Coherent Sheaves and Semistable Bundles on Curves

Adam Monteleone

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Throughout this talk let k be a field, and let C be a smooth, geometrically connected, projective curve over k. The category of coherent sheaves of $\mathcal{O}_C$-modules over C will be denoted by $\mathsf{Coh}(C)$ and the subcategory of locally free sheaves of rank r and degree d will be denoted $\mathsf{Vect}_{r,d}(C)$. We will employ the usual conflation of locally free sheaves and vector bundles. Recall the inclusions $\mathsf{Vect}_{r,d}(C) \subseteq \mathsf{Vect}(C) \subseteq \mathsf{Coh}(C) \subseteq \mathsf{QCoh}(C)$.

1 Coherent Sheaves on Projective Curves

We would like to understand the moduli of coherent sheaves on a smooth projective curve, the aim of this talk is to see that this reduces to the study of semistable and stable bundles on curves.

Let $\mathcal{F} \in \mathsf{Coh}(C)$ be a coherent sheaf of $\mathcal{O}_C$-modules on C, we define the torsion subsheaf of $\mathcal{F}$ to be

$$\mathcal{F}_{\text{tors}}(U) := \mathcal{F}(U)_{\text{tors}} \quad \text{where} \quad U \subseteq C.$$

Since $\mathcal{F}_{\text{tors}} \subseteq \mathcal{F}$ and $\mathsf{Coh}(C)$ is an abelian category there is an exact sequence

$$0 \longrightarrow \mathcal{F}_{\text{tors}} \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}_{\text{free}} \longrightarrow 0$$

where $\mathcal{F}_{\text{free}} := \mathcal{F}/\mathcal{F}_{\text{tors}}$. To see that this sequence splits take the long exact sequence in $\operatorname{Hom}(\mathcal{F}_{\text{free}}, -)$

$$\begin{tikzcd}
\operatorname{Hom}_{\mathcal{O}_C}(\mathcal{F}_{\text{free}}, \mathcal{F}_{\text{tors}}) \arrow[r] & \operatorname{Hom}_{\mathcal{O}_C}(\mathcal{F}_{\text{free}}, \mathcal{F}) \arrow[r] & \operatorname{Hom}_{\mathcal{O}_C}(\mathcal{F}_{\text{free}}, \mathcal{F}_{\text{free}}) \arrow[dll, rounded corners] \\
\operatorname{Ext}^1_{\mathcal{O}_C}(\mathcal{F}_{\text{free}}, \mathcal{F}_{\text{tors}}) \arrow[r] & \operatorname{Ext}^1_{\mathcal{O}_C}(\mathcal{F}_{\text{free}}, \mathcal{F}) \arrow[r] & \operatorname{Ext}^1_{\mathcal{O}_C}(\mathcal{F}_{\text{free}}, \mathcal{F}_{\text{free}})
\end{tikzcd}$$

Moreover

$$\operatorname{Ext}^1_{\mathcal{O}_C}(\mathcal{F}_{\text{free}}, \mathcal{F}_{\text{tors}}) = H^1(C, \mathcal{F}_{\text{tors}} \otimes \mathcal{F}_{\text{free}}^\vee) = 0,$$

as $\mathcal{F}_{\text{tors}} \otimes \mathcal{F}_{\text{free}}^\vee$ has a zero-dimensional support and sheaves supported at finitely many points have vanishing higher cohomology by Grothendieck vanishing [[Har77], Thm III.2.7]. Therefore the first line of our long exact sequence becomes the short exact sequence

$$0 \longrightarrow \operatorname{Hom}(\mathcal{F}_{\text{free}}, \mathcal{F}_{\text{tors}}) \longrightarrow \operatorname{Hom}(\mathcal{F}_{\text{free}}, \mathcal{F}) \twoheadrightarrow \operatorname{Hom}(\mathcal{F}_{\text{free}}, \mathcal{F}_{\text{free}}) \longrightarrow 0$$

The morphism $\mathrm{id}_{\mathcal{F}_{\text{free}}} : \mathcal{F}_{\text{free}} \to \mathcal{F}_{\text{free}}$ then lifts non-canonically to a section $s : \mathcal{F}_{\text{free}} \to \mathcal{F}$ which by the splitting lemma splits the exact sequence. We have shown the following

Proposition 1.1. *Every coherent sheaf $\mathcal{F} \in \mathsf{Coh}(C)$ is (non-canonically) isomorphic to a direct sum*

$$\mathcal{F} \xrightarrow{\sim} \mathcal{F}_{\text{free}} \oplus \mathcal{F}_{\text{tors}}.$$

From this decomposition we conclude that to understand a coherent sheaf $\mathcal{F}$ on a smooth curve C over k it suffices to understand locally free sheaves (vector bundles) and torsion sheaves on a curve. Using this decomposition we can now extend notions of degree and rank to vector bundles. Let $\mathcal{E} \in \mathsf{Vect}(C)$ then we define the determinant line bundle of a vector bundle $\mathcal{E}$ to be its highest nonzero exterior power. The degree of a vector bundle is then defined as

$$\deg(\mathcal{E}) := \deg(\det(\mathcal{E})) \quad \text{where} \quad \det(\mathcal{E}) := \bigwedge^r \mathcal{E}.$$

Moreover this definition of degree is extended to all $\mathcal{F} \in \mathsf{Coh}(C)$ by setting

$$\deg(\mathcal{F}) := \dim_k(H^0(C, \mathcal{F}_{\text{tors}})) + \deg(\mathcal{F}_{\text{free}}).$$

Alternatively, one could define the degree via the Riemann-Roch Theorem for coherent sheaves over C

$$\deg(\mathcal{F}) = \chi(C, \mathcal{F}) - \operatorname{rank}(\mathcal{F})\chi(C, \mathcal{O}_C),$$

where $\chi(C, \mathcal{F}) = \sum_{i \geq 0} (-1)^i h^i(C, \mathcal{F})$ is the Euler characteristic of $\mathcal{F}$. The rank of $\mathcal{F}$ is then defined by

$$\operatorname{rank}(\mathcal{F}) := \dim_{\kappa(\eta)}(\mathcal{F} \otimes \kappa(\eta)) \quad \text{where } \eta \in C \text{ is the generic point and } \kappa(\eta) = K(C).$$

This last definition can be equivalently stated as $\operatorname{rank}(\mathcal{F}) = \operatorname{rank}(\mathcal{F}_{\text{free}})$ as torsion vanishes at the generic point.

Proposition 1.2. *The functions* $\deg : K_0(C) \to \mathbb{Z}$ *and* $\operatorname{rank} : K_0(C) \to \mathbb{Z}$ *are group homomorphisms in particular for an exact sequence of coherent sheaves*

$$0 \longrightarrow \mathcal{F}' \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}'' \longrightarrow 0,$$

we have the additive formulas

$$\deg(\mathcal{F}) = \deg(\mathcal{F}') + \deg(\mathcal{F}'') \quad \textit{and} \quad \operatorname{rank}(\mathcal{F}) = \operatorname{rank}(\mathcal{F}') + \operatorname{rank}(\mathcal{F}'').$$

Proof sketch. The additivity of rank follows from the fact that taking stalks at the generic point is exact. The Euler characteristic of a coherent sheaf can be shown to be additive by taking the long exact sequence in cohomology and applying the rank-nullity theorem. The additivity of degree then follows from the Riemann-Roch theorem for coherent sheaves.

Proposition 1.3. *Let* $\mathcal{E}_1 \in \mathsf{Vect}_{r_1,d_1}(C)$ *and* $\mathcal{E}_2 \in \mathsf{Vect}_{r_2,d_2}(C)$ *be vector bundles on* C *then*

$$\deg(\mathcal{E}_1 \otimes \mathcal{E}_2) = r_2 \deg(\mathcal{E}_1) + r_1 \deg(\mathcal{E}_2).$$

Proof. It can be shown that there is a canonical isomorphism $\det(\mathcal{E}_1 \otimes \mathcal{E}_2) \cong (\det \mathcal{E}_1)^{\otimes r_2} \otimes (\det \mathcal{E}_2)^{\otimes r_1}$. It follows

$$\begin{aligned}
\deg(\mathcal{E}_1 \otimes \mathcal{E}_2) &= \deg(\det(\mathcal{E}_1 \otimes \mathcal{E}_2)) \\
&= \deg\left((\det \mathcal{E}_1)^{\otimes r_2} \otimes (\det \mathcal{E}_2)^{\otimes r_1}\right) \\
&= r_2 \deg(\det \mathcal{E}_1) + r_1 \deg(\det \mathcal{E}_2) \\
&= r_2 \deg(\mathcal{E}_1) + r_1 \deg(\mathcal{E}_2),
\end{aligned}$$

where we have used the fact that for line bundles $\deg(\mathcal{L} \otimes \mathcal{L}') = \deg(\mathcal{L}) + \deg(\mathcal{L}')$ see [[Har77], Prop II.6.13]. □

Definition 1.4. The *slope* of a coherent sheaf $\mathcal{F} \in \mathsf{Coh}(C)$ is defined to be

$$\mu(\mathcal{F}) := \frac{\deg(\mathcal{F})}{\operatorname{rank}(\mathcal{F})} \in \mathbb{Q} \cup \{\infty\},$$

with the convention that $\mu(\mathcal{F}) := \infty$ if $\mathcal{F}$ is torsion.

The reason for the terminology "slope" will become evident in example 2.8 when we compute the Harder-Narasimhan polygon of a sheaf.

Example 1.5. If $C = \mathbb{P}^1_{\mathbb{C}}$ then

$$\begin{aligned}
&\mu(\mathcal{O}_{\mathbb{P}^1}(n)) = \frac{\deg(\mathcal{O}_{\mathbb{P}^1}(n))}{\operatorname{rank}(\mathcal{O}_{\mathbb{P}^1}(n))} = \frac{n}{1} = n \\
&\mu(\mathbb{C}(x)) = \infty \quad \text{since } \mathbb{C}(x) \text{ is a torsion sheaf supported at } x, \text{ so } \operatorname{rank}(\mathbb{C}(x)) = 0, \\
&\mu(\mathcal{O}_{\mathbb{P}^1}(n) \oplus \mathbb{C}(x)) = \frac{\deg(\mathcal{O}_{\mathbb{P}^1}(n) \oplus \mathbb{C}(x))}{\operatorname{rank}(\mathcal{O}_{\mathbb{P}^1}(n) \oplus \mathbb{C}(x))} = \frac{\deg(\mathcal{O}_{\mathbb{P}^1}(n)) + \deg(\mathbb{C}(x))}{\operatorname{rank}(\mathcal{O}_{\mathbb{P}^1}(n)) + \operatorname{rank}(\mathbb{C}(x))} = n + 1.
\end{aligned}$$

Example 1.6. If $C = E$ is an elliptic curve over $\mathbb{C}$ with $O \in E$ the marked point corresponding to the identity element then

$$\begin{aligned}
\mu(\mathcal{O}_E) &= \frac{\deg(\mathcal{O}_E)}{\operatorname{rank}(\mathcal{O}_E)} = \frac{0}{1} = 0 \\
\mu(\mathcal{L}(3O)) &= \frac{\deg(\mathcal{L}(3O))}{\operatorname{rank}(\mathcal{L}(3O))} = \frac{3}{1} = 3 \\
\mu(\mathcal{L}(nP)) &= \frac{\deg(\mathcal{L}(nP))}{\operatorname{rank}(\mathcal{L}(nP))} = \frac{n}{1} = n.
\end{aligned}$$

Example 1.7. If C is a hyperelliptic curve over $\mathbb{C}$ of genus $g(C) = 2$ then

$$\mu(\omega_C^{\otimes 3}) = \frac{\deg(\omega_C^{\otimes 3})}{\operatorname{rank}(\omega_C^{\otimes 3})} = \frac{3\deg(\omega_C))}{1} = \frac{3(2g(C) - 2)}{1} = 6.$$

2 Vector Bundles on Curves

By the previous section we have seen that studying coherent sheaves of $\mathcal{O}_C$-modules has reduced to the study of locally free sheaves/vector bundles over C. To begin our study of vector bundles on a curve C we consider the case where $C = \mathbb{P}^1_k$, here we have a well known theorem of Grothendieck and Birkhoff which states that every vector bundle on $\mathbb{P}^1_k$ is built from direct sums of line bundles $\mathcal{O}_{\mathbb{P}^1}(n)$.

Theorem 2.1 (Birkhoff–Grothendieck)**.** *If $\mathcal{E} \in \mathsf{Vect}_{r,d}(\mathbb{P}^1_k)$ then there exist unique integers $d_1, d_2, \dots, d_r \in \mathbb{Z}$ such that*

$$\mathcal{E} \cong \bigoplus_{i=1}^{r} \mathcal{O}_{\mathbb{P}^1}(d_i) \quad \textit{where } d_1 \geq d_2 \geq \cdots \geq d_r.$$

Proof. We proceed by induction. For $\mathrm{rank}(\mathcal{E}) = 1$ we have by Hartshorne [[Har77], II.6.4] every line bundle in $\mathrm{Pic}(C)$ is of the form $\mathcal{O}_{\mathbb{P}^1}(n)$. If $\mathrm{rank}(\mathcal{E}) > 1$ for $n \in \mathbb{Z}$ we have

$$\begin{aligned}
\mathrm{Hom}(\mathcal{O}_{\mathbb{P}^1}(n), \mathcal{E}) &\cong H^0(\mathbb{P}^1_k, \mathcal{E}(-n)) \\
&\cong H^1(\mathbb{P}^1_k, \mathcal{E} \otimes \omega_{\mathbb{P}^1} \otimes \mathcal{O}_{\mathbb{P}^1}(n)))^\vee && \text{by Serre duality [[Har77], Thm III.7.6]} \\
&\cong H^1(\mathbb{P}^1_k, \mathcal{E}^\vee(n-2))^\vee \\
&\cong 0 \quad \text{for } n >> 0 && \text{by Serre vanishing [[Har77], Prop III.5.3].}
\end{aligned}$$

Let $d_1 \in \mathbb{Z}$ be the maximal integer for which $\mathrm{Hom}(\mathcal{O}_{\mathbb{P}^1}(n), \mathcal{E}) \neq 0$. Choose a nonzero morphism $\varphi \in \mathrm{Hom}(\mathcal{O}_{\mathbb{P}^1}(d_1), \mathcal{E})$ since $\mathcal{O}_{\mathbb{P}^1}(d_1)$ is torsion free any nonzero map is injective, so $\ker(\varphi) = 0$ and the map $\varphi : \mathcal{O}_{\mathbb{P}^1}(d_1) \to \mathcal{E}$ is injective. Therefore we can form the short exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^1}(d_1) \xrightarrow{\varphi} \mathcal{E} \longrightarrow \mathrm{coker}(\varphi) \longrightarrow 0$$

The cokernel $\mathrm{coker}(\varphi)$ is locally free. If it were not then $\mathrm{coker}(\varphi)_{\mathrm{tors}} \neq 0$ and so we could restrict to a morphism from a skyscraper sheaf $k(x) \to \mathrm{coker}(\varphi)$. Now recall the exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^1}(-1) \longrightarrow \mathcal{O}_{\mathbb{P}^1} \longrightarrow k(x) \longrightarrow 0$$

twisting by $\mathcal{O}(d_1 + 1)$ we obtain

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^1}(d_1) \longrightarrow \mathcal{O}_{\mathbb{P}^1}(d_1 + 1) \longrightarrow k(x) \longrightarrow 0$$

composition gives a morphism $\mathcal{O}_{\mathbb{P}^1}(d_1 + 1) \to \mathrm{coker}(\varphi)$. Moreover applying $\mathrm{Hom}(\mathcal{O}_{\mathbb{P}^1}(d_1 + 1), -)$ yields the long exact sequence

$$\begin{array}{c}
\mathrm{Hom}(\mathcal{O}(d_1{+}1), \mathcal{O}(a)) \longrightarrow \mathrm{Hom}(\mathcal{O}(d_1{+}1), \mathcal{E}) \longrightarrow \mathrm{Hom}(\mathcal{O}(d_1{+}1), \mathrm{coker}(\varphi)) \\
\longrightarrow \mathrm{Ext}^1(\mathcal{O}(d_1{+}1), \mathcal{O}(a)) \longrightarrow \mathrm{Ext}^1(\mathcal{O}(d_1{+}1), \mathcal{E}) \longrightarrow \mathrm{Ext}^1(\mathcal{O}(d_1{+}1), \mathrm{coker}(\varphi))
\end{array}$$

Since $\mathrm{Ext}^1(\mathcal{O}(d_1 + 1), \mathcal{O}(d_1)) \cong H^1(\mathbb{P}^1_k, \mathcal{O}_{\mathbb{P}^1}(-1)) = 0$, the map $\mathcal{O}_{\mathbb{P}^1}(d_1 + 1) \to \mathrm{coker}(\varphi)$ lifts a nonzero map $\mathcal{O}_{\mathbb{P}^1}(d_1 + 1) \to \mathcal{E}$ but this contradicts the maximality of our choice of d_1. Hence $\mathrm{coker}(\varphi)$ is torsion free so it is locally free with $\mathrm{rank}(\mathrm{coker}(\varphi)) = \mathrm{rank}(\mathcal{E}) - 1$. By the induction hypothesis the theorem holds for all sheaves of rank less than $\mathrm{rank}(\mathcal{E})$, we conclude

$$\mathrm{coker}(\varphi) \cong \bigoplus_{i=2}^{r} \mathcal{O}_{\mathbb{P}^1}(d_i) \quad \text{with } d_i \in \mathbb{Z}.$$

We claim $d_1 \geq d_2 \geq \cdots \geq d_r$. Consider the short exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^1}(d_1) \xrightarrow{\varphi} \mathcal{E} \longrightarrow \bigoplus_{i=2}^r \mathcal{O}_{\mathbb{P}^1}(d_i) \longrightarrow 0$$

Tensoring the exact sequence by $- \otimes \mathcal{O}_{\mathbb{P}^1}(-d_1 - 1)$ we obtain

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^1}(-1) \longrightarrow \mathcal{E}(-d_1 - 1) \longrightarrow \bigoplus_{i=2}^r \mathcal{O}_{\mathbb{P}^1}(d_i - d_1 - 1) \longrightarrow 0$$

Taking the long exact sequence in sheaf cohomology we conclude

$$H^i(\mathbb{P}^1_k, \bigoplus_{i=2}^{r} \mathcal{O}_{\mathbb{P}^1}(d_i - d_1 - 1)) = 0,$$

which is true if and only if $d_i - d_1 - 1 < 0$ hence $d_i \leq d_1$ for all $i \geq 2$.

Finally to obtain the theorem it suffices to show that the exact sequence splits. Applying $\mathrm{Hom}(-, \mathcal{O}_{\mathbb{P}^1}(d_1))$ and using the result

$$\mathrm{Ext}^1(\mathcal{E}, \mathcal{O}_{\mathbb{P}^1}(d_1)) = \bigoplus_{i \geq 2} H^1(\mathbb{P}^1_k, \mathcal{O}_{\mathbb{P}^1}(d_1 - d_i)) = 0,$$

as $d_1 - d_i \geq 0$ there exists a $\lambda \in \mathrm{Hom}(\mathcal{O}(d_1), \mathcal{O}(d_1))$ that lifts to a retraction $r \in \mathrm{Hom}(\mathcal{E}, \mathcal{O}_{\mathbb{P}^1})$ of the short exact sequence. Applying the splitting lemma the desired isomorphism follows

$$\mathcal{E} \cong \mathcal{O}_{\mathbb{P}^1}(d_1) \oplus \left(\bigoplus_{i=2}^{r} \mathcal{O}_{\mathbb{P}^1}(d_i) \right) \cong \bigoplus_{i=1}^{r} \mathcal{O}_{\mathbb{P}^1}(d_i) \quad \text{with} \quad d_1 \geq d_2 \geq \cdots \geq d_r.$$

We omit the proof of uniqueness but remark that it follows from the fact $\mathrm{Hom}(\mathcal{O}_{\mathbb{P}^1}(a), \mathcal{O}_{\mathbb{P}^1}(b)) = 0$ iff $b < a$. $\square$

Definition 2.2. A nonzero coherent sheaf $\mathcal{F} \in \mathsf{Coh}(C)$ is said to be

1. *μ-semistable* if $\mu(\mathcal{F}') \leq \mu(\mathcal{F})$ for every nonzero subsheaf $0 \neq \mathcal{F}' \subseteq \mathcal{F}$;
2. *μ-stable* if $\mu(\mathcal{F}') < \mu(\mathcal{F})$ for every nonzero subsheaf $0 \neq \mathcal{F}' \subsetneq \mathcal{F}$.

Often when it is clear from context what notion of stability is being discussed the slope μ is omitted.

Example 2.3. If $C = \mathbb{P}^1_k$ a vector bundle $\mathcal{E}$ is stable if and only if $\mathcal{E}$ is isomorphic to $\mathcal{O}_{\mathbb{P}^1}(n)$ for some $n \in \mathbb{Z}$.

Example 2.4. If $C = \mathbb{P}^1_k$ a vector bundle $\mathcal{E}$ is semistable if and only if $\mathcal{E} \cong \mathcal{O}_{\mathbb{P}^1}(n)^{\oplus r}$ for some $r \in \mathbb{Z}_{\geq 0}$.

Example 2.5. For a fixed $n \in \mathbb{Z}$ the bundle $\mathcal{O}_{\mathbb{P}^1}(n) \oplus \mathcal{O}_{\mathbb{P}^1}(n)$ is semistable but not stable, note that

$$\mu(\mathcal{O}_{\mathbb{P}^1}(n) \oplus \mathcal{O}_{\mathbb{P}^1}(n)) = \frac{2n}{2} = n = \mu(\mathcal{O}_{\mathbb{P}^1}(n)).$$

Theorem 2.6 ([HN75] Harder–Narasimhan)**.** *Every vector bundle $\mathcal{E} \in \mathsf{Vect}_{r,d}(C)$ admits a unique filtration*

$$\mathcal{E}_\bullet : \qquad 0 = \mathcal{E}_0 \subseteq \mathcal{E}_1 \subseteq \cdots \subseteq \mathcal{E}_n = \mathcal{E},$$

such that each factor $\mathcal{E}_i/\mathcal{E}_{i-1}$ is semistable and

$$\mu(\mathcal{E}_1/\mathcal{E}_0) > \mu(\mathcal{E}_2/\mathcal{E}_1) > \cdots > \mu(\mathcal{E}_n/\mathcal{E}_{n-1}).$$

Remark 2.7. This theorem generalises the Birkhoff-Grothendieck theorem and if we specialize to the case where $C = \mathbb{P}^1_k$ taking direct sums of quotients recovers the theorem.

We define the *charge* of a coherent sheaf to be

$$Z(\mathcal{F}) := -\deg(\mathcal{F}) + \mathrm{rank}(\mathcal{F}) i \in \mathbb{C}.$$

Example 2.8. For the vector bundle $\mathcal{E} = \mathcal{O}(3) \oplus \mathcal{O}(1) \oplus \mathcal{O}(-2) \in \mathsf{Coh}(\mathbb{P}^1)$ the Harder-Narasimhan filtration $\mathcal{E}_\bullet \in \mathbf{Fil}^{\mathrm{fin}}(\mathsf{Coh}(C))$[1] is given by

$$\mathcal{E}_\bullet : \qquad 0 \subseteq \mathcal{O}(3) \subseteq \mathcal{O}(3) \oplus \mathcal{O}(1) \subseteq \mathcal{O}(3) \oplus \mathcal{O}(1) \oplus \mathcal{O}(-2) = \mathcal{E}.$$

Computing the slopes we verify:

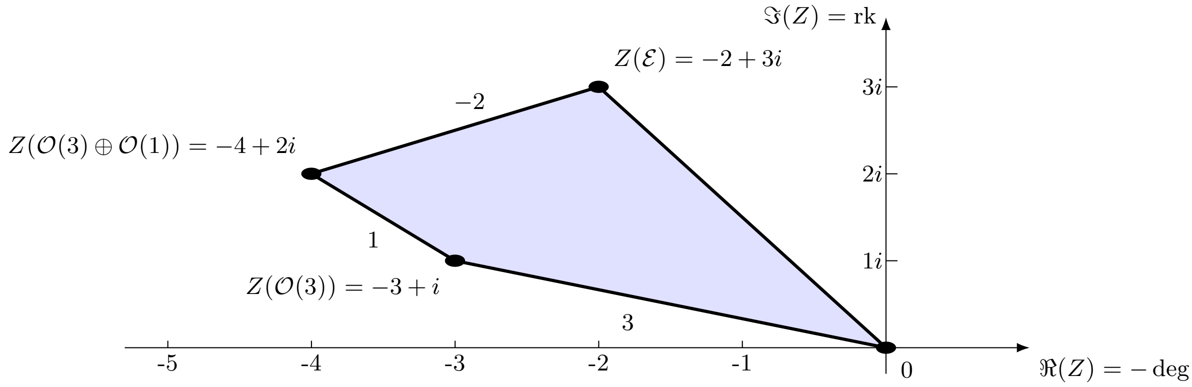
$$\mu(\mathcal{O}(3)/0) = 3 \ > \ \mu(\mathcal{O}(3) \oplus \mathcal{O}(1)/\mathcal{O}(3)) = 1 \ > \ \mu(\mathcal{O}(3) \oplus \mathcal{O}(1) \oplus (-2)/\mathcal{O}(3) \oplus \mathcal{O}(1)) = -2.$$

Moreover computing the charges of each of the coherent sheaves we find

$$Z(\mathcal{O}(3)) = -3 + i, \quad Z(\mathcal{O}(3) \oplus \mathcal{O}(1)) = -4 + 2i, \quad \text{and} \quad Z(\mathcal{O}(3) \oplus \mathcal{O}(1) \oplus \mathcal{O}(-2)) = -2 + 3i.$$

[1] The category $\mathbf{Fil}^{\mathrm{fin}}(\mathcal{A})$ consists of objects which are finite filtrations of the objects of an abelian category $\mathcal{A}$.

From these computations we see that the associated Harder-Narasimhan polygon $\mathrm{HNP}(\mathcal{E}_\bullet)$ is given by



Remark 2.9. Because of our choice of convention when defining Z the slope of a polygon edge does not necessarily coincide with the actual polygon slope instead it is related by $-1/\mu$.

To understand vector bundles on curves it suffices to understand the semistable ones by the Harder-Narasimhan theorem. In addition it turns out that every semistable vector bundle $\mathcal{E}$ on C admits a Jordan–Hölder filtration $\mathcal{E}_\bullet$ where the factors $\mathrm{gr}_i := \mathcal{E}_i/\mathcal{E}_{i-1}$ are stable vector bundles with the same slope as $\mathcal{E}$. While this filtration is not unique, the factors are unique up to permutation. By combining this with the HN filtration, we can filter every vector bundle by stable vector bundles with the same slope as $\mathcal{E}$.

Theorem 2.10 (Jordan–Hölder Filtration)**.** *Let $\mathcal{F}$ be a semistable vector bundle on C suppose that*

$$0 = \mathcal{E}_0 \subseteq \mathcal{E}_1 \subseteq \mathcal{E}_2 \subseteq \cdots \subseteq \mathcal{E}_n = \mathcal{E} \quad \textit{and} \quad 0 = \mathcal{E}'_0 \subseteq \mathcal{E}'_1 \subseteq \mathcal{E}'_2 \subseteq \cdots \subseteq \mathcal{E}'_{n'} = \mathcal{E},$$

are filtrations such that the factors $\mathrm{gr}_i = \mathcal{E}_i/\mathcal{E}_{i-1}$ and $\mathrm{gr}'_j = \mathcal{E}'_j/\mathcal{E}'_{j-1}$ are stable vector bundles with

$$\mu(\mathcal{E}) = \mu(\mathrm{gr}_i) = \mu(\mathrm{gr}'_j),$$

then $n = n'$ and there exists a permutation $\sigma \in S_n$ such that $\mathrm{gr}_i = \mathrm{gr}'_{\sigma(i)}$.

This theorem make the following definition well-defined.

Definition 2.11. The associated graded of a semistable vector bundle $\mathcal{E}$ is defined by

$$\mathrm{gr}(\mathcal{E}) := \bigoplus_{i=1}^{n} \mathrm{gr}_i,$$

where $\mathrm{gr}_i = \mathcal{E}_i/\mathcal{E}_{i-1}$ are the factors with respect to any Jordan-Hölder filtration $0 = \mathcal{E}_0 \subseteq \mathcal{E}_1 \subseteq \cdots \subseteq \mathcal{E}_n = \mathcal{E}$.

Definition 2.12. Let $\mathcal{E}, \mathcal{E}' \in \mathsf{Vect}^{\mathrm{ss}}_{r,d}(C)$ then $\mathcal{E}$ and $\mathcal{E}'$ are called *S-equivalent* (or *Seshadri equivalent*) written $\mathcal{E} \sim_S \mathcal{E}'$ if

$$\mathrm{gr}(\mathcal{E}) \cong \mathrm{gr}(\mathcal{E}').$$

Given a curve C studying coherent sheaves on C has now reduced to the study of the moduli space of semistable bundles on C up to S-equivalence. There is a coarse moduli space for the moduli space for the moduli functor parameterizing S-equivalence classes of semistable bundles on C.

Example 2.13. The Birkhoff-Grothendieck theorem (Theorem 2.1) implies

$$M^{\mathrm{ss}}_{r,d}(\mathbb{P}^1) = \begin{cases} \{*\} & \text{if } r \mid d; \\ \emptyset & \text{otherwise.} \end{cases}$$

In [Ati57] Michael Atiyah studied indecomposable vector bundles on an elliptic curve E in the 1950s, one of the things he found was a nice relationship between rank, degree and semistability when $\gcd(r,d) = 1$.

Example 2.14 ([Ati57])**.** Let E be an elliptic curve and let $m := \gcd(r,d)$. Then

$$M^{\mathrm{ss}}_{r,d}(E) = \begin{cases} \mathrm{Jac}(E) & \text{if } \gcd(r,d) = 1; \\ \mathrm{Sym}^m(E) & \text{otherwise.} \end{cases}$$

References

[Ati57] M. F. Atiyah. Vector bundles over an elliptic curve. *Proceedings of the London Mathematical Society*, s3-7(1):414–452, 1957.

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