Archimedean and Unramified Parameters of L-Groups

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1 Introduction

Throughout these notes we will assume the setup of the previous talks in the seminar. However in order to make some of the statements in these notes self contained we have included an appendix which outlines the necessary background from the first 14 sections of Gross' paper [Gro99] needed to understand sections 15 and 16. We encourage the reader who has not read through these sections of Gross' paper to look through appendix A before proceeding.

2 Preliminary Setup

Let G be connected, reductive group over $\mathbb{Q}$ satisfying condition A.1. Let $\overline{\mathbb{Q}}$ be a fixed algebraic closure of $\mathbb{Q}$, fix a $T \subset B \subset G$ with $W(T) := N_G(T)/T$, defined up to conjugacy. Associated to G we have the root datum

$$\varphi(G) = (X^\bullet(T), \Delta^\bullet, X_\bullet(T), \Delta_\bullet),$$

acted upon by $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. Let k be the splitting field of G, as the quasi-split inner form G_0 of G is split over k.

Let $\hat{G}$ denote the dual group of G, unique connected reductive group over $\mathbb{Z}$. This gives us a pinning $\hat{G} \supset \hat{B} \supset \hat{T}$ with $e_\alpha : \mathbb{G}_a \xrightarrow{\sim} \hat{U}_\alpha$ for $\alpha \in \Delta_\bullet$ over $\mathbb{Z}$. We have the root datum attached to $\hat{G}$ the root datum (which in terms of T):

$$\varphi(\hat{G}) \cong (X_\bullet(T), \Delta_\bullet, X^\bullet(T), \Delta^\bullet).$$

Let V be an irreducible representation of G over $\mathbb{Q}$. Let $K \subset G(\hat{\mathbb{Q}})$ open, compact subgroup. The $\mathbb{Q}$-vector space of modular forms on G with coefficient in V (restricted to K) is

$$M(V,K) := \{f : G(\mathbb{A})/(G(\mathbb{R})_+ \times K) \to V | f(\gamma g) = \gamma f(g) \text{ for } \gamma \in G(\mathbb{Q})\}.$$

where $\mathbb{A} := \mathbb{R} \times \hat{\mathbb{Q}}$. The $\mathbb{Q}$-vector space of $M(V,K)$ admits an inner product. Let $A \in \mathrm{End}_{\mathbb{Q}}(M(V,K))$ the $\mathbb{Q}$-subalgebra generated by $\mathrm{End}_G(V)$,

$$\{T(g_\infty) | g_\infty \in G(\mathbb{R})\}, \text{ and } \{T(\hat{g}) | \hat{g} \in G(\mathbb{A}_f)\}.$$

Let $\pi_0(G(\mathbb{R})) = G(\mathbb{R})/G(\mathbb{R})_+$, let H_K be the Hecke algebra associated to K. Define $E := Z(\mathrm{End}_A(N))$, which is a number field, in fact it is a CM field. For a simple A - submodule $N \subset M(V,K)$ we can obtain (by [[Gro99], Chapter 7]) two characters, and it is to these characters that we will associate with local parameters.

$$\varphi_\infty : \pi_0(G(\mathbb{R})) \to \langle \pm 1 \rangle \subset E^\times \rightsquigarrow \text{Archimedean Parameters,} \tag{1}$$

$$\hat{\varphi} : Z(H_K) \to E^\times \rightsquigarrow \text{Unramified Parameters.} \tag{2}$$

3 Archimedean Parameters

Let $N \subset M(V,K)$ be a simple submodule, we want to associate a local parameter, which will be a point of the variety of conjugacy classes fixed by τ

$$\varphi_\infty \mapsto h_\infty \in \mathrm{C}\ell_\tau(\mathbb{Z}),$$

where τ is complex conjugation.

Proposition 3.1 ([Gro99] Prop 2.4)**.** *Let $\theta : X^\bullet \to X^\bullet$ be the involution defined by $\theta := w \times \tau$ where $w \in W(T)$ is* *the longest element (i.e the element that sends each positive root to its negative). There is an isomorphism*

$$(X^{\bullet})^{\theta=1}/(1+\theta)X^{\bullet} \cong \mathrm{Hom}(\pi_0(G(\mathbb{R})), E^{\times}),$$

where $(X^{\bullet})^{\theta=1} := \{\chi \in X^{\bullet} | \theta(\chi) = \chi\}$.

Via this isomorphism we can view associate to the sign character φ_∞ an element (which we also denote by φ_∞) in the 2-group $(X^{\bullet})^{\theta=1}/(1+\theta)X^{\bullet}$. Let $\xi \in (X^{\bullet})^{\theta=1}$ be a lift of φ_∞ and let $\chi \in (X_{\bullet})^{\theta=1}$ then the inner product between characters and cocharacters can be used to define an inner product on the Tate cohomology given by

$$\langle \varphi_\infty, \chi \rangle := \langle \xi, \chi \rangle \bmod 2.$$

To see such an inner product is well defined, we let ξ' be another lift then $\xi' = \xi + (1+\theta)\lambda$, where $\lambda \in X^{\bullet}(T)$ then

$$\begin{aligned} \langle \xi', \chi \rangle &= \langle \xi, \chi \rangle + \langle (1+\theta)\lambda, \chi \rangle && (3)\\ &= \langle \xi, \chi \rangle + \langle \lambda, \chi \rangle + \langle \theta\lambda, \chi \rangle && (4)\\ &= \langle \xi, \chi \rangle + \langle \lambda, \chi \rangle + \langle \theta\lambda, \theta\chi \rangle && (5)\\ &= \langle \xi, \chi \rangle + 2\langle \lambda, \chi \rangle && (6)\\ &\equiv \langle \xi, \chi \rangle \bmod 2. && (7) \end{aligned}$$

Before stating the main result of this section, we need to recall the condition that for G there exists a character (cocharacter of $\hat{T}$) $\eta \in X^{\bullet}(T)$ fixed by $\mathrm{Gal}(k/\mathbb{Q})$ such that

$$\langle \eta, \alpha \rangle = 1, \text{ for all } \alpha \in \Delta_{\bullet}(T).$$

We assume this condition is satisfied however in general it may not be. Some examples of $\eta : \mathbb{G}_m \to \hat{T}$ are given below:

$$\begin{cases} \eta = 0, & \text{if G is a Torus;} \\ \eta = \frac{1}{2}\sum_{\beta>0} \beta^{\vee}, & \text{if } G \text{ is simply connected, with } \beta^{\vee} \text{ a positive coroot.} \end{cases}$$

Proposition 3.2 ([Gro99], Prop 15.2)**.** *Let $N \subset M(V,K)$ be a simple A-submodule, then there is a unique class $h_\infty := h_\infty(\varphi_\infty) \in \mathrm{C}\ell_\tau(\mathbb{Z})$ that satisfies*

1. $h_\infty^2 = 1$ *in* ${}^L G(\mathbb{Z})$;
2. $\mathrm{Tr}(h_\infty | \hat{\mathfrak{g}}) = \mathrm{Tr}(\theta | X^{\bullet} = \mathrm{Lie}(\hat{T}))$;
3. $\chi(h_\infty) = (-1)^{\langle \eta + \varphi_\infty, \chi \rangle}$ *for all* $\chi \in \mathrm{Hom}({}^L G, \mathbb{G}_m)$.

The class $h_\infty(\varphi_\infty)$ is fixed by the action $\mathrm{Gal}(k/\mathbb{Q})^{\tau}/\langle \tau \rangle \cong \mathrm{Gal}(k^+/\mathbb{Q})$ *on* $C\ell_\tau$.

Proof. Lift $\varphi_\infty \in (X^{\bullet})^{\theta=1}/(1+\theta)X^{\bullet}$ to an element $\xi \in (X^{\bullet})^{\theta=1}$ and view ξ as a cocharacter of $\hat{T}$ fixed by $W \rtimes \mathrm{Gal}(k/\mathbb{Q})$. Define the involution from the character group of $\pi_0(G(\mathbb{R}))$, that is $h : (X^{\bullet})^{\theta=1}/(1+\theta)X^{\bullet} \to^L G(\mathbb{Z})$ where

$$h_\infty := (\eta(-1) \cdot \xi(-1), \tau) \in^L G(\mathbb{Z}).$$

To show (1), we directly we use the fact that the Tate cohomology is a 2-group and τ acts trivially to compute

$$h_\infty^2 = (\eta(-1)\xi(-1)\tau(\eta(-1)\xi(-1)), \tau^2) = (1,1). \tag{8}$$

For (3), note that $\chi \circ \eta : \mathbb{G}_m \to \mathbb{G}_m$ is of the form $z \mapsto z^{\langle \chi, \eta \rangle}$. Therefore we have

$$\chi(h_\infty) := \chi(\eta(-1)\xi(-1)) = \chi(\eta(-1))\chi(\xi(-1)) = (-1)^{\langle \eta, \chi \rangle}(-1)^{\langle \xi, \chi \rangle} = (-1)^{\langle \eta + \varphi_\infty, \chi \rangle}. \tag{9}$$

We will also omit the proof (as Gross does in [Gro99]) of the second condition) however Gross claims that one can prove (2) analogously to how it is proved in [Gro97] for G simply connected.[1] $\square$

4 Unramified Parameters

Let p be a rational prime, unramified in k. Let $\mathfrak{p}$ be a factor of p in k, with corresponding Frobenius in $\mathrm{Gal}(k/\mathbb{Q}_p)$ given by

$$\sigma(\mathfrak{p}) : k \to k \text{ where } x \mapsto x^p \mod \mathfrak{p}.$$

Assume that $K_p := K \cap G(\mathbb{Q}_p) \subset G(\mathbb{Q}_p)$ is hyperspecial maximal compact subgroup (For the notion of a hyperspecial group see the appendix). Define the *spherical Hecke algebra at* p to be the commutative $\mathbb{Z}[1/p]$-algebra of $\mathbb{Z}[1/p]$-valued compactly supported functions on the double coset space $G(\mathbb{Q}_p)//K_p$, under convolution. Explicitly,

$$H_p \;:=\; \Big\{ F : G(\mathbb{Q}_p)//K_p \longrightarrow \mathbb{Z}[1/p] \;\Big|\; F \text{ is compactly supported and } K_p\text{-bi-invariant} \Big\},$$

equipped with the convolution product

$$(F_1 * F_2)(g) \;:=\; \int_{G(\mathbb{Q}_p)} F_1(x)\, F_2(x^{-1}g)\, dx, \qquad g \in G(\mathbb{Q}_p),$$

where the Haar measure dx is normalised so that $\mu(K_p) = 1$. For an $N \subset M(V, K)$ there is a homomorphism of $\mathbb{Z}[1/p]$-algebras $\varphi_p : H_p \to \mathcal{O}_E[1/p]$ by [[Gro97] Prop 8.19 & Eq. 7.5] to which we assign an unramified parameter,

$$(N, \varphi_p) \mapsto h_{\mathfrak{p}} \in \mathrm{C}\ell_{\sigma(\mathfrak{p})}(\mathcal{O}_E[1/p]),$$

denoted $h_{\mathfrak{p}} \in \mathrm{C}\ell_{\sigma(\mathfrak{p})}(\mathcal{O}_E[1/p])$. Since K_p is hyperspecial, the group G is quasi-split over $\mathbb{Q}_p$, and split over the maximal unramified extension of $\mathbb{Q}_p$. Let $\mathcal{G}$ be a model for G over $\mathbb{Z}_p$ with $\underline{G}(\mathbb{Z}_p) = K_p$ and good reduction mod p. Let $\underline{T} \subset \underline{H} \subset \underline{G}$ be a maximal torus contained in a Borel subgroup over $\mathbb{Z}_p$, and let W_s be the Weyl group of the maximal split torus $\underline{T}_s \subset T$. We have an isomorphism given by evaluating the cocharacter at p

$$X_\bullet(\underline{T})^{\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q})} = X_\bullet(\underline{T}_s) \xrightarrow{\sim} \underline{T}(\mathbb{Q}_p)/\underline{T}(\mathbb{Z}_p) \text{ which sends } \lambda \mapsto \lambda(p).$$

Recall $\eta : \underline{T}(\mathbb{Q}_p) \to \mathbb{Q}_p^\times$ we define a homomorphism

$$\eta_{\mathfrak{p}} : \underline{T}(\mathbb{Q}_p)/\underline{T}(\mathbb{Z}_p) \to \mathbb{Z}[1/p]^\times \text{ that sends } \lambda(p) \mapsto p^{-\langle \lambda, \eta \rangle}$$

Let $\underline{U} \subset \underline{B}$ be unipotent radical over $\mathbb{Z}_p$. The Satake transform $\mathcal{S} : H_p \xrightarrow{\sim} H(\underline{T}(\mathbb{Q}_p)//\underline{T}(\mathbb{Z}_p))^{W^{\sigma(\mathfrak{p})}}$

$$S[f(t)] := \eta_{\mathfrak{p}}(t) \int_{U(\mathbb{Q}_p)} f(tu) du,$$

where du is the Haar measure with $\underline{U}(\mathbb{Z}_p)$ normalized to have volume 1. Note that there is an isomorphism $H(\underline{T}(\mathbb{Q}_p)//\underline{T}(\mathbb{Z}_p))^{W^{\sigma(\mathfrak{p})}} \cong \mathbb{Z}[1/p][X_\bullet^{\sigma(\mathfrak{p})}]^{W^{\sigma(\mathfrak{p})}}$. Recall $\mathrm{C}\ell_{\sigma(\mathfrak{p})} := \mathrm{Spec}(\mathbb{Z}[1/p][X_\bullet^{\sigma(\mathfrak{p})}]^{W^{\sigma(\mathfrak{p})}})$, we can compose

$$\mathbb{Z}[1/p][X_\bullet^{\sigma(\mathfrak{p})}]^{W^{\sigma(\mathfrak{p})}} \xrightarrow{\mathcal{S}^{-1}} H_p \xrightarrow{\varphi_p} \mathcal{O}_E[1/p]$$

Therefore choosing a φ_p and taking the composite $\varphi_p \circ \mathcal{S}^{-1}$ gives a homomorphism of $\mathbb{Z}[1/p]$-algebras

$$\mathrm{Hom}(\mathbb{Z}[1/p][X_\bullet^{\sigma(\mathfrak{p})}]^{W^{\sigma(\mathfrak{p})}}, \mathcal{O}_E[1/p]).$$

Recall $\mathrm{C}\ell_{\sigma(\mathfrak{p})} := \mathrm{Spec}(\mathbb{Z}[1/p][X_\bullet^{\sigma(\mathfrak{p})}]^{W^{\sigma(\mathfrak{p})}})$, therefore from the functor-of-points we have a $\mathcal{O}_E[1/p]$-valued point $h_{\mathfrak{p}}(\varphi_p) \in \mathrm{C}\ell_{\sigma(\mathfrak{p})}$ and this point is the *unramified parameter associated to* φ_p.

[1] I tried to do the analogous proof, but don't see how it should work as Gross uses particular facts about the decomposition of $\hat{g}$ for the case he considers. If you know how the general proof goes, feel free to tell me. I would be very interested in understanding how it works! :)

A Algebraic Modular Forms Conventions and Gross' Condition

Let $\hat{\mathbb{Z}} := \varprojlim \mathbb{Z}/n\mathbb{Z}$ and $\hat{\mathbb{Q}} := \hat{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Q}$. Then $\hat{\mathbb{Q}}$ is the $\mathbb{Q}$-algebra of finite adéles, and $\mathbf{A} := \mathbb{R} \times \hat{\mathbb{Q}}$ is the ring of Adéles of $\mathbb{Q}$. Let G be a connected, reductive group over $\mathbb{Q}$ and let S be the maximal split torus in the center of G, with S' denoting the maximal quotient of G which is a split torus. We write $G(\mathbb{Q})$ for the $\mathbb{Q}$-rational points of G, and $G(\mathbf{A}) := G(\mathbb{R}) \times G(\hat{\mathbb{Q}})$ for the group of adélic points. An element $g \in G(\mathbf{A})$ has components $g_\infty \in G(\mathbb{R})$ and $\hat{g} \in G(\hat{\mathbb{Q}})$. We let $G(\mathbb{R})_+$ denote the connected component of the identity in the Lie group $G(\mathbb{R})$ and let $\pi_0(G(\mathbb{R})) := G(\mathbb{R})/G(\mathbb{R})_+$. Let V be an irreducible representation of the algebraic group G over $\mathbb{Q}$.

Gross gives a number of equivalent conditions on G which insure that every subgroup $\Gamma \subset G(\mathbb{Q})$ is finite.

Proposition A.1. *[[Gro99], Prop 1.4] The following conditions are all equivalent:*

1. *Every arithmetic subgroup $\Gamma \subset G(\mathbb{Q})$ is finite.*
2. *$\Gamma = \{e\}$ is an arithmetic subgroup of $G(\mathbb{Q})$.*
3. *$G(\mathbb{Q})$ is a discrete subgroup of the locally compact group $G(\hat{\mathbb{Q}})$.*
4. *$G(\mathbb{Q})$ is a discrete subgroup of the locally compact group $G(\hat{\mathbb{Q}})$ and the quotient space $G(\mathbb{Q}) \setminus G(\hat{\mathbb{Q}})$.*
5. *S is a maximal split torus in G over $\mathbb{R}$.*
6. *The Lie group $G(\mathbb{R})_1 = G(\mathbb{R}) \cap G(\mathbf{A})_1$ is a maximal compact subgroup of $G(\mathbb{R})$.*
7. *For every irreducible representation V of G there is a character $\mu : G \to \mathbb{G}_m$ and a positive definite symmetric bilinear form $\langle -, - \rangle : V \times V \to \mathbb{Q}$ which satisfy*

$$\langle \gamma v, \gamma v' \rangle = \mu(\gamma)\langle v, v' \rangle,$$

   for all $\gamma \in G(\mathbb{Q})$ and $v, v' \in V$.

Assume G satisfies the conditions of the above propositions. Fix an irreducible representation V of G over $\mathbb{Q}$. Let $D := \mathrm{End}_G(V)$, and let $F := Z(\mathrm{End}_G(V))$, be the center of D. Let $\mu : G \to \mathbb{G}_m$ be the character of G determined by V. The $\mathbb{Q}$-vector space of modular forms on G with coefficients in V is the left D-vector space of functions

$$M(V) := \{f : G(\mathbf{A}) \to V | f \text{ is locally constant on } G(\mathbf{A}) \text{ and } f(\gamma g) = \gamma f(g) \text{ for } \gamma \in G(\mathbb{Q})\}.$$

Since each f in $M(V)$ is locally constant, it is constant on the cosets of an open subgroup of the form $G(\mathbb{R})_+ \times K$, where $K \subset G(\hat{\mathbb{Q}})$ is an open compact subgroup. Hence $M(V)$ is the direct limit of the subspaces

$$M(V, K) := \{f : G(\mathbf{A})/(G(\mathbb{R})_+ \times K) \to V | f(\gamma g) = \gamma f(g) \text{ for } \gamma \in G(\mathbb{Q})\}.$$

Let $\Sigma_K := G(\hat{\mathbb{Q}}) \setminus G(\mathbf{A})/(G(\mathbb{R})_+ \times K)$ denote the double coset space for a fixed K.

Proposition A.2 ([Gro99], Prop 4.3)**.** *The space Σ_K is finite, and the D-vector space $M(V, K)$ is finite dimensional.*

We can equip $M(V, K)$ with an inner product. To do this, we first by the above proposition choose the inner product in (7). Then fix representative $\{g_\alpha\}$ for the classes in Σ_K. For each α define the arithmetic subgroup

$$\Gamma_\alpha := G(\mathbb{Q}) \cap g_\alpha(G(\mathbb{R})_+ \times K)g_\alpha^{-1} \subset G(\mathbb{Q}).$$

Second, we note that $\mu : G \to \mathbb{G}_m$ takes positive values on $G(\mathbb{R})$. Then we define $\mu_{\mathbf{A}} : G(\mathbf{A}) \to \mathbb{Q}^\times$ to be the composition of μ with projection onto the first factor (note this takes values in $\mathbb{Q}_+^\times$). Therefore, for modular forms $f, f' \in M(V, K)$ we define the inner product $\langle -, - \rangle_{M(V,K)} : M(V, K) \times M(V, K) \to \mathbb{Q}$ by the formula

$$\langle f, f' \rangle_{M(V,K)} := \sum_\alpha \frac{1}{|\Gamma_\alpha| \mu_{\mathbf{A}}(g_\alpha)} \langle f(g_\alpha), f'(g_\alpha) \rangle.$$

If $g_\infty \in G(\mathbb{R})$ we can define the linear map $T : M(V,K) \to M(V,K)$, given by

$$T_{g_\infty}(f(g)) := f(gg_\infty),$$

which defines an automorphism of M, which depends only on the image of g_∞ in $G(\mathbb{R})/G(\mathbb{R})^+$. This gives an action of the 2-group $\pi_0(G(\mathbb{R}))$ on M, via self adjoint operators:

$$T'_{g_\infty} = T_{g_\infty^{-1}} = T(g_\infty).$$

A more interesting family of operators comes from the Hecke algebra H_K of all locally constant, compactly supported functions $F : G(\hat{\mathbb{Q}}) \to \mathbb{Q}$ which is K-bi-invariant. Write

$$K\hat{g}K = \coprod_{i\in I} \hat{g}_i K$$

as a disjoint union of a finite number of single cosets; the number is finite as $K\hat{g}K$ is compact and each $\hat{g}_iK$ is open. Now define, the linear operator $T : M(V,K) \to M(V,K)$ by

$$T_{\hat{g}}(f(g)) := \sum f(gg_iK),$$

then $T_{\hat{g}}f$ is still right K-invariant, as left multiplication by K permutes the right cosets $\hat{g}_iK$. Hence $T_{\hat{g}}(f) \in M(V,K)$. Extending linearly to H_K gives a homomorphism of $\mathbb{Q}$ algebras $H_K \to \mathrm{End}_{\mathbb{Q}}(M(V,K))$.

Let $M(V,K)$ be equipped with the inner product $\langle -, - \rangle_{M(V,K)}$ of the previous section, and let $A \subset \mathrm{End}(M)$ be the $\mathbb{Q}$-algebra defined as the span of the operators in $D = \mathrm{End}_G(V)$, as well as the automorphisms T_{g_∞}, and the endomorphisms $T_{\hat{g}}$. M is a semisimple A-module. Let $N \subset M$ be a simple A-submodule. Then $\mathrm{End}_A(N)$ is a finite division algebra, of finite dimension over $\mathbb{Q}$, and $E := Z(\mathrm{End}_A(N))$ is a number field containing $F := Z(\mathrm{End}_G(V))$. From the above we have a homomorphism of commutative $\mathbb{Q}$-algebras.

B Hyperspecial Subgroups & Models

Let G be an affine algebraic group over a non-Archimedean local field F and let $G \to \mathrm{GL}_n$ be a faithful representation. Let B be a Dedekind domain with $\mathrm{Frac}(B) = F$. For example if $B = \mathbb{Z}_p$ for a fixed prime p then $\mathrm{Frac}(\mathbb{Z}_p) = \mathbb{Q}_p$. Let X be an affine scheme over B, then the *generic fiber* of X is the fiber X_F. If F is local and $B = \mathcal{O}_F$ then B has a unique prime ideal $\mathfrak{p}$, we let $\kappa(x) = A/\mathfrak{p}$ be the residue field at $x = [\mathfrak{p}]$. The scheme $X_{\kappa(x)}$ is the *special fiber* of X.

Example B.1. If $F = \mathbb{Q}_p$ then $B = \mathbb{Z}_p$ and so $\mathfrak{p} = (p)$ with residue field $\kappa(x) = \mathbb{Z}_p/(p) \cong \mathbb{F}_p$. The generic fiber of X is the fiber $X_{\mathbb{Q}_p}$ and the special fiber is the fiber $X_{\mathbb{F}_p}$.

Given a scheme Y of finite type over F we say that we can find a model for Y if we can find another scheme $\mathcal{Y}$ such that Y is the generic fiber of $\mathcal{Y}$, that is $\mathcal{Y}_F = \mathrm{Spec}(F) \times_{\mathrm{Spec}(B)} \mathcal{Y} \cong Y$.

Definition B.2. A *model* of $\mathcal{Y}$ over a Dedekind domain B is an affine scheme of finite type over B isomorphic to $\mathrm{Spec}(A)$ where $A \subseteq \mathscr{O}_Y$ is a B-algebra and $A \otimes_B F \cong \mathscr{O}_Y$.

When G is a group scheme we always assume that models $\mathcal{G}$ of G are again group schemes and that the generic fiber of the multiplication map $\mathcal{G} \times \mathcal{G} \to \mathcal{G}$ and the inversion map $\mathcal{G} \to \mathcal{G}$ are the multiplication map and inversion map on G, respectively. That is, we assume the group scheme structures on $\mathcal{G}$ and G are compatible. We can view GL_n as an affine group scheme over B, and it is a model of the generic fiber $(\mathrm{GL}_n)_F$.

Definition B.3. Let F be a non-Archimedean local field. A reductive group G over F is *unramified* if it is quasi-split and there is a finite degree unramified extension E/F such that G_E is split.

Theorem B.4 ([GH24], Thm 2.4.3)**.** *A reductive group G over F is unramified if and only if there exists a model $\mathcal{G}$ of G over $\mathcal{O}_F$ such that the special fiber of $\mathcal{G}$ is reductive.*

Proposition B.5 ([GH24], Thm 2.4.3)**.** *If $\mathcal{G}$ is a model of G over $\mathcal{O}_F$ such that the special fiber of $\mathcal{G}$ is reductive then $\mathcal{G}(\mathcal{O}_F)$ is a maximal compact subgroup of $G(F)$.*

Definition B.6. A subgroup of $G(F)$ of the form $\mathcal{G}(\mathcal{O}_F)$ for a model $\mathcal{G}$ of G over $\mathcal{O}_F$ with reductive special fiber is called a *hyperspecial group.*

Example B.7. When $G = \mathrm{GL}_n$, it is clear that $\mathrm{GL}_n(\mathbb{Z}_p)$ is a hyperspecial subgroup of $\mathrm{GL}_n(\mathbb{Q}_p)$. It turns out that all maximal compact subgroups of $\mathrm{GL}_n(\mathbb{Q}_p)$ are conjugate to $\mathrm{GL}_n(\mathbb{Z}_p)$.

The following theorem still holds even when G is ramified.

Theorem B.8 ([GH24], Thm 2.4.5)**.** *If G is a reductive group over a non-Archimedean local field F then*

1. *Every compact subgroup of $G(F)$ is contained in a maximal compact subgroup.*
2. *Maximal compact subgroups of $G(F)$ are open.*
3. *Every maximal compact subgroup $K \leq G(F)$ is of the form $\mathcal{G}(\mathcal{O}_F)$ where $\mathcal{G}$ is a smooth model of G.*

References

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