Power series division

Sometimes we'll want to use polynomial long division to simplify a fraction, but either the numerator and/or denominator isn't a polynomial. In this case, we may be able to replace the non-polynomial with its power series expansion, which will be a polynomial.

The simplest way to do this for the non-polynomial is to find a similar, known power series expansion and then modify it to match the non-polynomial function. Once we have polynomial expressions for both the numerator and denominator, we'll do polynomial long division until we have the number of non-zero terms we've been asked for.

Example

Use power series division to find the first three non-zero terms of the Maclaurin series of the given function.

$$y = \frac{x}{e^{3x}}$$

In order to use long division, we need polynomials in the numerator and denominator of our function. The numerator is already a polynomial, but we need to find a power series expansion for e^{3x} so that we can change it into a polynomial.

We know that the expanded version of the Maclaurin series for e^x is

$$e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \dots$$

Since we have e^{3x} instead of e^x, we'll need to modify the series, letting $x = 3x$, such that the expanded series will be

$$e^{3x} = 1 + 3x + \frac{1}{2}(3x)^2 + \frac{1}{6}(3x)^3 + \dots$$

$$e^{3x} = 1 + 3x + \frac{9}{2}x^2 + \frac{9}{2}x^3 + \dots$$

Now that both the numerator and denominator are represented as polynomials, we'll do the long division.

$$\begin{array}{r|l}
 & x - 3x^2 + \frac{9}{2}x^3 - \frac{9}{2}x^4 \\
\hline
1 + 3x + \frac{9}{2}x^2 + \frac{9}{2}x^3 + \dots & x \\
 & -\left(x + 3x^2 + \frac{9}{2}x^3 + \frac{9}{2}x^4 + \dots\right) \\
\cline{2-2}
 & -3x^2 - \frac{9}{2}x^3 - \frac{9}{2}x^4 + \dots \\
 & -\left(-3x^2 - 9x^3 - \frac{27}{2}x^4 - \frac{27}{2}x^5 + \dots\right) \\
\cline{2-2}
 & \frac{9}{2}x^3 + 9x^4 + \frac{27}{2}x^5 + \dots \\
 & -\left(\frac{9}{2}x^3 + \frac{27}{2}x^4 + \frac{81}{4}x^5 + \frac{81}{4}x^6 + \dots\right) \\
\cline{2-2}
 & -\frac{9}{2}x^4 - \frac{27}{4}x^5 - \frac{81}{4}x^6 + \dots
\end{array}$$

$$-\frac{9}{2}x^4 - \frac{27}{2}x^5 - \frac{81}{4}x^6 - \frac{81}{4}x^7 + \dots$$

Remember, we only need to find the first three non-zero terms. We'll take the first three terms from our quotient and say that the first three non-zero terms are

$$y = \frac{x}{e^{3x}} \approx x - 3x^2 + \frac{9}{2}x^3$$