Solving with conjugate method

If substitution and factoring haven't worked, or don't apply, the next technique we should consider to evaluate a limit is the conjugate method.

Most often, we'll use conjugate method when our function is a fraction, and either the numerator and/or denominator contains exactly two terms. For instance, given the limit problem

$$\lim_{h\to 0}\frac{\sqrt{4+h}-2}{h}$$

we notice right away that substitution leads to a 0 denominator, and there's no factoring to be done in either the numerator or denominator. However, our function is a fraction, and the numerator contains exactly two terms, $\sqrt{4+h}$ and -2, so conjugate method might be a good technique for evaluating this limit.

Because the numerator contains two terms, we want to find the conjugate of the numerator. The **conjugate** of an expression is an expression with the same two terms, but with the opposite sign between the terms. For instance, the conjugate of $\sqrt{4+h}-2$ is $\sqrt{4+h}+2$. It's the same two terms, but the sign in between the terms has been flipped from $-$ to $+$. As another example, the conjugate of $-3+\sqrt{x}$ would be $-3-\sqrt{x}$.

Once we've found the appropriate conjugate, we multiply both the numerator and denominator by the conjugate we've found. The reason we do this is because it should simplify the function and, hopefully, allow us to evaluate the limit with substitution.

Example

Evaluate the limit.

$$\lim_{h\to0}\frac{\sqrt{4+h}-2}{h}$$

As we've said, substitution and factoring both lead to dead-ends when we try to use them to evaluate this limit, so we'll try conjugate method, instead.

The conjugate of $\sqrt{4+h}-2$ is $\sqrt{4+h}+2$, so we'll multiply both the numerator and denominator by that conjugate.

$$\lim_{h\to0}\frac{\sqrt{4+h}-2}{h}\left(\frac{\sqrt{4+h}+2}{\sqrt{4+h}+2}\right)$$

It's okay to multiply by the conjugate like this, because, since we're multiplying both the numerator and denominator by the same value, it's as if we're multiplying by 1, which doesn't change the value of the function. We're just rewriting the function in a fancy way, but we're not actually changing its value.

$$\lim_{h\to0}\frac{\sqrt{4+h}\sqrt{4+h}+2\sqrt{4+h}-2\sqrt{4+h}-4}{h(\sqrt{4+h}+2)}$$

$$\lim_{h\to0}\frac{4+h+2\sqrt{4+h}-2\sqrt{4+h}-4}{h(\sqrt{4+h}+2)}$$

The two middle terms in the numerator cancel each other. This is why the conjugate method is helpful in simplifying certain functions.

$$\lim_{h \to 0} \frac{4 + h - 4}{h(\sqrt{4 + h} + 2)}$$

$$\lim_{h \to 0} \frac{h}{h(\sqrt{4 + h} + 2)}$$

The common factor of h can be canceled from both the numerator and denominator.

$$\lim_{h \to 0} \frac{1}{\sqrt{4 + h} + 2}$$

Now that we've simplified the function using conjugate method, we'll try substitution to evaluate the limit as $h \to 0$.

$$\frac{1}{\sqrt{4 + 0} + 2}$$

$$\frac{1}{2 + 2}$$

$$\frac{1}{4}$$