The University of Melbourne
Mathematical Physics Seminar

CATEGORICAL SYMMETRIES IN PHYSICS: Group Cohomology, Projective Representations and Central Extensions

Lecture Notes

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1 Group Cohomology

Suppose G is a group and M is an abelian group.

Definition 1.1. *Let n be a positive integer a n-cochain of G in M is a set map $f : G^n \to M$. We define $C^n(G;M)$ to be the abelian group of all $n-$cochains of G in M where*

- *Group multiplication is pointwise multiplication of functions*

$$f_1 f_2 : (g_1,..,g_n) \mapsto f_1(g_1,..,g_n)f_2(g_1,..,g_n) \quad \forall f,g \in C^n(G;M)$$

- *The Identity function sends all group elements to the identity in M that is $1 : (g_1,..,g_n) \mapsto 1_M$*
- *Inverses are given by applying the inversion map to the image of f. $f^{-1} : (g_1,...,g_n) \mapsto (f(g_1,...,g_n))^-1$*

A 0-cochain is defined to be an element in M.

Remark. The group $C^n(G;M)$ is abelian follows from the group M being abelian.

Definition 1.2. *The coboundary of a n-cochain f is the $n+1$-cochain $\delta^n f$ defined by*

$$(\delta^n f)(g_1,...,g_{n+1}) = f(g_2,..,g_{n+1}) \left[\prod_{i=1}^{n} f(g_1,...,g_i g_{i+1}..,g_{n+1})^{(-1)^i}\right] f(g_1,...,g_n)^{(-1)^{n+1}}$$

for all $(g_1,...,g_{n+1}) \in G^{n+1}$

Some quick computations show:

- For $n = 0$ we have $(\delta^0 f)(g_1) = f(g_1)$
- For $n = 1$ we have $(\delta f)(g_1,g_2) = \frac{f(g_2)f(g_1)}{f(g_1 g_2)}$
- For $n = 2$ we have $(\delta^2 f)(g_1,g_2,g_3) = \frac{f(g_2,g_3)f(g_1,g_2 g_3)}{f(g_1 g_2,g_3)f(g_1,g_2)}$

Lemma 1.1. *For all n-cochains f and g, we have:*

1. *δ factorises as $\delta^n(fg) = (\delta^n f)(\delta^n g)$*
2. *The coboundary of the coboundary is the identity. $\delta^{n+1}(\delta^n f) = 1$*

Proof. Showing (1) is a straightforward computation from the definition of the operator.
For any $f \in C^n(G;M)$ we define the map $\hat{f} : G^{n+1} \to M$ by setting[1]

$$\hat{f}(g_1,..g_{n+1}) := f(g_1^{-1}g_2,...,g_n^{-1}g_{n+1}) \quad \forall (g_1,..,g_{n+1}) \in G^{n+1}$$

Notice that $\hat{f}$ is scale invariant and so satisfies

$$\hat{f}(gg_1,...,gg_{n+1}) = g\hat{f}(g_1,..,g_{n+1})$$

For any $\omega \in C^{n+1}(G;M)$ define the $n+2$-cochain $\partial^n \omega : G^{n+2} \to M$ by setting

$$\partial^n(g_1,...g_{n+2}) = \prod_{i=1}^{n+2} \omega(g_1,...,\hat{g}_i,...g_{n+2})^{(-1)^{i+1}}$$

where $\hat{g}_i$ means the variable g_i has been omitted. We cite the following result from section 2 of [2] that

$$\delta^n f = \partial^n \hat{f}(1,x_1,x_1x_2,..,x_1...x_n)$$

So we can directly compute $\partial^{n+1}\partial^n \hat{f} = 1$. $\square$

From the above lemma it follows that the coboundary map δ is a homomorphism.

Definition 1.3. *Let n be a positive integer. Then we define the set of $n-$cocycles as $Z^n(G;M) := \ker(\delta^n)$ and the set of $n-$coboundaries as $B^n(G;M) := \mathrm{Im}(\delta^{n-1})$.*

It follows from the previous lemma that $\mathrm{Im}(\delta^{n-1}) \subseteq \ker(\delta^n)$ and therefore $B^n(G;M)$ is a subgroup of $Z^n(G;M)$

[1] Whilst obtaining this result by direct computation is possible we choose a less masochistic approach.

Definition 1.4. *Let n be a non-negative integer then the $n-th$ cohomology group is defined to be the quotient group*

$$H^n(G;M) = \frac{Z^n(G;M)}{B^n(G;M)}$$

*where the elements of $H^n(G;M)$ are called cohomology classes. If we have two cocycles in the same cohomology class they are said to be **cohomologous**.*

Example 1.1. *We have an element ω is in $Z^2(G;M)$ if and only if $\delta\omega = 1$. Therefore we have that ω is a $2-$cocycle if and only if*

$$\frac{\omega(g_2,g_3)\omega(g_1,g_2g_3)}{\omega(g_1g_2,g_3)\omega(g_1,g_2)} = 1$$

Suppose $\omega \in B^2(G;M) \subseteq Z^2(G;M)$ then there exists a $1-$cochain $f: G \to M$ such that $\omega(g_1,g_2) = \delta f$

$$\omega(g_1,g_2) = \frac{f(g_2)f(g_1)}{f(g_1g_2)}$$

Therefore two 2-cocycles ω and ω' are cohomologous if and only there is a $1-$cochain f such that[2]

$$\omega(g_1,g_2)' = \frac{f(g_2)f(g_1)}{f(g_1g_2)}\omega(g_1,g_2)$$

As we can see in table 1 the group cohomology can be determined for a range of groups. Section 4 will focus on the explicit computation of $H^2(G;M)$ utilizing the unique central extensions of a projective representation. For the reader interested in computing these groups in full generality they are referred to chapter 6 of Weibel [3].

Group Cohomology $(G;M)$	$H^0(G;M)$	$H^1(G;M)$	$H^2(G;M)$	$H^3(G;M)$
$(\mathbb{Z}/n\mathbb{Z};\mathbb{Z})$	$\mathbb{Z}$	1	$\mathbb{Z}/n\mathbb{Z}$	1
$(\mathbb{Z}/n\mathbb{Z};U(1))$	$U(1)$	$\mathbb{Z}/n\mathbb{Z}$	1	$\mathbb{Z}/n\mathbb{Z}$
$(\mathbb{Z}/m\mathbb{Z}\times\mathbb{Z}/n\mathbb{Z};\mathbb{Z})$	$\mathbb{Z}$	1	$\mathbb{Z}/m\mathbb{Z}\times\mathbb{Z}/n\mathbb{Z}$	1
$(\mathbb{Z}/m\mathbb{Z}\times\mathbb{Z}/n\mathbb{Z};U(1))$	$U(1)$	$\mathbb{Z}/m\mathbb{Z}\times\mathbb{Z}/n\mathbb{Z}$	$\mathbb{Z}/\gcd(m,n)\mathbb{Z}$	$\mathbb{Z}/m\mathbb{Z}\times\mathbb{Z}/n\mathbb{Z}\times\mathbb{Z}/\gcd(m,n)\mathbb{Z}$
$(S_3;U(1))$	$U(1)$	$\mathbb{Z}/2\mathbb{Z}$	1	$\mathbb{Z}/6\mathbb{Z}$
$(U(1);\mathbb{Z})$	$\mathbb{Z}$	1	$\mathbb{Z}$	1
$(U(1);U(1))$	$U(1)$	1	$\mathbb{Z}$	1
$(SU(2);\mathbb{Z})$	$\mathbb{Z}$	1	1	1
$(SU(2);U(1))$	$\mathbb{Z}$	1	1	$\mathbb{Z}$
$(SO(3);\mathbb{Z})$	$\mathbb{Z}$	1	1	$\mathbb{Z}/2\mathbb{Z}$

Table 1: The group cohomology of G for $n \leq 3$ of physically relevant groups as described in Chen et al [1]
.

[2]Two forms are cohomologous if and only if they differ by an exact form. In De-Rham cohomology this is stated as $\omega - \omega' = df$

2 Projective Representations

Definition 2.1. *A projective representation of a group G is the pair $(\tilde{\rho}, V)$ where $\tilde{\rho}$ is a group homomorphism from G to the projective linear group*[3] $\mathrm{PGL}(V) = \mathrm{GL}(V)/F^*$ *and V is a $\mathbb{K}-$vector space. That is*

$$\tilde{\rho} : G \to \mathrm{PGL}(V) \textit{ such that } \tilde{\rho}(g)\tilde{\rho}(h) = \rho(\tilde{gh})$$

Notice that we could have defined equivalently defined a projective representation $\tilde{\rho}$ to be a collection of (linear) group representations $\rho : G \to \mathrm{GL}(V)$ that satisfy

$$\rho(g)\rho(h) = c(g,h)\rho(gh) \text{ with } c(g,h) \in \mathbb{F}^{\times}$$

The constant $c(g,h)$ is known as the Schur multiplier. In performing the above we have reduced the study of projective representations back to linear transformations by introducing a gauge freedom. Therefore it makes sense to talk about ρ as being a c-representation.
Now we derive a surprising result connecting the theory of projective representations with the second group cohomology group.
Let $\tilde{\rho}$ be a projective representation with corresponding $\omega-$representation $\rho : G \to \mathrm{GL}(V)$.

$$\begin{aligned}
\omega(g_1g_2, g_3)\rho(g_1g_2g_3) &= \rho(g_1g_2)\rho(g_3) \\
\omega(g_1, g_2)\omega(g_1g_2, g_3)\rho(g_1g_2g_3) &= \omega(g_1, g_2)\rho(g_1g_2)\rho(g_3) \\
&= \rho(g_1)\rho(g_2)\rho(g_3) \\
&= \omega(g_2, g_3)\rho(g_1)\rho(g_2g_3) \\
&= \omega(g_1, g_2g_3)\omega(g_2, g_3)\rho(g_1g_2g_3) \quad \forall g_1, g_2, g_3
\end{aligned}$$

Equating coefficients and moving everything to one side we obtain

$$\frac{\omega(g_2, g_3)\omega(g_1, g_2g_3)}{\omega(g_1g_2, g_3)\omega(g_1, g_2)} = 1$$

Therefore we conclude the Schur multiplier ω is a 2-cocycle, that is $\omega \in Z^2(G, \mathbb{K}^{\times})$. Now let ρ and ρ' be an ω-representation and $\omega'-$representation respectively for a projective representation $\tilde{\rho}$ such that they are both sections of $\tilde{\rho}$ meaning $\pi(\rho(g)) = \pi(\rho'(g))$ for all $g \in G$.
Then for each $g \in G$ there is an $f(g) \in \mathbb{K}^{\times}$ such that $\rho'(g) = f(g)\rho(g)$ but for all $g_1, g_2 \in G$ we have

$$\begin{aligned}
\omega'(g_1, g_2)f(g_1g_2)\rho(g_1g_2) &= \omega'(g_1, g_2)\rho'(g_1g_2) \\
&= \rho'(g_1)\rho'(g_2) \\
&= f(g_1)f(g_2)\rho(g_1)\rho(g_2) \\
&= f(g_1)f(g_2)\omega(g_1, g_2)\rho(g_1g_2)
\end{aligned}$$

Therefore we that the two Schur multiplier satisfy

$$\omega'(g_1, g_2) = \frac{f(g_1)f(g_2)}{f(g_1g_2)}\omega(g_1g_2)$$

and hence ω and ω' are cohomologous 2- cocyles. Therefore we have shown that the cohomology classes $[\omega]$ of ω is independent of your choice of linear representations ρ.

[3] $PGL(V)$ is not a matrix group. Whereas $GL(V)$ the group of invertible linear transformations of V over $\mathbb{F}$ and $\mathbb{F}^*$ is the normal subgroup of non-zero scalar multiples of the identity transformation.

3 Central Extensions

Definition 3.1. *An exact sequence of groups is a sequence of groups with group homomorphisms* (G_i, f_i)

$$1 \to G_1 \xrightarrow{f_1} G_1 \xrightarrow{f_2} .. \xrightarrow{f_n} G_n \to 1$$

such that $\mathrm{Im}(f_{i-1}) = \ker(f_i)$ *for* $i = 1, ..., n$.

When $n = 2$ the above is called a short exact sequence.

Definition 3.2. *An extension of a group* Q[4] *by a group* N *is a short exact sequence*

$$1 \to N \xrightarrow{i} G \to Q \xrightarrow{\pi} 1$$

If G is a finite group this is called a finite extension of the group Q. Moreover if $i(N) \subset Z(G)$ then we say that the sequence is a central extension of Q by N.

Example 3.1. *Let* V *be a* $\mathbb{K}-$*vector space. Then the exact sequence below is an example of a central extension.*

$$1 \to \mathbb{K}^\times \xrightarrow{i} \mathrm{GL}(V) \xrightarrow{\pi} \mathrm{PGL}(V) \to 1$$

Definition 3.3. *Let* $1 \to N \xrightarrow{i_1} G_1 \xrightarrow{p_1} Q \to 1$ *and* $1 \to N \xrightarrow{i_2} G_2 \xrightarrow{p_2} Q \to 1$ *be extensions of the group* Q *by* N*. We say that the extensions* (G_1, p_1) *and* (G_2, p_2) *are equivalent if there exists a morphism of extensions* (id_N, β, id_Q) *of exact sequences*

$$\begin{tikzcd}
1 \arrow[r] & N \arrow[r, "i_1"] \arrow[d, equal] & G_1 \arrow[r, "p_1"] \arrow[d, "\beta"] & Q \arrow[r] \arrow[d, equal] & 1 \\
1 \arrow[r] & N' \arrow[r, "i_2"] & G_2 \arrow[r, "p_2"] & Q \arrow[r] & 1
\end{tikzcd}$$

Remark: By the five lemma the above homomorphism $\beta : G_1 \to G_2$ is an isomorphism.

Theorem 3.1. *Two central extensions* (G_1, p_1) *and* (G_2, p_2) *of the group* Q *by an abelian group* N *are equivalent if and only if, there associated cohomology classes* $\omega_{G_1,p_1}, \omega_{(G_2,p_2)} \in H^2(Q; N)$ *are equal.*

Remark. Let $\mathrm{CExt}(Q, N)/\sim$ be the set of all equivalence classes of central extensions of Q by an abelian group N. We have the following bijection Φ between unique central extensions of Q by N and cohomologous classes of Schur multipliers within $H^2(Q; N)$.

$$\Phi : \mathrm{CExt}(Q, N)/\sim\ \to H^2(Q; N)$$

[4] Q is used here since the group is typically a quotient group.

4 Computing $H^2(G;M)$

To rigorously compute the n-the group cohomology group $H^n(G;M)$ for an arbitrary group G with coefficients in M we would ordinarily use methods of homological algebra. However such methods require a lot of pre-requisite knowledge and take a lot of effort to develop. Therefore we will here instead take a more informal approach using the above correspondence to determine the the group structure of $H^2(G;M)$ via its equivalence classes.

Proposition 4.1. $H^2(\mathbb{Z}/n\mathbb{Z};U(1)) \cong 1$.

Consider the the following presentation of the cyclic group

$$\mathbb{Z}/n\mathbb{Z} = \langle g | g^n = 1 \rangle$$

where we have two arbitrary elements of the form $g^s \in \mathbb{Z}/n\mathbb{Z}$ and $g^t \in \mathbb{Z}/n\mathbb{Z}$ where $s,t \in \{0,..,n-1\}$

$$\rho(g^s)\rho(g^t) = \omega\rho(g^{s+t})$$

Now when $s = t = 0$ we have $\rho(1)\rho(1) = \omega(g^s, g^t)\rho(1)$ which gives us $\rho(1) = \omega(g^s, g^t)$. Here we use the gauge freedom to set $\rho(1) := 1$.

Now suppose $s + t = N = 0$ then

$$1 = \rho(1) = \rho(g^N) = \rho(g \cdot g^{n-1}) = \prod_{j=2}^{n-1} \left[\frac{1}{\omega(g, g_j)}\right] \rho(g)^n$$

where we have made use of the following $\rho(g)\rho(g) = \omega(1,1)\rho(g^2)$. As a multiplication of phases is a phase by the closure of $U(1)$ we have for some ϕ that $\rho(g)^n = e^{i\phi} \in U(1)$. Now we can define the projective representation as $\tilde{\rho}(g) := e^{\frac{-i\phi}{n}}\rho(g)$ which we see produces the desired relation.

$$(\tilde{\rho}(g))^n = (e^{-\frac{i\phi}{n}})^n(\rho(g))^n = (e^{-\frac{i\phi}{n}})^{-n}(e^{\frac{i\phi}{n}})^n = 1.$$

Therefore all representations here are equivalent up to a phase and so we have a single unique extension.
Note: In the following we omit the ρ for notational convenience.

Proposition 4.2. $H^2(\mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z};U(1)) \cong \mathbb{Z}/n\mathbb{Z}$

Consider the the following presentation of the cyclic group

$$\mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z} = \langle g, h | g^n = h^n = 1 \rangle$$

Clearly $(gh)^n = 1$. Consider the following product $P := ghg^{n-1}h^{n-1}$ then we evaluate P in two ways using $gh = \omega(g,h)hg$.

$$P = ghg^{n-1}h^{n-1} = \omega(g,h)hgg^{n-1}h^{n-1} = \omega(g,h)$$

and now taking P again and repeatedly using $\omega(g,h)^{-1}gh = hg$ we obtain

$$P = ghg^{n-1}h^{n-1} = \omega(g,h)^{-(n-1)}$$

Equating both sides we have that $\omega^n = 1$. Therefore we can define $\tilde{\rho}$ in n ways as ω to be an n-th root of unity yielding n unique central extensions.

Proposition 4.3. $H^2(\mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/m\mathbb{Z};U(1)) \cong \mathbb{Z}/\gcd(m,n)\mathbb{Z}$

Similar to the previous calculation let $P_1 = ghg^{m-1}h^{n-1}$ and use $gh = \omega(g,h)hg$ to obtain

$$P_1 = ghg^{m-1}h^{n-1} = \omega(g,h)hgg^{m-1}h^{n-1} = \omega(g,h)$$

Now evaluating P_1 using $\omega(g,h)^{-1}gh = hg$ and taking $n < m$ without loss of generality.

$$P_1 = gh(\omega(g,h)^{-1}hg)g^{m-1}h^{n-1} = \omega(g,h)^{-1}gh(h^{n-1}g^{n-1})g^{m-n} = \omega(g,h)^{-(m-1)}$$

Equating, we obtain $\omega(g,h)^n = 1$ and thus $\omega(g,h)$ is an n-th root of unity. Similarly we consider the product $P_2 = hgh^{n-1}g^{m-1}$ then we obtain

$$P_2 = \omega(g,h)^{-1}ghh^{n-1}g^{m-1} = \omega(g,h)^{-1}$$

and once again using $\omega(g,h)^{-1}gh = hg$, taking $n < m$ without loss of generality

$$P_2 = hgh^{n-1}g^{m-1} = hg(\omega(g,h)^{-1}gh)h^{n-1}g^{m-1} = \omega(g,h)^{-(m-1)}hg(g^{m-1}h^{m-1})h^{n-m} = \omega(g,h)^{-(m-1)}$$

Equating both sides of P_2 gives $\omega(g,h)^m = 1$.Thus for both conditions to be satisfied we conclude $\omega(g,h)$ is a $d-$th root of unity where $d = \gcd(m,n)$. This gives d ways of defining a projective representation up to a phase.

References

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