Homological Projective Duality

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Abstract

Over the past two decades, Homological Projective Duality (HPD) has been a tremendously successful framework for analyzing semiorthogonal decompositions of derived categories. This thesis aims to provide an expository survey of Homological Projective Duality, as well as explore some conjectural extensions of HPD. Assuming a conjecture of Davis on the existence of a Fourier transform for dg-categories tensored over a fixed derived category of coherent sheaves, we show that this transform naturally gives rise to a duality that looks like an affinization of Kuznetsov's Homological Projective Duality. In the equivariant setting, we recover Kuznetsov's Homological Projective Duality and give a description of the Kuznetsov component in terms of this transform.

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Introduction

Derived categories have become an indispensable tool in the study of algebraic geometry, representation theory, and mathematical physics since their introduction by Jean-Louis Verdier and Alexander Grothendieck in the 1960s. Throughout this thesis, we will focus on the derived category associated to a projective variety X. This category can be seen as a kind of invariant associated to X which encodes a vast amount of information about the geometry of X. In fact, it has even been shown that in some cases, that from the derived category $\mathrm{D}(X)$, along with some extra information, one can actually reconstruct X; see [BO01].

In chapter I we describe how for a given X one can compute $\mathrm{D}(X)$. That is, we introduce the notion of a semiorthogonal decomposition for the derived category. This allows one to decompose $\mathrm{D}(X)$ into simpler pieces such that every object in $\mathrm{D}(X)$ admits a filtration through these simpler pieces. Although as we will see in Theorem 5.3 and Proposition 5.2, it is not always the case that $\mathrm{D}(X)$ admits a nontrivial semiorthogonal decomposition.

In chapter II we give a survey of homological projective duality (HPD) which is a duality of certain decompositions of derived categories, introduced by Kuznetsov in [Kuz05]. In particular if $\mathrm{D}(X)$ and $\mathrm{D}(Y)$ are HP-dual, then the derived categories of their linear sections admit semiorthogonal decompositions which are related and share a special subcategory, the Kuznetsov component. By considering the simpler case of rectangular Lefschetz decompositions we motivate the general theorem of HPD. We show explicitly how the duality occurs for $\mathbf{Gr}(2,4)$ and $\mathbf{Gr}(2,5)$ and illustrate the decomposition using rectangular box diagrams. We conclude this chapter by describing the results of [JLX17], in which HPD is shown to hold for more general sections. Notably this allows us to obtain from HPD the decompositions of an intersection of varieties, such as an intersection of two quadrics.

In the chapter III we briefly describe the contemporary program of homological geometry, before giving a conjecture on the existence a Fourier transform on the level of derived categories (with dg-enhancement). Such a transform is motivated by work currently in-progress with Dougal Davis. Assuming a conjecture in this work we explore the formal implications in section 2 and in section 3. In section 2 we assume this conjecture is true and show that it implies we have a duality of the decompositions of affine plane bundles similar to that of HPD. In section 3 we assume an equivariant version of the conjecture where there is a $\mathbb{G}_m$-action and describe how Kuznetsov's HPD theorem is recovered giving an expression of the Kuznetsov component in terms of this action.

Contents

Chapter I

Semiorthogonal Decompositions

1 The Derived Category of Coherent Sheaves

Throughout this thesis we denote by k an algebraically closed field of characteristic zero. By a variety X over k we mean a separated geometrically integral scheme X of finite type over k.[1]

Derived Category of an Abelian Category

We begin by giving a quick introduction to derived categories in algebraic geometry and therefore we only provide the necessary details to begin working. For complete proofs and details on the construction of the derived category or derived functors see [Wei94], [GM02] or [Rot08].

Let $\mathcal{A}$ be an abelian category and let $\mathbf{Kom}(\mathcal{A})$ denote the category of cochain complexes of $\mathcal{A}$. The homotopy category associated to $\mathbf{Kom}(\mathcal{A})$ we denote $\mathbf{K}(\mathcal{A})$. The derived category of an abelian category $\mathcal{A}$, which we denote $\mathrm{D}(\mathcal{A})$ is defined as follows:

$$\mathrm{Obj}(\mathrm{D}(\mathcal{A})) := \mathrm{Obj}(\mathbf{Kom}(\mathcal{A})). \tag{I.1}$$

For two complexes $A^\bullet, B^\bullet \in \mathrm{D}(\mathcal{A})$ the set of morphisms $\mathrm{Hom}_{\mathrm{D}(\mathcal{A})}(A^\bullet, B^\bullet)$ is the set of all equivalence classes of diagrams of the form

$$\begin{tikzcd} & C^\bullet \arrow[dl, "\text{qis}"'] \arrow[dr] & \\ A^\bullet & & B^\bullet \end{tikzcd}$$

where $C^\bullet \to A^\bullet$ is a quasi-isomorphism. Two such diagrams are equivalent if they are dominated in the homotopy category $\mathbf{K}(\mathcal{A})$ by a third one of the same kind. That is there exists a commutative diagram in $\mathbf{K}(\mathcal{A})$ of the form:

$$\begin{tikzcd} & & C^\bullet \arrow[dl] \arrow[dr] \arrow[ddll, dashed, bend right, "\text{qis}"'] & & \\ & C_1^\bullet \arrow[dl] \arrow[drrr] & & C_2^\bullet \arrow[dlll] \arrow[dr] & \\ A^\bullet & & & & B^\bullet \end{tikzcd}$$

[1] We will abuse notation and on occasion write '=' for canonically isomorphic, apologies.

where the compositions $C^\bullet \to C_1^\bullet \to A^\bullet$ and $C^\bullet \to C_2^\bullet \to A^\bullet$ are homotopy equivalent. To define composition consider the two morphisms

$$\begin{tikzcd} & C_1^\bullet \arrow[dl, "\text{qis}"'] \arrow[dr] & \\ A^\bullet & & B^\bullet \end{tikzcd} \qquad \begin{tikzcd} & C_2^\bullet \arrow[dl, "\text{qis}"'] \arrow[dr] & \\ B^\bullet & & C^\bullet \end{tikzcd}$$

we define their composite to be given by a commutative diagram (in the homotopy category $\mathbf{K}(\mathcal{A})$) of the form

$$\begin{tikzcd} & & C_3^\bullet \arrow[dl, "\text{qis}"'] \arrow[dr] & & \\ & C_1^\bullet \arrow[dl, "\text{qis}"'] \arrow[dr] & & C_2^\bullet \arrow[dl, "\text{qis}"'] \arrow[dr] & \\ A^\bullet & & B^\bullet & & C^\bullet \end{tikzcd}$$

It remains to show that such a diagram as claimed always exists and is unique up to equivalence, for further details see [GM02].

Remark 1.1. The derived category $\mathrm{D}(\mathcal{A})$ can also be characterized as the localization of the category $\mathbf{Kom}(\mathcal{A})$ at quasi-isomorphisms.

An important construction that will be used repeatedly throughout this thesis is the mapping cone of a morphism $f : A^\bullet \to B^\bullet$.

Definition 1.2. Let $f : A^\bullet \to B^\bullet$ be a morphism of complexes. Its mapping cone is the complex $\mathrm{Cone}(f)$ where

$$\mathrm{Cone}(f)^i := A^{i+1} \oplus B^i \text{ and } d^i_{\mathrm{Cone}(f)} := \begin{bmatrix} -d_A^{i+1} & 0 \\ f^{i+1} & d_B^i \end{bmatrix}. \tag{I.2}$$

with the natural injective map $\tau : B^\bullet \to \mathrm{Cone}(f)$ and projection map $\pi : \mathrm{Cone}(f) \to A^\bullet[1]$ respectively[2], which gives rise to the long exact sequence in cohomology

$$H^i(A^\bullet) \longrightarrow H^i(B^\bullet) \longrightarrow H^i(\mathrm{Cone}(f)) \longrightarrow H^{i+1}(A^\bullet).$$

In general if $\mathcal{A}$ is an abelian category then the derived category $\mathrm{D}(\mathcal{A})$ is additive but not necessarily abelian[3]. Instead the derived category $\mathrm{D}(\mathcal{A})$ is a triangulated category, (see appendix A) with additive autoequivalence given by the shift functor $[1] : \mathrm{D}(\mathcal{A}) \to \mathrm{D}(\mathcal{A})$ where $A^\bullet[1]$ is the shifted complex defined by

$$(A^\bullet[1])^i := A^{i+1} \text{ with differentials } d^i_{A[1]} := -d_A^{i+1}. \tag{I.3}$$

and $f : A^\bullet \to B^\bullet$ is defined by $f[1]^i := f^{i+1} : A^\bullet[1] \to B^\bullet[1]$. Of course composing $[1]$ with itself and its inverse we set

$$A^\bullet[k]^i := A^{k+i} \text{ and } d^i_{A[k]} := (-1)^k d_A^{i+k} \text{ for } k \in \mathbb{Z}. \tag{I.4}$$

[2]Where $\mathcal{A}^\bullet[1]$ is the complex $\mathcal{A}^\bullet$ shifted to the left by 1.

[3]One can show $\mathrm{D}(\mathcal{A})$ is abelian if and only if $\mathcal{A}$ is semisimple.

The collection of distinguished triangles in $\mathrm{D}(\mathcal{A})$ are triangles

$$A_1^\bullet \longrightarrow A_2^\bullet \longrightarrow A_3^\bullet \longrightarrow A_1^\bullet[1]$$

that are isomorphic to a triangle of the form

$$A^\bullet \xrightarrow{\ f\ } B^\bullet \xrightarrow{\ \tau\ } \mathrm{Cone}(f) \xrightarrow{\ \pi\ } A_1^\bullet[1]$$

with $f : A^\bullet \to B^\bullet$ a morphism of complexes. Verdier showed in his thesis [Ver96] that such distinguished triangles satisfy axioms (TR1),(TR2),(TR3) and (TR4) (see 1.1 for the axioms) and hence showed that $\mathrm{D}(\mathcal{A})$ is a triangulated category.

Derived Categories in Algebraic Geometry

Derived categories can be defined for an arbitrary scheme. However we will be most interested in the derived category of coherent sheaves for a smooth projective variety X over k. The following definition will be for convenience and is not always standard.

Definition 1.3. Let X be a scheme. The *derived category* $\mathrm{D}(X)$ is defined to be the **bounded** derived category of the abelian category of coherent sheaves, $\mathbf{Coh}(X)$. That is,

$$\mathrm{D}(X) := \mathrm{D}^b(\mathbf{Coh}(X)). \tag{I.5}$$

Definition 1.4. Two schemes X and Y defined over k are called *derived equivalent* if there exists a k-linear exact equivalence from $\mathrm{D}(X)$ to $\mathrm{D}(Y)$.

Remark 1.5. Since $\mathbf{Coh}(X)$ contains no non-trivial injective objects, we compute derived functors by passing to the category of quasi-coherent sheaves $\mathbf{Qcoh}(X)$.

Remark 1.6. A useful consequence from the construction of the derived category and derived functors is

$$\mathrm{Hom}_{\mathrm{D}(X)}(A^\bullet, B^\bullet[i]) = \mathrm{Ext}^i_{\mathbf{Coh}(X)}(A^\bullet, B^\bullet).$$

Theorem 1.7 ([Ver96] Thm. 2.2.6). *Let X be a Noetherian scheme then $\mathrm{D}(X)$ is a triangulated Category.*

Definition 1.8. Let X be a smooth projective variety of dimension n. Then one defines the exact functor S_X as the composition

$$\mathrm{D}(X) \xrightarrow{\ \omega_X \otimes (-)\ } \mathrm{D}(X) \xrightarrow{\ [n]\ } \mathrm{D}(X)$$

It turns out that this functor gives a Serre functor (see appendix def 2.1). This gives a notion of Serre Duality on the level of derived categories

Theorem 1.9. *(Serre Duality) Let X be a smooth projective variety over k then*

$$S_X : \mathrm{D}(X) \to \mathrm{D}(X) \text{ where } S_X(-) = \omega_X \otimes (-)[n] \tag{I.6}$$

is a Serre functor. Hence $\mathrm{Ext}^i(\mathcal{E}, \mathcal{F}) = \mathrm{Hom}(\mathcal{E}, \mathcal{F}[i])$ *implies* $\mathrm{Ext}^i(\mathcal{E}, \mathcal{F}) \simeq \mathrm{Ext}^{n-i}(\mathcal{F}, \mathcal{E} \otimes \omega_X)^*$.

Example 1.10. If X is Calabi-Yau i.e $\omega_X = \mathcal{O}_X$ then $S_X = [n]$.

Fourier Mukai Transforms

Definition 1.11. Let X and Y be smooth projective varieties and denote the two projecions by $q : X \times Y \to X$ and $p : X \times Y \to Y$. Let $\mathcal{P} \in \mathrm{D}^b(X \times Y)$, then the induced Fourier-Mukai transform is the functor

$$\Phi_{\mathcal{P}} : \mathrm{D}(X) \to \mathrm{D}(Y) \text{ where } \mathcal{E}^\bullet \mapsto Rp_*(Lq^*\mathcal{E}^\bullet \otimes^{\mathbb{L}} \mathcal{P}). \tag{I.7}$$

The complex $\mathcal{P}$ is called the Fourier-Mukai kernel of the Fourier-Mukai transform $\Phi_{\mathcal{P}}$.

Remark 1.12. $Lq^* = q^*$ since the projection map q is flat and the left derived tensor product $\otimes^{\mathbb{L}}$ coincides with the ordinary tensor product when the kernel $\mathcal{P}$ is a complex of vector bundles.

Composing two Fourier-Mukai functors gives another Fourier Mukai functor up to isomorphism

Proposition 1.13 ([Muk81]). *Let $\Phi_{\mathcal{P}} : \mathrm{D}(X) \to \mathrm{D}(Y)$ and $\Phi_{\mathcal{Q}} : \mathrm{D}(Y) \to \mathrm{D}(Z)$ be Fourier-Mukai functors. The composition*

$$\mathrm{D}(X) \xrightarrow{\Phi_{\mathcal{P}}} \mathrm{D}(Y) \xrightarrow{\Phi_{\mathcal{Q}}} \mathrm{D}(Z)$$

is isomorphic to the Fourier-Mukai transform $\Phi_{\mathcal{R}} : \mathrm{D}(X) \to \mathrm{D}(Z)$.

When computing Fourier-Mukai transforms the following two results are frequently applied

Theorem 1.14 ([Huy06]). *(Projection Formula) Let $f : X \to Y$ be a proper morphism of projective schemes over k. For any $\mathcal{F}^\bullet \in \mathrm{D}(X), \mathcal{E}^\bullet \in \mathrm{D}(Y)$ there exists a natural isomorphism*

$$Rf_*(\mathcal{F}^\bullet) \otimes^{\mathbb{L}} \mathcal{E}^\bullet \xrightarrow{\sim} Rf_*(\mathcal{F}^\bullet \otimes Lf^*(\mathcal{E}^\bullet)). \tag{I.8}$$

Theorem 1.15 ([Huy06]). *Let $f : X \to Y$ be a morphism of projective schemes and let $\mathcal{F}^\bullet, \mathcal{E}^\bullet \in \mathrm{D}(Y)$. Then there exists a natural isomorphism*

$$Lf^*(\mathcal{F}^\bullet) \otimes^{\mathbb{L}} Lf^*(\mathcal{E}^\bullet) \xrightarrow{\sim} Lf^*(\mathcal{F}^\bullet \otimes \mathcal{E}^\bullet). \tag{I.9}$$

Example 1.16. Let $\Gamma_f : X \to X \times Y$ be the diagonal map, where $\mathcal{O}_{\Gamma_f} = (\Gamma_f)_*\mathcal{O}_X$. Moreover let $q : X \times Y \to X$ and $p : X \times Y \to Y$ be projections onto the X and Y factors respectively, then from the pullback diagram

$$\begin{tikzcd}
X \arrow[dr, "\Gamma_f"] \arrow[drr, bend left, "f"] \arrow[ddr, bend right, "\mathrm{id}_X"'] & & \\
& X \times Y \arrow[r, "p"] \arrow[d, "q"'] & Y \arrow[d] \\
& X \arrow[r, "f"] & Y
\end{tikzcd}$$

we can compute the following Fourier-Mukai transform

$$\begin{aligned}
\Phi_{\mathcal{O}_{\Gamma_f}}(\mathcal{F}^\bullet) &= Rp_*(Lq^* \otimes^{\mathbb{L}} \mathcal{O}_\Delta) && && \text{(I.10)} \\
&\cong p_*(q^* \otimes (\Gamma_f)_*\mathcal{O}_X) && \text{(remark 1.12)} && \text{(I.11)} \\
&\cong p_*(\Gamma_f)_*((\Gamma_f)^*q^*\mathcal{F}^\bullet \otimes_{\mathcal{O}_X} \mathcal{O}_X) && \text{(projection formula)} && \text{(I.12)} \\
&\cong (p \circ \Gamma_f)_*(q \circ \Gamma_f)^*(\mathcal{F}^\bullet) && && \text{(I.13)} \\
&\cong f_*(\mathrm{id}_X)^*(\mathcal{F}^\bullet) && && \text{(I.14)} \\
&\cong f_*(\mathcal{F}^\bullet). && && \text{(I.15)}
\end{aligned}$$

Setting the map $f = \mathrm{id}$ in the above example we recover the well known result $\Phi_{\mathcal{O}_\Delta}(\mathcal{F}) = \mathcal{F}$.

We now come to Orlov's celebrated result.

Theorem 1.17 ([Orl03]). *(Orlov's Theorem) Let X and Y be two smooth projective varieties and let*

$$F : \mathrm{D}(X) \to \mathrm{D}(Y) \tag{I.16}$$

be a fully faithful exact functor. If F admits a right and left adjoint then there exists an object $\mathcal{P} \in \mathrm{D}(X \times Y)$ unique up to isomorphism such that F is isomorphic to $\Phi_{\mathcal{P}}$:

$$F \simeq \Phi_{\mathcal{P}}. \tag{I.17}$$

Orlov's theorem is most often applied to equivalences:

Corollary 1.18. *Let $F : \mathrm{D}(X) \xrightarrow{\sim} \mathrm{D}(Y)$ be an equivalence between the derived category of two smooth projective varieties. Then F is isomorphic to a Fourier-Mukai transform $\Phi_{\mathcal{P}}$ associated to a certain object $\mathcal{P} \in \mathrm{D}(X \times Y)$, which is unique up to isomorphism.*

Recall the notion of a spanning class for a triangulated category (definition 2.3, Appendix A).

Proposition 1.19. *Let X be a smooth projective variety over k then the set of skyscraper sheaves $\{\mathcal{O}_x \in \mathrm{D}(X) \mid x \in X\}$ forms a spanning class of $\mathrm{D}(X)$.*

Proof. For any object $\mathcal{F} \in \mathrm{D}(X)$ and any $x \in X$, there is a spectral sequence

$$E_2^{p,q} = \mathrm{Ext}_X^p(H^{-q}(\mathcal{F}), \mathcal{O}_x) \implies \mathrm{Hom}_{\mathrm{D}(X)}^{p+q}(\mathcal{F}, \mathcal{O}_x), \tag{I.18}$$

Suppose $\mathcal{F}$ is not isomorphic to zero. Let q_0 be the maximal value of q such that $H^q(\mathcal{F})$ is non-zero and assume x is a closed point in the support of $H^{q_0}(\mathcal{F})$. Then there is a non-zero element of $E_2^{0,-q_0} = \mathrm{Ext}_x^0(H^{q_0}(\mathcal{F}), \mathcal{O}_x)$ which survives to give an element of $\mathrm{Hom}_{\mathrm{D}(X)}^{q_0}(\mathcal{F}, \mathcal{O}_x)$, proving (2). Serre duality then gives a non-zero element of $\mathrm{Hom}^i(\mathcal{O}_x, \mathcal{F})$ where $i = \dim(X) - q_0$, proving (1). □

Theorem 1.20 ([BO95a]). *Let X and Y be smooth projective varieties. The Fourier-Mukai transform $\Phi_{\mathcal{P}} : \mathrm{D}(X) \to \mathrm{D}(Y)$ is fully faithful if and only if for any two closed points $x, y \in X$ one has*

$$\mathrm{Hom}(\Phi_{\mathcal{P}}(\mathcal{O}_x), \Phi_{\mathcal{P}}(\mathcal{O}_y)[i]) = \begin{cases} k & \text{if } x = y \text{ and } i = 0; \\ 0 & \text{if } x \neq y \text{ or } i < 0 \text{ or } i > \dim(X). \end{cases} \tag{I.19}$$

This section so far has been a summary of almost all the important results needed to work with derived categories and derived functors. However we have not given some of the more basic compatibilities of derived functors in algebraic geometry. For the rigorous approach to derived functors and their compatibilities see either [Con00] or [LH09]. For a thorough survey of Fourier-Mukai transforms in algebraic geometry, see [Huy06].

We conclude with a surprising result that motivates why one should consider semiorthogonal decompositions as opposed to the standard decompositions of a triangulated category.

Recall the definition of a decomposable category [Appendix A, Def. 1.9] from the appendix. A triangulated category is then said to be indecomposable if it is not decomposable. Equivalently, in the case where we are considering $\mathrm{D}(X)$, we make the following definition.

Definition 1.21. The derived category $\mathrm{D}(X)$ is indecomposable (as a triangulated category) if for any pair of full triangulated subcategories $\mathcal{A}$ and $\mathcal{B}$ the following are satisfied

1. for every $\mathcal{F} \in \mathcal{T}$, there exist objects $A \in \mathcal{A}$ and $B \in \mathcal{B}$ such that $\mathcal{F}$ is a product of A and B.

2. For any pair of objects $A \in \mathcal{A}$ and $B \in \mathcal{B}$, we have for all $i \in \mathbb{Z}$, A and B are orthogonal

$$\mathrm{Hom}_{\mathrm{D}(X)}(A, B[i]) = \mathrm{Hom}_{\mathrm{D}(X)}(B, A[i]) = 0, \tag{I.20}$$

   then either $\mathcal{F} = 0$ for all $A \in \mathcal{A}$, or $\mathcal{F} = 0$ for all $B \in \mathcal{B}$.

This following theorem characterises when the derived category is indecomposable as a triangulated category.

Theorem 1.22 ([Bri19], Ex. 3.2). *Let X be a noetherian scheme. Then X is connected if and only if $\mathrm{D}(X)$ is indecomposable.*

Proof. Suppose X is connected and let $\mathcal{A}$ and $\mathcal{B}$ be full subcategories of $\mathrm{D}(X)$ satisfying conditions (a) and (b) of the definition. For any integral closed sub-scheme Y of X, the sheaf $\mathcal{O}_Y$ is indecomposable and is therefore isomorphic to some object of $\mathcal{A}$ or $\mathcal{B}$, without loss of generality suppose $\mathcal{O}_Y$ is an object in $\mathcal{A}$). For any point $y \in Y$, we must have $\mathcal{O}_y$ is isomorphic to an object $\mathcal{A}$ since otherwise (2) would imply that $\mathrm{Hom}_{\mathrm{D}(X)}(\mathcal{O}_Y, \mathcal{O}_y) = 0$, which is not the case.

Let X_1 be the union of integral sub-schemes Y such that $\mathcal{O}_Y$ is isomorphic to an object of $\mathcal{A}$. Let X_2 be the union of $Y \subset X$ such that $\mathcal{O}_Y$ is isomorphic to an object of $\mathcal{B}$. Then X_1 and X_2 are closed subsets of X and $X = X_1 \cup X_2$. If a point $x \in X$ lies in $X_1 \cap X_2$, then $\mathcal{O}_x$ is isomorphic to an object of $\mathcal{A}$ and to $\mathcal{B}$. This contradicts (2), thus X_1 and X_2 are disjoint. Since X is connected, either $X_1 = \emptyset$ or $X_2 = \emptyset$ (without loss of generality take $X_2 = \emptyset$ so $X = X_1$). Then (2) implies for any $\mathcal{F} \in \mathcal{B}$ one has

$$\mathrm{Hom}^i_{\mathrm{D}(X)}(\mathcal{F}, \mathcal{O}_x) = 0 \text{ for all } i \in \mathbb{Z} \text{ and } x \in X, \tag{I.21}$$

and since the $\mathcal{O}_x$ form a spanning class from proposition 2.3 it follows that $\mathcal{F} = 0$. For the converse suppose X is not connected, e.g. $X = X_1 \sqcup X_2$, then we let $\mathcal{A} = \mathrm{D}(X_1)$ and $\mathcal{B} = \mathrm{D}(X_2)$, applying lemma 3.9 [Huy06] allows us to decompose any $\mathcal{F}$ as $\mathcal{F} = \mathcal{F}_1 \oplus \mathcal{F}_2$ where $\mathcal{F}_1 \in \mathcal{A}$ and $\mathcal{F}_2 \in B$ have disjoint support. It follows that $\mathrm{D}(X)$ satisfies the definition of a decomposable triangulated category. □

2 Semiorthogonal Decompositions

Let X be a Noetherian scheme. From the concluding theorem in the previous section have the bounded derived category $\mathrm{D}(X)$ cannot be decomposed into triangulated subcategories if X is connected. This is problematic as for the derived category $\mathrm{D}(X)$ of almost all the projective varieties we care about are indecomposable. It turns out that the solution to this problem is to instead introduce a weaker notion of decomposition known as a semiorthogonal decomposition.

Semiorthogonal decompositions were first introduced by Bondal and Orlov in [BO95b] and they essentially decompose the category $\mathrm{D}(X)$ into a sequence of semi-orthogonal subcategories. That is if $\mathrm{D}(X)$ admits a semiorthogonal decomposition into $\mathcal{A}_1, \dots, \mathcal{A}_n$ then every complex $\mathcal{F}^\bullet \in \mathrm{D}(X)$ admits a filtration through each of the subcategories $\mathcal{A}_1, \dots \mathcal{A}_n$.

Definition 2.1. Let $\mathcal{A} \subset \mathcal{T}$ be a full triangulated subcategory. We define the right orthogonal of $\mathcal{A}$ in $\mathcal{T}$ to be

$$\mathcal{A}^\perp = \{T \in \mathcal{T} | \operatorname{Hom}(A[i], T) = 0, \text{ for all } A \in \mathcal{A},\ i \in \mathbb{Z}\}. \tag{I.22}$$

Similarly we define the left orthogonal of $\mathcal{A}$ in $\mathcal{T}$ as

$${}^\perp\mathcal{A} = \{T \in \mathcal{T} | \operatorname{Hom}(T, A[i]) = 0, \text{ for all } A \in \mathcal{A},\ i \in \mathbb{Z}\}. \tag{I.23}$$

Definition 2.2. Let $\mathcal{T}$ be a triangulated category. If $\mathcal{T}$ admits a sequence of full triangulated subcategories $\mathcal{A}_1, \dots, \mathcal{A}_n$ such that

1. (Semi-orthogonality) $\operatorname{Hom}_{\mathcal{T}}(\mathcal{A}_i, \mathcal{A}_j) = 0$ for $i > j$.
2. (Generation) the smallest triangulated subcategory of $\mathcal{T}$ containing $\mathcal{A}_1, \dots, \mathcal{A}_n$ is $\mathcal{T}$.

then $\mathcal{T}$ has a semiorthogonal decomposition by $\mathcal{A}_1, \dots, \mathcal{A}_n$ denoted by $\mathcal{T} = \langle \mathcal{A}_1, \dots \mathcal{A}_n \rangle$.

Remark 2.3. The generation condition above is sometimes stated in the following equivalent way. For every $T \in \mathcal{T}$ there exists a chain of morphisms

$$0 = T_n \longrightarrow T_{n-1} \longrightarrow \dots \longrightarrow T_1 \longrightarrow T_0 = T$$

such that the cone of the morphism $T_k \to T_{k-1}$ is contained in $\mathcal{A}_k$ for $k = 1, \dots, n$. In other words there exists a diagram

$$\begin{tikzcd}
0 = T_n \arrow[rr] & & T_{n-1} \arrow[r] \arrow[dl] & \dots \arrow[r] & T_2 \arrow[rr] & & T_1 \arrow[rr] \arrow[dl] & & T \arrow[dl] \\
& A_n \arrow[ul, dashed, "{[+1]}"] & & \dots & & A_2 \arrow[ul, dashed, "{[+1]}"] & & A_1 \arrow[ul, dashed, "{[+1]}"] &
\end{tikzcd} \tag{I.24}$$

where all triangles are distinguished, (dashed arrows have degree 1) and $A_k \in \mathcal{A}_k$. Thus every object $T \in \mathcal{T}$ admits a descending filtration with factors in $\mathcal{A}_1, \dots \mathcal{A}_n$ respectively.

The most well known example of a semiorthogonal decomposition is the semiorthogonal decomposition of the derived category of $\mathbb{P}^n$ given by Beilinson's collection

Example 2.4. (Beilinson's Collection)
Line bundles $\mathcal{O}_{\mathbb{P}^n}(k)$ for $0 \leq k \leq n$ on $\mathbb{P}^n$ form a strong full exceptional collection in the derived category of $\mathbb{P}^n$ (theorem 3.9), from this it follows

$$\mathrm{D}(\mathbb{P}^n) = \Big\langle \mathcal{O}_{\mathbb{P}^n}, \mathcal{O}_{\mathbb{P}^n}(1), \mathcal{O}_{\mathbb{P}^n}(2), \dots, \mathcal{O}_{\mathbb{P}^n}(n) \Big\rangle. \tag{I.25}$$

From the proof of Beilinson's collection theorem 3.9 we obtain explicitly via $\Phi_{\mathcal{O}_\Delta}(\mathcal{F}) = \mathcal{F}$ that any coherent sheaf $\mathcal{F} \in \mathrm{D}(\mathbb{P}^n)$ is quasi-isomorphic to a complex of the form

$$F_n \otimes \mathcal{O}_{\mathbb{P}^n}(-n) \longrightarrow \dots \longrightarrow F_1 \otimes \mathcal{O}_{\mathbb{P}^n}(-1) \longrightarrow F_0 \otimes \mathcal{O}_{\mathbb{P}^n}$$

where $F_i = R\operatorname{Hom}(\Lambda^i \mathcal{T}_{\mathbb{P}^n}, \mathcal{F}(i))$.

Example 2.5. (Twisted Beilinson's collection) In fact it can be shown (corollary 3.10) that twisting the above line bundles in the previous example by any fixed integer $d \in \mathbb{Z}$ yields another semiorthogonal decomposition of $\mathbb{P}^n$

$$\mathrm{D}(\mathbb{P}^n) = \Big\langle \mathcal{O}(d), \mathcal{O}(d+1), \dots \mathcal{O}(d+n) \Big\rangle. \tag{I.26}$$

Therefore in particular we see that a semiorthogonal decomposition of $\mathrm{D}(X)$ is in general not unique.

Example 2.6. Let V be a 5 dimensional k vector space then the Grassmannian $\mathbf{Gr}(2,V)$ with its Plücker embedding $\mathbf{Gr}(2,V) \to \mathbb{P}(\Lambda^2 V)$ admits a semiorthogonal decomposition of the form

$$\mathrm{D}(\mathbf{Gr}(2,V)) = \Big\langle \mathcal{O}, \mathcal{U}^\vee, \mathcal{O}(1), \mathcal{U}^\vee(1), \mathcal{O}(2), \mathcal{U}^\vee(2), \mathcal{O}(3), \mathcal{U}^\vee(3), \mathcal{O}(4), \mathcal{U}^\vee(4) \Big\rangle, \tag{I.27}$$

where $\mathcal{U}$ is the rank 2 tautological bundle on $\mathbf{Gr}(2,V)$.

The Grassmannian $\mathbf{Gr}(2,5)$ will return again in later sections as it is self-dual in the Pfaffian-Grassmannian correspondence since $\mathbf{Pf}(2,5) = \mathbf{Gr}(2,5)$.

Proposition 2.7. *Let X and Y be smooth projective varieties. If $f : X \to Y$ is a birational morphism then $\mathrm{D}(X)$ admits a semiorthogonal decomposition given by*

$$\mathrm{D}(X) = \langle \mathscr{C}, \mathrm{D}(Y) \rangle, \tag{I.28}$$

where $\mathscr{C} = \{a \in \mathrm{D}(X) | Rf_* a = 0\}$.

Proof. The left derived functor $Lf^* : \mathrm{D}(Y) \to \mathrm{D}(X)$ gives rise to adjunction $Rf_* Lf^* = \mathrm{id}$ as X and Y are smooth. For $a, b \in \mathrm{D}(X)$ we have

$$\operatorname{Hom}(Lf^* a, Lf^* b) = \operatorname{Hom}(a, Rf_* Lf^* b) = \operatorname{Hom}(a, b), \tag{I.29}$$

hence Lf^* is fully faithful. Therefore $\operatorname{Hom}(Lf^* a, c) = \operatorname{Hom}(a, Rf_* c) = 0$ for $a \in \mathrm{D}(Y)$ and $c \in \mathscr{C}$. So $\mathrm{D}(Y)$ is semiorthogonal to $\mathscr{C}$. To prove generation, let $a \in \mathrm{D}(X)$ we show that every a is built from $\mathscr{C}$ and $\mathrm{D}(Y)$. Consider the counit map $\varepsilon : Lf^* Rf_* \to \mathrm{id}$, this can be extended to an exact triangle

$$Lf^* Rf_* a \xrightarrow{\quad \varepsilon \quad} a \longrightarrow \operatorname{Cone}(\varepsilon) \longrightarrow Lf^* Rf_* a[1]$$

Applying the derived functor Rf_* and the fact that $Rf_* Lf^* = \mathrm{id}$ yields

$$Rf_* a \xrightarrow{\quad \sim \quad} Rf_* a \longrightarrow Rf_* \operatorname{Cone}(\varepsilon) \longrightarrow a[1]$$

Since the first term is an isomorphism we must have that $Rf_* \operatorname{Cone}(\varepsilon) = 0$ hence $\operatorname{Cone}(\epsilon) \in \mathscr{C}$. $\square$

We now give some standard properties of semiorthogonal decompositions.

Lemma 2.8. *If $\mathcal{T} = \langle \mathcal{A}_1, \dots \mathcal{A}_n \rangle$ is a semiorthogonal decomposition and $T \in \mathcal{T}$ then the filtration diagram I.24 for T is unique and functorial. That is for any $T \to T'$ there exists a unique collection of morphisms $T_i \to T_i'$, $A_i \to A_i'$ which combine into a morphism of diagrams from T to T'.*

Proof. From the diagram I.24, $T_1 \in \langle \mathcal{A}_2, \dots, \mathcal{A}_n \rangle$. Any morphism $T_1 \to T_0 \to A'[1]$ must be 0 since $A'[1] \in \langle \mathcal{A}_2, \dots, \mathcal{A}_n \rangle^{\perp}$ due to the semi-orthogonal condition $\mathrm{Hom}(T_1, A'[k]) = 0$ for all $k \in \mathbb{Z}$. It follows that there exists a lift $T_1 \to T_1'$

$$\begin{tikzcd} T_1 \arrow[r] \arrow[d, dashed] & T_0 \arrow[r] \arrow[d, equal] & A_1 \\ T_1' \arrow[r] & T_0' \arrow[r] & A_1' \end{tikzcd}$$

Uniqueness of this lift follows from the fact that $\mathrm{Hom}(T_1, A[-1]) = 0$. By the (TR3) axiom we can extend this diagram to a morphism of triangles $T_1 \to T \to A_1$ to $T_1' \to T' \to A_1'$. Proceeding by induction proves the lemma. ☐

Definition 2.9. A full triangulated subcategory $\mathcal{A}$ of a triangulated category $\mathcal{T}$ is called right admissible if for the inclusion functor $i : \mathcal{A} \to \mathcal{T}$ there exists a right adjoint $i^! : \mathcal{T} \to \mathcal{A}$, and left admissible if there exists a left adjoint $i^* : \mathcal{T} \to \mathcal{A}$. A subcategory $\mathcal{A}$ of $\mathcal{T}$ is admissible if it is both left and right admissible.

Remark 2.10. An admissible subcategory contains is equivalent to a recollement.

Proposition 2.11. *Let X be a smooth projective variety and let $\mathcal{A}$ be a full triangulated subcategory of $\mathrm{D}(X)$, then $\mathrm{D}(X) = \langle \mathcal{A}^{\perp}, \mathcal{A} \rangle$.*

Proof. Follows immediately from the definition of $\mathcal{A}^{\perp}$ and the fact that $\mathrm{D}(X)$ is admissible. ☐

Lemma 2.12. *[Bon90] If $\mathcal{T} = \langle \mathcal{A}, \mathcal{B} \rangle$ is a semiorthogonal decomposition then $\mathcal{A}$ is left admissible and $\mathcal{B}$ is right admissible.*

Lemma 2.13. *[Bon90] If $\mathcal{A}_1, \dots, \mathcal{A}_n$ is a semiorthogonal sequence in a triangulated category $\mathcal{T}$ such that $\mathcal{A}_1, \dots, \mathcal{A}_k$ are left admissible and $\mathcal{A}_{k+1}, \dots \mathcal{A}_n$ are right admissible then*

$$\langle \mathcal{A}_1, \dots, \mathcal{A}_k, {}^{\perp}\langle \mathcal{A}_1, \dots, \mathcal{A}_k \rangle \cap \langle \mathcal{A}_{k+1}, \dots, \mathcal{A}_n \rangle^{\perp}, \mathcal{A}_{k+1}, \dots \mathcal{A}_n \rangle, \tag{I.30}$$

is a semiorthogonal decomposition.

Note that if $\mathrm{D}(X) = \langle \mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_n \rangle$ is a semiorthogonal decomposition then

$$\mathcal{A}_i = \langle A_{i+1}, \dots, \mathcal{A}_n \rangle^{\perp} \cap^{\perp} \langle \mathcal{A}_1, \dots, \mathcal{A}_{i-1} \rangle. \tag{I.31}$$

Lemma 2.14. *[Bon90] If X is a smooth projective variety and if $\mathrm{D}(X) = \langle \mathcal{A}_1, \dots \mathcal{A}_n \rangle$ is a semiorthogonal decomposition then each subcategory $\mathcal{A}_i \subset \mathrm{D}(X)$ for $1 \leq i \leq n$ is admissible.*

3 Exceptional Collections on Projective Space

Exceptional collections are a rich source of semiorthogonal decompositions. In this section we prove that $\mathbb{P}^n$ admits a exceptional collection that spans the derived category.

Definition 3.1. Let $\mathcal{D}$ be a k-linear triangulated category. An object $E \in \mathcal{D}$ is an *exceptional object* if

$$\operatorname{Hom}(E, E[m]) = \begin{cases} k \text{ if } m = 0; \\ 0 \text{ if } m \neq 0. \end{cases} \tag{I.32}$$

Example 3.2. The line bundles $\mathcal{O}_{\mathbb{P}^n}(i)$ on $\mathbb{P}^n_k$ are exceptional objects in the derived category $\mathrm{D}(\mathbb{P}^n)$.

$$\operatorname{Hom}_{\mathrm{D}(\mathbb{P}^n)}(\mathcal{O}(i), \mathcal{O}(i)[m]) \simeq H^m(\mathbb{P}^n, \mathcal{O}) = \begin{cases} k \text{ if } m = 0, \\ 0 \text{ otherwise.} \end{cases} \tag{I.33}$$

A sequence of exceptional objects $E_1, \ldots, E_n$ may be arranged to yield an exceptional sequence.

Definition 3.3. An exceptional sequence is a sequence $E_1, \ldots, E_n$ of exceptional objects that satisfy $\operatorname{Hom}(E_i, E_j[m]) = 0$ for all $i > j$ and all $m \in \mathbb{Z}$.

Definition 3.4. An exceptional sequence $E_1, E_2, \ldots, E_n$ is called a full exceptional sequence if it spans the derived category $\mathrm{D}^b(X)$.

Proposition 3.5. *Every full exceptional sequence $E_1, \ldots, E_n$ of $\mathrm{D}(X)$ gives rise to a semiorthogonal decomposition of the form*

$$\mathrm{D}(X) = \Big\langle E_1, E_2, \ldots, E_n \Big\rangle. \tag{I.34}$$

Proof. Since exceptional objects are orthogonal the categories $\operatorname{Hom}_{\mathrm{D}(X)}(E_i, E_j) = 0$ for $i > j$ hence the categories generated by each exceptional object satisfies the semiorthogonality condition. Generation follows from the fact that the exceptional collection is full. □

Remark 3.6. A semiorthogonal decomposition induced by a full exceptional sequence is sometimes referred an orthogonal decomposition of $\mathrm{D}^b(X)$.

Remark 3.7. The adjective full is necessary even if we only considered exceptional collections of $\mathrm{D}(X)$ of length $n = \dim(X)$. Since it is known that there exist projective varieties X which admit exceptional collections of length n that span $K_0(X)$ but not $\mathrm{D}(X)$. This failure to span is due to the presence of phantom subcategories living in $\mathrm{D}(X)$. Recently Krah in [Kra23] found an example of such phantom, taking X is $\mathbb{P}^2$ blown up at 10 points.

In order to prove that Beilinson's collection is a full exceptional collection on $\mathbb{P}^n$, we must first prove Beilinson's resolution of the diagonal, see [Bei78].

Let V be a complex vector space of dimension $n+1$, and let $\mathbb{P}^n = \mathbb{P}(V)$ be the corresponding projective space. Let $\iota : \mathbb{P}^n \to \mathbb{P}^n \times \mathbb{P}^n$ be the diagonal embedding. The sheaf $\iota_* \mathcal{O}_\Delta$ has a canonical resolution by locally free sheaves on $\mathbb{P}^n \times \mathbb{P}^n$. This resolution is a projective space analogue of the Koszul resolution.

Lemma 3.8 ([Bei78]). *(Beilinsons's Resolution)*
The sheaf $\mathcal{O}_\Delta$ has a canonical resolution by locally free sheaves on $\mathbb{P}^n \times \mathbb{P}^n$, given by

$$0 \longrightarrow \mathcal{O}(-n) \boxtimes \Omega^n(n) \longrightarrow \ldots \longrightarrow \mathcal{O}(-1) \boxtimes \Omega^1(1) \longrightarrow \mathcal{O}_{\mathbb{P}^n \times \mathbb{P}^n} \longrightarrow \iota_* \mathcal{O}_\Delta \longrightarrow 0$$

Proof. Let $v \in V$ be a non-zero vector and let $\overline{v} = \mathbb{C} \cdot v$ be the one dimensional vector subspace viewed as a point in $\mathbb{P}^n$. Recall $\mathcal{O}_{\mathbb{P}^n}(-1)$ is the tautological bundle on $\mathbb{P}^n$ with fiber $L = \mathbb{C} \cdot v$. An dual vector $v^* \in V^*$ gives for each $v \in V$ a linear function on L and hence a global section of $\mathcal{O}_{\mathbb{P}^n}(1)$. In this way we obtain a canonical isomorphism $V^* \xrightarrow{\sim} H(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$.

Let $\mathrm{id} \in V^* \otimes V \cong \mathrm{Hom}(V, V)$, due to the above isomorphism we can view id as being equivalent to a global section of the sheaf $V \otimes \mathcal{O}_{\mathbb{P}^n}(1)$. The map $f \mapsto f \cdot \mathrm{id}$ gives rise to the following short exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^n} \longrightarrow V \otimes \mathcal{O}_{\mathbb{P}^n}(1) \longrightarrow \mathcal{T} \longrightarrow 0$$

The quotient sheaf $\mathcal{T}$ is canonically isomorphic to the tangent sheaf on $\mathbb{P}^n$. Its geometric fiber over $\overline{v} \in \mathbb{P}^n$ is $V/\mathbb{C}v \cong T_{\overline{v}}\mathbb{P}^n$. Tensoring the above exact sequence by $\mathcal{O}_{\mathbb{P}^n}(-1)$ gives the Euler sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^n}(-1) \longrightarrow V \otimes \mathcal{O}_{\mathbb{P}^n} \longrightarrow \mathcal{Q} \longrightarrow 0$$

Taking the long exact sequence in cohomology we have that $H^0(\mathbb{P}^n, \mathcal{Q}) \cong H^0(\mathbb{P}^n, V \otimes \mathcal{O}_{\mathbb{P}^n}) \cong V$ since $H^0(\mathcal{O}_{\mathbb{P}^n}(-1)) = 0$. Let $\pi_i : \mathbb{P}^n \times \mathbb{P}^n$ for $i = 1, 2$ be the projection map onto the i-th factor. Applying the Kunneth formula and the previous isomorphisms yields

$$\begin{aligned} H^0(\mathbb{P}^n \times \mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1) \boxtimes \mathcal{Q}) &\cong H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1)) \otimes H^0(\mathbb{P}^n, \mathcal{Q}) && \text{(I.35)} \\ &\cong V^* \otimes H^0(\mathbb{P}^n, V \otimes \mathcal{O}_{\mathbb{P}^n})) && \text{(I.36)} \\ &\cong V^* \otimes V && \text{(I.37)} \\ &\cong \mathrm{Hom}_{\mathbb{C}}(V, V). && \text{(I.38)} \end{aligned}$$

Now, let s in $\mathcal{O}_{\mathbb{P}^n}(1) \boxtimes \mathcal{Q}$ be the global section which corresponds to the identity morphism $\mathrm{id} \in \mathrm{Hom}(V, V)$. In order to describe this section explicitly we write

$$\mathcal{O}_{\mathbb{P}^n}(1) \boxtimes \mathcal{Q} \cong \mathcal{H}\mathrm{om}(\pi_1^* \mathcal{O}_{\mathbb{P}^n}(-1), \pi_2^* \mathcal{Q}). \tag{I.39}$$

The section s on the left hand side corresponds on the right hand side to a sheaf homomorphism which we denote $\hat{s} : \pi_1^* \mathcal{O}_{\mathbb{P}^n}(-1) \to \pi_2^* \mathcal{Q}$ defined as follows: Given $\overline{v}, \overline{w} \in \mathbb{P}^n$, we have that for the geometric fibers at $\overline{v}$ and $\overline{w}$

$$\mathcal{O}_{\mathbb{P}^n}(-1)|_{\overline{v}} = \mathbb{C} \cdot v \text{ and } \mathcal{Q}|_{\overline{w}} = V/\mathbb{C} \cdot w, \tag{I.40}$$

Hence giving the morphism $\hat{s}$ amounts to giving for each pair $\overline{v}, \overline{w} \in \mathbb{P}^n$ a linear map $\mathbb{C}{\cdot}v \to V/\mathbb{C}{\cdot}w$. We have

$$\hat{s}(\overline{v}, \overline{w}) : \mathbb{C} \cdot v \to V/\mathbb{C} \cdot w \text{ where } c \cdot v \mapsto c \cdot w \bmod \mathbb{C} \cdot w. \tag{I.41}$$

This section $\hat{s}(\overline{v}, \overline{w})$ vanishes if and only if v and w are linearly independent if and only if $\overline{v} = \overline{w}$ in $\mathbb{P}^n$. Thus the zero locus of s is the diagonal $\Delta \subset \mathbb{P}^n \times \mathbb{P}^n$. Furthermore contraction with $s \in \mathcal{O}_{\mathbb{P}^n} \boxtimes \mathcal{Q}$ defines for each $k \geq 1$ a sheaf homomorphism

$$i_s : \bigwedge^k (\mathcal{O}_{\mathbb{P}^n}(-1) \boxtimes \mathcal{Q}^*) \to \bigwedge^{k-1} (\mathcal{O}_{\mathbb{P}^n}(-1) \boxtimes \mathcal{Q}^*). \tag{I.42}$$

Thus we obtain the a locally free resolution of the sheaf $\mathcal{O}_\Delta = \mathcal{O}_{\mathbb{P}^n \times \mathbb{P}^n}/\mathcal{I}_\Delta$ of length n. Now to get the desired form in the lemma. Recall the standard identity $\Lambda^k(\mathcal{F} \otimes \mathcal{L}) \cong \Lambda^k \mathcal{F} \otimes \mathcal{L}(k)$ where $\mathcal{F}$ is a coherent sheaf and $\mathcal{L}$ is a line bundle. Applying this identity and using the fact that

$\mathcal{Q}^* \cong \mathcal{T}^* \otimes \mathcal{O}_{\mathbb{P}^n}(1) = \Omega^1_{\mathbb{P}^n}(1)$ where $\Omega^1_{\mathbb{P}^n}$ is the cotangent sheaf on $\mathbb{P}^n$. It follows that

$$\bigwedge^k Q^* \cong \bigwedge^k (\mathcal{T}^* \otimes \mathcal{O}_{\mathbb{P}^n}(1)) \cong \bigwedge^k \mathcal{T}^* \otimes \mathcal{O}_{\mathbb{P}^n}(k) = \Omega^k(k). \tag{I.43}$$

Therefore for each k, applying the same identity and commutativity isomorphism of the box tensor we obtain

$$\bigwedge^k (\mathcal{O}_{\mathbb{P}^n}(-1) \boxtimes \mathcal{Q}) \cong \bigwedge^k \mathcal{Q}^* \boxtimes \mathcal{O}_{\mathbb{P}^n}(-1)^{\otimes k} \cong \mathcal{O}_{\mathbb{P}^n}(-k) \boxtimes \Omega^k(k). \tag{I.44}$$

Therefore we have completed the construction of the resolution of the diagonal sheaf $\mathcal{O}_\Delta$

$$0 \longrightarrow \mathcal{O}(-n) \boxtimes \Omega^n(n) \longrightarrow \dots \longrightarrow \mathcal{O}(-1) \boxtimes \Omega^1(1) \longrightarrow \mathcal{O}_{\mathbb{P}^n \times \mathbb{P}^n} \longrightarrow \mathcal{O}_\Delta \longrightarrow 0$$

□

Theorem 3.9 ([Bei78]). *(Beilinson's collection on $\mathbb{P}^n$)*
The sequence of line bundles $\mathcal{O}(-n), \dots, \mathcal{O}(-1), \mathcal{O}$ form a full exceptional sequence and therefore $\mathbb{P}^n$ admits a semiorthogonal decomposition

$$\mathrm{D}(\mathbb{P}^n) = \Big\langle \mathcal{O}(-n), \mathcal{O}(-n+1), \dots \mathcal{O}(-1), \mathcal{O} \Big\rangle. \tag{I.45}$$

Proof. We know that $\mathcal{O}(-n), \mathcal{O}(-n+1) \dots, \mathcal{O}$, are an exceptional sequence of objects from the example 3.2. It suffices to show generation, that is if $\mathcal{F} \in \mathrm{D}(\mathbb{P}^n)$ then

$$\mathcal{F} \in \langle \mathcal{O}(-n), \mathcal{O}(-n+1) \dots, \mathcal{O} \rangle. \tag{I.46}$$

Let $\mathcal{F} \in \mathrm{D}(\mathbb{P}^n)$ and let $q, p : \mathbb{P}^n \times \mathbb{P}^n \to \mathbb{P}^n$ be projection onto the first and second factor respectively

$$\begin{tikzcd} \mathbb{P}^n \times \mathbb{P}^n \arrow[r, "p"] \arrow[d, "q"'] & \mathbb{P}^n \arrow[d, "q'"] \\ \mathbb{P}^n \arrow[r, "p'"] & \mathrm{Spec}(k) \end{tikzcd}$$

Now from lemma 3.8 there exists a resolution of the diagonal $L^\bullet \to \mathcal{O}_\Delta$. Splitting this resolution into exact sequences for $i = 0, \dots, n-1$ we have

$$0 \longrightarrow M_{i+1} \longrightarrow \bigwedge^i (\mathcal{O}(-1) \boxtimes \Omega(1)) \longrightarrow M_i \longrightarrow 0$$

These short exact sequences can be regarded as distinguished triangles in $\mathrm{D}(\mathbb{P}^n \times \mathbb{P}^n)$. Applying the Fourier-Mukai transform to each of the distinguished triangles above yields

$$\Phi_{M_{i+1}}(\mathcal{F}) \to \Phi_{\mathcal{O}(-i)\boxtimes\Omega^i(i)}(\mathcal{F}) \to \Phi_{M_i}(\mathcal{F}) \to \Phi_{M_{i+1}}(\mathcal{F})[1]. \tag{I.47}$$

Computing the Fourier-Mukai transform of the middle term

$$\begin{aligned} \Phi_{\mathcal{O}(-i)\boxtimes\Omega^i(i)}(\mathcal{F}) &= Rq_*(\mathcal{O}(-1) \boxtimes \Omega^i(i) \otimes Lp^*\mathcal{F}) && \text{(I.48)} \\ &\cong q_*(q^*\mathcal{O}(-i) \otimes p^*\Omega^i(i) \otimes p^*\mathcal{F}) && \text{(I.49)} \\ &\cong q_*(q^*\mathcal{O}(-i) \otimes p^*(\Omega^i(i) \otimes \mathcal{F})) && \text{(I.50)} \\ &\cong \mathcal{O}(-i) \otimes q_*(p^*(\Omega^i(i) \otimes \mathcal{F})) \qquad \text{(projection formula)} && \text{(I.51)} \\ &\cong \mathcal{O}(-i) \otimes (q')^* p'_*(\Omega^i(i) \otimes \mathcal{F})) \qquad \text{(flat base change)} && \text{(I.52)} \\ &\cong \mathcal{O}(-i) \otimes R\Gamma(\Omega^i(i) \otimes \mathcal{F}). && \text{(I.53)} \end{aligned}$$

Since $R\Gamma(\Omega^i(i)\otimes\mathcal{F})$ is a vector space and the category of vector spaces split we have for each $i\in\mathbb{Z}$, that $\Phi_{\mathcal{O}(-i)\boxtimes\Omega^i(i)}(\mathcal{F})\in\langle\mathcal{O}(-i)\rangle$. By induction this proves that $\Phi_{M_i}(\mathcal{F})\in\langle\mathcal{O}(-n),\dots,\mathcal{O}(-i)\rangle$ for all i and eventually

$$\mathcal{F}=\Phi_{\mathcal{O}_\Delta}(\mathcal{F})=\Phi_{L^\bullet}(\mathcal{F})\in\langle\mathcal{O}(-n),\dots,\mathcal{O}(-1),\mathcal{O}\rangle. \tag{I.54}$$

□

Twisting the exceptional collection by $\mathcal{O}(a)$ for any fixed $a\in\mathbb{Z}$ yields another exceptional collection for $\mathbb{P}^n$,

Corollary 3.10. *(Twisted Beilinson's Collection)*
Fix an integer $a\in\mathbb{Z}$ then $\mathcal{O}(-n+a),\mathcal{O}(-n+a+1),\dots\mathcal{O}(a)$ is an exceptional sequence and therefore

$$\mathrm{D}(\mathbb{P}^n)=\Big\langle\mathcal{O}(-n+a),\mathcal{O}(-n+a+1),\dots\mathcal{O}(a)\Big\rangle. \tag{I.55}$$

Proof. For $a\le j<i\le a+n$ and all ℓ

$$\mathrm{Hom}(\mathcal{O}(i),\mathcal{O}(j)[\ell])\simeq H^\ell(\mathbb{P}^n,\mathcal{O}(j-i))=0. \tag{I.56}$$

By the theorem we have that $\mathcal{O}(-n),\mathcal{O}(-n+1),\dots,\mathcal{O}(-1),\mathcal{O}$ is a full exceptional sequence. Since the tensor product functor $-\otimes\mathcal{O}(a)$ is an equivalence the image under this equivalence $\mathcal{O}(-n+a),\mathcal{O}(-n+a+1)\dots,\mathcal{O}$ remains full and hence we have that $\mathcal{O}(-n+a),\mathcal{O}(-n+a+1),\dots,\mathcal{O}(a)$ is a full exceptional sequence. □

Using the above resolution and commutativity of the derived tensor product we get another orthogonal decomposition of the derived category of $\mathbb{P}^n$ given by

$$\mathrm{D}(\mathbb{P}^n)=\Big\langle\mathcal{O},\Omega^1(1),\dots\Omega^n(n)\Big\rangle. \tag{I.57}$$

Orlov proved a relative version of Beilinson's result for projective bundles. The argument above used to prove Beilinsons collection can also be used to prove the relative the case, mutatis mutandis.

Theorem 3.11 ([Orl93]). *(Projective Bundle Formula)*
Let E be a vector bundle of rank r over a smooth projective variety X. Let $\pi:\mathbb{P}(E)\to X$ denote the natural projection map then for $i=0,1,\dots,r-1$ the functors $\pi^*(-)\otimes\mathcal{O}_{\mathbb{P}(E)}(i):\mathrm{D}(X)\to\mathrm{D}(\mathbb{P}(E))$ are fully faithful and yield the semiorthogonal decomposition

$$\mathrm{D}(\mathbb{P}(E))=\Big\langle\pi^*\mathrm{D}(X),\pi^*\mathrm{D}(X)(1),\dots,\pi^*\mathrm{D}(X)(r-1)\Big\rangle, \tag{I.58}$$

where $\pi^*D(X)(i):=\pi^*D(X)\otimes\mathcal{O}_{\mathbb{P}(E)}(i)$.

4 Decompositions for Quadrics and Grassmannians

In this we consider semiorthogonal decompositions of Quadrics and Grassmannians. These examples will recur throughout the thesis.

At the end of the previous section we proved Beilinson's collection for $\mathbb{P}^n$. We begin by considering the derived category of the simplest non-trivial quadric $\mathbb{P}^1 \times \mathbb{P}^1$ embedded in $\mathbb{P}^3$.

Proposition 4.1. *The quadric in $\mathbb{P}^3$ identified as $\mathbb{P}^1 \times \mathbb{P}^1$ has a semiorthogonal decomposition given by*

$$\mathrm{D}(\mathbb{P}^1 \times \mathbb{P}^1) = \Big\langle \mathcal{O}, \mathcal{O}(1,0), \mathcal{O}(0,1), \mathcal{O}(1,1) \Big\rangle, \tag{I.59}$$

where $\mathcal{O}(i,j) := \mathcal{O}_{\mathbb{P}^1}(i) \boxtimes \mathcal{O}_{\mathbb{P}^1}(j)$.

Proof. Let $X = \mathbb{P}^1 \times \mathbb{P}^1$ To show $\mathcal{O}(i,j)$ for $0 \leq i,j \leq 1$ are exceptional and the sequence $\mathcal{O}, \mathcal{O}(1,0), \mathcal{O}(0,1), \mathcal{O}(1,1)$ is semi-orthogonal take $0 \leq k,l \leq 1$ then by the Kunneth formula we have

$$\mathrm{Hom}(\mathcal{O}(i,j), \mathcal{O}(k,l)[m]) = H^m(X, \mathcal{O}(k-i, l-j)) = \begin{cases} k & \text{if } m = 0 \text{ and } (i,j) = (k,l); \\ 0 & \text{otherwise.} \end{cases} \tag{I.60}$$

Let $\iota_i : \mathbb{P}^1 \hookrightarrow \mathbb{P}^1 \times \mathbb{P}^1$ be the diagonal embedding of $\mathbb{P}^1$ with image Δ_i for $i = 1,2$. Let Δ_X be the image of the diagonal mapping of $\iota_X : X \hookrightarrow X \times X$. We have that $\mathcal{O}_{X \times X} \simeq \mathcal{O}_X \boxtimes \mathcal{O}_X$. Now for $i = 1,2$ we have have the resolution

$$0 \longrightarrow \mathcal{O}(-1) \boxtimes \Omega(1) \longrightarrow \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1} \longrightarrow \mathcal{O}_{\Delta_i} \longrightarrow 0$$

Using the fact that $\Omega^1(1) \cong \mathcal{O}_{\mathbb{P}^1}(-2) \otimes \mathcal{O}(1) \cong \mathcal{O}(-1)$ and tensoring these two resolutions together yields a resolution for $\mathcal{O}_X \simeq \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1} \boxtimes \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}$ given by

$$0 \longrightarrow \mathcal{O}(-1,-1) \longrightarrow \mathcal{O}(-1,0) \oplus \mathcal{O}(0,-1) \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_{\Delta_X}$$

With the resolution of the diagonal applying the exact same argument as we did for generation in the proof of Beilinson's collection theorem 3.9 yields the result. $\square$

Theorem 4.2 ([Kap88]). *Let $Q^n \subset \mathbb{P}^{n+1}$ be a smooth quadric. If* $\operatorname{char} k \neq 2$ *we have a semi-orthogonal decomposition for the odd dimensional quadric given by*

$$\mathrm{D}(Q^n) = \Big\langle \mathbf{S}, \mathcal{O}, \mathcal{O}(1), \dots \mathcal{O}(n-1) \Big\rangle, \tag{I.61}$$

and for the even the dimensional quadric a semiorthogonal decomposition is given by

$$\mathrm{D}(Q^n) = \Big\langle \mathbf{S}^-, \mathbf{S}^+, \mathcal{O}, \mathcal{O}(1), \dots \mathcal{O}(n-1) \Big\rangle, \tag{I.62}$$

where $\mathbf{S}$ and $\mathbf{S}^{\pm}$ are spinor bundles.

Theorem 4.3 ([BO95a]). *Intersection of even dimensional quadrics*
Let $Q_1 \cap Q_2$ be the intersection of quadrics with $\dim(Q_1 \cap Q_2) = 2g-1$ *in $\mathbb{P}^{2g+1}$ where $g \geq 2$. Then the derived category of $Q_1 \cap Q_2$ admits a semiorthogonal decomposition given by*

$$\mathrm{D}(Q_1 \cap Q_2) = \Big\langle \mathrm{D}(C), \mathcal{O}_{Q_1 \cap Q_2}, \dots \mathcal{O}_{Q_1 \cap Q_2}(2g-3) \Big\rangle. \tag{I.63}$$

where $\mathrm{D}(C)$ *is the derived category of a hyperelliptic curve with genus g, and* $\dim(Q_1 \cap Q_2) = 2g-1$.

Theorem 4.4 ([Kuz08a]). *Intersection of two odd dimensional quadrics*
Let $Q_1 \cap Q_2$ be the intersection of quadrics with $\dim(Q_1 \cap Q_2) = 2g-2$ in $\mathbb{P}^{2g+1}$ where $g \geq 2$. Then the derived category of the intersection of two even dimensional quadrics admits a semiorthogonal decomposition given by

$$\mathrm{D}(Q_1 \cap Q_2) = \Big\langle \mathrm{D}(\mathcal{C}), \mathcal{O}_{Q_1 \cap Q_2}, \dots \mathcal{O}_{Q_1 \cap Q_2}(2g-4) \Big\rangle. \tag{I.64}$$

where $\mathrm{D}(\mathcal{C})$ is the derived category of a curve with $\mathbb{Z}/2\mathbb{Z}$-stack structure at critical points.

Theorem 4.5 ([Kap88]). *(Kapranov's collection) Let $\mathbf{Gr}(k,n)$ be the Grassmannian of k-dimensional subspaces in a vector space of dimension n then there is a semiorthogonal decomposition*

$$\mathrm{D}(\mathbf{Gr}(k,n)) = \Big\langle \Sigma^\alpha \mathcal{U}^\vee \Big\rangle_{\alpha \in R(k,n-k)} \tag{I.65}$$

where $\mathcal{U}$ is the rank k tautological subbundle, $R(k,n-k)$ is the $k \times (n-k)$ rectangle, α is a Young diagram and Σ^α is the associated Schur functor.

In [Kuz08b], Kuznetsov refined this collection for when $k=2$ and found that even dimensional Grassmannians $\mathbf{Gr}(2,2m)$ admit a full exceptional collection of the form

$$\begin{pmatrix} S^{m-1}\mathcal{U}^\vee & \cdots & S^{m-1}\mathcal{U}^\vee(m-1) & & & \\ S^{m-2}\mathcal{U}^\vee & \cdots & S^{m-2}\mathcal{U}^\vee(m-1) & S^{m-2}\mathcal{U}^\vee(m) & \cdots & S^{m-2}\mathcal{U}^\vee(2m-1) \\ \dots & \cdots & \dots & \dots & \cdots & \dots \\ \mathcal{U}^\vee & \cdots & \mathcal{U}^\vee(m-1) & \mathcal{U}^\vee(m) & \cdots & \mathcal{U}^\vee(2m-1) \\ \mathcal{O} & \cdots & \mathcal{O}(m-1) & \mathcal{O}(m) & \cdots & \mathcal{O}(2m-1) \end{pmatrix} \tag{I.66}$$

and for rank 2 Grassmannians in odd dimensions $\mathbf{Gr}(2,2m+1)$ admits the exceptional collection

$$\begin{pmatrix} S^{m-1}\mathcal{U}^\vee & \cdots & S^{m-1}\mathcal{U}^\vee(m-1) & S^{m-1}\mathcal{U}^\vee(m) & \cdots & S^{m-1}\mathcal{U}^\vee(2m) \\ \dots & \cdots & \dots & \dots & \cdots & \dots \\ \mathcal{U}^\vee & \cdots & \mathcal{U}^\vee(m-1) & \mathcal{U}^\vee(m) & \cdots & \mathcal{U}^\vee(2m) \\ \mathcal{O} & \cdots & \mathcal{O}(m-1) & \mathcal{O}(m) & \cdots & \mathcal{O}(2m) \end{pmatrix} \tag{I.67}$$

Remark 4.6. From the matrices it is clear that this gives rise to a Lefschetz decomposition.

The above matrices are useful as they easily allow us to compute semiorthogonal decompositions for Grassmannians of the form $\mathbf{Gr}(2,n)$.

Example 4.7. The Grassmannian $\mathbf{Gr}(2,4)$ admits a semiorthogonal decomposition given by

$$\mathrm{D}(\mathbf{Gr}(2,4)) = \Big\langle \mathcal{O}, \mathcal{U}^\vee, \mathcal{O}(1), \mathcal{U}^\vee(1), \mathcal{O}(2), \mathcal{O}(3) \Big\rangle, \tag{I.68}$$

where $\mathcal{U}$ is the rank 2 tautological bundle.

Remark 4.8. Note that $\mathbf{Gr}(2,4)$ is also a quadric.

5 Decompositions for Curves, Surfaces and Blowups

In this section we look at semiorthogonal decompositions of curves and surfaces. First we consider varieties that do not admit semiorthogonal decompositions. It will turn out that these form the building blocks of semiorthogonal decompositions.

Proposition 5.1 ([Oka11], Thm. 1.1). *Let C be a nonsingular projective curve over an algebraically closed field k. If $g(C) \geq 1$ then $\mathrm{D}(C)$ admits no non-trivial semiorthogonal decomposition.*

Proof. Suppose $\mathrm{D}(C) = \langle \mathcal{A}, \mathcal{B} \rangle$. Let $x \in C$ be a closed point. It follows from [[GKR03], Lem. 7.2] that for every skyscraper sheaf $\mathcal{O}_x$ there exists $A \in \mathcal{A}$ and $B \in \mathcal{B}$ which are sheaves (or more precisely complexes concentrated in degree 0) such that we have the extension

$$0 \longrightarrow B \longrightarrow \mathcal{O}_x \longrightarrow A \longrightarrow 0$$

where $\mathcal{O}_x \in \mathcal{A}$ or $\mathcal{O}_x \in \mathcal{B}$. This gives a decomposition of the closed points $C(x) = C_{\mathcal{A}}(x) \coprod C_{\mathcal{B}}(x)$. Of course if $C_{\mathcal{B}}(x) = C(x)$ then $\mathcal{A}$ is trivial and we are done (since closed points form a spanning class proposition 1.19).

Assume $C_{\mathcal{B}}(x) \neq C(x)$, then $C_{\mathcal{A}}(x)$ is not empty. It follows that any coherent sheaf in $\mathcal{B}$ must be torsion, since otherwise the support of the sheaf would be the entire variety C and so there exists a non-trivial morphism from a torsion free sheaf to a closed point which belongs to $\mathcal{A}$, contradiction. It remains to show if $\mathcal{F} \in \mathrm{D}(C)$ is torsion free then $\mathcal{F} \in \mathcal{A}$. Let $\mathcal{F} \in \mathrm{D}(C)$ be torsion free then there exists an exact sequence

$$0 \longrightarrow \mathcal{F}_2 \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}_1 \longrightarrow 0$$

where $\mathcal{F}_2 \in \mathcal{B}$ is torsion and $\mathcal{F}_1 \in \mathcal{A}$. Since $\mathcal{F}_2$ is torsion and $\mathcal{F}$ is torsion free, it must be that $\mathcal{F}_2$ is trivial and hence $\mathcal{F} \cong \mathcal{F}_1$. Therefore $\mathcal{F}_1 \in \mathcal{A}$ is torsion free and so every torsion free sheaf in $\mathrm{D}(C)$ must be contained in the subcategory $\mathcal{A}$. However proposition 2.4 torsion free sheaves in $\mathrm{D}(C)$ form a spanning class, so we have a contradiction. $\square$

Recently it was proved in [Lin21] that for curves, taking the derived category of their symmetric product taken i times where i is less than the genus is indecomposable.

Theorem 5.2 ([Lin21], Thm. 1.9). *Let C be a smooth projective curve of genus $g(C) \geq 2$ then the derived category of the i-th symmetric product of C is indecomposable i.e $\mathrm{D}(\mathrm{Sym}^i(C))$ admits no non-trivial semiorthogonal decomposition for $i \leq g - 1$.*

Moving on to surfaces we begin by giving showing that Calabi-Yau varieties do not admit semiorthogonal decompositions. Bridgeland used Serre functors to prove

Theorem 5.3 ([Bri19]). *Let X be a smooth projective n-dimensional variety over k (Note X here is connected). If X is Calabi-Yau then $\mathrm{D}(X)$ admits no non-trivial semiorthogonal decomposition.*

Proof. By remark 1.10 the Serre functor on $\mathrm{D}(X)$ for X a smooth projective Calabi-Yau variety, is given by $S : \mathrm{D}(X) \to \mathrm{D}(X)$ where $S = [n]$.

Assume $\mathrm{D}(X)$ admits a Semiorthogonal Decomposition $\mathrm{D} = \langle \mathcal{A}, \mathcal{B} \rangle$. Then for all $A \in \mathcal{A}$ and $B \in \mathcal{B}$ we must have $\mathrm{Hom}(B, A[i]) = 0$ for all $i \in \mathbb{Z}$. By Serre duality 1.9 we have

$$\mathrm{Hom}(B, A[i]) \cong \mathrm{Hom}(A[i], S(B))^* = \mathrm{Hom}(A[i], B[n])^* = \mathrm{Hom}(A, B[n-i])^*, \tag{I.69}$$

letting $j = n - i$ implies $\operatorname{Hom}(A, B[j]) \cong 0$ for all $j \in \mathbb{Z}$. Therefore $\mathrm{D}(X) = \langle \mathcal{B}, \mathcal{A} \rangle$ is also a semi-orthogonal decomposition and hence $\mathrm{D}(X) = \langle \mathcal{A}, \mathcal{B} \rangle$ is a decomposable triangulated category. However by proposition 1.22 if X is connected (which it is by hypothesis) then no such decomposition can exist, contradiction. □

The above results show that the derived categoy of a curves or a Calabi-Yau variety is in general not amenable to be studied through semiorthogonal decompositions and instead require a different approach altogether (for instance via spherical objects, see [ST00]). However such derived categories still play a role in the study of semiorthogonal decompositions since they will often arise as the interesting piece (Kuznetsov component) of semiorthogonal decompositions which we encounter.

Hypersurfaces. Let X be a smooth hypersurface in $\mathbb{P}^{n+1}$ of degree d where $d \leq n + 1$. Then $\mathrm{D}(X)$ is admits a Serre functor $S_X : \mathrm{D}(X) \to \mathrm{D}(X)$ that is given by $E \mapsto E \otimes \mathcal{O}_X(d - (n + 2))[n]$ where we have used the adjunction formula $\omega_X = \mathcal{O}_X(d-(n+2))$. The line bundle $\mathcal{O}_X(i) := \mathcal{O}(i)|_X$ is exceptional if and only if $H^m(X, \mathcal{O}_X) = 0$ for all $m > 0$ but this follows from Bott vanishing of a hypersurface when $d \leq n + 1$. Now

$$\operatorname{Hom}(\mathcal{O}_X(j), \mathcal{O}_X(i)[m]) \cong \operatorname{Ext}^m(\mathcal{O}_X(j), \mathcal{O}_X(i)) \cong H^m(X, \mathcal{O}_X(i - j)), \tag{I.70}$$

By Bott vanishing again we have that

$$\mathcal{O}_X(i) \in \langle \mathcal{O}_X(j) \rangle^{\perp} \text{ if and only if } j - (n + 2 - d) < i < j. \tag{I.71}$$

so the longest sequence of line bundles with $\operatorname{Hom}(\mathcal{O}_X(j), \mathcal{O}_X(i)[m]) = 0$ for all $j > i$ is $\mathcal{O}_X, \mathcal{O}_X(1), \ldots, \mathcal{O}_X(n + 1 - d)$ and any line bundle twists of it. Therefore we have shown

Proposition 5.4. *Let $X \subset \mathbb{P}^{n+1}$ be a smooth hypersurface of degree d where $d \leq n + 1$. Then the derived category of the smooth hypersurface X admits a semiorthogonal decomposition of the form*

$$\mathrm{D}(X) = \Big\langle \mathcal{K}_X, \mathcal{O}_X, \ldots, \mathcal{O}_X(n + 1 - d) \Big\rangle. \tag{I.72}$$

where $\mathcal{K}_X := \langle \mathcal{O}_X, \ldots, \mathcal{O}_X(n + 1 - d) \rangle^{\perp}$ is the right orthogonal or the Kuznetsov component of X.

Example 5.5. Let $S \subset \mathbb{P}^3$ be a cubic surface then $\mathrm{D}(S) = \langle \mathcal{K}_S, \mathcal{O}_S \rangle$.

Example 5.6. Let $Y \subset \mathbb{P}^4$ be a cubic threefold then $\mathrm{D}(Y) = \langle \mathcal{K}_Y, \mathcal{O}_Y, \mathcal{O}_Y(1) \rangle$.

Example 5.7. Let $X \subset \mathbb{P}^5$ be a cubic fourfold then $\mathrm{D}(X) = \langle \mathcal{K}_X, \mathcal{O}_X, \mathcal{O}_X(1), \mathcal{O}_X(2) \rangle$.

In [Kuz10] Kuznetsov determines the residual/Kuznetsov component of the cubic fourfold in different cases. He conjectures a criterion which characterises the rationality of the cubic fourfold.

Conjecture 5.8 ([Kuz10], Conj. 1.1). Let $X \subset \mathbb{P}^5$ be a smooth cubic fourfold over k then by the above it has semiorthogonal decomposition of the form

$$\mathrm{D}(X) = \langle \mathcal{K}_X, \mathcal{O}_X, \mathcal{O}_X(1), \mathcal{O}_X(2) \rangle. \tag{I.73}$$

The cubic fourfold X is rational if and only if there exists a K3 surface S such that there exists an exact k-linear equivalence $\mathcal{K}_X \simeq \mathrm{D}(S)$.

It is known that Gushel-Mukai varieties can also have the derived category of a K3 surface as their Kuznetsov component. Quite remarkably Perry and Kuznetsov showed in [KP18] that these Kuznetsov components aren't just similar, they coincide.

Theorem 5.9. *[KP18] Let X be an ordinary Gushel-Mukai fourfold containing a plane of type* $\mathbf{Gr}(2,3)$. *Then there is a cubic fourfold X' such that $\mathcal{K}_X \simeq \mathcal{K}_{X'}$.*

Blow-ups. Let $\pi : \mathrm{Bl}_Z(X) \to X$ be the blow-up of a smooth projective variety X in a smooth subvariety $Z \subset X$ of codimension c. Let E denote the exceptional divisor and consider the following blow-up diagram

$$\begin{tikzcd} E \arrow[r, hook, "j"] \arrow[d, "p"'] & \mathrm{Bl}_Z(X) \arrow[d, "\pi"] \\ Z \arrow[r, hook, "i"] & X \end{tikzcd}$$

Theorem 5.10 ([Orl93], Orlov's Blow-up Formula). *With the notation and setup as above. There exists a natural semiorthogonal decomposition*

$$\mathrm{D}(\mathrm{Bl}_Z(X)) = \Big\langle \mathrm{D}(Z)_{-c+1}, \dots, \mathrm{D}(Z)_{-1}, \mathrm{D}(X) \Big\rangle, \tag{I.74}$$

Here, $\mathrm{D}(Z)_{-k} \simeq \mathrm{D}(Z)$ *is the image of the fully faithful functor* $j_*p^* : \mathrm{D}(Z) \hookrightarrow \mathrm{D}(\mathrm{Bl}_Z(X))$, *where* $F \mapsto j_*(p^*F \otimes O_E(kE))$, *where* $p : E \simeq \mathbb{P}(\mathcal{N}_{Z/X}) \to Z$ *is the exceptional divisor and* $j : E \hookrightarrow \mathrm{Bl}_Z(X)$ *is its natural closed embedding*

Example 5.11. Let S be a cubic surface where S is $\mathbb{P}^2$ blown up at 6 points. So $\pi : \mathrm{Bl}_Z(\mathbb{P}^2) \to \mathbb{P}^2$ with Z the six points $x_1, x_2, \dots, x_6 \in \mathbb{P}^2$. With the notation of the blow-up diagram

$$\begin{tikzcd} E \arrow[r, hook, "j"] \arrow[d, "p"'] & \mathrm{Bl}_Z(\mathbb{P}^2) \arrow[d, "\pi"] \\ Z \arrow[r, hook, "i"] & \mathbb{P}^2 \end{tikzcd}$$

where $p : E \simeq \mathbb{P}(\mathcal{N}_{Z/\mathbb{P}^2}) \to Z$ is the map from the exceptional divisor and $j : Z \hookrightarrow \mathbb{P}^2$ is a closed embedding. By Beilinsons collection (theorem 3.9) we have the semiorthogonal decomposition $\mathrm{D}(\mathbb{P}^2) = \langle \mathcal{O}(-2), \mathcal{O}(-1), \mathcal{O} \rangle$. Applying Orlov's blow-up formula above yields the semiorthogonal decomposition

$$\mathrm{D}(S) = \Big\langle \bigoplus_{i=1}^{6} \mathcal{O}_{E_i}(-1), \mathcal{O}_S(-2), \mathcal{O}_S(-1), \mathcal{O}_S \Big\rangle, \tag{I.75}$$

where $\mathcal{O}_S(i) = \pi^*\mathcal{O}(i)$ and E_i are the six exceptional lines.

Example 5.12. Let Y be a 3-fold and X be the blow-up of Y at a point. Then we have the exceptional divisor $E \cong \mathbb{P}^2$. Applying Orlov's blow-up formula again we obtain

$$\mathrm{D}(X) = \Big\langle \mathrm{D}(Y), \mathcal{O}_E(E), \mathcal{O}_E(2E) \Big\rangle. \tag{I.76}$$

Remark 5.13. It will be the case that HPD can in certain special cases be viewed as an Orlov-type theorem for the derived category where X admits a fibration by a projective bundle, see 1.

Remark 5.14. It is worth mentioning that currently there is no known analogue of Orlov's formula for the derived category of a weighted blow-up.

6 Mutations and Derived Categories over a Base

In this section we briefly introduce mutation functors on derived categories as they allow us to mutate semiorthogonal decompositions. The remainder of this section is devoted to establishing technical machinery that will be used later, such as Ext-boundedness, and base change results for semiorthogonal decompositions.

Mutation Functors

Assume that $\mathcal{A} \subset \mathcal{T}$ is an admissible subcategory then $\mathcal{T} = \langle \mathcal{A}, \mathcal{A}^{\perp} \rangle$ and $\mathcal{T} = \langle \mathcal{A}^{\perp}, \mathcal{A} \rangle$ are semiorthogonal decompositions. Hence ${}^{\perp}\mathcal{A}$ is right admissible and $\mathcal{A}^{\perp}$ is left admissible. Let $i_{{}^{\perp}\mathcal{A}} :^{\perp} \mathcal{A} \to \mathcal{T}$ and $i_{\mathcal{A}^{\perp}} : \mathcal{A}^{\perp} \to \mathcal{T}$ the inclusion functors

Definition 6.1. The functor $\mathbb{L}_{\mathcal{A}} := i_{\mathcal{A}^{\perp}} i^*_{\mathcal{A}^{\perp}}$ is called the left mutation through $\mathcal{A}$, where $i_{\mathcal{A}^{\perp}} : \mathcal{A}^{\perp} \to \mathcal{T}$ is the inclusion functor of the right orthogonal and $i^*_{\mathcal{A}^{\perp}}$ is its left adjoint. The functor $\mathbb{R}_{\mathcal{A}} := i_{{}^{\perp}\mathcal{A}} i^!_{{}^{\perp}\mathcal{A}}$ is called the right mutation through A where $i^!_{{}^{\perp}\mathcal{A}}$ is the right adjoint to the inclusion functor.

One can also define right mutation functors in the obvious way, however we will not require them in what follows. The following proposition contains the basic properties of mutation functors that we will need.

Proposition 6.2 ([Bon90],[BK90]). *Let $\mathcal{A}$ and $\mathcal{A}_1, \ldots, \mathcal{A}_n$ be admissible subcategories of a triangulated category $\mathcal{T}$ where $n \geq 2$ is an integer. Let k be an integer $2 \leq k \leq n$ then*

1. *$\mathbb{L}_{\mathcal{A}}|_{\mathcal{A}} = 0$ is the zero functor, $\mathbb{L}_{\mathcal{A}}|_{{}^{\perp}\mathcal{A}} :^{\perp} \mathcal{A} \to \mathcal{A}^{\perp}$ is an equivalence of categories and $(\mathbb{L}_{\mathcal{A}})|_{\mathcal{A}^{\perp}} = \mathrm{id}_{\mathcal{A}^{\perp}} : \mathcal{A}^{\perp} \to \mathcal{A}^{\perp}$ is equal to the identity functor.*

2. *For any $b \in \mathcal{T}$, there is a distinguished triangle.*
$$i_{\mathcal{A}} i^!_{\mathcal{A}} b \longrightarrow b \longrightarrow \mathbb{L}_{\mathcal{A}} b \longrightarrow i_{\mathcal{A}} i^!_{\mathcal{A}}(b)[1]$$

3. *If $\mathcal{A}_1, \ldots, \mathcal{A}_n$ is a semiorthogonal sequence then $\mathbb{L}_{\langle \mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n \rangle} = \mathbb{L}_{\mathcal{A}_1} \circ \mathbb{L}_{\mathcal{A}_2} \ldots \mathbb{L}_{\mathcal{A}_n}$.*

4. *If $\mathcal{A}_1, \ldots, \mathcal{A}_{k-1}, \mathcal{A}_K, \mathcal{A}_{k+1}, \ldots \mathcal{A}_n$ is a semiorthogonal sequence inside $\mathcal{T}$, then*
$$\mathcal{A}_1, \ldots, \mathbb{L}_{\mathcal{A}_{k-1}}(\mathcal{A}_k), \mathcal{A}_{k-1}, \mathcal{A}_{k+1}, \ldots, \mathcal{A}_n, \tag{I.77}$$
is also a semiorthogonal sequence, and it generates the same subcategory
$$\langle \mathcal{A}_1, \ldots, \mathcal{A}_{k-1}, \mathcal{A}_k, \mathcal{A}_{k+1}, \ldots, \mathcal{A}_n \rangle = \langle \mathcal{A}_1, \ldots, \mathbb{L}_{\mathcal{A}_{k-1}}(\mathcal{A}_k), \mathcal{A}_{k-1}, \mathcal{A}_{k+1}, \ldots, \mathcal{A}_n \rangle. \tag{I.78}$$

5. *Let $F : \mathcal{T} \to \mathcal{T}$ be an autoequivalence, then $F \circ \mathbb{L}_{\mathcal{A}} \simeq \mathbb{L}_{F(\mathcal{A})} \circ F$.*

Ext-Amplitude.

We briefly describe the notion of Ext-boundedness in the derived category $\mathrm{D}(X)$ and how it relates to the smoothness of X. This will be of particular importance in section II.4 when relating classical projective duality to homological projective duality.

Let $f : X \to Y$ be a morphism of algebraic varieties. For any subset $I \subset \mathbb{Z}$ we denote by $\mathrm{D}^I(X)$ the full subcategory of $\mathrm{D}(X)$ consisting of all objects $F \in \mathrm{D}(X)$with $\mathcal{H}^k(F) = 0$ for $k \notin I$.

Definition 6.3 ([Kuz05], Def. 2.24). A triangulated category $\mathcal{T}$ is Ext-bounded, if for any objects $F, G \in \mathcal{T}$ the set $\{n \in \mathbb{Z} | \operatorname{Hom}(F, G[n]) \neq 0\}$, is finite.

Lemma 6.4 ([Kuz06], Lem. 2.25). *The following conditions for an algebraic variety X are equivalent*

1. *X is smooth;*

2. *$\mathrm{D}(X) = \mathrm{D}^{perf}(X)$, where $\mathrm{D}^{perf}(X)$ is the derived category of perfect complexes over X;*

3. *the bounded derived category $\mathrm{D}(X)$ is Ext-bounded.*

Proposition 6.5 ([Kuz05], Lem. 2.26). *If $\mathcal{T} = \langle \mathcal{A}, \mathcal{B} \rangle$ is a semiorthogonal decomposition and both $\mathcal{A}$ and $\mathcal{B}$ are Ext-bounded and either $\mathcal{A}$ or $\mathcal{B}$ is admissible then $\mathcal{T}$ is Ext-bounded.*

Derived Categories over a Base

Assume S is a smooth algebraic variety, then a triangulated category $\mathcal{T}$ is called S-linear if it admits a module structure over the tensor triangulated category $\mathrm{D}(S)$. Let $f : X \to S$ be a map of smooth varieties, then $\mathrm{D}(X)$ is naturally equipped with $\mathrm{D}(S)$-module structure from the derived pullback.

Definition 6.6. An admissible subcategory $\mathcal{A} \subset \mathrm{D}(X)$ is called S-linear if for any $a \in \mathcal{A}$ and $F \in \mathrm{D}(S)$, we have $a \otimes f^*F \in \mathcal{A}$.

We would like to consider an S-linear triangulated subcategory as a family of categories over S so that under certain conditions we can pullback the family along the base change $\phi : T \to S$ to get a family of categories over T with desired properties.

Definition 6.7. A base change $\phi : T \to S$ is called faithful with respect to a morphism $f : X \to S$ if the Cartesian square

$$\begin{tikzcd} X_T \arrow[r, "\phi_T"] \arrow[d, "f_T"'] & X \arrow[d, "f"] \\ T \arrow[r, "\phi"] & S \end{tikzcd}$$

is exact Cartesian, i.e. the natural transformation $f^* \circ \phi_* \to (\phi_T)_* \circ f_T^*$ is an isomorphism. A base change $\phi : T \to S$ is called fully faithful with respect to a pair (X, Y) if ϕ is faithful with respect to morphisms $f : X \to S$, $g : Y \to S$ and $f \times_S g : X \times_S Y \to S$.

Lemma 6.8. *Let $f : X \to S$ be a morphism and $\phi : T \to S$ a base change.*

1. *If ϕ is flat, then it is faithful.*

2. *If T and X are smooth and X_T has expected dimension, i.e $\dim(X_T) = \dim(X) - \dim(T) - \dim(S)$, then $\phi : T \to S$ is faithful with respect to the morphisms $f : X \to S$.*

3. *If $\phi : T \to S$ is a closed embedding and $T \subset S$ is a locally complete intersection, and both S and X are Cohen-Macaulay and X_T has expected dimension. i.e $\dim(X_T) = \dim(X) + \dim(T) - \dim(S)$, then $\phi : T \to S$ is faithful with respect to the morphisms $f : X \to S$.*

The power of faithful base change is that it preserves S-linear fully faithful Fourier-Mulai transforms and S-linear semiorthogonal decomposition. Let us fix $S = P$ be a smooth projective variety.

Proposition 6.9. *If $\phi : T \to P$ is a faithful base change for a pair (X, Y) where $f : X \to P$ and $g : Y \to P$ and varieties X and Y are projective over P and smooth, and $K \in \mathrm{D}(X \times_P Y)$ is a kernel such that $\Phi_K : \mathrm{D}(X) \to \mathrm{D}(Y)$ is fully faithful. Then $\Phi_{K_T} : \mathrm{D}(X_T) \to \mathrm{D}(Y_T)$ is fully faithful, where the Fourier-Mukai kernel is $K_T := \phi_T^* K$.*

Proposition 6.10. *If $f : X \to P$ a map between smooth varieties, $\mathrm{D}(X) = \langle \mathcal{A}_1, \ldots, \mathcal{A}_n \rangle$ is a semiorthogonal decomposition by admissible P-linear subcategories. Let $\phi : T \to P$ is a faithful base change for f, then we have a T-linear semiorthogonal decomposition*

$$\mathrm{D}(X_T) = \langle \mathcal{A}_{1T}, \ldots, \mathcal{A}_{nT} \rangle. \tag{I.79}$$

where $\mathcal{A}_{kT}$ is the base-change of $\mathcal{A}_k$ to T which depends only on $\mathcal{A}_k$, i.e independent of the embedding $\mathcal{A}_k \subset \mathrm{D}(X)$ and satisfies $\phi_T^*(a) \in \mathcal{A}_{kT}$ for any $a \in \mathcal{A}_{kT}$, and $\phi_{T*}(b) \in \mathcal{A}_k$ for $b \in \mathcal{A}_{kT}$ with proper support over X.

Exterior Products. The exterior products of admissible subcategories of bounded derived categories of general algebraic varieties can be defined conveniently using base-change. Here we restrict ourselves to the case of product $X \times Y$ of two smooth quasi-projective varieties, say X, Y, Denote by $\pi_X : X \times Y \to X$ respectively $\pi_Y : X \times Y \to Y$. Notice $\mathrm{D}(X)$ and $\mathrm{D}(Y)$ are isomorphic to their respective categories of perfect complexes. Assume we have semiorthogonal decomposition by admissible subcategories

$$\mathrm{D}(X) = \langle \mathcal{A}_1, \ldots, \mathcal{A}_m \rangle, \text{ and } \mathrm{D}(Y) = \langle \mathcal{B}_1, \ldots \mathcal{B}_n \rangle \tag{I.80}$$

Define

$$\mathcal{A}_i \boxtimes \mathrm{D}(Y) := \langle \pi_X^* \mathcal{A}_i \otimes \pi_Y^* \mathrm{D}(Y) \rangle \subset \mathrm{D}(X \times Y), \ i = 1, \ldots, n \tag{I.81}$$

by which we mean $\mathcal{A}_i \boxtimes \mathrm{D}(Y)$ is the minimal triangulated subcategory of $\mathrm{D}(X \times Y)$ which is closed under taking summands and contains all objects of the form $\pi_X^* a \otimes \pi_Y^* F$ for $a \in \mathcal{A}_i$ and $F \in \mathrm{D}(Y)$. We define $\mathrm{D}(X) \boxtimes \mathcal{B}_j$ similarly then we have

Proposition 6.11 ([JLX17], Prop. 2.10). *Base Change for semiorthogonal Decompositions*
With the notations and assumption as above, there are Y-linear and respectively X-linear semiorthogonal decompositions

$$\mathrm{D}(X \times Y) = \langle \mathcal{A}_i \boxtimes \mathrm{D}(Y) \rangle_{1 \leq i \leq m} \textit{ and } \mathrm{D}(X \times Y) = \langle \mathrm{D}(X) \boxtimes \mathcal{B}_j \rangle_{1 \leq j \leq n}. \tag{I.82}$$

Furthermore

$$\mathrm{D}(X \times Y) = \langle \mathcal{A}_i \boxtimes \mathcal{B}_j \rangle_{1 \leq i \leq m, 1 \leq j \leq n}, \tag{I.83}$$

where the exterior product is defined as follows

$$\mathcal{A}_i \boxtimes \mathcal{B}_j := \mathcal{A}_i \boxtimes \mathrm{D}(Y) \cap \mathrm{D}(X) \boxtimes \mathcal{B}_j \subset \mathrm{D}(X \times Y). \tag{I.84}$$

Moreover we have semiorthogonal decompositions

$$\mathcal{A}_i \boxtimes \mathrm{D}(Y) = \langle \mathcal{A}_i \boxtimes \mathcal{B}_1, \ldots, \mathcal{A}_i \boxtimes \mathcal{B}_n \rangle \textit{ and } \mathrm{D}(X) \boxtimes \mathcal{B}_j = \langle \mathcal{A}_1 \boxtimes \mathcal{B}_j, \ldots, \mathcal{A}_m \boxtimes \mathcal{B}_j \rangle. \tag{I.85}$$

7 Lefschetz Decompositions

Definition 7.1. Let X be an algebraic variety with line bundle $\mathscr{L}$. A Lefschetz decomposition of $\mathrm{D}(X)$ with respect to $\mathscr{L}$ is a semiorthogonal decomposition of the form

$$\mathrm{D}(X) = \langle \mathcal{A}_0, \mathcal{A}_1(1), \dots, \mathcal{A}_{n-1}(n-1) \rangle, \tag{I.86}$$

where $0 \subset \mathcal{A}_{n-1} \subset \mathcal{A}_{n-2} \subset \cdots \subset \mathcal{A}_1 \subset \mathcal{A}_0 \subset \mathrm{D}(X)$ is an ascending chain of admissible subcategories of $\mathrm{D}(X)$ and $\mathcal{A}_k(k) := \{A(k) | A \in \mathcal{A}_k\}$. In other words each component of the decomposition is a subcategory of the previous category twisted by $\mathscr{L}$.

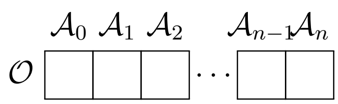
Remark 7.2. The integer $n = \min\{i \in \mathbb{Z}_{\geq 0} | \mathcal{A}_i = 0\}$ is the length of the decomposition.

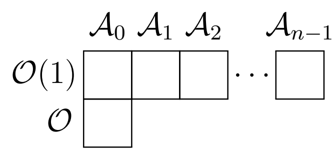
Remark 7.3. The component subcategories of each of the $\mathcal{A}_i$ are referred to as blocks, see ex. 7.5.

Definition 7.4. A Lefschetz Decomposition of $(X, \mathcal{O}_X(1))$ is called rectangular if $\mathcal{A}_0 = \cdots = \mathcal{A}_{n-1}$.

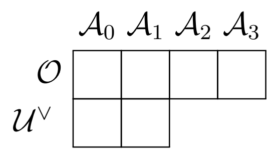
Many semi-orthogonal decompositions encountered in previous sections were indeed Lefschetz decompositions. For instance

Example 7.5. For $X = \mathbb{P}^n$ and $\mathscr{L} = \mathcal{O}(1)$ Beilinson's collection give a semiorthogonal decomposition $\mathrm{D}(\mathbb{P}^n) = \langle \mathcal{O}, \mathcal{O}(1), \dots, \mathcal{O}(n) \rangle$ we have a Lefschetz decomposition of length $n+1$ with $\mathcal{A}_0 = \cdots = \mathcal{A}_n = \langle \mathcal{O} \rangle$. There is also a Lefschetz decomposition of length n with $\mathcal{A}_0 = \langle \mathcal{O}, \mathcal{O}(1) \rangle$ and $\mathcal{A}_1 = \cdots = \mathcal{A}_{n-1} = \langle \mathcal{O}(1) \rangle$. These can be visualized by the diagrams below

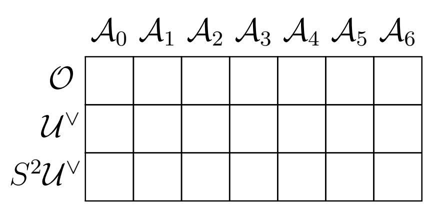




Example 7.6. Consider $\mathbf{Gr}(2,4)$ where $\mathrm{D}(\mathbf{Gr}(2,4)) = \langle \mathcal{O}, \mathcal{U}^\vee, \mathcal{O}(1), \mathcal{U}^\vee(1), \mathcal{O}(2), \mathcal{O}(3) \rangle$ admits a Lefschetz decomposition given by $\mathcal{A}_0 = \mathcal{A}_1 = \langle \mathcal{O}, \mathcal{U}^\vee \rangle$ and $\mathcal{A}_2 = \mathcal{A}_3 = \langle \mathcal{O} \rangle$.



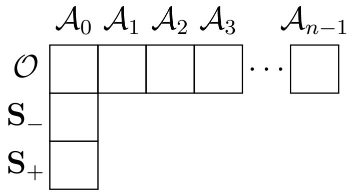
Example 7.7. Consider $\mathbf{Gr}(2,7)$ where $\mathrm{D}(\mathbf{Gr}(2,7)) = \langle \mathcal{O}, \mathcal{U}^\vee, S^2\mathcal{U}^\vee, \dots \mathcal{O}(6), \mathcal{U}^\vee(6), S^2\mathcal{U}^\vee(6) \rangle$ admits a Lefschetz decomposition given by $\mathcal{A}_0 = \mathcal{A}_1 = \cdots = \mathcal{A}_6 = \langle \mathcal{O}, \mathcal{U}^\vee, S^2\mathcal{U}^\vee \rangle$.

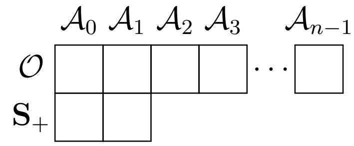


Remark 7.8. Notice the correspondence between the above and the matrices in I.66 and I.67.

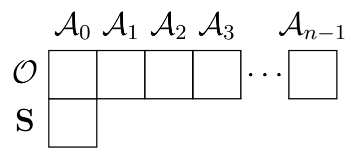
Remark 7.9. The above examples naturally extend to decompositions of $\mathbf{Gr}(2,2n)$ and $\mathbf{Gr}(2,2n+1)$.

Example 7.10. (Even Dimensional Quadric) The quadric Q^n with semiorthogonal decomposition $\mathrm{D}(Q^n) = \langle \mathbf{S}_+, \mathbf{S}_-, \mathcal{O}, \ldots \mathcal{O}(n-1)\rangle$ admits[4] a Lefschetz decomposition given by $\mathcal{A}_0 = \langle \mathbf{S}_+, \mathbf{S}_-, \mathcal{O}\rangle$ and $\mathcal{A}_1 = \cdots = \mathcal{A}_{n-1} = \langle \mathcal{O}\rangle$. Moreover can mutate S_- past $\mathcal{O}$ to obtain a new decomposition $\mathrm{D}(Q^n) = \langle \mathbf{S}_+, \mathcal{O}, \mathbf{S}_+(1), \mathcal{O}(1), \ldots, \mathcal{O}(n-1)\rangle$ which has Lefschetz decomposition $\mathcal{A}_0 = \mathcal{A}_1 = \langle \mathbf{S}_+, \mathcal{O}\rangle$ and $\mathcal{A}_2 = \cdots = \mathcal{A}_{n-1} = \langle \mathcal{O}\rangle$. This decomposition and the mutated decomposition have the form[5]





Example 7.11. The odd dimensional quadric Q^n with $\mathrm{D}(Q^n) = \langle \mathbf{S}, \mathcal{O}, \ldots, \mathcal{O}(n-1)\rangle$ admits a Lefschetz decomposition given by $\mathcal{A}_0 = \langle \mathbf{S}, \mathcal{O}\rangle$ and $\mathcal{A}_1 = \cdots = \mathcal{A}_{n-1} = \langle \mathcal{O}\rangle$.



Dual Lefschetz Decompositions. In the next chapter we will be working intricately with the dual Lefschetz decomposition, we spend the remaining part of this chapter constructing it. Suppose we are given a smooth projective variety X with a Lefschetz decomposition

$$\mathrm{D}(X) = \Big\langle \mathcal{A}_0, \mathcal{A}_1(1), \ldots, \mathcal{A}_{n-1}(n-1)\Big\rangle, \tag{I.87}$$

. Let $\mathfrak{a}_k$ denote the right orthogonal to $\mathcal{A}_{k+1}$ in $\mathcal{A}_k$ for $0 \leq k \leq n-1$, that is

$$\mathfrak{a}_k := \{E \in \mathcal{A}_k | \operatorname{Hom}(A[i], E) = 0, \text{ for all } A \in \mathcal{A}_{k+1},\ i \in \mathbb{Z}\}. \tag{I.88}$$

The categories $\mathfrak{a}_0, \mathfrak{a}_1, \ldots, \mathfrak{a}_{n-1}$ are are called the primitive categories of the Lefschetz decomposition I.87. By definition we have the following semiorthogonal decomposition each component category

$$\mathcal{A}_k = \Big\langle \mathfrak{a}_k, \mathfrak{a}_{k+1}, \ldots, \mathfrak{a}_{n-1}\Big\rangle. \tag{I.89}$$

If the Lefschetz decomposition I.87 is rectangular then $\mathfrak{a}_0 = \mathfrak{a}_1 = \cdots = \mathfrak{a}_{n-2} = 0$ and $\mathfrak{a}_{n-1} = \mathcal{A}_{n-1}$.

Moreover a Lefschetz decomposition can be determined by specifying the first component $\mathcal{A}_0$ since all other $\mathcal{A}_k$'s can be reconstructed inductively from $\mathcal{A}_0$ via

$$\mathcal{A}_k =^{\perp} \mathcal{A}_0(-k) \cap \mathcal{A}_{k-1}, \tag{I.90}$$

Since $\mathrm{D}(X)$ is admissible then each of the primitive categories $\mathfrak{a}_k$ are admissible subcategories of $\mathcal{A}_k$. Let $i_0 : \mathcal{A}_0 \to \mathrm{D}(X)$ be corresponding the inclusion functor and let $i_0^* : \mathrm{D}(X) \to \mathcal{A}_0$ be its left adjoint. Pulling back primitive categories of the Lefschetz decomposition gives a semiorthogonal decomposition.

[4]note we have swapped the sign convention for the spinors.

[5]Taking $n = 4$ we get another Lefschetz decomposition for $\mathbf{Gr}(2, 4)$.

Lemma 7.12 ([JLX17], Lem 2.12). *Given the above setup, for $k = 1, \dots, n$ the sequence $\mathfrak{a}_0(1), \dots, \mathfrak{a}_{k-1}(k)$ pulled back along i_0^* is a semiorthogonal sequence in $\mathcal{A}_0$. Moreover since $\mathcal{A}_n = 0$ we have*

$$\Big\langle \mathcal{A}_0(1), \mathcal{A}_1(2), \dots, \mathcal{A}_{k-1}(k) \Big\rangle = \Big\langle i_0^*(\mathfrak{a}_0(1)), i_0^*(\mathfrak{a}_1(1)), \dots, i_0^*(\mathfrak{a}_{k-1}(k)), \mathcal{A}_1(1), \dots \mathcal{A}_k(k) \Big\rangle, \tag{I.91}$$

and in particular taking $k = n$ we obtain a semiorthogonal decomposition of $\mathcal{A}_0$ in terms of the primitive subcategories

$$\mathcal{A}_0 = \Big\langle \mathfrak{a}_0(1), \mathfrak{a}_1(2), \dots, \mathfrak{a}_{n-1}(n) \Big\rangle. \tag{I.92}$$

where each primitive category is embedded via pullback along the inclusion $i_0^* : \mathrm{D}(X) \to \mathcal{A}_0$.

Proof. Recall that left mutation functor is given by $\mathbb{L}_{\mathcal{A}} = i_{\mathcal{A}^\perp} i^*_{\mathcal{A}^\perp} : \mathrm{D}(X) \to \mathrm{D}(X)$ and the properties in proposition 6.2. From the definition of primitive subcategory we have $\mathcal{A}_k = \langle \mathfrak{a}_k(k+1), \mathcal{A}_{k+1} \rangle$ twisting by $(k+1)$ yields $\mathcal{A}_k(k+1) = \langle \mathfrak{a}_k(k+1), \mathcal{A}_{k+1}(k+1) \rangle$. Therefore combining this with properties 3 and 4 from proposition 6.2 we can mutate all primitive subcategories past the $\mathcal{A}_n(n)$

$$\Big\langle \mathcal{A}_0(1), \mathcal{A}_1(2), \dots, \mathcal{A}_{k-1}(k) \Big\rangle = \Big\langle \mathfrak{a}_0(1), \mathcal{A}_1(1), \mathfrak{a}_1(2), \mathcal{A}_1(2), \dots, \mathfrak{a}_{k-1}(k), \mathcal{A}_k(k) \Big\rangle \tag{I.93}$$

$$= \Big\langle \mathfrak{a}_0(1), \mathbb{L}_{\mathcal{A}_1(1)} \mathfrak{a}_1(2), \mathcal{A}_1(2), \dots, \mathfrak{a}_{k-1}(k), \mathcal{A}_k(k) \Big\rangle \tag{I.94}$$

$$= \Big\langle \mathfrak{a}_0(1), \mathbb{L}_{\mathcal{A}_1(1)} \mathfrak{a}_1(2), \dots, \mathbb{L}_{\langle \mathcal{A}_1(1), \dots \mathcal{A}_k(k) \rangle} \mathfrak{a}_{k-1}(k), \mathcal{A}_1(1), \dots, \mathcal{A}_k(k) \Big\rangle \tag{I.95}$$

$$= \Big\langle i_0^*(\mathfrak{a}_0(1)), i_0^*(\mathfrak{a}_1(1)), \dots, i_0^*(\mathfrak{a}_{k-1}(k)), \mathcal{A}_1(1), \dots \mathcal{A}_k(k) \Big\rangle, \tag{I.96}$$

where in the last step we suppress the i_0 and hence from $\mathfrak{a}_k(k+1) \subset \mathcal{A}_0$, it follows that $i_0 i_0^* = \mathbb{L}_{\mathcal{A}_1(1), \dots, \mathcal{A}_{k-1}(k-1)}$ coincide for all k. ☐

Now we have a new semiorthogonal decomposition for $\mathcal{A}_0$. We can use this to generate the following dual sequence of ascending subcategories

$$\mathcal{B}^k := \Big\langle \mathfrak{a}_0(1), \mathfrak{a}_1(2), \dots \mathfrak{a}_{k-1}(k) \Big\rangle \subset \mathcal{A}_0 \text{ for } 1 \leq k \leq n. \tag{I.97}$$

where $\mathcal{B}^1 \subset \cdots \subset \mathcal{B}^{n-1} = \mathcal{A}_0$, defining $\mathcal{B}^k = 0$ for $k \leq 0$ and $\mathcal{B}^k = \mathcal{A}_0$ for $k \geq n-1$. So we intuitively we can regard $\mathcal{B}^k$ as the complement of $\mathcal{A}_k$ inside $\mathcal{A}_0$[6]. Soon we will see that these $\mathcal{B}^k$'s form the subcategories of the Lefschetz decomposition of the Homological Projective Dual Variety.

Definition 7.13. Let X be a smooth projective variety over k with a length n Lefschetz decomposition

$$\mathrm{D}(X) = \Big\langle \mathcal{A}_0, \mathcal{A}_1(1), \dots, \mathcal{A}_{n-2}(n-2), \mathcal{A}_{n-1}(n-1) \Big\rangle, \tag{I.98}$$

The dual Lefschetz decomposition is the unique decomposition of the same length of the form

$$\mathrm{D}(X) = \Big\langle \mathcal{B}^0(-n+1), \mathcal{B}^1(-n+2), \dots, \mathcal{B}^{n-2}(-1), \mathcal{B}^{n-1} \Big\rangle, \tag{I.99}$$

where $\mathcal{B}^0 \subset \mathcal{B}^1 \subset \cdots \subset \mathcal{B}^{n-1}$ and $\mathcal{B}^{n-1} = \mathcal{A}_0$. Hence $\mathcal{B}^{n-1}$ generates the dual decomposition.[7][8]

[6] which is another reason for dropping the notation i_0^*.

[7] There is also another 'homological notation' for dual Lefschetz decomposition they are related by $\mathcal{B}^k \equiv \mathcal{B}_{n-k-1}$.

[8] It is worth pointing out that there is a more general definition of Lefschetz dual, where the dual Lefschetz decomposition does not have to have the same length. Of course using this definition the dual is not unique.

Chapter II

Homological Projective Duality

1 Derived Category of the Universal Hyperplane Section

Following Thomas in [Tho17], we begin by proving a much simpler special case of Kuznetsov's Homological Projective Duality theorem. This special case can be seen as generalization of Orlov's projective bundle and blowup formula. The setup is as follows.

Fix $(X, \mathcal{O}_X(1))$, a variety with a semi-ample line bundle, and a basepoint-free linear system $V \subseteq H^0(\mathcal{O}_X(1))$. This corresponds to a morphism $f : X \to \mathbb{P}(V^*)$ where f is non-degenerate i.e the image of f is not contained in any hyperplane. Naturally $f : X \to \mathbb{P}(V^*)$ defines a map into the dual projective space where the image of $g : \mathcal{H} \to \mathbb{P}(V)$ is the parameter space of all hyperplanes containing X. That is the variety $\mathcal{H}$ is the universal family of hyperplanes, where

$$\mathcal{H} := \{(x, s) \in X \times \mathbb{P}(V) | s(x) = 0\} \subset X \times \mathbb{P}(V) \subseteq \mathbb{P}(V^*) \times \mathbb{P}(V), \tag{II.1}$$

is the incidence hyperplane. Let $L \subset \mathbb{P}(V)$ be an ℓ dimensional linear subspace and let $L^\perp \subset V^*$ be the corresponding annihilator subspace. So that $\mathbb{P}(L)$ parameterizes the subfamily of hyperplanes containing $\mathbb{P}(L^\perp)$ in $\mathbb{P}(V^*)$. Let the fibered products

$$X_{L^\perp} := X \times_{\mathbb{P}(V^*)} \mathbb{P}(L^\perp) \quad \text{and} \quad \mathcal{H}_L := \mathcal{H} \times_{\mathbb{P}(V)} \mathbb{P}(L). \tag{II.2}$$

denote the family of sections of X by $\mathbb{P}(L^\perp)$ and the dual family of hyperplane sections of X, respectively. Since $X_{L^\perp}$ can be viewed as the baselocus of the linear system $L \subseteq H^0(\mathcal{O}_X(1))$, it is contained in every fiber of $\mathcal{H}_L \to \mathbb{P}(L)$, giving the diagram

$$\begin{tikzcd}
X_{L^\perp} \times \mathbb{P}(L) \arrow[r, hook, "j"] \arrow[d, "p"] & \mathcal{H}_L \arrow[r, hook, "\iota"] \arrow[d, "\pi"] & X \times \mathbb{P}(L) \arrow[dl, "\rho"] \\
X_{L^\perp} \arrow[r, hook, "i"] & X &
\end{tikzcd}$$

Remark 1.1. Notice the analogy with the blowup diagram (see figure 5).

Remark 1.2. Since $\pi^{-1}(x) = \{H \in \mathbb{P}(L) | x \in H\} = \mathbb{P}(\ker(\mathrm{ev}_x : L \to \mathcal{O}_X(1)))$, the map π has generic fibre $\mathbb{P}^{\ell-2}$ and over the baselocus/"blow-up locus" has fiber $\pi^{-1}(x) = \mathbb{P}(L) = \mathbb{P}^{\ell-1}$.

Definition 1.3. With setup as above. The linear subspace $L \subset V$ is called admissible if

$$\dim(X_{L^\perp}) = \dim(X) - \ell. \tag{II.3}$$

Theorem 1.4 ([Tho17], Prop. 3.6). *If the subspace L is admissible i.e $\mathbb{P}(L^{\perp})$ intersects X with expected dimension. Then the functors $j_*p^* : \mathrm{D}(X_{L^{\perp}}) \to \mathrm{D}(\mathcal{H}_L)$ and $\pi^* : \mathrm{D}(X) \to \mathrm{D}(\mathcal{H}_L)$ are full and faithful embedding and so the universal hyperplane section admits a semiorthogonal decomposition of the form*

$$\mathrm{D}(\mathcal{H}_L) = \Big\langle j_*p^*\mathrm{D}(X_{L^{\perp}}), \pi^*\mathrm{D}(X)(0,1), \dots, \pi^*\mathrm{D}(0, \ell-1)\Big\rangle, \tag{II.4}$$

where by (i,j) we mean the by $\mathcal{O}(i,j)$, the restriction of $\mathcal{O}_X(i) \boxtimes \mathcal{O}_{\mathbb{P}(V)}(j)$ to $\mathcal{H}_L \subset X \times \mathbb{P}(L)$.

Proof. The baselocus $X_{L^{\perp}} \subset X$ is cut out by sections of $\mathcal{O}_X(1)$, one for each element of a basis of L. Invariantly, it is cut out by the section

$$\sigma \in H^0(\mathcal{O}_X(1) \otimes L^*) \cong \mathrm{Hom}(L, H^0(\mathcal{O}_X(1))), \tag{II.5}$$

corresponding to the inclusion $L \subseteq V \subseteq H^0(\mathcal{O}_X(1))$. Since L is admissible σ is a regular section of $\mathcal{O}_X(1) \otimes L^*$.

Now consider the dual of the Euler sequence for L

$$0 \longrightarrow \Omega_{\mathbb{P}(L)}(1) \longrightarrow \mathcal{O}_{\mathbb{P}(L)} \otimes L^* \longrightarrow \mathcal{O}_{\mathbb{P}(L)}(1) \longrightarrow 0$$

Tensoring by $\mathcal{O}_X(1)$ and pulling $\mathcal{O}_X(1) \otimes L^*$ back to $X \times \mathbb{P}(L)$ we get the exact sequence

$$0 \longrightarrow \mathcal{O}_X(1) \boxtimes \Omega_{\mathbb{P}(L)}(1) \longrightarrow \mathcal{O}_X(1) \boxtimes L^* \xrightarrow{\mathrm{ev}} \mathcal{O}_X(1) \boxtimes \mathcal{O}_{\mathbb{P}(L)}(1) \longrightarrow 0$$

The section σ sits inside the exact the above short exact sequence and corresponds to the section

$$\mathrm{ev} \circ \sigma \in H^0(\mathcal{O}_{X \times \mathbb{P}(L)}(1,1)) \cong \mathrm{Hom}(L, H^0(\mathcal{O}_X(1))), \tag{II.6}$$

which also corresponds to the inclusion $L \subseteq H^0(X, \mathcal{O}_X(1))$. It's zero locus is therefore $\mathcal{H}_L \subset X \times \mathbb{P}(L)$ on restriction to which σ lifts canonically to a section $\tilde{\sigma}$ the kernel of the above short exact sequence,

$$\tilde{\sigma} \in H^0(\Omega_{\mathbb{P}(L)}(1,1)|_{\mathcal{H}_L}). \tag{II.7}$$

Since L is of expected dimension the section $\tilde{\sigma}$ is a regular section cutting out $X_{L^{\perp}} \times \mathbb{P}(L) \subset \mathcal{H}_L$, with the normal bundle of $j : X_{L^{\perp}} \times \mathbb{P}(L) \to \mathcal{H}_L$ given by

$$N_j \cong \mathcal{O}_{X_{L^{\perp}}}(1) \boxtimes \Omega_{\mathbb{P}(L)}(1). \tag{II.8}$$

Now to show j_*p^* is fully faithful, Consider the exact triangle given by completing the counit morphism $\varepsilon : j^*j_* \to \mathrm{id}$ we get

$$j^*j_*p^*E \xrightarrow{\varepsilon} p^*E \longrightarrow \mathrm{Cone}(\varepsilon) \longrightarrow j^*j_*p^*E[1]$$

for $E \in \mathrm{D}(X_{L^{\perp}})$. Taking $R\,\mathrm{Hom}(-, p^*E)$ gives the triangle

$$R\,\mathrm{Hom}(p^*E, p^*E) \longrightarrow R\,\mathrm{Hom}(j^*j_*p^*E, p^*E) \longrightarrow R\,\mathrm{Hom}(\mathrm{Cone}(\varepsilon), p^*E)$$

It is therefore sufficient to show $\mathrm{Cone}(\varepsilon) \cong 0$. Since then by the isomorphism in the above triangle and adjunction we conclude

$$R\,\mathrm{Hom}(j_*p^*E, j_*p^*E) \cong R\,\mathrm{Hom}(j^*j_*p^*E, p^*E) \cong R\,\mathrm{Hom}(p^*E, p^*E) \cong R\,\mathrm{Hom}(E, E), \tag{II.9}$$

for all $E \in X_{L^\perp}$. To show that $R\operatorname{Hom}(\operatorname{Cone}(\varepsilon), p^*E) \cong 0$ observe

$$R\operatorname{Hom}(j^*j_*p^*E, p^*E) \cong R\operatorname{Hom}(p^*E, p^*E), \tag{II.10}$$

for any $E \in \mathrm{D}(X_{L^\perp})$. Then computing $R\operatorname{Hom}(\operatorname{Cone}(\varepsilon), p^*E)$, we have that by a standard argument on Fourier-Mukai kernels (c.f e.g [[Huy06], Prop 11.8]) the $\operatorname{Cone}(j^*j_* \to \mathrm{id})$ is an iterated extension of the functors

$$\Lambda^r N_j^*[r] \otimes (-), \quad \text{for } 1 \leq r \leq \ell - 1. \tag{II.11}$$

Therefore the cone is an iterated extension of the groups

$$\begin{aligned}
R\operatorname{Hom}(\operatorname{Cone}(\varepsilon), p^*E) &\cong R\operatorname{Hom}(\Lambda^r N_j^* \otimes p^*E, p^*E)[-r] && \text{(II.12)}\\
&\cong R\operatorname{Hom}(p^*E, p^*E \otimes \Lambda^r N_j)[-r] && \text{(II.13)}\\
&\cong R\operatorname{Hom}(E, p_*(p^*E \otimes \Lambda^r N_j))[-r] && \text{(II.14)}\\
&\cong R\operatorname{Hom}(E, E \otimes p_*(\Lambda^r N_j))[-r] && \text{(II.15)}\\
&\cong 0. && \text{(II.16)}
\end{aligned}$$

However because $N_j \cong \mathcal{O}_{X_{L^\perp}}(1) \boxtimes \Omega_{\mathbb{P}(L)}(1)$ it follows

$$p_*(\Lambda^r N_j) = p_*(\mathcal{O}_{X_{L^\perp}}(r) \boxtimes \Omega^r_{\mathbb{P}(L)}(r)) = 0, \tag{II.17}$$

as $\Omega^r_{\mathbb{P}^{\ell-1}}(r)$ is acyclic for $1 \leq r \leq \ell - 1$.

To prove semiorthogonality we begin by showing

$$\pi^*\mathrm{D}(X)(0,k) \in^\perp \left\langle j_*p^*\mathrm{D}(X_{L^\perp}) \right\rangle \text{ for } 1 \leq n < k \leq \ell - 1. \tag{II.18}$$

Let $E \in \mathrm{D}(X_{L^\perp})$ and $F \in \mathrm{D}(X)$, making use of adjunction, the projection formula, and the commutativity of diagram 1 we obtain

$$\begin{aligned}
R\operatorname{Hom}(\pi^*F(0,k), \pi^*E(0,n)) &\cong R\operatorname{Hom}(j^*\pi^*F(0,k), p^*E) && \text{(II.19)}\\
&\cong R\operatorname{Hom}(p^*i^*F(0,k), p^*E) && \text{(II.20)}\\
&\cong R\operatorname{Hom}(p^*i^*F, p^*E \otimes \mathcal{O}(0,-k)) && \text{(II.21)}\\
&\cong R\operatorname{Hom}(i^*F, p_*(p^*E \otimes \mathcal{O}(0,-k)) && \text{(II.22)}\\
&\cong R\operatorname{Hom}(i^*F, E \otimes p_*\mathcal{O}(0,-k)) && \text{(II.23)}\\
&\cong 0, && \text{(II.24)}
\end{aligned}$$

since $p_*\mathcal{O}(0,-k) \cong 0$. Next we will show that

$$\pi^*\mathrm{D}(X)(0,k) \in^\perp \left\langle \pi^*\mathrm{D}(X)(0,n) \right\rangle \text{ for } 1 \leq n < k \leq \ell - 1. \tag{II.25}$$

Let $E, F \in \mathrm{D}(X)$ by the commutativity of the diagram 1 and adjunction

$$\begin{aligned}
R\operatorname{Hom}(\pi^*F(0,k), \pi^*E(0,n)) &\cong R\operatorname{Hom}(\iota^*\rho^*F(0,k), \iota^*\rho^*E(0,n)) && \text{(II.26)}\\
&\cong R\operatorname{Hom}(\rho^*F, \iota_*\iota^*\rho^*E(0,n-k)). && \text{(II.27)}
\end{aligned}$$

Since $\mathcal{H}_L \subset X \times \mathbb{P}(V)$ is a divisor with bidegree (1,1), there is an exact triangle

$$\ldots \longrightarrow \rho^*E(-1,n-k-1) \longrightarrow \rho^*E(0,n-k) \longrightarrow \iota_*\iota^*\rho^*E(0,n-k) \xrightarrow{+1} \ldots$$

and applying the functor $R\operatorname{Hom}(\rho^*F, -)$ then yields an exact triangle. The first term is

$$R\operatorname{Hom}(\rho^*F, \rho^*E(0, n-k)) \cong 0, \tag{II.28}$$

since $\rho_*\mathcal{O}(0, n-k) = 0$, and the second term is

$$R\operatorname{Hom}(\rho^*F, \rho^*E(-1, n-k-1)) \cong 0, \tag{II.29}$$

as $\rho_*\mathcal{O}(0, n-k-1) = 0$. This implies that the third term is also trivial

$$R\operatorname{Hom}(\rho^*F, \iota_*\iota^*\rho^*E(0, n-k)) \cong 0. \tag{II.30}$$

Therefore equation II.26 allows us to conclude $R\operatorname{Hom}(\pi^*F(0,k), \pi^*E(0,n)) \cong 0$.

To see that $\pi^*\mathrm{D}(X)(0,k)$ is fully faithfully embedded in $\mathrm{D}(\mathcal{H}_L)$. Let $E, F \in \mathrm{D}(X)$ then

$$\begin{aligned} R\operatorname{Hom}(\pi^*F(0,k), \pi^*E(0,k)) &\cong R\operatorname{Hom}(\iota^*\rho^*F(0,k), \iota^*\rho^*E(0,k)) \text{ (as } \pi = \rho \circ \iota) & \text{(II.31)}\\ &\cong R\operatorname{Hom}(\rho^*F, \iota_*\iota^*\rho^*E), & \text{(II.32)} \end{aligned}$$

Again since $\mathcal{H}_L \subset X \times \mathbb{P}(V)$ is a divisor with bi-degree (1,1), we have the exact triangle

$$\dots \longrightarrow \rho^*E(-1,-1) \longrightarrow \rho^*E \longrightarrow \iota_*\iota^*\rho^*E \xrightarrow{+1} \dots$$

Applying the functor $R\operatorname{Hom}(\rho^*F, -)$ we get an exact triangle with the left most term

$$R\operatorname{Hom}(\rho^*F, \rho^*E(-1,-1)) \cong 0, \tag{II.33}$$

since $\rho_*\mathcal{O}(0,-1) = 0$. Therefore the exact triangle gives an isomorphism from which we deduce

$$R\operatorname{Hom}(\pi^*F(0,k), \pi^*E(0,k)) \cong R\operatorname{Hom}(\rho^*F, \iota_*\iota^*\rho^*E) \cong R\operatorname{Hom}(\rho^*F, \rho^*E) \cong R\operatorname{Hom}(F,E). \tag{II.34}$$

It remains to show that the semi-orthogonal sequence generates $\mathrm{D}(\mathcal{H}_L)$. Suppose

$$E \in {}^{\perp}\Big\langle \mathrm{D}(X_{\perp}), \pi^*\mathrm{D}(X)(0,1), \dots, \pi^*\mathrm{D}(X)(0,\ell-1)\Big\rangle, \tag{II.35}$$

and consider the case where $x \notin X_{L^{\perp}}$, so that $\pi^{-1}(x) \cong \mathbb{P}^{\ell-2}$ and π is locally flat. Thus the pushforward to $\mathcal{H}_L$ of $\mathcal{O}_{\pi^{-1}(x)}$ is $\pi^*\mathcal{O}_x \in \mathrm{D}(X)$, and

$$R\operatorname{Hom}(E_x, \mathcal{O}(k)) \cong R\operatorname{Hom}(E, (\pi^*\mathcal{O}_x)(k)) = 0, \tag{II.36}$$

since by assumption E is in the left orthogonal. As the sheaves $\mathcal{O}(k)$, from $1 \le k \le \ell - 1$, span $\mathrm{D}(\mathbb{P}^{\ell-2})$ by Beilinson's collection theorem 3.9, we have that $E_x = 0$.

Now consider the case where $x \in X_{L^{\perp}}$, so that $\pi^{-1}(x) \cong \mathbb{P}^{\ell-1}$. Since π is no longer flat near $\pi^{-1}(x)$ we instead use the commutativity of diagram 5 to compute $\pi^*\mathcal{O}_x$:

$$\pi^*\mathcal{O}_x = \iota^*\rho^*\mathcal{O}_x = \iota^*\iota_*j_*\mathcal{O}_{\{x\}\times\mathbb{P}(L)}, \tag{II.37}$$

Since $\iota : \mathcal{H}_L \to X \times \mathbb{P}(L)$ is a (1,1) divisor, we have an exact triangle $\mathrm{id}(-1,-1)[1] \to \iota^*\iota_* \to \mathrm{id}$. Applied to $j_*\mathcal{O}_{\{x\}\times\mathbb{P}(L)}$ this gives

$$\dots \longrightarrow j_*\mathcal{O}_{\{x\}\times\mathbb{P}(L)}(-1,-1)[1] \longrightarrow \pi^*\mathcal{O}_x \longrightarrow j_*\mathcal{O}_{\{x\}\times\mathbb{P}(L)} \xrightarrow{+1} \dots$$

Now tensoring this exact triangle by $\mathcal{O}(0,k)$ and applying $R\operatorname{Hom}(E_x,-)$ gives the exact triangle

$$\ldots \longrightarrow R\operatorname{Hom}(E_x,\mathcal{O}_{\mathbb{P}(L)}(k-1))[1] \longrightarrow 0 \longrightarrow R\operatorname{Hom}(E_x,\mathcal{O}_{\mathbb{P}(L)}(k)) \xrightarrow{+1} \ldots$$

for $1 \leq k \leq \ell-1$, because $R\operatorname{Hom}(E,(\pi^*\mathcal{O}_x)(k)) = 0$ but by assumption also gives

$$0 = R\operatorname{Hom}(E, j_*p^*\mathcal{O}_x) = R\operatorname{Hom}(E,\mathcal{O}_{\{x\}\times\mathbb{P}(L)}) = R\operatorname{Hom}(E_x,\mathcal{O}_{\mathbb{P}(L)}). \tag{II.38}$$

By the above triangle this gives the vanishing

$$R\operatorname{Hom}(E_x,\mathcal{O}_{\mathbb{P}(L)}(k)) = 0, \text{ for } 0 \leq k \leq \ell-1, \tag{II.39}$$

which by Beilinson implies $E_x = 0$. We claim that this implies $E = 0$. Replacing E by a quasi-isomorphic finite complex $E^\bullet$ of locally free sheaves, we have that its restriction to p is an exact complex of vector spaces. In particular the final map is onto and so by the Nakayama lemma, the final map in $E^\bullet$ is also onto in a neighbourhood of x. Therefore locally we have

$$0 \longrightarrow E_U^n \xrightarrow{f_n} E_U^{n-1} \xrightarrow{f_{n-1}} \ldots \xrightarrow{f_2} E_U^1 \xrightarrow{f_1} E_U^0 \longrightarrow 0$$

and this complex is quasi-isomorphic to the same complex with the second last term replaced by $\ker(f_1)$ and the final term replaced by 0. By induction we eventually have that E is quasi-isomorphic to 0 in a neighbourhood of x for every x and hence $E = 0$. □

Notice the similarities between the decomposition of the above and Orlov's projective bundle and blow up formula: Recall Orlov's projective bundle formula had the suggestive form

$$\mathrm{D}(X) = \Big\langle \mathrm{D}(B), \mathrm{D}(B)(1), \ldots, \mathrm{D}(B)(n-1) \Big\rangle, \tag{II.40}$$

for $X = \mathbb{P}(E) \xrightarrow{\pi} B$ a projective $\mathbb{P}^{n-1}$-bundle of a rank n vector bundle E over B. Similarly with the notation of the diagram below

$$\begin{tikzcd} E \arrow[r, hook, "j"] \arrow[d, "p"'] & \mathrm{Bl}_Z(X) \arrow[d, "\pi"] \\ Z \arrow[r, hook, "i"] & X \end{tikzcd}$$

we have that Orlov's blowup formula 5.10 has the form

$$\mathrm{D}(\mathrm{Bl}_Z(X)) = \Big\langle \mathrm{D}(X), \mathrm{D}(Z), \mathrm{D}(Z)(-E), \ldots, \mathrm{D}(Z)(-(n-2)E) \Big\rangle. \tag{II.41}$$

Using the same HPD argument above in 1.4 one can show generation for both of Orlov's theorems. Then to show the required functors are full and faithful reduces to a cohomology computation. As does showing the semi-orthogonality of the sequence.

2 HPD for Rectangular Lefschetz Decompositions

Throughout this section let X be a variety with $\mathcal{O}_X(1)$ a line bundle on X, and suppose $\mathrm{D}(X)$ admits a rectangular Lefschetz decomposition of the form

$$\mathrm{D}(X) = \Big\langle \mathcal{A}, \mathcal{A}(1), \ldots, \mathcal{A}(i-1) \Big\rangle. \tag{II.42}$$

We want to know for a given hyperplane $H \subset \mathbb{P}(V^*)$ such that H intersects X generically can we determine the derived category $\mathrm{D}(X_H)$ in term of $\mathrm{D}(X)$.

We begin by proving a vanishing lemma.

Lemma 2.1 ([Tho17], Lem. 4.3). *The functor $\pi^* : \mathrm{D}(X) \to \mathrm{D}(\mathcal{H}_L)$ restricted to $\mathcal{A} \subset \mathrm{D}(X)$ defines an embedding $\pi^*|_{\mathcal{A}} : \mathcal{A} \to \mathrm{D}(\mathcal{H}_L)$. Then*

$$R\operatorname{Hom}_{\mathcal{H}_L}(\mathcal{A}(\alpha, \beta), \mathcal{A}) = 0, \tag{II.43}$$

for $\mathcal{A}(0, \ell-1)$ $\mathcal{A}(i-1, 0)$ and for $\mathcal{A}(\alpha, \beta)$ when either $\alpha = 1, \ldots, i-2$ or $\beta = 1, \ldots, \ell-2$. Moreover there exists a semiorthogonal sequence of $\mathrm{D}(\mathcal{H}_L)$ given by

$$\mathcal{A}(1) \boxtimes \mathrm{D}(\mathbb{P}(L)), \mathcal{A}(2) \boxtimes \mathrm{D}(\mathbb{P}(L)), \ldots, \mathcal{A}(i-1) \boxtimes \mathrm{D}(\mathbb{P}(L)). \tag{II.44}$$

Remark 2.2. II.44 Note that this is not a semiorthogonal decomposition as it does not guarantee generation of $\mathrm{D}(\mathcal{H}_L)$.

Proof. Since $\mathcal{H}_L \subset X \times \mathbb{P}(L)$ is a divisor with bidegree $(1,1)$, is has an exact sequence which gives rise to the following exact triangle on $\mathrm{D}(X \times \mathbb{P}(L))$

$$\rho^*\mathcal{A}(-1,-1) \longrightarrow \rho^*\mathcal{A} \longrightarrow \iota_*\iota^*\rho^*\mathcal{A} \longrightarrow \rho^*\mathcal{A}(-1,-1)[1]$$

Applying $R\operatorname{Hom}_{X\times\mathbb{P}(L)}(\rho^*\mathcal{A}(\alpha,\beta), -)$ and using adjunction we obtain the exact triangle with first two terms

$$R\operatorname{Hom}_{X\times\mathbb{P}(L)}(\rho^*\mathcal{A}(\alpha+1,\beta+1), \rho^*\mathcal{A}) \to R\operatorname{Hom}_{X\times\mathbb{P}(L)}(\rho^*\mathcal{A}(\alpha,\beta), \rho^*\mathcal{A}) \tag{II.45}$$

and with cone $R\operatorname{Hom}_{X\times\mathbb{P}(L)}(\rho^*\mathcal{A}(\alpha,\beta), \iota_*\iota^*\rho^*\mathcal{A})$. Using adjunction, and fact that $\pi = \rho \circ \iota$, by the commutativity of diagram 1, yields

$$R\operatorname{Hom}_{X\times\mathbb{P}(L)}(\rho^*\mathcal{A}(\alpha,\beta), \iota_*\iota^*\rho^*\mathcal{A}) = R\operatorname{Hom}_{\mathcal{H}_L}(\iota^*\rho^*\mathcal{A}(\alpha,\beta), \iota^*\rho^*\mathcal{A}) \tag{II.46}$$

$$= R\operatorname{Hom}_{\mathcal{H}_L}(\pi^*\mathcal{A}(\alpha,\beta), \pi^*\mathcal{A}). \tag{II.47}$$

It suffices to determine when the cone of the morphism in II.45 is isomorphic to zero. Using the Kunneth Formula and adjunction on II.45 yields,

$$R\operatorname{Hom}_X(\mathcal{A}(\alpha+1), \mathcal{A}) \otimes R\Gamma(\mathcal{O}_{\mathbb{P}(L)}(-\beta-1)) \longrightarrow R\operatorname{Hom}_X(\mathcal{A}(\alpha), \mathcal{A}) \otimes R\Gamma(\mathcal{O}_{\mathbb{P}(L)}(-\beta)).$$

The first term vanishes for $0 \leq \alpha < i-1$ and the second term vanishes for $0 \leq \beta < \ell - 1$. The third term vanishes for $\alpha = \beta = 0$ and the fourth for $0 < \beta \leq \ell - 1$. Therefore both terms are zero and hence cone is 0 for the desired α, β in the lemma. The semiorthogonal decomposition on $\mathrm{D}(\mathcal{H}_L)$ follows from Beilinsons result, theorem 3.9. □

Let $H \subset X$ be a give hyperplane. Since H is just a degree 1 hypersurface by proposition 5.4 we have that $\mathrm{D}(H)$ admits a semiorthogonal decomposition given by

$$\mathrm{D}(H) = \Big\langle \mathcal{K}_H, \mathcal{A}(1), \mathcal{A}(2), \ldots, \mathcal{A}(i-1) \Big\rangle, \tag{II.48}$$

where we define $\mathcal{K}_H = \langle \mathcal{A}(1), \mathcal{A}(2), \ldots, \mathcal{A}(i-1)\rangle^{\perp}$. It follows from lemma 2.1 that we have that for any linear subspace $L \subseteq V$ we have

$$\mathrm{D}(\mathcal{H}_L) = \Big\langle \mathcal{K}_{\mathcal{H}_L}, \mathcal{A}(1) \boxtimes \mathrm{D}(\mathbb{P}(L)), \mathcal{A}(2) \boxtimes \mathrm{D}(\mathbb{P}(L)), \ldots, \mathcal{A}(i-1) \boxtimes \mathrm{D}(\mathbb{P}(L)) \Big\rangle, \tag{II.49}$$

where by Beilinson's collection 3.9 $\mathrm{D}(\mathbb{P}(L)) = \langle \mathcal{O}, \mathcal{O}(1), \ldots, \mathcal{O}(\ell-1)\rangle$ and with

$$\mathcal{K}_{\mathcal{H}_L} := \Big\langle \mathcal{A}(1) \boxtimes \mathrm{D}(\mathbb{P}(L)), \ldots, \mathcal{A}(i-1) \boxtimes \mathrm{D}(\mathbb{P}(L)) \Big\rangle^{\perp}. \tag{II.50}$$

This defines $\mathcal{K}_{\mathcal{H}}$ by taking $L = V$ above. In fact $\mathcal{K}_{\mathcal{H}_L}$ can be seen as the base change of $\mathcal{K}_H$ along $\mathrm{D}(\mathbb{P}(L)) \to \mathrm{D}(\mathbb{P}(V))$ that is

$$\begin{tikzcd}
\mathcal{K}_{\mathcal{H}_L} := \mathcal{K}_{\mathcal{H}} \times_{\mathrm{D}(\mathbb{P}(V))} \mathrm{D}(\mathbb{P}(L)) \arrow[r, hook] \arrow[d] & \mathrm{D}(\mathbb{P}(L)) \arrow[d] \\
\mathcal{K}_{\mathcal{H}} \arrow[r, hook] & \mathrm{D}(\mathbb{P}(V))
\end{tikzcd}$$

Definition 2.3. For $X \to \mathbb{P}(V)$ with a rectangular Lefschetz collection. The category $\mathcal{K}_{\mathcal{H}}$ is homological projective dual of the category $\mathrm{D}(X)$.

The following result describes how semiorthogonal decompositions of $\mathrm{D}(X_{L^\perp})$ relate to the ambient variety $\mathrm{D}(X)$.

Theorem 2.4 ([Tho17], Thm. 4.7). *HPD for Rectangular Lefschetz Decompositions*
Recall $\ell = \dim(L)$ *where* $L \subset V$. *Projecting the subcategories of* $\mathrm{D}(\mathcal{H}_L)$ *of theorem 1.4 into* $\mathcal{K}_{\mathcal{H}_L}$ *or* $\mathrm{D}(X_{L^\perp})$ *gives the following semiorthogonal decompositions:*

1. *If* $\ell < i$ *then*
$$\mathrm{D}(X_{L^\perp}) = \Big\langle \mathcal{K}_{\mathcal{H}_L}, \mathcal{A}(1,0), \mathcal{A}(2,0), \ldots, \mathcal{A}(i-\ell,0) \Big\rangle. \tag{II.51}$$
2. *If* $\ell = i$ *then* $\mathcal{K}_{\mathcal{H}_L} \cong \mathrm{D}(X_{L^\perp})$.
3. *If* $\ell > i$ *then*
$$\mathcal{K}_{\mathcal{H}_L} = \Big\langle \mathrm{D}(X_{L^\perp}), \mathcal{A}(0,1), \mathcal{A}(0,2), \ldots, \mathcal{A}(0,\ell-i) \Big\rangle. \tag{II.52}$$

Remark 2.5. Taking $L = V$ above we have that we have that $\mathcal{K}_{\mathcal{H}}$ has a rectangular Lefschetz decomposition of the form

$$\mathcal{K}_{\mathcal{H}} = \Big\langle \mathcal{A}(0,0), \mathcal{A}(0,1), \ldots \mathcal{A}(0,j-1) \Big\rangle, \tag{II.53}$$

where $j = \dim(V) - i$. Notice that this is indeed a Dual Lefschetz decomposition to that of $\mathrm{D}(X)$.

Remark 2.6. Since we won't be proving more general forms of HPD in this thesis its worth stating that proofs typically involve complicated inductions. This is called "playing" the chessboard game and can be found in the proof [[Tho17], Thm. 4.7].

3 The Fundamental Theorem of Homological Projective Duality

Setup. Let X be a smooth projective variety an effective line bundle $\mathcal{O}_X(1)$ on X that admits a Lefschetz decomposition

$$\mathrm{D}(X) = \Big\langle \mathcal{A}_0, \mathcal{A}_1(1), \dots, \mathcal{A}_{i-1}(i-1) \Big\rangle, \tag{II.54}$$

Let $V^* \subset H^0(\mathcal{O}_X(1))$ such that $N := \dim(V) > i$. Assume that V^* generates $\mathcal{O}_X(1)$ so we have a regular morphism $f : X \to \mathbb{P}(V)$. Set $j := N - 1 - \max\{k | \mathcal{A}_k = \mathcal{A}_0\}$.

Definition 3.1 ([Kuz05], Def. 6.1). An algebraic variety Y with a projective morphism $g : Y \to \mathbb{P}(V^*)$ is called Homologically Projective Dual to $f : X \to \mathbb{P}(V)$ with respect to II.54, if there exists an object $\mathcal{P} \in \mathrm{D}(Y \times_{\mathbb{P}(V^*)} \mathcal{H}_L)$ such that the associated Fourier-Mukai transform $\Phi_{\mathcal{P}} : \mathrm{D}(Y) \to \mathrm{D}(\mathcal{H}_L)$ is fully faithful and gives the semiorthogonal decomposition

$$\mathrm{D}(\mathcal{H}_L) = \Big\langle \Phi_{\mathcal{P}}(\mathrm{D}(Y)), \mathcal{A}_1(1) \boxtimes \mathrm{D}(\mathbb{P}(V^*)), \dots, \mathcal{A}_{i-1}(i-1) \boxtimes \mathrm{D}(\mathbb{P}(V^*)) \Big\rangle. \tag{II.55}$$

For every linear subspace $L \subset V^*$ we define the corresponding linear sections of X and Y:

$$X_L := X \times_{\mathbb{P}(V)} \mathbb{P}(L^{\perp}) \text{ and } \quad Y_L := Y \times_{\mathbb{P}(V^*)} \mathbb{P}(L), \tag{II.56}$$

where $L^{\perp} \subset V$ is the annihilator of $L \subset V^*$. Let $\mathcal{H}_L \subset X \times \mathbb{P}(V^*)$ denote the universal hyperplane section.

Definition 3.2 ([Kuz05], Def. 6.2). A subspace $L \subset V^*$ is called admissible, if

1. $\dim(X_L) = \dim(X) - \dim(L)$,
2. $\dim(Y_L) = \dim(Y) + \dim(L) - N$.

Theorem 3.3 ([Kuz05], Thm. 6.3). *The Fundamental Theorem of Homological Projective Duality. Suppose X is defined as above with Lefschetz decomposition*

$$\mathrm{D}(X) = \Big\langle \mathcal{A}_0, \mathcal{A}_1(1), \dots \mathcal{A}_{i-1}(i-1) \Big\rangle, \tag{II.57}$$

If Y is Homologically Projectively Dual to X then Y is smooth and $\mathrm{D}(Y)$ admits a dual Lefschetz decomposition

$$\mathrm{D}(Y) = \Big\langle \mathcal{B}_{j-1}(-j+1), \dots, \mathcal{B}_1(-1), \mathcal{B}_0 \Big\rangle, \tag{II.58}$$

where $0 \subset \mathcal{B}_{j-1} \subset \cdots \subset \mathcal{B}_1 \subset \mathcal{B}_0 \subset \mathrm{D}(Y)$ with the same set of primitive subcategories. Moreover for any admissible linear subspace $L \subset V^*$, with $\dim(L) = r$, there exists a category $\mathcal{K}_L$ and semiorthogonal decompositions

$$\mathrm{D}(X_L) = \Big\langle \mathcal{K}_L, \mathcal{A}_r(1), \dots, \mathcal{A}_{i-1}(i-r) \Big\rangle, \tag{II.59}$$

$$\mathrm{D}(Y_L) = \Big\langle \mathcal{B}_{j-1}(N-r-j), \dots, \mathcal{B}_{N-r}(-1), \mathcal{K}_L \Big\rangle. \tag{II.60}$$

Remark 3.4. The category $\mathcal{K}$ is now referred to as the Kuznetsov component of the decomposition.

Remark 3.5. Although this is how HPD was originally stated its common now to "normalize" the decompositions by by twisting by $r-1$ and $r-N+1$ so that II.60 and II.59 have the equivalent form $\mathrm{D}(X_L) = \langle \mathcal{K}_L, \mathcal{A}_r(r), \dots, \mathcal{A}_{i-1}(i-1) \rangle$ and $\mathrm{D}(Y_L) = \langle \mathcal{B}_{j-1}(j-1), \dots, \mathcal{B}_{N-r}(-N+r), \mathcal{K}_L \rangle$.

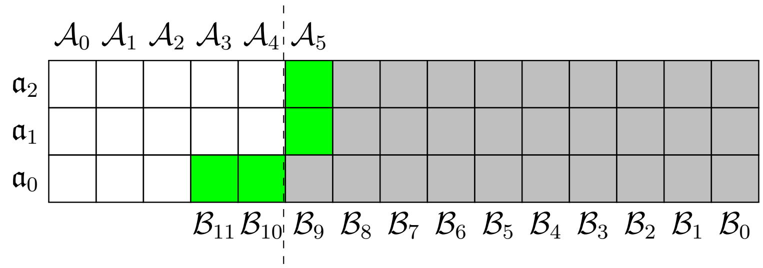


Figure II.1: Theorem 3.3 illustrated for $\mathbf{Gr}(2,6)$ and its HP-dual $\mathbf{Pf}(4,6)$.

For any HP-dual pair, theorem 3.3 can be better understood by drawing the associated box diagrams for each variety. By construction the primitive subcategories will match up in such a way that we can concatenate diagrams to obtain a rectangle. The choice of linear section is then represented by a divider (the dashed vertical line) aligned with the vertical lines of the rectangle. From the rectangle we can then re-interpret the variables in theorem 3.3 combinatorially.

- The length of the rectangle is N.
- The integer i is the maximum number of white boxes in a row.
- The integer j is the maximum number of grey boxes in a row.
- The integer r is the position of the divider starting from the far left at 0 and incrementing by one up to N.

Restriction Rules. We now re-interpret the second half of theorem 3.3 and introduce a combinatorial rule that shows how component subcategories coming from the ambient variety restrict to hyperplane sections for $\mathrm{D}(X_L)$ and $\mathrm{D}(Y_L)$. Note that our convention is that the decomposition on the left will be shaded white and the decomposition to the right shaded grey.

- Shade all white boxes to the right of the divider green.
- Shade all grey boxes to the left of the divider green.
- The white boxes that are green are the subcategories of the Lefschetz decomposition that remain in the semiorthogonal decomposition of $\mathrm{D}(X_L)$.
- The grey boxes that are green are the subcategories of the Lefschetz decomposition that remain in the semiorthogonal decomposition of $\mathrm{D}(Y_L)$.

Therefore for the rectangle in figure II.1 we easily determine $N = 15$, $i = 6$, $j = 12$ and $r = 5$. In particular for dimension 5 hyperplane sections of X we obtain

$$\mathrm{D}(X_L) = \Big\langle \mathcal{K}, \mathcal{A}_5(1) \Big\rangle, \tag{II.61}$$

$$\mathrm{D}(Y_L) = \Big\langle \mathcal{B}_{11}(-2), \mathcal{B}_{10}(-1), \mathcal{K} \Big\rangle. \tag{II.62}$$

Remark 3.6. Of course these rules does not allow us to determine the Kuznetsov component $\mathcal{K}$ only the objects that make up its orthogonal. In general determining $\mathcal{K}$ is highly nontrivial.

We now compute some examples applying the rectangle algorithm and restriction rules. Note however that we will use the normalized HPD theorem see remark 3.5

Example 3.7. Let $X = \mathbf{Gr}(2, W) \subset \mathbb{P}(\Lambda^2 W)$ be embedded via the Plücker embedding and let $\dim(W) = 4$. Recall that $\mathrm{D}(X)$ has a semiorthogonal decomposition

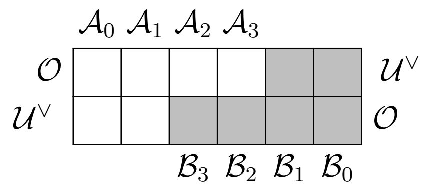
$$\mathrm{D}(\mathbf{Gr}(2, W)) = \Big\langle \mathcal{A}_0, \mathcal{A}_1(1), \mathcal{A}_2(2), \mathcal{A}_3(3) \Big\rangle. \tag{II.63}$$

which admits a Lefschetz decomposition where $\mathcal{A}_0 = \mathcal{A}_1 = \langle \mathcal{O}, \mathcal{U}^\vee \rangle$ and $\mathcal{A}_2 = \mathcal{A}_3 = \langle \mathcal{O} \rangle$.

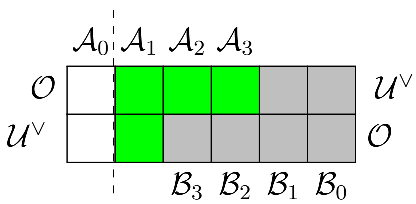
The Homological projective dual of $\mathbf{Gr}(2, W)$ is $Y = \mathbf{Gr}(2, W^*) \subset \mathbb{P}(\Lambda^2 W^*)$. We have $\dim(X) = \dim(Y) = 4$. Since Y is the HP-dual of X it admits a dual Lefschetz decomposition

$$\mathrm{D}(\mathbf{Gr}(2, W^*)) = \langle \mathcal{B}_3(3), \mathcal{B}_2(2), \mathcal{B}_1(1), \mathcal{B}_0 \rangle. \tag{II.64}$$

where $\mathcal{B}_0 = \mathcal{B}_1 = \langle \mathcal{O}, \mathcal{U}^\vee \rangle$ and $\mathcal{B}_2 = \mathcal{B}_3 = \langle \mathcal{O} \rangle$. Therefore we can form a rectangle from decomposition and its dual consisting of the concatenation of the boxes for $\mathrm{D}(X)$ and $\mathrm{D}(Y)$.

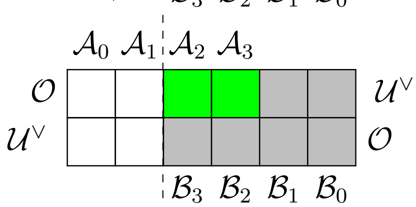


Iterating the dimension $r = 1, \ldots, 6$, and applying the rule we obtain



- For $r = 1$, $\dim(X_L) = 3$, and $Y_L = \emptyset$ hence

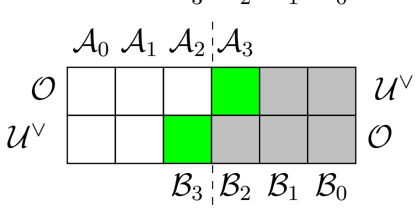
  $$\mathrm{D}(X_L) = \langle \mathcal{O}(1), \mathcal{U}^\vee(1), \mathcal{O}(2), \mathcal{O}(3) \rangle \text{ and } \mathrm{D}(Y_L) = \emptyset.$$



- For $r = 2$, $\dim(X_L) = 2$, $\dim(Y_L) = 0$. Here X_L is identified as the quadric $\mathbb{P}^1 \times \mathbb{P}^1$ and $Y_L = \{\mathrm{pt}\} \sqcup \{\mathrm{pt}\}$.

  $$\mathrm{D}(X_L) = \langle \mathrm{D}(Y_L), \mathcal{A}_2(2), \mathcal{A}_3(3) \rangle,$$

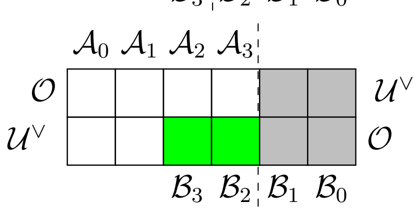
  where $\mathrm{D}(Y_L) = \langle \mathbf{S}^+, \mathbf{S}^- \rangle$.



- For $r = 3$, $\dim(X_L) = 1$, $\dim(Y_L) = 1$ hence

  $$\mathrm{D}(X_L) = \langle \mathcal{K}, \mathcal{A}_3(3) \rangle \text{ and } \mathrm{D}(Y_L) = \langle \mathcal{K}, \mathcal{B}_3(3) \rangle.$$

  where $\mathcal{K}$ is generated by one exceptional object.



- For $r = 4$, $\dim(X_L) = 0$, $\dim(Y_L) = 2$. Here $X_L = \{\mathrm{pt}\} \sqcup \{\mathrm{pt}\}$ with $\mathrm{D}(X_L) = \langle \mathbf{S}^+, \mathbf{S}^- \rangle$ and $Y_L \cong \mathbb{P}^1 \times \mathbb{P}^1$,

  $$\mathrm{D}(Y_L) = \langle \mathrm{D}(X_L), \mathcal{B}_3(3), \mathcal{B}_2(2) \rangle.$$

The remaining cases $r = 5, 6$ follow by symmetry.

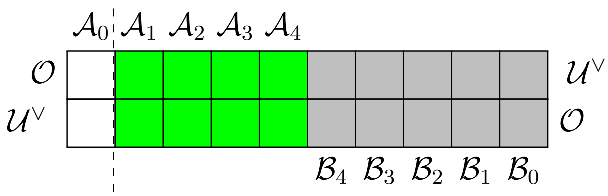
Example 3.8. Let $X = \mathbf{Gr}(2, W) \subset \mathbb{P}(\Lambda^2 W)$ be embedded via the Plücker embedding and let $\dim(W) = 5$. We have that $\mathrm{D}(X)$ admits a Lefschetz decomposition

$$\mathrm{D}(\mathbf{Gr}(2, W)) = \Big\langle \mathcal{A}_0, \mathcal{A}_1(1), \mathcal{A}_2(2), \mathcal{A}_3(3), \mathcal{A}_4(4) \Big\rangle, \tag{II.65}$$

with $\mathcal{A}_0 = \cdots = \mathcal{A}_4 = \langle \mathcal{O}, \mathcal{U}^\vee \rangle$, a rectangular decomposition. The Homological projective dual of $\mathbf{Gr}(2, W)$ is $Y = \mathbf{Gr}(2, W^*) \subset \mathbb{P}^2(\Lambda^2 W^*)$ with $\dim(X) = \dim(Y) = 6$. Since Y is the HP-dual of X it admits a dual Lefschetz decomposition where $\mathcal{B}_0 = \cdots = \mathcal{B}_4 = \langle \mathcal{O}, \mathcal{U}^\vee \rangle$. Therefore

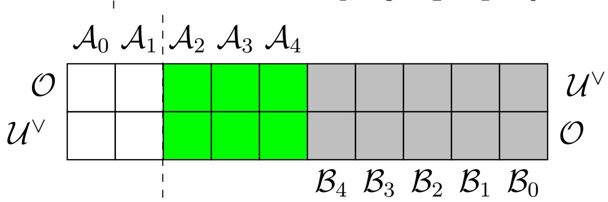
$$\mathrm{D}(\mathbf{Gr}(2, W^*)) = \Big\langle \mathcal{B}_4(4), \mathcal{B}_3(3), \mathcal{B}_2(2), \mathcal{B}_1(1), \mathcal{B}_0 \Big\rangle. \tag{II.66}$$

Therefore we can again form a rectangle from decomposition and its dual consisting of the concatenation of the boxes for $\mathrm{D}(X)$ and $\mathrm{D}(Y)$. Iterating through $r = 1, \ldots, 10$ we obtain:



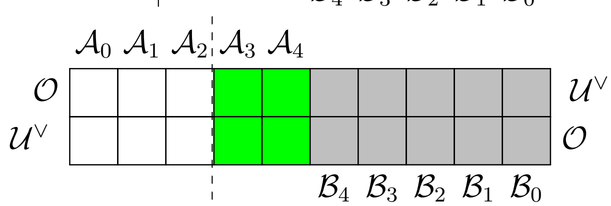
- For $r = 1$, generically $\dim(X_L) = 5$, and $Y_L = \emptyset$ hence

$$\mathrm{D}(X_L) = \langle \mathcal{A}_1(1), \mathcal{A}_2(3), \mathcal{A}_3(3) \rangle \text{ and } \mathrm{D}(Y_L) = \emptyset.$$



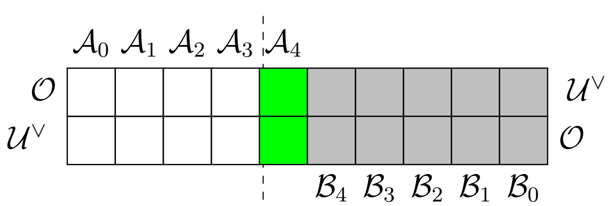
- For $r = 2$, generically $\dim(X_L) = 2$, and $Y_L = \emptyset$. Therefore

$$\mathrm{D}(X_L) = \langle \mathcal{A}_2(2), \mathcal{A}_3(3), \mathcal{A}_4(4) \rangle \text{ and } \mathrm{D}(Y_L) = \emptyset.$$



- For $r = 3$, generically $\dim(X_L) = 3$ and $Y_L = \emptyset$ hence

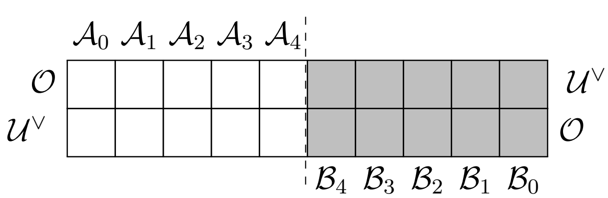
$$\mathrm{D}(X_L) = \langle \mathcal{A}_3(3), \mathcal{A}_4(4) \rangle \text{ and } \mathrm{D}(Y_L) = \emptyset.$$



- For $r = 4$, generically $\dim(X_L) = 2$ and $\dim(Y_L) = 0$. In fact X_L is a del Pezzo surface of degree 5 and $Y_L = \{x_1, x_2, x_3, x_4, x_5\}$

$$\mathrm{D}(X_L) = \langle \mathrm{D}(Y_L), \mathcal{A}_4(4) \rangle,$$

and $\mathrm{D}(Y_L)$ is generated by five exceptional objects.



- For $r = 5$, generically $\dim(X_L) = 1$, and $Y_L = 1$. In fact X_L and Y_L are elliptic curves and hence

$$\mathrm{D}(X_L) = \mathrm{D}(Y_L) = \mathrm{D}(C),$$

where C is an elliptic curve.

The remaining cases $r = 6, \ldots, 10$ follow by symmetry.

4 Properties of Homological Projective Duality

In this section we prove some basic properties about homological projective duality. We also characterize the relationship between homological projective duality and classical projective duality

An essential thing that Homological Projective Duality must satisfy is that is indeed a duality.

Theorem 4.1 ([Kuz05], Thm. 7.3). *If $g : Y \to \mathbb{P}(V^*)$ is Homologically Projectively Dual to $f : X \to \mathbb{P}(V)$ then $f : X \to \mathbb{P}(V)$ is Homologically Projectively Dual to $g : Y \to \mathbb{P}(V^*)$.*

It is natural to wonder what the relationship between a HP-dual pair X and Y is. In the case where X admits a rectangular Lefschetz decomposition this can be answered precisely.

Proposition 4.2 ([Kuz05], Prop. 7.4). *If $g \to \mathbb{P}(V^*)$ is homologically projectively dual to $f : X \to \mathbb{P}(V)$ with respect to a rectangular Lefschetz decomposition with i terms, then the number of terms j in the dual Lefschetz decomposition of $\mathrm{D}(Y)$ and the dimension of Y equal*

$$j = N - i \text{ and } \dim(Y) = \dim(X) + N - 2i, \tag{II.67}$$

where $N = \dim(V)$.

Proof. The formula for j follows immediately from

$$j = N - 1 - \max\{k|\mathcal{B}_k = \mathcal{B}_0\} = N - 1 - \max\{k|\mathcal{A}_k = \mathcal{A}_0\} = N - 1 - (i-1) = N - i. \tag{II.68}$$

The formula for dimension is then obtained by setting observing when we have a rectangular decomposition and the dimension r of the admissible subspace $L \subset V$ is equal to i then by theorem 3.3 (or by looking at the rectangles, as in the case of $\mathbf{Gr}(2,5)$) we see that there are no restricted objects that survive and hence we have an equivalence of categories $\mathrm{D}(X_L) = \mathrm{D}(Y_L)$. Therefore it follows that $\dim(X_L) = \dim(Y_L)$. Since $\dim(X_L) = \dim(X) - i$ and $\dim(Y_L) = \dim(Y) - (N - i)$, the desired result follows

$$\dim(Y) = \dim(Y_L) + N - i = \dim(X_L) + N - i = \dim(X_L) + N - 2i. \tag{II.69}$$

□

Disjoint Unions. Let X be an algebraic variety such that X is a disjoint union $X = X' \sqcup X''$. Then its derived category is a completely orthogonal direct sum, $\mathrm{D}(X) = \mathrm{D}(X') \oplus \mathrm{D}(X'')$ where $\mathrm{D}(X')$ and $\mathrm{D}(X'')$ admit Lefschetz decompositions of the form

$$\mathrm{D}(X') = \Big\langle \mathcal{A}'_0, \mathcal{A}'_1(1), \dots, \mathcal{A}_{i'-1}(i'-1) \Big\rangle \text{ and } \mathrm{D}(X'') = \Big\langle \mathcal{A}''_0, \mathcal{A}''_1(1), \dots, \mathcal{A}_{i''-1}(i''-1) \Big\rangle, \tag{II.70}$$

with by orthogonality yields the resulting Lefschetz decomposition

$$\mathrm{D}(X) = \mathrm{D}(X' \sqcup X'') = \Big\langle \mathcal{A}'_0 \oplus \mathcal{A}''_0, (\mathcal{A}'_1 \oplus \mathcal{A}''_1)(1), \dots, (\mathcal{A}_{i'-1}(i-1) \oplus \mathcal{A}_{i-1})(i-1) \Big\rangle, \tag{II.71}$$

where $i = \max\{i', i''\}$.

Proposition 4.3 ([Kuz05], Prop. 7.1). *If Y' and Y'' are Homologically Projectively Dual to X' and X'' respectively then $Y = Y' \sqcup Y''$ is Homologically Projectively Dual to $X = X' \sqcup X''$*

Proof. Since the universal hyperplane of $X = X' \sqcup X''$ is also a disjoint union we can combine semiorthogonal decompositions

$$\mathrm{D}(\mathcal{H}'_L) = \Big\langle \mathrm{D}(Y'), \mathcal{A}'_1 \boxtimes \mathrm{D}(\mathbb{P}(V^*)), \dots, \mathcal{A}'_{i-1}(i'-1) \boxtimes \mathrm{D}(\mathbb{P}(V^*)) \Big\rangle, \tag{II.72}$$

and

$$\mathrm{D}(\mathcal{H}''_L) = \Big\langle \mathrm{D}(Y'), \mathcal{A}'_1 \boxtimes \mathrm{D}(\mathbb{P}(V^*)), \dots, \mathcal{A}'_{i-1}(i'-1) \boxtimes \mathrm{D}(\mathbb{P}(V^*)) \Big\rangle, \tag{II.73}$$

to obtain

$$\mathrm{D}(\mathcal{H}_L) = \Big\langle \mathrm{D}(Y') \oplus \mathrm{D}(Y''), (\mathcal{A}'_1 \oplus \mathcal{A}''_1)(1) \boxtimes \mathrm{D}(\mathbb{P}(V^*)), \dots, (\mathcal{A}'_{i-1} \oplus \mathcal{A}''_{i-1})(i-1) \boxtimes \mathrm{D}(\mathbb{P}(V^*)) \Big\rangle, \tag{II.74}$$

which shows that $Y = Y' \sqcup Y''$ is homological Projectively Dual to $X = X' \sqcup X''$. ☐

Therefore we have shown that homological projective duality commutes with disjoint unions.

Products. Continuing with the notation above. If $X = X' \times F$ is a product variety and $\mathcal{O}_X(1) = p^*\mathcal{O}_{X'}(1)$, where $p : X' \times F \to X'$ is projection along F. Then we have a Lefschetz decomposition

$$\mathrm{D}(X) = \Big\langle \mathcal{A}'_0 \boxtimes \mathrm{D}(F), (\mathcal{A}'_1 \boxtimes \mathrm{D}(F))(1), \dots, (\mathcal{A}'_{i'-1} \boxtimes \mathrm{D}(F))(i'-1) \Big\rangle. \tag{II.75}$$

Proposition 4.4 ([Kuz05], Prop. 7.2). *If Y' and X' are homologically projective dual varieties then $Y = Y' \times F$ is homologically projective dual to $X = X' \times F$.*

Proof. The assumptions imply that the universal hyperplane section of X can also be represented as $\mathcal{H}_L = \mathcal{H}'_L \times F$. Applying the exterior product to semiorthogonal decomposition

$$\mathrm{D}(\mathcal{H}'_L) = \langle \mathrm{D}(Y'), \mathcal{A}'_1(1) \boxtimes \mathrm{D}(\mathbb{P}(V^*)), \dots, \mathcal{A}'_{i-1}(i'-1) \boxtimes \mathrm{D}(\mathbb{P}(V^*) \rangle, \tag{II.76}$$

with $\mathrm{D}(F)$ yields

$$\mathrm{D}(\mathcal{H}_L) = \Big\langle \mathrm{D}(Y') \boxtimes \mathrm{D}(F), (\mathcal{A}_1 \boxtimes \mathrm{D}(F))(1) \boxtimes \mathrm{D}(\mathbb{P}(V^*)), \dots, (\mathcal{A}'_{i'-1} \boxtimes \mathrm{D}(F))(i'-1) \boxtimes \mathrm{D}(\mathbb{P}(V^*)) \Big\rangle. \tag{II.77}$$

From which it follows that $Y = Y' \times F$ is homologically projectively dual to $X = X' \times F$. ☐

Remark 4.5. Recently Kuznetsov and Perry categorified the notion of a join for varieties which admit Lefschetz decompositions. These categorified joins, or categorical joins, also turn out to behave nicely with respect to homological projective duality. In fact homological projective duality commutes with categorical joins see [KP20] theorem 4.1.

HPD and Classical Projective Duality.We now relate the classical projective dual of X and the homological projective dual. Recall that the projective classical dual of X is given as follows:

Given an embedding $f : X \to \mathbb{P}(V)$. The classical projective dual variety to X denoted $X^\vee \subset \mathbb{P}(V^*)$ is the set of all points $H \in \mathbb{P}(V^*)$ such that the corresponding hyperplane section X_H of X is singular.

Theorem 4.6 ([Kuz05], Thm. 7.9). *Suppose $g : Y \to \mathbb{P}(V^*)$ is HP-dual to $f : X \to \mathbb{P}(V)$ then*

$$X^\vee = \{H \in \mathbb{P}(V^*) | X_H \text{ is singular}\} = \{\text{critical values of } g : Y \to \mathbb{P}(V^*)\} = \operatorname{sing}(g). \tag{II.78}$$

Proof. Consider the universal hyperplane section $\mathcal{H}_L$ of X and the maps $f_1 : \mathcal{H}_L \to \mathbb{P}(V^*)$ and $g : Y \to \mathbb{P}(V^*)$. Note also that $X^\vee = \operatorname{sing}(f_1)$ is the set of critical values of the map f_1 by the definition of the classical projective dual. Thus we have to check that $\operatorname{sing}(f_1) = \operatorname{sing}(g)$.

Assume that $\operatorname{sing}(g) \not\subseteq \operatorname{sing}(f_1)$. Let $H \in \mathbb{P}(V^*)$ such that $H \in \operatorname{sing}(g) \setminus \operatorname{sing}(f_1)$. Then it is clear that there exists a smooth hypersurface $D \subset \mathbb{P}(V^*)$ such that $H \in D$, and $Y_D := Y \times_{\mathbb{P}(V^*)} D$ has a singularity over H with $\dim(Y_D) = \dim(Y) - 1$. Let $T = D \setminus \operatorname{sing}(f_1)$. Then $H \in T$, $Y_T := Y \times_{\mathbb{P}(V^*)} T$ has a singularity over H, and $\dim(Y_T) = \dim(Y) - 1$. On the other hand, f_1 is smooth over T, hence $\mathcal{H}_{LT} := \mathcal{H}_L \times_{\mathbb{P}(V^*)} T$ is smooth and both $\mathcal{H}_{LT}$ and $\mathcal{H}_{LT} \times_T Y_T$ have expected dimension. Therefore the base change $T \to \mathbb{P}(V^*)$ is faithful for the pair $(\mathcal{H}_L, Y)$ and we obtain by the faithful base change theorem 6.11 a semiorthogonal decomposition

$$\mathrm{D}(\mathcal{H}_{LT}) = \Big\langle \mathrm{D}(Y_T), \mathcal{A}_1(1) \boxtimes \mathrm{D}(T), \dots, \mathcal{A}_{i-1}(i-1) \boxtimes \mathrm{D}(T) \Big\rangle, \tag{II.79}$$

but the category $\mathrm{D}(\mathcal{H}_{LT})$ is Ext-bounded since $\mathcal{H}_{LT}$ is smooth by proposition 6.4, while the category $\mathrm{D}(Y_T)$ is not Ext-bounded since Y_T is singular. This is a contradiction, hence $\operatorname{sing}(g) \subseteq \operatorname{sing}(f_1)$.

Similarly for the converse, assume $\operatorname{sing}(f_1) \not\subseteq \operatorname{sing}(g)$. Let $H \in \mathbb{P}(V^*)$ be a point in $\operatorname{sing}(f_1)$ such that $H \notin \operatorname{sing}(g)$ has a singularity over H, and $\dim(\mathcal{H}_{LD}) = \dim(\mathcal{H}_L) - 1$. Let $T = D \setminus \operatorname{sing}(g)$ then $H \in T, \mathcal{H}_{LT} := \mathcal{H}_L \times \mathbb{P}(V^*) T$ and $Y_T = Y \times_{\mathbb{P}(V^*)} T$ is smooth and both Y_T and $\mathcal{H}_{LT} \times_T Y_T$ have expected dimension. Therefore the base change $T \to \mathbb{P}(V^*)$ is faithful for the pair $\mathcal{H}_L$ and Y and we again obtain the by the faithful base change theorem 6.11 a semiorthogonal decomposition

$$\mathrm{D}(\mathcal{H}_{LT}) = \Big\langle \mathrm{D}(Y_T), \mathcal{A}_1(1) \boxtimes \mathrm{D}(T), \dots, \mathcal{A}_{i-1}(i-1) \boxtimes \mathrm{D}(T) \Big\rangle \tag{II.80}$$

Now note that the category $\mathrm{D}(Y_T)$ is Ext-bounded since Y_T is smooth by proposition 6.4 and categories $\mathcal{A}_1(1) \boxtimes \mathrm{D}(T), \dots, \mathcal{A}_{i-1}(i-1) \boxtimes \mathrm{D}(T)$ are Ext-bounded because T is smooth again by proposition 6.4. Therefore the category $\mathrm{D}(\mathcal{H}_{LT})$ is Ext bounded by lemma 6.5. But this contradicts that $\mathcal{H}_{LT}$ is singular. □

It is worth pointing that unlike the classical projective dual of X. The homological projective dual always exists, at least categorically. Even if $\mathrm{D}(Y)$ isn't the category of any honest variety Y. In fact taking $X = \mathbf{Gr}(2, 6)$ its dual is not an ordinary variety but the noncommutative Pfaffian $X^\vee = \mathbf{Pf}(4, 6)$ with diagram II.1. Therefore it wouldn't be unreasonable to say that HPD is a phenomena that naturally belongs in the realm of noncommutative algebraic geometry.

Remark 4.7. When Y is indeed a variety we say that X has a geometric HP-dual Y

5 Homological Projective Dual Varieties

Many of the semiorthogonal decompositions encountered in chapter I are well known examples of homological projective duality. Additionally some of these varieties are known to exhibit other type of dualities as well such i.e Pfaffian-Grassmanian duality. In this section we use the framework of HPD which to describe some more examples and explore these dualities from a new perspective. In particular we focus on the quadric and Grassmannian.

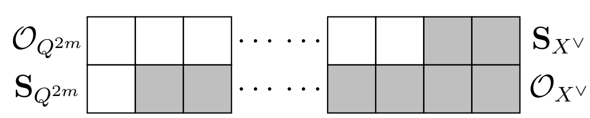
Example 5.1. (Odd Dimensional Quadrics)
Recall from 7.11 the Lefschetz decomposition for an odd dimensional quadric. Consider now for a fixed non-negative integer $m \in \mathbb{Z}_{\geq 0}$ the quadric $X = Q^{2m-1} \subset \mathbb{P}^{2m}$ which a has Lefschetz decomposition given by

$$\mathrm{D}(Q^{2m-1}) = \Big\langle \mathcal{A}_0, \mathcal{A}_1(1), \ldots, \mathcal{A}_{2m-2}(2m-2) \Big\rangle, \tag{II.81}$$

where $\mathcal{A}_0 = \langle \mathcal{O}_{Q^{2m-1}}, \mathbf{S}_{Q^{2m-1}} \rangle$ and $\mathcal{A}_1 = \cdots = \mathcal{A}_{2m-2} = \langle \mathcal{O}_{Q^{2m-1}} \rangle$. Then the homological projective dual is the double covering $X^\vee \to \check{\mathbb{P}}^{2m}$ ramified in the dual quadric $\overline{Q}^{2m-1} \to \mathbb{P}^{2m}$ with dual Lefschetz decomposition

$$\mathrm{D}(X^\vee) = \Big\langle \mathcal{B}_{2m-1}(2m-1), \ldots, \mathcal{B}_1(1), \mathcal{B}_0 \Big\rangle, \tag{II.82}$$

where $\mathcal{B}_0 = \mathcal{B}_1 = \langle \mathcal{O}_{X^\vee}, \mathbf{S}_{X^\vee} \rangle$ and $\mathcal{B}_2 = \cdots = \mathcal{B}_{2m-1} = \langle \mathcal{O}_{X^\vee} \rangle$. Concatenating the Lefschetz decomposition of this HP-dual pair gives us the rectangle



From which we can apply the combinatorial rules from section 3 we can determine all the information about the form of our decompositions for hyperplane sections of $X = Q^{2m}$ and $X^\vee$.

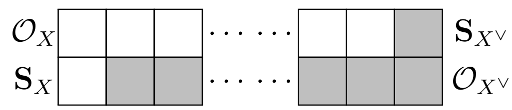
Example 5.2. (Odd Dimensional Ramified Quadric)
Recall from 7.11 the Lefschetz decomposition for an odd dimensional quadric. Consider now for a fixed non-negative integer $m \in \mathbb{Z}_{\geq 0}$, a quadric $X = Q^{2m-1} \subset \mathbb{P}^{2m-1}$ which is a ramified double covering of a quadric $Q_1^{2m-1} \subset \mathbb{P}^{2m-1}$ which has Lefschetz decomposition given by

$$\mathrm{D}(X) = \Big\langle \mathcal{A}_0, \mathcal{A}_1(1), \ldots, \mathcal{A}_{2m-2}(2m-2) \Big\rangle, \tag{II.83}$$

where $\mathcal{A}_0 = \langle \mathcal{O}_X, \mathbf{S}_X \rangle$ and $\mathcal{A}_1 = \cdots = \mathcal{A}_{2m-2} = \langle \mathcal{O}_X \rangle$. Then the homological projective dual variety is the double covering $X^\vee \to \check{\mathbb{P}}^{2m}$ ramified in the dual quadric $\overline{Q}_1^{2m-1} \to \check{\mathbb{P}}^{2m}$ with dual Lefschetz decomposition

$$\mathrm{D}(X^\vee) = \Big\langle \mathcal{B}_{2m-2}(2m-2), \ldots, \mathcal{B}_1(1), \mathcal{B}_0 \Big\rangle, \tag{II.84}$$

where $\mathcal{B}_0 = \langle \mathcal{O}_{X^\vee}, \mathbf{S}_{X^\vee} \rangle$ and $\mathcal{B}_1 = \cdots = \mathcal{B}_{2m-2} = \langle \mathcal{O}_{X^\vee} \rangle$. Concatenating the Lefschetz decomposition of this HP-dual pair gives us the rectangle



Applying the combinatorial rules from section 3 we can determine all the information about the form of our decompositions for hyperplane sections of X and $X^\vee$.

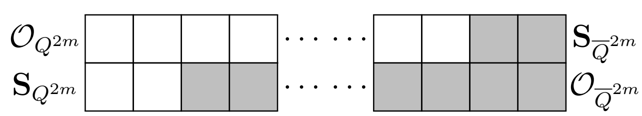
Example 5.3. (Even Dimensional Quadric)
Similarly to previous case, recall from 7.10 the Lefschetz decomposition for an even dimensional quadric. Consider now for a fixed non-negative integer $m \in \mathbb{Z}_{\geq 0}$ the quadric $Q^{2m} \subset \mathbb{P}^{2m+1}$ which a has Lefschetz decomposition given by

$$\mathrm{D}(Q^{2m}) = \Big\langle \mathcal{A}_0, \mathcal{A}_1(1), \ldots, \mathcal{A}_{2m-1}(2m-1) \Big\rangle, \tag{II.85}$$

where $\mathcal{A}_0 = \mathcal{A}_1 = \langle \mathcal{O}_{Q^{2m}}, \mathbf{S}_{Q^{2m}} \rangle$ and $\mathcal{A}_2 = \cdots = \mathcal{A}_{2m-1} = \langle \mathcal{O} \rangle$. Then the homological projective dual of $Q^{2m} \to \mathbb{P}^{2m+1}$ is $X^\vee = \overline{Q}^{2m} \to \check{\mathbb{P}}^{2m+1}$ with dual Lefschetz decomposition

$$\mathrm{D}(\overline{Q}^{2m}) = \Big\langle \mathcal{B}_{2m-1}(2m-1), \ldots, \mathcal{B}_1(1), \mathcal{B}_0 \Big\rangle, \tag{II.86}$$

where $\mathcal{B}_0 = \mathcal{B}_1 = \langle \mathcal{O}_{\overline{Q}^{2m}}, \mathbf{S}_{\overline{Q}^{2m}} \rangle$ and $\mathcal{B}_2 = \cdots = \mathcal{B}_{2m-1} = \langle \mathcal{O} \rangle$. Concatenating the Lefschetz decomposition of this HP-dual pair gives us the rectangle



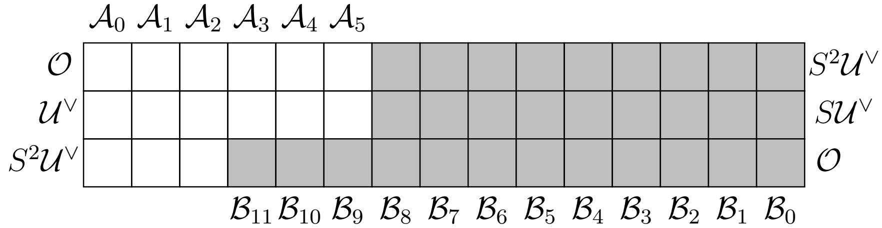
From which we can apply the combinatorial rules from section 3 we can determine all the information about the form of our decompositions for hyperplane sections of Q^{2m} and $\overline{Q}^{2m}$.

Now we consider the HP-duals of the Grassmannians $\mathbf{Gr}(2,6)$ and $\mathbf{Gr}(2,7)$.

Example 5.4. (Pfaffian-Grassmannian Duality I)
Let W be a k-linear vector space of dimension 6 then $X = \mathbf{Gr}(2, W)$ with the Plucker embedding $X \to \mathbb{P}(\wedge^2 W)$ has a natural Lefscchetz decomposition

$$\mathrm{D}(X) = \langle \mathcal{A}_0, \mathcal{A}_1(1), \ldots, \mathcal{A}_5(5) \rangle, \tag{II.87}$$

where $\mathcal{A}_0 = \mathcal{A}_1 = \mathcal{A}_2 = \langle S^2\mathcal{U}^\vee, \mathcal{U}^\vee, \mathcal{O} \rangle$, $\quad \mathcal{A}_3 = \mathcal{A}_4 = \mathcal{A}_5 = \langle \mathcal{U}, \mathcal{O} \rangle$ where $\mathcal{U}$ is the tautological rank 2 bundle on $\mathbf{Gr}(2, W)$ and $S^2\mathcal{U}$ is the symmetric product. Then the above decomposition is HP dual to a noncommutative resolution $(Y, \mathscr{R}_Y)$ of the Pfaffian variety $Y = \mathbf{Pf}(10, W) \subset \mathbb{P}(\wedge^2 W^*)$, where $\mathbf{Pf}(2k, W)$is the loci inside $\mathbb{P}(\wedge^2 W^*)$ where the skew symmetric form has rank smaller than or equal to $2k$.



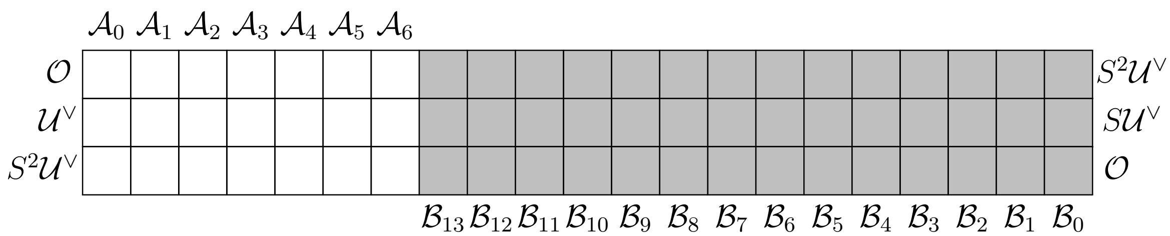
(II.88)

Let L be a generic $\mathbb{P}^5 \subset \mathbb{P}(\wedge^2 W^*)$, then we get a decomposition of Pfaffian cubic fourfold Y_L with Kuznetsov component of the decomposition equivalent to the derived category of the K3 surface $X_{L^\perp}$, see figure II.1 for this amazing example.

Example 5.5. (Pfaffian-Grassmannian Duality II)
Let W be a k-linear vector space of dimension 7 then $X = \mathbf{Gr}(2, W)$ with the Plücker embedding $X \to \mathbb{P}(\wedge^2 W)$ has a rectangular Lefschetz decomposition

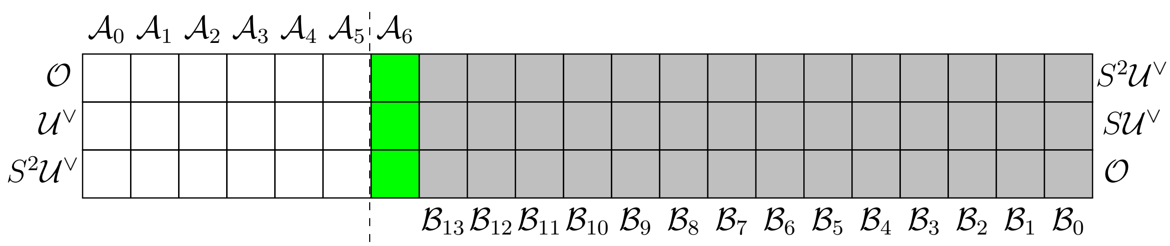
$$\mathrm{D}(X) = \langle \mathcal{A}_0, \mathcal{A}_1(1), \mathcal{A}_2(2), \mathcal{A}_3(3), \mathcal{A}_4(4), \mathcal{A}_5(5), \mathcal{A}_6(6) \rangle, \tag{II.89}$$

where $\mathcal{A}_0 = \cdots = \mathcal{A}_6 = \langle S^2\mathcal{U}^\vee, \mathcal{U}^\vee, \mathcal{O} \rangle$. Then X with the above decomposition is HP-dual to a noncommutative resolution $(Y, \mathscr{R}_Y)$ of the Pfaffian variety $Y = \mathbf{Pf}(12, W) \subset \mathbb{P}(\wedge^2 W^*)$, where $\mathbf{Pf}(12, W)$is the loci inside $\mathbb{P}(\wedge^2 W^*)$ where the skew symmetric form has rank smaller than or equal to 12.



(II.90)

If we take L to be a generic $\mathbb{P}(L) = \mathbb{P}^6 \subset \mathbb{P}(\wedge^2 W^*)$, then we have



(II.91)

In fact we get two Calabi-Yau threefolds $X_{L^\perp}$ and Y_L which are not birational equivalent but are derived equivalent $\mathrm{D}(X_{L^\perp}) \simeq \mathrm{D}(Y_L)$. This was actually the first example of a pair of provably non birational Calabi-Yau threefolds which are derived equivalent.

6 Generalization of HPD to Higher Degree Sections

In 2015 it was shown by Jiang et al. in [JLX17] that homological projective duality was a more general phenomena than just between linear sections of HP-dual pairs. In this section we describe the main theorem of this paper and demonstrate its powerful applications in constructing new semiorthogonal decompositions for varieties which are nonlinear sections of HP-dual pairs. This allows us to obtain in a slick way the semiorthogonal decomposition for the intersection of two quadrics or the intersection of a quadric and a Grassmannian (a.k.a. a Gushel-Mukai variety[1]) which previously could not be done the ordinary HPD theorem 3.3.

In the section we will use the notation of [JLX17]. Importantly note that the use the cohomological convention for dual Lefschetz decompositions which we purposefully introduced back in section I.7, see I.97. Therefore we restate the HPD theorem, theorem 3.3 in the notation of [JLX17].

Let V be an $n+1$ dimensional k-vector space then $\mathbb{P}^n = \mathbb{P}(V)$ Suppose that $f : X \to \mathbb{P}(V)$ and $f^\vee : X^\vee \to \mathbb{P}(V^*)$ are a HP-dual pair with dual Lefschetz decompositions.

$$\mathrm{D}(X) = \Big\langle \mathcal{A}_0, \mathcal{A}_1(1), \dots \mathcal{A}_{i-1}(i-1) \Big\rangle, \quad \mathrm{D}(X^\vee) = \Big\langle \mathcal{A}^1(-n+1), \dots \mathcal{A}^{n-1}(-1), \mathcal{A}^n \Big\rangle, \tag{II.92}$$

where the $\mathcal{A}$'s are complementary building blocks of each others components. Then for $L \subset V$ admissible there is a residual category of complete linear sections $X_L := X \times_{\mathbb{P}(V)} \mathbb{P}(L)$ and $X_L^\vee := X^\vee \times_{\mathbb{P}(V^*)} \mathbb{P}(L^\perp)$ where the codimension of $P(L) \subset \mathbb{P}(V)$ is ℓ and $L^\perp \subset V^*$ is the (annihilator) subspace of L. Then

$$\mathrm{D}(X_L) = \Big\langle \mathcal{K}, \mathcal{A}_{\ell-1}(\ell-1), \dots, \mathcal{A}_{i-1}(i-1) \Big\rangle, \tag{II.93}$$

there is also a similar decomposition of

$$\mathrm{D}(X^\vee) = \Big\langle \mathcal{A}^1(-\ell+2), \dots, \mathcal{A}^{\ell-2}(-1), \mathcal{A}^{\ell-1}, \mathcal{K} \Big\rangle, \tag{II.94}$$

Now we will replace the Homological Projective Dual pair $(\mathbb{P}(L), \mathbb{P}(L^\perp))$ with another pair of HP dual spaces $(Q, Q^\vee)$ where $Q \to \mathbb{P}(V)$ and $Q^\vee \to \mathbb{P}(V^*)$ provided they are admissible, i.e intersect with expected dimension. Denote the associated Lefschetz decompositions

$$\mathrm{D}(Q) = \langle \mathcal{B}^1(-n+1), \dots, \mathcal{B}^{n-1}(-1), \mathcal{B}^n \rangle, \quad \mathrm{D}(Q^\vee) = \langle \mathcal{B}_0, \mathcal{B}_1(1), \dots, \mathcal{B}_{\ell-1}(\ell-1) \rangle. \tag{II.95}$$

Theorem 6.1. *[[JLX17], Thm. 3.12] The Generalized Fundamental Theorem of HPD.*
If the homologically dual pairs $(X, X^\vee)$ and $(Q, Q^\vee)$ are admissible then we have the following dual Lefschetz decompositions of their intersections

$$\mathrm{D}(X_Q) = \Big\langle \mathcal{K}_Q(X), \langle (\mathcal{A}_k \boxtimes \mathcal{B}^k) \otimes \mathcal{O}(k) \rangle_{k \in \mathbb{Z}} \Big\rangle, \tag{II.96}$$

with $X_Q = X \times_{\mathbb{P}(V)} Q$ and dually $X^\vee_{Q^\vee} = X^\vee \times_{\mathbb{P}(V^*)} Q^\vee$ has decomposition

$$\mathrm{D}(X^\vee_{Q^\vee}) = \Big\langle \langle (\mathcal{A}^k \boxtimes \mathcal{B}_k) \otimes \mathcal{O}(-\ell+1+k) \rangle_{k \in \mathbb{Z}}, \mathcal{K}_{Q^\vee}(X^\vee) \Big\rangle, \tag{II.97}$$

where there is an equivalence of Kuznetsov components given by $\mathcal{K}_Q(X) \simeq \mathcal{K}_{Q^\vee}(X^\vee)$.

[1] Morally, I should emphasize that it is an ordinary GM variety. Meaning the intersection is away from the cone point.

If $f : X \to \mathbb{P}(V)$ is non degenerate i.e ($V^* \subset \Gamma(X, \mathcal{O}_X(1))$) and we take the homological projective dual pair $(\mathbb{P}(L), \mathbb{P}(L^\perp))$ of linear spaces where $\mathbb{P}(L) \subset \mathbb{P}(V)$ has codimension ℓ. That is since $\mathrm{D}(\mathbb{P}(L^\perp))$ is rectangular i,e $B_k = B_0$ for $k \in [0, \ell - 1]$ and $\mathcal{B}^k = 0$ for $k \leq \ell - 1$ and $\mathcal{B}^k = B_0$ for $k \geq \ell$. From Beilinsons decomposition we have that $\mathcal{B}_0 = \mathcal{O}$. Therefore

$$\mathrm{D}(X_{\mathbb{P}(L)}) = \Big\langle \mathcal{K}, \mathcal{A}_\ell(\ell) \boxtimes \mathcal{B}_0, \dots, \mathcal{A}_{i-1}(i-1) \boxtimes \mathcal{B}_0 \Big\rangle \tag{II.98}$$

$$= \Big\langle \mathcal{K}, \mathcal{A}_\ell(\ell), \dots \mathcal{A}_{i-1}(i-1) \Big\rangle. \tag{II.99}$$

and similarly for the dual decomposition

$$\mathrm{D}(X^\vee_{\mathbb{P}(L^\perp)}) = \Big\langle \mathcal{A}^1 \boxtimes \mathcal{B}_0(-\ell+2), \dots, \mathcal{A}^{\ell-1} \boxtimes \mathcal{B}_0, \mathcal{K} \Big\rangle \tag{II.100}$$

$$= \Big\langle \mathcal{A}^1(-\ell+2), \dots, \mathcal{A}^{\ell-2}(-1), \mathcal{A}^{\ell-1}, \mathcal{K} \Big\rangle. \tag{II.101}$$

Hence we recover Kuznetsov's fundamental theorem of homological projective duality 3.3 as a special case.

We now give examples of how this generalized homological projective duality theorem can be applied to obtain new semiorthogonal decompositions.

Example 6.2. (Intersections of Quadrics) Consider the 5-dimensional quadric $Q_1^5 \subset \mathbb{P}^6$ with the associated Lefschetz decomposition of its derived category which we know from 4.2

$$\mathrm{D}(Q_1^5) = \Big\langle \langle \mathbf{S}_X, \mathcal{O}_X \rangle, \mathcal{O}(1), \mathcal{O}(2), \mathcal{O}(3), \mathcal{O}(4) \Big\rangle, \tag{II.102}$$

Recall from the previous section the homological projective dual is a 6 dimensional quadric $X^\vee = Q_1^6$ with a ramified double cover to the dual projective space $Q_1^6 \to \check{\mathbb{P}}^6$ ramified over the dual quadric $\overline{Q}_1^5 \subset \check{\mathbb{P}}^6$.

$$\mathrm{D}(Q_1^6) = \Big\langle \langle \mathbf{S}_{X^\vee}, \mathcal{O}_{X^\vee} \rangle, \langle \mathbf{S}_{X^*}, \mathcal{O}_{X^*} \rangle(1), \mathcal{O}(2), \mathcal{O}(3), \mathcal{O}(4), \mathcal{O}(5) \Big\rangle \tag{II.103}$$

Now consider the same setup for the quadric $Y = Q_2^5 \to \mathbb{P}^6$ and its homological projective dual quadric $Y^\vee = Q_2^6 \to \check{\mathbb{P}}^6$ where we assume the intersection has expected dimension. Consider now the HPD pairs (Q_1^5, Q_1^6) and (Q_2^5, Q_2^6). Assuming Q_1^5 and Q_2^6 intersect with expected dimension then applying theorem 6.1 we obtain

$$\mathrm{D}(Q_1^5 \cap Q_2^5) = \mathrm{D}(Q_1^5 \times_{\mathbb{P}^6} Q_2^5) = \Big\langle \mathcal{K}, \mathcal{O}(1), \mathcal{O}(2), \mathcal{O}(3) \Big\rangle \tag{II.104}$$

and this agrees with the decomposition in 4.4 where we can identify the Kuznetsov component with $\mathrm{D}(C)$ where C is an orbifold $\mathbb{P}^1$ with a $\mathbb{Z}/2\mathbb{Z}$-stack structure over 7 points. The second decomposition of our theorem implies there is a decomposition of Q_1^6 with respect to Q_2^6.

$$\mathrm{D}(Q_1^6 \cap Q_2^6) = \Big\langle \langle \mathbf{S}_{X^\vee}, \mathcal{O}_{X^\vee} \rangle(-5), \mathcal{O}(-4), \mathcal{O}(-3), \mathcal{O}(-2), \langle \mathbf{S}_{X^\vee}, \mathcal{O} \rangle(-1), \mathrm{D}(C) \Big\rangle \tag{II.105}$$

with $Q_1^6 \cap Q_2^6$ a smooth 6-dimensional manifold admitting a degree 4 finite surjection onto $\mathbb{P}^6$.

Example 6.3. Consider the above example of the intersection of two even dimensional quadrics, but now in $\mathbb{P}^7$ that is $Q_1^6 \subset \mathbb{P}^7$ and $Q_2^6 \subset \mathbb{P}^7$. The theorem 6.1 implies there is a decomposition

$$\mathrm{D}(Q_1^6 \cap Q_2^6) = \Big\langle \mathcal{K}, \mathcal{O}(1), \mathcal{O}(2), \mathcal{O}(3), \mathcal{O}(4) \Big\rangle. \tag{II.106}$$

This agrees with 4.3 from which we know $\mathcal{K} = \mathrm{D}(C')$ where C' is a hyperelliptic curve ramified over 8 points over $\mathbb{P}^1$. Then the second decomposition of the theorem implies that for the associated intersection of dual quadrics:

$$\mathrm{D}(\overline{Q}_1^6 \cap \overline{Q}_2^6) = \Big\langle \mathcal{O}(-4), \mathcal{O}(-3), \mathcal{O}(-2), \mathcal{O}(-1), \mathrm{D}(C') \Big\rangle. \tag{II.107}$$

Example 6.4. (Quadric Sections and Pfaffian-Grassmannian Duality)
Let $X = \mathbf{Gr}(2,5) \subset \mathbb{P}(\wedge^2 \mathscr{C}^5) = \mathbb{P}^9$, with rectangular Lefschetz decomposition

$$\mathrm{D}(X) = \Big\langle \mathcal{A}_0, \mathcal{A}_0(1), \mathcal{A}_0(2), \mathcal{A}_0(3), \mathcal{A}_0(4) \Big\rangle \tag{II.108}$$

where $\mathcal{A}_0 = \langle \mathcal{U}^\vee, \mathcal{O} \rangle$. Then it is Homologically Projectively dual to $X^\vee = \mathbf{Pf}(2,5) = \mathbf{Gr}(2,5) \subset \check{\mathbb{P}}^9$ which has dual rectangular Lefschetz decomposition.

Consider the Homologically dual pair of Quadrics $Y = Q^8 \subset \mathbb{P}^9$ and $Y^\vee = \overline{Q}^8 \subset \check{\mathbb{P}}^9$ with

then as long as $\mathbf{Gr}(2,5) \cap Q^8$ and $\mathbf{Pf}(2,5) \cap \overline{\mathbb{Q}}^8$ are intersecting generically applying theorem 6.1 we have

$$\mathrm{D}(\mathbf{Gr}(2,5) \cap Q^8) = \Big\langle \mathcal{K}, \mathcal{A}_0, \mathcal{A}_0(1), \mathcal{A}_0(2) \Big\rangle, \tag{II.109}$$

Similarly for the dual intersection we have

$$\mathrm{D}(\mathbf{Pf}(2,5) \cap \overline{Q}^8) = \Big\langle \mathcal{A}^0(-2), \mathcal{A}^0(-1), \mathcal{A}^0, \mathcal{K} \Big\rangle, \tag{II.110}$$

which share equivalent Kuznetsov components $\mathcal{K}$.

Example 6.5. (Gushel-Mukai Varieties)
Consider the same exact setup Now consider the homological dual pair $X = Q^9 \to \mathbb{P}^9$, $X^\vee = \check{Q}^9 \to \check{\mathbb{P}}^9$ to be 9-dimensional quadric with ramified double covering over $\mathbb{P}^9$ and $\check{\mathbb{P}}^9$ ramified over Q^8 and $\overline{Q}^8$ dual respectively, then

Then we have

$$\mathrm{D}(\mathbf{Gr}(2,5) \cap Q^9) = \langle \mathcal{K}, \mathcal{A}_0, \mathcal{A}_0(1), \mathcal{A}_0(2), \mathcal{A}_0(3) \rangle, \tag{II.111}$$

and

$$\mathrm{D}(\mathbf{Pf}(2,5) \cap \overline{Q}^9) = \langle \mathcal{A}^0(-3), \mathcal{A}^0(-2), \mathcal{A}^0(-1), \mathcal{K} \rangle. \tag{II.112}$$

This generalized theorem of HPD was also proven using a slightly different, more general approach by Kuznetsov and Perry see [KP19].

Chapter III

Homological Geometry

1 Homological Projective Geometry

The previous chapter was dedicated to homological projective duality, which is a duality on the level of derived categories that is analogous to the classical Lefschetz duality for the cohomology of hyperplane sections. Surprisingly more of these classical results that have homological counterparts are being realized, leading to the development of a new area of research called Homological Projective Geometry. In this section we give a brief overview of some the key developments of homological projective geometry:

Categorical Resolutions: Kuznetsov defines the categorical analogue of a resolution as

Definition 1.1 ([Kuz15], Def. 3.1). A categorical resolution of singularities of a scheme Y is a smooth triangulated category $\mathcal{T}$ and an adjoint pair of triangulated functors $\pi_* : \mathcal{T} \to \mathrm{D}(Y)$ and $\pi^* : \mathrm{D}(Y) \to \mathcal{T}$ such that $\pi_* \circ \pi^* \cong \mathrm{id}_{\mathrm{D}^{\mathrm{perf}}(Y)}$. In particular the functor π^* is fully faithful.

This is a particular useful notion with respect to HPD since there are homological projective dual categories which are the categorical resolution of a variety. The most well known example being the Pfaffian-Grassmannian duality, see [Kuz15], Conj. 4.4.

Categorical Joins: In [KP20] Kuznetsov and Perry extend the classical notion of a join to the level of derived categories [[KP20], Def. 1.3], where the categorical join is a noncommutative categorical resolution of singularities of the classical join of X_1 and X_2. Recall that classical projective duality commutes with joins, i.e for $X_1 \subset \mathbb{P}(V_1)$ and $X_2 \subset \mathbb{P}(V_2)$ we have

$$\mathbf{J}(X_1, X_2)^* = \mathbf{J}(X_1^*, X_2^*). \tag{III.1}$$

It turns out that HPD and joins are compatible. More precisely, the homological projective dual category of the categorical join is naturally equivalent to the categorical join of the homological projective dual categories.

Theorem 1.2 ([KP20]). *Let $X_1 \to \mathbb{P}(V_1)$ and $X_2 \to \mathbb{P}(V_2)$ be varieties with Lefschetz decompositions that have length less than $\dim(V_1)$ and $\dim(V_2)$ respectively . Let $X_1^\vee \to \mathbb{P}(V_1^\vee)$ and $X_2^\vee \to \mathbb{P}(V_2^\vee)$ be the HPD varieties. Then there is an equivalence*

$$\mathcal{J}(X_1, X_2)^\vee \simeq \mathcal{J}(X_1^\vee, X_2^\vee), \tag{III.2}$$

of Lefschetz categories [[KP20], Def. 2.6] over $\mathbb{P}(V_1^* \oplus V_2^*)$.

Kuznetsov and Perry have also defined the notion of a **categorical cone** for noncommutative HPD pairs, see [[KP19], Def. 3.6]. They obtain a categorification of another classical relation between cones and classical projective duals in terms of HPD see [[KP19], Thm. 4.1].

2 The Transform Conjecture and Affine HPD

In the current work-in-progress [DM24], Davis-M. construct a transform that is compatible with respect to semiorthogonal decompositions. We now consider this functor/transform, however the explicit construction and related technicalities would take us to far away from the subject of homological projective duality. Therefore in this section we assume the existence of such a functor with the properties of conjecture 2.1 and see what its existence would imply.

Fix X a smooth algebraic variety and let E be a locally free sheaf on X. Let $\mathcal{D}_E$ denote the $(\infty, 2)$-category of dg-categories tensored over the derived category $\mathrm{D}(E)$.

Conjecture 2.1 ([DM24]). There exists a well defined functor

$$\mathscr{F}_E : \mathcal{D}_E \to \mathcal{D}_{E^*}, \tag{III.3}$$

where E^* is dual bundle of E such that $\mathscr{F}_E$ satisfies the following properties:

1. For every sub-bundle $U \subseteq E$ over X the transform $\mathscr{F}_E : \mathrm{D}_E \to \mathrm{D}_{E^*}$ sends

$$\mathscr{F}_E(\mathrm{D}(U)) = \mathrm{D}(U^\perp), \tag{III.4}$$

   where $U^\perp \subset E^*$ is the orthogonal bundle of U.

2. Let $f : X \to Y$ be a sufficiently nice morphism of schemes, then

$$\mathscr{F}_{f^*E}(\mathrm{D}) = \mathscr{F}_E(\mathrm{D}). \tag{III.5}$$

   for every dg-derived category $\mathrm{D} \in \mathcal{D}_{f^*E}$. That is the following diagram commutes

$$\begin{tikzcd} \mathcal{D}_{f^*E} \arrow[r, "\mathscr{F}_{f^*E}"] \arrow[d] & \mathcal{D}_{f^*E} \arrow[d] \\ \mathcal{D}_E \arrow[r, "\mathscr{F}_E"] & \mathcal{D}_E \end{tikzcd}$$

   where $\mathcal{D}_{f^*E} \to \mathcal{D}_E$ is the forgetful functor.

3. The functor is compatible with semiorthogonal decompositions. If $\mathcal{C}$ is a dg derived category tensored over $\mathrm{D}(E)$ which admits a semiorthogonal decomposition $\mathcal{C} = \langle \mathcal{A}, \mathcal{B} \rangle$. Then applying the functor $\mathcal{F}_E : \mathcal{D}_E \to \mathcal{D}_E$ yields the semiorthogonal decomposition

$$\mathscr{F}_E(\mathcal{C}) = \Big\langle \mathscr{F}_E(\mathcal{A}), \mathscr{F}_E(\mathcal{B}) \Big\rangle. \tag{III.6}$$

For the remainder of this section, we tacitly assume the above conjecture and take properties 1, 2 and 3 as axioms to see what it's truth would necessarily imply about semiorthogonal decompositions we might encounter throughout the study of derived categories.

Corollary 2.2. *Suppose $\mathcal{A} \in \mathcal{D}_E$ admits a semiorthogonal decomposition $\mathcal{A} = \langle \mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_n \rangle$ then*

$$\mathscr{F}_E(\mathcal{A}) = \Big\langle \mathscr{F}_E(\mathcal{A}_1), \mathscr{F}_E(\mathcal{A}_2), \dots, \mathscr{F}_E(\mathcal{A}_n) \Big\rangle. \tag{III.7}$$

Proof. Immediately follows by induction from property 3 of conjecture 2.1. □

Proposition 2.3. *Suppose $X \hookrightarrow \mathbb{P}(V)$ is a closed immersion with V a $(n+1)$-dimensional k-vector space dual to $V^* = H^0(X, \mathcal{O}_X(1))$. then*

$$\mathscr{F}_{g^*V}\, \mathrm{D}(\mathrm{Tot}(\mathcal{O}_X(-1))) = \mathscr{F}_V\, \mathrm{D}(\tilde{X}), \tag{III.8}$$

where $\tilde{X} = \{(x, f) \in X \times V^* | f(x) = 0\}$.

Proof. Since $X \hookrightarrow \mathbb{P}(V)$ is a closed immersion where V is a k-vector space. It follows that there is a map $\mathrm{Tot}(\mathcal{O}_X(-1)) \to V$ which contracts the zero section to a point. Let $g : X \to \mathrm{Spec}(k)$ then $\mathrm{D}(\mathrm{Tot}(\mathcal{O}_X(-1)))$ is also tensored over $\mathrm{D}(\mathrm{Tot}(g^*V)) = \mathrm{D}(X \times V)$ and hence $\mathrm{Tot}(\mathcal{O}_X(-1)) \to X \times V$ is a sub-bundle of the trivial bundle. Applying property (2) and property (1) of conjecture 2.1 yields

$$\mathscr{F}_{g^*V}\, \mathrm{D}(\mathrm{Tot}(\mathcal{O}_X(-1))) = \mathscr{F}_V\, \mathrm{D}(\mathrm{Tot}(\mathcal{O}_X(-1))) = \mathrm{D}(\mathrm{Tot}(\mathcal{O}_X(-1))^\perp) = \mathrm{D}(\tilde{X}). \tag{III.9}$$

where we have recognized that $\mathrm{Tot}(\mathcal{O}_X(-1))^\perp$ is precisely $\tilde{X}$. □

Remark 2.4. The relative case also follows by the exact same argument.

Observe that $\tilde{X}$ is very close to the universal hyperplane section in homological projective duality. Compare

$$\tilde{X} = \{(x, f) \in X \times V^* | f(x) = 0\} \text{ and } \mathcal{H} = \{(x, s) \in \mathbb{P}(V) \times \mathbb{P}(V^*) | s(x) = 0\}, \tag{III.10}$$

So in a way property 3 in the conjecture predicts that semiorthogonal decompositions of $\mathrm{D}(\mathrm{Tot}(\mathcal{O}(-1))$ and of $\mathrm{D}(\tilde{X})$ should be regarded as an affine analogue to $\mathrm{D}(X)$ and $\mathrm{D}(\mathcal{H})$ in HPD.

$$\mathrm{D}(X_L) \xleftrightarrow{HPD} \mathrm{D}(\mathcal{H}_L) \qquad\qquad \mathrm{D}(\mathrm{Tot}(\mathcal{O}_X(-1)) \xleftrightarrow{?} \mathrm{D}(\tilde{X})$$

We now continue to compute semiorthogonal decompositions of derived categories using the formal properties of conjecture 2.1.

Proposition 2.5. *Let V be a 3 dimensional k-vector space and let $U \subset V$ be a 2 dimensional subspace. Let $X = \mathbb{P}^1 = \mathbb{P}(U)$ be linearly embedded in $\mathbb{P}^2 = \mathbb{P}(V)$ then Orlov's formula gives*

$$\mathrm{D}(\mathrm{Tot}(\mathcal{O}_X(-1))) = \Big\langle \mathcal{O}_X(-1), \mathrm{D}(\mathbb{A}^2) \Big\rangle. \tag{III.11}$$

Here $\mathcal{O}_X(-1)$ is an exceptional object supported on the zero section $X \subseteq \mathrm{Tot}(\mathcal{O}_X(-1))$. We claim that the affine dual admits a semiorthogonal decomposition

$$\mathrm{D}(\tilde{X}) = \Big\langle \mathrm{D}(V^*), \mathrm{D}(U^\perp) \Big\rangle. \tag{III.12}$$

Proof. Applying the transform $\mathscr{F} := \mathscr{F}_V$ and identifying $U \cong \mathbb{A}^2$ we compute

$$
\begin{aligned}
\mathrm{D}(\tilde{X}) &= \mathscr{F}\,\mathrm{D}(\mathrm{Tot}(\mathcal{O}_X(-1))) && \text{(III.13)}\\
&= \mathscr{F}\left\langle \mathcal{O}_X(-1), \mathrm{D}(\mathbb{A}^2)\right\rangle && \text{(III.14)}\\
&= \left\langle \mathscr{F}\,\mathcal{O}_X(-1), \mathscr{F}\,\mathrm{D}(\mathbb{A}^2)\right\rangle \quad \text{(property 3)} && \text{(III.15)}\\
&= \left\langle \mathrm{D}(V^*), \mathrm{D}(U^{\perp})\right\rangle, && \text{(III.16)}
\end{aligned}
$$

where by property (1) and (2) $\mathscr{F}_V\,\mathcal{O}_X(-1)$ is equivalent to $\mathrm{D}(V^*)$. □

Proposition 2.6. *Fix positive integers $m, n \in \mathbb{Z}$ where $m < n$. Let V be a $n+1$ dimensional k-vector space and let $U \subset V$ be a $m+1$ dimensional subspace. Let $X = \mathbb{P}^m = \mathbb{P}(U)$ be linearly embedded in $\mathbb{P}^n = \mathbb{P}(V)$ then by Orlov's formula*

$$\mathrm{D}(\mathrm{Tot}(\mathcal{O}_X(-1))) = \left\langle \mathcal{O}_{\mathbb{P}^m}(-m), \cdots, \mathcal{O}_{\mathbb{P}^m}(-1), \mathrm{D}(\mathbb{A}^{m+1})\right\rangle. \tag{III.17}$$

Here $\mathcal{O}_{\mathbb{P}^m}(-1)$ is an exceptional object supported on the zero section $\mathbb{P}^m \subseteq \mathrm{Tot}(\mathcal{O}_{\mathbb{P}^m}(-1))$. Then affine dual admits a semiorthogonal decomposition

$$\mathrm{D}(\tilde{X}) = \Big\langle \underbrace{\mathrm{D}(V^*), \ldots, \mathrm{D}(V^*)}_{m\ \textit{times}}, \mathrm{D}(U^{\perp})\Big\rangle. \tag{III.18}$$

Proof. Applying the transform $\mathcal{F} := \mathcal{F}_V$ from conjecture 2.1 and identifying $U \cong \mathbb{A}^2$ we compute

$$
\begin{aligned}
\mathrm{D}(\tilde{X}) &= \mathscr{F}\,\mathrm{D}(\mathrm{Tot}(\mathcal{O}_X(-1))) && \text{(III.19)}\\
&= \mathscr{F}\left\langle \mathcal{O}_{\mathbb{P}^m}(-m), \cdots, \mathcal{O}_{\mathbb{P}^m}(-1), \mathrm{D}(\mathbb{A}^{m+1})\right\rangle && \text{(III.20)}\\
&= \left\langle \mathscr{F}\,\mathcal{O}_{\mathbb{P}^m}(-m), \cdots, \mathscr{F}\,\mathcal{O}_{\mathbb{P}^m}(-1), \mathscr{F}\,\mathrm{D}(\mathbb{A}^{m+1})\right\rangle \text{ by corollary 2.2,} && \text{(III.21)}\\
&= \Big\langle \underbrace{\mathrm{D}(V^*), \ldots, \mathrm{D}(V^*)}_{m \text{ times}}, \mathrm{D}(U^{\perp})\Big\rangle \text{ by property 1.} && \text{(III.22)}
\end{aligned}
$$

and by property (1) and (2) $\mathscr{F}_V\,\mathcal{O}_{\mathbb{P}^m}(-m)$ is equivalent to $\mathrm{D}(V^*)$. □

Example 2.7. It would be interesting to see what the above formalism allows you to conclude when applying it to Atiyah flops. For instance let V be a 2-dimensional k-vector space, take $X \cong \mathbb{P}^1 \times \mathbb{P}^1 = \mathbb{P}(V) \times \mathbb{P}(V) \subset \mathbb{P}^3$. Here

$$X' = \mathrm{Tot}(\mathcal{O}_X(-1)) \cong \mathrm{Tot}(\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(-1,-1)), \tag{III.23}$$

and $\tilde{X} = \{(x, f) \in Q \times V^* | f(x) = 0\}$. A straightforward application of Orlov's blow-up formula yields

$$\mathrm{D}(\mathrm{Tot}(\mathcal{O}_X(-1))) = \left\langle \mathcal{O}(-1,-1), \mathcal{O}(-1,0), \mathrm{D}(X^+)\right\rangle, \tag{III.24}$$

and applying Orlov's formula again to the rght hand side of the flop diagram 2.7, gives the decomposition

$$\mathrm{D}(\mathrm{Tot}(\mathcal{O}_X(-1))) = \left\langle \mathcal{O}_X(-1,-1), \mathcal{O}_X(0,-1), \mathrm{D}(X^-)\right\rangle, \tag{III.25}$$

where $X^+ \cong \mathrm{Tot}(\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)) \cong X^-$. Here X^+ and X^- denote the two sides of the Atiyah flop with setup given by the diagram

$$\begin{tikzcd}
 & & \mathbb{P}^1 \times \mathbb{P}^1 \arrow[dll, "p_1"'] \arrow[d, hook, "j"] \arrow[drr, "p_2"] & & \\
 & & X' \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & & \\
\mathbb{P}^1 \arrow[r, hook, "i_1"] & X^+ & & X^- & \mathbb{P}^1 \arrow[l, hook', "i_2"']
\end{tikzcd}$$

Consider the first decomposition, $\mathrm{D}(X') = \langle \mathcal{O}(-1,-1), \mathcal{O}(-1,0), \mathrm{D}(X^+) \rangle$ then by conjecture 2.1 we have the functor $\mathscr{F} := \mathscr{F}_V$. We compute

$$\begin{aligned}
\mathrm{D}(\tilde{X}) &= \mathscr{F}(\mathrm{D}(X')) && \text{(III.26)}\\
&= \mathscr{F}\left\langle \mathcal{O}(-1,-1), \mathcal{O}(-1,0), \mathrm{D}(X^+) \right\rangle && \text{(III.27)}\\
&= \left\langle \mathscr{F}\,\mathcal{O}(-1,-1), \mathscr{F}\,\mathcal{O}(-1,0), \mathscr{F}\,\mathrm{D}(X^+) \right\rangle && \text{(III.28)}\\
&= \left\langle \mathrm{D}(V^*), \mathrm{D}(V^*), \mathscr{F}\,\mathrm{D}(X^+) \right\rangle. && \text{(III.29)}
\end{aligned}$$

Similarly for the second semiorthogonal decomposition applying the functor yields

$$\mathrm{D}(X') = \left\langle \mathcal{O}(-1,-1), \mathcal{O}(0,-1), \mathrm{D}(X^+) \right\rangle, \tag{III.30}$$

then applying the functor we similarly obtain

$$\begin{aligned}
\mathrm{D}(\tilde{X}) &= \mathscr{F}(\mathrm{D}(X')) && \text{(III.31)}\\
&= \mathscr{F}\left\langle \mathcal{O}(-1,-1), \mathcal{O}(-1,0), \mathrm{D}(X^-) \right\rangle && \text{(III.32)}\\
&= \left\langle \mathscr{F}\,\mathcal{O}(-1,-1), \mathscr{F}\,\mathcal{O}(0,-1), \mathscr{F}\,\mathrm{D}(X^-) \right\rangle && \text{(III.33)}\\
&= \left\langle \mathrm{D}(V^*), \mathrm{D}(V^*), \mathscr{F}\,\mathrm{D}(X^-) \right\rangle. && \text{(III.34)}
\end{aligned}$$

And so we obtain two decompositions of the affine dual $\mathrm{D}(\tilde{X})$. However since

$$X^+ \cong \mathrm{Tot}(\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)) \cong X^-, \tag{III.35}$$

we must have that

$$\mathscr{F}\,\mathrm{D}(X^+) \cong \mathscr{F}\,\mathrm{D}(X^-) \cong \mathscr{F}\,\mathrm{D}(\mathrm{Tot}(\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1))). \tag{III.36}$$

Therefore $\mathrm{D}(\tilde{X})$ admits a semiorthogonal decomposition of the form

$$\mathrm{D}(\tilde{X}) = \left\langle \mathrm{D}(V^*), \mathrm{D}(V^*), \mathscr{F}\,\mathrm{D}(X^-) \right\rangle. \tag{III.37}$$

3 Equivariant Transform and HPD

In this section we consider a equivariant version of conjecture 2.1 and show how it can be used to recover Kuznetsov's homological projective duality.

Fix X a smooth algebraic variety and let E be a locally free sheaf on X. Denote by $\mathcal{D}_E^{\mathbb{G}_\mathrm{m}}$ the $(\infty, 2)$-category of dg-derived categories tensored over the derived category of $\mathbb{G}_\mathrm{m}$-equivariant sheaves on E, denoted $\mathrm{D}_{\mathbb{G}_\mathrm{m}}(E)$.

Conjecture 3.1 ([DM24]). There exists a well defined functor

$$\mathscr{F}_E : \mathcal{D}_E^{\mathbb{G}_\mathrm{m}} \to \mathcal{D}_E^{\mathbb{G}_\mathrm{m}}, \tag{III.38}$$

where E^* is dual bundle of E such that $\mathscr{F}_E$ satisfies the following properties:

1. For every sub-bundle $U \subseteq E$ over X the transform $\mathscr{F}_E : \mathcal{D}_E^{\mathbb{G}_\mathrm{m}} \to \mathcal{D}_E^{\mathbb{G}_\mathrm{m}}$ sends

   $$\mathscr{F}_E(\mathrm{D}_{\mathbb{G}_\mathrm{m}}(U)) = \mathrm{D}_{\mathbb{G}_\mathrm{m}}(U^\perp), \tag{III.39}$$

   where $U^\perp \subset E^*$ is the orthogonal bundle of U.

2. Let $f : X \to Y$ be a sufficiently nice morphism of schemes, then

   $$\mathscr{F}_{f^*E}(\mathrm{D}) = \mathscr{F}_E(\mathrm{D}). \tag{III.40}$$

   for every dg-derived category $\mathrm{D} \in \mathcal{D}_{f^*E}^{\mathbb{G}_\mathrm{m}}$. That is the following diagram commutes

   $$\begin{tikzcd} \mathcal{D}_{f^*E} \arrow[r, "\mathscr{F}_{f^*E}"] \arrow[d] & \mathcal{D}_{f^*E} \arrow[d] \\ \mathcal{D}_E \arrow[r, "\mathscr{F}_E"] & \mathcal{D}_E \end{tikzcd}$$

   where $\mathcal{D}_{f^*E}^{\mathbb{G}_\mathrm{m}} \to \mathcal{D}_E^{\mathbb{G}_\mathrm{m}}$ is the forgetful functor.

3. If $\mathcal{C}$ is a dg-category in $\mathcal{D}_E^{\mathbb{G}_\mathrm{m}}$ which admits a semiorthogonal decomposition $\mathcal{C} = \langle \mathcal{A}, \mathcal{B} \rangle$. Then applying the functor $\mathcal{F} : \mathcal{D}_E^{\mathbb{G}_\mathrm{m}} \to \mathcal{D}_E^{\mathbb{G}_\mathrm{m}}$ yields the semiorthogonal decomposition

   $$\mathscr{F}_E(\mathcal{C}) = \Big\langle \mathscr{F}_E(\mathcal{A}), \mathscr{F}_E(\mathcal{B}) \Big\rangle. \tag{III.41}$$

4. Let $\mathcal{A} \in \mathcal{D}_E^{\mathbb{G}_\mathrm{m}}$ and B is a dg-category. If $\mathrm{D}_{\mathbb{G}_\mathrm{m}}(E)$ acts on $\mathcal{A} \otimes \mathcal{B}$ through $\mathcal{A}$ only then

   $$\mathscr{F}_E(\mathcal{A} \otimes \mathcal{B}) = \mathscr{F}_E(\mathcal{A}) \otimes \mathcal{B}. \tag{III.42}$$

We also assume the analogous property holds when acting only through $\mathcal{B}$.

For the remainder of this section we assume conjecture 3.1 and take the properties of $\mathcal{F}_E$ as axioms.

Let X be a smooth algebraic variety with fixed polarization $\mathcal{O}_X(1)$ such that we have a morphism $X \to \mathbb{P}(V)$. Suppose in addition to that, X admits a rectangular[1] Lefschetz decomposition

$$\mathrm{D}(X) = \Big\langle \mathcal{A}, \mathcal{A}(1), \ldots, \mathcal{A}(i-1) \Big\rangle, \tag{III.43}$$

Let $\pi : \mathrm{Tot}(\mathcal{O}_X(-1)) \to X$ and $\iota : X \to \mathrm{Tot}(\mathcal{O}_X(-1))$ be projection and inclusion respectively. Let

$$\tilde{X} = \{(x, f) \in X \times V^* | f(x) = 0\}. \tag{III.44}$$

Proposition 3.2. *With the above setup, Their is rectangular Lefschetz decomposition of* $\mathrm{D}(\mathrm{Tot}(\mathcal{O}(-1)))$ *of the form.*

$$\mathrm{D}(\mathrm{Tot}(\mathcal{O}_X(-1)) = \Big\langle \pi^*\mathcal{A}, \mathcal{A} \otimes \mathrm{D}_{\mathbb{G}_\mathrm{m}}(\{pt\}), \ldots, \mathcal{A}(i-2) \otimes \mathrm{D}_{\mathbb{G}_\mathrm{m}}(\{pt\}) \Big\rangle, \tag{III.45}$$

where $\mathcal{A}_k(k-1)$ *is embedded by pushing forward the zero section* $\iota : X \subset \mathrm{Tot}(\mathcal{O}_X(-1))$.

Proof. (sketch) Semiorthogonality follows from applying the Kunneth formula and using the semiorthogonality of the Lefschetz decomposition for $\mathrm{D}(X)$. Generation follows from the fact that $\mathcal{A}$'s generate $\mathrm{D}(X)$. □

Remark 3.3. Note that $\otimes \mathrm{D}_{\mathbb{G}_\mathrm{m}}(\{pt\})$ is the induced grading on categories since categories in $\mathcal{D}_V^{\mathbb{G}_\mathrm{m}}$ are tensored over the dg-derived category of equivariant coherent sheaves over V.

Now considering the functor $\mathscr{F} := \mathscr{F}_V$ from conjecture 3.1 and the decomposition from proposition III.45 we compute:

$$\begin{aligned}
\mathrm{D}_{\mathbb{G}_\mathrm{m}}(\tilde{X}) &= \mathscr{F}(\mathrm{D}(\mathrm{Tot}(\mathcal{O}_X(-1)))) && \text{(III.46)}\\
&= \mathscr{F} \Big\langle \pi^*\mathcal{A}_0, \mathcal{A}_1 \otimes \mathrm{D}_{\mathbb{G}_\mathrm{m}}(\{\mathrm{pt}\}), \ldots, \mathcal{A}_{i-1}(i-2) \otimes \mathrm{D}_{\mathbb{G}_\mathrm{m}}(\{pt\}) \Big\rangle && \text{(III.47)}\\
&= \Big\langle \mathscr{F}(\pi^*\mathcal{A}_0), \mathscr{F}(\mathcal{A}_1 \otimes \mathrm{D}_{\mathbb{G}_\mathrm{m}}(\{\mathrm{pt}\})), \ldots, \mathscr{F}(\mathcal{A}_{i-1}(i-2) \otimes \mathrm{D}_{\mathbb{G}_\mathrm{m}}(\{pt\})) \Big\rangle && \text{(III.48)}
\end{aligned}$$

where we have used property (3). Applying property (4) it follows

$$\begin{aligned}
&= \Big\langle \mathscr{F}(\pi^*\mathcal{A}_0), \mathcal{A}_1 \otimes \mathscr{F}_V(\mathcal{O}_X(-1)), \ldots, \mathcal{A}_{i-1}(i-2) \otimes \mathscr{F}_V(\mathcal{O}_X(-1)) \Big\rangle && \text{(III.49)}\\
&= \Big\langle \mathscr{F}(\pi^*\mathcal{A}_0), \mathcal{A}_1 \otimes \mathrm{D}_{\mathbb{G}_\mathrm{m}}(V^*), \ldots, \mathcal{A}_{i-1}(i-2) \otimes \mathrm{D}_{\mathbb{G}_\mathrm{m}}(V^*) \Big\rangle, && \text{(III.50)}
\end{aligned}$$

Recall from the previous section that $\mathrm{D}(\mathrm{Tot}(\mathcal{O}_X(-1)))$ is the derived category of coherent sheaves on the affinization of the universal hyperplane section $\mathcal{H} \to V^*$ of X. Therefore $V^*/\mathbb{G}_\mathrm{m} \subset \tilde{X}/\mathbb{G}_\mathrm{m}$ and restricting away from the origin we have that the universal hyperplane is given by the fibered product $\mathcal{H} := \mathbb{P}(V^*) \times \tilde{X}/\mathbb{G}_\mathrm{m}$, with

$$\begin{tikzcd}
\mathcal{H} \arrow[r] \arrow[d] & \tilde{X}/\mathbb{G}_m \arrow[d] \\
\mathbb{P}(V^*) \arrow[r] & V^*/\mathbb{G}_m
\end{tikzcd}$$

[1]Although this isn't strictly necessary, we assume it here to make things clearer and to draw the analogy to section 2

In particular we would like to restrict the semi-orthogonal decomposition $\mathrm{D}(\tilde{X}/\mathbb{G}_{\mathrm{m}})$ away from the origin. We claim the following square where we restrict $\mathrm{D}(\tilde{X})$ away from the origin, is a Cartesian square:

$$\begin{tikzcd}
\mathrm{D}(\mathcal{H}) := \mathrm{D}(\mathbb{P}(V^*)) \boxtimes \mathrm{D}(\tilde{X}/\mathbb{G}_{\mathrm{m}}) \arrow[r] \arrow[d] & \mathrm{D}(\tilde{X}/\mathbb{G}_m) \arrow[d] \\
\mathrm{D}(\mathbb{P}(V^*)) \arrow[r] & \mathrm{D}(V^*/\mathbb{G}_m)
\end{tikzcd}$$

Since $\mathrm{D}(\tilde{X}/\mathbb{G}_{\mathrm{m}}) = \mathrm{D}_{\mathbb{G}_{\mathrm{m}}}(\tilde{X})$ we have by equation III.50 and faithful base change that

$$\mathrm{D}(\mathcal{H}) = \Big\langle \mathscr{F}_V(\pi^*\mathcal{A}_0)|_{\mathbb{P}(V^*)}, \mathcal{A} \boxtimes \mathrm{D}(\mathbb{P}(V^*)), \ldots, \mathcal{A}(i-2) \boxtimes \mathrm{D}(\mathbb{P}(V^*)) \Big\rangle, \tag{III.51}$$

And comparing this with $\mathrm{D}(\mathcal{H})$ with the equation III.55 in section 3. We take $L = V^*$

$$\mathrm{D}(\mathcal{H}) = \Big\langle \mathcal{K}_{\mathcal{H}}, \mathcal{A}(1) \boxtimes \mathrm{D}(\mathbb{P}(V^*)), \mathcal{A}(2) \boxtimes \mathrm{D}(\mathbb{P}(V^*)), \ldots, \mathcal{A}(i-1) \boxtimes \mathrm{D}(\mathbb{P}(V^*)) \Big\rangle, \tag{III.52}$$

where an almost identical decomposition. Twisting our semiorthogonal decomposition by $(1,0)$ yields

$$\mathrm{D}(\mathcal{H}) = \Big\langle \mathscr{F}_V(\pi^*\mathcal{A}_0)|_{\mathbb{P}(V^*)}(1), \mathcal{A}(1) \boxtimes \mathrm{D}(\mathbb{P}(V^*)), \ldots, \mathcal{A}(i-1) \boxtimes \mathrm{D}(\mathbb{P}(V^*)) \Big\rangle, \tag{III.53}$$

and so up to a twist by $(1,0)$ we have derived that the Kuznetsov component is equivalent to our $\mathscr{F}_V(\pi^*\mathcal{A}_0)|_{\mathbb{P}(V^*)}(1)$. That is

$$\mathcal{K}_{\mathcal{H}} = \mathscr{F}_V(\pi^*\mathcal{A}_0)|_{\mathbb{P}(V^*)}(1), \tag{III.54}$$

where $\mathscr{F}_V(\pi^*\mathcal{A}_0)|_{\mathbb{P}(V^*)}(1)$ is Homological Projective dual category of $\mathrm{D}(X)$ with $X \to \mathbb{P}(V)$.

Remark 3.4. The existence of such a functor would enable one, at least in theory to be able to compute the Kuznetsov component from it's definition.

Remark 3.5. Similar to equation one can take a subspace $L \subset V^*$ and find in almost the same way, the corresponding result to equation III.55.

$$\mathrm{D}(\mathcal{H}_L) = \Big\langle \mathcal{K}_{\mathcal{H}_L}, \mathcal{A}(1) \boxtimes \mathrm{D}(\mathbb{P}(L)), \mathcal{A}(2) \boxtimes \mathrm{D}(\mathbb{P}(L)), \ldots, \mathcal{A}(i-1) \boxtimes \mathrm{D}(\mathbb{P}(L)) \Big\rangle. \tag{III.55}$$

.

4 Fano Scheme of the Intersection of Two Quadrics

The original motivation for introducing this transform in the previous sections was towards understanding the derived category of the Fano scheme of k-planes in the intersection of two quadrics. Since the cohomology of this variety was recently computed by Chen-Vilonen-Xue in [CVX21] and [CVX20] by applying a fourier transform in order to relate the cohomology to the cohomology of a variety which was better understood.

Let Q_1 and Q_2 be a smooth intersection of quadrics in $\mathbb{P}^{2g+1}$, so that the associated curve C is a hyperelliptic curve. Let $\mathrm{Sym}^i(C)$ denote the i-th symmetric power of C, which is a smooth projective variety of dimension i. Recall theorem 5.2 which states that it's derived category is indecomposable for $i \leq g-1$.

Conjecture 4.1 ([BBF⁺24], Conjecture A). Let $Q_1 \cap Q_2$ be a smooth intersection of quadrics in $\mathbb{P}^{2g+1}$, and let C be the associated hyperelliptic curve. For all $k = 0, \dots, g-2$ there exists a semiorthogonal decomposition

$$\mathrm{D}(\mathrm{F}_k(Q_1 \cap Q_2)) = \left\langle \binom{2g-4-k-i}{k+1-i} + 2\binom{2g-4-k-i}{k-i} \text{copies of } \mathrm{D}(\mathrm{Sym}^i(C)) \right\rangle. \tag{III.56}$$

where $i = 0, \cdots, k, k+1$ and $\dim(\mathrm{F}_k(Q_1 \cap Q_2) = (k+1)(2g-2k-1)$.

Remark 4.2. Note that for $k = g-1$ we have $F_{g-1}(Q_1 \cap Q_2) \cong \mathrm{Jac}(C)$ which by [[Rei72], Theorem 4.8] admits an indecomposable derived category.

Remark 4.3. When $k = g-2$ this conjecture reduces to the well known BGMN conjecture see [[BR76], Thm. 1]. Here $\mathrm{F}_{g-2}(Q_1 \cap Q_2) \cong \mathrm{M}_C(2, \mathcal{L})$ is the moduli space of rank two bundles on C with fixed determinant of odd degree.

Conjecture 4.4 ([BBF⁺24], Conjecture D). Let $Q_1 \cap Q_2$ be a smooth intersection of quadrics $\mathbb{P}^{2g}$, and let $\mathcal{C}$ be the associated stacky curve. For all $k = 0, \dots, g-2$ there exists a semiorthogonal decomposition

$$\mathrm{D}(\mathrm{F}_k(Q_1 \cap Q_2)) = \left\langle \binom{2g-3-k-i}{k+1-i} \text{copies of } \mathrm{D}(\mathrm{Sym}^i(\mathcal{C})) \right\rangle, \tag{III.57}$$

where $i = 0, \cdots, k, k+1$ and $\dim(\mathrm{F}_k(Q_1 \cap Q_2) = (k+1)(2g-2k-2)$.

The stacky symmetric power, is defined in [Fon23], Chapter 2, for the specific stacky curve that appears for $Q_1 \cap Q_2$. Unlike the case for Conjecture 4.1, the derived category $\mathrm{D}(\mathrm{Sym}^i(\mathcal{C}))$ admits a full exceptional collection and therefore admits a semiorthogonal decomposition.

The idea is that the transform in Conjecture 3.1 or Conjecture 2.1 could be applied to the semiorthogonal decompositions featured to Conjecture 4.1 or Conjecture 4.4. Then by property (3) the transform this semiorthogonal decomposition would be preserved. The hope being that proving the transformed decomposition on the target variety is tractable. This is the derived category analogue to the approach that was taken by Chen-Vilonen-Xue in [CVX20] and [CVX21].

5 Other Approaches: String Theory, Landau-Ginzburg Models and Variational GIT

Throughout this thesis we have analyzed semiorthogonal decompositions of derived categories through the lens of homological projective duality. However there are other approaches (especially from theoretical physics), that are known to be intimately related to HPD and even in some cases coincide with HPD. In fact HPD has even provided insight into these adjacent fields of research.

In theoretical physics it was believed that the geometric phases corresponding to Gauged Linear Sigma Models (GLSMs) always had geometries that were birationally equivalent. This was demonstrated to be false with several counterexamples featured in [CDH+09], [DS08] and [Sha10]. Indeed what geometric phases actually coincided with is homological projective duality. Reversing the correspondence, implied that one can study HPD via geometric phases of these GLSMs. This approach has yielded conjectures for the existence of such HP-dual pairs, such as in [GLZ23]. Similarly Ballard et al. in [BDF+14] used variational GIT, along with Landau-Ginzburg models to determine the HP-dual of the d-th degree Veronese embedding. Then in 2017 Jørgen Rennemo in [Ren17] adapted Ballard et al.'s approach to reprove Kuznetsov's fundamental theorem of homological projective duality.

Conclusion and Outlook

In this thesis we began by examining the derived category of many important varieties that naturally occur in algebraic geometry. We explored how, in some cases, the derived category can be reduced into simpler pieces using semiorthogonal decompositions. In particular, we proved Beilinson's theorem for $\mathbb{P}^n$ as well as semiorthogonal decompositions for degree d hypersurfaces. We concluded the first chapter by looking at special kinds of semiorthogonal decompositions known as Lefschetz decompositions and showed how many examples of semiorthogonal decompositions naturally occur as Lefschetz decompositions.

In Chapter II, we provided an expository account of the phenomenon of Homological Projective Duality (HPD) as it appears in algebraic geometry. We described the homological projective duals of quadrics and Grassmannians. We also described an approach to the HPD theorem using rectangles and simple rules. Finally we concluded chapter II by giving a recent generalization of HPD to higher-degree sections. Using this more general form of HPD, we computed some explicit examples of dualities involving nonlinear sections.

In Chapter III, we assumed a conjecture of Davis and demonstrated through examples how this conjecture would give rise to a duality which looks like an affinization of HPD. Then we assumed a variant of this conjecture and proved that for rectangular Lefschetz decompositions, one indeed recovers HPD as defined by Kuznetsov when restricting away from the origin. We then described the motivation for such a conjecture and how, if it is true, it might shed light on the problem of proving the semiorthogonal decomposition for the Fano scheme of the intersection of two quadrics.

There are some promising future directions one could take this work. Indeed computing more examples using the conjectural Fourier transform of Davis might provide further insight into this affine analogue of HPD. One could also show the conjecture is true by construction. Then given the given explicit form of the transform, applying it to the semiorthogonal decomposition in conjecture 4.1 and conjecture 4.4 might result in transforming the problem into proving a decomposition on a variety that is better understood.

Appendix A

Triangulated Categories in Algebraic Geometry

1 Triangulated Categories

Triangulated Categories as we will define were first introduced and developed in 1962-1963 by Dieter Puppe and Jean-Louis Verdier respectively. Puppe was an algebraic topologist primarily interested in triangulated categories as an abstraction of the stable homotopy category whilst Verdier, who at the time was a PhD student of Grothendieck, was an algebraic geometer interested in how the derived category admits this triangulated structure. Both Puppe and Verdier gave similar definitions of a triangulated category at around the same time with the only notable difference being Verdier's inclusion of the octahedral axiom (TR4)[1]

Definition 1.1. Let $\mathcal{T}$ be a category equipped with an automorphism $\Sigma : \mathcal{T} \to \mathcal{T}$. A triangle (A, B, C) where $A, B, C \in \mathcal{T}$ is an ordered triple (u, v, w) of morphisms where $u : A \to B$, $v : B \to C$ and $w : C \to \Sigma A$. That is a triangle

$$A \xrightarrow{u} B \xrightarrow{v} C \xrightarrow{w} \Sigma A.$$

Definition 1.2. A morphism of triangles is a triple (f, g, h) forming a commutative diagram in $\mathcal{T}$:

$$\begin{array}{ccccccc} A & \xrightarrow{u} & B & \xrightarrow{v} & C & \xrightarrow{w} & \Sigma A \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle \Sigma f} \\ A & \xrightarrow{u'} & B & \xrightarrow{v'} & C & \xrightarrow{w'} & \Sigma A' \end{array}$$

An isomorphism of triangles is then defined in the natural way.

Definition 1.3. Let $\mathcal{T}$ be an additive category. Then $\mathcal{T}$ is a **triangulated category** if $\mathcal{T}$ admits an additive equivalence $\Sigma : \mathcal{T} \to \mathcal{T}$, called the translation or shift functor, and a collection of distinguished (or exact) triangles in $\mathcal{T}$, which are triangles (u, v, w) that satisfy the following four axioms:

(TR1) i) Every morphism $u : A \to B$ can be embedded in an exact triangle (u, v, w).
ii)Any triangle of the form

$$A \xrightarrow{id} A \longrightarrow 0 \longrightarrow \Sigma A,$$

[1]hence why (TR4) is sometimes referred to as the Verdier axiom.

is exact.
iii) if (u, v, w) is a triangle on (A, B, C), isomorphic to an exact triangle (u', v', w') on (A', B', C') then (u, v, w) is also exact.

$$\begin{tikzcd} A \arrow[r, "u"] \arrow[d, "\cong"] & B \arrow[r, "v"] \arrow[d, "\cong"] & C \arrow[r, "w"] \arrow[d, "\cong"] & \Sigma A \arrow[d, "\cong"] \\ A \arrow[r, "u'"] & B \arrow[r, "v'"] & C \arrow[r, "w'"] & \Sigma A' \end{tikzcd}$$

(TR2) (Rotation). If (u, v, w) is an exact triangle on (A, B, C)

$$\begin{tikzcd} & C \arrow[dl, "w"'] & \\ A \arrow[rr, "u"] & & B \arrow[ul, "v"'] \end{tikzcd}$$

then both its "rotates" $(-\Sigma^{-1}w, u, v)$ and $(v, w, -\Sigma u)$ are exact triangles on $(-\Sigma^{-1}w, u, v)$ and $(B, C, \Sigma A)$

$$\begin{tikzcd} & \Sigma A \arrow[dl, "-\Sigma u"'] & \\ B \arrow[rr, "v"] & & C \arrow[ul, "w"'] \end{tikzcd} \qquad \begin{tikzcd} & B \arrow[dl, "v"'] & \\ \Sigma^{-1}C \arrow[rr, "-\Sigma^{-1}w"] & & A \arrow[ul, "u"'] \end{tikzcd}$$

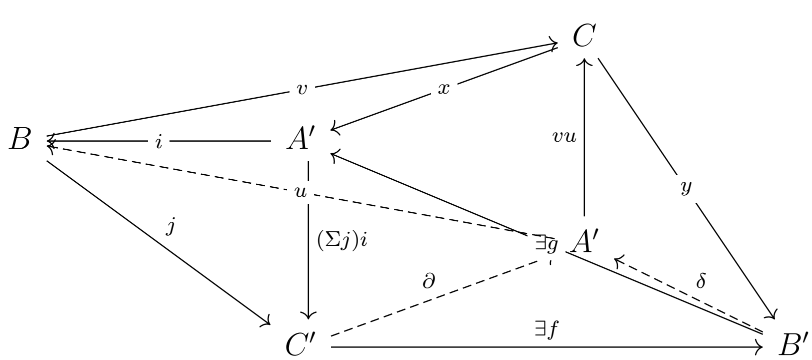
(TR3) Suppose there exists a commutative diagram of distinguished triangles (A, B, C) and (A', B', C') with vertical arrows $f : A \to A'$ and $g : B \to B'$: Then the diagram can be completed to a morphism of triangles, by a non-unique morphism $h : C \to C'$.

$$\begin{tikzcd} A \arrow[r, "u"] \arrow[d, "f"] & B \arrow[r, "v"] \arrow[d, "g"] & C \arrow[r, "w"] \arrow[d, dashed, "\exists h"] & \Sigma A \arrow[d, "\Sigma f"] \\ A' \arrow[r, "u'"] & B' \arrow[r, "v'"] & C' \arrow[r, "w'"] & \Sigma A' \end{tikzcd}$$

(TR4)(**Octahedral axiom/Verdier axiom**) Given objects $A, B, C, A', B', C' \in \mathcal{T}$, if there are three exact triangles: (u, j, ∂) on (A, B, C'), (v, x, i) on (B, C, A') and (vu, y, δ) on (A, C, B').

$$\begin{tikzcd} & C \arrow[dl, "\partial"'] & \\ A \arrow[rr, "u"] & & B \arrow[ul, "j"'] \end{tikzcd} \qquad \begin{tikzcd} & C' \arrow[dl, "i"'] & \\ A \arrow[rr, "v"] & & B' \arrow[ul, "x"'] \end{tikzcd} \qquad \begin{tikzcd} & B' \arrow[dl, "\delta"'] & \\ A \arrow[rr, "vu"] & & C \arrow[ul, "y"'] \end{tikzcd}$$

Then there is a fourth exact triangle $(f, g, (\Sigma j)i)$ on (C', B', A') such that in the following octahedron the four exact triangles form four of the faces and the remaining faces commute.



Remark 1.4. There is some uncertainty as to whether TR4 is the "right" axiom for the definition of a triangulated category, and whether instead it is more natural to impose that the morphisms of distinguished triangles should admit mapping cones which also form distinguished triangles. In [Nee01] Neeman calls such a condition (TR4') and in particular shows that (TR4') implies the octahedral axiom (TR4), with the converse implication proven in [Nee91]. For the reader interested in further such discussion see [Nee91] and the relevant sections of [May01].

Definition 1.5. An additive functor $F : \mathcal{T} \to \mathcal{T}'$ between triangulated categories $\mathcal{T}$ and $\mathcal{T}'$ is called exact if the following conditions are satisfied

1. There exists a functorial isomoprhism $F \circ T_{\mathcal{T}} \xrightarrow{\sim} T_{\mathcal{T}'} \circ F$.
2. Any distinguished triangle

$$A \longrightarrow B \longrightarrow C \longrightarrow A[1]$$

   in $\mathcal{T}$ is mapped to a distinguished triangle

$$F(A) \longrightarrow F(B) \longrightarrow F(C) \longrightarrow F(A)[1]$$

   in $\mathcal{T}'$ where $F(A[1])$ is identified with $F(A)[1]$ via the functor isomorphism in 1).

Definition 1.6. A subcategory $\mathcal{T}' \subset \mathcal{T}$ of a triangulated category is a triangulated subcategory if $\mathcal{T}'$ admits the structure of a triangulated category such that the inclusion $i : \mathcal{T}' \to \mathcal{T}$ is exact.

Proposition 1.7. *Let $\mathcal{T}' \subset \mathcal{T}$ be a full subcategory. $\mathcal{T}$ is a triangulated subcategory if and only if $\mathcal{T}'$ is invariant under the shift functor $\Sigma : \mathcal{T} \to \mathcal{T}$ and for any distinguished triangle*

$$A \longrightarrow B \longrightarrow C \longrightarrow A[1]$$

we have that $C \cong D$ for $D \in \mathcal{T}$.

Definition 1.8. Two triangulated categories $\mathcal{T}$ and $\mathcal{T}'$ are equivalent if there exists an exact equivalence $F : \mathcal{T} \to \mathcal{T}'$.

Definition 1.9. A triangulated category $\mathcal{T}$ is decomposable into triangulated subcategories $\mathcal{A} \subset \mathcal{T}$ and $\mathcal{B} \subset \mathcal{T}$ if the following three conditions are satisfied:

1. The categories $\mathcal{A}$ and $\mathcal{B}$ contain objects non-isomorphic to 0.
2. For every object $F \in \mathcal{T}$, there exists a distinguished triangle

$$A \longrightarrow F \longrightarrow B \longrightarrow A[1]$$

   where $A \in \mathcal{A}$ and $B \in \mathcal{B}$.
3. For every pair of objects $B_1 \in \mathcal{T}_1$ and $B_2 \in \mathcal{T}_2$, there exist no morphisms in $\mathcal{T}$ between them, i.e.,

$$\mathrm{Hom}(B_1, B_2) = \mathrm{Hom}(B_2, B_1) = 0. \tag{A.1}$$

2 Properties of Triangulated Categories

Definition 2.1. Let $\mathcal{A}$ be a k-linear category. A Serre functor $S : \mathcal{A} \to \mathcal{A}$ is an additive functor that is also an autoequivalence such that for any two objects $A, B \in \mathcal{T}$ there exists an isomorphism

$$\eta_{A,B} : \mathrm{Hom}(A, B) \to \mathrm{Hom}(B, S(A))^*. \tag{A.2}$$

Proposition 2.2 ([BK90]). *Any Serre functor on a triangulated category over a field k is exact.*

Definition 2.3. Let $\mathcal{T}$ be a triangulated category. A subclass $\Omega \subset \mathcal{T}$ of the objects of $\mathcal{T}$ is called a spanning class of $\mathcal{T}$ if for any object $B \in \mathcal{T}$:

1. $\mathrm{Hom}_{\mathcal{T}}(A, B[i]) = 0$ for all $A \in \Omega$ and all $i \in \mathbb{Z}$, then $B \cong 0$.
2. $\mathrm{Hom}_{\mathcal{T}}(B[i], A) = 0$ for all $A \in \Omega$ and all $i \in \mathbb{Z}$, then $B \cong 0$.

Proposition 2.4 ([Huy06], Cor. 3.19). *If X is a smooth projective variety and L is an amble line bundle on X, then the powers $L^i, i \in \mathbb{Z}$, form a spanning class in* $\mathrm{D}(X)$.

Example 2.5. Let $\mathcal{E}^\bullet \in \mathrm{D}(X)$ be any object and

$$\mathcal{E}^{\bullet\perp} := \{\mathcal{F}^\bullet \in \mathrm{D}(X) \mid \mathrm{Hom}(\mathcal{E}^\bullet, \mathcal{F}^\bullet[i]) = 0 \text{ for all } i \in \mathbb{Z}\}. \tag{A.3}$$

Then $\Omega = \{\mathcal{E}^\bullet\} \cup \mathcal{E}^{\bullet\perp} \subset \mathrm{D}(X)$ is a spanning class.

Remark 2.6. If the triangulated category admits a Serre functor, conditions (1) and (2) above are equivalent.

Proposition 2.7 ([Orl97]). *Let $F : \mathcal{T} \to \mathcal{T}'$ be an exact functor between triangulated categories with left and right adjoints: $G \dashv F \dashv H$. Suppose Ω is the spanning class of $\mathcal{T}$ such that for all objects $A, B \in \Omega$ and all $i \in \mathbb{Z}$ the natural homomorphisms*

$$F : \mathrm{Hom}(A, B[i]) \to \mathrm{Hom}(F(A), F(B[i])) \tag{A.4}$$

are bijective. Then F is fully faithful.

Proposition 2.8 ([Huy06], Cor. 1.56). *Let $F : \mathcal{T} \to \mathcal{T}'$ be an exact functor between triangulated categories $\mathcal{T}$ and $\mathcal{T}'$ with left adjoint $F \dashv H$. Furthermore assume that Ω is a spanning class of $\mathcal{T}$ satisfying the conditions:*

1. *for all $A, B \in \Omega$ the natural morphisms*
$$\mathrm{Hom}(A, B[i]) \to \mathrm{Hom}(F(A), F(B)[i]), \tag{A.5}$$
are bijective for all $i \in \mathbb{Z}$;
2. *the categories $\mathcal{T}$ and $\mathcal{T}'$ admit Serre functors $S_{\mathcal{T}}$ and respectively $S_{\mathcal{T}'}$ such that*
$$F(S_{\mathcal{T}}(A)) = S_{\mathcal{T}'}(F(A)), \tag{A.6}$$
for all $A \in \Omega$;
3. *the category $\mathcal{T}'$ is indecomposable and $\mathcal{T}$ is non-trivial;*

then F is an equivalence.

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