

# Quantum Cohomology and the Gromov-Witten Theory of $\mathbb{P}^2$

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## 1 Introduction

We would like to know the answer to the following question:

**Question 1.1.** *What is the number of rational curves  $N_d$  of degree  $d$  that pass through  $(3d - 1)$  points which are in general position in the plane?*

One way to proceed is to get our hands dirty and see if we can spot some kind of pattern. Well, the case when  $n = 1$  reduces to a basic question that one first encounters early in school, namely

**Question 1.2.** *How many lines pass between two points in the plane?*

The answer is of course 1. So the number of degree 1 rational curves that pass through 2 marked points which are in general position is 1. The next iteration of the question would be

**Question 1.3.** *How many conics pass between five points in the plane?*

Once again the answer is classically known to be 1. The answer to this question has been known at least since the time of Apollonius ( $\sim 200\text{B.C.}$ ). We can show this by construction, take points  $(x_i, y_i)$  with  $1 \leq i \leq 5$  in general position and consider the determinant

$$f(X, Y) = \begin{vmatrix} 1 & X & Y & X^2 & Y^2 & XY \\ 1 & x_1 & y_1 & x_1^2 & y_1^2 & x_1 y_1 \\ 1 & x_2 & y_2 & x_2^2 & y_2^2 & x_2 y_2 \\ 1 & x_3 & y_3 & x_3^2 & y_3^2 & x_3 y_3 \\ 1 & x_4 & y_4 & x_4^2 & y_4^2 & x_4 y_4 \\ 1 & x_5 & y_5 & x_5^2 & y_5^2 & x_5 y_5 \end{vmatrix}. \quad (1)$$

This determinant is a polynomial of degree at most 2 and  $f(x_i, y_i) = 0$  for  $i = 1, \dots, 5$  is clear since it would yield two identical rows.

**Question 1.4.** *How many cubics pass between 8 points in the plane?*

This is a much more complicated question whose answer turns out to be 12, as computed by Chasles/Steiner in the early 19th century. A sketch of the argument goes as follows:

Consider a line with coordinate  $t$  in the space of cubics,  $f(x, y) + tg(x, y) = 0$  where  $\deg(f) = \deg(g) = 3$ . Regarding this family over  $\mathbb{P}^1_{[t]}$  can compute the Euler characteristic of the total space  $\mathcal{S}$  in two ways. By Bezout's theorem  $f$  and  $g$  intersect at nine points, giving rise to nine lines. If  $P \in \mathbb{P}^2$  is not one of those nine points then there is one cubic in the family that contains such a point. The total space is recognized as the blow-up of  $\mathbb{P}^2$  at 9 points, and hence has Euler characteristic  $\chi(\mathcal{S}) = \chi(\mathbb{P}^2 \setminus 9\mathbb{P}^0 \sqcup 9\mathbb{P}^1) = 3 - 9 \cdot 1 + 9 \cdot 2 = 12$ . Considering the family fiberwise, the generic fiber is a smooth cubic (torus) and the nodal cubic has  $\chi(C_{\text{nodal}}) = 1$  therefore  $\chi(\mathcal{S}) = n_{\text{nodal}}\chi(C_{\text{nodal}}) + n_{\text{smooth}}\chi(C_{\text{smooth}}) = n_{\text{nodal}}$ , equating then gives the desired result.

Summarising we have found that  $N_1 = N_2 = 1$  and  $N_3 = 12$ . It wasn't until the late 19th century that  $N_4 = 620$  was computed by Schubert/Zeuthen, and it wasn't until the mid 20th century  $N_5 = 87304$ . So our pattern is therefore given by

$$1, 1, 12, 620, 87304, \dots$$

and it is not all clear what  $N_6$  should be, and that was the state of things... until Maxim Kontsevich discovered the general recursive formula finding  $N_d$  for all  $d \geq 0$  in 1993. Obtaining this recursion will be the subject of these notes on the way we will introduce build on the basics of Gromov-Witten theory introduced thus far in the seminar and in particular give motivation for the introduction of quantum cohomology.

## 2 Quantum Cohomology

Let  $X$  be a smooth, projective, homogeneous variety. Begin by observing that on  $H^*(X)$  there is an associative cup product and a bilinear non-degenerate pairing  $\langle -, - \rangle : H^*(X) \times H^*(X) \rightarrow H^*(X)$  given by

$$\langle \alpha, \beta \rangle := \int \alpha \cup \beta. \quad (2)$$

The cup product and the non-degenerate bilinear pairing  $\langle -, - \rangle$  give  $H^*(X)$  the structure of a Frobenius algebra, with unit the fundamental class  $1_X \in H^0(X)$ . Now we generalize this structure by defining the multiplication:

$$\alpha_1 *_{\beta} \alpha_2 = \text{ev}_{3*}(\text{ev}_1^*(\alpha_1) \cup \text{ev}_2^*(\alpha_2)), \quad (3)$$

where the moduli space considered is  $\overline{\mathcal{M}}_{0,3}(X, \beta)$ . We introduce a formal parameter  $q^{\beta}$  for each element  $\beta \in H_2(X; \mathbb{Z})_+$  with rule  $q^{\beta_1} q^{\beta_2} = q^{\beta_1 + \beta_2}$ . Thus we define

$$\alpha_1 * \alpha_2 = \sum_{\beta \in H_2(X)} (\alpha_1 *_{\beta} \alpha_2) q^{\beta} \quad (4)$$

and extend this product  $\mathbb{Q}[[H_2(X; \mathbb{Z})_+]]$ -linearly to  $H^*(X) \otimes \mathbb{Q}[[H_2(X; \mathbb{Z})_+]]$ . The coefficients are the genus-zero, three point Gromov-Witten invariants. That is, if  $\{T_i\}_{i=0}^m$  is a basis for  $H^*(X)$  then we can write the small quantum product as

$$\alpha_1 * \alpha_2 = \sum_{\beta \in H_2(X)} \langle \alpha_1, \alpha_2, T_i \rangle_{0,3,\beta} g^{ij} T_j q^{\beta} \text{ where } g_{ef} = \int_X T_e \cup T_f, \text{ and } g^{ef} := (g_{ef})^{-1}. \quad (5)$$

The resulting structure on this vector space  $QH_s^*(X)$  is called the **small quantum cohomology** of  $X$ .

**Theorem 2.1.** *The small quantum cohomology  $QH_s^*(X)$  is a Frobenius algebra with the same unit of  $H^*(X)$ .*

**Example 2.2.**  $QH_s^*(\mathbb{P}^n) \cong \mathbb{Q}[H][[q]]/(H^{n+1} - q)$  where  $H$  is the class of a hyperplane in  $H^2(\mathbb{P}^n)$ .

The small quantum cohomology ring of  $\mathbb{P}^n$  is a deformation of the usual cohomology ring that contains the 3-point information. Therefore the gromov witten numbers  $N_d$  do not appear in the small quantum cohomology of  $\mathbb{P}^2$ . If we want to go beyond the information of 3-point functions and consider  $n$ -point functions, we need to study an object known as the big quantum cohomology associated to  $X$ . We define the Gromov-Witten potential:

$$\Phi(\gamma) := \sum_{n \geq 3} \sum_{\beta} \frac{\langle \gamma^n \rangle_{0,n,\beta}}{n!} q^{\beta} \quad (6)$$

Here we use the notation that  $\gamma^n = \gamma, \dots, \gamma$ ,  $n$ -times. Let  $\gamma = \sum_i y_i T_i$ , we obtain a formal power series in the  $y_i$  given by:

$$\Phi(y_0, \dots, y_m) = \sum_{\substack{n_0 + \dots + n_m = n \\ \beta \in H_2(X; \mathbb{Z})}} \langle T_0^{n_0}, \dots, T_m^{n_m} \rangle_{0,n,\beta} \frac{y_0^{n_0}}{n_0!} \dots \frac{y_m^{n_m}}{n_m!}. \quad (7)$$

*Remark 2.3.* By [[FP97], Lem 15] we have that for any  $n$ , there are only finitely many  $\beta$  such that  $\langle \gamma^n \rangle_{0,n,\beta}$  is nonzero.

**Example 2.4.** The Gromov-Witten potential for  $\mathbb{P}^2$  with  $\gamma = y_0 1 + y_1 H + y_2 H^2 \in H^*(\mathbb{P}^2)$  is given by

$$\Phi(\gamma) = \frac{1}{2}(y_0 y_1^2 + y_0^2 y_2) + \sum_{d=1}^{\infty} N_d \frac{y_2^{3d-1}}{(3d-1)!} e^{dy_1} \quad (8)$$

It will be convenient to define the following bit of notation  $\Phi_{ijk} := \partial_i \partial_j \partial_k \Phi$  with  $0 \leq i, j, k \leq m$ .

**Definition 2.5.** Let  $X$  be a smooth projective variety and let  $\{T_i\}_i$  be a basis for  $H^*(X)$ . The big quantum product on  $H^*(X)[[y_0, \dots, y_m, q]]$  is defined on the basis  $T_0, \dots, T_m$  as

$$T_i *_b T_j := \sum_{e,f} \Phi_{ije} T^e, \text{ where } T^e = \sum_{f=1} g^{ef} T_f \quad (9)$$

where the product is then extended  $\mathbb{Q}[[y_0, \dots, y_m]]$ -linearly to a product on the  $\mathbb{Q}[[y_0, \dots, y_m]]$ -module  $H^*(X) \otimes_{\mathbb{Z}} \mathbb{Q}[[y_0, \dots, y_m]]$  making it a  $\mathbb{Q}[[y_0, \dots, y_m]]$ -algebra.

*Remark 2.6.* One needs to check that the product defined above is indeed well-defined. That is, given a basis  $T'_0, \dots, T'_m$ , there is a linear change of coordinates from  $H^*(X) \otimes \mathbb{Q}[[y_0, \dots, y_m]]$  to  $H^*(X) \otimes \mathbb{Q}[[y'_0, \dots, y'_m]]$  identifying the two product structures.

**Example 2.7.** The big quantum product for  $\mathbb{P}^2$  with basis  $T_0 = 1, T_1 = [L]$  and  $T_2 = [pt]$  for  $H^*(\mathbb{P}^2)$  is given by

$$T^0 = T_2 = [pt] \quad T^1 = T_1 = [L] \quad \text{and} \quad T^2 = T_0 = 1 = [\mathbb{P}^2]. \quad (10)$$

Using the potential for  $\mathbb{P}^2$  defined above, we find

$$[L] * [L] = \sum_{k=0}^2 \Phi_k T^k = \Phi_{110}[pt] + \Phi_{111}[L] + \Phi_{112}[\mathbb{P}^2] = \left( \int_{\mathbb{P}^2} [L] \cup [L] \right) [pt] + \Phi_{111}[L] + \Phi_{112}[\mathbb{P}^2]. \quad (11)$$

Using  $\Phi_{110} = 1 = \int_{\mathbb{P}^2} [L] \cup [L]$  we see that we now have additional higher order terms to the classical cup product.

It's natural to ask what properties such a multiplication has, in particular whether it's associative or commutative. Well, commutativity is clear as the product is symmetric in the subscripts i.e  $\phi_{ijk} = \Phi_{jik}$

$$T_j * T_i = \sum_k \Phi_{jik} T^k = \sum_k \Phi_{ijk} T^k = T_i * T_j. \quad (12)$$

Associativity enforces that

$$(T_i * T_j) * T_k = \sum_{e,f} \Phi_{ije} g^{ef} T_f * T_k = \sum_{e,f} \sum_{c,d} \Phi_{ije} g^{ef} \Phi_{fkc} g^{cd} T_d \quad (13)$$

$$T_i * (T_j * T_k) = \sum_{e,f} \Phi_{jke} g^{ef} T_i * T_f = \sum_{e,f} \sum_{c,d} \Phi_{jke} g^{ef} \Phi_{ifc} g^{cd} T_d, \quad (14)$$

and since the matrix  $g^{cd}$  is non-singular the equality of  $(T_i * T_j) * T_k = T_i * (T_j * T_k)$  is equivalent to the following non-linear partial differential equation

$$\sum_{e,f} \Phi_{ije} g^{ef} \Phi_{fkl} = \sum_{e,f} \Phi_{jke} g^{ef} \Phi_{ifl}, \quad (15)$$

called the Witten-Dijkgraaf-Verlinde-Verline (WDVV) equation. A formal solution to this nonlinear PDE is called a Gromov-Witten potential. One can also show  $T_0 = 1$  is a unit for the  $*$ -multiplication. Altogether we have

**Definition 2.8.** The big quantum product endows  $H^*(X) \otimes \mathbb{Q}[[y_0, \dots, y_m]]$  with a multiplication. The ring

$$QH^*(X) := H^*(X) \otimes \mathbb{Q}[[y_0, \dots, y_m]], \quad (16)$$

is called the **big quantum cohomology ring** of  $X$ .

**Theorem 2.9** ([FP97], Thm 4). *The big quantum cohomology ring  $QH^*(X)$  is a commutative, associative  $\mathbb{Q}[[y_0, \dots, y_m]]$ -algebra with unit  $T_0$ , i.e a Frobenius algebra.*

**Example 2.10.** When  $X = \mathbb{P}^2$  the big quantum cohomology ring of  $X$  has the identification

$$QH^*(\mathbb{P}^2) \cong \frac{\mathbb{Q}[[y_0, y_1, y_2]][Z]}{Z^3 - \Phi_{111}Z^2 - 2\Phi_{112}Z - \Phi_{122}}, \quad (17)$$

which we can compare with the cohomology ring  $H^*(\mathbb{P}^2) \cong \mathbb{Q}[Z]/Z^3$  with basis  $\{1, Z, Z^2\}$ .

### 3 The Gromov-Witten Theory of $\mathbb{P}^2$

Recall the Gromov-Witten potential for  $\mathbb{P}^2$ , where  $\gamma = y_0 1 + y_1 H + y_2 H^2 \in H^*(X)$  is given by

$$\Phi(\gamma) = \frac{1}{2}(y_0^2 y_2 + y_0 y_1^2) + \sum_{d=1}^{\infty} N_d \frac{e^{dy_1} y_2^{3d-1}}{(3d-1)!} q^d. \quad (18)$$

Using the associativity of the big quantum cohomology product, one can derive the WDVV equation for  $\mathbb{P}^2$ , which has the form

$$\Phi_{222} = \Phi_{112}^2 - \Phi_{111} \Phi_{122}. \quad (19)$$

Computing the term on the left hand side gives

$$\Phi_{222} = \sum_{d=1}^{\infty} N_d \frac{e^{dy_1} y_2^{3d-4}}{(3d-4)!} q^d. \quad (20)$$

On the right hand side we compute

$$\Phi_{112}^2 = \left( \sum_{d=1}^{\infty} d^2 N_d \frac{e^{dy_1} y_2^{3d-2}}{(3d-2)!} q^d \right)^2 \quad (21)$$

$$= \sum_{d=1}^{\infty} \sum_{d=d_1+d_2} d_1^2 d_2^2 N_{d_1} N_{d_2} \frac{e^{dy_1} y_2^{3d-4}}{(3d_1-2)!(3d_2-2)!} q^d, \quad (22)$$

and

$$\Phi_{111} \Phi_{122} = \partial_1^3 \Phi \cdot \partial_1 \partial_2^2 \Phi \quad (23)$$

$$= \left( \sum_{d=1}^{\infty} d^3 N_d \frac{e^{dy_1} y_2^{3d-1}}{(3d-1)!} q^d \right) \left( \sum_{d=1}^{\infty} d N_d \frac{e^{dy_1} y_2^{3d-3}}{(3d-3)!} q^d \right) \quad (24)$$

$$= \sum_{d=1}^{\infty} \sum_{d=d_1+d_2} d_1^3 d_2 N_{d_1} N_{d_2} \frac{e^{dy_1} y_2^{3d-3}}{(3d_1-1)!(3d_2-3)!}. \quad (25)$$

Equating coefficients of both sides for a fixed value of  $d$  we obtain

$$\frac{N_d}{(3d-4)!} = \sum_{d=d_1+d_2} N_{d_1} N_{d_2} \left[ \frac{d_1^2 d_2^2}{(3d_1-1)!(3d_2-2)!} + \frac{d_1^3 d_2}{(3d_1-1)!(3d_2-3)!} \right]. \quad (26)$$

Rearranging and using  $d_2 = d - d_1$ , we derive Kontsevich's formula for the Gromov-Witten invariants of  $\mathbb{P}^2$

**Theorem 3.1** (Kontsevich 1993). *The number of rational curves  $N_d$  of degree  $d$  that pass through  $(3d-1)$  points which are in general position in the  $\mathbb{P}^2$ , satisfies the recursion relation*

$$N_d = \sum_{d=d_1+d_2} N_{d_1} N_{d_2} \left[ d_1^2 d_2^2 \binom{3d-4}{3d_1-2} - d_1 d_2 \binom{3d-4}{3d_1-1} \right]. \quad (27)$$

where  $N_1 = 1$ .

Computing  $N_d$  for  $d = 1, \dots, 8$  using Kontsevich's formula gives

$$N_1 = 1, \quad N_2 = 1, \quad N_3 = 12, \quad N_4 = 620, \quad N_5 = 87,304,$$

$$N_6 = 26,312,976, \quad N_7 = 14,616,808,192,$$

$$N_8 = 13,525,751,027,392.$$

## References

[FP97] W. Fulton and R. Pandharipande. Notes on stable maps and quantum cohomology, 1997.