Topic: Midpoint rule error bound

Question: Calculate the area under the curve. Then, use the Midpoint Rule with $n = 4$ to approximate the same area. Compare the actual area to the result to determine the error of the Midpoint Rule approximation of the area.

$$\int_0^2 x^3 + x^2 + x + 1 \ dx$$

Answer choices:

A Actual area is $\frac{44}{3}$ $MRAM_4$ is $\frac{115}{8}$ Error is $\frac{7}{24}$

B Actual area is $\frac{115}{8}$ $MRAM_4$ is $\frac{44}{3}$ Error is $\frac{7}{24}$

C Actual area is $\frac{32}{3}$ $MRAM_4$ is $\frac{21}{2}$ Error is $\frac{1}{6}$

D Actual area is $\frac{1{,}315}{128}$ $MRAM_4$ is $\frac{32}{3}$ Error is $\frac{151}{384}$

Solution: C

The question asks us to calculate the area under the curve, and then approximate the same area using the Midpoint Rule, with $n = 4$, and compare the results by identifying the error.

$$\int_0^2 x^3 + x^2 + x + 1 \ dx$$

From the integral, we know that the function is

$$f(x) = x^3 + x^2 + x + 1$$

Let's begin by integrating $f(x)$ using the power rule and evaluating the integral. This will give the actual area under the curve.

$$\int_0^2 x^3 + x^2 + x + 1 \ dx$$

$$\left(\frac{x^4}{4} + \frac{x^3}{3} + \frac{x^2}{2} + x\right)\Bigg|_0^2$$

$$\left(\frac{2^4}{4} + \frac{2^3}{3} + \frac{2^2}{2} + 2\right) - \left(\frac{0^4}{4} + \frac{0^3}{3} + \frac{0^2}{2} + 0\right)$$

$$\frac{16}{4} + \frac{8}{3} + \frac{4}{2} + 2 = \frac{32}{3}$$

Now, we'll estimate the area under the curve using the Midpoint Rule, with $n = 4$. The table below shows the interval divided into 4 subintervals, the midpoint of each interval, and the function values at each midpoint. The work is shown below the table.

$[a,b]$	$[0,0.5]$	$[0.5,1]$	$[1,1.5]$	$[1.5,2]$
$a+\dfrac{b-a}{2}$	0.25	0.75	1.25	1.75
$f\left(a+\dfrac{b-a}{2}\right)$	$\dfrac{85}{64}$	$\dfrac{175}{64}$	$\dfrac{369}{64}$	$\dfrac{715}{64}$

$$f\left(\frac{1}{4}\right)=\left(\frac{1}{4}\right)^3+\left(\frac{1}{4}\right)^2+\frac{1}{4}+1=\frac{85}{64}$$

$$f\left(\frac{3}{4}\right)=\left(\frac{3}{4}\right)^3+\left(\frac{3}{4}\right)^2+\frac{3}{4}+1=\frac{175}{64}$$

$$f\left(\frac{5}{4}\right)=\left(\frac{5}{4}\right)^3+\left(\frac{5}{4}\right)^2+\frac{5}{4}+1=\frac{369}{64}$$

$$f\left(\frac{7}{4}\right)=\left(\frac{7}{4}\right)^3+\left(\frac{7}{4}\right)^2+\frac{7}{4}+1=\frac{715}{64}$$

The subinterval widths are all $1/2$, so to find the Midpoint Rule approximation of the area, multiply each function value by $1/2$ and add the areas.

$$\left(\frac{85}{64}\right)\left(\frac{1}{2}\right)+\left(\frac{175}{64}\right)\left(\frac{1}{2}\right)+\left(\frac{369}{64}\right)\left(\frac{1}{2}\right)+\left(\frac{715}{64}\right)\left(\frac{1}{2}\right)=\frac{1{,}344}{128}=\frac{21}{2}$$

The error is the actual area minus the estimated area.

$$\frac{32}{3}-\frac{21}{2}=\frac{1}{6}$$

Topic: Midpoint rule error bound

Question: Calculate the error bound of the Midpoint Rule, with $n = 4$.

$$\int_0^2 x^3 + x^2 + x + 1 \ dx$$

Answer choices:

A $\quad |E_M| \le \frac{5}{23}$

B $\quad |E_M| \le \frac{4}{21}$

C $\quad |E_M| \le \frac{7}{24}$

D $\quad |E_M| \le \frac{5}{24}$

Solution: C

The question asks us to calculate the error bound of the Midpoint Rule, with $n = 4$, for the area under the curve.

$$\int_0^2 x^3 + x^2 + x + 1 \ dx$$

To find the error bound of the Midpoint Rule on the interval $[a, b]$, we use this formula.

$$|E_M| \le k\frac{(b-a)^3}{24n^2}$$

Where $|E_M|$ denotes the maximum error of the Midpoint Rule, k is a constant based on the function, which we will find, a is the lower limit of the interval, b is the upper limit of the interval, and n is the number of subintervals.

First, let's find k. The value k is often denoted by the notation $M_{f''}$ which means the maximum absolute value of the function's second derivative in the interval. Let's find k for the function and interval in this problem.

$$f(x) = x^3 + x^2 + x + 1$$

$$f'(x) = 3x^2 + 2x + 1$$

$$f''(x) = 6x + 2$$

The second derivative, $f''(x)$, will never equal 0 on $[0,2]$, so to find k, find the maximum value on the interval. Evaluate $f''(x)$ at the endpoints.

$$f''(0) = 2, \ \ f''(2) = 14, \ \ k = 14$$

Now in the expression

$$|E_M| \leq k\frac{(b-a)^3}{24n^2}$$

$k = 14$, $a = 0$, $b = 2$, and $n = 4$.

Evaluate the error bound.

$$k\frac{(b-a)^3}{24n^2} = (14)\frac{(2-0)^3}{(24)(4)^2} = \frac{(14)(8)}{(24)(16)} = \frac{7}{24}$$

Therefore,

$$|E_M| \leq \frac{7}{24}$$

Topic: Midpoint rule error bound

Question: Find n to get the accuracy of the Midpoint Rule of the approximation of the area under the curve to be within 0.00001.

$$\int_0^2 x^3+x^2+x+1\ dx$$

Answer choices:

A $\quad n \geq 683$

B $\quad n \geq 684$

C $\quad n \geq 632$

D $\quad n \geq 633$

Solution: B

The question asks us to find n to get the accuracy of the Midpoint Rule to within 0.00001.

$$\int_0^2 x^3 + x^2 + x + 1\ dx$$

To find the error bound of the Midpoint Rule on the interval $[a, b]$, we use this formula.

$$|E_M| \le k\frac{(b-a)^3}{24n^2}$$

Where $|E_M|$ denotes the maximum error of the Midpoint Rule, k is a constant based on the function, which we will find, a is the lower limit of the interval, b is the upper limit of the interval, and n is the number of subintervals.

First, let's find k. The value k if often denoted by the notation $M_{f''}$ which means the maximum absolute value of the function's second derivative in the interval. Let's find k for the function and interval in this problem.

$$f(x) = x^3 + x^2 + x + 1$$

$$f'(x) = 3x^2 + 2x + 1$$

$$f''(x) = 6x + 2$$

The second derivative, $f''(x)$, will never equal 0 on $[0,2]$, so to find k, find the maximum value on the interval. Evaluate $f''(x)$ at the endpoints.

$$f''(0) = 2, \;\; f''(2) = 14, \;\; k = 14$$

Now in the expression

$$|E_M| \le k\frac{(b-a)^3}{24n^2}$$

$k = 14$, $a = 0$, and $b = 2$.

We will find the value of n. Let's simplify the expression first.

$$|E_M| \le (14)\frac{(2-0)^3}{24n^2}$$

$$|E_M| \le \frac{14(8)}{24n^2}$$

$$|E_M| \le \frac{14}{3n^2}$$

Since we want the error to be less than 0.00001, we set the maximum error bound expression to be less than 0.00001.

$$\frac{14}{3n^2} \le 0.00001$$

Multiply by $3n^2$ and divide by 0.00003.

$$\frac{14}{3n^2} \le 0.00001$$

$$14 \le (0.00003)n^2$$

$$\frac{14}{0.00003} \le n^2$$

Square root both sides of the inequality, ignoring the possibility that n could be negative.

$$\sqrt{\frac{14}{0.00003}} \le \sqrt{n^2}$$

$$n \ge 683.1300511$$

We found an interval for n. However, since n is the number of subintervals, n has to be a whole number. Thus, to be accurate to within 0.0001, $n \ge 684$.