Coalescence Proof Search for Classical Logic

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Introduction

Outline:

- Additive Linear Logic (ALL)
- *Coalescence* proof search for ALL
- Classical Logic (CL)
- Proof search in CL through *coalescence* and *additive stratification*
- Dimensionality of CL formulae and optimisations

Additive Linear Logic (ALL)

$$A, B, C \quad ::= \quad 0 \mid 1 \mid a \mid \overline{a} \mid A + B \mid A \times B$$

- $0, a, +$ are duals of $1, \overline{a}, \times$ respectively

Additive Linear Logic (ALL)

Some examples:

- $(a + \overline{b}) \times 1$ is the dual of $(\overline{a} \times b) + 0$ and vice versa
- $(a + (\overline{b} \times c \times 0)) \times (b + 1)$
- etc.

Additive Linear Logic (ALL)

$$\frac{}{\vdash a, \overline{a}}\ ax \qquad \frac{}{\vdash 1, A}\ 1 \qquad \frac{\vdash A, C}{\vdash A + B, C}\ +_1$$

$$\frac{\vdash B, C}{\vdash A + B, C}\ +_2 \qquad \frac{\vdash A, C \qquad \vdash B, C}{\vdash A \times B, C}\ \times$$

- All sequents are comprised of a pair of terms, maintained by deduction rules
- $\llcorner\!\lrcorner + 0, \llcorner\!\lrcorner \times 1$ are identities
- Sequents are not necessarily commutative, idempotent etc.

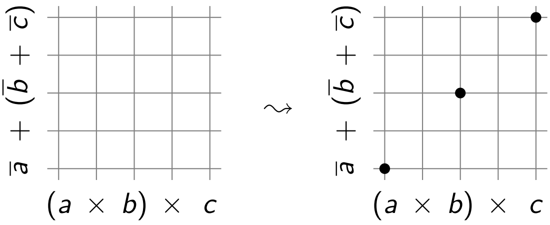
Additive Linear Logic (ALL)

Example: $\vdash a \times b, \overline{a} + \overline{b}$

$$\dfrac{\dfrac{\dfrac{}{\vdash a, \overline{a}}\,ax}{\vdash a, \overline{a} + \overline{b}}\,+ \qquad \dfrac{\dfrac{}{\vdash b, \overline{b}}\,ax}{\vdash b, \overline{a} + \overline{b}}\,+}{\vdash a \times b, \overline{a} + \overline{b}}\,\times$$

Coalescence Proof Search

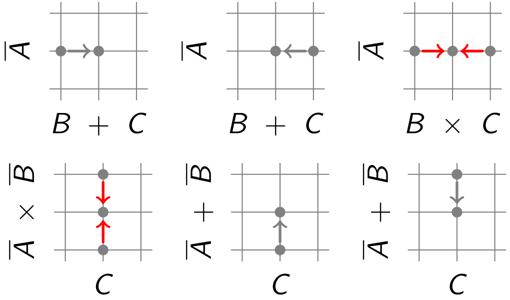
Initialization:



- See Heijltjes & Hughes (2015)
- Corresponds to axiom rule for ALL

Coalescence Proof Search

Transitions:



Coalescence Proof Search

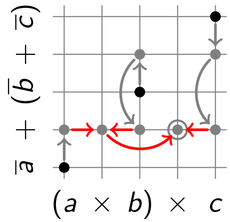
Corresponding sequent rules for ALL:

$$\frac{\vdash A, B}{\vdash A, B + C} \qquad \frac{\vdash A, C}{\vdash A, B + C} \qquad \frac{\vdash A, B \qquad \vdash A, C}{\vdash A, B \times C}$$

$$\frac{\vdash A, C \qquad \vdash B, C}{\vdash A \times B, C} \qquad \frac{\vdash A, C}{\vdash A + B, C} \qquad \frac{\vdash B, C}{\vdash A + B, C}$$

Coalescence Proof Search

Example:



Classical Logic (CL)

$$A, B, C \quad ::= \quad \top \mid \bot \mid a \mid \bar{a} \mid A \vee B \mid A \wedge B$$

- $\bot, a, \vee$ are duals of $\top, \bar{a}, \wedge$
- Similar to ALL formulae

Classical Logic (CL)

Examples:

- $(a \vee \overline{a}) \wedge (b \vee \overline{b})$ will be a recurrent example
- $\overline{a \vee \overline{b}} \leftrightsquigarrow \overline{a} \wedge b$ and associated De Morgan laws
- $\overline{A} \vee B \leftrightsquigarrow A \implies B$ and other useful syntax

Classical Logic (CL)

Sequent calculus with *additive* rules:

$$\frac{}{\vdash \top}\,\top \qquad \frac{\vdash \Gamma, A}{\vdash \Gamma, A \vee B}\,\vee \qquad \frac{\vdash \Gamma}{\vdash \Gamma, A}\,w$$

$$\frac{}{\vdash a, \overline{a}}\,ax \qquad \frac{\vdash \Gamma, A \qquad \vdash \Gamma, B}{\vdash \Gamma, A \wedge B}\,\wedge \qquad \frac{\vdash \Gamma, A, A}{\vdash \Gamma, A}\,c$$

- A, B, C are formulae, Γ, Δ, Σ are sequents
- $\wedge, \vee$ rules preserve the number of terms in a sequent, c, w rules do not
- Sequents are commutative, *morally equivalent* up to idempotence

Classical Logic (CL)

Example:

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash a, \neg a}\;ax}{\vdash a \vee \neg a, \neg a}\;\vee}{\vdash a \vee \neg a, a \vee \neg a}\;\vee}{\vdash a \vee \neg a}\;c \qquad \dfrac{}{\vdash \top}\;\top}{\vdash (a \vee \neg a) \wedge \top}\;\wedge
$$

Additive Stratification

$$
\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash A_1}\ \top, ax}{\vdash \Gamma_1}\ w \qquad \dots \qquad \dfrac{\dfrac{}{\vdash A_n}\ \top, ax}{\vdash \Gamma_n}\ w}{\vdash P \dots P}\ \wedge, \vee}{\vdash P}\ c
$$

- Does not work for *multiplicative* $\vee, \wedge$ rules, see Brünnler (2003)

Additive Stratification

$$\dfrac{\dfrac{\vdash \Gamma, A}{\vdash \Gamma, A \vee B}\,\vee}{\vdash \Gamma, A \vee B,\ C}\,w \quad \rightsquigarrow \quad \dfrac{\dfrac{\vdash \Gamma, A}{\vdash \Gamma, A, C}\,w}{\vdash \Gamma, A \vee B, C}\,\vee$$

$$\dfrac{\dfrac{\vdash \Gamma, A \qquad \vdash \Gamma, B}{\vdash \Gamma, A \wedge B}\,\wedge}{\vdash \Gamma, A \wedge B, C}\,w \quad \rightsquigarrow \quad \dfrac{\dfrac{\vdash \Gamma, A}{\vdash \Gamma, A, C}\,w \qquad \dfrac{\vdash \Gamma, B}{\vdash \Gamma, B, C}\,w}{\vdash \Gamma, A \wedge B, C}\,\wedge$$

$$\dfrac{\dfrac{\vdash \Gamma, A, A}{\vdash \Gamma, A}\,c}{\vdash \Gamma, A, B}\,w \quad \rightsquigarrow \quad \dfrac{\dfrac{\vdash \Gamma, A, A}{\vdash \Gamma, A, A, B}\,w}{\vdash \Gamma, A, B}\,c$$

Additive Stratification

$$\dfrac{\dfrac{\vdash \Gamma, A, A}{\vdash \Gamma, A}\,c}{\vdash \Gamma, A \vee B}\,\vee \qquad\leadsto\qquad \dfrac{\dfrac{\dfrac{\vdash \Gamma, A, A}{\vdash \Gamma, A \vee B, A}\,\vee}{\vdash \Gamma, A \vee B, A \vee B}\,\vee}{\vdash \Gamma, A \vee B}\,c$$

$$\dfrac{\dfrac{\vdash \Gamma, A, A}{\vdash \Gamma, A}\,c \qquad \vdash \Gamma, B}{\vdash \Gamma, A \wedge B}\,\wedge \;\leadsto\; \dfrac{\dfrac{\dfrac{\vdash \Gamma, A, A \qquad \dfrac{\vdash \Gamma, B}{\vdash \Gamma, A, B}\,w}{\vdash \Gamma, A, A \wedge B}\,\wedge \qquad \dfrac{\vdash \Gamma, B}{\vdash \Gamma, B, A \wedge B}\,w}{\vdash \Gamma, A \wedge B, A \wedge B}\,\wedge}{\vdash \Gamma, A \wedge B}\,c$$

Additive Stratification

Previous Example:

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash a, \neg a}\,ax}{\vdash a \vee \neg a, \neg a}\,\vee}{\vdash a \vee \neg a, a \vee \neg a}\,\vee}{\vdash a \vee \neg a}\,c \qquad \dfrac{}{\vdash \top}\,\top}{\vdash (a \vee \neg a) \wedge \top}\,\wedge
$$

Additive Stratification

Example:

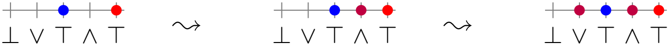
$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash a, \neg a}\,ax}{\vdash a \vee \neg a, \neg a}\,\vee}{\vdash a \vee \neg a, a \vee \neg a}\,\vee \qquad \dfrac{\dfrac{}{\vdash \top}\,\top}{\vdash \top, a \vee \neg a}\,w}{\vdash (a \vee \neg a) \wedge \top, a \vee \neg a}\,\wedge \qquad \dfrac{\dfrac{}{\vdash \top}\,\top}{\vdash (a \vee \neg a) \wedge \top, \top}\,w}{\dfrac{\vdash (a \vee \neg a) \wedge \top, (a \vee \neg a) \wedge \top}{\vdash (a \vee \neg a) \wedge \top}\,c}\,\wedge$$

Coalescence Proof Search in CL

- No longer polynomially bounded
- Coalescence on $|A|^n$
- Increase n until $\ldots$?

Coalescence Proof Search in CL

Example: $\bot \vee (\top \wedge \top)$, $n := 1$

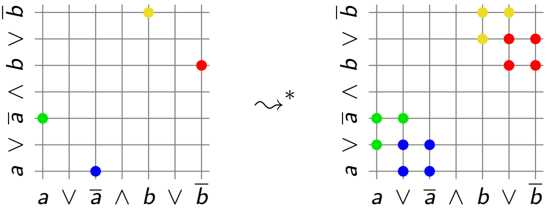


Coalescence Proof Search in CL

Example: $a \vee \bar{a}$, $n := 2$

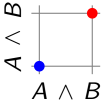
Coalescence Proof Search in CL

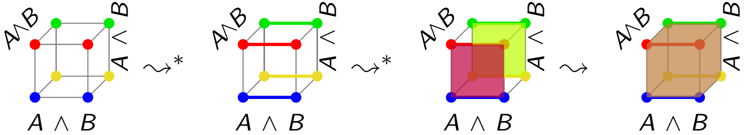
Example $(a \vee \overline{a}) \wedge (b \vee \overline{b})$, $n := 2$



Coalescence Proof Search in CL

Example $(a \vee \overline{a}) \wedge (b \vee \overline{b})$, $n := 3$, $A := a \vee \overline{a}$, $B := b \vee \overline{b}$



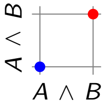


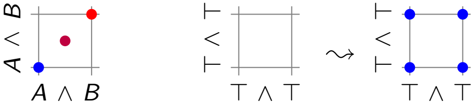
Dimensionality

$$
\begin{aligned}
\mathcal{D}^1 \quad &::= \quad \top \mid \bot \mid \mathcal{D}^1 \vee \mathcal{D}^1 \mid \mathcal{D}^1 \wedge \mathcal{D}^1 \\
\mathcal{D}^2 \quad &::= \quad ALL \\
&\dots
\end{aligned}
$$

- Difficult to categorise further
- Unsatisfactory results for $(a \vee \overline{a}) \wedge \cdots \wedge (z \vee \overline{z})$

Satisfying Optimisations



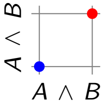


Questions

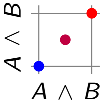
- Upper bound on dimensionality?
  - Exact or approximate?
- Good datatype representation for:
  - Matrix?
  - Tokens?

Datatypes

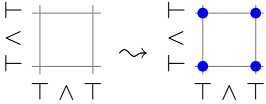
Tuple:



Set:

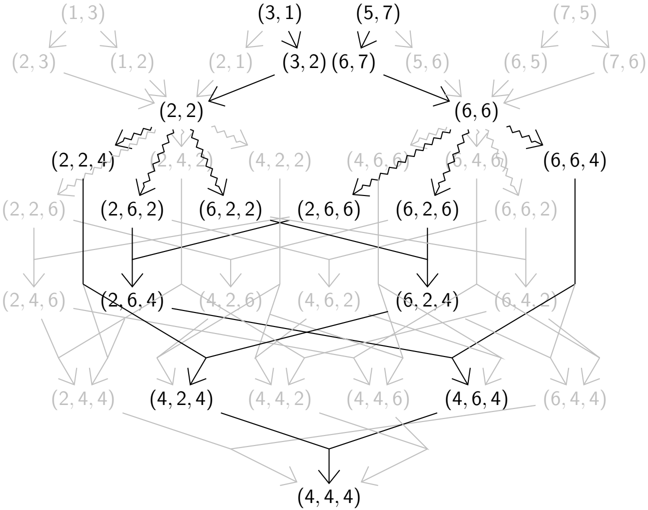


Multiset:



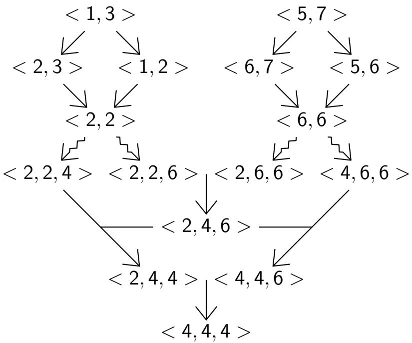
Datatypes

Tuple:



Datatypes

Multiset:



- Store tokens as sorted tuples
- Matrix is upper-triangular (in n-dimensions)

Datatypes

Set:

$$
\begin{tikzcd}
 & \{1,3\} \arrow[dl] \arrow[dr] & & & & \{5,7\} \arrow[dl] \arrow[dr] & \\
\{2,3\} \arrow[dr] & & \{1,2\} \arrow[dl] & & \{6,7\} \arrow[dr] & & \{5,6\} \arrow[dl] \\
 & \{2\} \arrow[ddrr] & & & & \{6\} \arrow[ddll] & \\
 & & & & & & \\
 & & & \{4\} & & &
\end{tikzcd}
$$

Datatypes

Set:

$$
\begin{tikzcd}[column sep=small]
& \{a, \neg a\} \arrow[dl] \arrow[dr] & & & & \{b, \neg b\} \arrow[dl] \arrow[dr] & \\
\{a \vee \neg a, \neg a\} \arrow[dr] & & \{a, a \vee \neg a\} \arrow[dl] & & \{b \vee \neg b, \neg b\} \arrow[dr] & & \{b, b \vee \neg b\} \arrow[dl] \\
& \{a \vee \neg a\} \arrow[drr, dash] & & & & \{b \vee \neg b\} \arrow[dll, dash] & \\
& & & \{(a \vee \neg a) \wedge (b \vee \neg b)\} & & &
\end{tikzcd}
$$

- How should the matrix be encoded?

Dimensionality

$$
\begin{aligned}
\mathcal{D}^1 \quad &::= \quad \top \mid \bot \mid \mathcal{D}^1 \vee \mathcal{D}^1 \mid \mathcal{D}^1 \wedge \mathcal{D}^1 \\
\mathcal{D}^2 \quad &::= \quad ALL \mid A \vee \overline{A} \mid \mathcal{D}^{n \leq 2} \wedge \mathcal{D}^2 \mid \mathcal{D}^{n \geq 2} \vee \mathcal{D}^2 \\
\mathcal{D}^3 \quad &::= \quad (A \wedge B) \vee (\overline{A} \wedge B) \vee (A \wedge \overline{B}) \vee (\overline{A} \wedge \overline{B}) \\
& \qquad \mid \mathcal{D}^{n \leq 3} \wedge \mathcal{D}^3 \mid \mathcal{D}^{n \geq 3} \vee \mathcal{D}^3 \\
&\ldots
\end{aligned}
$$

- $A \in \mathcal{D}^n, B \in \mathcal{D}^m \implies A \vee B \in \mathcal{D}^{min(n,m)},\ A \wedge B \in \mathcal{D}^{max(n,m)}$

Dimensionality Bounds

- Number of variables
- Dimensionality of largest *disjunctive normal form* term
- Exact bound is equivalent to proof search?

Questions?

References

Brünnler, K. (2003), 'Two restrictions on contraction', *Logic Journal of the IGPL* **11**(5), 525–529.

Heijltjes, W. & Hughes, D. J. (2015), Complexity bounds for sum-product logic via additive proof nets and petri nets, *in* '2015 30th Annual ACM/IEEE Symposium on Logic in Computer Science', IEEE, pp. 80–91.