Vertical motion

Vertical motion problems are just what they sound like: problems that deal with an object dropped straight down from some height, or thrown straight up in the air so that they go up and then come back down. The motion of the object is vertical.

Problems like these require you to know the relationship between position $x(t)$, velocity $v(t)$, and acceleration $a(t)$. The important thing to know is that the derivative of position is velocity, and the derivative of velocity is acceleration.

$$x(t)$$

$$x'(t) = v(t)$$

$$x''(t) = v'(t) = a(t)$$

We can also describe the above relationship using integrals instead of derivatives, and we see that the integral of acceleration is velocity, and the integral of velocity is position.

$$a(t)$$

$$\int a(t)\ dt = v(t)$$

$$\int \int a(t)\ dt = \int v(t) = x(t)$$

Example

A pumpkin is thrown straight up into the air from the ground with an initial velocity $v(t_0) = 24$ m/s. The acceleration is due to gravity only, so $a(t) = -9.8$ m/s^2. The velocity of the pumpkin is modeled by $v(t) = at + v_0$.

What is the maximum height the pumpkin reaches, and how much time passes before the pumpkin hits the ground again?

Before we start, we need to figure out what's happening to the pumpkin. Here's what we know: The path of the pumpkin has three important points. The first point occurs when it's initially thrown into the air, where

$$t_0 = 0$$

$$x(t_0) = 0 \qquad \text{or} \qquad x(0) = 0$$

$$v(t_0) = 24 \qquad \text{or} \qquad v(0) = 24$$

At the start, time and position are both 0 because it's the beginning of the pumpkin's flight and it hasn't moved yet. Initial velocity is given in the question as $v(t_0) = 24$.

The second important point occurs when the pumpkin reaches its maximum height, where we know its velocity is $v(t_1) = 0$, because this is the transition moment when the pumpkin stops traveling up and starts traveling down. If the pumpkin is traveling neither up nor down, then it has no direction, and therefore velocity is 0. We don't know the time t_1 or the position $x(t_1)$. In other words,

$$t_1 = ?$$

$$x(t_1) = ?$$

$$v(t_1) = 0$$

The first part of the question asks us to solve for $x(t_1)$.

The third important point occurs when the pumpkin hits the ground, where the position will be 0 again, and $s(t_2) = 0$. We don't know the time t_2 or the velocity $v(t_2)$. So

$$t_2 = ?$$

$$s(t_2) = 0$$

$$v(t_2) = ?$$

The second part of the question asks us to solve for t_2.

Now that we've identified what we know, we need to solve for the position equation. The problem tells us that $v(t) = at + v_0$, $a(t) = -9.8$ and $v(t_0) = 24$. Plugging these values into $v(t)$, we get

$$v(t) = -9.8t + 24$$

Now we can integrate $v(t)$ to get $x(t)$.

$$x(t) = \int -9.8t + 24\ dt$$

$$x(t) = \int -9.8t\ dt + \int 24\ dt$$

$$x(t) = -9.8\int t\ dt + 24\int 1\ dt$$

$$x(t) = -9.8\left(\frac{t^2}{2}\right) + 24t + C$$

$$x(t) = -4.9t^2 + 24t + C$$

Since we know $x(0) = 0$ we can plug this into the position function we just found in order to solve for C.

$$0 = -4.9(0)^2 + 24(0) + C$$

$$C = 0$$

Therefore, the position function is

$$x(t) = -4.9t^2 + 24t$$

With the position function in hand, we can start working on solving for the maximum height of the pumpkin. We'll start by finding t_1. Since we know that $v(t_1) = 0$ and $v(t) = -9.8t + 24$, we get

$$0 = -9.8t_1 + 24$$

$$-24 = -9.8t_1$$

$$t_1 \approx 2.45$$

Now that we know that $t_1 \approx 2.45$ when the pumpkin reaches maximum height, we can use $x(t) = -4.9t^2 + 24t$ to solve for $x(t_1)$.

$$x(2.45) = -4.9(2.45)^2 + 24(2.45)$$

$$x(2.45) \approx 29.39$$

The maximum height the pumpkin reaches is 29.39 m, and that answers the first part of the question.

To find out how much time passes before the pumpkin hits the ground, we have to remember that $x(t_2) = 0$ and $x(t) = -4.9t^2 + 24t$. We can plug in this information and solve for t_2.

$$0 = -4.9t_2^2 + 24t_2$$

$$0 = -4.9t(t_2 - 4.9)$$

$$t_2 = 0 \text{ or } t_2 = 4.9$$

We know $t_0 = 0$ corresponds to the pumpkin's initial position, which means $t_2 = 4.9$ has to correspond to the pumpkin's final position.

The amount of time that passes before the pumpkin hits the ground again is 4.9 s, which answers the second part of the question.