Natural Proof Search for Classical Logic

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Abstract

We investigate a natural algorithm for proof search within classical logic.

1 Classical Logic

Definition 1.1. Formulae

A *formula* within classical logic is constructed as follows:

$$A, B, C ::= \bot \mid \top \mid a \mid \neg a \mid A \vee B \mid A \wedge B$$
$$\Gamma, \Delta, \Sigma ::= A_1 \dots A_n$$

where $\vee, \wedge$ are additive linear logic disjunction and conjunction respectively and Γ, Δ, Σ are contexts..

Example.

Definition 1.2. Sequent Proofs

Within *classical logic*, a *sequent proof* is constructed from the following rules:

$$\frac{}{\vdash \top}\ \top \qquad \frac{\vdash \Gamma, A}{\vdash \Gamma, A \vee B}\ \vee\mathrm{R} \qquad \frac{\vdash \Gamma}{\vdash \Gamma, A}\ w$$

$$\frac{}{\vdash a, \neg a}\ ax \qquad \frac{\vdash \Gamma, A \qquad \vdash \Gamma, B}{\vdash \Gamma, A \wedge B}\ \wedge\mathrm{R} \qquad \frac{\vdash \Gamma, A, A}{\vdash \Gamma, A}\ c$$

where A, B, C are formulae and Γ, Δ, Σ are sequents. A sequent proof provides, without context, a proof of its conclusion and each line of the proof represents a tautology.

Example.

Remark 1.3. Within the context of weakening and contraction, *additive* and *multiplicative* rules are inter-derivable.

Definition 1.4. Derivations

Given *tops* $\Gamma_1 \dots \Gamma_n$ for the sequent proof $\vdash \Delta$, a *derivation* is a tree providing a proof of $\Gamma_1 \dots \Gamma_n \implies \Delta$.

A derivation is written as:

$$\dfrac{\vdash \Gamma_1 \qquad \dots \qquad \vdash \Gamma_n}{\vdash \Delta}\,\textit{[label]}$$

where the *label* describes which rules may be used within the derivation.

Corollary 1.5. Derivation Equivalence
A sequent proof is a derivation where all top derivations of the tree are $=\!=\; \top, ax$. Equivalence of derivations may be weakly defined up to equivalence of leaves and conclusion.

Example.

Definition 1.6. Additive Stratification
A proof tree is said to be *additively stratified* if $\vdash P$ is structured as follows:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash A_1}\,\top, ax}{\vdash \Gamma_1}\,w \qquad \dots \qquad \dfrac{\dfrac{}{\vdash A_n}\,\top, ax}{\vdash \Gamma_n}\,w}{\vdash P \dots P}\,\wedge, \vee}{\vdash P}\,c$$

That is, the inferences made in an additively stratified proof are strictly ordered by:

1. Top/Axiomatic
2. Weakening
3. Conjunction/Disjunction
4. Contraction

Example.

Theorem 1.7. Stratification Equivalence
Given $\vdash A$, there exists an additively stratified proof of A.

Proof. For each instance of a weakening below another inference, there exists an equivalent subproof that is additively stratified:

$$\dfrac{\dfrac{\vdash \Gamma, A, B}{\vdash \Gamma, A \vee B}\,\vee}{\vdash \Gamma, A \vee B,\ \mathrm{C}}\,w \quad \rightsquigarrow \quad \dfrac{\dfrac{\vdash \Gamma, A, B}{\vdash \Gamma, A, B, C}\,w}{\vdash \Gamma, A \vee B,\ \mathrm{C}}\,\vee$$

$$\dfrac{\dfrac{\vdash \Gamma, A \qquad \vdash \Gamma, B}{\vdash \Gamma, A \wedge B}\,\wedge}{\vdash \Gamma, A \wedge B, C}\,w \quad \rightsquigarrow \quad \dfrac{\dfrac{\vdash \Gamma, A}{\vdash \Gamma, A, C}\,w \qquad \dfrac{\vdash \Gamma, B}{\vdash \Gamma, B, C}\,w}{\vdash \Gamma, A \wedge B, C}\,\wedge$$

$$\dfrac{\dfrac{\vdash \Gamma, A, A}{\vdash \Gamma, A}\,c}{\vdash \Gamma, A, B}\,w \quad \rightsquigarrow \quad \dfrac{\dfrac{\vdash \Gamma, A, A}{\vdash \Gamma, A, A, B}\,w}{\vdash \Gamma, A, B}\,c$$

Similarly, for each instance of a contraction above another inference, there exists an equivalent subproof that is additively stratified:

$$\dfrac{\dfrac{\vdash \Gamma, A, A, B}{\vdash \Gamma, A, B}\,c}{\vdash \Gamma, A \vee B}\,\vee \qquad \rightsquigarrow \qquad \dfrac{\dfrac{\dfrac{\dfrac{\vdash \Gamma, A, A, B}{\vdash \Gamma, A, A, B, B}\,w}{\vdash \Gamma, A \vee B, A, B}\,\vee}{\vdash \Gamma, A \vee B, A \vee B}\,\vee}{\vdash \Gamma, A \vee B}\,c$$

$$\dfrac{\dfrac{\vdash \Gamma, A, A}{\vdash \Gamma, A}\,c \qquad \vdash \Gamma, B}{\vdash \Gamma, A \wedge B}\,\wedge \qquad \rightsquigarrow \qquad \dfrac{\dfrac{\dfrac{\vdash \Gamma, A, A \qquad \dfrac{\vdash \Gamma, B}{\vdash \Gamma, A, B}\,w}{\vdash \Gamma, A, A \wedge B}\,\wedge \qquad \dfrac{\vdash \Gamma, B}{\vdash \Gamma, B, A \wedge B}\,w}{\vdash \Gamma, A \wedge B, A \wedge B}\,\wedge}{\vdash \Gamma, A \wedge B}\,c$$

By induction from the leaves downwards on a finite height tree, apply the associated rule to each pair of inferences of the form (c above *inf*). Any given $\vdash P$ may be rewritten:

$$\dfrac{\dfrac{\dfrac{}{\vdash A_1}\,\top, ax}{\vdash \Gamma_1}\,\wedge, \vee, w \qquad \dots \qquad \dfrac{\dfrac{}{\vdash A_n}\,\top, ax}{\vdash \Gamma_n}\,\wedge, \vee, w}{\vdash P}\,c$$

Again, by induction from the root upwards on this partially stratified tree, apply the associated rule to each pair of inferences of the form (w below *inf*). $\vdash P$ may then be further rewritten:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash A_1}\,\top, ax}{\vdash \Gamma_1}\,w \qquad \dots \qquad \dfrac{\dfrac{}{\vdash A_n}\,\top, ax}{\vdash \Gamma_n}\,w}{\vdash P \dots P}\,\wedge, \vee}{\vdash P}\,c$$

□

2 Coalescence

Definition 2.1. Petri Nets

For the purposes required here, a *petri net* $\mathcal{N}$ is $(\mathcal{P}, \mathcal{F})$ where $f \in \mathcal{F} : \mathcal{P}^m \times \mathcal{P}$. In particular, $\mathcal{P}$ is a set of places and $\mathcal{F}$ a set of flows. A *configuration* is a set $\mathcal{C} \subset \mathcal{P}$ of tokens in places.

Example.

Definition 2.2. Firing Petri Nets

Given a petri net $\mathcal{N}$ and configuration $\mathcal{C}$, a *firing* of the net $\mathcal{N}$ is a new configuration generated by application of a transition $f \in \mathcal{F}$ on m tokens $c_1 \dots c_n \in \mathcal{C}$. In particular, $(\mathcal{N} = (\mathcal{P}, \mathcal{F}), \mathcal{C}) \mapsto (\mathcal{N}, \mathcal{C} \cup f_{right} \setminus f_{left})$ for some $f = (f_{left}, f_{right}) \in \mathcal{F}$

A petri net is said to be *exhaustively fired* if it is fired until there does not exist any such $f \in \mathcal{F}$ to fire.

Example.

Remark 2.3. Implementing fireable petri nets is straightforward using an n-dimensional boolean array of places visited and a collection of n-tuples representing tokens.

Definition 2.4. Coalescence
Given a formula P, the coalescence algorithm is as follows:

1. Set $n := 1$
2. Construct a n-dimensional petri net $\mathcal{N}$ from P where each subformula is a place, each conjunction and disjunction a flow
3. Construct a configuration C with a token at each place $p = (\ldots, a, \ldots, \neg a, \ldots)$ the intersection of a pair of tautological atoms
4. Exhaustively fire the petri net $\mathcal{N}$ using the *spawning* method.
5. If there exists a token in the configuration $\mathcal{C}^*$ at the root of the formula P, halt and return n
6. Otherwise, set $n := n + 1$ and go to step 2

Proposition 2.5. Coalescence Proof Search
The coalescence algorithm on P is exactly a proof search on P.

Proof. In particular, consider the additively stratified $\vdash P$ with $n-1$ contractions. The first configuration of tokens is precisely a proof all possible combinations of the *axiom* rule with $n - 3$ other terms through the *weakening*. Each flow transition followed when fired is an application of either the $\vee$ or $\wedge$ rule. Finally, a token at the root of the formula is $\vdash P \ldots P$, with the *contraction* rule applied implicitly.
If there exists an additively stratified proof of P, the coalescence algorithm will find it. Since for every proof there exists an additively stratified proof, coalescence is precisely proof search. □

3 Dimensionality

Definition 3.1. Dimensionality
Coalescence proof search produces a proof in n dimensions. Equivalently, an additively stratified sequent proof requires $n - 1$ contractions at the bottom of the proof. Given $\vdash P$, its dimensionality is defined $dim(P) ::= n$.

Proposition 3.2. Deducing Dimensionality
Given formulae P, Q such that $\exists \vdash P$ with $dim(P) = n$ and $\exists \vdash Q$ with $dim(Q) = m$, then:

$$dim(P \vee Q) = min(n, m)$$
$$dim(P \wedge Q) = max(n, m)$$

Proof. Omitted □

Definition 3.3. Satisfiability vs Proveability
Coalescence $O(l^n)$ vs SAT $O(n!)$

Definition 3.4. Dimensionality when not Proveable
$dim(P) ::=$ dimension when a token last moved up the tree

Definition 3.5. Paths in a Tree
Path from root to any leaf

Theorem 3.6. Bounds on Coalescence Dimensionality

$dim(P) \leq \#\{\vee \in P\}$

$dim(P) \leq max\{\#\{\vee \in path\} \, \forall \, path \in tree(P)\}$

Proof. $\square$