Vertical and horizontal tangent lines to the polar curve

We'll find equations of the vertical and horizontal tangent lines to a polar curve by following these steps:

1. Convert the polar equation into rectangular equations using the conversion formulas

$$x = r\cos\theta$$

$$y = r\sin\theta$$

2. Find the slope of the tangent line m using the formula

$$m = \frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}$$

3. Find horizontal tangent lines

   a. Set $m = 0$ and solve for θ

   b. Plug these values of θ into the original polar equation to find associated values of r

   c. Pair up values of r and θ to find the coordinate points where the polar equation has horizontal tangent lines

4. Find **vertical tangent lines**

   a. Find the values of θ where m is undefined

b. Plug these values of θ into the original polar equation to find associated values of r

c. Pair up values of r and θ to find the coordinate points where the polar equation has vertical tangent lines

Example

Find the points on the polar curve where the graph of the tangent line is vertical or horizontal.

$$r = 2\sin\theta$$

We'll convert the polar equation to a rectangular equation using

$$x = r\cos\theta$$

$$y = r\sin\theta$$

Plugging $r = 2\sin\theta$ into these conversion formulas, we get equations for x and y.

$$x = r\cos\theta$$

$$x = 2\sin\theta\cos\theta$$

and

$$y = r\sin\theta$$

$$y = 2\sin\theta\sin\theta$$

$$y = 2\sin^2\theta$$

We'll find the derivatives $dy/d\theta$ and $dx/d\theta$.

$$\frac{dy}{d\theta} = 4\sin\theta\cos\theta$$

$$\frac{dy}{d\theta} = 2(2\sin\theta\cos\theta)$$

Because $2\sin\theta\cos\theta = \sin(2\theta)$,

$$\frac{dy}{d\theta} = 2\sin(2\theta)$$

and

$$\frac{dx}{d\theta} = 2\cos\theta\cos\theta - 2\sin\theta\sin\theta$$

$$\frac{dx}{d\theta} = 2\left(\cos^2\theta - \sin^2\theta\right)$$

Because $\cos^2\theta - \sin^2\theta = \cos(2\theta)$,

$$\frac{dx}{d\theta} = 2\cos(2\theta)$$

Plugging both derivatives into the formula for dy/dx, we get

$$\frac{dy}{dx} = \frac{2\sin(2\theta)}{2\cos(2\theta)}$$

$$\frac{dy}{dx} = \frac{\sin(2\theta)}{\cos(2\theta)}$$

With an equation for dy/dx in hand, we're ready to find vertical and horizontal tangent lines.

Horizontal tangent lines exist where $dy/dx = 0$. In order for dy/dx to be 0, the numerator has to be 0.

$$\sin(2\theta) = 0$$

So

$$2\theta = 0$$

$$\theta = 0$$

or

$$2\theta = \pi$$

$$\theta = \frac{\pi}{2}$$

To find the r-values associated with these θ values, we'll plug them back into the original polar equation.

$$r = 2\sin\theta$$

$$r = 2\sin(0)$$

$$r = 2(0)$$

$$r = 0$$

and

$$r = 2\sin\theta$$

$$r = 2\sin\frac{\pi}{2}$$

$$r = 2(1)$$

$$r = 2$$

Putting our values together, we can say that $r = 2\sin\theta$ has horizontal tangent lines at $(0,0)$ and $(2,\pi/2)$.

Vertical tangent lines exist where dy/dx is undefined. In order for dy/dx to be undefined, the denominator has to be 0.

$$\cos(2\theta) = 0$$

So

$$2\theta = \frac{\pi}{2}$$

$$\theta = \frac{\pi}{4}$$

or

$$2\theta = \frac{3\pi}{2}$$

$$\theta = \frac{3\pi}{4}$$

To find the r-values associated with these θ values, we'll plug them back into the original polar equation.

$$r = 2\sin\theta$$

$$r = 2\sin\frac{\pi}{4}$$

$$r = 2\cdot\frac{\sqrt{2}}{2}$$

$$r = \sqrt{2}$$

and

$$r = 2\sin\theta$$

$$r = 2\sin\frac{3\pi}{4}$$

$$r = 2\cdot\left(-\frac{\sqrt{2}}{2}\right)$$

$$r = -\sqrt{2}$$

Putting our values together, we can say that $r = 2\sin\theta$ has vertical tangent lines at $\left(\sqrt{2}, \pi/4\right)$ and $\left(-\sqrt{2}, 3\pi/4\right)$.

We'll summarize our findings.

Horizontal tangent lines at $(0,0)$ and $(2,\pi/2)$

Vertical tangent lines at $\left(\sqrt{2}, \pi/4\right)$ and $\left(-\sqrt{2}, 3\pi/4\right)$