Calculus 2 Workbook Solutions

Maclaurin series

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MATH

MACLAURIN SERIES

1. Write the first four non-zero terms of the Maclaurin series and use it to estimate $f(\pi/9)$.

$$f(x)=\cos(3x)$$

Solution:

The common Maclaurin series for $f(x)=\cos x$ is

$$\cos x=1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\cdots+(-1)^n\frac{x^{2n}}{(2n)!}+\cdots$$

So the Maclaurin series for $f(x)=\cos(3x)$ is

$$\cos(3x)=1-\frac{(3x)^2}{2!}+\frac{(3x)^4}{4!}-\frac{(3x)^6}{6!}+\cdots+(-1)^n\frac{(3x)^{2n}}{(2n)!}+\cdots$$

$$\cos(3x)=1-\frac{9x^2}{2}+\frac{81x^4}{24}-\frac{729x^6}{720}+\cdots+(-1)^n\frac{(3x)^{2n}}{(2n)!}+\cdots$$

$$\cos(3x)=1-\frac{9x^2}{2}+\frac{27x^4}{8}-\frac{81x^6}{80}+\cdots+(-1)^n\frac{(3x)^{2n}}{(2n)!}+\cdots$$

Then $f(\pi/9)$ is

$$\cos\left(3\cdot\frac{\pi}{9}\right)=1-\frac{9\left(\frac{\pi}{9}\right)^2}{2}+\frac{27\left(\frac{\pi}{9}\right)^4}{8}-\frac{81\left(\frac{\pi}{9}\right)^6}{80}$$

$$\cos\left(\frac{\pi}{3}\right) \approx 1 - 0.548311 + 0.050108 - 0.001832$$

$$\cos\left(\frac{\pi}{3}\right) \approx 0.499965$$

$$\cos\left(\frac{\pi}{3}\right) \approx 0.500$$

■ 2. Write the first three non-zero terms of the Maclaurin series and use it to estimate $f(2\pi/3)$.

$$f(x) = \cos^2 x$$

Solution:

Find the first few terms of the series.

$n = 0$ $\qquad f(x) = \cos^2 x$ $\qquad f(0) = 1$

$n = 1$ $\qquad f'(x) = -2\cos x \sin x$ $\qquad f'(0) = 0$

$n = 2$ $\qquad f''(x) = 2 - 4\cos^2 x$ $\qquad f''(0) = -2$

$n = 3$ $\qquad f'''(x) = 8 \sin x \cos x$ $\qquad f'''(0) = 0$

$n = 4$ $\qquad f^{(4)}(x) = 16\cos^2 x - 8$ $\qquad f^{(4)}(0) = 8$

So the Maclaurin series for $f(x) = \cos^2 x$ is

$$f(x) = \frac{f(0)x^0}{0!} + \frac{f'(0)x^1}{1!} + \frac{f''(0)x^2}{2!} + \frac{f'''(0)x^3}{3!} + \cdots + \frac{f^{(n)}(0)x^n}{n!}$$

$$f_{(4)}(x) = \frac{f(0)x^0}{0!} + \frac{f'(0)x^1}{1!} + \frac{f''(0)x^2}{2!} + \frac{f'''(0)x^3}{3!} + \frac{f^{(4)}(0)x^4}{4!}$$

$$f_{(4)}(x) = \frac{1x^0}{0!} + \frac{0 \cdot x^1}{1!} + \frac{-2 \cdot x^2}{2!} + \frac{0 \cdot x^3}{3!} + \frac{8 \cdot x^4}{4!}$$

$$f_{(4)}(x) = 1 + 0x - x^2 + 0x^3 + \frac{x^4}{3}$$

$$f_{(4)}(x) = 1 - x^2 + \frac{x^4}{3}$$

Then $f(2\pi/3)$ is

$$f\left(\frac{2\pi}{3}\right) \approx 1 - \left(\frac{2\pi}{3}\right)^2 + \frac{\left(\frac{2\pi}{3}\right)^4}{3}$$

$$f\left(\frac{2\pi}{3}\right) \approx 1 - 4.386491 + 6.413767$$

$$f\left(\frac{2\pi}{3}\right) \approx 3.027276$$

■ 3. Write the first four non-zero terms of the Maclaurin series and use it to estimate $f(2)$.

$$f(x) = (x+4)^{\frac{3}{2}}$$

Solution:

Find the first few terms of the series.

$n = 0$ $\qquad f(x) = (x+4)^{\frac{3}{2}}$ $\qquad f(0) = (0+4)^{\frac{3}{2}} = 8$

$n = 1$ $\qquad f'(x) = \dfrac{3(x+4)^{\frac{1}{2}}}{2}$ $\qquad f'(0) = \dfrac{3(0+4)^{\frac{1}{2}}}{2} = 3$

$n = 2$ $\qquad f''(x) = \dfrac{3}{4(x+4)^{\frac{1}{2}}}$ $\qquad f''(0) = \dfrac{3}{4(0+4)^{\frac{1}{2}}} = \dfrac{3}{8}$

$n = 3$ $\qquad f'''(x) = \dfrac{-3}{8(x+4)^{\frac{3}{2}}}$ $\qquad f'''(0) = \dfrac{-3}{8(0+4)^{\frac{3}{2}}} = -\dfrac{3}{64}$

$n = 4$ $\qquad f^{(4)}(x) = \dfrac{9}{16(x+4)^{\frac{5}{2}}}$ $\qquad f^{(4)}(0) = \dfrac{9}{16(0+4)^{\frac{5}{2}}} = \dfrac{9}{512}$

So the Maclaurin series for $f(x) = (x+4)^{\frac{3}{2}}$ is

$$f(x) = \frac{f(0)x^0}{0!} + \frac{f'(0)x^1}{1!} + \frac{f''(0)x^2}{2!} + \frac{f'''(0)x^3}{3!} + \cdots + \frac{f^{(n)}(0)x^n}{n!}$$

$$f_{(4)}(x) = \frac{f(0)x^0}{0!} + \frac{f'(0)x^1}{1!} + \frac{f''(0)x^2}{2!} + \frac{f'''(0)x^3}{3!} + \frac{f^{(4)}(0)x^4}{4!}$$

$$f_{(4)}(x) = \frac{8x^0}{0!} + \frac{3 \cdot x^1}{1!} + \frac{3 \cdot x^2}{8 \cdot 2!} + \frac{-3 \cdot x^3}{64 \cdot 3!} + \frac{9 \cdot x^4}{512 \cdot 4!}$$

$$f_{(4)}(x) = 8 + 3x + \frac{3x^2}{16} - \frac{x^3}{128} + \frac{3x^4}{4{,}096}$$

$$f_{(3)}(x) = 8 + 3x + \frac{3x^2}{16} - \frac{x^3}{128}$$

Then $f(2)$ is

$$f(2) \approx 8 + 3x + \frac{3x^2}{16} - \frac{x^3}{128}$$

$$f(2) \approx 8 + 3(2) + \frac{3(2)^2}{16} - \frac{(2)^3}{128}$$

$$f(2) \approx 8 + 6 + \frac{12}{16} - \frac{8}{128}$$

$$f(2) \approx 14.6875$$

SUM OF THE MACLAURIN SERIES

1. Find the sum of the Maclaurin series.

$$\sum_{n=0}^{\infty} \frac{7(x+4)^n}{n!}$$

Solution:

Begin with the common series

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

and then start manipulating it until it matches the given series.

$$e^{x+4} = \sum_{n=0}^{\infty} \frac{(x+4)^n}{n!}$$

$$7e^{x+4} = \sum_{n=0}^{\infty} \frac{7(x+4)^n}{n!}$$

So the sum of the series is $7e^{x+4}$.

2. Find the sum of the Maclaurin series.

$$\sum_{n=0}^{\infty} \frac{6(-1)^n(x-\pi)^{2n+1}}{7(2n+1)!}$$

Solution:

Begin with the common series

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n(x)^{2n+1}}{(2n+1)!}$$

and then start manipulating it until it matches the given series.

$$\sin(x-\pi) = \sum_{n=0}^{\infty} \frac{(-1)^n(x-\pi)^{2n+1}}{(2n+1)!}$$

$$\frac{6}{7}\sin(x-\pi) = \sum_{n=0}^{\infty} \frac{6(-1)^n(x-\pi)^{2n+1}}{7(2n+1)!}$$

So the sum of the series is $(6/7)\sin(x-\pi)$.

■ 3. Find the sum of the Maclaurin series.

$$4 + \sum_{n=0}^{\infty} \frac{e(-1)^n(x+\pi)^{2n}}{3(2n)!}$$

Solution:

Begin with the common series

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$$

and then start manipulating it until it matches the given series.

$$\cos(x+\pi) = \sum_{n=0}^{\infty} \frac{(-1)^n (x+\pi)^{2n}}{(2n)!}$$

$$\frac{e}{3}\cos(x+\pi) = \sum_{n=0}^{\infty} \frac{e(-1)^n (x+\pi)^{2n}}{3(2n)!}$$

$$4+\frac{e}{3}\cos(x+\pi) = 4+\sum_{n=0}^{\infty} \frac{e(-1)^n (x+\pi)^{2n}}{3(2n)!}$$

So the sum of the series is

$$4+\frac{e}{3}\cos(x+\pi)$$

RADIUS AND INTERVAL OF CONVERGENCE OF A MACLAURIN SERIES

1. Find the radius of convergence of the Maclaurin series.

$$f(x) = \frac{5}{1 - x^3}$$

Solution:

Start with the common series.

$$\frac{1}{1-x} = \sum_{n=1}^{\infty} x^n$$

and then start manipulating it until it matches the given series.

$$\frac{5}{1-x} = 5\sum_{n=1}^{\infty} x^n = \sum_{n=1}^{\infty} 5x^n$$

$$\frac{5}{1-x^3} = \sum_{n=1}^{\infty} 5(x^3)^n = \sum_{n=1}^{\infty} 5x^{3n}$$

Use the ratio test.

$$L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$$

$$L = \lim_{n\to\infty} \left| \frac{5x^{3(n+1)}}{5x^{3n}} \right|$$

$$L = \lim_{n \to \infty} \left| \frac{5x^{3n+3}}{5x^{3n}} \right|$$

$$L = \lim_{n \to \infty} \left| x^3 \right| < 1$$

So the interval of convergence is

$$-1 < x^3 < 1$$

$$\sqrt[3]{-1} < x < \sqrt[3]{1}$$

$$-1 < x < 1$$

The interval of convergence spans from -1 to 1, which is 2 units wide. The radius of convergence is half that, which means the radius of convergence is 1.

■ 2. Find the radius of convergence of the Maclaurin series.

$$f(x) = 4\cos(x^2)$$

Solution:

Start with the common series.

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

and then start manipulating it until it matches the given series.

$$4\cos x = \sum_{n=0}^{\infty} \frac{4(-1)^n x^{2n}}{(2n)!}$$

$$4\cos(x^2) = \sum_{n=0}^{\infty} \frac{4(-1)^n (x^2)^{2n}}{(2n)!}$$

$$4\cos(x^2) = \sum_{n=0}^{\infty} \frac{4(-1)^n x^{4n}}{(2n)!}$$

Use the ratio test.

$$L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$$

$$L = \lim_{n\to\infty} \left| \frac{\frac{4(-1)^{n+1} x^{4(n+1)}}{(2(n+1))!}}{\frac{4(-1)^n x^{4n}}{(2n)!}} \right|$$

$$L = \lim_{n\to\infty} \left| \frac{\frac{4x^{4(n+1)}}{(2(n+1))!}}{\frac{4x^{4n}}{(2n)!}} \right|$$

$$L = \lim_{n\to\infty} \left| \frac{\frac{x^{4(n+1)}}{(2(n+1))!}}{\frac{x^{4n}}{(2n)!}} \right|$$

$$L = \lim_{n\to\infty}\left|\frac{x^{4n+4}}{(2n+2)!}\cdot\frac{(2n)!}{x^{4n}}\right|$$

$$L = \lim_{n\to\infty}\left|\frac{x^4}{(2n+2)(2n+1)(2n)!}\cdot\frac{(2n)!}{1}\right|$$

$$L = \lim_{n\to\infty}\left|\frac{x^4}{(2n+2)(2n+1)}\cdot\frac{1}{1}\right|$$

$$L = \lim_{n\to\infty}\left|\frac{x^4}{(2n+2)(2n+1)}\right|$$

$$L = x^4\lim_{n\to\infty}\left|\frac{1}{(2n+2)(2n+1)}\right|$$

$$L = x^4\cdot 0$$

$$L = 0$$

The series converges if $L<1$ and diverges if $L>1$, so this series converges everywhere, which means the interval of convergence is ∞, and therefore the radius of conversion is ∞, too.

■ 3. Find the radius of convergence of the Maclaurin series.

$$\sum_{n=1}^{\infty}\frac{x^n\cdot 3^n}{n}$$

Solution:

Use the ratio test.

$$L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$$

$$L = \lim_{n\to\infty} \left| \frac{\frac{x^{n+1} \cdot 3^{n+1}}{n+1}}{\frac{x^n \cdot 3^n}{n}} \right|$$

$$L = \lim_{n\to\infty} \left| \frac{x^{n+1} \cdot 3^{n+1}}{n+1} \cdot \frac{n}{x^n \cdot 3^n} \right|$$

$$L = \lim_{n\to\infty} \left| \frac{x \cdot 3}{n+1} \cdot \frac{n}{1} \right|$$

$$L = \left| 3x \right| \lim_{n\to\infty} \left| \frac{1}{n+1} \cdot \frac{n}{1} \right|$$

$$L = \left| 3x \right| \lim_{n\to\infty} \left| \frac{n}{n+1} \right|$$

$$L = \left| 3x \right| \cdot 1$$

$$L = \left| 3x \right|$$

So the interval of convergence is

$$|3x| < 1$$

$$-1 < 3x < 1$$

$$-\frac{1}{3} < x < \frac{1}{3}$$

The interval of convergence spans from $-1/3$ to $1/3$, which is a width of $2/3$. The radius of convergence is half that, so the radius of convergence is $1/3$.

INDEFINITE INTEGRAL AS AN INFINITE SERIES

■ 1. Use an infinite series to evaluate the indefinite integral.

$$\int x^2 \cos(x^3)\ dx$$

Solution:

Start with the known Maclaurin series

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

and then start manipulating it until it matches the given series, first by substituting x^3 for x,

$$\cos x^3 = \sum_{n=0}^{\infty} \frac{(-1)^n (x^3)^{2n}}{(2n)!}$$

$$\cos x^3 = \sum_{n=0}^{\infty} \frac{(-1)^n x^{6n}}{(2n)!}$$

and then multiplying by x^2.

$$x^2 \cos x^3 = x^2 \sum_{n=0}^{\infty} \frac{(-1)^n x^{6n}}{(2n)!}$$

$$x^2 \cos x^3 = \sum_{n=0}^{\infty} \frac{(-1)^n x^2 \cdot x^{6n}}{(2n)!}$$

$$x^2 \cos x^3 = \sum_{n=0}^{\infty} \frac{(-1)^n x^{6n+2}}{(2n)!}$$

Integrate the series.

$$\int x^2 \cos(x^3)\ dx = \int \sum_{n=0}^{\infty} \frac{(-1)^n x^{6n+2}}{(2n)!}\ dx$$

$$\int x^2 \cos(x^3)\ dx = \frac{(-1)^n}{(2n)!} \int \sum_{n=0}^{\infty} x^{6n+2}\ dx$$

$$\int x^2 \cos(x^3)\ dx = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \cdot \frac{x^{6n+3}}{6n+3} + C$$

$$\int x^2 \cos(x^3)\ dx = C + \sum_{n=0}^{\infty} \frac{(-1)^n x^{6n+3}}{(6n+3)(2n)!}$$

■ 2. Use an infinite series to evaluate the indefinite integral.

$$\int 4x^3 \sin(x^4)\ dx$$

Solution:

Start with the known Maclaurin series

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

and then start manipulating it until it matches the given series, first by substituting x^4 for x,

$$\sin(x^4) = \sum_{n=0}^{\infty} \frac{(-1)^n (x^4)^{2n+1}}{(2n+1)!}$$

$$\sin(x^4) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{8n+4}}{(2n+1)!}$$

and then multiplying by $4x^3$.

$$4x^3 \sin(x^4) = \sum_{n=0}^{\infty} \frac{4x^3(-1)^n x^{8n+4}}{(2n+1)!}$$

$$4x^3 \sin(x^4) = \sum_{n=0}^{\infty} \frac{4(-1)^n x^{8n+7}}{(2n+1)!}$$

Integrate the series.

$$\int 4x^3 \sin(x^4)\ dx = \int \sum_{n=0}^{\infty} \frac{4(-1)^n x^{8n+7}}{(2n+1)!}\ dx$$

$$\int 4x^3 \sin(x^4)\ dx = \frac{4(-1)^n}{(2n+1)!} \int \sum_{n=0}^{\infty} x^{8n+7}\ dx$$

$$\int 4x^3 \sin(x^4)\ dx = \sum_{n=0}^{\infty} \frac{4(-1)^n}{(2n+1)!} \cdot \frac{x^{8n+8}}{8n+8} + C$$

$$\int 4x^3 \sin(x^4)\ dx = C + \sum_{n=0}^{\infty} \frac{4(-1)^n x^{8n+8}}{(8n+8)(2n+1)!}$$

■ 3. Use an infinite series to evaluate the indefinite integral.

$$\int 2x \ln(1+x^2)\ dx$$

Solution:

Start with the known Maclaurin series

$$\ln(1+x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n}$$

and then start manipulating it until it matches the given series, first by substituting x^2 for x,

$$\ln(1+x^2) = \sum_{n=0}^{\infty} \frac{(-1)^n (x^2)^n}{n}$$

$$\ln(1+x^2) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n}$$

and then multiplying by $2x$.

$$2x \ln(1+x^2) = 2x \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n}$$

$$2x\ln(1+x^2) = \sum_{n=0}^{\infty} \frac{2x^1(-1)^n x^{2n}}{n}$$

$$2x\ln(1+x^2) = \sum_{n=0}^{\infty} \frac{2(-1)^n x^{2n+1}}{n}$$

Integrate the series.

$$\int 2x\ln(1+x^2)\ dx = \int \sum_{n=0}^{\infty} \frac{2(-1)^n x^{2n+1}}{n}\ dx$$

$$\int 2x\ln(1+x^2)\ dx = \frac{2(-1)^n}{n} \int \sum_{n=0}^{\infty} x^{2n+1}\ dx$$

$$\int 2x\ln(1+x^2)\ dx = \sum_{n=0}^{\infty} \frac{2(-1)^n}{n} \cdot \frac{x^{2n+2}}{2n+2} + C$$

$$\int 2x\ln(1+x^2)\ dx = C + \sum_{n=0}^{\infty} \frac{2(-1)^n x^{2n+2}}{2n^2+2n}$$

MACLAURIN SERIES TO ESTIMATE AN INDEFINITE INTEGRAL

1. Use a Maclaurin series to estimate the indefinite integral.

$$\int \frac{\sin(2x)}{4x} \ dx$$

Solution:

Begin with the common series

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n (x)^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \cdots$$

and then start manipulating it until it matches the given series, first by substituting $2x$ for x,

$$\sin(2x) = \sum_{n=0}^{\infty} \frac{(-1)^n (2x)^{2n+1}}{(2n+1)!} = 2x - \frac{(2x)^3}{3!} + \frac{(2x)^5}{5!} - \frac{(2x)^7}{7!} + \frac{(2x)^9}{9!} - \cdots$$

$$\sin(2x) = \sum_{n=0}^{\infty} \frac{(-1)^n (2x)^{2n+1}}{(2n+1)!} = 2x - \frac{8x^3}{6} + \frac{32x^5}{120} - \frac{128x^7}{5{,}040} + \frac{512x^9}{362{,}880} - \cdots$$

$$\sin(2x) = \sum_{n=0}^{\infty} \frac{(-1)^n (2x)^{2n+1}}{(2n+1)!} = 2x - \frac{4x^3}{3} + \frac{4x^5}{15} - \frac{8x^7}{315} + \frac{4x^9}{2{,}835} - \cdots$$

and then dividing by $4x$.

$$\frac{\sin(2x)}{4x} = \sum_{n=0}^{\infty} \frac{(-1)^n(2x)^{2n+1}}{4x(2n+1)!} = \frac{2x}{4x} - \frac{4x^3}{3 \cdot 4x} + \frac{4x^5}{15 \cdot 4x} - \frac{8x^7}{315 \cdot 4x} + \frac{4x^9}{2{,}835 \cdot 4x} - \cdots$$

$$\frac{\sin(2x)}{4x} = \sum_{n=0}^{\infty} \frac{(-1)^n(2x)^{2n+1}}{4x(2n+1)!} = \frac{1}{2} - \frac{x^2}{3} + \frac{x^4}{15} - \frac{2x^6}{315} + \frac{x^8}{2{,}835} - \cdots$$

$$\frac{\sin(2x)}{4x} = \sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n+1} x^{2n+1}}{2^2 x^1 (2n+1)!} = \frac{1}{2} - \frac{x^2}{3} + \frac{x^4}{15} - \frac{2x^6}{315} + \frac{x^8}{2{,}835} - \cdots$$

$$\frac{\sin(2x)}{4x} = \sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n-1} x^{2n}}{(2n+1)!} = \frac{1}{2} - \frac{x^2}{3} + \frac{x^4}{15} - \frac{2x^6}{315} + \frac{x^8}{2{,}835} - \cdots$$

Integrate the series term by term.

$$\int \frac{\sin(2x)}{4x}\ dx = \int \frac{1}{2} - \frac{x^2}{3} + \frac{x^4}{15} - \frac{2x^6}{315} + \frac{x^8}{2{,}835} - \dots\ dx$$

$$\int \frac{\sin(2x)}{4x}\ dx = C + \frac{x}{2} - \frac{x^{2+1}}{3 \cdot 3} + \frac{x^{4+1}}{15 \cdot 5} - \frac{2x^{6+1}}{315 \cdot 7} + \frac{x^{8+1}}{2{,}835 \cdot 9} - \dots$$

$$\int \frac{\sin(2x)}{4x}\ dx = C + \frac{x}{2} - \frac{x^3}{9} + \frac{x^5}{75} - \frac{2x^7}{2{,}205} + \frac{x^9}{25{,}515} - \dots$$

Then the indefinite integral can be expressed as

$$\int \frac{\sin(2x)}{4x}\ dx = \sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n-1} x^{2n+1}}{(2n+1)(2n+1)!} + C$$

■ 2. Use a Maclaurin series to estimate the indefinite integral.

$$\int \frac{\cos x}{x^2}\ dx$$

Solution:

Begin with the common series

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \ldots$$

and then start manipulating it until it matches the given series, by dividing through by x^2.

$$\frac{\cos x}{x^2} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{x^2(2n)!} = \frac{1}{x^2} - \frac{x^2}{2!x^2} + \frac{x^4}{4!x^2} - \frac{x^6}{6!x^2} + \frac{x^8}{8!x^2} - \cdots$$

$$\frac{\cos x}{x^2} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n-2}}{(2n)!} = \frac{1}{x^2} - \frac{1}{2} + \frac{x^2}{24} - \frac{x^4}{720} + \frac{x^6}{40{,}320} - \cdots$$

Integrate the series term by term.

$$\int \frac{\cos x}{x^2}\ dx = \int \frac{1}{x^2} - \frac{1}{2} + \frac{x^2}{24} - \frac{x^4}{720} + \frac{x^6}{40{,}320} - \ldots\ dx$$

$$\int \frac{\cos x}{x^2}\ dx = \int x^{-2} - \frac{1}{2} + \frac{x^2}{24} - \frac{x^4}{720} + \frac{x^6}{40{,}320} - \ldots\ dx$$

$$\int \frac{\cos x}{x^2}\ dx = C - \frac{1}{x} - \frac{x}{2} + \frac{x^3}{24 \cdot 3} - \frac{x^5}{720 \cdot 5} + \frac{x^7}{40{,}320 \cdot 7} - \ldots$$

$$\int \frac{\cos x}{x^2}\,dx = C - \frac{1}{x} - \frac{x}{2} + \frac{x^3}{72} - \frac{x^5}{3{,}600} + \frac{x^7}{282{,}240} - \dots$$

Then the indefinite integral can be expressed as this series:

$$\int \frac{\cos x}{x^2}\,dx = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n-1}}{(2n-1)(2n)!} + C$$

■ 3. Use a Maclaurin series to estimate the indefinite integral.

$$\int \frac{\arctan x}{x^2}\,dx$$

Solution:

Begin with the common series

$$\arctan x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \dots$$

and then start manipulating it until it matches the given series, by dividing through by x^2.

$$\frac{\arctan x}{x^2} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)x^2} = \frac{x}{x^2} - \frac{x^3}{3x^2} + \frac{x^5}{5x^2} - \frac{x^7}{7x^2} + \frac{x^9}{9x^2} - \cdots$$

$$\frac{\arctan x}{x^2} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n-1}}{(2n+1)} = \frac{1}{x} - \frac{x}{3} + \frac{x^3}{5} - \frac{x^5}{7} + \frac{x^7}{9} - \cdots$$

Integrate the series term by term.

$$\int \frac{\arctan x}{x^2}\,dx = \int \frac{1}{x} - \frac{x}{3} + \frac{x^3}{5} - \frac{x^5}{7} + \frac{x^7}{9} - \dots\,dx$$

$$\int \frac{\arctan x}{x^2}\,dx = C + \ln|x| - \frac{x^2}{3 \cdot 2} + \frac{x^4}{5 \cdot 4} - \frac{x^6}{7 \cdot 6} + \frac{x^8}{9 \cdot 8} - \dots$$

$$\int \frac{\arctan x}{x^2}\,dx = C + \ln|x| - \frac{x^2}{6} + \frac{x^4}{20} - \frac{x^6}{42} + \frac{x^8}{72} - \dots$$

Then the indefinite integral can be expressed as this series:

$$\int \frac{\arctan x}{x^2}dx = C + \ln|x| + \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n}}{2n(2n+1)}$$

MACLAURIN SERIES TO ESTIMATE A DEFINITE INTEGRAL

1. Use a Maclaurin series to estimate the value of the definite integral.

$$\int_0^3 3xe^{\frac{1}{2}x^2}\ dx$$

Solution:

Start with the common series

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

and then start manipulating it until it matches the given series, first by substituting $(1/2)x^2$ for x,

$$e^{\frac{1}{2}x^2} = \sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}x^2\right)^n}{n!} = \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n \frac{x^{2n}}{n!} = \sum_{n=0}^{\infty} \frac{x^{2n}}{2^n n!}$$

and then multiplying by $3x$.

$$3xe^{\frac{1}{2}x^2} = 3x\sum_{n=0}^{\infty} \frac{x^{2n}}{2^n n!} = \sum_{n=0}^{\infty} \frac{3x^1 \cdot x^{2n}}{2^n n!} = \sum_{n=0}^{\infty} \frac{3x^{2n+1}}{2^n n!}$$

Integrate the power series.

$$\int_0^3 3xe^{\frac{1}{2}x^2}\ dx = \int_0^3 \sum_{n=0}^{\infty} \frac{3x^{2n+1}}{2^n n!}\ dx$$

$$\int_0^3 3xe^{\frac{1}{2}x^2}\ dx = \frac{3}{2^n n!} \int_0^3 \sum_{n=0}^{\infty} x^{2n+1}\ dx$$

$$\int_0^3 3xe^{\frac{1}{2}x^2}\ dx = \sum_{n=0}^{\infty} \frac{3}{2^n n!} \cdot \frac{x^{2n+2}}{2n+2} \Bigg|_0^3$$

Expand the power series through its first eight terms.

$$\int_0^3 3xe^{\frac{1}{2}x^2}\ dx = \frac{3x^2}{2} + \frac{3x^4}{8} + \frac{3x^6}{48} + \frac{3x^8}{384} + \frac{3x^{10}}{3{,}840}$$

$$+\frac{3x^{12}}{46{,}080} + \frac{3x^{14}}{645{,}120} + \frac{3x^{16}}{10{,}321{,}920} \Bigg|_0^3$$

$$\int_0^3 3xe^{\frac{1}{2}x^2}\ dx = \frac{3(3)^2}{2} + \frac{3(3)^4}{8} + \frac{3(3)^6}{48} + \frac{3(3)^8}{384} + \frac{3(3)^{10}}{3{,}840}$$

$$+\frac{3(3)^{12}}{46{,}080} + \frac{3(3)^{14}}{645{,}120} + \frac{3(3)^{16}}{10{,}321{,}920}$$

$$-\left(\frac{3(0)^2}{2} + \frac{3(0)^4}{8} + \frac{3(0)^6}{48} + \frac{3(0)^8}{384} + \frac{3(0)^{10}}{3{,}840}\right.$$

$$\left.+\frac{3(0)^{12}}{46{,}080} + \frac{3(0)^{14}}{645{,}120} \frac{3(0)^{16}}{10{,}321{,}920}\right)$$

$$\int_0^3 3xe^{\frac{1}{2}x^2}\ dx \approx 13.5 + 30.375 + 45.5625 + 51.257813 + 46.132031$$

$$+34.599023 + 22.242229 + 12.511254$$

$$\int_0^3 3xe^{\frac{1}{2}x^2}\ dx \approx 256.180$$

■ 2. Use a Maclaurin series to estimate the value of the definite integral.

$$\int_0^{\sqrt{\pi/2}} 12\cos(x^2)\ dx$$

Solution:

Start with the common series

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

and then start manipulating it until it matches the given series, first by substituting x^2 for x,

$$\cos(x^2) = \sum_{n=0}^{\infty} \frac{(-1)^n (x^2)^{2n}}{(2n)!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n}}{(2n)!}$$

and then multiplying by 12.

$$12\cos(x^2) = 12\sum_{n=0}^{\infty}\frac{(-1)^n x^{4n}}{(2n)!} = \sum_{n=0}^{\infty}\frac{12\cdot(-1)^n x^{4n}}{(2n)!}$$

Integrate the power series.

$$\int_0^{\sqrt{\pi/2}} 12\cos(x^2)\ dx = \int_0^{\sqrt{\pi/2}} \sum_{n=0}^{\infty}\frac{12\cdot(-1)^n x^{4n}}{(2n)!}\ dx$$

$$\int_0^{\sqrt{\pi/2}} 12\cos(x^2)\ dx = \frac{12\cdot(-1)^n}{(2n)!}\int_0^{\sqrt{\pi/2}} \sum_{n=0}^{\infty} x^{4n}\ dx$$

$$\int_0^{\sqrt{\pi/2}} 12\cos(x^2)\ dx = \frac{12\cdot(-1)^n}{(2n)!}\cdot\frac{x^{4n+1}}{4n+1}\Bigg|_0^{\sqrt{\pi/2}}$$

Expand the power series through its first six terms.

$$\int_0^{\sqrt{\pi/2}} 12\cos(x^2)\ dx = \frac{12(-1)^0}{(2\cdot 0)!}\cdot\frac{x^{4(0)+1}}{4(0)+1} + \frac{12(-1)^1}{(2\cdot 1)!}\cdot\frac{x^{4(1)+1}}{4(1)+1}$$

$$+\frac{12(-1)^2}{(2\cdot 2)!}\cdot\frac{x^{4(2)+1}}{4(2)+1} + \frac{12(-1)^3}{(2\cdot 3)!}\cdot\frac{x^{4(3)+1}}{4(3)+1} + \frac{12(-1)^4}{(2\cdot 4)!}\cdot\frac{x^{4(4)+1}}{4(4)+1}$$

$$+\frac{12(-1)^5}{(2\cdot 5)!}\cdot\frac{x^{4(5)+1}}{4(5)+1}\Bigg|_0^{\sqrt{\pi/2}}$$

$$\int_0^{\sqrt{\pi/2}} 12\cos(x^2)\ dx = \frac{12x^1}{1} - \frac{12x^5}{10} + \frac{12x^9}{216} - \frac{12x^{13}}{9{,}360} + \frac{x^{17}}{57{,}120} - \frac{12x^{21}}{76{,}204{,}800}\Bigg|_0^{\sqrt{\pi/2}}$$

$$\int_0^{\sqrt{\pi/2}} 12\cos(x^2)\ dx = 12x - \frac{6x^5}{5} + \frac{x^9}{18} - \frac{x^{13}}{780} + \frac{x^{17}}{57{,}120} - \frac{x^{21}}{6{,}350{,}400}\Bigg|_0^{\sqrt{\pi/2}}$$

$$\int_0^{\sqrt{\pi/2}} 12\cos(x^2)\ dx$$

$$= 12\sqrt{\pi/2} - \frac{6\sqrt{\pi/2}^5}{5} + \frac{\sqrt{\pi/2}^9}{18} - \frac{\sqrt{\pi/2}^{13}}{780} + \frac{\sqrt{\pi/2}^{17}}{57{,}120} - \frac{\sqrt{\pi/2}^{21}}{6{,}350{,}400}$$

$$-\left(12(0) - \frac{6(0)^5}{5} + \frac{0^9}{18} - \frac{0^{13}}{780} + \frac{0^{17}}{57{,}120} - \frac{0^{21}}{6{,}350{,}400}\right)$$

$$\int_0^{\sqrt{\pi/2}} 12\cos(x^2)\ dx \approx 15.039770 - 3.710914 + 0.423903$$

$$-0.024137 + 0.000813 - 0.000018$$

$$\int_0^{\sqrt{\pi/2}} 12\cos(x^2)\ dx \approx 11.729417$$

■ 3. Use a Maclaurin series to estimate the value of the definite integral.

$$\int_0^{\sqrt[3]{\pi}} 15\sin(x^3)\ dx$$

Solution:

Start with the common series

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

and then start manipulating it until it matches the given series, first by substituting x^3 for x,

$$\sin(x^3) = \sum_{n=0}^{\infty} \frac{(-1)^n (x^3)^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{6n+3}}{(2n+1)!}$$

and then multiplying by 15.

$$15\sin(x^3) = 15\sum_{n=0}^{\infty} \frac{(-1)^n x^{6n+3}}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{15(-1)^n x^{6n+3}}{(2n+1)!}$$

Integrate the power series.

$$\int_0^{\sqrt[3]{\pi}} 15\sin(x^3)\ dx = \int_0^{\sqrt[3]{\pi}} \sum_{n=0}^{\infty} \frac{15(-1)^n x^{6n+3}}{(2n+1)!}\ dx$$

$$\int_0^{\sqrt[3]{\pi}} 15\sin(x^3)\ dx = \frac{15(-1)^n}{(2n+1)!} \int_0^{\sqrt[3]{\pi}} \sum_{n=0}^{\infty} x^{6n+3}\ dx$$

$$\int_0^{\sqrt[3]{\pi}} 15\sin(x^3)\ dx = \frac{15(-1)^n}{(2n+1)!} \cdot \frac{x^{6n+4}}{6n+4} \Big|_0^{\sqrt[3]{\pi}}$$

Expand the power series through its first six terms.

$$\int_0^{\sqrt[3]{\pi}} 15\sin(x^3)\ dx = \frac{15(-1)^0}{(2(0)+1)!} \cdot \frac{x^{6(0)+4}}{6(0)+4} + \frac{15(-1)^1}{(2(1)+1)!} \cdot \frac{x^{6(1)+4}}{6(1)+4}$$

$$+\frac{15(-1)^2}{(2(2)+1)!}\cdot\frac{x^{6(2)+4}}{6(2)+4}+\frac{15(-1)^3}{(2(3)+1)!}\cdot\frac{x^{6(3)+4}}{6(3)+4}$$

$$+\frac{15(-1)^4}{(2(4)+1)!}\cdot\frac{x^{6(4)+4}}{6(4)+4}+\frac{15(-1)^5}{(2(5)+1)!}\cdot\frac{x^{6(5)+4}}{6(5)+4}\Bigg|_0^{\sqrt[3]{\pi}}$$

$$\int_0^{\sqrt[3]{\pi}} 15\sin(x^3)\ dx=\frac{15x^4}{4}-\frac{x^{10}}{4}+\frac{x^{16}}{128}-\frac{x^{22}}{7{,}392}+\frac{x^{28}}{677{,}376}-\frac{x^{34}}{90{,}478{,}080}\Bigg|_0^{\sqrt[3]{\pi}}$$

$$\int_0^{\sqrt[3]{\pi}} 15\sin(x^3)\ dx=\frac{15\sqrt[3]{\pi}^4}{4}-\frac{\sqrt[3]{\pi}^{10}}{4}+\frac{\sqrt[3]{\pi}^{16}}{128}-\frac{\sqrt[3]{\pi}^{22}}{7{,}392}+\frac{\sqrt[3]{\pi}^{28}}{677{,}376}-\frac{\sqrt[3]{\pi}^{34}}{90{,}478{,}080}$$

$$-\left(\frac{15(0)^4}{4}-\frac{0^{10}}{4}+\frac{0^{16}}{128}-\frac{0^{22}}{7{,}392}+\frac{0^{28}}{677{,}376}-\frac{0^{34}}{90{,}478{,}080}\right)$$

$$\int_0^{\sqrt[3]{\pi}} 15\sin(x^3)\ dx\approx 17.254317-11.352885+3.501515$$

$$-0.598417+0.064452-0.004762$$

$$\int_0^{\sqrt[3]{\pi}} 15\sin(x^3)\ dx\approx 8.86422$$

MACLAURIN SERIES TO EVALUATE A LIMIT

■ 1. Use a Maclaurin series to evaluate the limit.

$$\lim_{x \to 0} \frac{e^{2x} - 1 - 2x}{x^2}$$

Solution:

The Maclaurin series expansion of e^x is

$$e^x = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \cdots$$

So the Maclaurin series expansion of e^{2x} is

$$e^{2x} = 1 + 2x + \frac{1}{2}(2x)^2 + \frac{1}{6}(2x)^3 + \frac{1}{24}(2x)^4 + \cdots$$

$$e^{2x} = 1 + 2x + \frac{1}{2}(4x^2) + \frac{1}{6}(8x^3) + \frac{1}{24}(16x^4) + \cdots$$

$$e^{2x} = 1 + 2x + 2x^2 + \frac{4}{3}x^3 + \frac{2}{3}x^4 + \cdots$$

Subtract 1 and subtract $2x$.

$$e^{2x} - 1 - 2x = 1 + 2x + 2x^2 + \frac{4}{3}x^3 + \frac{2}{3}x^4 + \cdots - 1 - 2x$$

$$e^{2x} - 1 - 2x = 2x^2 + \frac{4}{3}x^3 + \frac{2}{3}x^4 + \cdots$$

Divide by x^2.

$$\frac{e^{2x}-1-2x}{x^2}=\frac{2x^2}{x^2}+\frac{\frac{4}{3}x^3}{x^2}+\frac{\frac{2}{3}x^4}{x^2}+\cdots$$

$$\frac{e^{2x}-1-2x}{x^2}=2+\frac{4}{3}x+\frac{2}{3}x^2+\cdots$$

Substitute into the given limit and then evaluate.

$$\lim_{x\to 0}\frac{e^{2x}-1-2x}{x^2}=\lim_{x\to 0}\left(2+\frac{4}{3}x+\frac{2}{3}x^2+\cdots\right)$$

$$\lim_{x\to 0}\frac{e^{2x}-1-2x}{x^2}=2+\frac{4}{3}(0)+\frac{2}{3}(0)^2+\cdots$$

$$\lim_{x\to 0}\frac{e^{2x}-1-2x}{x^2}=2$$

■ 2. Use a Maclaurin series to evaluate the limit.

$$\lim_{x\to 0}\frac{\arctan x-x}{x^3}$$

Solution:

The Maclaurin series for the expansion of $\arctan x$ is

$$\arctan x=x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+\frac{x^9}{9}-\cdots$$

Subtract x.

$$\arctan x - x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \cdots - x$$

$$\arctan x - x = -\frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \cdots$$

Divide by x^3.

$$\frac{\arctan x - x}{x^3} = -\frac{x^3}{x^3 \cdot 3} + \frac{x^5}{x^3 \cdot 5} - \frac{x^7}{x^3 \cdot 7} + \frac{x^9}{x^3 \cdot 9} - \cdots$$

$$\frac{\arctan x - x}{x^3} = -\frac{1}{3} + \frac{x^2}{5} - \frac{x^4}{7} + \frac{x^6}{9} - \cdots$$

Substitute into the given limit and then evaluate.

$$\lim_{x \to 0} \frac{\arctan x - x}{x^3} = \lim_{x \to 0} \left(-\frac{1}{3} + \frac{x^2}{5} - \frac{x^4}{7} + \frac{x^6}{9} - \cdots \right)$$

$$\lim_{x \to 0} \frac{\arctan x - x}{x^3} = -\frac{1}{3} + \frac{0^2}{5} - \frac{0^4}{7} + \frac{0^6}{9} - \cdots$$

$$\lim_{x \to 0} \frac{\arctan x - x}{x^3} = -\frac{1}{3}$$

■ 3. Use a Maclaurin series to evaluate the limit.

$$\lim_{x \to 0} \frac{\cos(3x) + \frac{9}{2}x^2 - 1}{x^4}$$

Solution:

The Maclaurin series for the expansion of $\cos x$ is

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots$$

So the Maclaurin series expansion of $\cos(3x)$ is

$$\cos(3x) = 1 - \frac{(3x)^2}{2!} + \frac{(3x)^4}{4!} - \frac{(3x)^6}{6!} + \frac{(3x)^8}{8!} - \cdots$$

$$\cos(3x) = 1 - \frac{9x^2}{2!} + \frac{81x^4}{4!} - \frac{729x^6}{6!} + \frac{6561x^8}{8!} - \cdots$$

Add $(9/2)x^2$ and subtract 1.

$$\cos(3x) + \frac{9}{2}x^2 - 1 = 1 - \frac{9x^2}{2!} + \frac{81x^4}{4!} - \frac{729x^6}{6!} + \frac{6561x^8}{8!} - \cdots + \frac{9}{2}x^2 - 1$$

$$\cos(3x) + \frac{9}{2}x^2 - 1 = \frac{81x^4}{4!} - \frac{729x^6}{6!} + \frac{6561x^8}{8!} - \cdots$$

Divide by x^4.

$$\frac{\cos(3x) + \frac{9}{2}x^2 - 1}{x^4} = \frac{81x^4}{x^4 \cdot 4!} - \frac{729x^6}{x^4 \cdot 6!} + \frac{6561x^8}{x^4 \cdot 8!} - \cdots$$

$$\frac{\cos(3x) + \frac{9}{2}x^2 - 1}{x^4} = \frac{81}{4!} - \frac{729x^2}{6!} + \frac{6561x^4}{8!} - \cdots$$

Substitute the terms into the given limit and then evaluate.

$$\lim_{x\to0}\frac{\cos(3x)+\frac{9}{2}x^2-1}{x^4}=\lim_{x\to0}\left(\frac{81}{4!}-\frac{729x^2}{6!}+\frac{6561x^4}{8!}-\cdots\right)$$

$$\lim_{x\to0}\frac{\cos(3x)+\frac{9}{2}x^2-1}{x^4}=\frac{81}{4!}-\frac{729(0)^2}{6!}+\frac{6561(0)^4}{8!}-\cdots$$

$$\lim_{x\to0}\frac{\cos(3x)+\frac{9}{2}x^2-1}{x^4}=\frac{81}{4!}$$

$$\lim_{x\to0}\frac{\cos(3x)+\frac{9}{2}x^2-1}{x^4}=\frac{27}{8}$$

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