frac{32}{3}$

D $-\frac{3}{5}$

Solution: A

The question asks us to find the area between the function $f(x) = 4 - x^2$ and the x-axis over the interval $[1,2]$. A quick look at the graph of $f(x)$ reveals that the function is above the x-axis over the entire interval. Thus, we can clearly expect a positive answer for the area.

We know that the limit process to find an area in an interval is

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\ \Delta x$$

We must find Δx and x_i.

Step 1: Recall that in the interval $[a, b]$, divided into n subdivisions,

$$\Delta x = \frac{b-a}{n}$$

The interval is $[1,2]$. Divide the region in the interval into n rectangles to find Δx.

$$\Delta x = \frac{2-1}{n} = \frac{1}{n}$$

Step 2: Find x_i by adding the left bound to $i\Delta x$. The left bound is 1.

$$x_i = 1 + i\Delta x = 1 + \frac{i}{n}$$

Step 3: Write the limit of the sum to find the area. The formula is

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\ \Delta x$$

Using the information from Step 1 and Step 2, we have

$$\lim_{n\to\infty}\sum_{i=1}^{n} f\left(1+\frac{i}{n}\right)\frac{1}{n}$$

Next, substitute the $f(x)=4-x^2$ into the summation, recalling that the x-value is $1+(i/n)$.

$$\lim_{n\to\infty}\sum_{i=1}^{n}\left[4-\left(1+\frac{i}{n}\right)^2\right]\frac{1}{n}$$

Next, square the expression in the summation, and since the summation is in terms of i, remove $1/n$ and place it in front of the summation.

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\left[4-\left(1+\frac{2i}{n}+\frac{i^2}{n^2}\right)\right]$$

Distribute the negative inside the summation.

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\left(4-1-\frac{2i}{n}-\frac{i^2}{n^2}\right)$$

Now, recall that in calculating the summation involving i, we have the following identities.

For a constant term, $\sum_{i=1}^{n} a = an$

For a term containing i, $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$

For a term containing i^2, $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$

Substitute these expressions into the limit of the summation, taking the sum.

$$\lim_{n\to\infty} \frac{1}{n}\left[4n - 1n - \frac{2}{n} \times \frac{n(n+1)}{2} - \frac{1}{n^2} \times \frac{n(n+1)(2n+1)}{6}\right]$$

$$\lim_{n\to\infty} \frac{1}{n}\left[3n - (n+1) - \frac{(n+1)(2n+1)}{6n}\right]$$

Distribute and multiply to remove the parentheses.

$$\lim_{n\to\infty} \frac{1}{n}\left[3n - n - 1 - \frac{2n^2+3n+1}{6n}\right]$$

Distribute the negative and separate the fraction into individual terms.

$$\lim_{n\to\infty} \frac{1}{n}\left[2n - 1 - \frac{2n^2}{6n} - \frac{3n}{6n} - \frac{1}{6n}\right]$$

$$\lim_{n\to\infty} \left[\frac{2n}{n} - \frac{1}{n} - \frac{2n^2}{6n^2} - \frac{3n}{6n^2} - \frac{1}{6n^2}\right]$$

$$\lim_{n\to\infty} \left[2 - \frac{1}{n} - \frac{1}{3} - \frac{1}{2n} - \frac{1}{6n^2}\right]$$

Take the limit.

$$2 - 0 - \frac{1}{3} - 0 - 0 = \frac{5}{3}$$

The area between $f(x) = 4 - x^2$ and the x-axis in the interval $[1,2]$ is $5/3$.

Topic: Limit process to find area on [a,b]

Question: Use the limit process to find the net area of the region between the function and the x-axis over the given interval.

$$f(x) = 4 - x^2$$

on the interval $[1,3]$

Answer choices:

A $\frac{2}{3}$

B 4

C $-\frac{2}{3}$

D $\frac{3}{2}$

Solution: C

The question asks us to find the area between the function $f(x)=4-x^2$ and the x-axis over the interval $[1,3]$. A quick look at the graph of $f(x)$ reveals that the function is above the x-axis on the interval $[1,2]$, but below the x-axis on the interval $[2,3]$. Thus, we can expect a net area as an answer. If the area below the x-axis is larger than the area above the x-axis, then the net area will be a negative value.

We know that the limit process to find an area in an interval is

$$\lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\ \Delta x$$

We must find Δx and x_i.

Step 1: Recall that in the interval $[a,b]$, divided into n subdivisions,

$$\Delta x=\frac{b-a}{n}$$

The interval is $[1,3]$. Divide the region in the interval into n rectangles to find Δx.

$$\Delta x=\frac{3-1}{n}=\frac{2}{n}$$

Step 2: Find x_i by adding the left bound to $i\Delta x$. The left bound is 1.

$$x_i=1+i\Delta x=1+\frac{2i}{n}$$

Step 3: Write the limit of the sum to find the area. The formula is

$$\lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\ \Delta x$$

Using the information from Step 1 and Step 2, we have

$$\lim_{n\to\infty}\sum_{i=1}^{n} f\left(1+\frac{2i}{n}\right)\frac{2}{n}$$

Next, substitute the $f(x)=4-x^2$ into the summation, recalling that the x-value is $1+(2i/n)$.

$$\lim_{n\to\infty}\sum_{i=1}^{n}\left[4-\left(1+\frac{2i}{n}\right)^2\right]\frac{2}{n}$$

Next, square the expression in the summation, and since the summation is in terms of i, remove $2/n$ and place it in front of the summation.

$$\lim_{n\to\infty}\frac{2}{n}\sum_{i=1}^{n}\left[4-\left(1+\frac{4i}{n}+\frac{4i^2}{n^2}\right)\right]$$

Distribute the negative inside the summation.

$$\lim_{n\to\infty}\frac{2}{n}\sum_{i=1}^{n}\left(4-1-\frac{4i}{n}-\frac{4i^2}{n^2}\right)$$

Now, recall that in calculating the summation involving i, we have the following identities.

For a constant term, $\sum_{i=1}^{n} a = an$

For a term containing i, $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$

For a term containing i^2, $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$

Substitute these expressions into the limit of the summation, taking the sum.

$$\lim_{n\to\infty} \frac{2}{n}\left[4n - 1n - \frac{4}{n} \times \frac{n(n+1)}{2} - \frac{4}{n^2} \times \frac{n(n+1)(2n+1)}{6}\right]$$

$$\lim_{n\to\infty} \frac{2}{n}\left[3n - 2(n+1) - \frac{2}{3n}(n+1)(2n+1)\right]$$

Distribute and multiply to remove the parentheses.

$$\lim_{n\to\infty} \frac{2}{n}\left[3n - 2n - 2 - \frac{2}{3n}\left(2n^2 + 3n + 1\right)\right]$$

Distribute the negative and separate the fraction into individual terms.

$$\lim_{n\to\infty} \frac{2}{n}\left[n - 2 - \frac{4n^2}{3n} - \frac{6n}{3n} - \frac{2}{3n}\right]$$

$$\lim_{n\to\infty} \frac{2}{n}\left[n - 2 - \frac{4n}{3} - 2 - \frac{2}{3n}\right]$$

$$\lim_{n\to\infty} \left[\frac{2n}{n} - \frac{4}{n} - \frac{8n}{3n} - \frac{4}{n} - \frac{4}{3n^2}\right]$$

$$\lim_{n\to\infty}\left[2-\frac{4}{n}-\frac{8}{3}-\frac{4}{n}-\frac{4}{3n^2}\right]$$

Take the limit.

$$2-0-\frac{8}{3}-0-0=-\frac{2}{3}$$

The area between $f(x)=4-x^2$ and the x-axis in the interval $[1,3]$ is $-2/3$. This means the area below the x-axis was larger than the area above the x-axis.

Topic: Limit process to find area on [a,b]

Question: Use the limit process to find the net area of the region between the function and the x-axis over the given interval.

$$g(x) = x^2 - 5x + 7$$

on the interval $[1,4]$

Answer choices:

A $-\frac{8}{3}$

B $\frac{8}{3}$

C $-\frac{9}{2}$

D $\frac{9}{2}$

Solution: D

The question asks us to find the area between the function $g(x) = x^2 - 5x + 7$ and the x-axis over the interval $[1,4]$. A quick look at the graph of $g(x)$ reveals that the function is above the x-axis in the entire interval $[1,4]$. Thus, we can clearly expect a positive answer for the area.

We know that the limit process to find an area in an interval is

$$\lim_{n\to\infty} \sum_{i=1}^{n} g(x_i)\ \Delta x$$

We must find Δx and x_i.

Step 1: Recall that in the interval $[a, b]$, divided into n subdivisions,

$$\Delta x = \frac{b-a}{n}$$

The interval is $[1,4]$. Divide the region in the interval into n rectangles to find Δx.

$$\Delta x = \frac{4-1}{n} = \frac{3}{n}$$

Step 2: Find x_i by adding the left bound to $i\Delta x$. The left bound is 1.

$$x_i = 1 + i\Delta x = 1 + \frac{3i}{n}$$

Step 3: Write the limit of the sum to find the area. The formula is

$$\lim_{n\to\infty} \sum_{i=1}^{n} g(x_i)\ \Delta x$$

Using the information from Step 1 and Step 2, we have

$$\lim_{n\to\infty}\sum_{i=1}^{n} g\left(1+\frac{3i}{n}\right)\frac{3}{n}$$

Next, substitute the $g(x)=x^2-5x+7$ into the summation, recalling that the x-value is $1+(3i/n)$.

$$\lim_{n\to\infty}\sum_{i=1}^{n}\left[\left(1+\frac{3i}{n}\right)^2-5\left(1+\frac{3i}{n}\right)+7\right]\frac{3}{n}$$

Next, square the first expression in the summation, distribute the middle term, and since the summation is in terms of i, remove $3/n$ and place it in front of the summation.

$$\lim_{n\to\infty}\frac{3}{n}\sum_{i=1}^{n}\left[1+\frac{6i}{n}+\frac{9i^2}{n^2}-5-\frac{15i}{n}+7\right]$$

Now, recall that in calculating the summation involving i, we have the following identities.

For a constant term, $\sum_{i=1}^{n} a = an$

For a term containing i, $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$

For a term containing i^2, $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$

Substitute these expressions into the limit of the summation, taking the sum.

$$\lim_{n\to\infty} \frac{3}{n}\left[n + \frac{6}{n} \times \frac{n(n+1)}{2} + \frac{9}{n^2} \times \frac{n(n+1)(2n+1)}{6} - 5n - \frac{15}{n} \times \frac{n(n+1)}{2} + 7n\right]$$

$$\lim_{n\to\infty} \frac{3}{n}\left[n + 3(n+1) + \frac{3}{2n}(n+1)(2n+1) - 5n - \frac{15}{2}(n+1) + 7n\right]$$

$$\lim_{n\to\infty} \frac{3}{n}\left[n + 3n + 3 + \frac{3}{2n}(2n^2 + 3n + 1) - 5n - \frac{15n}{2} - \frac{15}{2} + 7n\right]$$

$$\lim_{n\to\infty} \frac{3}{n}\left(n + 3n + 3 + \frac{6n^2}{2n} + \frac{9n}{2n} + \frac{3}{2n} - 5n - \frac{15n}{2} - \frac{15}{2} + 7n\right)$$

$$\lim_{n\to\infty} \frac{3}{n}\left(n + 3n + 3 + 3n + \frac{9}{2} + \frac{3}{2n} - 5n - \frac{15n}{2} - \frac{15}{2} + 7n\right)$$

Consolidate the n terms, then the constants, and then distribute the $3/n$ across the terms in the parentheses.

$$\lim_{n\to\infty} \frac{3}{n}\left(\frac{3n}{2} + \frac{3}{2n} + 3 + \frac{9}{2} - \frac{15}{2}\right)$$

$$\lim_{n\to\infty} \frac{3}{n}\left(\frac{3n}{2} + \frac{3}{2n}\right)$$

$$\lim_{n\to\infty} \frac{9n}{2n} + \frac{9}{2n^2}$$

$$\lim_{n\to\infty} \frac{9}{2} + \frac{9}{2n^2}$$

Take the limit.

$$\frac{9}{2} + 0$$

$$\frac{9}{2}$$

The area between $g(x) = x^2 - 5x + 7$ and the x-axis on the interval $[1,4]$ is $9/2$.