Comparison test

The comparison test for convergence lets us determine the convergence or divergence of the given series a_n by *comparing* it to a similar, but simpler comparison series b_n.

We're usually trying to find a comparison series that's a geometric or p-series, since it's very easy to determine the convergence of a geometric or p-series.

We can use the comparison test to show that

the original series a_n is **diverging** if

the original series a_n is greater than or equal to the comparison series b_n and both series are positive, $a_n \geq b_n \geq 0$, and

the comparison series b_n is diverging

Note: If $a_n < b_n$, the test is inconclusive

the original series is **converging** if

the original series a_n is less than or equal to the comparison series b_n and both series are positive, $0 \leq a_n \leq b_n$, and

the comparison series b_n is converging

Note: If $b_n < a_n$, the test is inconclusive

Example

Use the comparison test to say whether or not the series converges.

$$\sum_{n=1}^{\infty} \frac{n}{\sqrt{n^5}+n}$$

We need to find a series that's similar to the original series, but simpler. The original series is

$$a_n = \frac{n}{\sqrt{n^5}+n}$$

For the comparison series, we'll use the same numerator as the original series, since it's already pretty simple. Looking at the denominator, we can see that the first term $\sqrt{n^5}$ carries more weight and will affect our series more than the second term n, so we'll just use the first term from the original denominator for the denominator of our comparison series, and the comparison series is

$$b_n = \frac{n}{\sqrt{n^5}}$$

$$b_n = \frac{n}{n^{\frac{5}{2}}}$$

$$b_n = n^{1-\frac{5}{2}}$$

$$b_n = n^{-\frac{3}{2}}$$

$$b_n = \frac{1}{n^{\frac{3}{2}}}$$

We can see that this simplified version of b_n is just a p-series, where $p = 3/2$. We'll use the p-series test for convergence to say whether or not b_n converges. Remember, the p-series test says that the series will

converge when $p > 1$

diverge when $p \le 1$

Since $p = 3/2$ in b_n, we know that b_n converges.

That means we need to show that $0 \le a_n \le b_n$ to prove that the original series a_n is also converging. If we can't show that $0 \le a_n \le b_n$, then the test is inconclusive with this particular comparison series.

Let's try to verify that $0 \le a_n \le b_n$ by checking a few points for both a_n and b_n, like $n = 1$, $n = 4$ and $n = 9$.

		a_n		b_n
$n = 1$	$\frac{1}{\sqrt{(1)^5 + (1)}}$	$\frac{1}{2}$	$\frac{1}{(1)^{\frac{3}{2}}}$	1
$n = 4$	$\frac{4}{\sqrt{(4)^5 + (4)}}$	$\frac{1}{9}$	$\frac{1}{(4)^{\frac{3}{2}}}$	$\frac{1}{8}$
$n = 9$	$\frac{9}{\sqrt{(9)^5 + (9)}}$	$\frac{1}{28}$	$\frac{1}{(9)^{\frac{3}{2}}}$	$\frac{1}{27}$

Looking at these three terms, we can see that $0 \le a_n \le b_n$, since a_n is always positive and always smaller than b_n.

Therefore, we can say that the original series a_n converges.