Calculus 1 Workbook Solutions

Applied optimization

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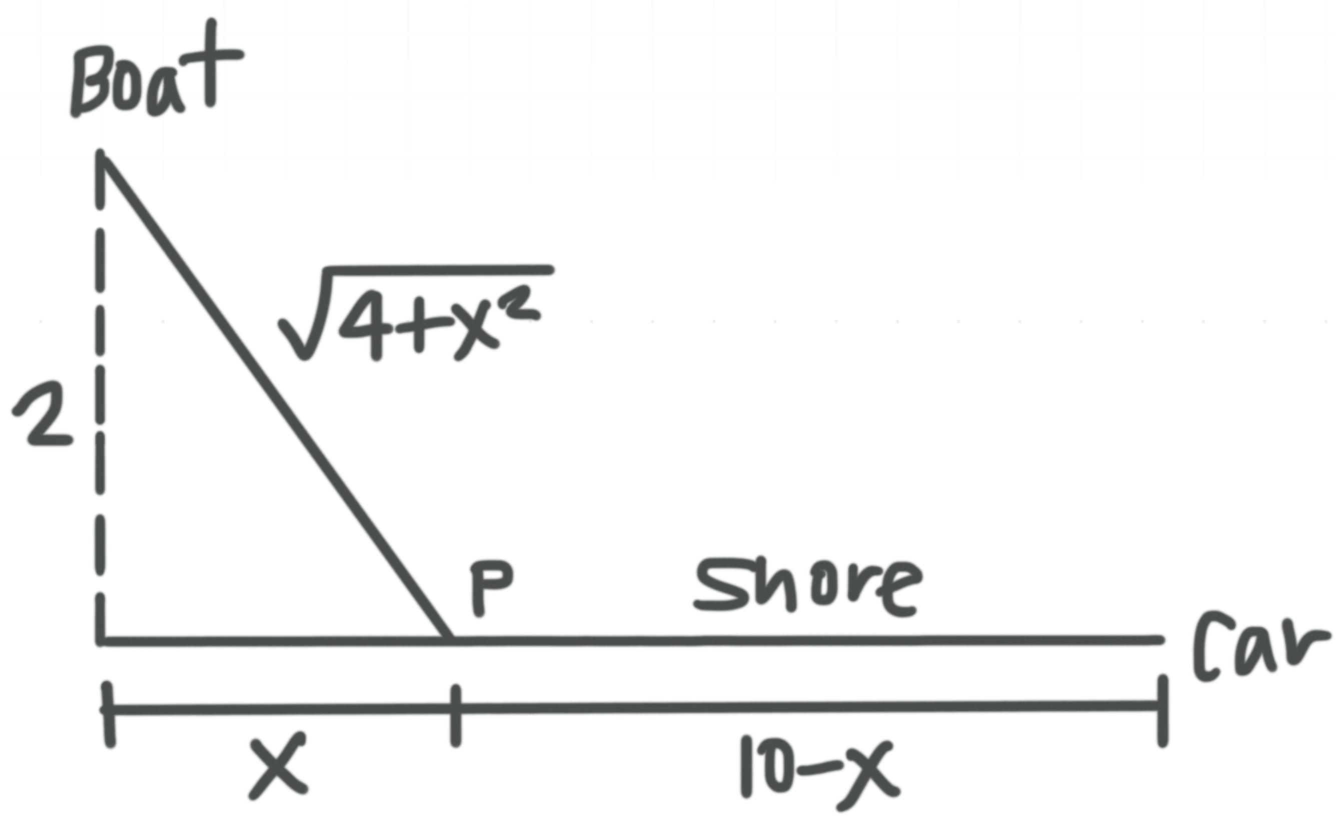
APPLIED OPTIMIZATION

1. A boater finds herself 2 miles from the nearest point to a straight shoreline, which is 10 miles down the shore from where she parked her car. She plans to row to shore and then walk to her car. If she can walk 4 miles per hour but only row 3 miles per hour, toward what point on the shore should she row in order to reach her car in in the least amount of time?

Solution:

Draw a diagram.



From the diagram, the distance to point P is $\sqrt{4+x^2}$ and the distance from point P to the car is $10-x$. So the total time to reach the car is

$$T = \frac{\sqrt{4+x^2}}{3} + \frac{10-x}{4} \text{ with } 0 \le x \le 10$$

We set $0 \le x \le 10$ because x will be 0 if she rows directly to the shore, and x will be 10 if she rows directly to the car. Find the derivative of T.

$$dT = \frac{x}{3\sqrt{4+x^2}} - \frac{1}{4}$$

Set $dT = 0$ and solve for x.

$$\frac{x}{3\sqrt{4+x^2}} = \frac{1}{4}$$

$$\frac{4x}{3} = \sqrt{4+x^2}$$

$$\frac{16x^2}{9} = 4 + x^2$$

$$\frac{7}{9}x^2 = 4$$

$$x^2 = \frac{36}{7}$$

$$x = \frac{6}{\sqrt{7}} \approx 2.2678$$

If she rows to point P, where $x = 2.2678$ miles down the shoreline, it will take her

$$T = \frac{\sqrt{4 + (2.2678)^2}}{3} + \frac{10 - (2.2678)}{4} \approx 2.9409 \text{ hours}$$

If she rows directly to the shore, where $x = 0$, it will take her

$$T = \frac{2}{3} + \frac{10}{4} \approx 3.167 \text{ hours}$$

Find the distance directly to her car using the Pythagorean Theorem.

$$d^2 = 2^2 + 10^2 = 104$$

$$d = 2\sqrt{26} \text{ miles}$$

If she rows directly to her car, where $x = 10$,

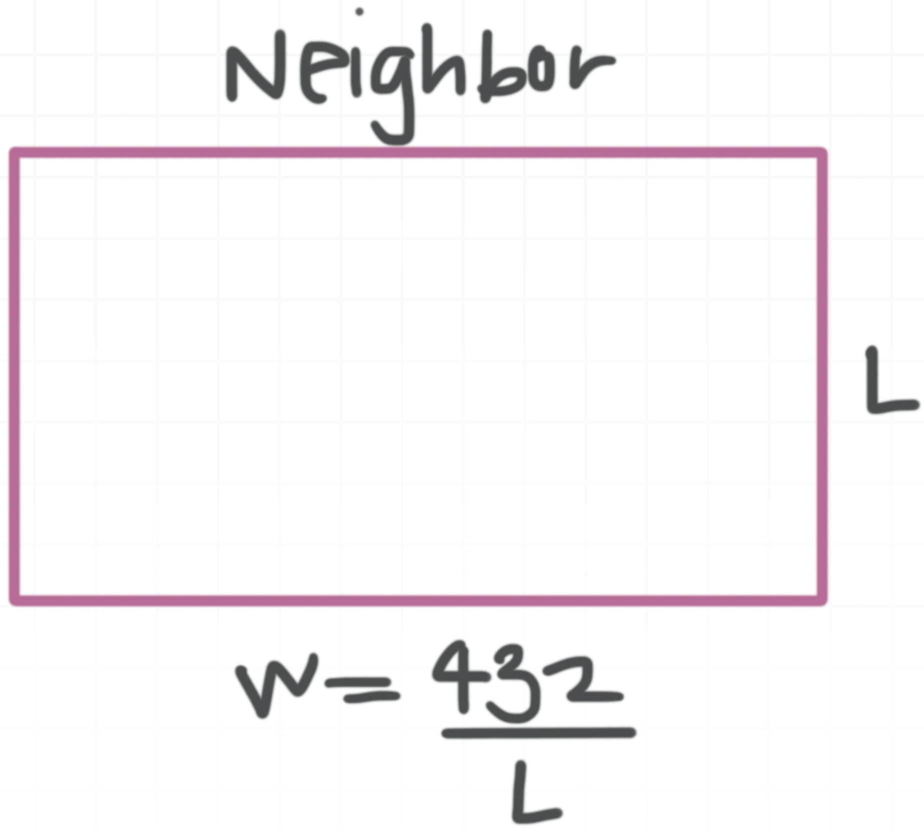
$$T = \frac{2\sqrt{26}}{3} = 3.399 \text{ hours}$$

She will minimize her time by rowing to a point that is 2.2678 miles down shore toward her car.

2. Mr. Quizna wants to build in a completely fenced-in rectangular garden. The fence will be built so that one side is adjacent to his neighbor's property. The neighbor agrees to pay for half of that part of the fence because it borders his property. The garden will contain 432 square meters. What dimensions should Mr. Quizna select for his garden in order to minimize his cost?

Solution:

Draw a diagram.



The area is

$$A = L \cdot W$$

$$432 = L \cdot W$$

$$W = \frac{432}{L}$$

Let C be the total cost and y be the cost per meter. Then,

$$C = 2L \cdot y + \frac{432}{L} \cdot y + \frac{432}{L} \cdot \frac{y}{2} = 2yL + 648yL^{-1} = y(2L + 648L^{-1})$$

Take the derivative of the cost equation.

$$dC = y(2 - 648L^{-2})$$

Set the derivative equal to 0 and solve for L.

$$2 = \frac{648}{L^2}$$

$$2L^2 = 648$$

$$L^2 = 324$$

$$L = 18$$

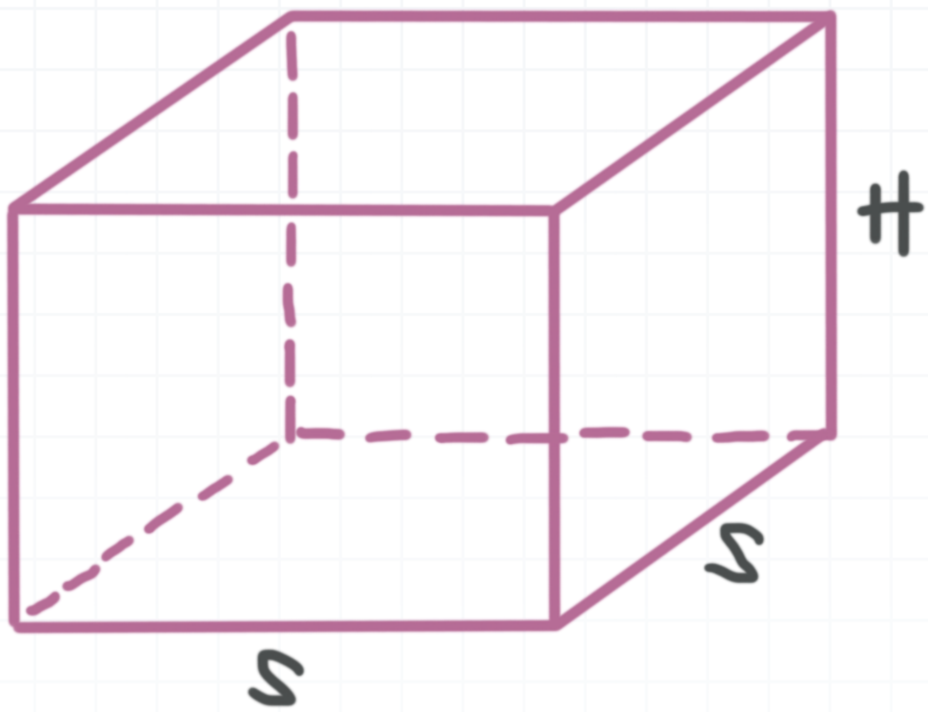
The length of the garden should be $L = 18$ meters and the width of the garden should be

$$W = \frac{432}{L} = \frac{432}{18} = 24 \text{ meters}$$

■ 3. A company is designing shipping crates and wants the volume of each crate to be 6 cubic feet, and each crate's base to be a square between 1.5 feet and 2.0 feet per side. The material for the bottom of the crate costs $\$5$ per square foot, the sides $\$3$ per square foot, and the top $\$1$ per square foot. What dimensions will minimize the cost of the shipping crates?

Solution:

Draw a diagram.



Based on the given information,

$$V = S \cdot S \cdot H$$

$$6 = S \cdot S \cdot H$$

$$H = \frac{6}{S^2}$$

The surface area of the bottom is S^2, the surface area of the top is S^2, and the surface area of the four sides is

$$4 \cdot S \cdot H$$

$$4 \cdot S \cdot \frac{6}{S^2} = \frac{24}{S}$$

Create a cost function.

$$C = 5 \cdot S^2 + 1 \cdot S^2 + 3 \cdot \frac{24}{S} = 6S^2 + \frac{72}{S} = 6S^2 + 72S^{-1}$$

Differentiate the cost function.

$$dC = 12S - \frac{72}{S^2}$$

Set the derivative equal to 0 and solve for S.

$$12S = \frac{72}{S^2}$$

$$12S^3 = 72$$

$$S^3 = 6$$

$$S = \sqrt[3]{6}$$

$$S \approx 1.82$$

Use the first derivative test with test values of $S = 1$ and $S = 2$ to confirm that the critical point represents a minimum.

$$C'(1) = 12(1) - \frac{72}{1^2}$$

$$C'(1) = 12 - 72$$

$$C'(1) = -60$$

and

$$C'(2) = 12(2) - \frac{72}{2^2}$$

$$C'(2) = 24 - 18$$

$$C'(2) = 6$$

Since we get a negative value to the left of the critical point and a positive value to the right of it, the function has a minimum at the critical point. The dimensions that will give the minimum cost are $S=1.82$ feet and $H=1.81$ feet.

■ 4. We want to construct a cylindrical can with a bottom and no top, that has a volume of 50 cm^3. Find the dimensions of the can that minimize its surface area.

Solution:

The volume of the cylinder is

$$V=\pi r^2h$$

$$50=\pi r^2h$$

$$h=\frac{50}{\pi r^2}$$

Its surface area is

$$A=2\pi rh+\pi r^2$$

$$A=2\pi r\frac{50}{\pi r^2}+\pi r^2$$

$$A=\frac{100}{r}+\pi r^2$$

Differentiate the surface area function.

$$dA = -\frac{100}{r^2} + 2\pi r$$

Set the derivative equal to 0 and solve for r.

$$0 = -\frac{100}{r^2} + 2\pi r$$

$$2\pi r = \frac{100}{r^2}$$

$$2\pi r^3 = 100$$

$$r = \sqrt[3]{\frac{50}{\pi}}$$

$$r \approx 2.52$$

Use the first derivative test with test values of $r = 2$ and $r = 3$ to confirm that the critical point represents a minimum.

$$A'(2) = -\frac{100}{2^2} + 2\pi(2)$$

$$A'(2) = -25 + 4\pi$$

$$A'(2) = -12.4$$

and

$$A'(3) = -\frac{100}{3^2} + 2\pi(3)$$

$$A'(2) = 7.7$$

Since we get a negative value to the left of the critical point and a positive value to the right of it, the function has a minimum at the critical point.

$$h = \frac{50}{\pi(2.52)^2}$$

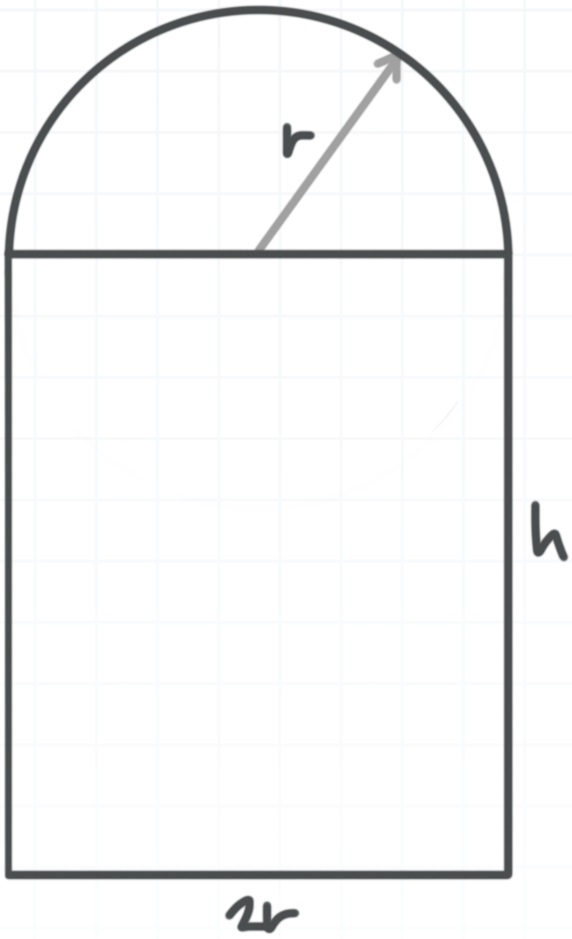
$$h \approx 2.51$$

So $r = 2.52$ ft and $h = 2.51$ ft will minimize the can's surface area.

■ 5. We're building a rectangular window with a a semicircular top. If we have 16 meters of framing material, what dimensions should we use in order to maximize the size of the window in order to let in the most light?

Solution:

Draw a diagram.



The perimeter of the window is the length of the three sides of the rectangular portion, plus half the circumference of a circle of radius r.

$$P = h + h + 2r + \frac{2\pi r}{2}$$

$$P = 2h + 2r + \pi r$$

The perimeter needs to be made with 16 meters of framing material.

$$16 = 2h + 2r + \pi r$$

$$16 - 2r - \pi r = 2h$$

$$h = 8 - r - \frac{\pi r}{2}$$

The area is the area of the rectangle plus the area of the half circle.

$$A = 2rh + \frac{1}{2}\pi r^2$$

$$A = 2r\left(8 - r - \frac{\pi r}{2}\right) + \frac{1}{2}\pi r^2$$

$$A = 16r - 2r^2 - \pi r^2 + \frac{1}{2}\pi r^2$$

$$A = 16r - 2r^2 - \frac{1}{2}\pi r^2$$

Differentiate the area function.

$$dA = 16 - 4r - \pi r$$

$$dA = 16 - r(4 + \pi)$$

Set the derivative equal to 0 and solve for r.

$$0 = 16 - r(4 + \pi)$$

$$r = \frac{16}{4 + \pi}$$

$$r \approx 2.24$$

Use the first derivative test with test values of $r = 2$ and $r = 3$ to confirm that the critical point represents a minimum.

$$A'(2) = 16 - 2(4 + \pi)$$

$$A'(2) = 1.72$$

and

$$A'(3) = 16 - 3(4 + \pi)$$

$$A'(3) = -5.4$$

Since we get a positive value to the left of the critical point and a negative value to the right of it, the function has a maximum at the critical point.

$$h = 8 - 2.24 - \frac{2.24\pi}{2}$$

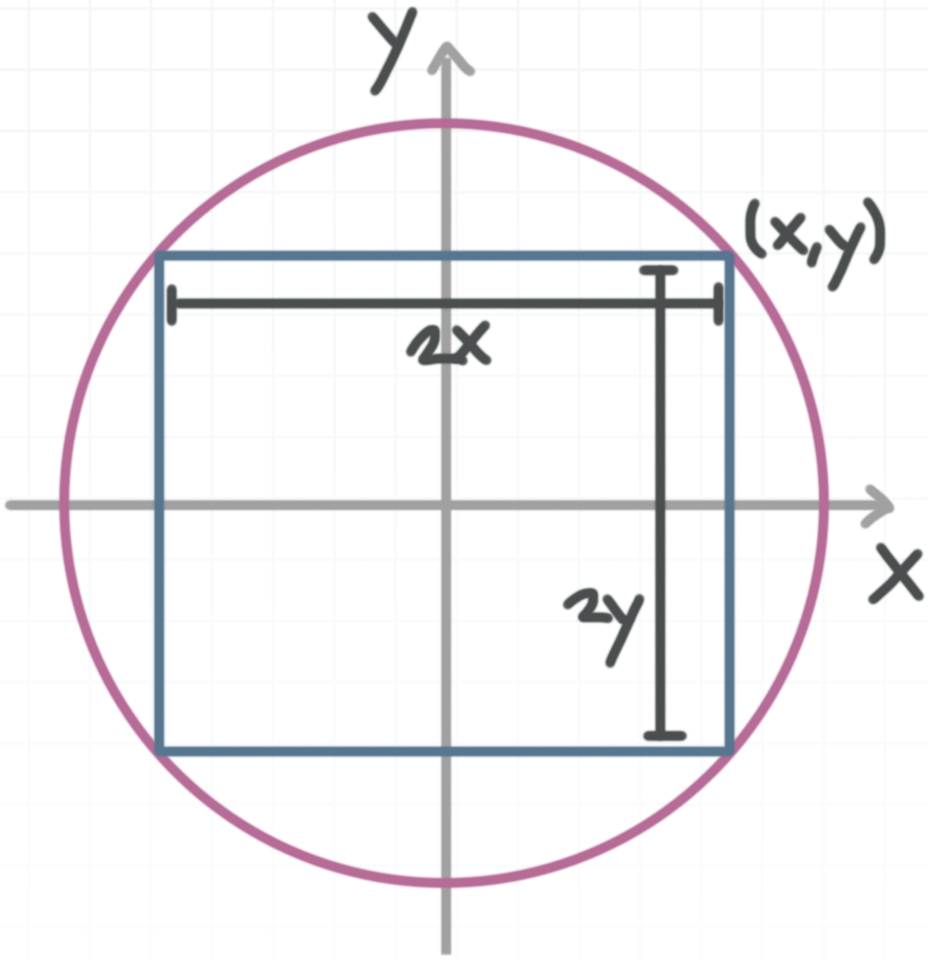
$$h \approx 2.24$$

In order to maximize the area of the window, the semicircle must have a radius of $r = 2.24$ and the rectangle must have dimensions 2.24×4.48 meters.

■ 6. Determine the dimensions of the rectangle with maximum area that can be inscribed in a circle of radius 3 cm.

Solution:

Draw a diagram.



The equation of the circle is

$$x^2 + y^2 = 9$$

$$y = \sqrt{9 - x^2}$$

with $-3 \le x \le 3$ and $-3 \le y \le 3$. The area of the rectangle is

$$A = (2x)(2y)$$

$$A = 4xy$$

$$A = 4x\sqrt{9 - x^2}$$

Differentiate the area function.

$$dA = 4\sqrt{9 - x^2} + 4x(-2x)\frac{1}{2\sqrt{9 - x^2}}$$

$$dA = 4\sqrt{9 - x^2} - \frac{4x^2}{\sqrt{9 - x^2}}$$

Set the derivative equal to 0 and solve for x.

$$0 = 4\sqrt{9 - x^2} - \frac{4x^2}{\sqrt{9 - x^2}}$$

$$4\sqrt{9 - x^2} = \frac{4x^2}{\sqrt{9 - x^2}}$$

$$9 - x^2 = x^2$$

$$9 = 2x^2$$

$$x^2 = \frac{9}{2}$$

$$x = \sqrt{\frac{9}{2}}$$

$$x \approx 2.12$$

Use the first derivative test with test values of $x = 2$ and $x = 3$ to confirm that the critical point represents a maximum.

$$A'(2) = 4\sqrt{9 - 2^2} - \frac{4(2)^2}{\sqrt{9 - 2^2}}$$

$$A'(2) = 4\sqrt{5} - \frac{16}{\sqrt{5}}$$

$$A'(2) \approx 1.79$$

and

$$A'(2.5) = 4\sqrt{9 - 2.5^2} - \frac{4(2.5)^2}{\sqrt{9 - 2.5^2}}$$

$$A'(2.5) \approx -8.44$$

Since we get a positive value to the left of the critical point and a negative value to the right of it, the function has a maximum at the critical point.

$$y = \sqrt{9 - x^2}$$

$$y = \sqrt{9 - \left(\sqrt{\frac{9}{2}}\right)^2}$$

$$y = \sqrt{9 - \frac{9}{2}}$$

$$y = \sqrt{\frac{9}{2}}$$

$$y \approx 2.12$$

We defined the dimensions of the rectangle originally as $2x \times 2y$, so the rectangle will have maximum area when its dimensions are

$$2x \times 2y = 2(2.12) \times 2(2.12) = 4.24 \times 4.24$$

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