Topic: Radius and interval of convergence of a Taylor series

Question: Find the radius of convergence of the Taylor series.

$$2 + 2(x-1) + 4(x-1)^2 + 6(x-1)^3$$

Answer choices:

A 2

B 1

C 4

D ∞

Solution: B

To find the radius of convergence of the given Taylor series, we'll need to transform it into its power series representation, and then use the ratio test to find the radius of convergence of that power series.

To get a power series representation of the series, we need to remember that the formula for the Taylor polynomial of f at a is

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \ldots + \frac{f^n(a)}{n!}(x-a)^n$$

Which means that the power series representation of the function will be the last term,

$$\sum_{n=1}^{\infty} \frac{f^n(a)}{n!}(x-a)^n$$

We can rewrite the given series as

$$2 + 2(x-1) + 4(x-1)^2 + 6(x-1)^3$$

$$2(x-1)^0 + 2(x-1)^1 + 4(x-1)^2 + 6(x-1)^3$$

$$2(x-1)^0 + 2(1)(x-1)^1 + 2(2)(x-1)^2 + 2(3)(x-1)^3$$

The first term doesn't follow the same pattern as the second and third terms, so we can pull it out in front of the sum and represent the series as

$$2 + \sum_{n=1}^{\infty} 2n(x-1)^n$$

To find the radius of convergence of this series, we'll first identify a_n and a_{n+1}.

$$a_n = 2n(x-1)^n$$

$$a_{n+1} = 2(n+1)(x-1)^{n+1}$$

Now we can use the ratio test to find the radius of convergence. The ratio test tells us that a series converges if $L < 1$ when

$$L = \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right|$$

Plugging the values we found for a_n and a_{n+1} into this formula for L, we get

$$L = \lim_{n \to \infty} \left| \frac{2(n+1)(x-1)^{n+1}}{2n(x-1)^n} \right|$$

$$L = \lim_{n \to \infty} \left| \frac{(n+1)(x-1)^{n+1-n}}{n} \right|$$

$$L = \lim_{n \to \infty} \left| \frac{(n+1)(x-1)}{n} \right|$$

The limit only effects n, not x, so we can pull $(x-1)$ out in front of the limit, as long as we keep it inside absolute value bars.

$$L = \left| x-1 \right| \lim_{n \to \infty} \left| \frac{n+1}{n} \right|$$

$$L = \left|x-1\right| \lim_{n\to\infty} \left| \frac{n+1}{n} \left(\frac{\frac{1}{n}}{\frac{1}{n}} \right) \right|$$

$$L = \left|x-1\right| \lim_{n\to\infty} \left| \frac{\frac{n}{n}+\frac{1}{n}}{\frac{n}{n}} \right|$$

$$L = \left|x-1\right| \lim_{n\to\infty} \left| \frac{1+\frac{1}{n}}{1} \right|$$

$$L = \left|x-1\right| \left| \frac{1+\frac{1}{\infty}}{1} \right|$$

$$L = \left|x-1\right| \left| \frac{1+0}{1} \right|$$

$$L = \left|x-1\right| \left|1\right|$$

$$L = \left|x-1\right|$$

Now we can set $L < 1$ to find the radius of convergence.

$$\left|x-1\right| < 1$$

Since this inequality is already in the form

$$|x-a| < R$$

and since R is the radius of convergence, we can say that the radius of convergence of the series is $R=1$.

Topic: Radius and interval of convergence of a Taylor series

Question: Find the radius of convergence of the Taylor series.

$$3 - 3(x-1) + 9(x-1)^2 - 27(x-1)^3$$

Answer choices:

A $\frac{1}{3}$

B 3

C $\frac{1}{2}$

D 1

Solution: A

To find the radius of convergence of the given Taylor series, we'll need to transform it into its power series representation, and then use the ratio test to find the radius of convergence of that power series.

To get a power series representation of the series, we need to remember that the formula for the taylor polynomial of f at a is

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \dots + \frac{f^n(a)}{n!}(x-a)^n$$

Which means that the power series representation of the function will be the last term,

$$\sum_{n=1}^{\infty} \frac{f^n(a)}{n!}(x-a)^n$$

We can rewrite the given series as

$$3 - 3(x-1) + 9(x-1)^2 - 27(x-1)^3$$

$$3(x-1)^0 - 3(x-1)^1 + 9(x-1)^2 - 27(x-1)^3$$

$$3(x-1)^0 - 3^1(x-1)^1 + 3^2(x-1)^2 - 3^3(x-1)^3$$

The first term doesn't follow the same pattern as the other terms, so we can pull it out in front of the sum and represent the series as

$$3 + \sum_{n=1}^{\infty} (-3)^n(x-1)^n$$

To find the radius of convergence of this series, we'll first identify a_n and a_{n+1}.

$$a_n = (-3)^n(x-1)^n$$

$$a_{n+1} = (-3)^{n+1}(x-1)^{n+1}$$

Now we can use the ratio test to find the radius of convergence. The ratio test tells us that a series converges if $L < 1$ when

$$L = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|$$

Plugging the values we found for a_n and a_{n+1} into this formula for L, we get

$$L = \lim_{n\to\infty}\left|\frac{(-3)^{n+1}(x-1)^{n+1}}{(-3)^n(x-1)^n}\right|$$

$$L = \lim_{n\to\infty}\left|(-3)^{n+1-n}(x-1)^{n+1-n}\right|$$

$$L = \lim_{n\to\infty}\left|-3(x-1)\right|$$

The limit only effects n, not x, so we can pull $(x-1)$ out in front of the limit, as long as we keep it inside absolute value bars.

$$L = \left|x-1\right|\lim_{n\to\infty}\left|-3\right|$$

$$L = \left|x-1\right|\left|-3\right|$$

$$L = 3\left|x-1\right|$$

Now, we can set $L < 1$ to find the radius of convergence.

$$3\left|x-1\right| < 1$$

$$\left|x-1\right| < \frac{1}{3}$$

Comparing this to

$$|x-a| < R$$

where R is the radius of convergence, we can say that the radius of convergence of the series is $R = 1/3$.

Topic: Radius and interval of convergence of a Taylor series

Question: Find the radius of convergence of the Taylor series.

$$2+(x-2)+\frac{1}{2}(x-2)^2+\frac{1}{3}(x-2)^3$$

Answer choices:

A $\frac{1}{3}$

B 2

C $\frac{1}{2}$

D 1

Solution: D

To find the radius of convergence of the given Taylor series, we'll need to transform it into its power series representation, and then use the ratio test to find the radius of convergence of that power series.

To get a power series representation of the series, we need to remember that the formula for the taylor polynomial of f at a is

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \ldots + \frac{f^n(a)}{n!}(x-a)^n$$

Which means that the power series representation of the function will be the last term,

$$\sum_{n=1}^{\infty} \frac{f^n(a)}{n!}(x-a)^n$$

We can rewrite the given series as

$$2 + (x-2) + \frac{1}{2}(x-2)^2 + \frac{1}{3}(x-2)^3$$

$$2(x-2)^0 + (x-2)^1 + \frac{1}{2}(x-2)^2 + \frac{1}{3}(x-2)^3$$

$$2(x-2)^0 + \frac{1}{1}(x-2)^1 + \frac{1}{2}(x-2)^2 + \frac{1}{3}(x-2)^3$$

The first term doesn't follow the same pattern as the other terms, so we can pull it out in front of the sum and represent the series as

$$2 + \sum_{n=1}^{\infty} \frac{1}{n}(x-2)^n$$

To find the radius of convergence of this series, we'll first identify a_n and a_{n+1}.

$$a_n = \frac{1}{n}(x-2)^n$$

$$a_{n+1} = \frac{1}{n+1}(x-2)^{n+1}$$

Now we can use the ratio test to find the radius of convergence. The ratio test tells us that a series converges if $L < 1$ when

$$L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$$

Plugging the values we found for a_n and a_{n+1} into this formula for L, we get

$$L = \lim_{n\to\infty} \left| \frac{\frac{1}{n+1}(x-2)^{n+1}}{\frac{1}{n}(x-2)^n} \right|$$

$$L = \lim_{n\to\infty} \left| \frac{\frac{1}{n+1}(x-2)^{n+1-n}}{\frac{1}{n}} \right|$$

$$L = \lim_{n\to\infty} \left| \frac{\frac{1}{n+1}}{\frac{1}{n}}(x-2) \right|$$

$$L = \lim_{n\to\infty} \left| \frac{1}{n+1} \cdot \frac{n}{1} \cdot (x-2) \right|$$

$$L = \lim_{n \to \infty} \left| \frac{n}{n+1} \cdot (x-2) \right|$$

The limit only effects n, not x, so we can pull $(x-2)$ out in front of the limit, as long as we keep it inside absolute value bars.

$$L = |x-2| \lim_{n \to \infty} \left| \frac{n}{n+1} \right|$$

$$L = |x-2| \lim_{n \to \infty} \left| \frac{n}{n+1} \left(\frac{\frac{1}{n}}{\frac{1}{n}} \right) \right|$$

$$L = |x-2| \lim_{n \to \infty} \left| \frac{\frac{n}{n}}{\frac{n}{n} + \frac{1}{n}} \right|$$

$$L = |x-2| \lim_{n \to \infty} \left| \frac{1}{1 + \frac{1}{n}} \right|$$

$$L = |x-2| \left| \frac{1}{1 + \frac{1}{\infty}} \right|$$

$$L = |x-2| \left| \frac{1}{1+0} \right|$$

$$L = |x-2| \, |1|$$

$$L = |x-2|$$

Now we can set $L < 1$ to find the radius of convergence.

$$\left| x - 2 \right| < 1$$

Comparing this to

$$|x - a| < R$$

where R is the radius of convergence, we can say that the radius of convergence of the series is $R = 1$.