Inverse hyperbolic integrals

Inverse hyperbolic functions follow standard rules for integration. Remember, an inverse hyperbolic function can be written two ways. For example, inverse hyperbolic sine can be written as

arcsinh or as

$\sinh^{-1}$

Some people argue that the arcsinh form should be used instead of $\sinh^{-1}$ because $\sinh^{-1}$ can be misinterpreted as $1/\sinh$. Whichever form you prefer, you see both, so you should be able to recognize both and understand that they mean the same thing.

The general rules for the six inverse hyperbolic functions are

$$\int \text{arcsinh}(ax)\ dx = x\text{arcsinh}(ax) - \frac{\sqrt{a^2x^2+1}}{a} + C$$

$$\int \text{arccosh}(ax)\ dx = x\text{arccosh}(ax) - \frac{\sqrt{ax+1}\sqrt{ax-1}}{a} + C$$

$$\int \text{arctanh}(ax)\ dx = x\text{arctanh}(ax) + \frac{\ln(1-a^2x^2)}{2a} + C$$

$$\int \text{arccoth}(ax)\ dx = x\text{arccoth}(ax) + \frac{\ln(a^2x^2-1)}{2a} + C$$

$$\int \text{arcsech}(ax)\ dx = x\text{arcsech}(ax) - \frac{2}{a}\arctan\sqrt{\frac{1-ax}{1+ax}} + C$$

$$\int \text{arccsch}(ax)\ dx = x\text{arccsch}(ax) + \frac{1}{a}\text{arccoth}\sqrt{\frac{1}{a^2x^2}+1} + C$$

We also have a few other standard inverse hyperbolic integrals that are based on the standard inverse hyperbolic derivatives. In the following formulas, u represents a function.

$$\int \frac{1}{\sqrt{a^2+u^2}}\ du = \text{arcsinh}\left(\frac{u}{a}\right) + C \qquad \text{where } a > 0$$

$$\int \frac{1}{\sqrt{u^2-a^2}}\ du = \text{arccosh}\left(\frac{u}{a}\right) + C \qquad \text{where } u > a > 0$$

$$\int \frac{1}{a^2-u^2}\ du = \frac{1}{a}\text{arctanh}\left(\frac{u}{a}\right) + C \qquad \text{if } u^2 < a^2$$

$$\int \frac{1}{a^2-u^2}\ du = \frac{1}{a}\text{arccoth}\left(\frac{u}{a}\right) + C \qquad \text{if } u^2 > a^2$$

$$\int \frac{1}{u\sqrt{a^2-u^2}}\ du = -\frac{1}{a}\text{arcsech}\left(\frac{u}{a}\right) + C \qquad \text{where } 0 < u < a$$

$$\int \frac{1}{u\sqrt{a^2+u^2}}\ du = -\frac{1}{a}\text{arccsch}\left(\frac{u}{a}\right) + C \qquad \text{where } u \neq 0$$

Example

Evaluate the integral.

$$\int -7\text{arcsech}(5x)\ dx$$

We'll simplify by factoring -7 out of the integral.

$$-7\int \text{arcsech}(5x)\ dx$$

We'll use

$$\int \text{arcsech}(ax)\ dx = x\text{arcsech}(ax) - \frac{2}{a}\arctan\sqrt{\frac{1-ax}{1+ax}} + C$$

to integrate, and get

$$-7x\text{arcsech}(5x) + \frac{14}{5}\arctan\sqrt{\frac{1-5x}{1+5x}} + C$$

Example

Evaluate the integral.

$$\int \frac{1}{\sqrt{9+x^2}} - \text{arccsch}(4x) + 3x^2\ dx$$

First, break the integral into parts.

$$\int \frac{1}{\sqrt{9+x^2}}\ dx + \int -\text{arccsch}(4x)\ dx + \int 3x^2\ dx$$

$$\int \frac{1}{\sqrt{9+x^2}}\ dx - \int \text{arccsch}(4x)\ dx + 3\int x^2\ dx$$

Now we'll integrate using the formulas from this section, and we'll get

$$\text{arcsinh}\left(\frac{x}{3}\right) - x\text{arccsch}(4x) - \frac{1}{4}\text{arccoth}\sqrt{\frac{1}{4^2x^2}+1} + \frac{3}{3}x^3 + C$$

$$\text{arcsinh}\left(\frac{x}{3}\right) - x\text{arccsch}(4x) - \frac{1}{4}\text{arccoth}\sqrt{\frac{1}{16x^2}+\frac{16x^2}{16x^2}} + x^3 + C$$

$$\text{arcsinh}\left(\frac{x}{3}\right) - x\text{arccsch}(4x) - \frac{1}{4}\text{arccoth}\sqrt{\frac{1+16x^2}{16x^2}} + x^3 + C$$

$$\text{arcsinh}\left(\frac{x}{3}\right) - x\text{arccsch}(4x) - \frac{1}{4}\text{arccoth}\left(\frac{\sqrt{16x^2+1}}{4x}\right) + x^3 + C$$
