Topic: Rationalizing substitutions

Question: Rewrite the integral using partial fractions. Do not solve it.

$$\int \frac{\sqrt{x+4}}{x}\ dx$$

Answer choices:

A $\quad 2\int \frac{1}{u-2}\ du - 2\int \frac{1}{u+2}\ du$

B $\quad 2\int \frac{1}{u+2}\ du - 2\int \frac{1}{u-2}\ du$

C $\quad 2u + 2\int \frac{1}{u-2}\ du - 2\int \frac{1}{u+2}\ du$

D $\quad 2u + 2\int \frac{1}{u+2}\ du - 2\int \frac{1}{u-2}\ du$

Solution: C

In order to use partial fractions to evaluate an integral, we need a rational, proper function. Rational functions can't include radicals, so we have to eliminate the radical before we can more forward. The easiest way to eliminate the radical is to use u-substitution. We'll let

$$u = \sqrt{x+4}$$

$$du = \frac{1}{2\sqrt{x+4}}\ dx$$

$$dx = 2\sqrt{x+4}\ du$$

And now we'll substitute into the integral.

$$\int \frac{\sqrt{x+4}}{x}\ dx = \int \frac{u}{x}\left(2\sqrt{x+4}\right)\ du$$

$$\int \frac{u}{x}(2u)\ du$$

$$\int \frac{2u^2}{x}\ du$$

Solving $u = \sqrt{x+4}$ for x so that we can replace the remaining x in the integral, we get

$$u = \sqrt{x+4}$$

$$u^2 = x + 4$$

$$x = u^2 - 4$$

So

$$\int \frac{2u^2}{u^2-4}\ du$$

Since we've completely removed the radical and the new integral is in terms of the new variable, we're finished with our rationalizing substitution.

Normally here we would go straight to our partial fractions decomposition, but first we have to make the integral proper. Remember, in order to use partial fractions, the function must be proper, which means that the degree of the numerator must be less than the degree of the denominator. Currently we have u^2 in both the numerator and denominator, so their degrees are equal.

We'll use polynomial long division to make the function proper, dividing u^2-4 into u^2. We'll get

$$1+\frac{4}{u^2-4}$$

Plugging this into the integral, we get

$$2\int 1+\frac{4}{u^2-4}\ du$$

$$2\int 1\ du+2\int \frac{4}{u^2-4}\ du$$

$$2u+\int \frac{8}{(u-2)(u+2)}\ du$$

Use partial fractions to simplify the remaining integral.

$$\frac{8}{(u-2)(u+2)}=\frac{A}{u-2}+\frac{B}{u+2}$$

$$\left[\frac{8}{(u-2)(u+2)}=\frac{A}{u-2}+\frac{B}{u+2}\right](u-2)(u+2)$$

$$8=A(u+2)+B(u-2)$$

If we set $u=2$, then

$$8=A(2+2)+B(2-2)$$

$$8=A(4)+B(0)$$

$$A=2$$

And if we set $u=-2$, then

$$8=A(-2+2)+B(-2-2)$$

$$8=A(0)+B(-4)$$

$$B=-2$$

Plugging the values for both of the constants back into the partial fractions decomposition, and putting the decomposition back into the integral, we get

$$\int \frac{8}{(u-2)(u+2)}\ du=\int \frac{2}{u-2}+\frac{-2}{u+2}\ du$$

$$\int \frac{2}{u-2}-\frac{2}{u+2}\ du$$

$$2u + \int \frac{2}{u-2} - \frac{2}{u+2}\ du$$

$$2u + 2\int \frac{1}{u-2}\ du - 2\int \frac{1}{u+2}\ du$$

Topic: Rationalizing substitutions

Question: Rewrite the integral using partial fractions. Do not solve it.

$$\int \frac{\sqrt{x+9}}{x}\ dx$$

Answer choices:

A $\quad 2u + 3\int \frac{1}{u+3}\ du - 3\int \frac{1}{u-3}\ du$

B $\quad 2u + 3\int \frac{1}{u-3}\ du - 3\int \frac{1}{u+3}\ du$

C $\quad 3\int \frac{1}{u+3}\ du - 3\int \frac{1}{u-3}\ du$

D $\quad 3\int \frac{1}{u-3}\ du - 3\int \frac{1}{u+3}\ du$

Solution: B

In order to use partial fractions to evaluate an integral, we need a rational, proper function. Rational functions can't include radicals, so we have to eliminate the radical before we can more forward. The easiest way to eliminate the radical is to use u-substitution. We'll let

$$u = \sqrt{x+9}$$

$$du = \frac{1}{2\sqrt{x+9}}\ dx$$

$$dx = 2\sqrt{x+9}\ du$$

And now we'll substitute into the integral.

$$\int \frac{\sqrt{x+9}}{x}\ dx = \int \frac{u}{x}\left(2\sqrt{x+9}\right)\ du$$

$$\int \frac{u}{x}(2u)\ du$$

$$\int \frac{2u^2}{x}\ du$$

Solving $u = \sqrt{x+9}$ for x so that we can replace the remaining x in the integral, we get

$$u = \sqrt{x+9}$$

$$u^2 = x+9$$

$$x = u^2 - 9$$

So

$$\int \frac{2u^2}{u^2-9}\ du$$

Since we've completely removed the radical and the new integral is in terms of the new variable, we're finished with our rationalizing substitution.

Normally here we would go straight to our partial fractions decomposition, but first we have to make the integral proper. Remember, in order to use partial fractions, the function must be proper, which means that the degree of the numerator must be less than the degree of the denominator. Currently we have u^2 in both the numerator and denominator, so their degrees are equal.

We'll use polynomial long division to make the function proper, dividing u^2-9 into u^2. We'll get

$$1+\frac{9}{u^2-9}$$

Plugging this into the integral, we get

$$2\int 1+\frac{9}{u^2-9}\ du$$

$$2\int 1\ du+2\int \frac{9}{u^2-9}\ du$$

$$2u+\int \frac{18}{(u-3)(u+3)}\ du$$

Use partial fractions to simplify the remaining integral.

$$\frac{18}{(u-3)(u+3)}=\frac{A}{u-3}+\frac{B}{u+3}$$

$$\left[\frac{18}{(u-3)(u+3)}=\frac{A}{u-3}+\frac{B}{u+3}\right](u-3)(u+3)$$

$$18=A(u+3)+B(u-3)$$

If we set $u=3$, then

$$18=A(3+3)+B(3-3)$$

$$18=A(6)+B(0)$$

$$A=3$$

And if we set $u=-3$, then

$$18=A(-3+3)+B(-3-3)$$

$$18=A(0)+B(-6)$$

$$B=-3$$

Plugging the values for both of the constants back into the partial fractions decomposition, and putting the decomposition back into the integral, we get

$$\int \frac{18}{(u-3)(u+3)}\ du=\int \frac{3}{u-3}+\frac{-3}{u+3}\ du$$

$$\int \frac{3}{u-3}-\frac{3}{u+3}\ du$$

$$2u + \int \frac{3}{u-3} - \frac{3}{u+3}\ du$$

$$2u + 3\int \frac{1}{u-3}\ du - 3\int \frac{1}{u+3}\ du$$

Topic: Rationalizing substitutions

Question: Rewrite the integral using partial fractions. Do not solve it.

$$\int \frac{\sqrt{x+1}}{x^2}\, dx$$

Answer choices:

A $\quad \frac{1}{2}\int \frac{1}{u+1}\, du - \frac{1}{2}\int \frac{1}{u-1}\, du$

B $\quad -\frac{1}{2}\int \frac{1}{u+1}\, du + \frac{1}{2}\int \frac{1}{u-1}\, du$

C $\quad \frac{1}{2}\int \frac{1}{u+1}\, du - \frac{1}{2}\int \frac{1}{(u+1)^2}\, du - \frac{1}{2}\int \frac{1}{u-1}\, du - \frac{1}{2}\int \frac{1}{(u-1)^2}\, du$

D $\quad -\frac{1}{2}\int \frac{1}{u+1}\, du + \frac{1}{2}\int \frac{1}{(u+1)^2}\, du + \frac{1}{2}\int \frac{1}{u-1}\, du + \frac{1}{2}\int \frac{1}{(u-1)^2}\, du$

Solution: D

In order to use partial fractions to evaluate an integral, we need a rational, proper function. Rational functions can't include radicals, so we have to eliminate the radical before we can more forward. The easiest way to eliminate the radical is to use u-substitution. We'll let

$$u = \sqrt{x+1}$$

$$du = \frac{1}{2\sqrt{x+1}}\ dx$$

$$dx = 2\sqrt{x+1}\ du$$

And now we'll substitute into the integral.

$$\int \frac{\sqrt{x+1}}{x^2}\ dx = \int \frac{u}{x^2}\left(2\sqrt{x+1}\right)\ du$$

$$\int \frac{u}{x^2}(2u)\ du$$

$$\int \frac{2u^2}{x^2}\ du$$

Solving $u = \sqrt{x+1}$ for x so that we can replace the remaining x in the integral, we get

$$u = \sqrt{x+1}$$

$$u^2 = x + 1$$

$$x = u^2 - 1$$

So

$$\int \frac{2u^2}{\left(u^2-1\right)^2}\ du$$

$$\int \frac{2u^2}{(u+1)^2(u-1)^2}\ du$$

Use partial fractions with repeated linear factors to simplify the remaining integral.

$$\frac{2u^2}{(u+1)^2(u-1)^2} = \frac{A}{u+1} + \frac{B}{(u+1)^2} + \frac{C}{u-1} + \frac{D}{(u-1)^2}$$

$$\left[\frac{2u^2}{(u+1)^2(u-1)^2} = \frac{A}{u+1} + \frac{B}{(u+1)^2} + \frac{C}{u-1} + \frac{D}{(u-1)^2}\right](u+1)^2(u-1)^2$$

$$2u^2 = A(u+1)(u-1)^2 + B(u-1)^2 + C(u+1)^2(u-1) + D(u+1)^2$$

$$2u^2 = A(u+1)(u^2-2u+1) + B(u^2-2u+1)$$

$$+C(u^2+2u+1)(u-1) + D(u^2+2u+1)$$

$$2u^2 = A(u^3-2u^2+u+u^2-2u+1) + B(u^2-2u+1)$$

$$+C(u^3-u^2+2u^2-2u+u-1) + D(u^2+2u+1)$$

$$2u^2 = A(u^3-u^2-u+1) + B(u^2-2u+1)$$

$$+C(u^3+u^2-u-1) + D(u^2+2u+1)$$

$$2u^2 = Au^3 - Au^2 - Au + A + Bu^2 - 2Bu + B$$

$$+Cu^3 + Cu^2 - Cu - C + Du^2 + 2Du + D$$

$$2u^2 = (A + C)u^3 + (-A + B + C + D)u^2$$

$$+(-A - 2B - C + 2D)u + (A + B - C + D)$$

Equating coefficients on both sides, we get

[1] $A + C = 0$

[2] $-A + B + C + D = 2$

[3] $-A - 2B - C + 2D = 0$

[4] $A + B - C + D = 0$

Solve [1] for A.

$$A + C = 0$$

$$A = -C$$

Then equations [2] through [4] become

[5] $-(-C) + B + C + D = 2$, or $B + 2C + D = 2$

[6] $-(-C) - 2B - C + 2D = 0$, or $-2B + 2D = 0$

[7] $-C + B - C + D = 0$, or $B - 2C + D = 0$

Solve [6] for B.

$$-2B + 2D = 0$$

$$-2B = -2D$$

$$B = D$$

Then we can substitute into equations [5] and [7] to make a system that's in terms of C and D only.

$$D + 2C + D = 2$$

$$2C + 2D = 2$$

[8] $C + D = 1$

and

$$D - 2C + D = 0$$

$$-2C + 2D = 0$$

[9] $-C + D = 0$

Add equations [8] and [9].

$$C + D + (-C + D) = 1 + 0$$

$$C + D - C + D = 1$$

$$D + D = 1$$

$$2D = 1$$

$$D = \frac{1}{2}$$

We know $B = D$, so $B = 1/2$. And $C + D = 1$, so $C + (1/2) = 1$, so $C = 1/2$. And we know $A = -C$, so $A = -1/2$. Then we can plug the coefficients into the partial fractions decomposition.

$$\frac{2u^2}{(u+1)^2(u-1)^2} = \frac{-\frac{1}{2}}{u+1} + \frac{\frac{1}{2}}{(u+1)^2} + \frac{\frac{1}{2}}{u-1} + \frac{\frac{1}{2}}{(u-1)^2}$$

Putting this back into the integral, we could write the simplified integral as

$$\int \frac{-\frac{1}{2}}{u+1} + \frac{\frac{1}{2}}{(u+1)^2} + \frac{\frac{1}{2}}{u-1} + \frac{\frac{1}{2}}{(u-1)^2}\ du$$

$$-\frac{1}{2}\int \frac{1}{u+1}\ du + \frac{1}{2}\int \frac{1}{(u+1)^2}\ du + \frac{1}{2}\int \frac{1}{u-1}\ du + \frac{1}{2}\int \frac{1}{(u-1)^2}\ du$$