Topic: Surface area of revolution of a polar curve

Question: Find the surface area generated by revolving the polar curve about the y-axis.

$$r = 5\sqrt{\sin(2\theta)}$$

on the interval $0 \leq \theta \leq \frac{\pi}{4}$

Answer choices:

A $25\sqrt{2}$

B $2\pi\sqrt{5}$

C $25\pi\sqrt{2}$

D 25π

Solution: C

The area of the surface generated by revolving a curve about the y-axis is given by

$$S=\int_{\alpha}^{\beta}2\pi r\cos\theta\sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}\ d\theta$$

We'll find the derivative of the given equation so that we can plug it into the surface area formula. We'll also calculate r^2 and hope to plug it in and avoid plugging in square roots.

$$\frac{dr}{d\theta}=5\cdot\frac{1}{2}(\sin(2\theta))^{-\frac{1}{2}}\cdot\cos(2\theta)\cdot 2$$

$$\frac{dr}{d\theta}=\frac{5\cos(2\theta)}{\sqrt{\sin(2\theta)}}$$

and

$$r=5\sqrt{\sin(2\theta)}$$

$$r^2=25\sin(2\theta)$$

To avoid plugging square roots into our formula, let's absorb the r into the square root. If it's r outside of the square root, it must have been r^2 inside the square root.

$$S=\int_{\alpha}^{\beta}2\pi\cos\theta\sqrt{r^2\left[r^2+\left(\frac{dr}{d\theta}\right)^2\right]}\ d\theta$$

$$S=\int_{\alpha}^{\beta}2\pi\cos\theta\sqrt{r^4+r^2\left(\frac{dr}{d\theta}\right)^2}\ d\theta$$

Now that all of our r's are raised to even powers, we can plug in the value we found for r^2 and avoid the square roots.

$$S=\int_0^{\frac{\pi}{4}}2\pi\cos\theta\sqrt{\left(25\sin(2\theta)\right)^2+\left(25\sin(2\theta)\right)\left(\frac{5\cos(2\theta)}{\sqrt{\sin(2\theta)}}\right)^2}\ d\theta$$

$$S=2\pi\int_0^{\frac{\pi}{4}}\cos\theta\sqrt{625\sin^2(2\theta)+25\sin(2\theta)\frac{25\cos^2(2\theta)}{\sin(2\theta)}}\ d\theta$$

$$S=2\pi\int_0^{\frac{\pi}{4}}\cos\theta\sqrt{625\sin^2(2\theta)+625\cos^2(2\theta)}\ d\theta$$

$$S=2\pi\int_0^{\frac{\pi}{4}}\cos\theta\sqrt{625\left(\sin^2(2\theta)+\cos^2(2\theta)\right)}\ d\theta$$

$$S=50\pi\int_0^{\frac{\pi}{4}}\cos\theta\sqrt{\sin^2(2\theta)+\cos^2(2\theta)}\ d\theta$$

Knowing that $\sin^2 x+\cos^2 x=1$, we can simplify the integral to

$$S=50\pi\int_0^{\frac{\pi}{4}}\cos\theta\sqrt{1}\ d\theta$$

$$S=50\pi\int_0^{\frac{\pi}{4}}\cos\theta\ d\theta$$

$$S=50\pi\sin\theta\Big|_0^{\frac{\pi}{4}}$$

$$S = 50\pi\left(\sin\frac{\pi}{4} - \sin 0\right)$$

$$S = 50\pi\left(\frac{\sqrt{2}}{2}\right)$$

$$S = 25\pi\sqrt{2}$$

Topic: Surface area of revolution of a polar curve

Question: Find surface area of revolution.

$$r = \sin\theta$$

about the x-axis

on the interval $0 \le \theta \le \pi$

Answer choices:

A $\quad \pi$

B $\quad \dfrac{\pi}{2}$

C $\quad \pi^2$

D $\quad \dfrac{\pi^2}{2}$

Solution: C

To find surface area of revolution when we rotate about the x-axis, we need to use the formula

$$S_x = \int_{\alpha}^{\beta} 2\pi r \sin\theta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta$$

To use the formula, we'll need to first find the derivative $dr/d\theta$. If $r = \sin\theta$, then

$$\frac{dr}{d\theta} = \cos\theta$$

Plugging this, and the given interval $0 \le \theta \le \pi$ into the formula, we get

$$S_x = \int_0^{\pi} 2\pi \sin\theta \sin\theta \sqrt{\sin^2\theta + \cos^2\theta}\ d\theta$$

Remembering the trigonometric identity $\sin^2\theta + \cos^2\theta = 1$, we can substitute into the integral.

$$S_x = 2\pi \int_0^{\pi} \sin^2\theta \sqrt{1}\ d\theta$$

$$S_x = 2\pi \int_0^{\pi} \sin^2\theta\ d\theta$$

Using the double-angle formula

$$\sin^2\theta = \frac{1}{2}(1 - \cos(2\theta))$$

we'll rewrite the integral as

$$S_x = 2\pi \int_0^{\pi} \frac{1}{2}(1 - \cos(2\theta))\ d\theta$$

$$S_x = \pi \int_0^{\pi} 1 - \cos(2\theta)\ d\theta$$

And then we'll integrate and evaluate over the interval.

$$S_x = \pi \left(\theta - \frac{1}{2}\sin(2\theta) \right) \Bigg|_0^{\pi}$$

$$S_x = \pi\theta - \frac{\pi}{2}\sin(2\theta) \Big|_0^{\pi}$$

$$S_x = \pi(\pi) - \frac{\pi}{2}\sin(2(\pi)) - \left[\pi(0) - \frac{\pi}{2}\sin(2(0)) \right]$$

$$S_x = \pi^2 - \frac{\pi}{2}(0) - (0) + \frac{\pi}{2}(0)$$

$$S_x = \pi^2$$

Topic: Surface area of revolution of a polar curve

Question: Find the surface area of revolution.

$$r = 2\cos\theta$$

about the y-axis

on the interval $0 \leq \theta \leq \pi$

Answer choices:

A $4\pi^2$

B 2π

C 4π

D $2\pi^2$

Solution: A

To find surface area of revolution when we rotate about the y-axis, we need to use the formula

$$S_y = \int_{\alpha}^{\beta} 2\pi r \cos\theta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\ d\theta$$

To use the formula, we'll need to first find the derivative $dr/d\theta$. If $r = 2\cos\theta$, then

$$\frac{dr}{d\theta} = -2\sin\theta$$

Plugging this, and the given interval $0 \le \theta \le \pi$ into the formula, we get

$$S_y = \int_0^{\pi} 2\pi(2\cos\theta)\cos\theta\sqrt{4\cos^2\theta + 4\sin^2\theta}\ d\theta$$

$$S_y = \int_0^{\pi} 4\pi\cos^2\theta\sqrt{4\cos^2\theta + 4\sin^2\theta}\ d\theta$$

$$S_y = \int_0^{\pi} 4\pi\cos^2\theta\sqrt{4\left(\cos^2\theta + \sin^2\theta\right)}\ d\theta$$

Remembering the trigonometric identity $\sin^2\theta + \cos^2\theta = 1$, we can substitute into the integral.

$$S_y = \int_0^{\pi} 4\pi\cos^2\theta\sqrt{4(1)}\ d\theta$$

$$S_y = 8\pi\int_0^{\pi} \cos^2\theta\ d\theta$$

Using the double-angle formula

$$\cos^2\theta = \frac{1}{2}(1 + \cos(2\theta))$$

we'll rewrite the integral as

$$S_y = 8\pi \int_0^\pi \frac{1}{2}(1 + \cos(2\theta))\ d\theta$$

$$S_y = 4\pi \int_0^\pi 1 + \cos(2\theta)\ d\theta$$

And then we'll integrate and evaluate over the interval.

$$S_y = 4\pi \left(\theta + \frac{1}{2}\sin(2\theta)\right)\Big|_0^\pi$$

$$S_y = 4\pi\theta + 2\pi\sin(2\theta)\Big|_0^\pi$$

$$S_y = 4\pi(\pi) + 2\pi\sin(2(\pi)) - \left[4\pi(0) + 2\pi\sin(2(0))\right]$$

$$S_y = 4\pi^2 + 2\pi(0) - 4\pi(0) - 2\pi(0)$$

$$S_y = 4\pi^2$$