Dg Categories and Functoriality of the Mapping Cone

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1 Introduction

The introduction of higher category theory and higher algebra have led to significant progress in our understanding of long standing open problems in algebraic geometry and topology. In particular ∞-categories in algebraic geometry allow us to track higher coherences giving us a more robust intersection theory for algebro-geometric objects which allows us to deal with the geometry of non-transversal intersections. Adding in higher coherences also allows us to resolve issues in the theory of derived categories of coherent sheaves, specifically in fixing the functoriality of the mapping cone. One unfortunate drawback of using the higher categorical approach standard is that it is often too abstract or formal for one to be able to derive anything explicit for concrete examples. This problem is one of many reasons why one might be interested in learning about dg categories. There is a phrase often repeated in the theory of dg categories:

A dg category over k where k is a field of characteristic 0 is equivalent to a stable k-linear infinity category.

The formal statement and proof of this folklore result feature in [[Coh16], Cor 5.5].

Throughout these notes R is a commutative ring and k is a field of characteristic 0. Where appropriate we adopt the convention of labelling the degree under the arrow of a morphism.[1] The content of these notes draws primarily from the exposition given in the unpublished notes [Dav21] and the book [Tab15].

Definition 1.1. A *graded category over R* is a category $\mathbf{A}$ equipped with a graded R-module structure

$$\mathrm{Hom}_{\mathbf{A}}(A,B) = \bigoplus_{n\in\mathbb{Z}} \mathrm{Hom}^n_{\mathbf{A}}(A,B),$$

on $\mathrm{Hom}_{\mathbf{A}}(A,B)$ for every $A, B \in \mathbf{A}$, such that

1. composition maps are graded and bilinear;
2. identities have degree 0, i.e $\mathrm{id}_{\mathbf{A}} \in \mathrm{Hom}^0_{\mathbf{A}}(A,A)$ for all $A \in \mathbf{A}$.

Definition 1.2. A *differential graded category* (or *dg category* for short) is a graded category $\mathbf{A}$ over R equipped with a k-linear degree 1 differential $d = (d^i_{A,B})_{i\in\mathbb{Z}}$ where

$$d^i_{A,B} : \mathrm{Hom}^i_{\mathbf{A}}(A,B) \to \mathrm{Hom}^{i+1}_{\mathbf{A}}(A,B)$$

for all A and B in $\mathbf{A}$, such that

1. $d^{i+1} \circ d^i = 0$;
2. $d^0_{A,A}(\mathrm{id}_A) = 0$ for all $A \in \mathbf{A}$;
3. $d^{m+n}_{E,G}(g \circ f) = d^n_{F,G}(g) \circ f + (-1)^n g \circ d^m_{E,F}(f)$ for $f \in \mathrm{Hom}^m_{\mathbf{A}}(A,B)$ and $g \in \mathrm{Hom}^n_{\mathbf{A}}(B,C)$.

If in addition the objects of $\mathbf{A}$ form a set, we say that $\mathbf{A}$ is a *small* dg category.

Remark 1.3. Alternatively, a dg category is a category enriched over the category of complexes of R-modules.

Remark 1.4. Remember that a complex of R-modules is different language for a dg R-module.

Definition 1.5. Let $\mathbf{A}$ be a dg category over k. A morphism $f : A \to B$ in $\mathbf{A}$ is called a *dg morphism*[2] if

1. the morphism $f : A \to B$ is degree 0, i.e., $f \in \mathrm{Hom}^0(A,B)$;

[1] The author hopes introducing this convention makes the exposition more concise.
[2] This definition was introduced in [Dav21] and is nonstandard.

f is a closed morphism, $df = 0$.

Definition 1.6. A dg morphism that is an isomorphism is called a *dg isomorphism*.

Definition 1.7. If $f : A \to B$ and $g : A \to B$ are degree n morphisms then a *homotopy* from f to g is a morphism $h \in \mathrm{Hom}^{n-1}(A, B)$ such that $dh = g - f$.

Below is a useful diagram (found in [Dav21]) used to keep track of the different morphisms in $\mathbf{A}$:

$$\begin{tikzcd}
\{\text{All morphisms}\} & \{\text{Closed Morphisms}\} \arrow[l, hook'] \arrow[r, hook] & \frac{\{\text{Closed morphisms}\}}{\text{Homotopy}} \\
\{\text{Deg 0 Morphisms}\} \arrow[u, hook] & \{\text{dg morphisms}\} \arrow[l, hook'] \arrow[r, hook] \arrow[u, hook] & \frac{\{\text{dg morphisms}\}}{\text{Homotopy}} \arrow[u, hook]
\end{tikzcd}$$

Example 1.8. Every dg k-algebra A is a gives rise to a dg category $\mathbf{A}$ by setting

$$\mathrm{Obj}(\mathbf{A}) := \{*\} \text{ and } \mathrm{Hom}_{\mathbf{A}}(*, *) = A.$$

Example 1.9. Let $C(R) := \mathbf{Kom}(R\text{-}\mathbf{Mod})$, we define the dg category $C_{\mathrm{dg}}(R)$ by

$$\mathrm{Obj}(\mathcal{C}_{\mathrm{dg}}(R)) := \mathrm{Obj}(\mathcal{C}(R)), \text{ and } \mathrm{Hom}_{\mathcal{C}_{\mathrm{dg}}}(A, B)^n := \prod_{i \in \mathbb{Z}} \mathrm{Hom}(A^i, B^{i+n}),$$

where the differential associated to the above complex is given by

$$d^n((f^i)_{i\in\mathbb{Z}}) := (d_B \circ f^i - (-1)^n f^{i+1} \circ d_A)_{i\in\mathbb{Z}}.$$

Remark 1.10. This example can be extended to $\mathcal{C}_{\mathrm{dg}}(\mathcal{A})$ where $\mathcal{A}$ is any k-linear category.

Example 1.11. Let $\mathbf{A}$ be a dg category then the opposite category $\mathbf{A}^{\mathrm{op}}$ naturally has a dg structure where we just set

$$\mathrm{Hom}_{\mathbf{A}^{\mathrm{op}}}(x, y) := \mathrm{Hom}_{\mathbf{A}}(x, y),$$

for every pair of objects $x, y \in \mathbf{A}$.

Given a dg category $\mathbf{A}$ we can always form the following categories: The category $Z^0(\mathbf{A})$ called the *zero cocycle category of* $\mathbf{A}$ is defined by taking

$$\mathrm{Obj}(Z^0(\mathbf{A})) := \mathrm{Obj}(\mathbf{A});$$

$$\mathrm{Hom}_{Z^0(\mathbf{A})}(A, B) := Z^0(\mathrm{Hom}_{\mathbf{A}}(A, B)),$$

where the functor Z^0 is the 0-cocycles functor. This essentially discards the higher coherences in our category.

Example 1.12. If $\mathbf{A} = \mathcal{C}_{\mathrm{dg}}(k)$ then $Z^0(\mathcal{C}_{\mathrm{dg}}(k)) = \mathcal{C}(k)$, that is taking the 0-cycles of a dg-category gives the abelian category of complexes of k-modules.

The *homotopy category of* $\mathbf{A}$, denoted $\mathrm{Ho}(\mathbf{A})$ is obtained by setting

$$\mathrm{Obj}(\mathrm{Ho}(\mathbf{A})) := \mathrm{Obj}(\mathbf{A});$$

$$\mathrm{Hom}_{\mathrm{Ho}(\mathbf{A})}(A, B) := H^0(\mathrm{Hom}_{\mathbf{A}}(A, B)).$$

Remark 1.13. The categories $\mathrm{Ho}(\mathbf{A})$ and $\mathcal{C}_{\mathrm{dg}}(\mathbf{A})$ agree up to homotopy.

Lastly, given $\mathbf{A}$ we can form the *graded homotopy category* $\mathrm{Ho}^\bullet(\mathbf{A})$ in the same way as above except by taking

$$\mathrm{Hom}_{\mathrm{Ho}^\bullet(A)}(A, B) := H^\bullet(\mathrm{Hom}_{\mathbf{A}}(A, B)).$$

Definition 1.14. A functor $F : \mathbf{A} \to \mathbf{B}$ is a *dg functor* if

$$F_{A,B} : \mathrm{Hom}_{\mathbf{A}}(A, B) \to \mathrm{Hom}_{\mathbf{B}}(F(A), F(B)),$$

is a morphism of complexes for every pair of objects $A, B \in \mathbf{A}$.

Definition 1.15. A dg functor $F : \mathbf{A} \to \mathbf{B}$ is *fully faithful*[3] if every morphism is a quasi-isomorphism of complexes.

Definition 1.16. A dg functor $F : \mathbf{A} \to \mathbf{B}$ is *essentially surjective* if every object $B \in \mathbf{B}$ is homotopy equivalent to an object of the form $F(A)$.

Definition 1.17. A dg functor $F : \mathbf{A} \to \mathbf{B}$ is a quasi-equivalence if it is fully faithful and essentially surjective.

Two dg categories $\mathbf{A}$ and $\mathbf{B}$ are *quasi-equivalent dg categories* if there exists a chain of quasi-isomorphisms relating them

$$\begin{tikzcd}
\mathbf{A} \arrow[dr, "\sim"] & & \mathbf{C}_2 \arrow[dl, "\sim"'] \arrow[dr, "\sim"] & & \cdots \arrow[dl, "\sim"'] \arrow[dr, "\sim"] & & \mathbf{C}_n \arrow[dl, "\sim"'] \arrow[dr, "\sim"] & \\
& \mathbf{C}_1 & & \mathbf{C}_3 & & \mathbf{C}_{n-1} & & \mathbf{B}
\end{tikzcd}$$

The quasi-equivalence relation is but one that is studied with respect to dg-categories. Morita equivalence and pretriangulated equivalence are other equivalence relations on $\mathrm{dgcat}(k)$ that one considers, especially relevant in the study of noncommutative algebraic goemtry.

Definition 1.18. Let $\mathbf{A}$ be a dg category. A right dg $\mathbf{A}$-module is a dg functor $M : \mathbf{A}^{\mathrm{op}} \to \mathcal{C}_{\mathrm{dg}}(R)$.

When one does algebraic geometry from a functorial point of view it is common to define modules of rings in a similar way, and so this definition can be seen as a natural extension of that one. In any case we have an action

$$M(B) \times \mathrm{Hom}_{\mathbf{A}}(A, B) \to M(A) \text{ where } (y, f) \mapsto y \cdot f,$$

with $y \cdot f := M_{B,A}(f)(y) \in M(A)$.

Let $\mathcal{C}(\mathbf{A})$ denote the category of right dg $\mathbf{A}$-modules. The category $\mathcal{C}(\mathbf{A})$ has a projective Quillen mdoel structure where weak equivalences are given by quasiisomorphisms and fibrations are given by surjections. We can invert quasi-isomorphisms to get the derived category of $\mathbf{A}$

$$\mathrm{D}(\mathbf{A}) := \mathrm{Ho}(\mathcal{C}(\mathbf{A})) = \mathcal{C}(\mathbf{A})[\mathrm{qiso}^{-1}].$$

2 Homological Algebra and Mapping Cones

Lets begin by looking at some of our favourite homological algebra constructions in a general dg category. Our objects of $\mathbf{A}$ are not complexes, so we have the definition of shifting of an object need to be modified appropriately.

Definition 2.1. Fix an $A \in \mathbf{A}$, for $n \in \mathbb{Z}$ an object $B \in \mathbf{A}$ is called the *shift* of A by n if it is equipped with closed morphisms $\varepsilon^n : A \to B$ of degree $-n$ and $\varepsilon^{-n} : B \to A$ of degree n such that

$$\varepsilon^n \circ \varepsilon^{-n} = \mathrm{id}_B \text{ and } \epsilon^{-n} \circ \epsilon^n = \mathrm{id}_A .$$

If a shift B of A by n exists we denote it by $A[n] := B$.

Proposition 2.2. *If $A[n]$ exists for every $A \in \mathbf{A}$ then there is a dg functor $[n] : \mathbf{A} \to \mathbf{A}$ it sends objects $A \mapsto A[n]$ and morphisms $f : A \to B$ of degree m to*

$$f[n] := (-1)^{mn} \varepsilon^n \circ f \circ \varepsilon^{-n}.$$

[3]Strongly full faithful is the term used what fully faithful ordinarily means in category theory.

Proof. The identity morphisms is preserved with $[n]$:

$$\mathrm{id}_A[n] := (-1)^{0\cdot n}\varepsilon_A^n \circ \mathrm{id}_A \circ \varepsilon^{-n}A = \varepsilon_A^n \circ \epsilon_A^{-n} = \mathrm{id}_{A[n]}\,.$$

Let $f \in \mathrm{Hom}^m(A,B)$ and $g \in \mathrm{Hom}^k(B,C)$ then

$$g[n]\circ f[n] := (-1)^{kn}\varepsilon_C^n \circ g \circ \varepsilon_B^n \circ (-1)^{mn}\varepsilon_B^n \circ f \circ \varepsilon_A^{-n} = (-1)^{(k+m)n}\varepsilon_C^n \circ gf \circ \varepsilon_A^{-n} = (g\circ f)[n].$$

So $[n]$ is a functor. To prove it is a dg functor we show it is compatible with the differential:

$$d(f[n]) := (-1)^{mn}d(\varepsilon_B^n \circ f \circ \varepsilon_A^{-n}) = (-1)^{mn}[d\varepsilon_B^n f \circ] = (-1)^{mn}(-1)^n\varepsilon_B^n \circ df \circ \varepsilon_A^{-n} = (-1)^{(m+1)n}\varepsilon_B^n \circ df \circ \varepsilon_A^{-n} = df[n],$$

where we repeatedly used the Leibniz rule and the fact that $d(\varepsilon^{\pm n}) = 0$.

□

Remark 2.3. The following identities follow from the above:

$$\mathrm{Hom}_{\mathbf{A}}(A[n],B) \cong \mathrm{Hom}_{\mathbf{A}}(A,B)[-n], \text{ and } \mathrm{Hom}_{\mathbf{A}}(A,B[n]) \cong \mathrm{Hom}_{\mathbf{A}}(A,B)[n].$$

Similar to shifts, we have to define what it means to have a cone for an arbitrary dg category $\mathbf{A}$.

Definition 2.4. Let $\mathbf{A}$ be a dg category, let f be a dg morphism. An object $C \in \mathbf{A}$ is a cone of f if it is equipped with morphisms

$$A \xrightarrow[-1]{i_f} C \xrightarrow[1]{p_f} A, \text{ and } B \xrightarrow[0]{j_f} C \xrightarrow[0]{s_f} B.$$

such that the following hold

1. the identities:
$$\begin{bmatrix} p_f \circ i_f & s_f \circ i_f \\ p_i \circ j_f & s_f \circ j_f \end{bmatrix} = \begin{bmatrix} \mathrm{id}_A & 0 \\ 0 & \mathrm{id}_B \end{bmatrix},$$
2. $i_f \circ p_f + j_f \circ s_f = \mathrm{id}_C$,
3. the morphisms j_f and p_f are closed with
$$d(i_f) = j_f \circ f \text{ and } d(s_f) = -f \circ p_f.$$

Remark 2.5. The first and second conditions are equivalent classically to saying that $C \cong A[1] \oplus B$ in the graded category $\mathbf{A}$ such that $\varepsilon^{-1} : A[1] \to C$ and $j_f : B \to C$ are inclusions of summands (and $p_f : C \to A[1]$ and $s_f : C \to B$ are projections.

Proposition 2.6. *A cone of a dg morphism is unique up to unique dg isomorphism.*

Proof. Let $(C(f), i_f, p_f, j_f, s_f)$ and $(C'(f), i'_f, p'_f, j'_f, s'_f)$ be two cones of the closed morphism $f : A \to B$. Define the degree 0 morphisms $\varphi := i'_f p_f + j'_f s_f : C \to C'$ and $\psi := i_f p'_f + j_f s'_f : C' \to C$, these morphisms are dg since

$$d\varphi = d(i'_f p_f) + d(j'_f s_f) = di'_f p_f + j'_f ds_f = j'_f f p_f + j'_f(-f p_f) = 0,$$

and an identical calculation shows $d\psi = 0$. To see that ψ and φ are mutual inverses observe

$$\psi \circ \varphi = (i_f p'_f + j_f s_f)(i'_f p_f + j'_f s_f) = i_f p_f + j_f s_f = \mathrm{id}_C,$$

a similar computation yields $\varphi \circ \psi = \mathrm{id}_{C'}$. Let $\theta : C \to C'$ be a dg morphism such that $p'_f\theta = p_f$ and $s'_f\theta = s_f$ then

$$\theta = id_{C'}\theta = (i'_f p'_f + j'_f s'_f)\theta = i'_f(p'_f\theta) + j'_f(s'_f\theta) = i'_f p_f + j'_f s_f = \varphi,$$

and hence $\mathrm{Cone}(f)$ is unique up to unique dg isomorphism φ. □

Functoriality of the Mapping Cone

Let $\mathbf{A}$ be a dg category such that $\mathrm{Cone}(f)$ exists for all dg morphisms f in $\mathbf{A}$. Define $\mathbf{A}^{[1],\mathrm{Ho}}$

$$\mathrm{Obj}(\mathbf{A}^{[1],\mathrm{Ho}}) := \{\text{dg morphisms } f : A \to B \text{ in } \mathbf{A}\}.$$

Given two objects (dg morphisms) $f : A \to B$ and $f' : A' \to B'$ then

$$\mathrm{Hom}(f, f')^n := \{(F, G, h) | F : A \underset{n}{\to} A', G : B \underset{n}{\to} B' \text{ and } h : A \underset{n-1}{\longrightarrow} B' \text{ are such that the diagram below commutes}\}$$

$$\begin{tikzcd} A \arrow[r, "F"] \arrow[d, "f"] \arrow[dr, "h"] & A' \arrow[d, "f'"] \\ B \arrow[r, "G"] & B' \end{tikzcd}$$

where h is a homotopy from $f'F$ to Gf. This complex has differential $d : \mathrm{Hom}(f, f')^n \to \mathrm{Hom}(f, f')^{n+1}$ given by

$$d(F, G, h) := (dF, dG, -dh + f'F + Gf).$$

Composition of morphisms in the category $\mathbf{A}^{[1],\mathrm{Ho}}$ are as follows: for $(F_1, G_1, h_1) \in \mathrm{Hom}(f_1, f_2)^m$ and the triple $(F_2, G_2, h_2) \in \mathrm{Hom}(f_2, f_3)^n$ we define

$$(F_2, G_2, h_2) \circ (F_1, G_1, h_1) := (F_2F_1, G_2G_1, h_2F_1 + (-1)^n G_2h_1).$$

which is equivalent to the following diagram commuting:

$$\begin{tikzcd} A_1 \arrow[r, "F_1"] \arrow[d, "f_1"] \arrow[dr, "h_1"] & A_2 \arrow[r, "F_2"] \arrow[d, "f_2"] \arrow[dr, "h_2"] & A_3 \arrow[d, "f_3"] \\ B_1 \arrow[r, "G_1"] & B_2 \arrow[r, "G_2"] & B_3 \end{tikzcd}$$

Theorem 2.7. *There is a dg functor* $\mathrm{Cone} : \mathbf{A}^{[1],\mathrm{Ho}} \to \mathbf{A}$ *which sends* $f \mapsto \mathrm{Cone}(f)$ *and which acts on morphisms by*

$$\mathrm{Cone}(F, G, h) := (-1)^n i_{f'} F p_f + j_{f'} G s_f + (-1)^n j_{f'} h p_f : \mathrm{Cone}(f) \to \mathrm{Cone}(f'),$$

where (F, G, h) *is of degree* n*.*

Proof. Let $(F_1, G_1, h_1) \in \mathrm{Hom}(f_1, f_2)^n$ and $(F_2, G_2, h_2) \in \mathrm{Hom}(f_2, f_3)^m$ then to show functoriality, we must show

$$\mathrm{Cone}(F_2, G_2, h_2) \circ \mathrm{Cone}(F_1, G_1, h_1) = \mathrm{Cone}(F_2F_1, G_2G_1, G_2h_1 + (-1)^m h_2F_1).$$

This is a straightforward computation:

$$\begin{aligned} \text{L.H.S} &:= \mathrm{Cone}(F_2, G_2, h_2) \circ \mathrm{Cone}(F_1, G_1, h_1) \\ &= ((-1)^m i_{f_3} F_2 p_{f_2} + j_{f_3} G_2 s_{f_2} + (-1)^m j_{f_3} h_2 p_{f_2}) \circ ((-1)^n i_{f_2} F_1 p_{f_1} + j_{f_2} G_1 s_{f_1} + (-1)^n j_{f_2} h_1 p_{f_1}) \\ &= (-1)^{m+n}\, i_{f_3} F_2 p_{f_2}\, i_{f_2} F_1 p_{f_1} + (-1)^m\, i_{f_3} F_2 p_{f_2}\, j_{f_2} G_1 s_{f_1} + (-1)^{m+n}\, i_{f_3} F_2 p_{f_2}\, j_{f_2} h_1 p_{f_1} \\ &\quad + (-1)^n\, j_{f_3} G_2 s_{f_2}\, i_{f_2} F_1 p_{f_1} + j_{f_3} G_2 s_{f_2}\, j_{f_2} G_1 s_{f_1} + (-1)^n\, j_{f_3} G_2 s_{f_2}\, j_{f_2} h_1 p_{f_1} \\ &\quad + (-1)^{m+n}\, j_{f_3} h_2 p_{f_2}\, i_{f_2} F_1 p_{f_1} + (-1)^m\, j_{f_3} h_2 p_{f_2}\, j_{f_2} G_1 s_{f_1} + (-1)^{m+n}\, j_{f_3} h_2 p_{f_2}\, j_{f_2} h_1 p_{f_1}. \end{aligned}$$

Applying the relations from definition 2.4, the above expression simplifies

$$
\begin{aligned}
&= (-1)^{m+n}\, i_{f_3} F_2 F_1 p_{f_1} + j_{f_3} G_2 G_1 s_{f_1} + (-1)^n\, j_{f_3} G_2 h_1 p_{f_1} + (-1)^{m+n}\, j_{f_3} h_2 F_1 p_{f_1} \\
&= (-1)^{m+n}\, i_{f_3} F_2 F_1 p_{f_1} + j_{f_3} G_2 G_1 s_{f_1} + (-1)^n\, j_{f_3} [G_2 h_1 + (-1)^m h_2 F_1] p_{f_1} \\
&= \mathrm{Cone}(F_2 F_1, G_2 G_1, G_2 h_1 + (-1)^m h_2 F_1).
\end{aligned}
$$

Applying Cone to the identity element and using relation 2.4 yields

$$
\mathrm{Cone}(\mathrm{id}_A, \mathrm{id}_B, 0) = (-1)^0 i_f p_f + j_f s_f = \mathrm{id}_C\,.
$$

Therefore Cone is a well defined functor. To verify it is a dg functor it remains to show that the it is compatible with differentials. That is that the following diagram commutes

$$
\begin{tikzcd}
\mathrm{Hom}(f_1, f_2)^n \arrow[r, "d^n_{\mathbf{A}^{[1],\mathrm{Ho}}}"] \arrow[d, "\mathrm{Cone}"'] & \mathrm{Hom}(f_1, f_2)^{n+1} \arrow[d, "\mathrm{Cone}"] \\
\mathrm{Hom}(\mathrm{Cone}(f_1), \mathrm{Cone}(f_2))^n \arrow[r, "d^n_{\mathbf{A}}"] & \mathrm{Hom}(\mathrm{Cone}(f_1), \mathrm{Cone}(f_2))^{n+1}
\end{tikzcd}
$$

Doing the computation yields:

$$
\begin{aligned}
d(\mathrm{Cone}(F, G, h)) &:= (-1)^n d(i_{f_2} F p_{f_1}) + d(j_{f_2} G s_{f_1}) + (-1)^n d(j_{f_2} h p_{f_1}) \\
&= (-1)^n [d(i_{f_2} F) p_{f_1}] + (-1)^{n+1} j_{f_2} d(G s_{f_1}) + (-1)^n [j_{f_2} d(h p_{f_1})] \\
&= (-1)^n [d i_{f_2} F p_{f_1} + (-1) i_{f_2} dF p_{f_1}] + (-1)^{n+1} [j_{f_2} dG s_{f_1} + (-1)^n j_{f_2} G d s_{f_1}] + (-1)^n j_{f_2} dh p_{f_1}
\end{aligned}
$$

Using the fact that $dh = Gf_1 - f_2 F$ and the relations from 2.4 we get cancellations

$$
\begin{aligned}
&= (-1)^n [j_{f_2} f_2 F p_{f_1} + (-1) i_{f_2} dF p_{f_1}] + (-1)^{n+1} [j_{f_2} dG s_{f_1} + (-1)^{n+1} j_{f_2} G f_1 p_{f_1}] + (-1)^n j_{f_2} dh p_{f_1} \\
&= (-1)^{n+1} i_{f_2} dF p_{f_1} + (-1)^{n+1} j_{f_2} dG s_{f_1} \\
&= \mathrm{Cone}(dF, dG, dh - f_2 F - G f_1).
\end{aligned}
$$

□

We now see what goes wrong when one tries to define the Cone functor on the homotopy category naively. There is a functor $\mathrm{Ho}(\mathbf{A}^{[n],\mathrm{Ho}}) \to \mathrm{Ho}(\mathbf{A})^{[1]}$ that sends $[F, G, h)] \mapsto [(F, G)]$ on morphisms. This functor is clearly not strongly fully faithful as it is not injective on morphisms since it forgets the choice of homotopy h from $f'F$ to Gf. More explicitly we see from our formula for $\mathrm{Cone}(F, G, h)$ that two morphisms define out of $\mathrm{Ho}(\mathbf{A})$ will only coincide if $h - h'$ is a coboundary, in which case it is eliminated inside the cone formula.

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