Sum of the geometric series

We already know from the last section that the standard form of a geometric series is

$$\sum_{n=1}^{\infty} ar^{n-1}$$

or

$$\sum_{n=0}^{\infty} ar^{n}$$

Given either of these forms, the geometric series test for convergence says that

if $|r| < 1$ then the series converges

if $|r| \geq 1$ then the series diverges

When a geometric series converges, we can find its sum.

Sum of a geometric series

We can use the values of a and r and the formula for the sum of a geometric series

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}$$

or

$$\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}$$

to find the sum of the geometric series.

Example

Calculate the sum of the geometric series.

$$\sum_{n=0}^{\infty} \frac{2^{n-1}}{3^n}$$

We showed in the last section that this series was geometric by rewriting it as

$$\sum_{n=0}^{\infty} \frac{2^{n-1}}{3^n}$$

$$\sum_{n=0}^{\infty} \frac{2^n 2^{-1}}{3^n}$$

$$\sum_{n=0}^{\infty} 2^{-1}\left(\frac{2^n}{3^n}\right)$$

$$\sum_{n=0}^{\infty} \frac{1}{2}\left(\frac{2}{3}\right)^n$$

Now that we have the series in the right form, we can say

$$\sum_{n=0}^{\infty} ar^n = \sum_{n=0}^{\infty} \frac{1}{2}\left(\frac{2}{3}\right)^n$$ where

$$a = \frac{1}{2}$$

$$r = \frac{2}{3}$$

Since the sum of a geometric series is given by

$$\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}$$

we can say that the sum is

$$\sum_{n=0}^{\infty} \frac{1}{2}\left(\frac{2}{3}\right)^n = \frac{\frac{1}{2}}{1-\frac{2}{3}}$$

$$\frac{\frac{1}{2}}{\frac{3}{3}-\frac{2}{3}}$$

$$\frac{\frac{1}{2}}{\frac{1}{3}}$$

$$\frac{1}{2}\cdot\frac{3}{1}$$

$$\frac{3}{2}$$

We could have also found the sum by expanding the series through its first few terms and identifying values for a and r.

$$\sum_{n=0}^{\infty}\frac{1}{2}\left(\frac{2}{3}\right)^n=\frac{1}{2}\left[\left(\frac{2}{3}\right)^0+\left(\frac{2}{3}\right)^1+\left(\frac{2}{3}\right)^2+\left(\frac{2}{3}\right)^3+\dots\right]$$

$$\sum_{n=0}^{\infty}\frac{1}{2}\left(\frac{2}{3}\right)^n=\frac{1}{2}\left(1+\frac{2}{3}+\frac{4}{9}+\frac{8}{27}+\dots\right)$$

So

$$a=\frac{1}{2}$$

$$r=\frac{2}{3}$$

and

$$\sum_{n=0}^{\infty}\frac{1}{2}\left(\frac{2}{3}\right)^n=\frac{\frac{1}{2}}{1-\frac{2}{3}}$$

$$\frac{\frac{1}{2}}{\frac{3}{3}-\frac{2}{3}}$$

$$\frac{\frac{1}{2}}{\frac{1}{3}}$$

$$\frac{1}{2}\cdot\frac{3}{1}$$

$$\frac{3}{2}$$