From Additive to Classical Proof Search

Adam Lassiter Willem Heijltjes
Department of Computer Science
University of Bath

Introduction

Outline:

- Additive Linear Logic (ALL)
- Proof search in ALL through *coalescence*
- Classical Logic (CL)
- Proof search in CL through *coalescence* and *additive stratification*
- Complexity bounds of proof search
- Dimensionality of CL formulae

Additive Linear Logic (ALL)

$$A, B, C \quad ::= \quad 0 \mid 1 \mid a \mid \overline{a} \mid A + B \mid A \times B$$

- $0, a, +$ are duals of $1, \overline{a}, \times$ respectively

Additive Linear Logic (ALL)

Some examples:

- $(a + \overline{b}) \times 1$ is the dual of $(\overline{a} \times b) + 0$ and vice versa
- $(a + (\overline{b} \times c \times 0)) \times (b + 1)$
- etc.

Additive Linear Logic (ALL)

$$\frac{}{\vdash a, \overline{a}}\, ax \qquad \frac{}{\vdash 1, A}\, 1 \qquad \frac{\vdash A, C}{\vdash A + B, C}\, +_1$$

$$\frac{\vdash B, C}{\vdash A + B, C}\, +_2 \qquad \frac{\vdash A, C \qquad \vdash B, C}{\vdash A \times B, C}\, \times$$

- All sequents are comprised of a pair of terms, maintained by deduction rules
- $\llcorner\!\lrcorner + 0, \llcorner\!\lrcorner \times 1$ are identities
- Sequents are not necessarily commutative, idempotent etc.

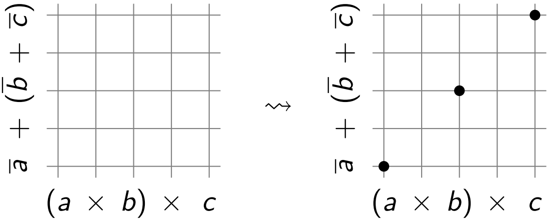
Additive Linear Logic (ALL)

Example: $\vdash a \times b, \overline{a} + \overline{b}$

$$\dfrac{\dfrac{\dfrac{}{\vdash a, \overline{a}}\,ax}{\vdash a, \overline{a} + \overline{b}}\,+ \qquad \dfrac{\dfrac{}{\vdash b, \overline{b}}\,ax}{\vdash b, \overline{a} + \overline{b}}\,+}{\vdash a \times b, \overline{a} + \overline{b}}\,\times$$

Coalescence

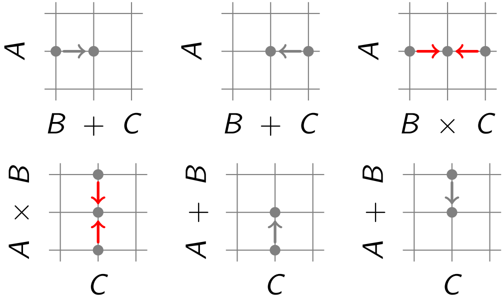
Spawning:



- Corresponds to axiom rule for ALL

Coalescence

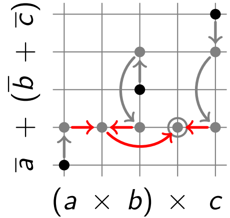
Transitions:



- Corresponds to sequent rules for ALL

Coalescence

Example:



- Algorithm is precisely ALL proof search, see Heijltjes & Hughes (2015)

Classical Logic (CL)

$$A, B, C \quad ::= \quad \top \mid \bot \mid a \mid \overline{a} \mid A \vee B \mid A \wedge B$$

- Analogous to `True`, `False` and boolean operations
- $\bot, a, \vee$ are duals of $\top, \overline{a}, \wedge$
- Similar to ALL formulae

Classical Logic (CL)

Examples:

- $(a \vee \overline{a}) \wedge (b \vee \overline{b})$ will be a recurrent example
- $\overline{a \vee \overline{b}} \leftrightsquigarrow \overline{a} \wedge b$ and associated De Morgan laws
- $\overline{A} \vee B \leftrightsquigarrow A \implies B$ and other useful syntax

Classical Logic (CL)

$$\frac{}{\vdash \top}\ \top \qquad \frac{\vdash \Gamma, A}{\vdash \Gamma, A \vee B}\ \vee \qquad \frac{\vdash \Gamma}{\vdash \Gamma, A}\ w$$

$$\frac{}{\vdash a, \overline{a}}\ ax \qquad \frac{\vdash \Gamma, A \qquad \vdash \Gamma, B}{\vdash \Gamma, A \wedge B}\ \wedge \qquad \frac{\vdash \Gamma, A, A}{\vdash \Gamma, A}\ c$$

- A, B, C are formulae, Γ, Δ, Σ are sequents
- $\wedge, \vee$ rules preserve the number of terms in a sequent, c, w rules do not
- Sequents are commutative, *morally equivalent* up to idempotence

Classical Logic (CL)

Example:

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\vdash a, \neg a$}
\RightLabel{$\vee$}
\UnaryInfC{$\vdash a \vee \neg a, \neg a$}
\RightLabel{$\vee$}
\UnaryInfC{$\vdash a \vee \neg a, a \vee \neg a$}
\RightLabel{c}
\UnaryInfC{$\vdash a \vee \neg a$}
\AxiomC{}
\RightLabel{$\top$}
\UnaryInfC{$\vdash \top$}
\RightLabel{$\wedge$}
\BinaryInfC{$\vdash (a \vee \neg a) \wedge \top$}
\end{prooftree}
$$

Additive Stratification

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash A_1}\;\top, ax}{\vdash \Gamma_1}\;w \qquad \ldots \qquad \dfrac{\dfrac{}{\vdash A_n}\;\top, ax}{\vdash \Gamma_n}\;w}{\vdash P \ldots P}\;\wedge, \vee}{\vdash P}\;c$$

- Rearrangement does not affect the number of terms in sequents
- Does not work for MLL rules, see Brunnler (2003)

Additive Stratification

$$\dfrac{\dfrac{\vdash \Gamma, A}{\vdash \Gamma, A \vee B}\,\vee}{\vdash \Gamma, A \vee B, C}\,w \quad\rightsquigarrow\quad \dfrac{\dfrac{\vdash \Gamma, A}{\vdash \Gamma, A, C}\,w}{\vdash \Gamma, A \vee B, C}\,\vee$$

$$\dfrac{\dfrac{\vdash \Gamma, A \qquad \vdash \Gamma, B}{\vdash \Gamma, A \wedge B}\,\wedge}{\vdash \Gamma, A \wedge B, C}\,w \quad\rightsquigarrow\quad \dfrac{\dfrac{\vdash \Gamma, A}{\vdash \Gamma, A, C}\,w \qquad \dfrac{\vdash \Gamma, B}{\vdash \Gamma, B, C}\,w}{\vdash \Gamma, A \wedge B, C}\,\wedge$$

$$\dfrac{\dfrac{\vdash \Gamma, A, A}{\vdash \Gamma, A}\,c}{\vdash \Gamma, A, B}\,w \quad\rightsquigarrow\quad \dfrac{\dfrac{\vdash \Gamma, A, A}{\vdash \Gamma, A, A, B}\,w}{\vdash \Gamma, A, B}\,c$$

Additive Stratification

$$\dfrac{\dfrac{\vdash \Gamma, A, A}{\vdash \Gamma, A}\,c}{\vdash \Gamma, A \vee B}\,\vee \quad \rightsquigarrow \quad \dfrac{\dfrac{\dfrac{\vdash \Gamma, A, A}{\vdash \Gamma, A \vee B, A}\,\vee}{\vdash \Gamma, A \vee B, A \vee B}\,\vee}{\vdash \Gamma, A \vee B}\,c$$

$$\dfrac{\dfrac{\vdash \Gamma, A, A}{\vdash \Gamma, A}\,c \quad \vdash \Gamma, B}{\vdash \Gamma, A \wedge B}\,\wedge \quad \rightsquigarrow \quad \dfrac{\dfrac{\dfrac{\vdash \Gamma, A, A \quad \dfrac{\vdash \Gamma, B}{\vdash \Gamma, A, B}\,w}{\vdash \Gamma, A, A \wedge B}\,\wedge \quad \dfrac{\vdash \Gamma, B}{\vdash \Gamma, B, A \wedge B}\,w}{\vdash \Gamma, A \wedge B, A \wedge B}\,\wedge}{\vdash \Gamma, A \wedge B}\,c$$

- Proof trees are described as *morally equivalent*, see Girard et al. (1989)

Additive Stratification

Previous Example:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash a, \neg a}\, ax}{\vdash a \vee \neg a, \neg a}\, \vee}{\vdash a \vee \neg a, a \vee \neg a}\, \vee}{\vdash a \vee \neg a}\, c \qquad \dfrac{}{\vdash \top}\, \top}{\vdash (a \vee \neg a) \wedge \top}\, \wedge$$

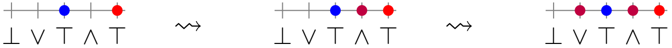
Additive Stratification

Example:

$$
\dfrac{
  \dfrac{
    \dfrac{
      \dfrac{
        \dfrac{}{\vdash a, \neg a}\,ax
      }{\vdash a \vee \neg a, \neg a}\,\vee
    }{\vdash a \vee \neg a, a \vee \neg a}\,\vee
    \qquad
    \dfrac{
      \dfrac{}{\vdash \top}\,\top
    }{\vdash \top, a \vee \neg a}\,w
  }{\vdash (a \vee \neg a) \wedge \top, a \vee \neg a}\,\wedge
  \qquad
  \dfrac{
    \dfrac{}{\vdash \top}\,\top
  }{\vdash (a \vee \neg a) \wedge \top, \top}\,w
}{
  \dfrac{\vdash (a \vee \neg a) \wedge \top, (a \vee \neg a) \wedge \top}{\vdash (a \vee \neg a) \wedge \top}\,c
}\,\wedge
$$

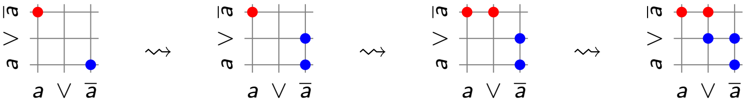
Coalescence in CL

Example: $\bot \vee (\top \wedge \top)$, $n := 1$



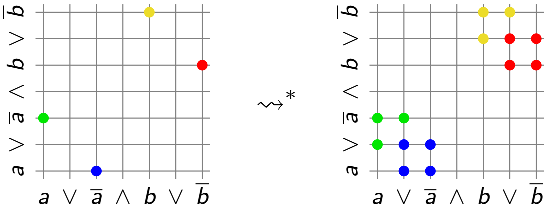
Coalescence in CL

Example: $a \vee \overline{a}$, $n := 2$



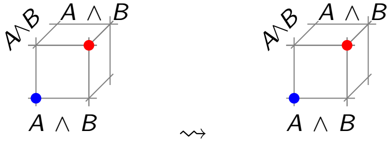
Coalescence in CL

Example $(a \vee \overline{a}) \wedge (b \vee \overline{b})$, $n := 2$



Coalescence in CL

Example $(a \vee \overline{a}) \wedge (b \vee \overline{b})$, $n := 3$, using the above proofs for $A := a \vee \overline{a}$, $B := b \vee \overline{b}$



Coalescence in CL

For a formula P in CL, the coalescence algorithm runs as follows:

1. Set $n := 1$
2. Construct the n-dimensional grid of possible n-term sequents $\vdash A_1 \ldots A_n \in |\mathrm{A}| \times \ldots_n \times |\mathrm{A}|$
3. Spawn tokens at all instances of axiom links $\vdash \Gamma, a, \overline{a}$
4. Exhaustively perform transitions given by the CL sequent calculus rules
5. Does there exist a token at $(P, P \ldots P) \equiv\ \vdash P, P \ldots P \equiv\ \vdash P$?
   5.1 Yes — Halt with a proof for P and dimensionality n
   5.2 No — Increment $n := n + 1$ and goto 2

Questions

Dimensionality

The dimensionality of a proof is then the dimensionality of our grid when the root is reached, equivalent to the number of contractions required in an equivalent sequent proof. A CL formula can thus be proved by an n-dimensional additively stratified proof, where n is the number of terms in a sequent before contraction. Through a natural transformation on steps of the algorithm to equivalent sequent proofs, coalescence up to $n = N$ is then exactly (additively stratified) proof search, with implicit weakening and contraction of all sequents up to N terms.

Conclusion

References

Brunnler, K. (2003), 'Two restrictions on contraction', *Logic Journal of the IGPL* **11**(5), 525–529.

Girard, J.-Y., Taylor, P. & Lafont, Y. (1989), *Proofs and types*, Vol. 7, Cambridge university press Cambridge.

Heijltjes, W. & Hughes, D. J. (2015), Complexity bounds for sum-product logic via additive proof nets and petri nets, *in* '2015 30th Annual ACM/IEEE Symposium on Logic in Computer Science', IEEE, pp. 80–91.