Calculus 2 Workbook Solutions

Power series

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POWER SERIES REPRESENTATION

■ 1. Find the power series representation of the function.

$$f(x) = \frac{3x}{7 + x^2}$$

Solution:

The standard form of a power series is

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

Manipulate the function until it's in the form of the standard power series.

$$\frac{1}{1-x} = \frac{3x}{7 + x^2}$$

$$\frac{1}{1-x} = (3x)\frac{1}{7 + x^2}$$

$$\frac{1}{1-x} = (3x)\frac{1}{7\left(1 + \frac{x^2}{7}\right)}$$

$$\frac{1}{1-x} = \left(\frac{3x}{7}\right)\frac{1}{\left(1 + \frac{x^2}{7}\right)}$$

$$\frac{1}{1-x}=\left(\frac{3x}{7}\right)\frac{1}{1-\left(-\frac{x^2}{7}\right)}$$

Then the power series representation of the function is

$$\frac{3x}{7}\sum_{n=0}^{\infty}\left(-\frac{x^2}{7}\right)^n$$

$$\sum_{n=0}^{\infty}\frac{3^1x^1}{7^1}\left(\frac{(-1)x^2}{7}\right)^n$$

$$\sum_{n=0}^{\infty}\frac{3^1x^1(-1)^n}{7^1}\left(\frac{x^{2n}}{7^n}\right)$$

$$\sum_{n=0}^{\infty}\frac{(-1)^n3x^{2n+1}}{7^{n+1}}$$

2. Find the power series representation of the function.

$$f(x)=\frac{5}{4-6x}$$

Solution:

The standard form of a power series is

$$\frac{1}{1-x}=\sum_{n=0}^{\infty}x^n$$

Manipulate the function until it's in the form of the standard power series.

$$\frac{1}{1-x}=\frac{5}{4-6x}$$

$$\frac{1}{1-x}=(5)\frac{1}{4-6x}$$

$$\frac{1}{1-x}=(5)\frac{1}{4\left(1-\frac{6x}{4}\right)}$$

$$\frac{1}{1-x}=(5)\frac{1}{4\left(1-\frac{3x}{2}\right)}$$

$$\frac{1}{1-x}=\left(\frac{5}{4}\right)\frac{1}{\left(1-\frac{3x}{2}\right)}$$

$$\frac{1}{1-x}=\left(\frac{5}{4}\right)\frac{1}{1-\frac{3x}{2}}$$

Then the power series representation of the function is

$$\frac{5}{4}\sum_{n=0}^{\infty}\left(\frac{3x}{2}\right)^n$$

$$\sum_{n=0}^{\infty}\frac{5^1}{4^1}\left(\frac{3x}{2}\right)^n$$

$$\sum_{n=0}^{\infty} \frac{5^1}{2^2} \left(\frac{3x}{2} \right)^n$$

$$\sum_{n=0}^{\infty} \frac{5(3x)^n}{2^{n+2}}$$

■ 3. Find the power series representation of the function.

$$f(x) = \frac{4}{x^2 - x^3}$$

Solution:

The standard form of a power series is

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

Manipulate the function until it's in the form of the standard power series.

$$\frac{1}{1-x} = \frac{4}{x^2 - x^3}$$

$$\frac{1}{1-x} = (4)\frac{1}{x^2 - x^3}$$

$$\frac{1}{1-x} = (4)\frac{1}{x^2(1-x)}$$

$$\frac{1}{1-x}=\left(\frac{4}{x^2}\right)\frac{1}{1-x}$$

Then the power series representation of the function is

$$\frac{4}{x^2}\sum_{n=0}^{\infty}x^n$$

$$\sum_{n=0}^{\infty}\frac{4^1}{x^2}(x)^n$$

$$\sum_{n=0}^{\infty}\frac{4^1x^n}{x^2}$$

$$\sum_{n=0}^{\infty}4x^{n-2}$$

■ 4. Find the power series representation of the function.

$$f(x)=\frac{5x^2}{1+x^3}$$

Solution:

The standard form of a power series is

$$\frac{1}{1-x}=\sum_{n=0}^{\infty}x^n$$

Manipulate the function until it's in the form of the standard power series.

$$\frac{1}{1-x}=\frac{5x^2}{1+x^3}$$

$$\frac{1}{1-x}=(5x^2)\frac{1}{1+x^3}$$

$$\frac{1}{1-x}=(5x^2)\frac{1}{1-(-x^3)}$$

Then the power series representation of the function is

$$5x^2\sum_{n=0}^{\infty}(-x^3)^n$$

$$\sum_{n=0}^{\infty}5^1x^2(-x^3)^n$$

$$\sum_{n=0}^{\infty}5^1x^2(-1)^n(x^3)^n$$

$$\sum_{n=0}^{\infty}5x^2(-1)^nx^{3n}$$

$$\sum_{n=0}^{\infty}(-1)^n5x^{3n+2}$$

5. Find the power series representation of the function.

$$f(x) = \frac{x}{8 - x}$$

Solution:

The standard form of a power series is

$$\frac{1}{1 - x} = \sum_{n=0}^{\infty} x^n$$

Manipulate the function until it's in the form of the standard power series.

$$\frac{1}{1 - x} = \frac{x}{8 - x}$$

$$\frac{1}{1 - x} = (x)\frac{1}{8 - x}$$

$$\frac{1}{1 - x} = (x)\frac{1}{8\left(1 - \frac{x}{8}\right)}$$

$$\frac{1}{1 - x} = \left(\frac{x}{8}\right)\frac{1}{1 - \frac{x}{8}}$$

Then the power series representation of the function is

$$\frac{x}{8}\sum_{n=0}^{\infty}\left(\frac{x}{8}\right)^n$$

$$\sum_{n=0}^{\infty}\frac{x^1}{8^1}\left(\frac{x}{8}\right)^n$$

$$\sum_{n=0}^{\infty} \frac{x^1}{8^1} \cdot \frac{x^n}{8^n}$$

$$\sum_{n=0}^{\infty} \frac{x^{n+1}}{8^{n+1}}$$

POWER SERIES MULTIPLICATION

■ 1. Use power series multiplication to find the first four non-zero terms of the Maclaurin series.

$$y = \cos(3x)e^{3x}$$

Solution:

The given series is the product of two other series.

$$y = \cos(3x)$$

$$y = e^{3x}$$

For the first one, start with the common series

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots$$

Substitute $3x$ for x.

$$\cos(3x) = 1 - \frac{(3x)^2}{2!} + \frac{(3x)^4}{4!} - \frac{(3x)^6}{6!} + \frac{(3x)^8}{8!} - \cdots$$

$$\cos(3x) = 1 - \frac{9x^2}{2} + \frac{81x^4}{24} - \frac{729x^6}{720} + \frac{6{,}561x^8}{40{,}320} - \cdots$$

$$\cos(3x) = 1 - \frac{9x^2}{2} + \frac{27x^4}{8} - \frac{81x^6}{80} + \frac{729x^8}{4{,}480} - \cdots$$

For the second, start with the common series

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots$$

Substitute $3x$ for x.

$$e^{3x} = 1 + \frac{3x}{1!} + \frac{(3x)^2}{2!} + \frac{(3x)^3}{3!} + \frac{(3x)^4}{4!} + \frac{(3x)^5}{5!} + \cdots$$

$$e^{3x} = 1 + 3x + \frac{9x^2}{2} + \frac{9x^3}{2} + \frac{27x^4}{8} + \frac{81x^5}{40} + \cdots$$

Multiply the series together.

$$\cos(3x)e^{3x} = \left(1 - \frac{9x^2}{2} + \frac{27x^4}{8} - \frac{81x^6}{80} + \frac{729x^8}{4,480} - \cdots\right)$$

$$\left(1 + 3x + \frac{9x^2}{2} + \frac{9x^3}{2} + \frac{27x^4}{8} + \frac{81x^5}{40} + \cdots\right)$$

Multiply every term in the first series by every term in the second series.

$$\cos(3x)e^{3x} = 1\left(1 + 3x + \frac{9x^2}{2} + \frac{9x^3}{2} + \frac{27x^4}{8} + \frac{81x^5}{40} + \cdots\right)$$

$$-\frac{9x^2}{2}\left(1 + 3x + \frac{9x^2}{2} + \frac{9x^3}{2} + \frac{27x^4}{8} + \frac{81x^5}{40} + \cdots\right)$$

$$+\frac{27x^4}{8}\left(1 + 3x + \frac{9x^2}{2} + \frac{9x^3}{2} + \frac{27x^4}{8} + \frac{81x^5}{40} + \cdots\right)$$

$$-\frac{81x^6}{80}\left(1+3x+\frac{9x^2}{2}+\frac{9x^3}{2}+\frac{27x^4}{8}+\frac{81x^5}{40}+\cdots\right)$$

$$+\cdots$$

$$\cos(3x)e^{3x}=1+3x+\frac{9x^2}{2}+\frac{9x^3}{2}+\frac{27x^4}{8}+\frac{81x^5}{40}+\cdots$$

$$-\frac{9x^2}{2}-\frac{27x^3}{2}-\frac{81x^4}{4}-\frac{81x^5}{4}-\frac{243x^6}{16}-\frac{729x^7}{80}-\cdots$$

$$+\frac{27x^4}{8}+\frac{81x^5}{8}+\frac{243x^6}{16}+\frac{243x^7}{16}+\frac{729x^8}{64}+\frac{2{,}187x^9}{320}+\cdots$$

$$-\frac{81x^6}{80}-\frac{243x^7}{80}-\frac{729x^8}{160}-\frac{729x^9}{160}-\frac{2{,}187x^{10}}{60}-\frac{6{,}561x^{11}}{3{,}200}-\cdots$$

To get the first four non-zero terms, we only need terms through x^4.

$$\cos(3x)e^{3x}=1+3x+\frac{9x^2}{2}-\frac{9x^2}{2}+\frac{9x^3}{2}-\frac{27x^3}{2}+\frac{27x^4}{8}-\frac{81x^4}{4}+\frac{27x^4}{8}+\ldots$$

$$\cos(3x)e^{3x}=1+3x-9x^3-\frac{27x^4}{4}+\ldots$$

■ 2. Use power series multiplication to find the first four non-zero terms of the Maclaurin series.

$$y=\arctan(2x)\sin x$$

Solution:

The given series is the product of two other series.

$$y = \arctan(2x)$$

$$y = \sin x$$

For the first one, start with the common series

$$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \cdots$$

Substitute $2x$ for x.

$$\arctan(2x) = 2x - \frac{(2x)^3}{3} + \frac{(2x)^5}{5} - \frac{(2x)^7}{7} + \frac{(2x)^9}{9} - \cdots$$

$$\arctan(2x) = 2x - \frac{8x^3}{3} + \frac{32x^5}{5} - \frac{128x^7}{7} + \frac{512x^9}{9} - \cdots$$

For the second, use the common series

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \cdots$$

$$\sin x = x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{5{,}040} + \frac{x^9}{362{,}880} - \cdots$$

Multiply the series together.

$$\arctan(2x)\sin x = \left(2x - \frac{8x^3}{3} + \frac{32x^5}{5} - \frac{128x^7}{7} + \frac{512x^9}{9} - \cdots\right)$$

$$\left(x-\frac{x^3}{6}+\frac{x^5}{120}-\frac{x^7}{5,040}+\frac{x^9}{362,880}-\cdots\right)$$

Multiply every term in the first series by every term in the second series.

$$\arctan(2x)\sin x=x\left(2x-\frac{8x^3}{3}+\frac{32x^5}{5}-\frac{128x^7}{7}+\frac{512x^9}{9}-\cdots\right)$$

$$-\frac{x^3}{6}\left(2x-\frac{8x^3}{3}+\frac{32x^5}{5}-\frac{128x^7}{7}+\frac{512x^9}{9}-\cdots\right)$$

$$+\frac{x^5}{120}\left(2x-\frac{8x^3}{3}+\frac{32x^5}{5}-\frac{128x^7}{7}+\frac{512x^9}{9}-\cdots\right)$$

$$-\frac{x^7}{5,040}\left(2x-\frac{8x^3}{3}+\frac{32x^5}{5}-\frac{128x^7}{7}+\frac{512x^9}{9}-\cdots\right)$$

$$+\frac{x^9}{362,880}\left(2x-\frac{8x^3}{3}+\frac{32x^5}{5}-\frac{128x^7}{7}+\frac{512x^9}{9}-\cdots\right)$$

$$-\cdots$$

$$\arctan(2x)\sin x=2x^2-\frac{8x^4}{3}+\frac{32x^6}{5}-\frac{128x^8}{7}+\frac{512x^{10}}{9}+\cdots$$

$$-\frac{x^4}{3}+\frac{4x^6}{9}-\frac{16x^8}{15}+\frac{128x^{10}}{42}-\cdots$$

$$+\frac{x^6}{60}-\frac{x^8}{45}+\frac{4x^{10}}{75}-\cdots$$

$$-\frac{x^8}{2{,}520}+\frac{8x^{10}}{15{,}120}-\cdots$$

$$+\frac{x^{10}}{181{,}440}-\cdots$$

To get the first four non-zero terms, we only need terms through x^8.

$$\arctan(2x)\sin x=2x^2-\frac{8x^4}{3}-\frac{x^4}{3}+\frac{32x^6}{5}+\frac{4x^6}{9}+\frac{x^6}{60}$$

$$-\frac{128x^8}{7}-\frac{16x^8}{15}-\frac{x^8}{45}-\frac{x^8}{2{,}520}+\cdots$$

$$\arctan(2x)\sin x=2x^2-3x^4+\frac{247x^6}{36}-\frac{155x^8}{8}+\cdots$$

■ 3. Use power series multiplication to find the first four non-zero terms of the Maclaurin series.

$$y=e^{-2x}\cos(2x)$$

Solution:

The given series is the product of two other series.

$$y=e^{-2x}$$

$$y=\cos(2x)$$

For the first one, start with the common series

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots$$

Substitute $-2x$ for x.

$$e^{-2x} = 1 + \frac{-2x}{1!} + \frac{(-2x)^2}{2!} + \frac{(-2x)^3}{3!} + \frac{(-2x)^4}{4!} + \frac{(-2x)^5}{5!} + \cdots$$

$$e^{-2x} = 1 - 2x + 2x^2 - \frac{4x^3}{3} + \frac{2x^4}{3} - \frac{4x^5}{15} + \cdots$$

For the second, use the common series

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots$$

$$\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} + \frac{x^8}{40{,}320} - \cdots$$

Substitute $2x$ for x.

$$\cos(2x) = 1 - \frac{(2x)^2}{2} + \frac{(2x)^4}{24} - \frac{(2x)^6}{720} + \frac{(2x)^8}{40{,}320} - \cdots$$

$$\cos(2x) = 1 - \frac{4x^2}{2} + \frac{16x^4}{24} - \frac{64x^6}{720} + \frac{256x^8}{40{,}320} - \cdots$$

$$\cos(2x) = 1 - 2x^2 + \frac{2x^4}{3} - \frac{4x^6}{45} + \frac{2x^8}{315} - \cdots$$

Multiply the series together.

$$e^{-2x}\cos(2x)=\left(1-2x+2x^2-\frac{4x^3}{3}+\frac{2x^4}{3}-\frac{4x^5}{15}+\cdots\right)$$

$$\left(1-2x^2+\frac{2x^4}{3}-\frac{4x^6}{45}+\frac{2x^8}{315}-\cdots\right)$$

Multiply every term in the first series by every term in the second series.

$$e^{-2x}\cos(2x)=1\left(1-2x^2+\frac{2x^4}{3}-\frac{4x^6}{45}+\frac{2x^8}{315}-\cdots\right)$$

$$-2x\left(1-2x^2+\frac{2x^4}{3}-\frac{4x^6}{45}+\frac{2x^8}{315}-\cdots\right)$$

$$+2x^2\left(1-2x^2+\frac{2x^4}{3}-\frac{4x^6}{45}+\frac{2x^8}{315}-\cdots\right)$$

$$-\frac{4x^3}{3}\left(1-2x^2+\frac{2x^4}{3}-\frac{4x^6}{45}+\frac{2x^8}{315}-\cdots\right)$$

$$+\frac{2x^4}{3}\left(1-2x^2+\frac{2x^4}{3}-\frac{4x^6}{45}+\frac{2x^8}{315}-\cdots\right)$$

$$-\frac{4x^5}{15}\left(1-2x^2+\frac{2x^4}{3}-\frac{4x^6}{45}+\frac{2x^8}{315}-\cdots\right)$$

$$+\cdots$$

$$e^{-2x}\cos(2x) = 1 - 2x^2 + \frac{2x^4}{3} - \frac{4x^6}{45} + \frac{2x^8}{315} - \cdots$$

$$-2x + 4x^3 - \frac{4x^5}{3} + \frac{8x^7}{45} - \frac{4x^9}{315} + \cdots$$

$$+2x^2 - 4x^4 + \frac{4x^6}{3} - \frac{8x^8}{45} + \frac{4x^{10}}{315} - \cdots$$

$$-\frac{4x^3}{3} + \frac{8x^5}{3} - \frac{8x^7}{9} + \frac{16x^9}{135} - \frac{8x^{11}}{945} + \cdots$$

$$+\frac{2x^4}{3} - \frac{4x^6}{3} + \frac{4x^8}{9} - \frac{8x^{10}}{135} + \frac{4x^{12}}{945} - \cdots$$

$$-\frac{4x^5}{15} + \frac{8x^7}{15} - \frac{8x^9}{45} + \frac{16x^{11}}{675} - \frac{8x^{13}}{4,725} + \cdots$$

To get the first four non-zero terms, we only need terms through x^4.

$$e^{-2x}\cos(2x) = 1 - 2x - 2x^2 + 2x^2 + 4x^3 - \frac{4x^3}{3} + \frac{2x^4}{3} - 4x^4 + \frac{2x^4}{3}$$

$$e^{-2x}\cos(2x) = 1 - 2x + \frac{8x^3}{3} - \frac{8x^4}{3}$$

■ 4. Use power series multiplication to find the first four non-zero terms of the Maclaurin series.

$$y = e^{5x}\ln(1 + 3x)$$

Solution:

The given series is the product of two other series.

$$y = e^{5x}$$

$$y = \ln(1 + 3x)$$

For the first one, start with the common series

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots$$

Substitute $5x$ for x.

$$e^{(5x)} = 1 + \frac{5x}{1!} + \frac{(5x)^2}{2!} + \frac{(5x)^3}{3!} + \frac{(5x)^4}{4!} + \frac{(5x)^5}{5!} + \cdots$$

$$e^{(5x)} = 1 + 5x + \frac{25x^2}{2} + \frac{125x^3}{6} + \frac{625x^4}{24} + \frac{3{,}125x^5}{120} + \cdots$$

For the second, use the common series

$$\ln(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \frac{x^6}{6} + \frac{x^7}{7} - \cdots$$

Substitute $3x$ for x.

$$\ln(1 + 3x) = 3x - \frac{(3x)^2}{2} + \frac{(3x)^3}{3} - \frac{(3x)^4}{4} + \frac{(3x)^5}{5} - \cdots$$

$$\ln(1 + 3x) = 3x - \frac{9x^2}{2} + \frac{27x^3}{3} - \frac{81x^4}{4} + \frac{273x^5}{5} - \cdots$$

Multiply the series together.

$$e^{5x}\ln(1+3x)=\left(1+5x+\frac{25x^2}{2}+\frac{125x^3}{6}+\frac{625x^4}{24}+\frac{3,125x^5}{120}+\cdots\right)$$

$$\left(3x-\frac{9x^2}{2}+\frac{27x^3}{3}-\frac{81x^4}{4}+\frac{273x^5}{5}-\cdots\right)$$

Multiply every term in the first series by every term in the second series.

$$e^{5x}\ln(1+3x)=1\left(3x-\frac{9x^2}{2}+\frac{27x^3}{3}-\frac{81x^4}{4}+\frac{273x^5}{5}-\cdots\right)$$

$$+5x\left(3x-\frac{9x^2}{2}+\frac{27x^3}{3}-\frac{81x^4}{4}+\frac{273x^5}{5}-\cdots\right)$$

$$+\frac{25x^2}{2}\left(3x-\frac{9x^2}{2}+\frac{27x^3}{3}-\frac{81x^4}{4}+\frac{273x^5}{5}-\cdots\right)$$

$$+\frac{125x^3}{6}\left(3x-\frac{9x^2}{2}+\frac{27x^3}{3}-\frac{81x^4}{4}+\frac{273x^5}{5}-\cdots\right)$$

$$+\frac{625x^4}{24}\left(3x-\frac{9x^2}{2}+\frac{27x^3}{3}-\frac{81x^4}{4}+\frac{273x^5}{5}-\cdots\right)$$

$$+\frac{3,125x^5}{120}\left(3x-\frac{9x^2}{2}+\frac{27x^3}{3}-\frac{81x^4}{4}+\frac{273x^5}{5}-\cdots\right)$$

$$e^{5x}\ln(1+3x)=3x-\frac{9x^2}{2}+\frac{27x^3}{3}-\frac{81x^4}{4}+\frac{273x^5}{5}-\cdots$$

$$+15x^2 - \frac{45x^3}{2} + \frac{135x^4}{3} - \frac{405x^5}{4} + \frac{1,365x^6}{5} - \cdots$$

$$+\frac{75x^3}{2} - \frac{225x^4}{4} + \frac{675x^5}{6} - \frac{2,025x^6}{8} + \frac{6,825x^7}{10} - \cdots$$

$$+\frac{125x^4}{2} - \frac{375x^5}{4} + \frac{375x^6}{2} - \frac{3,375x^7}{8} + \frac{11,375x^8}{10} - \cdots$$

$$+\frac{625x^5}{8} - \frac{1,875x^6}{16} + \frac{1,875x^7}{8} - \frac{16,875x^8}{32} + \frac{56,875x^9}{40} - \cdots$$

$$+\frac{3,125x^6}{40} - \frac{9,375x^7}{80} + \frac{9,375x^8}{40} - \frac{84,375x^9}{160} + \frac{284,375x^{10}}{200} - \cdots$$

To get the first four non-zero terms, we only need terms through x^4.

$$e^{5x}\ln(1+3x) = 3x - \frac{9x^2}{2} + 15x^2 + \frac{27x^3}{3} - \frac{45x^3}{2} + \frac{75x^3}{2} - \frac{81x^4}{4} + \frac{135x^4}{3} - \frac{225x^4}{4} + \frac{125x^4}{2}$$

$$e^{5x}\ln(1+3x) = 3x + \frac{21x^2}{2} + 24x^3 + 31x^4$$

■ 5. Use power series multiplication to find the first four non-zero terms of the Maclaurin series.

$$y = e^{3x} \cdot \frac{3}{1-x}$$

Solution:

The given series is the product of two other series.

$$y = e^{3x}$$

$$y = \frac{3}{1-x}$$

For the first one, start with the common series

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots$$

Substitute $3x$ for x.

$$e^{3x} = 1 + \frac{3x}{1!} + \frac{(3x)^2}{2!} + \frac{(3x)^3}{3!} + \frac{(3x)^4}{4!} + \frac{(3x)^5}{5!} + \cdots$$

$$e^{3x} = 1 + 3x + \frac{9x^2}{2} + \frac{9x^3}{2} + \frac{27x^4}{8} + \frac{81x^5}{40} + \cdots$$

For the second, use the common series

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + x^5 + x^6 + \cdots$$

Multiply through by 3.

$$\frac{3}{1-x} = 3 + 3x + 3x^2 + 3x^3 + 3x^4 + 3x^5 + 3x^6 + \cdots$$

Multiply the series together.

$$e^{3x} \cdot \frac{3}{1-x} = \left(1 + 3x + \frac{9x^2}{2} + \frac{9x^3}{2} + \frac{27x^4}{8} + \frac{81x^5}{40} + \cdots\right)$$

$$(3 + 3x + 3x^2 + 3x^3 + 3x^4 + 3x^5 + 3x^6 + \cdots)$$

Multiply every term in the first series by every term in the second series.

$$e^{3x} \cdot \frac{3}{1-x} = 1(3 + 3x + 3x^2 + 3x^3 + 3x^4 + 3x^5 + 3x^6 + \cdots)$$

$$+3x(3 + 3x + 3x^2 + 3x^3 + 3x^4 + 3x^5 + 3x^6 + \cdots)$$

$$+\frac{9x^2}{2}(3 + 3x + 3x^2 + 3x^3 + 3x^4 + 3x^5 + 3x^6 + \cdots)$$

$$+\frac{9x^3}{2}(3 + 3x + 3x^2 + 3x^3 + 3x^4 + 3x^5 + 3x^6 + \cdots)$$

$$+\frac{27x^4}{8}(3 + 3x + 3x^2 + 3x^3 + 3x^4 + 3x^5 + 3x^6 + \cdots)$$

$$+\frac{81x^5}{40}(3 + 3x + 3x^2 + 3x^3 + 3x^4 + 3x^5 + 3x^6 + \cdots)$$

$$e^{3x} \cdot \frac{3}{1-x} = 3 + 3x + 3x^2 + 3x^3 + 3x^4 + 3x^5 + 3x^6 + \cdots$$

$$+9x + 9x^2 + 9x^3 + 9x^4 + 9x^5 + 9x^6 + 9x^7 + \cdots$$

$$+\frac{27x^2}{2} + \frac{27x^3}{2} + \frac{27x^4}{2} + \frac{27x^5}{2} + \frac{27x^6}{2} + \frac{27x^7}{2} + \frac{27x^8}{2} + \cdots$$

$$+\frac{27x^3}{2} + \frac{27x^4}{2} + \frac{27x^5}{2} + \frac{27x^6}{2} + \frac{27x^7}{2} + \frac{27x^8}{2} + \frac{27x^9}{2} + \cdots$$

$$+\frac{81x^4}{8} + \frac{81x^5}{8} + \frac{81x^6}{8} + \frac{81x^7}{8} + \frac{81x^8}{8} + \frac{81x^9}{8} + \frac{81x^{10}}{8} + \cdots$$

$$+\frac{243x^5}{40}+\frac{243x^6}{40}+\frac{243x^7}{40}+\frac{243x^8}{40}+\frac{243x^9}{40}+\frac{243x^{10}}{40}+\frac{243x^{11}}{40}+\cdots$$

To get the first four non-zero terms, we only need terms through x^3.

$$e^{3x}\cdot\frac{3}{1-x}=3+3x+9x+3x^2+9x^2+\frac{27x^2}{2}+3x^3+9x^3+\frac{27x^3}{2}+\frac{27x^3}{2}$$

$$e^{3x}\cdot\frac{3}{1-x}=3+12x+\frac{51x^2}{2}+39x^3$$

POWER SERIES DIVISION

1. Use power series division to find the first four non-zero terms of the Maclaurin series.

$$y = \frac{e^{3x}}{x^2}$$

Solution:

Start with the common series

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

Substitute $3x$ for x.

$$e^{3x} = 1 + \frac{3x}{1!} + \frac{(3x)^2}{2!} + \frac{(3x)^3}{3!} + \dots$$

$$e^{3x} = 1 + \frac{3x}{1!} + \frac{9x^2}{2!} + \frac{27x^3}{3!} + \dots$$

Divide through by x^2.

$$\frac{e^{3x}}{x^2} = \frac{1}{x^2} + \frac{3x}{1!x^2} + \frac{9x^2}{2!x^2} + \frac{27x^3}{3!x^2} + \dots$$

$$\frac{e^{3x}}{x^2} = \frac{1}{x^2} + \frac{3}{x} + \frac{9}{2} + \frac{9x}{2} + \dots$$

2. Use power series division to find the first four non-zero terms of the Maclaurin series.

$$y = \frac{6x}{\ln(1+6x)}$$

Solution:

Start with the common series

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \frac{x^6}{6} + \frac{x^7}{7} - \cdots$$

Substitute $6x$ for x.

$$\ln(1+6x) = 6x - \frac{(6x)^2}{2} + \frac{(6x)^3}{3} - \frac{(6x)^4}{4} + \frac{(6x)^5}{5} - \frac{(6x)^6}{6} + \frac{(6x)^7}{7} - \cdots$$

$$\ln(1+6x) = 6x - \frac{36x^2}{2} + \frac{216x^3}{3} - \frac{1{,}296x^4}{4} + \frac{7{,}776x^5}{5} - \frac{46{,}656x^6}{6} + \cdots$$

$$\ln(1+6x) = 6x - 18x^2 + 72x^3 - 324x^4 + 1{,}552.2x^5 - 7{,}776x^6 + \cdots$$

Divide through by $6x$.

$$\frac{6x}{\ln(1+6x)} = \frac{6x}{6x} - \frac{6x}{18x^2} + \frac{6x}{72x^3} - \frac{6x}{324x^4} + \frac{60x}{15{,}522x^5} - \frac{6x}{7{,}776x^6} + \cdots$$

$$\frac{6x}{\ln(1+6x)} = 1 - \frac{1}{3x} + \frac{1}{12x^2} - \frac{1}{54x^3} + \frac{10}{2587x^4} - \frac{1}{1{,}296x^5} + \cdots$$

To get the first four non-zero terms, we only need

$$\frac{6x}{\ln(1+6x)} = 1 - \frac{1}{3x} + \frac{1}{12x^2} - \frac{1}{54x^3}$$

3. Use power series division to find the first four non-zero terms of the Maclaurin series.

$$y = \frac{\cos(2x)}{2x^3}$$

Solution:

Start with the common series

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots$$

Substitute $2x$ for x.

$$\cos(2x) = 1 - \frac{(2x)^2}{2!} + \frac{(2x)^4}{4!} - \frac{(2x)^6}{6!} + \frac{(2x)^8}{8!} - \cdots$$

$$\cos(2x) = 1 - \frac{4x^2}{2!} + \frac{16x^4}{4!} - \frac{64x^6}{6!} + \frac{256x^8}{8!} - \cdots$$

Divide through by $2x^3$.

$$\frac{\cos(2x)}{2x^3} = \frac{1}{2x^3} - \frac{4x^2}{2!2x^3} + \frac{16x^4}{4!2x^3} - \frac{64x^6}{6!2x^3} + \frac{256x^8}{8!2x^3} - \ldots$$

$$\frac{\cos(2x)}{2x^3} = \frac{1}{2x^3} - \frac{4x^2}{4x^3} + \frac{16x^4}{48x^3} - \frac{64x^6}{1{,}440x^3} + \frac{256x^8}{80{,}640x^3} - \dots$$

$$\frac{\cos(2x)}{2x^3} = \frac{1}{2x^3} - \frac{1}{x} + \frac{x}{3} - \frac{2x^3}{45} + \frac{x^5}{315} - \dots$$

To get the first four non-zero terms, we only need

$$\frac{\cos(2x)}{2x^3} = \frac{1}{2x^3} - \frac{1}{x} + \frac{x}{3} - \frac{2x^3}{45}$$

■ 4. Use power series division to find the first four non-zero terms of the Maclaurin series.

$$y = \frac{\sin(3x)}{3x^2}$$

Solution:

Start with the common series

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \cdots$$

Substitute $3x$ for x.

$$\sin(3x) = 3x - \frac{(3x)^3}{3!} + \frac{(3x)^5}{5!} - \frac{(3x)^7}{7!} + \frac{(3x)^9}{9!} - \cdots$$

$$\sin(3x) = 3x - \frac{27x^3}{6} + \frac{243x^5}{120} - \frac{2{,}187x^7}{5{,}040} + \frac{19{,}683x^9}{362{,}880} - \cdots$$

$$\sin(3x)=3x-\frac{9x^3}{2}+\frac{81x^5}{40}-\frac{243x^7}{560}+\frac{243x^9}{4{,}480}-\cdots$$

Divide this series by $3x^2$.

$$\frac{\sin(3x)}{3x^2}=\frac{3x}{3x^2}-\frac{9x^3}{2(3x^2)}+\frac{81x^5}{40(3x^2)}-\frac{243x^7}{560(3x^2)}+\frac{243x^9}{4{,}480(3x^2)}-\cdots$$

$$\frac{\sin(3x)}{3x^2}=x^{-1}-\frac{3}{2}x+\frac{27}{40}x^3-\frac{81}{560}x^5+\frac{81}{4{,}480}x^7-\cdots$$

To get the first four non-zero terms, we only need

$$\frac{\sin(3x)}{3x^2}=x^{-1}-\frac{3}{2}x+\frac{27}{40}x^3-\frac{81}{560}x^5+\ldots$$

■ 5. Use power series division to find the first four non-zero terms of the Maclaurin series.

$$y=\frac{\arctan(4x)}{4x^2}$$

Solution:

Start with the common series

$$\arctan x=x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+\frac{x^9}{9}-\cdots$$

Substitute $4x$ for x.

$$\arctan(4x)=4x-\frac{(4x)^3}{3}+\frac{(4x)^5}{5}-\frac{(4x)^7}{7}+\frac{(4x)^9}{9}-\cdots$$

$$\arctan(4x)=4x-\frac{64x^3}{3}+\frac{1{,}024x^5}{5}-\frac{16{,}384x^7}{7}+\frac{262{,}144x^9}{9}-\cdots$$

Divide through by $4x^2$.

$$\frac{\arctan(4x)}{4x^2}=\frac{4x}{4x^2}-\frac{64x^3}{3(4x^2)}+\frac{1{,}024x^5}{5(4x^2)}-\frac{16{,}384x^7}{7(4x^2)}+\frac{262{,}144x^9}{9(4x^2)}-\cdots$$

$$\frac{\arctan(4x)}{4x^2}=\frac{1}{x}-\frac{16x}{3}+\frac{256x^3}{5}-\frac{4{,}096x^5}{7}+\frac{65{,}536x^7}{9}-\cdots$$

To get the first four non-zero terms, we only need

$$\frac{\arctan(4x)}{4x^2}=\frac{1}{x}-\frac{16x}{3}+\frac{256x^3}{5}-\frac{4{,}096x^5}{7}+\cdots$$

POWER SERIES DIFFERENTIATION

1. Differentiate to find the power series representation of the function.

$$f(x) = \frac{5}{(3-x)^2}$$

Solution:

Integrate the given function using u-substitution.

$$\int \frac{5}{(3-x)^2}\ dx$$

$$u = 3 - x$$

$$du = -dx$$

$$dx = -du$$

$$\int \frac{5}{(3-x)^2}\ dx = \int \frac{5}{u^2}(-du) = \int -5u^{-2}\ du$$

$$\frac{-5u^{-1}}{-1} + C = \frac{5}{u} + C = \frac{5}{3-x} + C$$

Starting with the standard form of a power series,

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + x^5 + \cdots = \sum_{n=0}^{\infty} x^n$$

find the series that represents the integrated function.

$$\frac{1}{1-x}=\frac{5}{3-x}$$

$$\frac{1}{1-x}=(5)\frac{1}{3-x}$$

$$\frac{1}{1-x}=(5)\frac{1}{3\left(1-\frac{x}{3}\right)}$$

$$\frac{1}{1-x}=\left(\frac{5}{3}\right)\frac{1}{1-\frac{x}{3}}$$

The power series representation of this is

$$\frac{5}{3}\sum_{n=0}^{\infty}\left(\frac{x}{3}\right)^n=\sum_{n=0}^{\infty}\frac{5}{3}\left(\frac{x}{3}\right)^n=\sum_{n=0}^{\infty}\frac{5^1\cdot x^n}{3^1\cdot 3^n}=\sum_{n=0}^{\infty}\frac{5x^n}{3^{n+1}}$$

So the integrated function can be written as

$$\frac{5}{3-x}=\frac{5}{3}+\frac{5}{3}\left(\frac{x}{3}\right)+\frac{5}{3}\left(\frac{x}{3}\right)^2+\frac{5}{3}\left(\frac{x}{3}\right)^3+\frac{5}{3}\left(\frac{x}{3}\right)^4+\frac{5}{3}\left(\frac{x}{3}\right)^5+\cdots=\sum_{n=0}^{\infty}\frac{5x^n}{3^{n+1}}$$

$$\frac{5}{3-x}=\frac{5}{3}+\frac{5x}{9}+\frac{5x^2}{27}+\frac{5x^3}{81}+\frac{5x^4}{243}+\frac{5x^5}{729}+\cdots=\sum_{n=0}^{\infty}\frac{5x^n}{3^{n+1}}$$

Now differentiate this entire equation, include the function on the left, the terms in the middle, and the series on the right.

$$\frac{(3-x)(0)-5(-1)}{(3-x)^2}=\frac{5}{9}+\frac{10x}{27}+\frac{15x^2}{81}+\frac{20x^3}{243}+\frac{25x^4}{729}+\cdots=\sum_{n=0}^{\infty}\frac{5nx^{n-1}}{3^{n+1}}$$

$$\frac{5}{(3-x)^2}=\frac{5}{9}+\frac{10x}{27}+\frac{15x^2}{81}+\frac{20x^3}{243}+\frac{25x^4}{729}+\cdots=\sum_{n=0}^{\infty}\frac{5nx^{n-1}}{3^{n+1}}$$

So the power series representation of the original function is

$$\frac{5}{(3-x)^2}=\sum_{n=0}^{\infty}\frac{5nx^{n-1}}{3^{n+1}}$$

When $n=0$,

$$\frac{5(0)x^{0-1}}{3^{0+1}}=\frac{0}{3x}=0$$

so we can start the index at $n=1$ instead of $n=0$, without changing the value of the series.

$$\frac{5}{(3-x)^2}=\sum_{n=1}^{\infty}\frac{5nx^{n-1}}{3^{n+1}}$$

■ 2. Differentiate to find the power series representation of the function.

$$f(x)=\frac{3}{(4+x)^2}$$

Solution:

Integrate the given function using u-substitution.

$$\int \frac{3}{(4+x)^2}\ dx$$

$$u = 4 + x$$

$$du = dx$$

$$dx = du$$

$$\int \frac{3}{(4+x)^2}\ dx = \int \frac{3}{u^2}\ du = \int 3u^{-2}\ du$$

$$\frac{3u^{-1}}{-1} + C = -\frac{3}{u} + C = -\frac{3}{4+x} + C$$

Starting with the standard form of a power series,

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + x^5 + \ \cdots = \sum_{n=0}^{\infty} x^n$$

find the series that represents the integrated function.

$$\frac{1}{1-x} = -\frac{3}{4+x}$$

$$\frac{1}{1-x} = -\frac{3}{4-(-x)}$$

$$\frac{1}{1-x} = (-3)\frac{1}{4\left(1-\left(-\frac{x}{4}\right)\right)}$$

$$\frac{1}{1-x} = (-3)\frac{1}{4\left(1-\left(-\frac{x}{4}\right)\right)}$$

$$\frac{1}{1-x} = \left(-\frac{3}{4}\right)\frac{1}{1-\left(-\frac{x}{4}\right)}$$

The power series representation of this is

$$-\frac{3}{4}\sum_{n=0}^{\infty}\left(-\frac{x}{4}\right)^n = \sum_{n=0}^{\infty}-\frac{3}{4}\left(-\frac{x}{4}\right)^n = \sum_{n=0}^{\infty}\frac{-3^1\cdot(-1)^n x^n}{4^1\cdot 4^n} = \sum_{n=0}^{\infty}\frac{(-1)^{n+1}3x^n}{4^{n+1}}$$

So the original function can be written as

$$-\frac{3}{4+x} = -\frac{3}{4}\left(1+\left(-\frac{x}{4}\right)+\left(-\frac{x}{4}\right)^2+\left(-\frac{x}{4}\right)^3+\left(-\frac{x}{4}\right)^4+\left(-\frac{x}{4}\right)^5+\cdots\right)$$

$$= \sum_{n=0}^{\infty}\frac{(-1)^{n+1}3x^n}{4^{n+1}}$$

$$-\frac{3}{4+x} = -\frac{3}{4}\left(1-\frac{x}{4}+\frac{x^2}{16}-\frac{x^3}{64}+\frac{x^4}{256}-\frac{x^5}{1{,}024}+\cdots\right)$$

$$= \sum_{n=0}^{\infty}\frac{(-1)^{n+1}3x^n}{4^{n+1}}$$

$$-\frac{3}{4+x} = \left(-\frac{3}{4}\right)-\frac{x}{4}\left(-\frac{3}{4}\right)+\frac{x^2}{16}\left(-\frac{3}{4}\right)-\frac{x^3}{64}\left(-\frac{3}{4}\right)+\frac{x^4}{256}\left(-\frac{3}{4}\right)-\frac{x^5}{1{,}024}\left(-\frac{3}{4}\right)+\cdots$$

$$= \sum_{n=0}^{\infty}\frac{(-1)^{n+1}3x^n}{4^{n+1}}$$

$$-\frac{3}{4+x} = -\frac{3}{4}+\frac{3x}{16}-\frac{3x^2}{64}+\frac{3x^3}{256}-\frac{3x^4}{1{,}024}+\frac{3x^5}{4{,}096}-\cdots$$

$$=\sum_{n=0}^{\infty}\frac{(-1)^{n+1}3x^n}{4^{n+1}}$$

Now differentiate this entire equation, include the function on the left, the terms in the middle, and the series on the right.

$$-\frac{(4+x)(0)-3(1)}{(4+x)^2}=\frac{3}{16}-\frac{6x}{64}+\frac{9x^2}{256}-\frac{12x^3}{1,024}+\frac{15x^4}{4,096}-\cdots=\sum_{n=0}^{\infty}\frac{(-1)^{n+1}3nx^{n-1}}{4^{n+1}}$$

$$\frac{3}{(4+x)^2}=\frac{3}{16}-\frac{6x}{64}+\frac{9x^2}{256}-\frac{12x^3}{1,024}+\frac{15x^4}{4,096}-\cdots=\sum_{n=0}^{\infty}\frac{(-1)^{n+1}3nx^{n-1}}{4^{n+1}}$$

So the power series representation of the original function is

$$\frac{3}{(4+x)^2}=\sum_{n=0}^{\infty}\frac{(-1)^{n+1}3nx^{n-1}}{4^{n+1}}$$

$$\frac{3}{(4+x)^2}=\sum_{n+1=0}^{\infty}\frac{(-1)^{n+1+1}3(n+1)x^{n+1-1}}{4^{n+1+1}}$$

$$\frac{3}{(4+x)^2}=\sum_{n=-1}^{\infty}\frac{(-1)^{n+2}3(n+1)x^n}{4^{n+2}}$$

If we plug in $n=-1$, the start of the new index, we get a zero value because of the $n+1$ factor in the numerator. Which means we can start the index at $n=0$ instead.

$$\frac{3}{(4+x)^2}=\sum_{n=0}^{\infty}\frac{(-1)^{n+2}3(n+1)x^n}{4^{n+2}}$$

3. Differentiate to find the power series representation of the function.

$$f(x) = \frac{1}{(-5-x)^2}$$

Solution:

Integrate the given function using u-substitution.

$$\int \frac{1}{(-5-x)^2}\ dx$$

$$u = -5 - x$$

$$du = -dx$$

$$dx = -du$$

$$\int \frac{1}{(-5-x)^2}\ dx = \int \frac{1}{u^2}(-du) = -\int u^{-2}\ du$$

$$-\frac{u^{-1}}{-1} + C = \frac{1}{u} + C = \frac{1}{-5-x} + C$$

Starting with the standard form of a power series,

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + x^5 + \ \cdots = \sum_{n=0}^{\infty} x^n$$

find the series that represents the integrated function.

$$\frac{1}{1-x}=\frac{1}{-5-x}$$

$$\frac{1}{1-x}=-\frac{3}{1-(6+x)}$$

The power series representation of this is

$$\sum_{n=0}^{\infty}(x+6)^n$$

So the original function can be written as

$$\frac{1}{1-(x+6)}=1+(x+6)+(x+6)^2+(x+6)^3+(x+6)^4+(x+6)^5+\ \cdots=\sum_{n=0}^{\infty}(x+6)^n$$

$$\frac{1}{-5-x}=1+(x+6)+(x+6)^2+(x+6)^3+(x+6)^4+(x+6)^5+\ \cdots=\sum_{n=0}^{\infty}(x+6)^n$$

Now differentiate this entire equation, include the function on the left, the terms in the middle, and the series on the right.

$$\frac{(-5-x)(0)-1(-1)}{(-5-x)^2}=1+2(x+6)+3(x+6)^2+4(x+6)^3+5(x+6)^4+\ \cdots=\sum_{n=0}^{\infty}n(x+6)^{n-1}$$

$$\frac{1}{(-5-x)^2}=1+2(x+6)+3(x+6)^2+4(x+6)^3+5(x+6)^4+\ \cdots=\sum_{n=0}^{\infty}n(x+6)^{n-1}$$

So the power series representation of the original function is

$$\frac{1}{(-5-x)^2}=\sum_{n=0}^{\infty}n(x+6)^{n-1}$$

$$\frac{1}{(-5-x)^2}=\sum_{n+1=0}^{\infty}(n+1)(x+6)^{n+1-1}$$

$$\frac{1}{(-5-x)^2}=\sum_{n=-1}^{\infty}(n+1)(x+6)^{n}$$

■ 4. Differentiate to find the power series representation of the function.

$$f(x)=\frac{3}{(6-3x)^2}$$

Solution:

Integrate the given function using u-substitution.

$$\int \frac{3}{(6-3x)^2}\ dx$$

$$u=6-3x$$

$$du=-3dx$$

$$dx=-\frac{du}{3}$$

$$\int \frac{3}{(6-3x)^2}\ dx=\int \frac{3}{u^2}\left(-\frac{du}{3}\right)=-\int u^{-2}\ du$$

$$-\frac{u^{-1}}{-1}+C=\frac{1}{u}+C=\frac{1}{6-3x}+C$$

Starting with the standard form of a power series,

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + x^5 + \ \cdots = \sum_{n=0}^{\infty} x^n$$

find the series that represents the integrated function.

$$\frac{1}{1-x} = \frac{1}{6-3x}$$

$$\frac{1}{1-x} = \frac{1}{6\left(1-\frac{x}{2}\right)}$$

$$\frac{1}{1-x} = \frac{1}{6} \cdot \frac{1}{1-\frac{x}{2}}$$

The power series representation of this is

$$\frac{1}{6}\sum_{n=0}^{\infty}\left(\frac{x}{2}\right)^n = \sum_{n=0}^{\infty}\frac{1}{6}\left(\frac{x}{2}\right)^n$$

So the original function can be written as

$$\frac{1}{6-3x} = \frac{1}{6}\left(1 + \frac{x}{2} + \left(\frac{x}{2}\right)^2 + \left(\frac{x}{2}\right)^3 + \left(\frac{x}{2}\right)^4 + \left(\frac{x}{2}\right)^5 + \dots\right) = \sum_{n=0}^{\infty}\frac{1}{6}\left(\frac{x}{2}\right)^n$$

$$\frac{1}{6-3x} = \frac{1}{6}\left(1 + \frac{x}{2} + \frac{x^2}{4} + \frac{x^3}{8} + \frac{x^4}{16} + \frac{x^5}{32} + \dots\right) = \sum_{n=0}^{\infty}\frac{1}{6}\left(\frac{x}{2}\right)^n$$

$$\frac{1}{6-3x} = \frac{1}{6} + \frac{x}{12} + \frac{x^2}{24} + \frac{x^3}{48} + \frac{x^4}{96} + \frac{x^5}{192} + \dots = \sum_{n=0}^{\infty}\frac{1}{6}\left(\frac{x}{2}\right)^n$$

Now differentiate this entire equation, including the function on the left, the terms in the middle, and the series on the right.

$$\frac{(0)(6-3x)-(1)(-3)}{(6-3x)^2}=\frac{1}{12}+\frac{x}{12}+\frac{x^2}{16}+\frac{x^3}{24}+\frac{5x^4}{192}+\ldots=\sum_{n=0}^{\infty}\frac{1}{6}n\left(\frac{x}{2}\right)^{n-1}\cdot\frac{1}{2}$$

$$\frac{3}{(6-3x)^2}=\frac{1}{12}+\frac{x}{12}+\frac{x^2}{16}+\frac{x^3}{24}+\frac{5x^4}{192}+\ldots=\sum_{n=0}^{\infty}\frac{n}{12}\left(\frac{x}{2}\right)^{n-1}$$

So the power series representation of the original function is

$$\frac{3}{(6-3x)^2}=\sum_{n=0}^{\infty}\frac{n}{12}\left(\frac{x}{2}\right)^{n-1}$$

■ 5. Differentiate to find the power series representation of the function.

$$f(x)=\frac{2}{(1-2x)^2}$$

Solution:

Integrate the given function using u-substitution.

$$\int\frac{2}{(1-2x)^2}\ dx$$

$$u=1-2x$$

$$du = -2dx$$

$$dx = -\frac{du}{2}$$

$$\int \frac{2}{(1-2x)^2}\ dx = \int \frac{2}{u^2}\left(-\frac{du}{2}\right) = -\int u^{-2}\ du$$

$$-\frac{u^{-1}}{-1} + C = \frac{1}{u} + C = \frac{1}{1-2x} + C$$

Starting with the standard form of a power series,

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + x^5 + \ \cdots = \sum_{n=0}^{\infty} x^n$$

find the series that represents the integrated function.

$$\frac{1}{1-x} = \frac{1}{1-2x}$$

The power series representation of this is

$$\sum_{n=0}^{\infty} (2x)^n$$

So the original function can be written as

$$\frac{1}{1-2x} = 1 + (2x) + (2x)^2 + (2x)^3 + (2x)^4 + (2x)^5 + \ \cdots = \sum_{n=0}^{\infty} (2x)^n$$

$$\frac{1}{1-2x} = 1 + 2x + 4x^2 + 8x^3 + 16x^4 + 32x^5 + \ \cdots = \sum_{n=0}^{\infty} (2x)^n$$

Now differentiate this entire equation, include the function on the left, the terms in the middle, and the series on the right.

$$\frac{(1-2x)(0)-1(-2)}{(1-2x)^2} = 2 + 8x + 24x^2 + 64x^3 + 160x^4 + \ \cdots = \sum_{n=0}^{\infty} n(2x)^{n-1} \cdot 2$$

$$\frac{1}{(1-2x)^2} = 2 + 8x + 24x^2 + 64x^3 + 160x^4 + \ \cdots = \sum_{n=0}^{\infty} 2n(2x)^{n-1}$$

So the power series representation of the original function is

$$\frac{2}{(1-2x)^2} = \sum_{n=0}^{\infty} 4n(2x)^{n-1}$$

$$\frac{2}{(1-2x)^2} = \sum_{n+1=0}^{\infty} 4(n+1)(2x)^{n+1-1}$$

$$\frac{2}{(1-2x)^2} = \sum_{n=-1}^{\infty} 4(n+1)(2x)^{n}$$

RADIUS OF CONVERGENCE

1. Find the radius of convergence of the series.

$$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{4 \cdot 2^{2n}}$$

Solution:

Apply the ratio test.

$$L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$$

$$L = \lim_{n\to\infty} \left| \frac{\frac{(-1)^{n+1} x^{2(n+1)}}{4 \cdot 2^{2(n+1)}}}{\frac{(-1)^n x^{2n}}{4 \cdot 2^{2n}}} \right|$$

$$L = \lim_{n\to\infty} \left| \frac{\frac{x^{2(n+1)}}{4 \cdot 2^{2(n+1)}}}{\frac{x^{2n}}{4 \cdot 2^{2n}}} \right|$$

$$L = \lim_{n\to\infty} \left| \frac{x^{2n+2}}{4 \cdot 2^{2n+2}} \cdot \frac{4 \cdot 2^{2n}}{x^{2n}} \right|$$

$$L = \lim_{n\to\infty} \left| \frac{x^2}{2^2} \cdot \frac{1}{1} \right|$$

$$L = \lim_{n \to \infty} \left| \frac{x^2}{2^2} \right|$$

$$L = \frac{x^2}{4}$$

Then the interval of convergence is given by the inequality

$$\frac{x^2}{4} < 1$$

$$x^2 < 4$$

$$-2 < x < 2$$

The interval of convergence spans -2 to 2, which is 4 units wide. The radius of convergence will be half that, so the radius of convergence is 2.

■ 2. Find the radius of convergence of the series.

$$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

Solution:

Apply the ratio test.

$$L = \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right|$$

$$L = \lim_{n\to\infty} \left| \frac{\frac{(-1)^{n+1}x^{2(n+1)+1}}{(2(n+1)+1)!}}{\frac{(-1)^n x^{2n+1}}{(2n+1)!}} \right|$$

$$L = \lim_{n\to\infty} \left| \frac{\frac{x^{2n+3}}{(2n+3)!}}{\frac{x^{2n+1}}{(2n+1)!}} \right|$$

$$L = \lim_{n\to\infty} \left| \frac{x^{2n+3}}{(2n+3)!} \cdot \frac{(2n+1)!}{x^{2n+1}} \right|$$

$$L = \lim_{n\to\infty} \left| \frac{x^2}{(2n+3)(2n+2)(2n+1)!} \cdot \frac{(2n+1)!}{1} \right|$$

$$L = \lim_{n\to\infty} \left| \frac{x^2}{(2n+3)(2n+2)} \cdot \frac{1}{1} \right|$$

$$L = x^2 \lim_{n\to\infty} \left| \frac{1}{(2n+3)(2n+2)} \right|$$

$$L = x^2 \cdot 0$$

$$L = 0$$

The series converges if $L < 1$ and diverges if $L > 1$, which means the series converges everywhere, so the interval of convergence is ∞, and the radius of convergence is, too.

3. Find the radius of convergence of the series.

$$\sum_{n=0}^{\infty} \frac{x^n}{n+4}$$

Solution:

Apply the ratio test.

$$L = \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right|$$

$$L = \lim_{n \to \infty} \left| \frac{\frac{x^{n+1}}{n+1+4}}{\frac{x^n}{n+4}} \right|$$

$$L = \lim_{n \to \infty} \left| \frac{\frac{x^{n+1}}{n+5}}{\frac{x^n}{n+4}} \right|$$

$$L = \lim_{n \to \infty} \left| \frac{x^{n+1}}{n+5} \cdot \frac{n+4}{x^n} \right|$$

$$L = \lim_{n \to \infty} \left| \frac{x}{n+5} \cdot \frac{n+4}{1} \right|$$

$$L = |x| \lim_{n \to \infty} \left| \frac{n+4}{n+5} \right|$$

$$L = |x| \cdot 1$$

$$L = |x|$$

Then the interval of convergence is given by the inequality

$$|x| < 1$$

$$-1 < x < 1$$

The interval of convergence spans -1 to 1, which is 2 units wide. The radius of convergence will be half that, so the radius of convergence is 1.

■ 4. Find the radius of convergence of the series.

$$\sum_{n=0}^{\infty} \frac{3^n(x+2)^n}{n!}$$

Solution:

Apply the ratio test.

$$L = \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right|$$

$$L = \lim_{n \to \infty} \left| \frac{\frac{3^{n+1}(x+2)^{n+1}}{(n+1)!}}{\frac{3^n(x+2)^n}{n!}} \right|$$

$$L = \lim_{n\to\infty} \left| \frac{3^{n+1}(x+2)^{n+1}}{(n+1)!} \cdot \frac{n!}{3^n(x+2)^n} \right|$$

$$L = \lim_{n\to\infty} \left| \frac{3(x+2)}{(n+1)n!} \cdot \frac{n!}{1} \right|$$

$$L = \lim_{n\to\infty} \left| \frac{3(x+2)}{(n+1)} \cdot \frac{1}{1} \right|$$

$$L = \left| 3(x+2) \right| \lim_{n\to\infty} \left| \frac{1}{n+1} \right|$$

$$L = \left| 3(x+2) \right| \cdot 0$$

$$L = 0$$

The series converges if $L < 1$ and diverges if $L > 1$, which means the series converges everywhere, so the interval of convergence is ∞, and the radius of convergence is, too.

5. Find the radius of convergence of the series.

$$\sum_{n=0}^{\infty} \frac{3^n(x+2)^n}{n+1}$$

Solution:

Apply the ratio test.

$$L = \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right|$$

$$L = \lim_{n \to \infty} \left| \frac{\frac{3^{n+1}(x+2)^{n+1}}{n+1+1}}{\frac{3^n(x+2)^n}{n+1}} \right|$$

$$L = \lim_{n \to \infty} \left| \frac{3^{n+1}(x+2)^{n+1}}{n+2} \cdot \frac{n+1}{3^n(x+2)^n} \right|$$

$$L = \lim_{n \to \infty} \left| \frac{3(x+2)}{n+2} \cdot \frac{n+1}{1} \right|$$

$$L = \left| 3(x+2) \right| \lim_{n \to \infty} \left| \frac{n+1}{n+2} \right|$$

$$L = \left| 3(x+2) \right| \cdot 1$$

$$L = \left| 3(x+2) \right|$$

Then the interval of convergence is given by the inequality

$$\left| 3(x+2) \right| < 1$$

$$-1 < 3(x+2) < 1$$

$$-\frac{1}{3} < x + 2 < \frac{1}{3}$$

$$-\frac{7}{3} < x + 2 < -\frac{5}{3}$$

The interval of convergence spans $-7/3$ to $-5/3$, which is $2/3$ units wide. The radius of convergence will be half that, so the radius of convergence is $1/3$.

INTERVAL OF CONVERGENCE

1. Find the interval of convergence of the series.

$$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$$

Solution:

Apply the ratio test.

$$L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$$

$$L = \lim_{n\to\infty} \left| \frac{\frac{(-1)^{n+1} x^{2(n+1)+1}}{2(n+1)+1}}{\frac{(-1)^n x^{2n+1}}{2n+1}} \right|$$

$$L = \lim_{n\to\infty} \left| \frac{\frac{x^{2n+3}}{2n+3}}{\frac{x^{2n+1}}{2n+1}} \right|$$

$$L = \lim_{n\to\infty} \left| \frac{x^{2n+3}}{2n+3} \cdot \frac{2n+1}{x^{2n+1}} \right|$$

$$L = \lim_{n\to\infty} \left| \frac{x^2}{2n+3} \cdot \frac{2n+1}{1} \right|$$

$$L = x^2 \lim_{n\to\infty} \left| \frac{2n+1}{2n+3} \right|$$

$$L = x^2 \cdot 1$$

$$L = x^2$$

Then the interval of convergence is given by the inequality

$$x^2 < 1$$

$$-1 < x < 1$$

Check the endpoints of the interval.

At $x = -1$,

$$\lim_{n\to\infty} \frac{(-1)^n(-1)^{2n+1}}{2n+1} = \lim_{n\to\infty} \frac{(-1)^{3n+1}}{2n+1}$$

This converges by the alternating series test.

At $x = 1$,

$$\lim_{n\to\infty} \frac{(-1)^n(1)^{2n+1}}{2n+1} = \lim_{n\to\infty} \frac{(-1)^n}{2n+1}$$

This converges by the alternating series test.

So the interval of convergence is

$$-1 \le x \le 1$$

2. Find the interval of convergence of the series.

$$\sum_{n=0}^{\infty} \frac{(-1)^n(x-3)^n}{n+1}$$

Solution:

Apply the ratio test.

$$L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$$

$$L = \lim_{n\to\infty} \left| \frac{\frac{(-1)^{n+1}(x-3)^{n+1}}{n+1+1}}{\frac{(-1)^n(x-3)^n}{n+1}} \right|$$

$$L = \lim_{n\to\infty} \left| \frac{\frac{(x-3)^{n+1}}{n+2}}{\frac{(x-3)^n}{n+1}} \right|$$

$$L = \lim_{n\to\infty} \left| \frac{(x-3)^{n+1}}{n+2} \cdot \frac{n+1}{(x-3)^n} \right|$$

$$L = \lim_{n\to\infty} \left| \frac{x-3}{n+2} \cdot \frac{n+1}{1} \right|$$

$$L = |x-3| \lim_{n\to\infty} \left| \frac{n+1}{n+2} \right|$$

$$L = |x-3| \cdot 1$$

$$L = |x-3|$$

Then the interval of convergence is given by the inequality

$$|x-3| < 1$$

$$-1 < x-3 < 1$$

$$2 < x < 4$$

Check the endpoints of the interval.

At $x = 2$,

$$\lim_{n\to\infty} \frac{(-1)^n(2-3)^n}{n+1} = \lim_{n\to\infty} \frac{(-1)^n(-1)^n}{n+1} = \lim_{n\to\infty} \frac{(-1)^{2n}}{n+1} = \lim_{n\to\infty} \frac{1}{n+1}$$

This is a harmonic series that diverges.

At $x = 4$,

$$\lim_{n\to\infty} \frac{(-1)^n(4-3)^n}{n+1} = \lim_{n\to\infty} \frac{(-1)^n(1)^n}{n+1} = \lim_{n\to\infty} \frac{(-1)^n}{n+1}$$

This converges by the alternating series test.

So the interval of convergence is

$$2 < x \le 4$$

ESTIMATING DEFINITE INTEGRALS

1. Evaluate the definite integral as a power series, using the first four terms.

$$\int_0^2 \frac{24}{x^2+4}\,dx$$

Solution:

Rewrite the integral.

$$\int_0^2 \frac{24}{x^2+4}\,dx = 24\int_0^2 \frac{1}{x^2+4}\,dx = 24\int_0^2 \frac{1}{x^2+2^2}\,dx = 24\int_0^2 \frac{\frac{1}{4}}{\left(\frac{x}{2}\right)^2+1}\,dx$$

Integrate, then evaluate over the interval.

$$\frac{24}{2}\arctan\left(\frac{x}{2}\right)\Big|_0^2 = 12\arctan\left(\frac{2}{2}\right) = 12\arctan 1 = 12\left(\frac{\pi}{4}\right) = 3\pi$$

Write the original function in the same format as the common series.

$$\frac{1}{1+x} = \frac{24}{x^2+4}$$

$$\frac{1}{1+x} = (24)\frac{1}{x^2+4}$$

$$\frac{1}{1+x} = (24)\frac{1}{4\left(\left(\frac{x}{2}\right)^2 + 1\right)}$$

$$\frac{1}{1+x} = (6)\frac{1}{\left(\frac{x}{2}\right)^2 + 1}$$

Substitute into the common series.

$$\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n$$

$$\frac{24}{x^2+4} = 6\sum_{n=0}^{\infty} (-1)^n \left(\frac{x}{2}\right)^{2n} = \sum_{n=0}^{\infty} 6(-1)^n \frac{x^{2n}}{2^{2n}}$$

Now integrate this series, then evaluate over the interval.

$$\int_0^2 6(-1)^n \frac{x^{2n}}{2^{2n}}\ dx$$

$$\frac{6(-1)^n}{2^{2n}} \int_0^2 x^{2n}\ dx$$

$$\frac{6(-1)^n}{2^{2n}} \cdot \frac{x^{2n+1}}{2n+1} \Bigg|_0^2$$

$$\frac{6(-1)^n}{2^{2n}} \cdot \frac{2^{2n+1}}{2n+1}$$

$$\frac{6(-1)^n}{2^{2n}} \cdot \frac{0^{2n+1}}{2n+1}$$

$$\frac{6(-1)^n}{2^{2n}} \cdot \frac{2^{2n+1}}{2n+1}$$

$$\frac{6(-1)^n}{1} \cdot \frac{2}{2n+1}$$

$$\frac{12(-1)^n}{2n+1}$$

Then we can set up an equation with the integral and the new series.

$$\int_0^2 \frac{24}{x^2+4}\ dx = \sum_{n=0}^{\infty} \frac{12(-1)^n}{2n+1}$$

$$\int_0^2 \frac{24}{x^2+4}\ dx = \frac{12(-1)^0}{2(0)+1} + \frac{12(-1)^1}{2(1)+1} + \frac{12(-1)^2}{2(2)+1} + \frac{12(-1)^3}{2(3)+1} + \dots$$

$$\int_0^2 \frac{24}{x^2+4}\ dx = 12 - 4 + 2.4 - 1.714 + 1.333$$

Using the first four terms of the series,

$$\int_0^2 \frac{24}{x^2+4}\ dx \approx 12 - 4 + 2.4 - 1.714 \approx 8.686$$

■ 2. Evaluate the definite integral as a power series, using the first four terms.

$$\int_0^1 3x\cos(x^3)\ dx$$

Solution:

Rework the common series so that it matches the integrand. First substitute x^3 for x.

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

$$\cos(x^3) = \sum_{n=0}^{\infty} \frac{(-1)^n (x^3)^{2n}}{(2n)!}$$

$$\cos(x^3) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{6n}}{(2n)!}$$

Multiply by $3x$.

$$3x\cos(x^3) = \sum_{n=0}^{\infty} \frac{3x(-1)^n (x^3)^{2n}}{(2n)!}$$

$$3x\cos(x^3) = \sum_{n=0}^{\infty} \frac{3(-1)^n x^{6n+1}}{(2n)!}$$

Now integrate this series, then evaluate over the interval.

$$\int_0^1 \frac{3(-1)^n x^{6n+1}}{(2n)!}\ dx$$

$$\frac{3(-1)^n}{(2n)!}\int_0^1 x^{6n+1}\ dx$$

$$\frac{3(-1)^n}{(2n)!} \cdot \frac{x^{6n+2}}{6n+2}\Bigg|_0^1$$

$$\frac{3(-1)^n}{(2n)!} \cdot \frac{1^{6n+2}}{6n+2} - \frac{3(-1)^n}{(2n)!} \cdot \frac{0^{6n+2}}{6n+2}$$

$$\frac{3(-1)^n}{(2n)!(6n+2)}$$

Then we can set up an equation with the integral and the new series.

$$\int_0^1 3x\cos(x^3)\ dx = \sum_{n=0}^{\infty} \frac{3(-1)^n}{(2n)!(6n+2)}$$

$$\int_0^1 3x\cos(x^3)\ dx = \frac{3(-1)^0}{(2\cdot 0)!(6(0)+2)} + \frac{3(-1)^1}{(2\cdot 1)!(6(1)+2)}$$

$$+\frac{3(-1)^2}{(2\cdot 2)!(6(2)+2)} + \frac{3(-1)^3}{(2\cdot 3)!(6(3)+2)} + \dots$$

$$\int_0^1 3x\cos(x^3)\ dx \approx 1.5 - 0.1875 + 0.00893 - 0.000208$$

Using the first four terms of the series,

$$\int_0^1 3x\cos(x^3)\ dx \approx 1.5 - 0.1875 + 0.00893 - 0.000208 \approx 1.321222$$

3. Evaluate the definite integral as a power series, using the first four terms.

$$\int_0^1 4e^{x^2}\ dx$$

Solution:

Rework the common series so that it matches the integrand. First substitute x^2 for x.

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$e^{x^2} = \sum_{n=0}^{\infty} \frac{(x^2)^n}{n!}$$

$$e^{x^2} = \sum_{n=0}^{\infty} \frac{x^{2n}}{n!}$$

Multiply by 4.

$$4e^{x^2} = 4\sum_{n=0}^{\infty} \frac{(x^2)^n}{n!} = \sum_{n=0}^{\infty} \frac{4x^{2n}}{n!}$$

Now integrate this series, then evaluate over the interval.

$$\int \frac{4x^{2n}}{n!}\ dx = \frac{4}{n!}\int x^{2n}\ dx$$

$$\frac{4}{n!} \cdot \frac{x^{2n+1}}{2n+1}\Bigg|_0^1$$

$$\frac{4}{n!}\cdot\frac{1^{2n+1}}{2n+1}-\frac{4}{n!}\cdot\frac{0^{2n+1}}{2n+1}$$

$$\frac{4}{n!(2n+1)}$$

Then we can set up an equation with the integral and the new series.

$$\int_0^1 4e^{x^2}\ dx=\sum_{n=0}^{\infty}\frac{4}{n!(2n+1)}$$

$$\int_0^1 4e^{x^2}\ dx=\frac{4}{0!(2(0)+1)}+\frac{4}{1!(2(1)+1)}+\frac{4}{2!(2(2)+1)}$$

$$+\frac{4}{3!(2(3)+1)}+\frac{4}{4!(2(4)+1)}+\dots$$

$$\int_0^1 4e^{x^2}\ dx\approx 4+1.3333+0.4+0.0953+0.01852+\dots$$

Using the first four terms of the series,

$$\int_0^1 4e^{x^2}\ dx\approx 4+1.3333+0.4+0.0953+0.01852\approx 5.84712$$

ESTIMATING INDEFINITE INTEGRALS

1. Evaluate the indefinite integral as a power series.

$$\int x^2 \sin(x^2)\, dx$$

Solution:

Start with a common series, substitute x^2 for x,

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

$$\sin x^2 = \sum_{n=0}^{\infty} \frac{(-1)^n (x^2)^{2n+1}}{(2n+1)!}$$

$$\sin x^2 = \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+2}}{(2n+1)!}$$

and then multiply by x^2.

$$x^2 \sin x^2 = x^2 \sum_{n=0}^{\infty} \frac{(-1)^n (x^2)^{2n+1}}{(2n+1)!}$$

$$x^2 \sin x^2 = \sum_{n=0}^{\infty} \frac{(-1)^n x^2 \cdot x^{4n+2}}{(2n+1)!}$$

$$x^2 \sin x^2 = \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+4}}{(2n+1)!}$$

Integrate this series.

$$\int \frac{(-1)^n x^{4n+4}}{(2n+1)!}\ dx$$

$$\frac{(-1)^n}{(2n+1)!} \int x^{4n+4}\ dx$$

$$\frac{(-1)^n}{(2n+1)!} \cdot \frac{x^{4n+5}}{4n+5} + C$$

$$\frac{(-1)^n x^{4n+5}}{(4n+5)(2n+1)!} + C$$

$$\frac{(-1)^n x^{4n+5}}{(4n+5)(2n+1)!}$$

So this is the value of the integral, integrated as a power series.

$$\int x^2 \sin(x^2)\ dx = \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+5}}{(4n+5)(2n+1)!}$$

■ 2. Evaluate the indefinite integral as a power series.

$$\int \ln(1+2x)\ dx$$

Solution:

Start with a common series, substitute $2x$ for x.

$$\ln(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\frac{x^5}{5}-\cdots=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}x^n}{n}$$

$$\ln(1+2x)=2x-\frac{(2x)^2}{2}+\frac{(2x)^3}{3}-\frac{(2x)^4}{4}+\frac{(2x)^5}{5}-\cdots=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}(2x)^n}{n}$$

$$\ln(1+2x)=2x-\frac{4x^2}{2}+\frac{8x^3}{3}-\frac{16x^4}{4}+\frac{32x^5}{5}-\cdots=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}2^nx^n}{n}$$

Integrate this series.

$$\int\frac{(-1)^{n-1}2^nx^n}{n}\ dx$$

$$\frac{(-1)^{n-1}2^n}{n}\int x^n\ dx$$

$$\frac{(-1)^{n-1}2^n}{n}\left(\frac{x^{n+1}}{n+1}\right)+C$$

$$\frac{(-1)^{n-1}2^nx^{n+1}}{n(n+1)}+C$$

So this is the value of the integral, integrated as a power series.

$$\int\ln(1+2x)\ dx=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}2^nx^{n+1}}{n(n+1)}$$

3. Evaluate the indefinite integral as a power series.

$$\int x^2 \cos(x^3)\ dx$$

Solution:

Start with a common series, substitute x^3 for x,

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

$$\cos x^3 = \sum_{n=0}^{\infty} \frac{(-1)^n (x^3)^{2n}}{(2n)!}$$

$$\cos x^3 = \sum_{n=0}^{\infty} \frac{(-1)^n x^{6n}}{(2n)!}$$

and then multiply by x^2.

$$x^2 \cos x^3 = x^2 \sum_{n=0}^{\infty} \frac{(-1)^n x^{6n}}{(2n)!}$$

$$x^2 \cos x^3 = \sum_{n=0}^{\infty} \frac{(-1)^n x^{6n+2}}{(2n)!}$$

Integrate this series.

$$\int \frac{(-1)^n x^{6n+2}}{(2n)!}\ dx$$

$$\frac{(-1)^n}{(2n)!} \int x^{6n+2}\ dx$$

$$\frac{(-1)^n}{(2n)!} \cdot \frac{x^{6n+3}}{6n+3} + C$$

$$\frac{(-1)^n x^{6n+3}}{(2n)!(6n+3)} + C$$

$$\frac{(-1)^n x^{6n+3}}{(2n)!(6n+3)}$$

So this is the value of the integral, integrated as a power series.

$$\int x^2 \cos(x^3)\ dx = \sum_{n=0}^{\infty} \frac{(-1)^n x^{6n+3}}{(2n)!(6n+3)}$$

BINOMIAL SERIES

1. Use a binomial series to expand the function as a power series.

$$f(x) = (3+x)^5$$

Solution:

Begin with the binomial series.

$$(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n = 1 + kx + \frac{k(k-1)}{2!}x^2$$

$$+\frac{k(k-1)(k-2)}{3!}x^3 + \frac{k(k-1)(k-2)(k-3)}{4!}x^4 + \dots$$

Replace x with $x+2$ and k with 5.

$$(1+x+2)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n = 1 + \frac{k(x+2)}{1!} + \frac{k(k-1)}{2!}(x+2)^2$$

$$+\frac{k(k-1)(k-2)}{3!}(x+2)^3 + \frac{k(k-1)(k-2)(k-3)}{4!}(x+2)^4 + \dots$$

$$(3+x)^5 = \sum_{n=0}^{\infty} \binom{5}{n} x^n = \frac{1}{0!}(x+2)^0 + \frac{5}{1!}(x+2)^1 + \frac{5(5-1)}{2!}(x+2)^2$$

$$+\frac{5(5-1)(5-2)}{3!}(x+2)^3 + \frac{5(5-1)(5-2)(5-3)}{4!}(x+2)^4 + \dots$$

Simplify the right side.

$$(3+x)^5=\sum_{n=0}^{\infty}\binom{5}{n}x^n=\frac{1}{0!}(x+2)^0+\frac{5}{1!}(x+2)^1+\frac{5(4)}{2!}(x+2)^2$$

$$+\frac{5(4)(3)}{3!}(x+2)^3+\frac{5(4)(3)(2)}{4!}(x+2)^4+\dots$$

Match the terms to their corresponding n-values.

$n=0$ $\quad\frac{1}{0!}(x+2)^0$

$n=1$ $\quad\frac{5}{1!}(x+2)^1$

$n=2$ $\quad\frac{5(4)}{2!}(x+2)^2$

$n=3$ $\quad\frac{5(4)(3)}{3!}(x+2)^3$

$n=4$ $\quad\frac{5(4)(3)(2)}{4!}(x+2)^4$

Then the power series is

$$(3+x)^5=1+\sum_{n=1}^{\infty}\frac{5\cdot 4\cdot 3\cdot 2\cdot \dots \cdot (6-n)}{n!}(x+2)^n$$

■ 2. Use a binomial series to expand the function as a power series.

$$f(x)=(6-x)^4$$

Solution:

Begin with the binomial series.

$$(1+x)^k=\sum_{n=0}^{\infty}\binom{k}{n}x^n=1+kx+\frac{k(k-1)}{2!}x^2$$

$$+\frac{k(k-1)(k-2)}{3!}x^3+\frac{k(k-1)(k-2)(k-3)}{4!}x^4+\dots$$

Replace x with $-x-5$ and k with 4.

$$(1+(-x-5))^k=\sum_{n=0}^{\infty}\binom{k}{n}x^n=1+\frac{k}{1!}(-x-5)+\frac{k(k-1)}{2!}(-x-5)^2$$

$$+\frac{k(k-1)(k-2)}{3!}(-x-5)^3+\frac{k(k-1)(k-2)(k-3)}{4!}(-x-5)^4+\dots$$

$$(6-x)^4=\sum_{n=0}^{\infty}\binom{4}{n}x^n=\frac{1}{0!}(-x-5)^0+\frac{4}{1!}(-x-5)+\frac{4(4-1)}{2!}(-x-5)^2$$

$$+\frac{4(4-1)(4-2)}{3!}(-x-5)^3+\frac{4(4-1)(4-2)(4-3)}{4!}(-x-5)^4+\dots$$

Simplify the right side.

$$(6-x)^4=\sum_{n=0}^{\infty}\binom{4}{n}x^n=\frac{1}{0!}(-x-5)^0+\frac{4}{1!}(-x-5)^1+\frac{4(3)}{2!}(-x-5)^2$$

$$+\frac{4(3)(2)}{3!}(-x-5)^3+\frac{4(3)(2)(1)}{4!}(-x-5)^4+\dots$$

Match the terms to their corresponding n-values.

$n = 0$ $\qquad$ $\frac{1}{0!}(-x-5)^0$

$n = 1$ $\qquad$ $\frac{4}{1!}(-x-5)^1$

$n = 2$ $\qquad$ $\frac{4(3)}{2!}(-x-5)^2$

$n = 3$ $\qquad$ $\frac{4(3)(2)}{3!}(-x-5)^3$

$n = 4$ $\qquad$ $\frac{4(3)(2)(1)}{4!}(-x-5)^4$

Then the power series is

$$(6-x)^4 = 1 + \sum_{n=1}^{\infty} \frac{4 \cdot 3 \cdot 2 \cdot \ldots \cdot (5-n)}{n!}(-x-5)^n$$

3. Use a binomial series to expand the function as a power series.

$$f(x) = (-4+x)^5$$

Solution:

Begin with the binomial series.

$$(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n = 1 + kx + \frac{k(k-1)}{2!}x^2$$

$$+\frac{k(k-1)(k-2)}{3!}x^3 + \frac{k(k-1)(k-2)(k-3)}{4!}x^4 + \dots$$

Replace x with $x - 5$ and k with 5.

$$(1+x-5)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n = 1 + \frac{k(x+2)}{1!} + \frac{k(k-1)}{2!}(x-5)^2$$

$$+\frac{k(k-1)(k-2)}{3!}(x-5)^3 + \frac{k(k-1)(k-2)(k-3}{4!}(x-5)^4 + \dots$$

$$(-4+x)^5 = \sum_{n=0}^{\infty} \binom{5}{n} x^n = \frac{1}{0!}(x-5)^0 + \frac{5}{1!}(x-5)^1 + \frac{5(5-1)}{2!}(x-5)^2$$

$$+\frac{5(5-1)(5-2)}{3!}(x-5)^3 + \frac{5(5-1)(5-2)(5-3)}{4!}(x-5)^4 + \dots$$

Simplify the right side.

$$(-4+x)^5 = \sum_{n=0}^{\infty} \binom{5}{n} x^n = \frac{1}{0!}(x-5)^0 + \frac{5}{1!}(x-5)^1$$

$$+\frac{5(4)}{2!}(x-5)^2 + \frac{5(4)(3)}{3!}(x-5)^3 + \frac{5(4)(3)(2)}{4!}(x-5)^4 + \dots$$

Match the terms to their corresponding n-values.

$n = 0$ $\quad\quad$ $\frac{1}{0!}(x-5)^0$

$n=1$ $\quad\quad \dfrac{5}{1!}(x-5)^1$

$n=2$ $\quad\quad \dfrac{5(4)}{2!}(x-5)^2$

$n=3$ $\quad\quad \dfrac{5(4)(3)}{3!}(x-5)^3$

$n=4$ $\quad\quad \dfrac{5(4)(3)(2)}{4!}(x-5)^4$

Then the power series is

$$(-4+x)^5 = 1 + \sum_{n=1}^{\infty} \frac{5 \cdot 4 \cdot 3 \cdot 2 \cdot \ldots \cdot (6-n)}{n!}(x-5)^n$$

■ 4. Use a binomial series to expand the function as a power series.

$$f(x) = (7-x)^6$$

Solution:

Begin with the binomial series.

$$(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n = 1 + kx + \frac{k(k-1)}{2!}x^2$$

$$+\frac{k(k-1)(k-2)}{3!}x^3 + \frac{k(k-1)(k-2)(k-3)}{4!}x^4 + \ldots$$

Replace x with $-x+6$ and k with 6.

$$(1+(-x+6))^k=\sum_{n=0}^{\infty}\binom{k}{n}x^n=1+\frac{k}{1!}(-x+6)+\frac{k(k-1)}{2!}(-x+6)^2$$

$$+\frac{k(k-1)(k-2)}{3!}(-x+6)^3+\frac{k(k-1)(k-2)(k-3)}{4!}(-x+6)^4+\dots$$

$$(7-x)^6=\sum_{n=0}^{\infty}\binom{6}{n}x^n=\frac{1}{0!}(-x+6)^0+\frac{6}{1!}(-x+6)+\frac{6(6-1)}{2!}(-x+6)^2$$

$$+\frac{6(6-1)(6-2)}{3!}(-x+6)^3+\frac{6(6-1)(6-2)(6-3)}{4!}(-x+6)^4+\dots$$

Simplify the right side.

$$(7-x)^6=\sum_{n=0}^{\infty}\binom{6}{n}x^n=\frac{1}{0!}(-x+6)^0+\frac{6}{1!}(-x+6)+\frac{6(5)}{2!}(-x+6)^2$$

$$+\frac{6(5)(4)}{3!}(-x+6)^3+\frac{6(5)(4)(3)}{4!}(-x+6)^4+\dots$$

Match the terms to their corresponding n-values.

$n=0$ $\qquad \dfrac{1}{0!}(-x+6)^0$

$n=1$ $\qquad \dfrac{6}{1!}(-x+6)^1$

$n=2$ $\qquad \dfrac{6(5)}{2!}(-x+6)^2$

$n=3$ $\qquad \dfrac{6(5)(4)}{3!}(-x+6)^3$

$$n = 4 \qquad \frac{6(5)(4)(3)}{4!}(-x+6)^4$$

Then the power series is

$$(7-x)^6 = 1 + \sum_{n=1}^{\infty} \frac{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot \ldots \cdot (7-n)}{n!}(-x+6)^n$$

■ 5. Use a binomial series to expand the function as a power series.

$$f(x) = (8+x)^7$$

Solution:

Begin with the binomial series.

$$(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n = 1 + kx + \frac{k(k-1)}{2!}x^2$$

$$+\frac{k(k-1)(k-2)}{3!}x^3 + \frac{k(k-1)(k-2)(k-3)}{4!}x^4 + \ldots$$

Replace x with $x+7$ and k with 7.

$$(1+x+7)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n = 1 + \frac{k}{1!}(x+7) + \frac{k(k-1)}{2!}(x+7)^2$$

$$+\frac{k(k-1)(k-2)}{3!}(x+7)^3 + \frac{k(k-1)(k-2)(k-3)}{4!}(x+7)^4 + \ldots$$

$$(8+x)^7 = \sum_{n=0}^{\infty} \binom{7}{n} x^n = \frac{1}{0!}(x+7)^0 + \frac{7}{1!}(x+7)^1 + \frac{7(7-1)}{2!}(x+7)^2$$

$$+\frac{7(7-1)(7-2)}{3!}(x+7)^3 + \frac{7(7-1)(7-2)(7-3)}{4!}(x+7)^4 + \dots$$

Simplify the right side.

$$(8+x)^7 = \sum_{n=0}^{\infty} \binom{7}{n} x^n = \frac{1}{0!}(x+7)^0 + \frac{7}{1!}(x+7)^1$$

$$+\frac{7(6)}{2!}(x+7)^2 + \frac{7(6)(5)}{3!}(x+7)^3 + \frac{7(6)(5)(4)}{4!}(x+7)^4 + \dots$$

Match the terms to their corresponding n-values.

$n = 0$ $\qquad \dfrac{1}{0!}(x+7)^0$

$n = 1$ $\qquad \dfrac{7}{1!}(x+7)^1$

$n = 2$ $\qquad \dfrac{7(6)}{2!}(x+7)^2$

$n = 3$ $\qquad \dfrac{7(6)(5)}{3!}(x+7)^3$

$n = 4$ $\qquad \dfrac{7(6)(5)(4)}{4!}(x+7)^4$

Then the power series is

$$(8+x)^7 = 1 + \sum_{n=1}^{\infty} \frac{7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot \dots \cdot (8-n)}{n!}(x+7)^n$$

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