

# Precise definition of the limit

The precise definition of the limit is something we use as a proof for the existence of a limit.

## The precise definition

Let's start by stating that  $f(x)$  is a function on an open interval that contains  $x = a$ , but that the function doesn't necessarily exist at  $x = a$ . The **precise definition of the limit** of the function tells us that, at  $x = a$ , the limit is  $L$ ,

$$\lim_{x \rightarrow a} f(x) = L$$

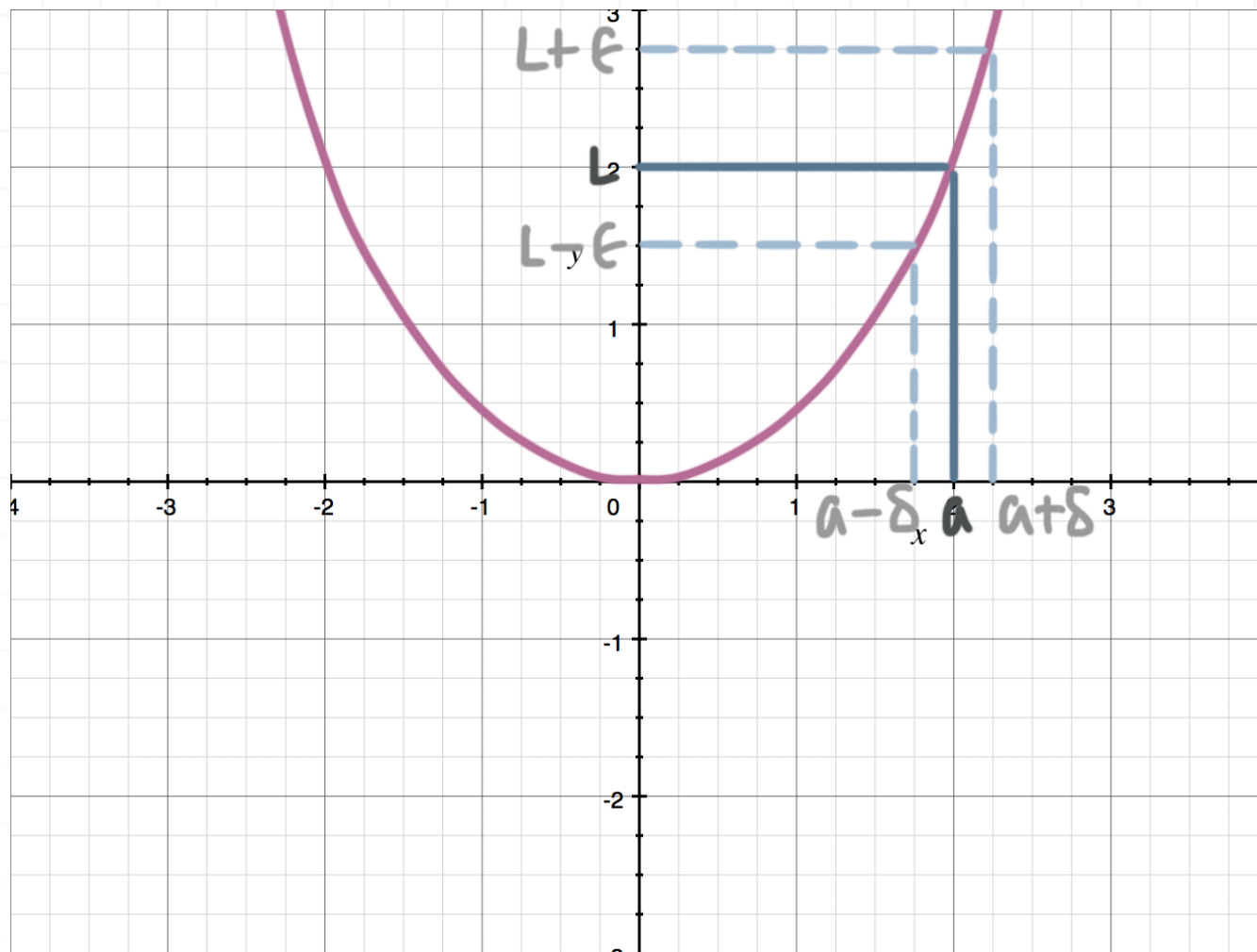
if for every number  $\epsilon > 0$  there is some number  $\delta > 0$  such that

$$|f(x) - L| < \epsilon \text{ whenever } 0 < |x - a| < \delta$$

What does all this mean? Well, since the open interval includes  $a$  but doesn't necessarily exist at  $a$ , we'll have to look at how the function behaves as it approaches  $a$ .  $L$  just represents the value of the limit.

When we're evaluating a limit, we're looking at the function as it approaches a specific point. In the graph,





that point is  $(a, L)$ . The precise definition of the limit proves that the limit exists and is  $L$ , as long as any number we pick between  $a - \delta$  and  $a + \delta$  will always return a value between  $L - \epsilon$  and  $L + \epsilon$ .

If this is true, then we know that if we pick a value that's closer and closer to  $a$ , the value we get back will be closer and closer to  $L$ . And that's the definition of the limit, that, as we approach  $x = a$ , the value of the function gets closer to  $L$ .

### Example

Using the precise definition of the limit, prove the following limit.

$$\lim_{x \rightarrow 4} (2x - 3) = 5$$



Substituting  $2x - 3$  for  $f(x)$ , 5 for  $L$ , and 4 for  $a$  into the definition, we get

$$|(2x - 3) - 5| < \epsilon \text{ whenever } 0 < |x - 4| < \delta$$

If we simplify  $|(2x - 3) - 5| < \epsilon$ , we get

$$|2x - 8| < \epsilon$$

$$2|x - 4| < \epsilon$$

$$|x - 4| < \frac{\epsilon}{2}$$

Notice now that the left side of this inequality looks just like the middle part of the inequality above that contains  $\delta$ . When this happens, we set  $\delta$  equal to the right-hand side of the last inequality, and we get

$$\delta = \frac{\epsilon}{2}$$

$$0 < |x - 4| < \delta = \frac{\epsilon}{2}$$

Going back to the beginning,

$$|(2x - 3) - 5| = |2x - 8|$$

$$|(2x - 3) - 5| = 2|x - 4|$$

and using the assumption that  $\delta = \epsilon/2$  and that  $0 < |x - 4| < \delta$ , by substitution, we get



$$|(2x - 3) - 5| = 2 \left| \frac{\epsilon}{2} \right|$$

$$|(2x - 3) - 5| = \epsilon$$

Since we started with  $|(2x - 3) - 5| < \epsilon$  and ended with  $\epsilon$ , we've shown that  $\epsilon = \epsilon$  and that

$$|(2x - 3) - 5| < \epsilon \text{ whenever } 0 < |x - 4| < \frac{\epsilon}{2}$$

Therefore,

$$\lim_{x \rightarrow 4} (2x - 3) = 5$$


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## Solving for delta

Sometimes we'll want to find  $\delta$ , given other values in the precise definition of the limit. When this is the case, we'll follow a specific set of steps in order to find the value of  $\delta$ .

### Example

Find  $\delta$  when  $f(x) = x^2$ , such that, if  $|x - 2| < \delta$  then  $|x^2 - 4| < 0.5$ .



We want to use the value for  $\epsilon$  to determine the  $\delta$  value by remembering that

$$0 < |x - a| < \delta \text{ then } |f(x) - L| < \epsilon$$

from the precise definition of the limit.

To solve for  $\delta$ , we'll take the epsilon value  $\epsilon = 0.5$  and the value of  $L$  to find the two  $y$ -values. This means we have  $4 + 0.5 = 4.5$  and  $4 - 0.5 = 3.5$ . Then we can plug these values into the function to get the associated  $x$ -values.

$$4.5 = x^2$$

$$x = 2.12$$

and

$$3.5 = x^2$$

$$x = 1.87$$

We'll find  $|x - a|$  with these two  $x$ -values and  $a = 2$ .

$$\text{For } x = 2.12: |x - a| = |2.12 - 2| = |0.12| = 0.12$$

$$\text{For } x = 1.87: |x - a| = |1.87 - 2| = |-0.13| = 0.13$$

If the two values are different, the smaller value will be the value we need to pick for  $\delta$ . Which means that for the function  $f(x) = x^2$ , such that if  $|x - 2| < \delta$  then  $|x^2 - 4| < 0.5$ , we know that  $\delta = 0.12$ .

