Natural Proof Search for Classical Logic

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Abstract

We investigate a natural algorithm for proof search within classical logic.

1 Classical Logic

Definition 1.1. Formulae
A *formula* within classical logic is constructed as follows:

$$A, B, C ::= \bot \mid \top \mid a \mid \neg\, a \mid A \vee B \mid A \wedge B$$
$$\Gamma, \Delta, \Sigma ::= A_1 \dots A_n$$

where $\vee, \wedge$ are additive linear logic disjunction and conjunction respectively and Γ, Δ, Σ are contexts..

Example.

Definition 1.2. Sequent Proofs
Within *classical logic*, a *sequent proof* is constructed from the following rules:

$$\frac{}{\vdash \top}\ \top \qquad \frac{\vdash \Gamma, A}{\vdash \Gamma, A \vee B}\ \vee\mathrm{R} \qquad \frac{\vdash \Gamma}{\vdash \Gamma, A}\ w$$

$$\frac{}{\vdash a, \neg\, a}\ ax \qquad \frac{\vdash \Gamma, A \qquad \vdash \Gamma, B}{\vdash \Gamma, A \wedge B}\ \wedge\mathrm{R} \qquad \frac{\vdash \Gamma, A, A}{\vdash \Gamma, A}\ c$$

where A, B, C are formulae and Γ, Δ, Σ are sequents. A sequent proof provides, without context, a proof of its conclusion and each line of the proof represents a tautology.

Example.

Remark 1.3. Within the context of weakening and contraction, *additive* and *multiplicative* rules are inter-derivable.

Definition 1.4. Derivations
Given *tops* $\Gamma_1 \dots \Gamma_n$ for the sequent proof $\vdash \Delta$, a *derivation* is a tree providing a proof of $\Gamma_1 \dots \Gamma_n \implies \Delta$.

A derivation is written as:

$$\dfrac{\vdash \Gamma_1 \qquad \ldots \qquad \vdash \Gamma_n}{\vdash \Delta} \text{[label]}$$

where the *label* describes which rules may be used within the derivation.

Corollary 1.5. Derivation Equivalence
A sequent proof is a derivation where all top derivations of the tree are $=\!=\ \top, ax$. Equivalence of derivations may be weakly defined up to equivalence of leaves and conclusion.

Example.

Definition 1.6. Additive Stratification
A proof tree is said to be *additively stratified* if $\vdash P$ is structured as follows:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash A_1}\top, ax}{\vdash \Gamma_1} w \qquad \ldots \qquad \dfrac{\dfrac{}{\vdash A_n}\top, ax}{\vdash \Gamma_n} w}{\dfrac{\vdash P \ldots P}{\vdash P} c} \wedge, \vee$$

That is, the inferences made in an additively stratified proof are strictly ordered by:

1. Top/Axiomatic
2. Weakening
3. Conjunction/Disjunction
4. Contraction

Example.

Theorem 1.7. Stratification Equivalence
Given $\vdash A$, there exists an additively stratified proof of A.

Proof. For each instance of a weakening below another inference, there exists an equivalent subproof that is additively stratified:

$$\dfrac{\dfrac{\vdash \Gamma, A, B}{\vdash \Gamma, A \vee B} \vee}{\vdash \Gamma, A \vee B, \mathrm{C}} w \quad \rightsquigarrow \quad \dfrac{\dfrac{\vdash \Gamma, A, B}{\vdash \Gamma, A, B, C} w}{\vdash \Gamma, A \vee B, \mathrm{C}} \vee$$

$$\dfrac{\dfrac{\vdash \Gamma, A \qquad \vdash \Gamma, B}{\vdash \Gamma, A \wedge B} \wedge}{\vdash \Gamma, A \wedge B, C} w \quad \rightsquigarrow \quad \dfrac{\dfrac{\vdash \Gamma, A}{\vdash \Gamma, A, C} w \qquad \dfrac{\vdash \Gamma, B}{\vdash \Gamma, B, C} w}{\vdash \Gamma, A \wedge B, C} \wedge$$

$$\dfrac{\dfrac{\vdash \Gamma, A, A}{\vdash \Gamma, A} c}{\vdash \Gamma, A, B} w \quad \rightsquigarrow \quad \dfrac{\dfrac{\vdash \Gamma, A, A}{\vdash \Gamma, A, A, B} w}{\vdash \Gamma, A, B} c$$

Similarly, for each instance of a contraction above another inference, there exists an equivalent subproof that is additively stratified:

$$\dfrac{\dfrac{\vdash \Gamma, A, A, B}{\vdash \Gamma, A, B}\,c}{\vdash \Gamma, A \vee B}\,\vee \qquad \rightsquigarrow \qquad \dfrac{\dfrac{\dfrac{\dfrac{\vdash \Gamma, A, A, B}{\vdash \Gamma, A, A, B, B}\,w}{\vdash \Gamma, A \vee B, A, B}\,\vee}{\vdash \Gamma, A \vee B, A \vee B}\,\vee}{\vdash \Gamma, A \vee B}\,c$$

$$\dfrac{\dfrac{\vdash \Gamma, A, A}{\vdash \Gamma, A}\,c \qquad \vdash \Gamma, B}{\vdash \Gamma, A \wedge B}\,\wedge \quad \rightsquigarrow \quad \dfrac{\dfrac{\dfrac{\vdash \Gamma, A, A \qquad \dfrac{\vdash \Gamma, B}{\vdash \Gamma, A, B}\,w}{\vdash \Gamma, A, A \wedge B}\,\wedge \qquad \dfrac{\vdash \Gamma, B}{\vdash \Gamma, B, A \wedge B}\,w}{\vdash \Gamma, A \wedge B, A \wedge B}\,\wedge}{\vdash \Gamma, A \wedge B}\,c$$

By induction from the leaves downwards on a finite height tree, apply the associated rule to each pair of inferences of the form (c above *inf*). Any given $\vdash P$ may be rewritten:

$$\dfrac{\dfrac{\dfrac{}{\vdash A_1}\,\top, ax}{\vdash \Gamma_1}\,\wedge, \vee, w \qquad \dots \qquad \dfrac{\dfrac{}{\vdash A_n}\,\top, ax}{\vdash \Gamma_n}\,\wedge, \vee, w}{\vdash P}\,c$$

Again, by induction from the root upwards on this partially stratified tree, apply the associated rule to each pair of inferences of the form (w below *inf*). $\vdash P$ may then be further rewritten:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash A_1}\,\top, ax}{\vdash \Gamma_1}\,w \qquad \dots \qquad \dfrac{\dfrac{}{\vdash A_n}\,\top, ax}{\vdash \Gamma_n}\,w}{\vdash P \dots P}\,\wedge, \vee}{\vdash P}\,c$$

□

2 Coalescence

Definition 2.1. Petri Nets
A petri net is ...

Example.

Definition 2.2. Coalescence
The coalescence algorithm is ...

Theorem 2.3. Coalescence Proof Search
The coalescence algorithm is a proof search ...

Proof. By relation of:

- $c \iff$ dimensionality
- $ax \iff$ 2-D token initialisation
- $w \iff$ n-D extension of tokens

- $\wedge$R, $\vee$R $\equiv$ petri net firing

$\square$