Natural Proof Search for Classical Logic

Adam Lassiter
Department of Computer Science
University of Bath

Willem Heijltjes
Department of Computer Science
University of Bath

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Abstract

We investigate a natural algorithm for proof search within classical logic and prove bounds on the complexity class of such a search.

1 Classical Logic

Definition 1.1 (Formulae)**.**
A *formula* within classical logic is constructed as follows:

$$\begin{array}{lcl} A, B, C & ::= & \top \mid \bot \mid a \mid \neg a \mid A \vee B \mid A \wedge B \\ \Gamma, \Delta, \Sigma & ::= & A \mid A, B \mid A, B, C \ldots \end{array}$$

where $\vee, \wedge$ are additive linear logic disjunction and conjunction respectively and Γ, Δ, Σ are contexts..

Example.

Definition 1.2 (Sequent Proofs)**.**
Within *classical logic*, a *sequent proof* is constructed from the following rules:

$$\frac{}{\vdash \top}\ \top \qquad \frac{\vdash \Gamma, A}{\vdash \Gamma, A \vee B}\ \vee\mathrm{R} \qquad \frac{\vdash \Gamma}{\vdash \Gamma, A}\ w$$

$$\frac{}{\vdash a, \neg a}\ ax \qquad \frac{\vdash \Gamma, A \qquad \vdash \Gamma, B}{\vdash \Gamma, A \wedge B}\ \wedge\mathrm{R} \qquad \frac{\vdash \Gamma, A, A}{\vdash \Gamma, A}\ c$$

where A, B, C are formulae and Γ, Δ, Σ are sequents. A sequent proof provides, without context, a proof of its conclusion and each line of the proof represents a tautology.

Example.

Remark 1.3.
Within the context of weakening and contraction, *additive* and *multiplicative* rules are interderivable.

Definition 1.4 (Derivations)**.**
Given *tops* $\Gamma_1 \ldots \Gamma_n$ for the sequent proof $\vdash \Delta$, a *derivation* is a tree providing a proof of $\Gamma_1 \ldots \Gamma_n \implies \Delta$.

A derivation is written as:

$$\frac{\vdash \Gamma_1 \qquad \ldots \qquad \vdash \Gamma_n}{\vdash \Delta}\ \textit{[label]}$$

where the *label* describes which rules may be used within the derivation.

Corollary 1.5 (Derivation Equivalence)**.**
A sequent proof is a derivation where all top derivations of the tree are $= \top, ax$. Equivalence of derivations may be weakly defined up to equivalence of leaves and conclusion.

Example.

Definition 1.6 (Additive Stratification)**.**
A proof tree is said to be *additively stratified* if $\vdash P$ is structured as follows:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash A_1}\ \top, ax}{\vdash \Gamma_1}\ w \qquad \ldots \qquad \dfrac{\dfrac{}{\vdash A_n}\ \top, ax}{\vdash \Gamma_n}\ w}{\vdash P \ldots P}\ \wedge, \vee}{\vdash P}\ c$$

That is, the inferences made in an additively stratified proof are strictly ordered by:

1. Top/Axiomatic
2. Weakening
3. Conjunction/Disjunction
4. Contraction

Example.

Proposition 1.7 (Stratification Equivalence)**.**
Given $\vdash A$, there exists an additively stratified proof of A.

Proof. For each instance of a weakening below another inference, there exists an equivalent subproof that is additively stratified:

$$\dfrac{\dfrac{\vdash \Gamma, A, B}{\vdash \Gamma, A \vee B}\ \vee}{\vdash \Gamma, A \vee B, \mathrm{C}}\ w \qquad \rightsquigarrow \qquad \dfrac{\dfrac{\vdash \Gamma, A, B}{\vdash \Gamma, A, B, C}\ w}{\vdash \Gamma, A \vee B, \mathrm{C}}\ \vee$$

$$
\dfrac{\dfrac{\vdash \Gamma, A \qquad \vdash \Gamma, B}{\vdash \Gamma, A \wedge B}\,\wedge}{\vdash \Gamma, A \wedge B, C}\,w
\quad \rightsquigarrow \quad
\dfrac{\dfrac{\vdash \Gamma, A}{\vdash \Gamma, A, C}\,w \qquad \dfrac{\vdash \Gamma, B}{\vdash \Gamma, B, C}\,w}{\vdash \Gamma, A \wedge B, C}\,\wedge
$$

$$
\dfrac{\dfrac{\vdash \Gamma, A, A}{\vdash \Gamma, A}\,c}{\vdash \Gamma, A, B}\,w
\quad \rightsquigarrow \quad
\dfrac{\dfrac{\vdash \Gamma, A, A}{\vdash \Gamma, A, A, B}\,w}{\vdash \Gamma, A, B}\,c
$$

Similarly, for each instance of a contraction above another inference, there exists an equivalent subproof that is additively stratified:

$$
\dfrac{\dfrac{\vdash \Gamma, A, A, B}{\vdash \Gamma, A, B}\,c}{\vdash \Gamma, A \vee B}\,\vee
\quad \rightsquigarrow \quad
\dfrac{\dfrac{\dfrac{\dfrac{\vdash \Gamma, A, A, B}{\vdash \Gamma, A, A, B, B}\,w}{\vdash \Gamma, A \vee B, A, B}\,\vee}{\vdash \Gamma, A \vee B, A \vee B}\,\vee}{\vdash \Gamma, A \vee B}\,c
$$

$$
\dfrac{\dfrac{\vdash \Gamma, A, A}{\vdash \Gamma, A}\,c \qquad \vdash \Gamma, B}{\vdash \Gamma, A \wedge B}\,\wedge
\quad \rightsquigarrow \quad
\dfrac{\dfrac{\dfrac{\vdash \Gamma, A, A \qquad \dfrac{\vdash \Gamma, B}{\vdash \Gamma, A, B}\,w}{\vdash \Gamma, A, A \wedge B}\,\wedge \qquad \dfrac{\vdash \Gamma, B}{\vdash \Gamma, B, A \wedge B}\,w}{\vdash \Gamma, A \wedge B, A \wedge B}\,\wedge}{\vdash \Gamma, A \wedge B}\,c
$$

By induction from the leaves downwards on a finite height tree, apply the associated rule to each pair of inferences of the form (c above *inf*). Any given $\vdash P$ may be rewritten:

$$
\dfrac{\dfrac{\dfrac{}{\vdash A_1}\,\top, ax}{\vdash \Gamma_1}\,\wedge,\vee,w \qquad \dots \qquad \dfrac{\dfrac{}{\vdash A_n}\,\top, ax}{\vdash \Gamma_n}\,\wedge,\vee,w}{\vdash P}\,c
$$

Again, by induction from the root upwards on this partially stratified tree, apply the associated rule to each pair of inferences of the form (w below *inf*). $\vdash P$ may then be further rewritten:

$$
\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash A_1}\,\top, ax}{\vdash \Gamma_1}\,w \qquad \dots \qquad \dfrac{\dfrac{}{\vdash A_n}\,\top, ax}{\vdash \Gamma_n}\,w}{\vdash P \dots P}\,\wedge,\vee}{\vdash P}\,c
$$

$\square$

Example.

2 Coalescence

Definition 2.1 (Petri Nets)**.**
For the purposes required here, a *petri net* $\mathcal{N}$ is $(\mathcal{P}, \mathcal{F})$ where $f \in \mathcal{F} : \mathcal{P}^m \times \mathcal{P}$. In particular, $\mathcal{P}$ is a set of places and $\mathcal{F}$ a set of flows. A *configuration* is a set $\mathcal{C} \subset \mathcal{P}$ of tokens in places.

Example.

Definition 2.2 (Firing Petri Nets)**.**
Given a petri net $\mathcal{N}$ and configuration $\mathcal{C}$, a *firing* of the net $\mathcal{N}$ is a new configuration generated by application of a transition $f \in \mathcal{F}$ on m tokens $c_1 \dots c_n \in \mathcal{C}$. In particular, $(\mathcal{N} = (\mathcal{P}, \mathcal{F}), \mathcal{C}) \mapsto (\mathcal{N}, \mathcal{C} \cup f_{right} \setminus f_{left})$ for some $f = (f_{left}, f_{right}) \in \mathcal{F}$

A petri net is said to be *exhaustively fired* if it is fired until there does not exist any such $f \in \mathcal{F}$ to fire.

Example.

Remark 2.3.
Implementing firable petri nets is straightforward using an n-dimensional boolean array of places visited and a collection of n-tuples representing tokens.

Definition 2.4 (Coalescence)**.**
Given a formula P, the coalescence algorithm is as follows:

1. Set $n := 1$
2. Construct a n-dimensional petri net $\mathcal{N}$ from P where each subformula is a place, each conjunction and disjunction a flow
3. Construct a configuration C with a token at each place $p = (\dots, a, \dots, \neg a, \dots)$ the intersection of a pair of tautological atoms
4. Exhaustively fire the petri net $\mathcal{N}$ using the *spawning* method.
5. If there exists a token in the configuration $\mathcal{C}^*$ at the root of the formula P, halt and return n
6. Otherwise, increment $n := n + 1$ and go to step 2

Example.

Proposition 2.5 (Coalescence Proof Search)**.**
The coalescence algorithm on P is exactly a proof search on P.

Proof. In particular, consider the additively stratified $\vdash P$ with $n - 1$ contractions (should such a proof exist). The first configuration of tokens is precisely a proof all possible combinations of the *axiom* rule with $n-3$ other terms through the *weakening*. Each flow transition followed when fired is an application of either the $\vee$ or $\wedge$ rule. Finally, a token at the root of the formula is $\vdash P \dots P$, with the *contraction* rule applied implicitly.

If there exists an additively stratified proof of P, the coalescence algorithm will find it.

Since for every proof there exists an additively stratified proof, coalescence is precisely proof search. $\square$

3 Dimensionality

Definition 3.1 (Dimensionality)**.**
Given a formula P, the coalescence proof search produces a proof in an n dimensional petri net. The dimensionality of P is then defined $\dim(P) ::= n$. Equivalently, an additively stratified sequent proof requires $n-1$ contractions at the bottom of the proof. Given $\vdash P$, its dimensionality is defined $\dim(P) ::= n$.

Examples.

Definition 3.2 (Classes of Formulae)**.**
Let A^i be the subclasses of formulae defined as:

$$\begin{aligned} A^1 &::= \top \mid \bot \mid A^1 \wedge A^1 \mid A^* \vee A^1 \\ A^2 &::= A^1 \mid \textit{Additive Linear Logic} \mid A^* \vee A^2 \mid P \vee \neg P \end{aligned}$$

where $P \in A^*$, such that A^n is the class of all formulae provable in n dimensions.

Remark (Satisfiability vs Provability)**.**
For any formula P, there exist four distinct classes: *true, false, satisfiably true, satisfiably false*, where satisfiable differs by finding a particular assignment of values to each variable. Coalescence searches for a proof of P in *true*, whereas the SAT problem addresses a proof of P in *true*. In particular, *true* is the complement class to *satisfiably false* and *false* complement to *satisfiably true*.

Definition 3.3 (Dimensionality when not Provable)**.**
Given a formula P such that there does not exist $\vdash P$, for all formulae Q such that there does exist $\vdash Q$, the dimensionality of P is defined as the least n such that:

$$\dim(P) ::= n \implies \begin{cases} \exists \vdash (P \vee Q) \implies \dim(P \vee Q) = \min(n, \dim(Q)) \\ \exists \vdash (P \wedge Q) \implies \dim(P \wedge Q) = \max(n, \dim(Q)) \end{cases}$$

Remark.
This definition of a 'partial dimensionality' amounts to finding the highest dimension in coalescence proof search that some useful deduction was made, outside of trivial cases.

Definition 3.4 (Paths in a Tree)**.**
Path from root to any leaf (effectively iterating leaves, but tracing parents)

Proposition 3.5 ($\vee$-Bound on Dimensionality)**.**

$$\begin{aligned} &\dim(P) \leq 1 + \#\{\vee \in P\} \\ &\dim(P) \leq 1 + \max\{\#\{\vee \in \mathit{path}\}\, \forall\, \mathit{path} \in \mathit{tree}(P)\} \end{aligned}$$

Proof. Omitted $\square$

Proposition 3.6 (ax-Bound on Dimensionality)**.**

$$\dim(P) \leq 1 + \#\{ax\text{-}Rule \in \vdash P\}$$
$$\dim(P) \leq 1 + \#\{vars \in P\}$$

Proof. The first case is trivial. Given a formula P and a constructed sequent proof $\vdash P$ through coalescence, additive stratification ensures that $\dim(P) = n \implies leaves(P) \geq n$. As each of the n branches of the tree moves 'upwards', they must either terminate at an ax-rule or branch, leaving $n + 1$ branches total. Subsequently, for each ax-rule forming a leaf of the tree, $\dim(P)$ increases by no more than one. Finally, consider a base case such as $P := \vdash a \vee \neg a$ where $\dim(P) = 2 \leq 1 + \#\{ax - Rule \in \vdash P\}$ (in this case, it is equal).

Second case omitted □