# Natural Proof Search for Classical Logic


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February 10, 2019


**Abstract**

**We investigate a natural algorithm for proof search within classical logic and prove bounds on the complexity class of such a search.**


## 1 Classical Logic

**Definition 1.1** (Formulae)**.**
A *formula* within classical logic is constructed as follows:

$$\begin{aligned} A, B, C &::= \bot \mid \top \mid a \mid \neg a \mid A \vee B \mid A \wedge B \\ \Gamma, \Delta, \Sigma &::= A_1 \ldots A_n \end{aligned}$$

where $\vee, \wedge$ are additive linear logic disjunction and conjunction respectively and $\Gamma, \Delta, \Sigma$ are contexts..

**Example.**

**Definition 1.2** (Sequent Proofs)**.**
Within *classical logic*, a *sequent proof* is constructed from the following rules:

$$\frac{}{\vdash \top}\ \top \qquad \frac{\vdash \Gamma, A}{\vdash \Gamma, A \vee B}\ \vee\mathrm{R} \qquad \frac{\vdash \Gamma}{\vdash \Gamma, A}\ w$$

$$\frac{}{\vdash a, \neg a}\ ax \qquad \frac{\vdash \Gamma, A \qquad \vdash \Gamma, B}{\vdash \Gamma, A \wedge B}\ \wedge\mathrm{R} \qquad \frac{\vdash \Gamma, A, A}{\vdash \Gamma, A}\ c$$

where $A, B, C$ are formulae and $\Gamma, \Delta, \Sigma$ are sequents. A sequent proof provides, without context, a proof of its conclusion and each line of the proof represents a tautology.

**Example.**

**Remark 1.3.**
Within the context of weakening and contraction, *additive* and *multiplicative* rules are inter-derivable.

**Definition 1.4** (Derivations)**.**
Given *tops* $\Gamma_1 \dots \Gamma_n$ for the sequent proof $\vdash \Delta$, a *derivation* is a tree providing a proof of $\Gamma_1 \dots \Gamma_n \implies \Delta$.

A derivation is written as:

$$\frac{\vdash \Gamma_1 \qquad \dots \qquad \vdash \Gamma_n}{\vdash \Delta}\,\textit{[label]}$$

where the *label* describes which rules may be used within the derivation.

**Corollary 1.5** (Derivation Equivalence)**.**
A sequent proof is a derivation where all top derivations of the tree are $= \top, ax$. Equivalence of derivations may be weakly defined up to equivalence of leaves and conclusion.

**Example.**

**Definition 1.6** (Additive Stratification)**.**
A proof tree is said to be *additively stratified* if $\vdash P$ is structured as follows:

$$\dfrac{\dfrac{\dfrac{}{\vdash A_1}\,\top, ax}{\vdash \Gamma_1}\,w \qquad \dots \qquad \dfrac{\dfrac{}{\vdash A_n}\,\top, ax}{\vdash \Gamma_n}\,w}{\dfrac{\vdash P \dots P}{\vdash P}\,c}\,\wedge, \vee$$

That is, the inferences made in an additively stratified proof are strictly ordered by:

1. Top/Axiomatic
2. Weakening
3. Conjunction/Disjunction
4. Contraction

**Example.**

**Proposition 1.7** (Stratification Equivalence)**.**
Given $\vdash A$, there exists an additively stratified proof of $A$.

*Proof.* For each instance of a weakening below another inference, there exists an equivalent subproof that is additively stratified:

$$\dfrac{\dfrac{\vdash \Gamma, A, B}{\vdash \Gamma, A \vee B}\,\vee}{\vdash \Gamma, A \vee B, \text{C}}\,w \qquad \rightsquigarrow \qquad \dfrac{\dfrac{\vdash \Gamma, A, B}{\vdash \Gamma, A, B, C}\,w}{\vdash \Gamma, A \vee B, \text{C}}\,\vee$$

$$\dfrac{\dfrac{\vdash \Gamma, A \qquad \vdash \Gamma, B}{\vdash \Gamma, A \wedge B}\,\wedge}{\vdash \Gamma, A \wedge B, C}\,w \quad\rightsquigarrow\quad \dfrac{\dfrac{\vdash \Gamma, A}{\vdash \Gamma, A, C}\,w \qquad \dfrac{\vdash \Gamma, B}{\vdash \Gamma, B, C}\,w}{\vdash \Gamma, A \wedge B, C}\,\wedge$$

$$\dfrac{\dfrac{\vdash \Gamma, A, A}{\vdash \Gamma, A}\,c}{\vdash \Gamma, A, B}\,w \quad\rightsquigarrow\quad \dfrac{\dfrac{\vdash \Gamma, A, A}{\vdash \Gamma, A, A, B}\,w}{\vdash \Gamma, A, B}\,c$$

Similarly, for each instance of a contraction above another inference, there exists an equivalent subproof that is additively stratified:

$$\dfrac{\dfrac{\vdash \Gamma, A, A, B}{\vdash \Gamma, A, B}\,c}{\vdash \Gamma, A \vee B}\,\vee \quad\rightsquigarrow\quad \dfrac{\dfrac{\dfrac{\dfrac{\vdash \Gamma, A, A, B}{\vdash \Gamma, A, A, B, B}\,w}{\vdash \Gamma, A \vee B, A, B}\,\vee}{\vdash \Gamma, A \vee B, A \vee B}\,\vee}{\vdash \Gamma, A \vee B}\,c$$

$$\dfrac{\dfrac{\vdash \Gamma, A, A}{\vdash \Gamma, A}\,c \qquad \vdash \Gamma, B}{\vdash \Gamma, A \wedge B}\,\wedge \quad\rightsquigarrow\quad \dfrac{\dfrac{\dfrac{\vdash \Gamma, A, A \qquad \dfrac{\vdash \Gamma, B}{\vdash \Gamma, A, B}\,w}{\vdash \Gamma, A, A \wedge B}\,\wedge \qquad \dfrac{\vdash \Gamma, B}{\vdash \Gamma, B, A \wedge B}\,w}{\vdash \Gamma, A \wedge B, A \wedge B}\,\wedge}{\vdash \Gamma, A \wedge B}\,c$$

By induction from the leaves downwards on a finite height tree, apply the associated rule to each pair of inferences of the form ($c$ above *inf*). Any given $\vdash P$ may be rewritten:

$$\dfrac{\dfrac{\dfrac{}{\vdash A_1}\,\top, ax}{\vdash \Gamma_1}\,\wedge, \vee, w \qquad \dots \qquad \dfrac{\dfrac{}{\vdash A_n}\,\top, ax}{\vdash \Gamma_n}\,\wedge, \vee, w}{\vdash P}\,c$$

Again, by induction from the root upwards on this partially stratified tree, apply the associated rule to each pair of inferences of the form ($w$ below *inf*). $\vdash P$ may then be further rewritten:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash A_1}\,\top, ax}{\vdash \Gamma_1}\,w \qquad \dots \qquad \dfrac{\dfrac{}{\vdash A_n}\,\top, ax}{\vdash \Gamma_n}\,w}{\vdash P \dots P}\,\wedge, \vee}{\vdash P}\,c$$

$\square$

**Example.**

## 2 Coalescence

**Definition 2.1** (Petri Nets)**.**
For the purposes required here, a *petri net* $\mathcal{N}$ is $(\mathcal{P}, \mathcal{F})$ where $f \in \mathcal{F} : \mathcal{P}^m \times \mathcal{P}$. In particular, $\mathcal{P}$ is a set of places and $\mathcal{F}$ a set of flows. A *configuration* is a set $\mathcal{C} \subset \mathcal{P}$ of tokens in places.

**Example.**

**Definition 2.2** (Firing Petri Nets)**.**
Given a petri net $\mathcal{N}$ and configuration $\mathcal{C}$, a *firing* of the net $\mathcal{N}$ is a new configuration generated by application of a transition $f \in \mathcal{F}$ on $m$ tokens $c_1 \dots c_n \in \mathcal{C}$. In particular, $(\mathcal{N} = (\mathcal{P}, \mathcal{F}), \mathcal{C}) \mapsto (\mathcal{N}, \mathcal{C} \cup f_{right} \setminus f_{left})$ for some $f = (f_{left}, f_{right}) \in \mathcal{F}$

A petri net is said to be *exhaustively fired* if it is fired until there does not exist any such $f \in \mathcal{F}$ to fire.

**Example.**

**Remark 2.3.**
Implementing fireable petri nets is straightforward using an $n$-dimensional boolean array of places visited and a collection of $n$-tuples representing tokens.

**Definition 2.4** (Coalescence)**.**
Given a formula $P$, the coalescence algorithm is as follows:

1. Set $n := 1$
2. Construct a $n$-dimensional petri net $\mathcal{N}$ from $P$ where each subformula is a place, each conjunction and disjunction a flow
3. Construct a configuration $C$ with a token at each place $p = (\dots, a, \dots, \neg a, \dots)$ the intersection of a pair of tautological atoms
4. Exhaustively fire the petri net $\mathcal{N}$ using the *spawning* method.
5. If there exists a token in the configuration $\mathcal{C}^*$ at the root of the formula $P$, halt and return $n$
6. Otherwise, increment $n := n + 1$ and go to step 2

**Example.**

**Proposition 2.5** (Coalescence Proof Search)**.**
The coalescence algorithm on $P$ is exactly a proof search on $P$.

*Proof.* In particular, consider the additively stratified $\vdash P$ with $n - 1$ contractions (should such a proof exist). The first configuration of tokens is precisely a proof all possible combinations of the *axiom* rule with $n - 3$ other terms through the *weakening*. Each flow transition followed when fired is an application of either the $\vee$ or $\wedge$ rule. Finally, a token at the root of the formula is $\vdash P \dots P$, with the *contraction* rule applied implicitly.

If there exists an additively stratified proof of $P$, the coalescence algorithm will find it.

Since for every proof there exists an additively stratified proof, coalescence is precisely proof search. □

## 3 Dimensionality

**Definition 3.1** (Dimensionality)**.**
Given a formula $P$, the coalescence proof search produces a proof in an $n$ dimensional petri net. The dimensionality of $P$ is then defined $\dim(P) ::= n$. Equivalently, an additively stratified sequent proof requires $n-1$ contractions at the bottom of the proof. Given $\vdash P$, its dimensionality is defined $\dim(P) ::= n$.

**Proposition 3.2** (Deducing Dimensionality)**.**
Given formulae $P, Q$ such that $\exists \vdash P$ with $\dim(P) = n$ and $\exists \vdash Q$ with $\dim(Q) = m$, then:

$$\begin{aligned}\dim(P \vee Q) &= \min(n, m)\\ \dim(P \wedge Q) &= \max(n, m)\end{aligned}$$

*Proof.* Omitted □

**Remark** (Satisfiability vs Provability)**.**
For any formula $P$, there exist four distinct classes: *true, false, satisfiably true, satisfiably false*, where satisfiable differs by finding a particular assignment of values to each variable. Coalescence searches for a proof of $P$ in *true*, whereas the SAT problem addresses a proof of $P$ in *true*. In particular, *true* is the complement class to *satisfiably false* and *false* complement to *satisfiably true*.

**Definition 3.3** (Dimensionality when not Provable)**.**
Given a formula $P$ such that there does not exist $\vdash P$, for all formulae $Q$ such that there does exist $\vdash Q$, the dimensionality of $P$ is defined as the least $n$ such that:

$$\dim(P) ::= n \implies \begin{cases} \exists \vdash (P \vee Q) \implies \dim(P \vee Q) = \min(n, \dim(Q)) \\ \exists \vdash (P \wedge Q) \implies \dim(P \wedge Q) = \max(n, \dim(Q)) \end{cases}$$

**Remark.**
This definition of a 'partial dimensionality' amounts to finding the highest dimension in coalescence proof search that some useful deduction was made, outside of trivial cases.

**Definition 3.4** (Paths in a Tree)**.**
Path from root to any leaf (effectively iterating leaves, but tracing parents)

**Proposition 3.5** ($\vee$-Bound on Dimensionality)**.**

$$\begin{aligned}&\dim(P) \leq 1 + \#\{\vee \in P\}\\ &\dim(P) \leq 1 + \max\{\#\{\vee \in \mathit{path}\}\, \forall\, \mathit{path} \in \mathit{tree}(P)\}\end{aligned}$$

*Proof.* Omitted □

**Proposition 3.6** ($ax$-Bound on Dimensionality)**.**

$$\dim(P) \leq 1 + \#\{ax\text{-}Rule \in \vdash P\}$$
$$\dim(P) \leq 1 + \#\{vars \in P\}$$

*Proof.* The first case is trivial. Given a formula $P$ and a constructed sequent proof $\vdash P$ through coalescence, additive stratification ensures that $\dim(P) = n \implies leaves(P) \geq n$. As each of the $n$ branches of the tree moves 'upwards', they must either terminate at an $ax$-rule or branch, leaving $n + 1$ branches total. Subsequently, for each $ax$-rule forming a leaf of the tree, $\dim(P)$ increases by no more than one. Finally, consider a base case such as $P :=\vdash a \vee \neg a$ where $\dim(P) = 2 \leq 1 + \#\{ax - Rule \in\vdash P\}$ (in this case, it is equal).

Second case omitted □