Taylor series

We already know how to use the power series

$$\frac{1}{1-x}=\sum_{n=0}^{\infty} x^n$$

to find a polynomial representation of a function. But sometimes we'll want to represent a function as a series, and we won't be able to easily relate the function to $1/(1-x)$.

Taylor series let us find a series representation for any function, whether or not we can relate it to the power series $1/(1-x)$.

In order to create a Taylor series representation for a function, we'll need

a - the value about which the function is defined

n - the degree to which we want to evaluate the function

Both of these are usually given in the problem. With a value for a and n, we can build the chart below.

n	$n!$	$f^{(n)}(x)$	$f^{(n)}(a)$	$\frac{f^{(n)}(a)}{n!}$
0	1	$f(x)$	$f(a)$	$f(a)$
1	1	$f'(x)$	$f'(a)$	$f'(a)$
2	2	$f''(x)$	$f''(a)$	$\frac{f''(a)}{2}$

3	6	$f'''(x)$	$f'''(a)$	$\frac{f'''(a)}{6}$
4	24	$f''''(x)$	$f''''(a)$	$\frac{f''''(a)}{24}$
		...		
n	$n!$	$f^{(n)}(x)$	$f^{(n)}(a)$	$\frac{f^{(n)}(a)}{n!}$

When we're done with the chart, the value in the far right column becomes the coefficient on each term in the Taylor polynomial, in the form

$$\frac{f^{(n)}(a)}{n!}(x-a)^n$$

The sum of all these terms is the Taylor series for the function.

Example

Find the fourth-degree Taylor polynomial about $a = 1$.

$$f(x) = x^4 + 5x^3 - x^2 + 3x - 1$$

Since we're looking for the fourth-degree polynomial, we can say that $n = 4$. As always, we'll start n at 0, so the values of n we'll include in our chart are $n = 0,1,2,3,4$. We can also include all of the corresponding values of $n!$.

n	$n!$	$f^{(n)}(x)$	$f^{(n)}(a)$	$\frac{f^{(n)}(a)}{n!}$
0	1	$f(x)$	$f(a)$	$f(a)$
1	1	$f'(x)$	$f'(a)$	$f'(a)$
2	2	$f''(x)$	$f''(a)$	$\frac{f''(a)}{2}$
3	6	$f'''(x)$	$f'''(a)$	$\frac{f'''(a)}{6}$
4	24	$f''''(x)$	$f''''(a)$	$\frac{f''''(a)}{24}$

To find the values for the third column, we'll put the original function in the first row, followed by its derivatives in the following rows.

n	$n!$	$f^{(n)}(x)$	$f^{(n)}(a)$	$\frac{f^{(n)}(a)}{n!}$
0	1	$x^4+5x^3-x^2+3x-1$	$f(a)$	$f(a)$
1	1	$4x^3+15x^2-2x+3$	$f'(a)$	$f'(a)$
2	2	$12x^2+30x-2$	$f''(a)$	$\frac{f''(a)}{2}$
3	6	$24x+30$	$f'''(a)$	$\frac{f'''(a)}{6}$
4	24	24	$f''''(a)$	$\frac{f''''(a)}{24}$

To find the values for the fourth column, $f^{(n)}(a)$, we'll evaluate the values in the third column at $a=1$.

n	$n!$	$f^{(n)}(x)$	$f^{(n)}(a)$	$\frac{f^{(n)}(a)}{n!}$
0	1	$x^4 + 5x^3 - x^2 + 3x - 1$	$(1)^4 + 5(1)^3 - (1)^2 + 3(1) - 1 = 7$	$f(a)$
1	1	$4x^3 + 15x^2 - 2x + 3$	$4(1)^3 + 15(1)^2 - 2(1) + 3 = 20$	$f'(a)$
2	2	$12x^2 + 30x - 2$	$12(1)^2 + 30(1) - 2 = 40$	$\frac{f''(a)}{2}$
3	6	$24x + 30$	$24(1) + 30 = 54$	$\frac{f'''(a)}{6}$
4	24	24	24	$\frac{f''''(a)}{24}$

To get the values for the last column, we'll divide the result of the fourth column by $n!$ from the second column.

n	$n!$	$f^{(n)}(x)$	$f^{(n)}(a)$	$\frac{f^{(n)}(a)}{n!}$
0	1	$x^4 + 5x^3 - x^2 + 3x - 1$	$(1)^4 + 5(1)^3 - (1)^2 + 3(1) - 1 = 7$	7
1	1	$4x^3 + 15x^2 - 2x + 3$	$4(1)^3 + 15(1)^2 - 2(1) + 3 = 20$	20
2	2	$12x^2 + 30x - 2$	$12(1)^2 + 30(1) - 2 = 40$	$\frac{40}{2} = 20$
3	6	$24x + 30$	$24(1) + 30 = 54$	$\frac{54}{6} = 9$
4	24	24	24	$\frac{24}{24} = 1$

With the whole chart filled in, we can build each term of the Taylor polynomial.

$n = 0$	$\frac{f^{(n)}(a)}{n!}(x-a)^n = 7(x-1)^0$	7
$n = 1$	$\frac{f^{(n)}(a)}{n!}(x-a)^n = 20(x-1)^1$	$20(x-1)$
$n = 2$	$\frac{f^{(n)}(a)}{n!}(x-a)^n = 20(x-1)^2$	$20(x-1)^2$
$n = 3$	$\frac{f^{(n)}(a)}{n!}(x-a)^n = 9(x-1)^3$	$9(x-1)^3$
$n = 4$	$\frac{f^{(n)}(a)}{n!}(x-a)^n = 1(x-1)^4$	$(x-1)^4$

Putting all of the terms together, we get the fourth-degree Taylor polynomial.

$$7 + 20(x-1) + 20(x-1)^2 + 9(x-1)^3 + (x-1)^4$$