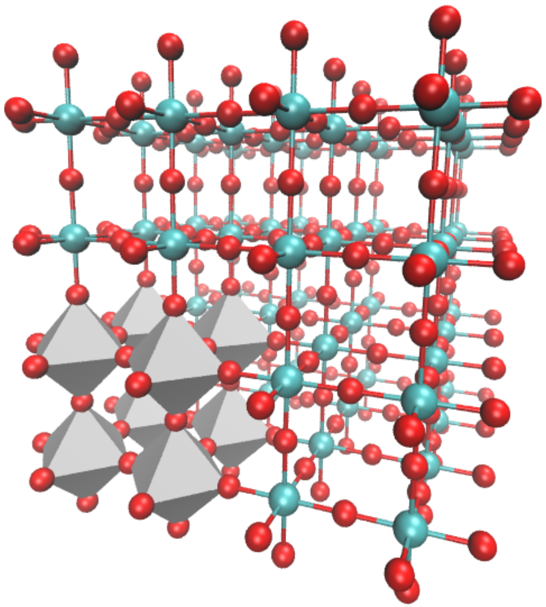
Linear Algebra



Prof. Lee M. Thompson

Operators

Operators

An operator $\hat{O}$ transforms a function into another

E.g. $\hat{x}$ multiplies by *x*

$$\hat{x}f(x) = xf(x)$$
$$\hat{x}\sin(x) = x\sin(x)$$

$\hat{D}_x$ *takes the derivative with respect to x*

$$\hat{D}_x f(x) = \frac{d}{dx}f(x)$$
$$\hat{D}_x \sin(x) = \cos(x)$$

$\hat{B}\hat{A}$ applies the operation $\hat{A}$ followed by $\hat{B}$

$$\hat{x}\hat{D}_x f(x) = x\frac{d}{dx}f(x)$$
$$\hat{x}\hat{D}_x \sin(x) = x\cos(x)$$

$$\hat{D}_x\hat{x}f(x) = \frac{d}{dx}(xf(x)) = f(x) + x\frac{d}{dx}f(x)$$
$$\hat{D}_x\hat{x}\sin(x) = \sin(x) + x\cos(x)$$

Generally $\hat{B}\hat{A} \neq \hat{A}\hat{B}$

Operators

Operators

$\hat{A}^n$ applies the operator $\hat{A}$ *n* times

$$\hat{D}_x^2 f(x) = \frac{d}{dx}\left(\frac{d}{dx}f(x)\right) = \frac{d^2}{dx^2}f(x)$$

$$\hat{D}_x^2 \sin(x) = \frac{d^2}{dx^2}\sin(x) = -\sin(x)$$

If a function $f(x)$ obeys the following equation:

$$\hat{A}f(x) = af(x)$$

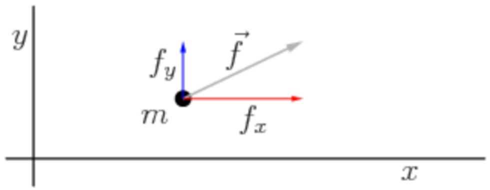
where $\hat{A}$ is an operator and *a* is a scalar, then:

- *a* is an eigenvalue of the operator $\hat{A}$
- $f(x)$ is an eigenfunction of the operator $\hat{A}$

Linear Algebra

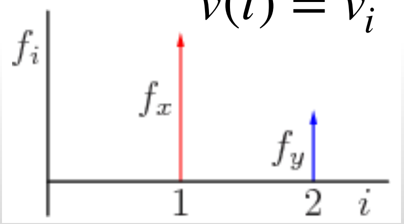
Discretization

- Functions ⇔ vectors
  - Vector (e.g. $\overrightarrow{p} = m\overrightarrow{v}$) can be visualized as an arrow with components in vector space



  - Alternatively, can plot vector as discretized plot of component against component value

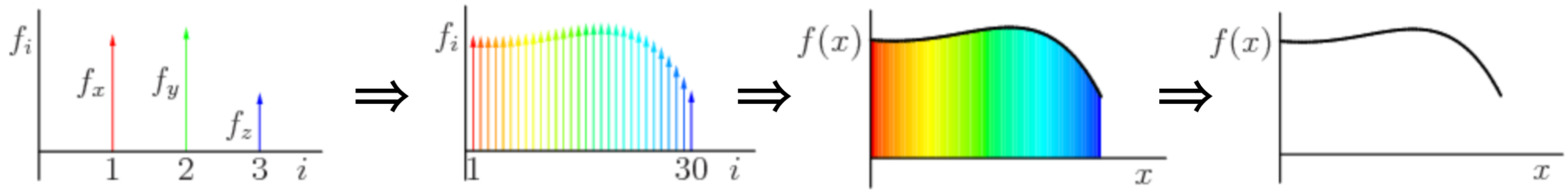
$$v(i) = v_i$$



Linear Algebra

Discretization

- Increasing number of dimensions in the vector space increases components
- A vector in an infinite dimension vector space defines a function



- Operators can act on vectors, **linear** operators are separable

$$\hat{O}\mathbf{v} = \mathbf{w} \qquad\qquad \hat{O}(x\mathbf{v} + y\mathbf{w}) = x\hat{O}\mathbf{v} + y\hat{O}\mathbf{w}$$

- Operator can be **represented** as a matrix

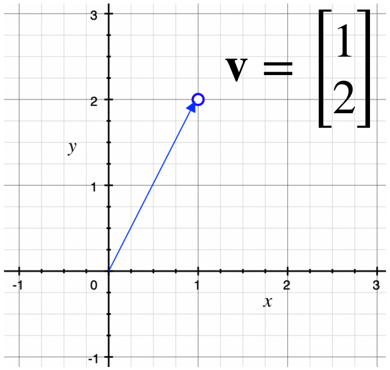
$$\mathbf{M} = \begin{bmatrix} m_{11} & m_{12} & \dots & m_{1n} \\ m_{21} & m_{22} & \dots & m_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ m_{n1} & m_{n2} & \dots & m_{nn} \end{bmatrix}$$

Vectors

Vectors

- Vectors have both magnitude and direction. Can be expressed with respect to basis vectors

$$\mathbf{v} = v_i\mathbf{e}_i + v_j\mathbf{e}_j + v_k\mathbf{e}_k + \ldots = \sum_i v_i\mathbf{e}_i$$



$$\mathbf{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

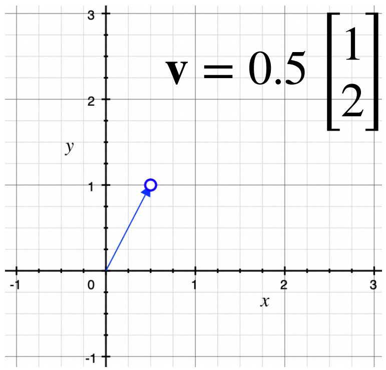
- Basis is not unique - we can choose a different set of basis vectors

$$\mathbf{v} = v_i'\mathbf{f}_i + v_j'\mathbf{f}_j + v_k'\mathbf{f}_k + \ldots = \sum_i v_i'\mathbf{f}_i$$

Vectors

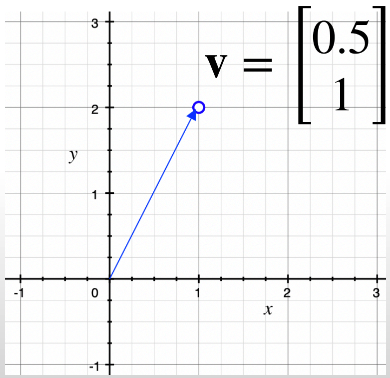
Vector operations

- Operations of vectors
  - Vector scalar multiplication



$$\mathbf{v} = 0.5\begin{bmatrix}1\\2\end{bmatrix} = \begin{bmatrix}0.5\\1\end{bmatrix}$$

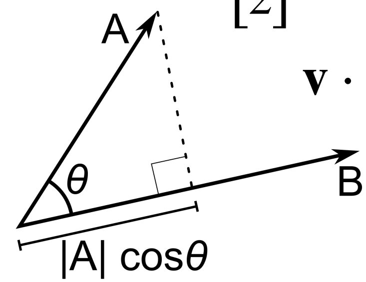
  - Vector addition



$$\mathbf{v} = \begin{bmatrix}0.5\\1\end{bmatrix} + \begin{bmatrix}0.5\\1\end{bmatrix} = \begin{bmatrix}1\\2\end{bmatrix}$$

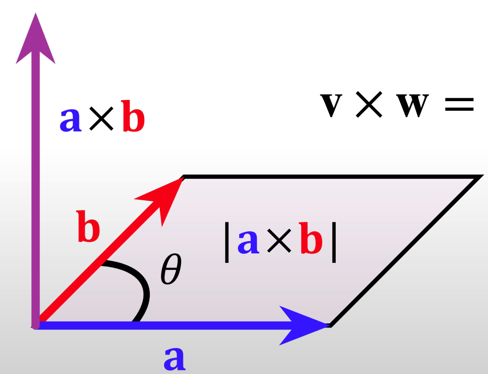
  - Vector dot product

$$\mathbf{v}\,.\,\mathbf{w} = [0.5 \quad 1]\begin{bmatrix}1\\2\end{bmatrix} = 0.5\times 1 + 1\times 2 = 2.5$$



$$\mathbf{v}\cdot\mathbf{w} = \sum_{ij}\left(v_i w_j\right)\mathbf{e}_i\cdot\mathbf{e}_j$$

  - Vector cross and triple product



$$\mathbf{v}\times\mathbf{w} = \begin{vmatrix}\mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3\\ v_1 & v_2 & v_3\\ w_1 & w_2 & w_3\end{vmatrix}$$

Triple product gives volume: $\mathbf{v}\,.\,(\mathbf{u}\times\mathbf{w})$

Vectors

Linear vector space

- A set of vectors {**v**,**w**,**x**…} forms a **linear vector space** if:

1. Closed under commutative and associative addition

$$\mathbf{v}+\mathbf{w}=\mathbf{w}+\mathbf{v}$$
$$(\mathbf{v}+\mathbf{w})+\mathbf{x}=\mathbf{v}+(\mathbf{w}+\mathbf{x})$$

2. Closed under scalar multiplication

$$\lambda(\mathbf{v}+\mathbf{w})=\lambda\mathbf{v}+\lambda\mathbf{w}$$
$$(\lambda+\mu)\mathbf{v}=\lambda\mathbf{v}+\mu\mathbf{v}$$
$$\lambda(\mu\mathbf{v})=(\lambda\mu)\mathbf{v}$$

3. A null vector **0** is defined

$$\mathbf{v}+\mathbf{0}=\mathbf{v}\quad\forall\mathbf{v}$$

4. Multiplication by unity leaves the vector unchanged

$$1\times\mathbf{v}=\mathbf{v}\quad\forall\mathbf{v}$$

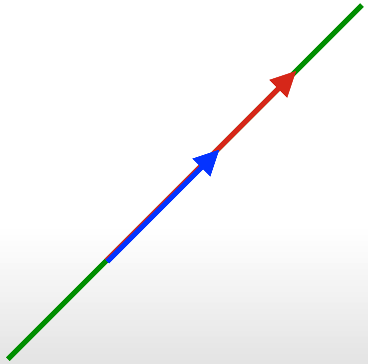
5. All vectors have corresponding negative vector **-v**

$$\mathbf{v}+(-\mathbf{v})=\mathbf{0}$$

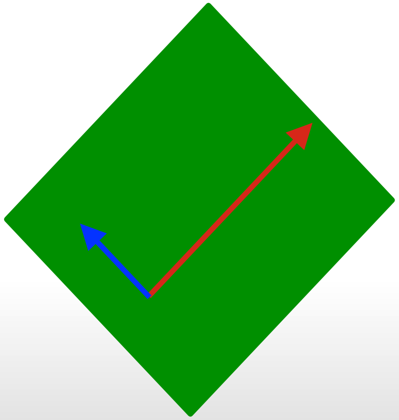
Vectors

Linear independence of vectors

- Vectors can be combined in a linear combination
  - We already defined arbitrary vectors as linear combinations of basis vectors
- Operators required are vector addition and vector-scalar multiplication
- N linear independent vectors required in N-dimensional space
- The **span** is the set of new vectors that can be constructed



$$
\begin{aligned}
\mathbf{v} &= c_1 \begin{bmatrix} 0.5 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} \\
&= c_1 \begin{bmatrix} 0.5 \\ 1 \end{bmatrix} + 2c_2 \begin{bmatrix} 0.5 \\ 1 \end{bmatrix} \\
&= (c_1 + 2c_2) \begin{bmatrix} 0.5 \\ 1 \end{bmatrix} = c_3 \begin{bmatrix} 0.5 \\ 1 \end{bmatrix}
\end{aligned}
$$

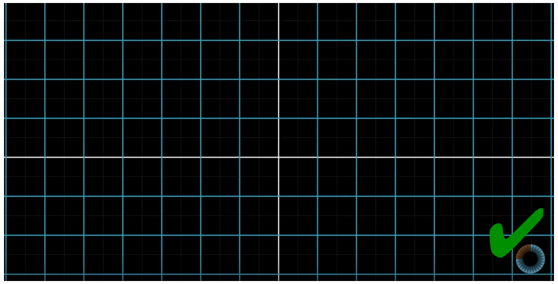


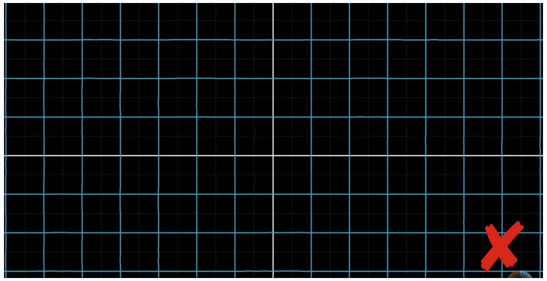
- Linearly dependence if $\nu \mathbf{v} + \omega \mathbf{w} + \chi \mathbf{x} = \mathbf{0}$ where {λ} not all zero

Basis Transformation

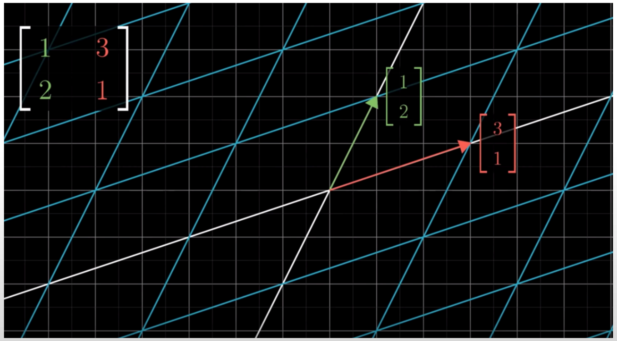
Linear transformations

- Linear transformations
  - Transform old basis vectors to new basis vector where:
    1. Origin is invariant
    2. Space does not become “curved”





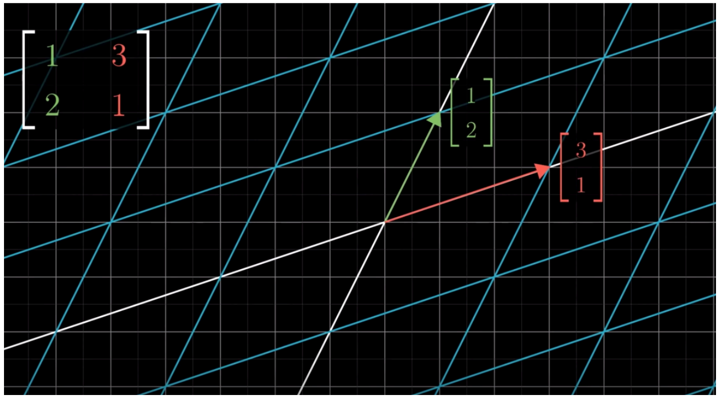
- Matrices describe the transformation
  - Columns tell us where each basis vector lands after transformation



Basis Transformation

Linear transformation

- What happens to a given vector under a basis transformation?
- Alias vs. alibi



$$\mathbf{v}' = \begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix} \mathbf{v} \qquad \mathbf{v} = \begin{bmatrix} 0.5 \\ 1 \end{bmatrix}$$

$$\mathbf{v}' = 0.5 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + 1 \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 3.5 \\ 2 \end{bmatrix}$$

- Matrix-vector multiplication encodes the transformation

$$\mathbf{v}' = \begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 0.5 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \times 0.5 + 3 \times 1 \\ 2 \times 0.5 + 1 \times 1 \end{bmatrix} = \begin{bmatrix} 3.5 \\ 2 \end{bmatrix}$$

Matrix Algebra

Matrix Multiplication

- Linear maps have linear maps
- I.e. how can we describe the total transformation of two consecutive transformations

$$\begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} \begin{bmatrix} n_{11} & n_{12} \\ n_{21} & n_{22} \end{bmatrix} = \begin{bmatrix} m_{11}n_{11} + m_{12}n_{21} & m_{11}n_{12} + m_{12}n_{22} \\ m_{21}n_{11} + m_{22}n_{21} & m_{21}n_{12} + m_{22}n_{22} \end{bmatrix}$$

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$$\begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} \begin{bmatrix} n_{11} & n_{12} \\ n_{21} & n_{22} \end{bmatrix} = \begin{bmatrix} m_{11}n_{11} + m_{12}n_{21} & m_{11}n_{12} + m_{12}n_{22} \\ m_{21}n_{11} + m_{22}n_{21} & m_{21}n_{12} + m_{22}n_{22} \end{bmatrix}$$

- Matrix multiplication may not commute

Matrix Addition

$$\begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} + \begin{bmatrix} n_{11} & n_{12} \\ n_{21} & n_{22} \end{bmatrix} = \begin{bmatrix} m_{11} + n_{11} & m_{12} + n_{12} \\ m_{21} + n_{21} & m_{22} + n_{22} \end{bmatrix}$$

Matrix-Scalar Multiplication

$$\lambda \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} = \begin{bmatrix} \lambda m_{11} & \lambda m_{12} \\ \lambda m_{21} & \lambda m_{22} \end{bmatrix}$$

Matrix Algebra

Unary Matrix Operations

- Transpose

$$\{\mathbf{M}^T : m_{ij} = m_{ji}\}$$

$$\begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix}^T = \begin{bmatrix} m_{11} & m_{21} \\ m_{12} & m_{22} \end{bmatrix}$$

$$(\mathbf{M}^T)^T = \mathbf{M}$$
$$(\mathbf{M} + \mathbf{N})^T = \mathbf{M}^T + \mathbf{N}^T$$
$$(\mathbf{MN})^T = \mathbf{N}^T\mathbf{M}^T$$
$$(\lambda\mathbf{M})^T = \lambda(\mathbf{M}^T)$$

- Adjoint

$$\{\mathbf{M}^\dagger : m_{ij} = m_{ji}^*\}$$

$$\begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix}^\dagger = \begin{bmatrix} m_{11}^* & m_{21}^* \\ m_{12}^* & m_{22}^* \end{bmatrix}$$

- Trace

$$\mathrm{tr}(\mathbf{M}) = \sum_i m_{ii}$$

$$\mathrm{tr}\left(\begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix}\right) = m_{11} + m_{22}$$

$$\mathrm{tr}(\mathbf{MN}) = \mathrm{tr}(\mathbf{NM})$$
$$\mathrm{tr}(\mathbf{M} + \mathbf{N}) = \mathrm{tr}(\mathbf{M}) + \mathrm{tr}(\mathbf{N})$$

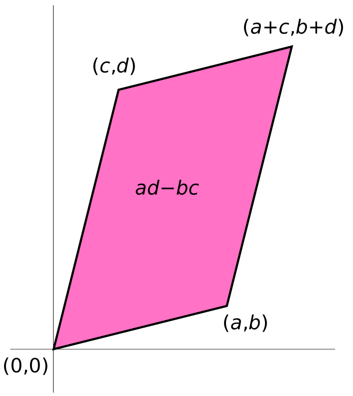
Matrix Algebra

Determinants

- c.f. triple product and volume elements
- Gives signed scale factor by which area/volume is transformed

$$\det(\mathbf{M}) = |\mathbf{M}| = \begin{vmatrix} m_{11} & m_{12} & \cdots & m_{1n} \\ m_{21} & m_{22} & \cdots & m_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ m_{n1} & m_{n2} & \cdots & m_{nn} \end{vmatrix} = \sum_{i=1}^{n!} (-1)^{p_i} \mathscr{P}_i m_{11} m_{22} \ldots m_{nn}$$

$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 1 & -1 & 1 \end{vmatrix} = -12$$



- Properties
  - Any two rows or columns equal, det(**M**)=0
  - Determinant changes sign on exchange of rows and columns
  - $|\mathbf{M}| = |\mathbf{M}^\dagger|^*$
  - $|\mathbf{MN}| = |\mathbf{M}||\mathbf{N}|$

Matrix Algebra

Inverse

- The inverse of a matrix is defined such that

$$\mathbf{M}^{-1}\mathbf{M} = \mathbf{I}$$

where **I** is the identity matrix

$$\mathbf{I} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

One way to compute the inverse is using

$$\mathbf{M}^{-1} = \frac{1}{\det(\mathbf{M})}\mathbf{C_M}^T$$

$\mathbf{C_M}^T$ is the cofactor matrix of $\mathbf{M}$

- Properties
  - $\mathbf{M}^{-1}$ is unique if it exists
  - $(\mathbf{M}^{-1})^{-1} = \mathbf{M}$
  - $(\mathbf{MN})^{-1} = \mathbf{N}^{-1}\mathbf{M}^{-1}$

Matrix Algebra

Types of matrices

- Diagonal matrix

  $\{\boldsymbol{\varepsilon} : m_{ij}\delta_{ij}\}$
- Unit matrix

  $\{\mathbf{1} : \delta_{ij}\}$
- Null matrix

  $\{\mathbf{0} : 0\}$
- Orthogonal matrix

  $\mathbf{M}^{-1} = \mathbf{M}^{T}$
- Unitary matrix

  $\mathbf{M}^{-1} = \mathbf{M}^{\dagger}$
- Symmetric matrix

  $\mathbf{M}^{T} = \mathbf{M}$
- Hermitian matrix

  $\mathbf{M}^{\dagger} = \mathbf{M}$
- Antisymmetric matrix

  $-\mathbf{M}^{T} = \mathbf{M}$
- Antihermitian matrix

  $-\mathbf{M}^{\dagger} = \mathbf{M}$
- Normal matrix

  $\mathbf{M}\mathbf{M}^{\dagger} = \mathbf{M}^{\dagger}\mathbf{M}$

Matrix Algebra

- Changing basis - Set of basis vectors transformed to new set by transformation matrix (which is necessarily unitary - preserves length and orthogonality of basis vectors)

$$\mathbf{UV} = \mathbf{W}$$

- Representation of operator (matrix) changes depending on basis. Always possible to find a representation which is diagonal

$$\mathbf{U}^{\dagger}\mathbf{VU} = \boldsymbol{\varepsilon}$$

- Eigenvalue equations - find the vectors (eigenvectors) that are unchanged under linear transformation except for a scaling factor (eigenvalues)

$$\mathbf{VU} = \mathbf{U}\boldsymbol{\varepsilon}$$

Simultaneous Equations

- Consider the following set of simultaneous equations

$$(1) \quad 2x + 3y = 1$$

$$(2) \quad x + y = 1$$

- We can rewrite using matrix notation and solve for the unknowns

$$\begin{bmatrix} 2 & 3 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

- Generally we have $\mathbf{Ax} = \mathbf{b}$ which transforms vector **x** in vector space **V** into vector **b** in vector space **W**
- If the inverse of **A** does not exist (the matrix is singular) then some subspace of **V** is mapped to **0** in **W** - **null space**
- N equations with N unknowns arises when the **nullity** of A is zero
- Inversion can be computationally demanding so can use alternative decomposition methods instead (LU, Cholesky)

Eigensystems

Determining eigenvectors and eigenvalues

- General eigenvalue equation

$$\mathbf{A}\mathbf{x} = \lambda\mathbf{x}$$

- Can rewrite as homogeneous set of simultaneous equations

$$\mathbf{A}\mathbf{x} - \lambda\mathbf{I}\mathbf{x} = (\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = \mathbf{0}$$

- For such a system, either **x**=**0** (trivial solution) or det(**A**-**λI**) = 0
- N roots of the characteristic determinant gives the set of eigenvalues of **A**
- Example: $\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix}$ $(\mathbf{A} - \lambda\mathbf{I}) = \begin{vmatrix} 1-\lambda & 2 \\ 3 & 1-\lambda \end{vmatrix} = 0$ $(1-\lambda)^2 - 6 = 0$ $\lambda = 1 - \pm\sqrt{6}$

$$\begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 1 + \sqrt{6}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \Rightarrow \begin{bmatrix} x_1 + 2x_2 \\ 3x_1 + x_2 \end{bmatrix} = \begin{bmatrix} (1+\sqrt{6})x_1 \\ (1+\sqrt{6})x_2 \end{bmatrix}$$

$$x_1 = \frac{\sqrt{6}}{3}x_2 = \sqrt{\frac{2}{3}}x_2 \quad x_1^2 + x_2^2 = 1 \Rightarrow \left(\frac{\sqrt{6}}{3}k\right)^2 + k^2 = 1 \Rightarrow k = \sqrt{\frac{3}{5}} \quad \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \sqrt{\frac{2}{5}} \\ \sqrt{\frac{3}{5}} \end{bmatrix}$$

Group Theory

Group theory and representations

A group G is a set of elements {g} with an operation ⊙ that satisfies the axioms:

1. closure: g⊙h=i where g, h and i are elements of G
2. associativity: (g⊙h)⊙i= g⊙(h⊙i) where g and e are elements of G
3. identity: g⊙e=g where g and e are elements of G
4. inverse: g⊙h=e where g, h and e are elements of G

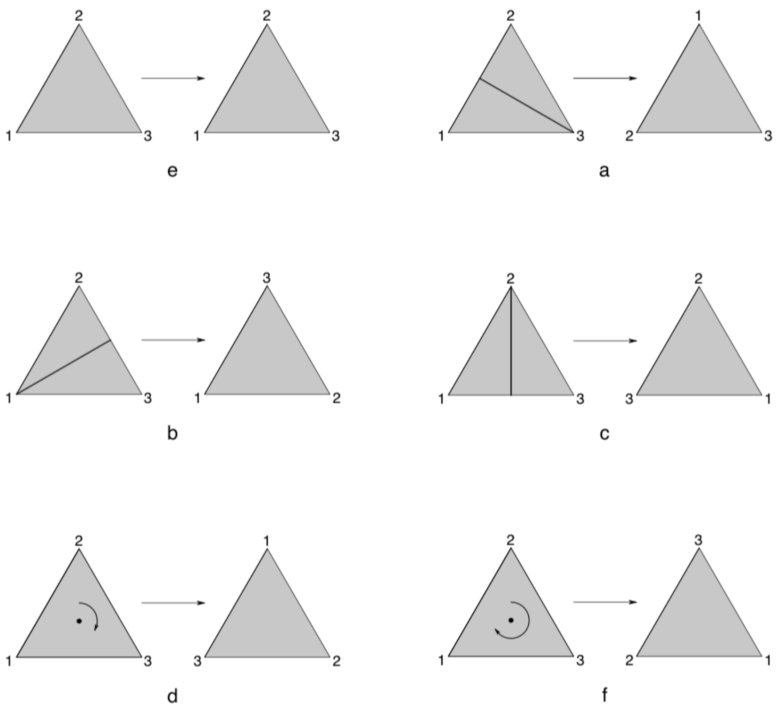
Group elements can be commutative (Abelian) but may not

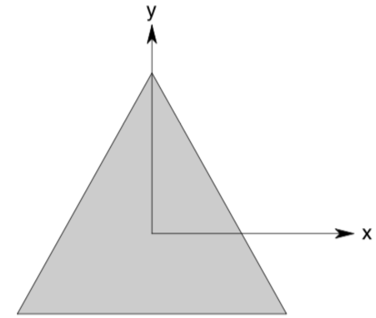
Examples of groups:

- $\mathbb{Z}$, set of integers under addition
- $\mathbb{Z}_n$, set of integers mod 2 under addition
- $\mathbb{Q}$, set of rational numbers under addition
- $\mathbb{R}^*$, set of non-zero real numbers under multiplication
- $GL(2,\mathbb{R})$, set of all 2×2 invertible matrices under matrix multiplication

Group Theory

Group theory and representations





$$\mathrm{D}_e = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathrm{D}_a = \frac{1}{2}\begin{pmatrix} 1 & -\sqrt{3} \\ -\sqrt{3} & -1 \end{pmatrix}, \quad \mathrm{D}_b = \frac{1}{2}\begin{pmatrix} 1 & \sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix},$$

$$\mathrm{D}_c = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathrm{D}_d = \frac{1}{2}\begin{pmatrix} -1 & -\sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix}, \quad \mathrm{D}_f = \frac{1}{2}\begin{pmatrix} -1 & \sqrt{3} \\ -\sqrt{3} & -1 \end{pmatrix}$$

Group Theory

Group theory and representations

Groups are not the only areas of abstract algebra (collections of objects with operations) that are useful:

- Rings
  - set with two operations - commutative $\odot$ and associative $\circledast$
  - $\mathbb{Z}$, with addition and multiplication
- Fields
  - set with two commutative operators
  - $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$
- Vector spaces
  - set of vectors *V* and a set of scalars *F*
  - *V* is a commutative group under vector addition
  - *F* is a field under multiplication
- Algebras
  - Vector space with a bilinear product
  - e.g. matrix multiplication M(m,n) × M(n,p) → M(m,p)

Summary

- Operators
- Functions
- Vectors
- Basis transformations
- Matrix Algebra
- Simultaneous equations
- Eigensystems
- Group theory