Alternating series test

The alternating series test for convergence lets us say whether an alternating series

$$\sum_{n=1}^{\infty} (-1)^{n-1} a_n$$

where $a_n > 0$

is converging or diverging.

The alternating series test for convergence tells us that

an alternating series converges if

$0 < a_{n+1} < a_n$ for all values of n, and

$$\lim_{n \to \infty} a_n = 0$$

When we use the alternating series test, we need to make sure that we separate the series a_n from the $(-1)^n$ part that makes it alternating.

Example

Use the alternating series test to say whether the series converges or diverges

$$\sum_{n=5}^{\infty} \frac{(-1)^{n-3}\sqrt{n}}{n+4}$$

First, we separate the series from the part that makes it alternating.

$$\sum_{n=5}^{\infty} (-1)^{n-3} \frac{\sqrt{n}}{n+4}$$

Matching this up to the standard form of an alternating series given above, we can say that the series is

$$a_n = \frac{\sqrt{n}}{n+4}$$

Now we need to show that

$0 < a_{n+1} < a_n$ for all values of n, and

$$\lim_{n\to\infty} a_n = 0$$

for the series a_n, in order to say that a_n converges. Remembering that this series starts at $n = 5$, let's check the first few terms of the series to see if it looks like $0 < a_{n+1} < a_n$.

		a_n		a_{n+1}
$n = 5$	$\frac{\sqrt{5}}{5+4}$	$\frac{\sqrt{5}}{9}$	$\frac{\sqrt{5+1}}{5+1+4}$	$\frac{\sqrt{6}}{10}$
$n = 6$	$\frac{\sqrt{6}}{6+4}$	$\frac{\sqrt{6}}{10}$	$\frac{\sqrt{6+1}}{6+1+4}$	$\frac{\sqrt{7}}{11}$
$n = 7$	$\frac{\sqrt{7}}{7+4}$	$\frac{\sqrt{7}}{11}$	$\frac{\sqrt{7+1}}{7+1+4}$	$\frac{\sqrt{8}}{12}$

$n=8$	$\frac{\sqrt{8}}{8+4}$	$\frac{\sqrt{8}}{12}$	$\frac{\sqrt{8+1}}{8+1+4}$	$\frac{\sqrt{9}}{13}$

We can see that the terms of a_n and a_{n+1} will always be positive, because there's no value of n, when $n \geq 5$, that will make either series negative. We can also see that a_{n+1} is always going to be smaller than a_n. If you're not convince by their fractional values in the table, compute the decimal values on your calculator to be sure.

If you can't be sure that $0 < a_{n+1} < a_n$ just by looking at the table, you can always take the derivative of a_n to double-check. If the derivative is negative, then you know the series is decreasing, which means that a_{n+1} will always be less than a_n.

$$\frac{d}{dx}\left(\frac{\sqrt{x}}{x+4}\right)$$

Using the quotient rule, we get

$$\frac{\frac{1}{2}(x)^{-\frac{1}{2}}(x+4)-(x)^{\frac{1}{2}}(1)}{(x+4)^2}$$

$$\frac{\frac{1}{2}x^{\frac{1}{2}}+2x^{-\frac{1}{2}}-x^{\frac{1}{2}}}{(x+4)^2}$$

$$\frac{-\frac{1}{2}x^{\frac{1}{2}}+2x^{-\frac{1}{2}}}{(x+4)^2}$$

$$\frac{-\frac{x^{\frac{1}{2}}}{2}+\frac{2}{x^{\frac{1}{2}}}}{(x+4)^2}$$

$$\frac{-\frac{x}{2x^{\frac{1}{2}}}+\frac{4}{2x^{\frac{1}{2}}}}{(x+4)^2}$$

$$\frac{\frac{4-x}{2x^{\frac{1}{2}}}}{(x+4)^2}$$

$$\frac{4-x}{2x^{\frac{1}{2}}}\cdot\frac{1}{(x+4)^2}$$

$$\frac{4-x}{2x^{\frac{1}{2}}(x+4)^2}$$

Looking at the derivative, we can see that for all values of the series (remember, the series starts at $n=5$), the derivative is negative because the numerator will be negative and the denominator will be positive. This confirms that the series is decreasing, and therefore that it converges.

The final step is to verify that $\lim_{n\to\infty} a_n = 0$.

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{\sqrt{n}}{n+4}$$

$$\lim_{n\to\infty} a_n = \frac{\sqrt{\infty}}{\infty+4}$$

$$\lim_{n\to\infty} a_n = \frac{\sqrt{\infty}}{\infty}$$

Since the numerator will be significantly smaller than the denominator, especially as n gets really big, we can say that

$$\lim_{n\to\infty} a_n = 0$$

Since we've shown that $0 < a_{n+1} < a_n$ and that $\lim_{n \to \infty} a_n = 0$, we can say that the series converges.