From Additive to Classical Proof Search

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Introduction

Outline:

- Additive Linear Logic (ALL)
- Proof search in ALL through *coalescence*
- Classical Logic (CL)
- Proof search in CL through *coalescence* and *additive stratification*
- Complexity bounds of proof search
- Dimensionality of CL formulae

Additive Linear Logic (ALL)

$$A, B, C \quad ::= \quad 0 \mid 1 \mid a \mid \overline{a} \mid A + B \mid A \times B$$

- $0, a, +$ are duals of $1, \overline{a}, \times$ respectively

Additive Linear Logic (ALL)

Some examples:

- $(a + \overline{b}) \times 1$ is the dual of $(\overline{a} \times b) + 0$ and vice versa
-

Additive Linear Logic (ALL)

$$\frac{}{\vdash a, \overline{a}}\ ax \qquad \frac{}{\vdash 1, A}\ 1 \qquad \frac{\vdash A, C}{\vdash A + B, C}\ +_1$$

$$\frac{\vdash B, C}{\vdash A + B, C}\ +_2 \qquad \frac{\vdash A, C \qquad \vdash B, C}{\vdash A \times B, C}\ \times$$

- All sequents are comprised of a pair of terms, maintained by deduction rules

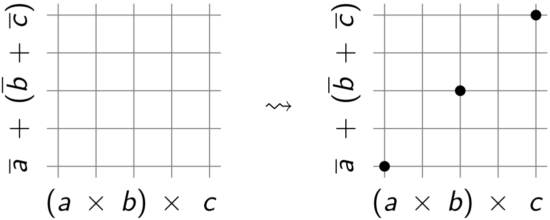
Additive Linear Logic (ALL)

Example: $\vdash a \times b, \overline{a} + \overline{b}$

$$
\dfrac{\dfrac{\dfrac{}{\vdash a, \overline{a}}\,ax}{\vdash a, \overline{a} + \overline{b}}\,+ \qquad \dfrac{\dfrac{}{\vdash b, \overline{b}}\,ax}{\vdash b, \overline{a} + \overline{b}}\,+}{\vdash a \times b, \overline{a} + \overline{b}}\,\times
$$

Coalescence

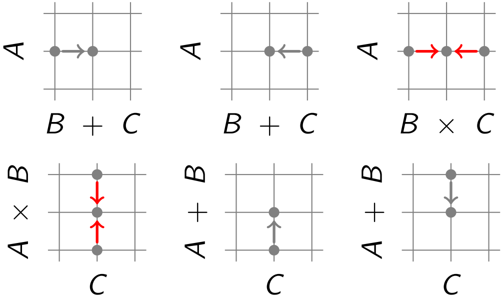
Spawning:



- Corresponds to axiom rule for ALL

Coalescence

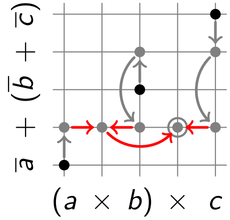
Transitions:



- Corresponds to sequent rules for ALL

Coalescence

Example:



- Algorithm succeeds if and only if there is an ALL proof of the given formula, see Heijltjes & Hughes (2015)

Classical Logic (CL)

$$A, B, C \quad ::= \quad \top \mid \bot \mid a \mid \overline{a} \mid A \vee B \mid A \wedge B$$

Classical Logic (CL)

$$\frac{}{\vdash \top}\ \top \qquad \frac{\vdash \Gamma, A}{\vdash \Gamma, A \vee B}\ \vee \qquad \frac{\vdash \Gamma}{\vdash \Gamma, A}\ w$$

$$\frac{}{\vdash a, \bar{a}}\ ax \qquad \frac{\vdash \Gamma, A \qquad \vdash \Gamma, B}{\vdash \Gamma, A \wedge B}\ \wedge \qquad \frac{\vdash \Gamma, A, A}{\vdash \Gamma, A}\ c$$

- A, B, C are formulae, Γ, Δ, Σ are sequents
- $\wedge, \vee$ rules preserve the number of terms in a sequent

Classical Logic (CL)

Example:

Additive Stratification

$$
\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash A_1}\ \top, ax}{\vdash \Gamma_1}\ w \qquad \dots \qquad \dfrac{\dfrac{}{\vdash A_n}\ \top, ax}{\vdash \Gamma_n}\ w}{\vdash P \dots P}\ \wedge, \vee}{\vdash P}\ c
$$

- Rearrangement does not affect the number of terms in sequents
- Does not work for MLL rules, see Brunnler (2003)

Additive Stratification

$$\dfrac{\dfrac{\vdash \Gamma, A}{\vdash \Gamma, A \vee B}\ \vee}{\vdash \Gamma, A \vee B,\ C}\ w \quad\rightsquigarrow\quad \dfrac{\dfrac{\vdash \Gamma, A}{\vdash \Gamma, A, C}\ w}{\vdash \Gamma, A \vee B, C}\ \vee$$

$$\dfrac{\dfrac{\vdash \Gamma, A \qquad \vdash \Gamma, B}{\vdash \Gamma, A \wedge B}\ \wedge}{\vdash \Gamma, A \wedge B, C}\ w \quad\rightsquigarrow\quad \dfrac{\dfrac{\vdash \Gamma, A}{\vdash \Gamma, A, C}\ w \qquad \dfrac{\vdash \Gamma, B}{\vdash \Gamma, B, C}\ w}{\vdash \Gamma, A \wedge B, C}\ \wedge$$

$$\dfrac{\dfrac{\vdash \Gamma, A, A}{\vdash \Gamma, A}\ c}{\vdash \Gamma, A, B}\ w \quad\rightsquigarrow\quad \dfrac{\dfrac{\vdash \Gamma, A, A}{\vdash \Gamma, A, A, B}\ w}{\vdash \Gamma, A, B}\ c$$

Additive Stratification

$$\dfrac{\dfrac{\vdash \Gamma, A, A}{\vdash \Gamma, A}\,c}{\vdash \Gamma, A \vee B}\,\vee \qquad \rightsquigarrow \qquad \dfrac{\dfrac{\dfrac{\vdash \Gamma, A, A}{\vdash \Gamma, A \vee B, A}\,\vee}{\vdash \Gamma, A \vee B, A \vee B}\,\vee}{\vdash \Gamma, A \vee B}\,c$$

$$\dfrac{\dfrac{\vdash \Gamma, A, A}{\vdash \Gamma, A}\,c \qquad \vdash \Gamma, B}{\vdash \Gamma, A \wedge B}\,\wedge \quad \rightsquigarrow \quad \dfrac{\dfrac{\dfrac{\vdash \Gamma, A, A \qquad \dfrac{\vdash \Gamma, B}{\vdash \Gamma, A, B}\,w}{\vdash \Gamma, A, A \wedge B}\,\wedge \qquad \dfrac{\vdash \Gamma, B}{\vdash \Gamma, B, A \wedge B}\,w}{\vdash \Gamma, A \wedge B, A \wedge B}\,\wedge}{\vdash \Gamma, A \wedge B}\,c$$

- Proof trees are described as *morally equivalent*, see Girard et al. (1989)

Additive Stratification

Example:

Coalescence in CL

We investigate how the coalescence technique applies to classical propositional logic. The main idea is that a classical formula A can be proved by additive rules applied to a sequent $\vdash A, \ldots, A$ with n copies of A. This is easily shown by a *stratification* of sequent proofs with additive rules, where all contractions are pushed to the bottom (conclusion), and weakenings to the top (axioms). Correspondingly, we generalize coalescence proof search to CL by applying it to a grid of n dimensions, for any n, where it was previously fixed at 2. The algorithm will start at dimension 1, and when it fails at dimension n it will try again at dimension $n + 1$, up to a theoretically determined upper bound.

Coalescence in CL

For a formula P in CL, the coalescence algorithm runs as follows:

1. Set $n := 1$
2. Construct the n-dimensional grid of possible n-term sequents $\vdash A_1 \ldots A_n \in |\mathsf{A}| \times \ldots_n \times |\mathsf{A}|$
3. Spawn tokens at all instances of axiom links $\vdash \Gamma, a, \overline{a}$
4. Exhaustively perform transitions given by the CL sequent calculus rules
5. Does there exist a token at $(P, P \ldots P) \equiv \vdash P, P \ldots P \equiv \vdash P$?
    5.1 Yes — Halt with a proof for P and dimensionality n
    5.2 No — Increment $n := n + 1$ and goto 2

Questions

Dimensionality

The dimensionality of a proof is then the dimensionality of our grid when the root is reached, equivalent to the number of contractions required in an equivalent sequent proof. A CL formula can thus be proved by an n-dimensional additively stratified proof, where n is the number of terms in a sequent before contraction. Through a natural transformation on steps of the algorithm to equivalent sequent proofs, coalescence up to $n = N$ is then exactly (additively stratified) proof search, with implicit weakening and contraction of all sequents up to N terms.

Conclusion

References

Brunnler, K. (2003), 'Two restrictions on contraction', *Logic Journal of the IGPL* **11**(5), 525–529.

Girard, J.-Y., Taylor, P. & Lafont, Y. (1989), *Proofs and types*, Vol. 7, Cambridge university press Cambridge.

Heijltjes, W. & Hughes, D. J. (2015), Complexity bounds for sum-product logic via additive proof nets and petri nets, *in* '2015 30th Annual ACM/IEEE Symposium on Logic in Computer Science', IEEE, pp. 80–91.