Topic: Washers, horizontal axis

Question: Use washers to find the volume of the solid generated by revolving the region bounded by the curves about the given axis.

$y = x^2$ and $y = (32x)^{\frac{1}{3}}$

about the x-axis

Answer choices:

A $\frac{64\pi}{5}$

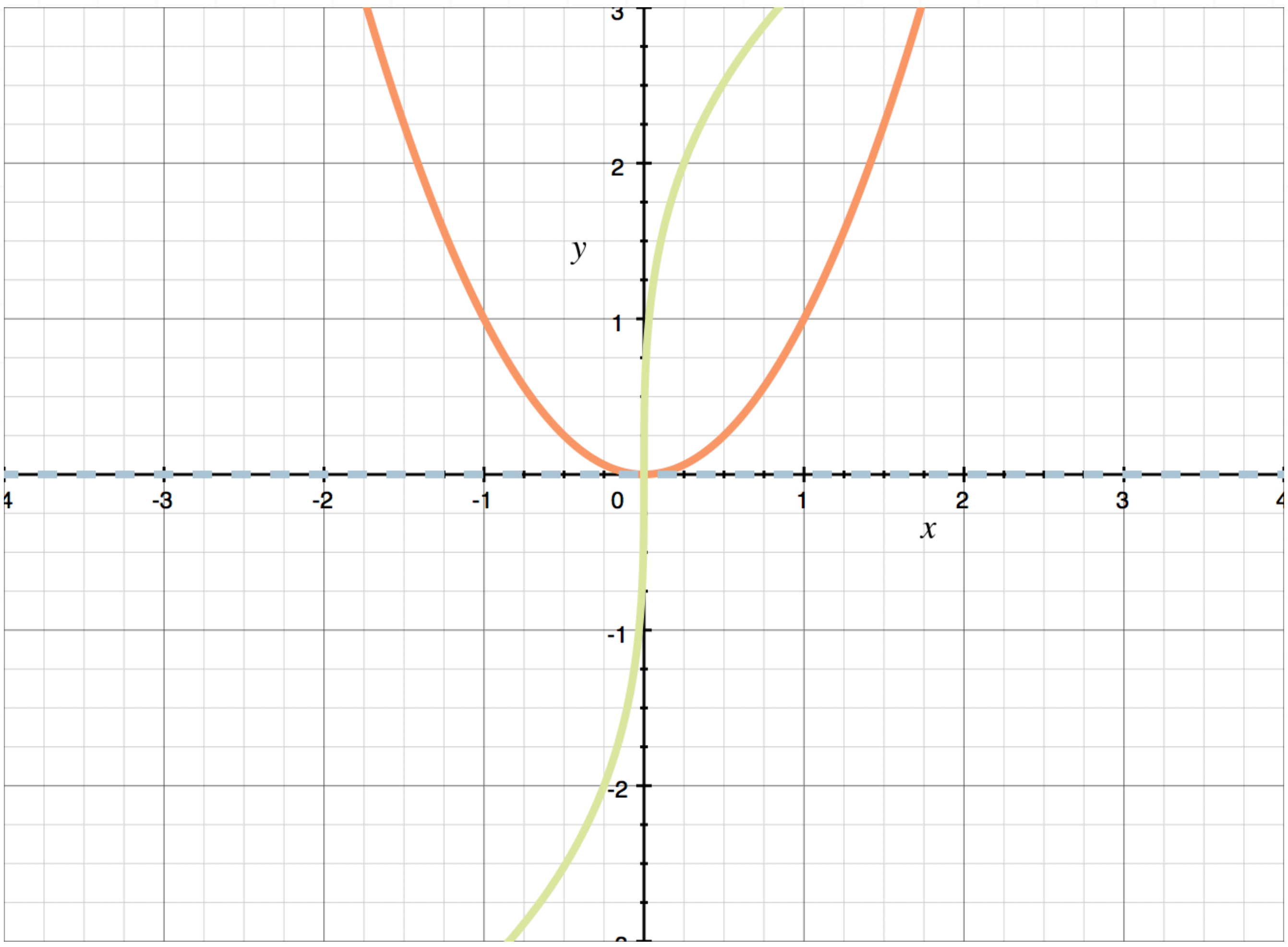
B 64π

C $\frac{64}{5}$

D $\frac{288\pi}{5}$

Solution: A

Before doing anything else, we always want to draw a picture of the area. If you don't know how to graph the function, just plug in values for x or y to get individual coordinate points, and plot them until you have a picture of each function.



Using washers means we'll take slices of our area that are perpendicular to the axis of rotation. Therefore, since the axis of rotation is horizontal, we'll take vertical slices of our area and rotate each of them around the axis to form washers.

Using washers around a horizontal axis, specifically the x-axis, tells us that we'll use the volume formula

$$V = \int_a^b \pi \left[f(x)\right]^2 - \pi \left[g(x)\right]^2 \ dx$$

We can see from the formula that we need our curves and our limits of integration defined in terms of x. The given curves are already defined for y in terms of x, so now we just need to find limits of integration, which will be the smallest and largest x-values for which the area is defined. Since these are just the two points of intersection, we can do this by looking at the graph, or we can set the curves equal to one another and solve for x.

$$x^2 = (32x)^{\frac{1}{3}}$$

$$x^6 = 32x$$

$$x^6 - 32x = 0$$

$$x\left(x^5 - 32\right) = 0$$

$$x = 0$$

or

$$x^5 - 32 = 0$$

$$x^5 = 32$$

$$\left(x^5\right)^{\frac{1}{5}} = (32)^{\frac{1}{5}}$$

$$x = 2$$

Now we know that our limits of integration are $a = 0$ and $b = 2$.

$f(x)$ is the radius of the curve that's further from the axis of revolution, and $g(x)$ is the radius of the curve that's closer to the axis of revolution.

To figure out which curve is further away and which one is closer, we can look at the graph or we can plug an x-value between the points of intersection (between $x = 0$ and $x = 2$) into both curves to see which function returns a larger value (this will be the further curve) and which one returns a smaller value (this will be the closer curve). Let's plug in $x = 1$ to check.

$$y = x^2$$

$$y = (1)^2$$

$$y = 1$$

and

$$y = (32x)^{\frac{1}{3}}$$

$$y = \left[32(1)\right]^{\frac{1}{3}}$$

$$y = (8 \cdot 4)^{\frac{1}{3}}$$

$$y = 8^{\frac{1}{3}}4^{\frac{1}{3}}$$

$$y = 2(4)^{\frac{1}{3}}$$

Since $y = (32x)^{\frac{1}{3}}$ returns a larger value than $y = x^2$, we can say

$$g(x) = x^2$$

and

$$f(x) = (32x)^{\frac{1}{3}}$$

Plugging everything we know into the volume formula, we get

$$V = \int_0^2 \pi \left[(32x)^{\frac{1}{3}}\right]^2 - \pi \left(x^2\right)^2 \ dx$$

$$V=\int_0^2 \pi\left[(32x)^{\frac{1}{3}}\right]^2-\pi x^4\ dx$$

$$V=\int_0^2 \pi(32x)^{\frac{2}{3}}-\pi x^4\ dx$$

$$V=\int_0^2 \pi\left(32^{\frac{2}{3}}x^{\frac{2}{3}}\right)-\pi x^4\ dx$$

$$V=\int_0^2 \pi\left(8^{\frac{2}{3}}4^{\frac{2}{3}}x^{\frac{2}{3}}\right)-\pi x^4\ dx$$

$$V=\int_0^2 4\pi\left(4^{\frac{2}{3}}x^{\frac{2}{3}}\right)-\pi x^4\ dx$$

$$V=\int_0^2 4\pi\left(16^{\frac{1}{3}}x^{\frac{2}{3}}\right)-\pi x^4\ dx$$

$$V=\int_0^2 4\pi\left(8^{\frac{1}{3}}2^{\frac{1}{3}}x^{\frac{2}{3}}\right)-\pi x^4\ dx$$

$$V=\int_0^2 8\pi\left(2^{\frac{1}{3}}x^{\frac{2}{3}}\right)-\pi x^4\ dx$$

$$V=\int_0^2 \left[8\pi(2)^{\frac{1}{3}}\right]x^{\frac{2}{3}}-\pi x^4\ dx$$

Integrate and then evaluate over the interval.

$$V=\left[\frac{3}{5}\left[8\pi(2)^{\frac{1}{3}}\right]x^{\frac{5}{3}}-\frac{\pi}{5}x^5\right]\Bigg|_0^2$$

$$V=\left[\frac{24\pi(2)^{\frac{1}{3}}}{5}x^{\frac{5}{3}}-\frac{\pi}{5}x^5\right]\Bigg|_0^2$$

$$V=\left[\frac{24\pi(2)^{\frac{1}{3}}}{5}(2)^{\frac{5}{3}}-\frac{\pi}{5}(2)^5\right]-\left[\frac{24\pi(2)^{\frac{1}{3}}}{5}(0)^{\frac{5}{3}}-\frac{\pi}{5}(0)^5\right]$$

$$V=\frac{24\pi(2)^{\frac{1}{3}}}{5}(32)^{\frac{1}{3}}-\frac{32\pi}{5}$$

$$V=\frac{24\pi(2)^{\frac{1}{3}}}{5}8^{\frac{1}{3}}4^{\frac{1}{3}}-\frac{32\pi}{5}$$

$$V=\frac{48\pi(2)^{\frac{1}{3}}(4)^{\frac{1}{3}}}{5}-\frac{32\pi}{5}$$

$$V=\frac{48\pi(2\cdot 4)^{\frac{1}{3}}}{5}-\frac{32\pi}{5}$$

$$V=\frac{48\pi(8)^{\frac{1}{3}}}{5}-\frac{32\pi}{5}$$

$$V=\frac{96\pi}{5}-\frac{32\pi}{5}$$

$$V=\frac{64\pi}{5}$$

Topic: Washers, horizontal axis

Question: Use washers to find the volume of the solid generated by revolving the region bounded by the curves about the given axis.

$y = x^2$ and $y = 0$ and $x = 1$

about $y = 1$

Answer choices:

A $\frac{2}{3}$

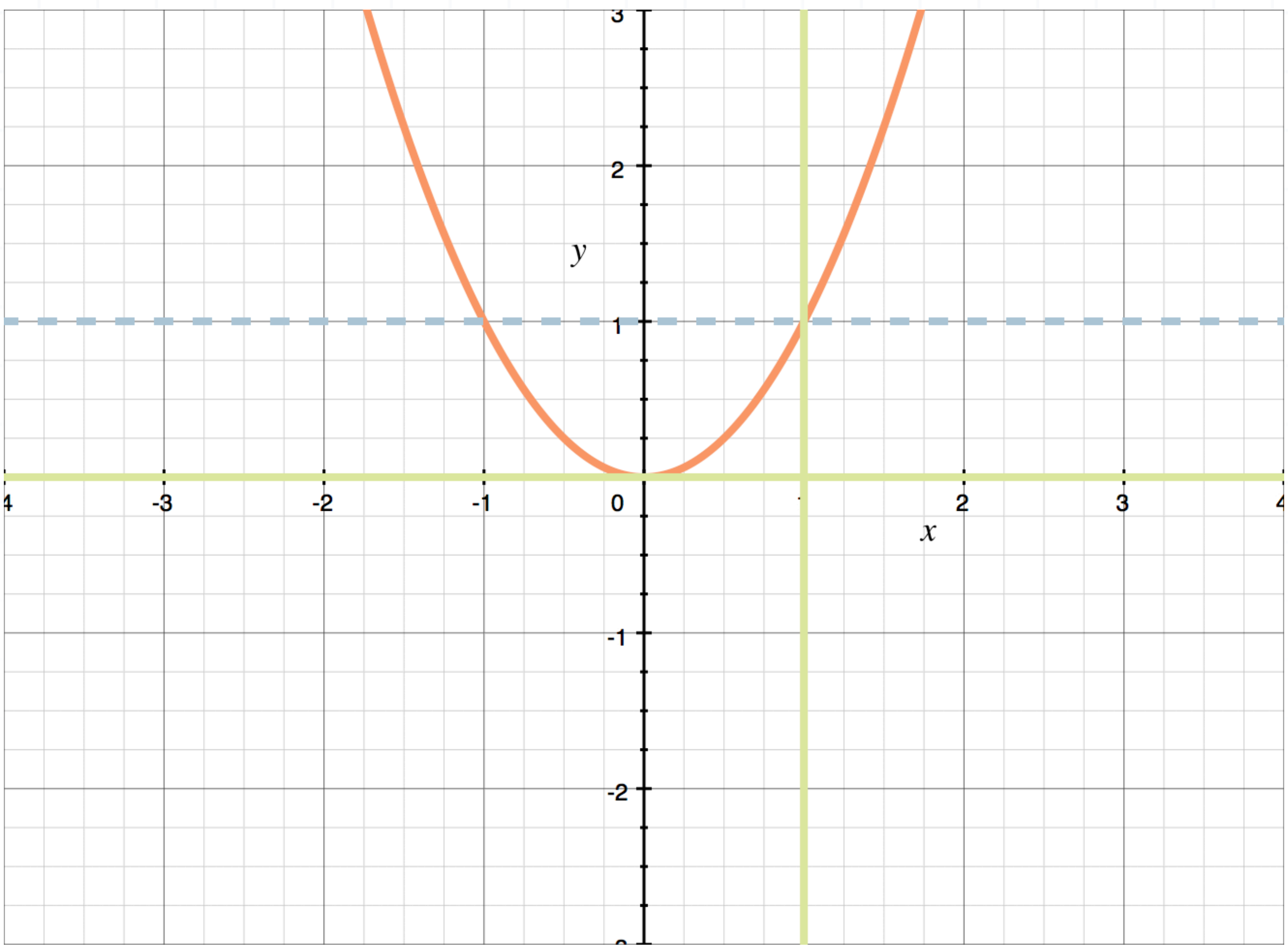
B $\frac{7\pi}{30}$

C $\frac{7\pi}{15}$

D $\frac{3}{2}$

Solution: C

Before doing anything else, we always want to draw a picture of the area. If you don't know how to graph the function, just plug in values for x or y to get individual coordinate points, and plot them until you have a picture of each function.



Using washers means we'll take slices of our area that are perpendicular to the axis of rotation. Therefore, since the axis of rotation is horizontal, we'll take vertical slices of our area and rotate each of them around the axis to form washers.

Using washers around a horizontal axis, specifically $y = 1$, tells us that we'll use the volume formula

$$V = \int_a^b \pi \left[k - g(x)\right]^2 - \pi \left[k - f(x)\right]^2 \ dx$$

We can see from the formula that we need our curves and our limits of integration defined in terms of x. The given curves are already defined for y in terms of x, so now we just need to find limits of integration, which will be the smallest and largest x-values for which the area is defined.

We can see from the graph that the largest x-value for which the area is defined is $x = 1$. This was given in the original problem. We can see that the smallest value for which it's defined is a point of intersection, so we can set the curves equal to one another and solve for x.

$$x^2 = 0$$

$$x = 0$$

Now we know that our limits of integration are $a = 0$ and $b = 1$.

$g(x)$ is the radius of the curve that's further from the axis of revolution, and $f(x)$ is the radius of the curve that's closer to the axis of revolution.

To figure out which curve is further away and which one is closer, we can look at the graph or we can plug an x-value between the points of intersection (between $x = 0$ and $x = 1$) into both curves to see which function returns a larger value (this will be the closer curve) and which one returns a smaller value (this will be the further curve). Let's plug in $x = 1/2$ to check.

$$y = x^2$$

$$y = \left(\frac{1}{2}\right)^2$$

$$y = \frac{1}{4}$$

and

$$y = 0$$

Since $y = x^2$ returns a larger value than $y = 0$, we can say

$$f(x) = x^2$$

and

$$g(x) = 0$$

Plugging everything we know into the volume formula, we get

$$V = \int_0^1 \pi (1-0)^2 - \pi \left(1 - x^2\right)^2 \ dx$$

$$V = \int_0^1 \pi - \pi \left(1 - 2x^2 + x^4\right) \ dx$$

$$V = \int_0^1 \pi - \pi + 2\pi x^2 - \pi x^4 \ dx$$

$$V = \int_0^1 2\pi x^2 - \pi x^4 \ dx$$

Integrate and then evaluate over the interval.

$$V = \left(\frac{2\pi}{3}x^3 - \frac{\pi}{5}x^5\right)\Bigg|_0^1$$

$$V = \left[\frac{2\pi}{3}(1)^3 - \frac{\pi}{5}(1)^5\right] - \left[\frac{2\pi}{3}(0)^3 - \frac{\pi}{5}(0)^5\right]$$

$$V = \frac{2\pi}{3} - \frac{\pi}{5}$$

$$V = \frac{10\pi}{15} - \frac{3\pi}{15}$$

$$V = \frac{7\pi}{15}$$

Topic: Washers, horizontal axis

Question: Use washers to find the volume of the solid formed by rotating the region bounded by the curves.

$y = x^2$ and $y = 0$

$x = 0$ and $x = 3$

about the line $y = -1$

Answer choices:

A $\quad V = 333\pi$ cubic units

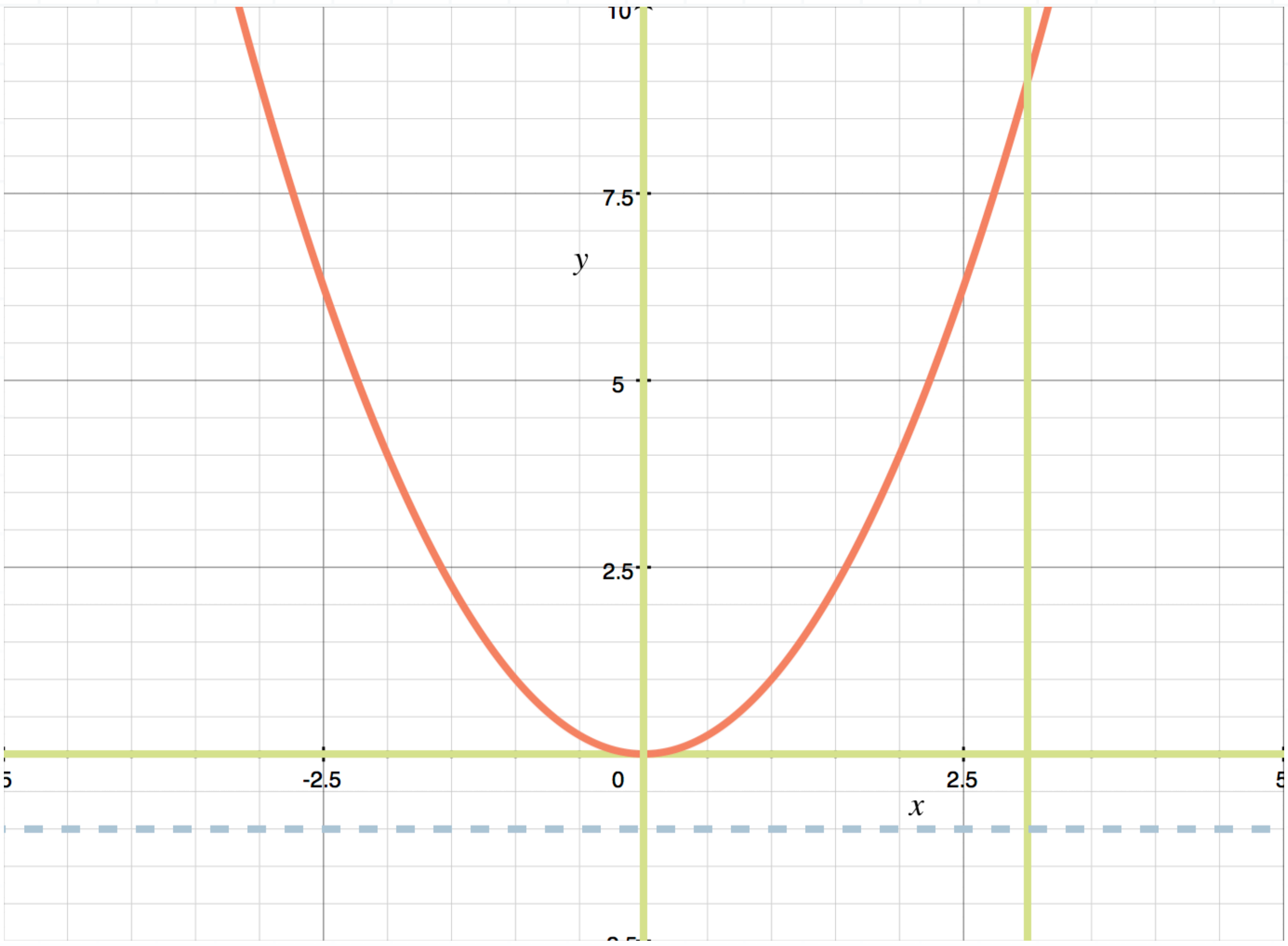
B $\quad V = \dfrac{333}{5}$ cubic units

C $\quad V = \dfrac{117}{4}\pi$ cubic units

D $\quad V = \dfrac{333}{5}\pi$ cubic units

Solution: D

The region enclosed by $y = x^2$, $y = 0$, $x = 0$ and $x = 3$ is



Because we're rotating about $y = -1$, and because our slices of volume must always be perpendicular to the axis of rotation, that means we'll be taking vertical slices of volume. Which means that the width of each infinitely thin slice of volume can be given by dx, which means we'll be integrating with respect to x. Therefore, the limits of integration will be given by $x = [0,3]$. The outer radius will be defined by $y = x^2$. So if the axis of rotation is $y = -k$, then the volume can be given by

$$V = \int_a^b \pi \left[k + f(x)\right]^2 - \pi \left[k + g(x)\right]^2 \ dx$$

$$V = \int_0^3 \pi \left(1 + x^2\right)^2 - \pi (1 + 0)^2 \ dx$$

$$V = \int_0^3 \pi \left(1 + 2x^2 + x^4\right) - \pi \ dx$$

$$V = \int_0^3 \pi + 2\pi x^2 + \pi x^4 - \pi \ dx$$

$$V = \int_0^3 2\pi x^2 + \pi x^4 \ dx$$

Integrate, then evaluate over the interval.

$$V = \frac{2}{3}\pi x^3 + \frac{1}{5}\pi x^5 \Big|_0^3$$

$$V = \frac{2}{3}\pi(3)^3 + \frac{1}{5}\pi(3)^5 - \left(\frac{2}{3}\pi(0)^3 + \frac{1}{5}\pi(0)^5\right)$$

$$V = 18\pi + \frac{243}{5}\pi$$

$$V = \frac{90}{5}\pi + \frac{243}{5}\pi$$

$$V = \frac{333}{5}\pi$$