

Fourier-Mukai Transforms and Derived Equivalences

Adam Monteleone

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1. DERIVED CATEGORIES

1.1. Introduction. The theory of derived categories, introduced by Grothendieck and Verdier, provides a powerful tool for studying the geometry of algebraic varieties via their categories of coherent sheaves. The derived category of a variety (or more generally, a scheme) is known to encode a large amount information of the underlying geometry and even in some cases allowing one to reconstruct the underlying variety [Section 4, [BO95a](#)].

In this talk we provide an overview of derived categories in algebraic geometry, describing some of the major machinery, such as the Fourier-Mukai transform, and the Bondal-Orlov full-faithfulness criterion. We then describe derived equivalences of smooth projective varieties by studying the special case of when our varieties are K3 surfaces.

1.2. Derived Categories. In this lecture we will primarily be interested in the derived category $D(\mathbf{Coh}(X))$ where X is a scheme. However the construction of the derived category is more general, and so it makes sense to define the derived category $D(\mathcal{A})$ with respect to an arbitrary abelian category \mathcal{A} .

Remark 1.1. If \mathcal{A} is an abelian category the category $\mathbf{Kom}(\mathcal{A})$ of cochain complexes with objects in \mathcal{A} is abelian.

Concretely, the derived category $D(\mathcal{A})$ of an abelian category \mathcal{A} has objects given by cochain complexes of \mathcal{A} ,

$$\mathrm{Obj}(D(\mathcal{A})) := \mathrm{Obj}(\mathbf{Kom}(\mathcal{A})).$$

The morphisms are slightly more complicated. For $A^\bullet, B^\bullet \in D(\mathcal{A})$ the collection of morphisms $\mathrm{Hom}_{D(\mathcal{A})}(A^\bullet, B^\bullet)$ are defined as all equivalence classes of diagrams

$$\begin{array}{ccc} & C^\bullet & \\ \swarrow \text{qis} & & \searrow \\ A^\bullet & & B^\bullet \end{array}$$

where $C^\bullet \rightarrow A^\bullet$ is a quasi-isomorphism. Two such diagrams are equivalent if they are dominated by a third such diagram in the homotopy category $\mathbf{K}(\mathcal{A})$.

$$\begin{array}{ccccc} & & C^\bullet & & \\ & \swarrow & & \searrow & \\ & C_1^\bullet & & C_2^\bullet & \\ & \swarrow & & \searrow & \\ A^\bullet & & & & B^\bullet \end{array}$$

(Note: In the original image, there are additional arrows from C_1^\bullet to B^\bullet and from C_2^\bullet to A^\bullet , and a dashed curved arrow labeled 'qis' from C^\bullet to A^\bullet .)

where the compositions $C^\bullet \rightarrow C_1^\bullet \rightarrow A^\bullet$ and $C^\bullet \rightarrow C_2^\bullet \rightarrow A^\bullet$ are homotopy equivalent. Note that the commutativity of this diagram is only required up to homotopy because the construction of the mapping cone is unique only up to homotopy. To define composition consider the two morphisms

$$\begin{array}{ccc} & C_1^\bullet & \\ \swarrow \text{qis} & & \searrow \\ A^\bullet & & B^\bullet \end{array} \quad \begin{array}{ccc} & C_2^\bullet & \\ \swarrow \text{qis} & & \searrow \\ B^\bullet & & C^\bullet \end{array}$$

we define their composite to be given by a commutative diagram (in the homotopy category $\mathbf{K}(\mathcal{A})$) of the form

$$\begin{array}{ccccc} & & C^\bullet & & \\ & \swarrow & & \searrow & \\ & C_1^\bullet & & C_2^\bullet & \\ & \swarrow & & \searrow & \\ A^\bullet & & B^\bullet & & C^\bullet \end{array}$$

(Note: In the original image, there are additional arrows labeled 'qis' from C_1^\bullet to A^\bullet and from C_2^\bullet to B^\bullet .)

It remains to show that such a diagram as claimed always exists and is unique up to equivalence.

The derived category $D(\mathcal{A})$ therefore essentially is the category of chain complexes with quasi-isomorphisms inverted. For instance in $\mathbf{Kom}(\mathbf{AbGrp})$ we have a quasi-isomorphism $f : A^\bullet \rightarrow B^\bullet$ corresponding to the diagram

$$\begin{array}{ccccccc} A^\bullet & \cdots & \longrightarrow & 0 & \longrightarrow & \mathbb{Z} & \xrightarrow{\times 2} & \mathbb{Z} & \longrightarrow & 0 & \longrightarrow & \cdots \\ & & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ B^\bullet & \cdots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \mathbb{Z}/2\mathbb{Z} & \longrightarrow & 0 & \longrightarrow & \cdots \end{array}$$

However there can be no quasi-isomorphism in the opposite direction as there are no non-trivial group homomorphisms $\mathbb{Z}/2\mathbb{Z} \rightarrow \mathbb{Z}$. In the derived category however we do have morphisms in both directions

$$\begin{array}{ccc} & A^\bullet & \\ \text{id} \swarrow & & \searrow f \\ A^\bullet & & B^\bullet \end{array} \qquad \begin{array}{ccc} & A^\bullet & \\ f \swarrow & & \searrow \text{id} \\ B^\bullet & & A^\bullet \end{array}$$

Definition 1.2. Let $f : A^\bullet \rightarrow B^\bullet$ be a morphism of complexes in $K(\mathcal{A})$ or $D(\mathcal{A})$ its mapping cone is the complex

$$c(f)^i := A^{i+1} \oplus B^i \text{ and } d_{C(f)}^i := \begin{bmatrix} -d_A^{i+1} & 0 \\ f^{i+1} & d_B^i \end{bmatrix}.$$

Let \mathcal{A} be an abelian category, the derived category $D(\mathcal{A})$ is triangulated.

Definition 1.3. An additive category \mathcal{T} is triangulated if it has an autoequivalence $\Sigma : \mathcal{T} \rightarrow \mathcal{T}$ and a collection of exact triangles

$$A \longrightarrow B \longrightarrow C \longrightarrow \Sigma A$$

that satisfy (TR1) - (TR4).

The derived category $D(\mathcal{A})$ is triangulated with autoequivalence given by shift functor $[1] : D(\mathcal{A}) \rightarrow D(\mathcal{A})$ with

$$(A^\bullet[1])^i := A^{i+1} \text{ and } d_{A^\bullet[1]}^i := -d_A^{i+1}.$$

We can compose A with itself to get the k -shifted complex

$$A^\bullet[k]^i := A^{k+i} \text{ with } d_{A^\bullet[k]}^i = (-1)^k d_A^{i+k},$$

The exact triangles in $D(\mathcal{A})$ are just triangles isomorphic to the triangle

$$A^\bullet \longrightarrow B^\bullet \longrightarrow \text{Cone}(f) \longrightarrow A^\bullet[1]$$

where the maps $B^\bullet \rightarrow \text{Cone}(f)$ and $\text{Cone}(f) \rightarrow A^\bullet[1]$ are just the natural inclusion $B^\bullet \rightarrow A^\bullet[1] \oplus B^\bullet$ and projection map $A^\bullet[1] \oplus B^\bullet \rightarrow A^\bullet[1]$.

Definition 1.4. Let X be a scheme, then $\mathbf{Coh}(X)$ is an abelian category. We denote by $D^b(\mathbf{Coh}(X))$ the bounded¹ derived category of coherent sheaves of X .

For notational convenience we make the following definition

$$D(X) := D^b(\mathbf{Coh}(X)).$$

Remark 1.5. The category $\mathbf{Coh}(X)$ does not have enough injectives so we usually pass to $\mathbf{Qcoh}(X)$ at least when X is Noetherian. Therefore in what follows we always assume X is Noetherian.

Definition 1.6. Let X and Y be schemes then X and Y are said to be derived equivalent or D-equivalent if there exists a k -linear exact equivalence,

$$D(X) \simeq D(Y).$$

Remark 1.7. It is well known that there exists non-birational Calabi-Yau's which are derived equivalent.

¹bounded just means each complex has finitely many non-zero terms.

1.3. Serre Functors.

Definition 1.8. Let X be a smooth projective variety of dimension n . Then one defines the exact functor S_X as the composition

$$D(X) \xrightarrow{\omega_X \otimes (-)} D(X) \xrightarrow{[n]} D(X).$$

This functor is an example of a Serre functor (definition B.1). This is a generalisation of the familiar notion of Serre duality to the level of derived categories.

Theorem 1.9. (*Serre Duality*) Let X be a smooth projective variety over k then

$$S_X : D(X) \rightarrow D(X) \text{ where } S_X(-) = \omega_X \otimes (-)[n].$$

is a Serre functor. Hence $\text{Ext}^i(\mathcal{E}, \mathcal{F}) = \text{Hom}(\mathcal{E}, \mathcal{F}[i])$ implies $\text{Ext}^i(\mathcal{E}, \mathcal{F}) \simeq \text{Ext}^{n-i}(\mathcal{F}, \mathcal{E} \otimes \omega_X)^*$.

Example 1.10. If X is Calabi-Yau i.e $\omega_X = \mathcal{O}_X$ then $S_X = [n]$.

2. FOURIER-MUKAI TRANSFORMS

Let X and Y be smooth projective varieties over k an algebraically closed field of characteristic 0. and denote the two projections by

$$\begin{array}{ccc} & X \times Y & \\ q \swarrow & & \searrow p \\ X & & Y \end{array}$$

Definition 2.1. Let $\mathcal{P} \in D^b(X \times Y)$. The induced Fourier-Mukai transform is the functor

$$\Phi_{\mathcal{P}} : D(X) \rightarrow D(Y), \text{ where } \mathcal{E}^\bullet \mapsto R p_*(L q^* \mathcal{E}^\bullet \otimes^{\mathbb{L}} \mathcal{P}),$$

where \mathcal{P} is called the Fourier-Mukai kernel of the Fourier-Mukai transform $\Phi_{\mathcal{P}}$.

Remark 2.2. $Lq^* = q^*$ since the projection map q is flat and the left derived tensor product $\otimes^{\mathbb{L}}$ coincides with the ordinary tensor product when the kernel \mathcal{P} is a complex of vector bundles.

Composing two Fourier-Mukai functors gives another Fourier Mukai functor up to isomorphism

Proposition 2.3 ([Muk81]). Let $\Phi_{\mathcal{P}} : D(X) \rightarrow D(Y)$ and $\Phi_{\mathcal{Q}} : D(Y) \rightarrow D(Z)$ be Fourier-Mukai functors. The composition

$$D(X) \xrightarrow{\Phi_{\mathcal{P}}} D(Y) \xrightarrow{\Phi_{\mathcal{Q}}} D(Z),$$

is isomorphic to the Fourier-Mukai transform $\Phi_{\mathcal{R}} : D(X) \rightarrow D(Z)$.

When computing Fourier-Mukai transforms the following two results are frequently applied

Theorem 2.4 ([Huy06]). (*Projection Formula*) Let $f : X \rightarrow Y$ be a proper morphism of projective schemes over k . For any $\mathcal{F}^\bullet \in D(X), \mathcal{E}^\bullet \in D(Y)$ there exists a natural isomorphism

$$Rf_*(\mathcal{F}^\bullet) \otimes^{\mathbb{L}} \mathcal{E}^\bullet \xrightarrow{\sim} Rf_*(\mathcal{F}^\bullet \otimes Lf^*(\mathcal{E}^\bullet)).$$

Theorem 2.5 ([Huy06]). Let $f : X \rightarrow Y$ be a morphism of projective schemes and let $\mathcal{F}^\bullet, \mathcal{E}^\bullet \in D(Y)$. Then there exists a natural isomorphism

$$Lf^*(\mathcal{F}^\bullet) \otimes^{\mathbb{L}} Lf^*(\mathcal{E}^\bullet) \xrightarrow{\sim} Lf^*(\mathcal{F}^\bullet \otimes \mathcal{E}^\bullet).$$

Example 2.6. Let $\Gamma_f : X \rightarrow X \times Y$ be the diagonal map, where $\mathcal{O}_{\Gamma_f} = (\Gamma_f)_* \mathcal{O}_X$. Moreover let $q : X \times Y \rightarrow X$ and $p : X \times Y \rightarrow Y$ be projections onto the X and Y factors respectively, then from the pullback diagram

$$\begin{array}{ccccc} X & & & & \\ & \searrow \Gamma_f & & \searrow f & \\ & X \times Y & \xrightarrow{p} & Y & \\ & \downarrow q & & \downarrow & \\ X & \xrightarrow{f} & Y & & \end{array}$$

(Note: The diagram above is a simplified representation of the pullback diagram shown in the image, which includes an additional arrow from X to X labeled id_X .)

we can compute the following Fourier-Mukai transform

$$\begin{aligned}
\Phi_{\mathcal{O}_{\Gamma_f}}(\mathcal{F}^\bullet) &= Rp_*(Lq^* \otimes^{\mathbb{L}} \mathcal{O}_\Delta) \\
&\cong p_*(q^* \otimes (\Gamma_f)_* \mathcal{O}_X) && \text{(remark 2.2)} \\
&\cong p_*(\Gamma_f)_*((\Gamma_f)^* q^* \mathcal{F}^\bullet \otimes_{\mathcal{O}_X} \mathcal{O}_X) && \text{(projection formula)} \\
&\cong (p \circ \Gamma_f)_*(q \circ \Gamma_f)^*(\mathcal{F}^\bullet) \\
&\cong f_*(\text{id}_X)^*(\mathcal{F}^\bullet) \\
&\cong f_*(\mathcal{F}^\bullet).
\end{aligned}$$

Setting the map $f = \text{id}$ in the above example we recover the well known result $\Phi_{\mathcal{O}_\Delta}(\mathcal{F}) = \mathcal{F}$.

We now come to Orlov's celebrated result.

Theorem 2.7 ([Orl03]). (*Orlov's Theorem*) Let X and Y be two smooth projective varieties and let

$$F : D(X) \rightarrow D(Y),$$

be a fully faithful exact functor. If F admits a right and left adjoint then there exists an object $\mathcal{P} \in D(X \times Y)$ unique up to isomorphism such that F is isomorphic to $\Phi_{\mathcal{P}}$, that is

$$F \simeq \Phi_{\mathcal{P}}.$$

Orlov's theorem is most often applied to equivalences:

Corollary 2.8. Let $F : D(X) \xrightarrow{\sim} D(Y)$ be an equivalence between the derived category of smooth projective varieties over k . Then F is isomorphic to a Fourier-Mukai transform $\Phi_{\mathcal{P}}$ associated to a certain object $\mathcal{P} \in D(X \times Y)$, which is unique up to isomorphism.

Theorem 2.9 ([BO95b]). Let X and Y be smooth projective varieties. The Fourier-Mukai transform $\Phi_{\mathcal{P}} : D(X) \rightarrow D(Y)$ is fully faithful if and only if for any two closed points $x, y \in X$ one has

$$\text{Hom}(\Phi_{\mathcal{P}}(\mathcal{O}_x), \Phi_{\mathcal{P}}(\mathcal{O}_y)[i]) = \begin{cases} k & \text{if } x = y \text{ and } i = 0; \\ 0 & \text{if } x \neq y \text{ or } i < 0 \text{ or } i > \dim(X). \end{cases}$$

Remark 2.10. This full-faithfulness criterion as well as Orlov's theorem have been considerably generalized. Recently Orlov's theorem was generalized to smooth proper tame stacks in [Pen24]. Similarly the full-faithfulness criterion was also generalized to smooth proper Deligne-Mumford stacks² in [HP24].

Corollary 2.11. Suppose $\Phi_{\mathcal{P}} : D(X) \rightarrow D(Y)$ is a fully faithful functor. Then the functor $\Phi_{\mathcal{P}}$ is an equivalence if and only if $\dim(X) = \dim(Y)$ and

$$\mathcal{P} \otimes q^* \omega_X \simeq \mathcal{P} \otimes p^* \omega_Y.$$

Cohomological Fourier-Mukai Transform. Let X and Y be smooth projective varieties over k . Suppose $\Phi_{\mathcal{P}} : D(X) \rightarrow D(Y)$ is a Fourier-Mukai transform. For any cohomology class $\alpha \in H^*(X \times Y, \mathbb{Q})$ we have

$$\Phi_{\alpha}^H : H^*(X; \mathbb{Q}) \rightarrow H^*(Y; \mathbb{Q}) \text{ given by } \beta \mapsto p_*(q^* \beta \cdot \alpha),$$

where $\alpha := v(\mathcal{P}) = \text{ch}(\mathcal{P}) \cdot \sqrt{\text{td}(X \times Y)}$.

Remark 2.12. The cohomological Fourier-Mukai transform is used extensively when proving criteria relating derived categories to the underlying cohomology of the variety. In particular it is used extensively in the proof of theorem 3.7.

This section so far has been a summary of almost all the important results needed to work with derived categories and derived functors. However we have not given some of the more basic compatibilities of derived functors in algebraic geometry. For the rigorous approach to dualities and derived categories, see either [Con00] or [LH09]. For a thorough survey of Fourier-Mukai transforms in algebraic geometry, see [Huy06].

²This generalization of the full faithfulness theorem to stacks will be the subject of Jack's talk next week.

3. DERIVED EQUIVALENCES OF K3 SURFACES

We now study the rich theory of derived equivalences of K3 surfaces. Recall the definition of a K3 surface:

Definition 3.1. A K3 surface is a compact complex surface X with trivial canonical bundle

$$\omega_X \simeq \mathcal{O}_X \text{ and } H^1(X, \mathcal{O}_X) = 0.$$

Remark 3.2. Every K3 surface is Kähler.

Remark 3.3. Algebraic K3 surfaces are dense in the moduli space of all K3 surfaces.

Question 1. *When are two K3 surfaces equivalent?*

Answer:

Theorem 3.4 ([PSS71]). (*Global Torelli Theorem*) *Two complex K3 surface X and Y are isomorphic if and only if there exists a Hodge isometry*

$$H^2(X; \mathbb{Z}) \rightarrow H^2(Y; \mathbb{Z}).$$

Remark 3.5. Originally proven by Pyatetski-Shapiro–Shafarevich in 1971 in the paper [PSS71] for algebraic K3 Surfaces. This result was then extended to all K3 surfaces by numerous authors, most notably in the work of Gritsenko–Hulek–Sankaran [GHS07].

It is natural to then formulate the derived analogue of question 1:

Question 2. *When are two K3 surfaces derived equivalent?*

Let X be a K3 surface then

$$H^*(X; \mathbb{Z}) = H^0(X; \mathbb{Z}) \oplus H^2(X; \mathbb{Z}) \oplus H^4(X; \mathbb{Z}).$$

The Mukai pairing $\langle -, - \rangle : H^*(X; \mathbb{Z}) \times H^*(X; \mathbb{Z}) \rightarrow H^4(X; \mathbb{Z}) \xrightarrow{\sim} \mathbb{Z}$ is defined as the pairing

$$\langle \alpha, \beta \rangle := \alpha_1 \beta_1 - \alpha_0 \beta_2 - \alpha_2 \beta_0 \in H^4(X; \mathbb{Z}) \xrightarrow{\sim} \mathbb{Z},$$

where $\alpha = (\alpha_0, \alpha_1, \alpha_2)$ and $\beta = (\beta_0, \beta_1, \beta_2)$ with $\alpha_i, \beta_i \in H^{2i}(X; \mathbb{Z})$.

Definition 3.6. A Mukai lattice for X a K3 surface denoted $\hat{H}(X; \mathbb{Z})$ is the integral cohomology ring $H^*(X; \mathbb{Z})$ along with the Mukai pairing $\langle -, - \rangle$ defined as above.

A K3 surface admits a weight two Hodge decomposition

$$\begin{aligned} \hat{H}^{2,0}(X) &\simeq H^{2,0}(X) \\ \hat{H}^{1,1}(X) &\simeq H^{1,1}(X) \oplus H^0(X) \oplus H^4(X) \\ \hat{H}^{0,2}(X) &\simeq H^{0,2}(X). \end{aligned}$$

Theorem 3.7 ([Muk84], [Orl97]). (*Derived Torelli Theorem*) *Let X and Y be K3 surfaces with Mukai lattices $\hat{H}(X; \mathbb{Z})$ and $\hat{H}(Y; \mathbb{Z})$ respectively, then there is a derived equivalence*

$$D(X) \simeq D(Y),$$

if and only if there exists a Hodge isometry between Mukai lattices $\hat{H}(X; \mathbb{Z}) \rightarrow \hat{H}(Y; \mathbb{Z})$.

Remark 3.8. Mukai initially proved the forward direction in [Muk84], later Orlov proved the converse in [Orl97].

Let X be a K3 surface we define the set of Fourier-Mukai partners

$$\text{FM}(X) = \{Y | D(X) \simeq D(Y)\} / \simeq$$

where \simeq quotient by autoequivalences.

Remark 3.9. Since $X \in \text{FM}(X)$ the set of Fourier-Mukai partners for X is never empty.

A famous example of a K3 surface is a Kummer surface X . Kummer surfaces are obtained from resolving an abelian surface A / \sim where we have taken a quotient by the involution corresponding to the inversion map.

Example 3.10. Let X be the Kummer surface associated to an abelian surface A , that is $X = \text{Km}(A)$ then

$$\text{FM}(X) = \{X\}.$$

Therefore given any two Kummer surfaces $X = \text{Km}(A)$ and $Y = \text{Km}(B)$ with A and B abelian surfaces then $D(A) \simeq D(B)$ if and only if $X \simeq Y$.

We conclude this talk with some recent results for derived equivalence of K3 surfaces in positive characteristic. Suppose k is an algebraically closed field with positive characteristic.

Theorem 3.11 (Theorem 6.1, [LO11]). *(Derived Torelli Theorem in Positive Characteristic)*

Let X and Y be K3 surfaces over k . If there exists a kernel $\mathcal{P} \in \mathbf{D}(X \times Y)$ such that we have a filtered equivalence $\mathbf{D}(X) \rightarrow \mathbf{D}(Y)$ then $X \simeq Y$.

Remark 3.12. See paragraph 2.11 in Lieblich–Olsson [LO11] for the precise definition of filtered equivalence.

Theorem 3.13 (Theorem 4.1, [LO11]). *If X and Y are K3 surfaces over a finite field \mathbb{F} and derived equivalent, (i.e. $\mathbf{D}(X) \simeq \mathbf{D}(Y)$) then X and Y have the same zeta-function and the same number of points over \mathbb{F} .*

$$\#X(\mathbb{F}) = \#Y(\mathbb{F}).$$

APPENDIX A. TRIANGULATED CATEGORIES

Triangulated Categories as we will define were first introduced and developed in 1962-1963 by Dieter Puppe and Jean-Louis Verdier respectively. Puppe was an algebraic topologist primarily interested in triangulated categories as an abstraction of the stable homotopy category whilst Verdier, who at the time was a PhD student of Grothendieck, was an algebraic geometer interested in how the derived category admits this triangulated structure. Both Puppe and Verdier gave similar definitions of a triangulated category at around the same time with the only notable difference being Verdier's inclusion of the octahedral axiom (TR4)³

Definition A.1. Let \mathcal{T} be a category equipped with an automorphism $\Sigma : \mathcal{T} \rightarrow \mathcal{T}$. A triangle (A, B, C) where $A, B, C \in \mathcal{T}$ is an ordered triple (u, v, w) of morphisms where $u : A \rightarrow B$, $v : B \rightarrow C$ and $w : C \rightarrow \Sigma A$. That is a triangle

$$A \xrightarrow{u} B \xrightarrow{v} C \xrightarrow{w} \Sigma A.$$

Definition A.2. A morphism of triangles is a triple (f, g, h) forming a commutative diagram in \mathcal{T} :

$$\begin{array}{ccccccc} A & \xrightarrow{u} & B & \xrightarrow{v} & C & \xrightarrow{w} & \Sigma A \\ \downarrow f & & \downarrow g & & \downarrow h & & \downarrow \Sigma f \\ A & \xrightarrow{u'} & B & \xrightarrow{v'} & C & \xrightarrow{w'} & \Sigma A' \end{array}$$

An isomorphism of triangles is then defined in the natural way.

Definition A.3. Let \mathcal{T} be an additive category. Then \mathcal{T} is a **triangulated category** if \mathcal{T} admits an additive equivalence $\Sigma : \mathcal{T} \rightarrow \mathcal{T}$, called the translation or shift functor, and a collection of distinguished (or exact) triangles in \mathcal{T} , which are triangles (u, v, w) that satisfy the following four axioms:

- (TR1) i) Every morphism $u : A \rightarrow B$ can be embedded in an exact triangle (u, v, w) .
 ii) Any triangle of the form

$$A \xrightarrow{id} A \longrightarrow 0 \longrightarrow \Sigma A,$$

is exact.

- iii) if (u, v, w) is a triangle on (A, B, C) , isomorphic to an exact triangle (u', v', w') on (A', B', C') then (u, v, w) is also exact.

$$\begin{array}{ccccccc} A & \xrightarrow{u} & B & \xrightarrow{v} & C & \xrightarrow{w} & \Sigma A \\ \downarrow \cong & & \downarrow \cong & & \downarrow \cong & & \downarrow \cong \\ A & \xrightarrow{u'} & B & \xrightarrow{v'} & C & \xrightarrow{w'} & \Sigma A' \end{array}$$

- (TR2) (Rotation). If (u, v, w) is an exact triangle on (A, B, C)

$$\begin{array}{ccc} & C & \\ w \swarrow & & \nwarrow v \\ A & \xrightarrow{u} & B \end{array}$$

then both its “rotates” $(-\Sigma^{-1}w, u, v)$ and $(v, w, -\Sigma u)$ are exact triangles on $(-\Sigma^{-1}w, u, v)$ and $(B, C, \Sigma A)$

$$\begin{array}{ccc} & \Sigma A & \\ -\Sigma u \swarrow & & \nwarrow w \\ B & \xrightarrow{v} & C \end{array} \quad \begin{array}{ccc} & B & \\ v \swarrow & & \nwarrow u \\ \Sigma^{-1}C & \xrightarrow{-\Sigma^{-1}w} & A \end{array}$$

- (TR3) Suppose there exists a commutative diagram of distinguished triangles (A, B, C) and (A', B', C') with vertical arrows $f : A \rightarrow A'$ and $g : B \rightarrow B'$: Then the diagram can be completed to a morphism of triangles, by a non-unique morphism $h : C \rightarrow C'$.

$$\begin{array}{ccccccc} A & \xrightarrow{u} & B & \xrightarrow{v} & C & \xrightarrow{w} & \Sigma A \\ \downarrow f & & \downarrow g & & \downarrow \exists h & & \downarrow \Sigma f \\ A' & \xrightarrow{u'} & B' & \xrightarrow{v'} & C' & \xrightarrow{w'} & \Sigma A' \end{array}$$

³hence why (TR4) is sometimes referred to as the Verdier axiom.

(TR4)(**Octahedral axiom/Verdier axiom**) Given objects $A, B, C, A', B', C' \in \mathcal{T}$, if there are three exact triangles: (u, j, ∂) on (A, B, C') , (v, x, i) on (B, C, A') and (vu, y, δ) on (A, C, B') .

$$\begin{array}{ccc}
 \begin{array}{ccc} & C & \\ \swarrow \partial & & \nwarrow j \\ A & \xrightarrow{u} & B \end{array} &
 \begin{array}{ccc} & C' & \\ \swarrow i & & \nwarrow x \\ A & \xrightarrow{v} & B' \end{array} &
 \begin{array}{ccc} & B' & \\ \swarrow \delta & & \nwarrow y \\ A & \xrightarrow{vu} & C \end{array}
 \end{array}$$

Then there is a fourth exact triangle $(f, g, (\Sigma j)i)$ on (C', B', A') such that in the following octahedron the four exact triangles form four of the faces and the remaining faces commute.

Remark A.4. There is some uncertainty as to whether TR4 is the “right” axiom for the definition of a triangulated category, and whether instead it is more natural to impose that the morphisms of distinguished triangles should admit mapping cones which also form distinguished triangles. In [Nee01] Neeman calls such a condition (TR4') and in particular shows that (TR4') implies the octahedral axiom (TR4), with the converse implication proven in [Nee91]. For the reader interested in further such discussion see [Nee91] and the relevant sections of [May01].

Definition A.5. An additive functor $F : \mathcal{T} \rightarrow \mathcal{T}'$ between triangulated categories \mathcal{T} and \mathcal{T}' is called exact if the following conditions are satisfied

- (1) There exists a functorial isomorphism $F \circ T_{\mathcal{T}} \xrightarrow{\sim} T_{\mathcal{T}'} \circ F$.
- (2) Any distinguished triangle

$$A \longrightarrow B \longrightarrow C \longrightarrow A[1]$$

in \mathcal{T} is mapped to a distinguished triangle

$$F(A) \longrightarrow F(B) \longrightarrow F(C) \longrightarrow F(A)[1]$$

in \mathcal{T}' where $F(A[1])$ is identified with $F(A)[1]$ via the functor isomorphism in 1).

Definition A.6. A subcategory $\mathcal{T}' \subset \mathcal{T}$ of a triangulated category is a triangulated subcategory if \mathcal{T}' admits the structure of a triangulated category such that the inclusion $i : \mathcal{T}' \rightarrow \mathcal{T}$ is exact.

Proposition A.7. Let $\mathcal{T}' \subset \mathcal{T}$ be a full subcategory. \mathcal{T}' is a triangulated subcategory if and only if \mathcal{T}' is invariant under the shift functor $\Sigma : \mathcal{T} \rightarrow \mathcal{T}$ and for any distinguished triangle

$$A \longrightarrow B \longrightarrow C \longrightarrow A[1]$$

we have that $C \cong D$ for $D \in \mathcal{T}'$.

Definition A.8. Two triangulated categories \mathcal{T} and \mathcal{T}' are equivalent if there exists an exact equivalence $F : \mathcal{T} \rightarrow \mathcal{T}'$.

Definition A.9. A triangulated category \mathcal{T} is decomposable into triangulated subcategories $\mathcal{A} \subset \mathcal{T}$ and $\mathcal{B} \subset \mathcal{T}$ if the following three conditions are satisfied:

- (1) The categories \mathcal{A} and \mathcal{B} contain objects non-isomorphic to 0.
- (2) For every object $F \in \mathcal{T}$, there exists a distinguished triangle

$$A \longrightarrow F \longrightarrow B \longrightarrow A[1]$$

where $A \in \mathcal{A}$ and $B \in \mathcal{B}$.

- (3) For every pair of objects $B_1 \in \mathcal{T}_1$ and $B_2 \in \mathcal{T}_2$, there exist no morphisms in \mathcal{T} between them, i.e.,

$$\text{Hom}(B_1, B_2) = \text{Hom}(B_2, B_1) = 0.$$

APPENDIX B. PROPERTIES OF TRIANGULATED CATEGORIES

Definition B.1. Let \mathcal{A} be a k -linear category. A Serre functor $S : \mathcal{A} \rightarrow \mathcal{A}$ is an additive functor that is also an autoequivalence such that for any two objects $A, B \in \mathcal{T}$ there exists an isomorphism

$$\eta_{A,B} : \text{Hom}(A, B) \rightarrow \text{Hom}(B, S(A))^*.$$

Proposition B.2 ([BK90]). *Any Serre functor on a triangulated category over a field k is exact.*

Definition B.3. Let \mathcal{T} be a triangulated category. A subclass $\Omega \subset \mathcal{T}$ of the objects of \mathcal{T} is called a spanning class of \mathcal{T} if for any object $B \in \mathcal{T}$:

- (1) $\text{Hom}_{\mathcal{T}}(A, B[i]) = 0$ for all $A \in \Omega$ and all $i \in \mathbb{Z}$, then $B \cong 0$.
- (2) $\text{Hom}_{\mathcal{T}}(B[i], A) = 0$ for all $A \in \Omega$ and all $i \in \mathbb{Z}$, then $B \cong 0$.

Proposition B.4 (Corollary. 3.19, [Huy06]). *If X is a smooth projective variety and L is an ample line bundle on X , then the powers $L^i, i \in \mathbb{Z}$, form a spanning class in $\text{D}(X)$.*

Example B.5. Let $\mathcal{E}^\bullet \in \text{D}(X)$ be any object and

$$\mathcal{E}^{\bullet\perp} := \{\mathcal{F}^\bullet \in \text{D}(X) \mid \text{Hom}(\mathcal{E}^\bullet, \mathcal{F}^\bullet[i]) = 0 \text{ for all } i \in \mathbb{Z}\}.$$

Then $\Omega = \{\mathcal{E}^\bullet\} \cup \mathcal{E}^{\bullet\perp} \subset \text{D}(X)$ is a spanning class.

Remark B.6. If the triangulated category admits a Serre functor, conditions (1) and (2) above are equivalent.

Proposition B.7 ([Orl97]). *Let $F : \mathcal{T} \rightarrow \mathcal{T}'$ be an exact functor between triangulated categories with left and right adjoints: $G \dashv F \dashv H$. Suppose Ω is the spanning class of \mathcal{T} such that for all objects $A, B \in \Omega$ and all $i \in \mathbb{Z}$ the natural homomorphisms*

$$F : \text{Hom}(A, B[i]) \rightarrow \text{Hom}(F(A), F(B[i]))$$

are bijective. Then F is fully faithful.

Proposition B.8 (Corollary, 1.56, [Huy06]). *Let $F : \mathcal{T} \rightarrow \mathcal{T}'$ be an exact functor between triangulated categories \mathcal{T} and \mathcal{T}' with left adjoint $F \dashv H$. Furthermore assume that Ω is a spanning class of \mathcal{T} satisfying the following conditions*

- (1) *For all $A, B \in \Omega$ the natural morphisms*

$$\text{Hom}(A, B[i]) \rightarrow \text{Hom}(F(A), F(B)[i]),$$

are bijective for all $i \in \mathbb{Z}$.

- (2) *The categories \mathcal{T} and \mathcal{T}' admit Serre functors $S_{\mathcal{T}}$ and respectively $S_{\mathcal{T}'}$ such that for all $A \in \Omega$,*

$$F(S_{\mathcal{T}}(A)) = S_{\mathcal{T}'}(F(A)).$$

- (3) *The category \mathcal{T}' is indecomposable and \mathcal{T} is non-trivial.*

Then F is an equivalence.

REFERENCES

- [BK90] Alexey Bondal and Mikhail Kapranov. Representable functors, serre functors, and mutations. *Mathematics of The USSR-izvestiya*, 35:519–541, 1990.
- [BO95a] Alexey Bondal and Dmitri Orlov. Semiorthogonal decomposition for algebraic varieties. 07 1995.
- [BO95b] Alexey Bondal and Dmitri Orlov. Semiorthogonal decomposition for algebraic varieties. 07 1995.
- [Con00] B. Conrad. *Grothendieck Duality and Base Change*. Number no. 1750 in Grothendieck Duality and Base Change. Springer, 2000.
- [GHS07] Valery A Gritsenko, Klaus Hulek, and Gregory K Sankaran. The kodaira dimension of the moduli of k3 surfaces. *Inventiones mathematicae*, 169(3):519–567, 2007.
- [HP24] Jack Hall and Kyle Priver. A generalized bondal-orlov full faithfulness criterion for deligne-mumford stacks, 2024.
- [Huy06] D. Huybrechts. *Fourier-Mukai Transforms in Algebraic Geometry*. Clarendon Press, 2006.
- [LH09] J. Lipman and M. Hashimoto. *Foundations of Grothendieck Duality for Diagrams of Schemes*. Lecture Notes in Mathematics. Springer Berlin Heidelberg, 2009.
- [LO11] Max Lieblich and Martin Olsson. Fourier-mukai partners of k3 surfaces in positive characteristic. *Annales Scientifiques de l'École Normale Supérieure*, 48, 12 2011.
- [May01] J.P. May. The additivity of traces in triangulated categories. *Advances in Mathematics*, 163(1):34–73, 2001.
- [Muk81] Shigeru Mukai. Duality between $D(X)$ and $D(\hat{X})$ with its application to Picard sheaves. *Nagoya Mathematical Journal*, 81(none):153 – 175, 1981.
- [Muk84] Shigeru Mukai. On the moduli space of bundles on k3 surfaces. in: *Vector bundles on Algebraic Varieties*, 48, 1984.
- [Nee91] Amnon Neeman. Some new axioms for triangulated categories. *Journal of Algebra*, 139:221–255, 1991.
- [Nee01] Amnon Neeman. *Triangulated Categories. (AM-148), Volume 148*. Princeton University Press, Princeton, 2001.
- [Orl97] Dmitri Orlov. Equivalences of derived categories and k3 surfaces. *Journal of Mathematical Sciences*, 84:1361–1381, 06 1997.
- [Orl03] D O Orlov. Derived categories of coherent sheaves and equivalences between them. *Russian Mathematical Surveys*, 58(3):511, jun 2003.
- [Pen24] Fei Peng. Equivalences of derived categories of sheaves on tame stacks, 2024.
- [PSS71] I. Pyatetski-Shapiro and I. Shafarevich. Torelli's theorem for algebraic surfaces of type k3. *Izvestiya Akademii Nauk SSSR, Seriya Matematicheskaya*, 35:530–572, 1971.