

# Petri Nets and Proof Search for Additive Classical Logic

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ABSTRACT

**We investigate efficient algorithms for proof search within additive classical logic.**

## 1 ADDITIVE CLASSICAL LOGIC

**Definition 1.1.** Logic Statements

A *statement* within additive classical logic is constructed as follows:

$$A, B, C ::= \perp \mid \top \mid a \mid \neg a \mid A \vee B \mid A \wedge B$$

$$\Gamma, \Delta, \Sigma ::= A_1 \dots A_n$$

where  $\vee, \wedge$  are additive disjunction and conjunction respectively.

**Example.**

**Definition 1.2.** Proof Trees

Within *additive classical logic*, a *proof tree* is constructed from the following deductions:

$$\frac{}{\vdash \top} \top R \qquad \frac{\vdash \Gamma, A \quad \vdash \Gamma, B}{\vdash \Gamma, A \wedge B} \wedge R$$

$$\frac{}{\vdash a, \neg a} a R \qquad \frac{\vdash \Gamma}{\vdash \Gamma, A} w R$$

$$\frac{\vdash \Gamma, A, B}{\vdash \Gamma, A \vee B} \vee R \qquad \frac{\vdash \Gamma, A, A}{\vdash \Gamma, A} c R$$

That is, a proof tree provides, without context, a proof of its bottom statement.

**Example.**

**Lemma 1.3.** *Additive linear logic* rules may be encoded within *multiplicative linear logic* and vice versa.

**Definition 1.4.** Subproofs

A *subproof* is a set of *tops*  $\Gamma_1 \dots \Gamma_n$  and a singleton *bottom*  $\Delta$  such that the tree provides a proof of  $\Gamma_1 \dots \Gamma_n \Rightarrow \Delta$ .

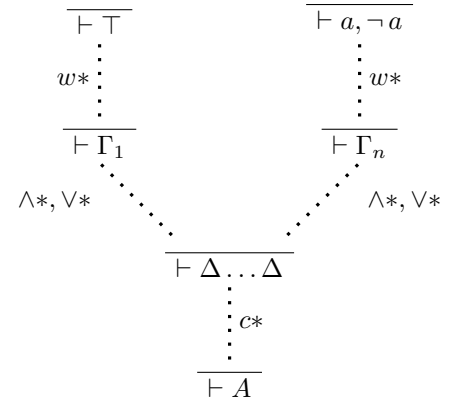
**Corollary 1.5.** Proof Tree Equivalence

Equivalence of subproofs is defined up to equivalence of tops and bottoms. Subsequently, proof trees may be manipulated by substituting equivalent subproofs.

**Example.**

**Definition 1.6.** Additive Stratification

A proof tree is said to be *additively stratified* if  $\vdash A$  is structured as follows:



That is, the deductions made in an additively stratified proof are strictly ordered by:

- Top/Atomic
- Weakening
- Conjunction/Disjunction
- Contraction

**Example.**

**Theorem 1.7.** Stratification Equivalence

Given  $\vdash A$ , there exists an additively stratified proof of  $A$ .

*Proof.* By examining each neighbouring pair of deductions.

1. For each instance of a weakening below another deduction, there exists an equivalent subproof that is additively stratified.
2. Similarly, for each instance of a contraction above another deduction, there exists an equivalent subproof that is additively stratified.
3. Finally, given any proof tree, the process of continued substitutions both halts and produces an additively stratified proof tree.

□

## 2 PETRI NETS

### **Definition 2.1.** Petri Nets

A petri net is ...

### **Example.**

### **Definition 2.2.** Coalescence

The coalescence algorithm is ...

### **Theorem 2.3.** Coalescence Proof Search

The coalescence algorithm is a proof search ...

*Proof.*    •  $cR \equiv$  dimensionality

- $aR \equiv$  2-D token initialisation
- $wR \equiv$  n-D extension of tokens
- $\wedge R, \vee R \equiv$  petri net firing

□