Fourier-Mukai Transforms and Derived Equivalences

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October 11, 2024

1. Derived Categories

1.1. **Introduction.** The theory of derived categories, introduced by Grothendieck and Verdier, provides a powerful tool for studying the geometry of algebraic varieties via their categories of coherent sheaves. In 1981, Mukai introduced the Fourier-Mukai transform while studying abelian varieties, and it has since become a fundamental tool used in understanding when the derived categories of two varieties are equivalent.

In this talk, I will setup the basic theory of derived categories, introducing the Fourier-Mukai transform and the Bondal-Orlov full faithfulness criterion. I will then discuss derived equivalences in the context of K3 surfaces, and in particular the work of Lieblich–Olsson [LO11].

1.2. **Derived Categories.** Although we are interested in the derived category $\mathrm{D}^b(\mathbf{Coh}(X))$, we can always take the derived category $\mathrm{D}(\mathcal{A})$ of any abelian category $\mathcal{A}$.

Remark 1.1. If $\mathcal{A}$ is an abelian category the category $\mathbf{Kom}(\mathcal{A})$ of cochain complexes with objects in $\mathcal{A}$ is abelian.

Concretely, the derived category $\mathrm{D}(\mathcal{A})$ of an abelian category $\mathcal{A}$ has objects given by cochain complexes of $\mathcal{A}$,

$$\mathrm{Obj}(\mathrm{D}(A)) := \mathrm{Obj}(\mathbf{Kom}(A)).$$

The morphisms are slightly more complicated. For $\mathcal{A}^\bullet, B^\bullet \in \mathrm{D}(\mathcal{A})$ the collection of morphisms $\mathrm{Hom}_{\mathrm{D}(\mathcal{A})}(A^\bullet, B^\bullet)$ are defined as all equivalence classes of diagrams

$$\begin{tikzcd} & C^\bullet \arrow[dl, "qis"'] \arrow[dr] & \\ A^\bullet & & B^\bullet \end{tikzcd}$$

where $C^\bullet \to A^\bullet$ is a quasi-isomorphism. Two such diagrams are equivalent if they are dominated by a third such diagram in the homotopy category $\mathrm{K}(\mathcal{A})$.

$$\begin{tikzcd} & & C^\bullet \arrow[dl] \arrow[dr] \arrow[dddll, dashed, bend right, "\text{qis}"'] & & \\ & C_1^\bullet \arrow[ddl] \arrow[ddrrr] & & C_2^\bullet \arrow[ddlll] \arrow[ddr] & \\ & & & & \\ A^\bullet & & & & B^\bullet \end{tikzcd}$$

where the compositions $C^\bullet \to C_1^\bullet \to A^\bullet$ and $C^\bullet \to C_2^\bullet \to A^\bullet$ are homotopy equivalent. Note that the commutativity of this diagram is only required up to homotopy because the construction of the mapping cone is unique only up to homotopy. To define composition consider the two morphisms

$$\begin{tikzcd} & C_1^\bullet \arrow[dl, "\text{qis}"'] \arrow[dr] & \\ A^\bullet & & B^\bullet \end{tikzcd} \qquad \begin{tikzcd} & C_2^\bullet \arrow[dl, "\text{qis}"'] \arrow[dr] & \\ B^\bullet & & C^\bullet \end{tikzcd}$$

we define their composite to be given by a commutative diagram (in the homotopy category $\mathbf{K}(\mathcal{A})$) of the form

$$\begin{tikzcd} & & C^\bullet \arrow[dl, "\text{qis}"'] \arrow[dr] & & \\ & C_1^\bullet \arrow[dl, "\text{qis}"'] \arrow[dr] & & C_2^\bullet \arrow[dl, "\text{qis}"'] \arrow[dr] & \\ A^\bullet & & B^\bullet & & C^\bullet \end{tikzcd}$$

It remains to show that such a diagram as claimed always exists and is unique up to equivalence.

The derived category $\mathrm{D}(\mathcal{A})$ therefore essentially is the category of chain complexes with quasi-isomorphisms inverted. For instance in $\mathbf{Kom}(\mathrm{AbGrp})$ we have a quasi-isomorphism $f : A^\bullet \to B^\bullet$ induced by the chain map

$$\begin{tikzcd} A^\bullet & \dots \arrow[r] & 0 \arrow[r] \arrow[d] & \mathbb{Z} \arrow[r, "\times 2"] \arrow[d] & \mathbb{Z} \arrow[r] \arrow[d] & 0 \arrow[r, no head] \arrow[d] & \dots \\ B^\bullet & \dots \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & \mathbb{Z}/2\mathbb{Z} \arrow[r] & 0 \arrow[r, no head] & \dots \end{tikzcd}$$

However there can be no quasi-isomorphism in the opposite direction as there are no non-trivial group homomorphisms $\mathbb{Z}/2\mathbb{Z} \to \mathbb{Z}$. In the derived category however we do have morphisms in both directions

$$\begin{tikzcd} & A^\bullet \arrow[dl, "\mathrm{id}"'] \arrow[dr, "f"] & \\ A^\bullet & & B^\bullet \end{tikzcd} \qquad \begin{tikzcd} & A^\bullet \arrow[dl, "f"'] \arrow[dr, "\mathrm{id}"] & \\ B^\bullet & & A^\bullet \end{tikzcd}$$

Definition 1.2. Let $f : A^\bullet \to B^\bullet$ be a morphism of complexes in $\mathrm{K}(\mathcal{A})$ or $\mathrm{D}(\mathcal{A})$ its mapping cone is the complex

$$c(f)^i := A^{i+1} \oplus B^i \text{ and } d^i_{C(f)} := \begin{bmatrix} -d_A^{i+1} & 0 \\ f^{i+1} & d_B^i \end{bmatrix}.$$

Let $\mathcal{A}$ be an abelian category, the derived category $\mathrm{D}(\mathcal{A})$ is triangulated.

Definition 1.3. An additive category $\mathcal{T}$ is triangulated if it has an autoequivalence $\Sigma : \mathcal{T} \to \mathcal{T}$ and a collection of exact triangles

$$A \longrightarrow B \longrightarrow C \longrightarrow \Sigma A$$

that satisfy (TR1) - (TR4).

The derived category $\mathrm{D}(\mathcal{A})$ is triangulated with autoequivalence given by shift functor $[1] : \mathrm{D}(\mathcal{A}) \to \mathrm{D}(\mathcal{A})$ with

$$(A^\bullet[1])^i := A^{i+1} \text{ and } d^i_{A^\bullet[1]} := -d_A^{i+1}.$$

We can compose A with itself to get the k-shifted complex

$$A^\bullet[k]^i := A^{k+i} \text{ with } d^i_{A^\bullet[k]} = (-1)^k d_A^{i+k},$$

The exact triangles in $\mathrm{D}(\mathcal{A})$ are just triangles isomorphic to the triangle

$$A^\bullet \longrightarrow B^\bullet \longrightarrow \mathrm{Cone}(f) \longrightarrow A^\bullet[1]$$

where the maps $B^\bullet \to \mathrm{Cone}(f)$ and $\mathrm{Cone}(f) \to C^\bullet$ are just the natural inclusion $B^\bullet \to A^\bullet[1] \oplus B^\bullet$ and projection map $A^\bullet[1] \oplus B^\bullet \to B^\bullet$.

Definition 1.4. Let X be a scheme, then $\mathbf{Coh}(X)$ is an abelian category. We denote by $\mathrm{D}^b(\mathbf{Coh}(X))$ the bounded[1] derived category of coherent sheaves of X.

For notational convenience we make the following definition

$$\mathrm{D}(X) := \mathrm{D}^b(\mathbf{Coh}(X)).$$

Remark 1.5. The category $\mathbf{Coh}(X)$ does not have enough injectives so we usually pass to $\mathbf{Qcoh}(X)$ at least when X is Noetherian. Therefore in what follows we always assume X is Noetherian.

Definition 1.6. Let X and Y be schemes then X and Y are said to be derived equivalent or D-equivalent if there exists a k-linear exact equivalence,

$$\mathrm{D}(X) \simeq \mathrm{D}(Y).$$

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[1]bounded just means each complex has finitely many non-zero terms.

1.3. **Serre Functors.**

Definition 1.7. Let X be a smooth projective variety of dimension n. Then one defines the exact functor S_X as the composition

$$\mathrm{D}(X) \xrightarrow{\omega_X \otimes (-)} \mathrm{D}(X) \xrightarrow{[n]} \mathrm{D}(X).$$

This functor is an example of a Serre functor (definition B.1). This is a generalisation of the familiar notion of Serre duality to the level of derived categories.

Theorem 1.8. *(Serre Duality) Let X be a smooth projective variety over k then*

$$S_X : \mathrm{D}(X) \to \mathrm{D}(X) \text{ where } S_X(-) = \omega_X \otimes (-)[n].$$

is a Serre functor. Hence $\mathrm{Ext}^i(\mathcal{E}, \mathcal{F}) = \mathrm{Hom}(\mathcal{E}, \mathcal{F}[i])$ *implies* $\mathrm{Ext}^i(\mathcal{E}, \mathcal{F}) \simeq \mathrm{Ext}^{n-i}(\mathcal{F}, \mathcal{E} \otimes \omega_X)^*$.

Example 1.9. If X is Calabi-Yau i.e $\omega_X = \mathcal{O}_X$ then $S_X = [n]$.

2. Fourier-Mukai Transforms

Let X and Y be smooth projective varieties over k an algebraically closed field of characteristic 0. and denote the two projections by

$$\begin{tikzcd} & X \times Y \arrow[dl, "q"'] \arrow[dr, "p"] & \\ X & & Y \end{tikzcd}$$

Definition 2.1. Let $\mathcal{P} \in \mathrm{D}^b(X \times Y)$. The induced Fourier-Mukai transform is the functor

$$\Phi_{\mathcal{P}} : \mathrm{D}(X) \to \mathrm{D}(Y), \text{ where } \mathcal{E}^\bullet \mapsto Rp_*(Lq^*\mathcal{E}^\bullet \otimes^{\mathbb{L}} \mathcal{P}),$$

where $\mathcal{P}$ is called the Fourier-Mukai kernel of the Fourier-Mukai transform $\Phi_{\mathcal{P}}$.

Remark 2.2. $Lq^* = q^*$ since the projection map q is flat and the left derived tensor product $\otimes^{\mathbb{L}}$ coincides with the ordinary tensor product when the kernel $\mathcal{P}$ is a complex of vector bundles.

Composing two Fourier-Mukai functors gives another Fourier Mukai functor up to isomorphism

Proposition 2.3 ([Muk81])**.** *Let $\Phi_{\mathcal{P}} : \mathrm{D}(X) \to \mathrm{D}(Y)$ and $\Phi_{\mathcal{Q}} : \mathrm{D}(Y) \to \mathrm{D}(Z)$ be Fourier-Mukai functors. The composition*

$$\mathrm{D}(X) \xrightarrow{\Phi_{\mathcal{P}}} \mathrm{D}(Y) \xrightarrow{\Phi_{\mathcal{Q}}} \mathrm{D}(Z),$$

is isomorphic to the Fourier-Mukai transform $\Phi_{\mathcal{R}} : \mathrm{D}(X) \to \mathrm{D}(Z)$.

When computing Fourier-Mukai transforms the following two results are frequently applied

Theorem 2.4 ([Huy06])**.** *(Projection Formula) Let $f : X \to Y$ be a proper morphism of projective schemes over k. For any $\mathcal{F}^\bullet \in \mathrm{D}(X), \mathcal{E}^\bullet \in \mathrm{D}(Y)$ there exists a natural isomorphism*

$$Rf_*(\mathcal{F}^\bullet) \otimes^{\mathbb{L}} \mathcal{E}^\bullet \xrightarrow{\sim} Rf_*(\mathcal{F}^\bullet \otimes Lf^*(\mathcal{E}^\bullet)).$$

Theorem 2.5 ([Huy06])**.** *Let $f : X \to Y$ be a morphism of projective schemes and let $\mathcal{F}^\bullet, \mathcal{E}^\bullet \in \mathrm{D}(Y)$. Then there exists a natural isomorphism*

$$Lf^*(\mathcal{F}^\bullet) \otimes^{\mathbb{L}} Lf^*(\mathcal{E}^\bullet) \xrightarrow{\sim} Lf^*(\mathcal{F}^\bullet \otimes \mathcal{E}^\bullet).$$

Example 2.6. Let $\Gamma_f : X \to X \times Y$ be the diagonal map, where $\mathcal{O}_{\Gamma_f} = (\Gamma_f)_*\mathcal{O}_X$. Moreover let $q : X \times Y \to X$ and $p : X \times Y \to Y$ be projections onto the X and Y factors respectively, then from the pullback diagram

$$\begin{tikzcd} X \arrow[dr, "\Gamma_f"] \arrow[drr, bend left, "f"] \arrow[ddr, bend right, "\mathrm{id}_X"'] & & \\ & X \times Y \arrow[r, "p"] \arrow[d, "q"] & Y \arrow[d] \\ & X \arrow[r, "f"] & Y \end{tikzcd}$$

we can compute the following Fourier-Mukai transform

$$\begin{aligned}\Phi_{\mathcal{O}_{\Gamma_f}}(\mathcal{F}^\bullet) &= Rp_*(Lq^* \otimes^{\mathbb{L}} \mathcal{O}_\Delta) \\ &\cong p_*(q^* \otimes (\Gamma_f)_*\mathcal{O}_X) && \text{(remark 2.2)} \\ &\cong p_*(\Gamma_f)_*((\Gamma_f)^* q^* \mathcal{F}^\bullet \otimes_{\mathcal{O}_X} \mathcal{O}_X) && \text{(projection formula)} \\ &\cong (p \circ \Gamma_f)_*(q \circ \Gamma_f)^*(\mathcal{F}^\bullet) \\ &\cong f_*(\mathrm{id}_X)^*(\mathcal{F}^\bullet) \\ &\cong f_*(\mathcal{F}^\bullet).\end{aligned}$$

Setting the map $f = \mathrm{id}$ in the above example we recover the well known result $\Phi_{\mathcal{O}_\Delta}(\mathcal{F}) = \mathcal{F}$.
We now come to Orlov's celebrated result.

Theorem 2.7 ([Orl03]). *(Orlov's Theorem) Let X and Y be two smooth projective varieties and let*

$$F : \mathrm{D}(X) \to \mathrm{D}(Y),$$

be a fully faithful exact functor. If F admits a right and left adjoint then there exists an object $\mathcal{P} \in \mathrm{D}(X \times Y)$ unique up to isomorphism such that F is isomorphic to $\Phi_{\mathcal{P}}$, that is

$$F \simeq \Phi_{\mathcal{P}}.$$

Orlov's theorem is most often applied to equivalences:

Corollary 2.8. *Let $F : \mathrm{D}(X) \xrightarrow{\sim} \mathrm{D}(Y)$ be an equivalence between the derived category of smooth projective varieties over k. Then F is isomorphic to a Fourier-Mukai transform $\Phi_{\mathcal{P}}$ associated to a certain object $\mathcal{P} \in \mathrm{D}(X \times Y)$, which is unique up to isomorphism.*

Theorem 2.9 ([BO95]). *Let X and Y be smooth projective varieties. The Fourier-Mukai transform $\Phi_{\mathcal{P}} : \mathrm{D}(X) \to \mathrm{D}(Y)$ is fully faithful if and only if for any two closed points $x, y \in X$ one has*

$$\mathrm{Hom}(\Phi_{\mathcal{P}}(\mathcal{O}_x), \Phi_{\mathcal{P}}(\mathcal{O}_y)[i]) = \begin{cases} k & \text{if } x = y \text{ and } i = 0; \\ 0 & \text{if } x \neq y \text{ or } i < 0 \text{ or } i > \dim(X). \end{cases}$$

Remark 2.10. This full-faithfulness criterion as well as Orlov's theorem have been considerably generalized. Recently Orlov's theorem was generalized to smooth proper tame stacks in [Pen24]. Similarly the full-faithfullness criterion was also generalized to smooth proper Deligne-Mumford stacks[2] in [HP24].

Corollary 2.11. *Suppose $\Phi_{\mathcal{P}} : \mathrm{D}(X) \to \mathrm{D}(Y)$ is a fully faithful functor. Then the functor $\Phi_{\mathcal{P}}$ is an equivalence if and only if $\dim(X) = \dim(Y)$ and*

$$\mathcal{P} \otimes q^*\omega_X \simeq \mathcal{P} \otimes p^*\omega_Y.$$

Cohomological Fourier-Mukai Transform. Let X and Y be smooth projective varieties over k. Suppose $\Phi_{\mathcal{P}} : \mathrm{D}(X) \to \mathrm{D}(Y)$ is a Fourier-Mukai transform. For any cohomology class $\alpha \in H^*(X \times Y, \mathbb{Q})$ we have

$$\Phi^H_\alpha : H^*(X; \mathbb{Q}) \to H^*(Y; \mathbb{Q}) \text{ given by } \beta \mapsto p_*(q^*\beta.\alpha),$$

where $\alpha := v(\mathcal{P}) = \mathrm{ch}(\mathcal{P}).\sqrt{\mathrm{td}(X \times Y)}.$

This section so far has been a summary of almost all the important results needed to work with derived categories and derived functors. However we have not given some of the more basic combaitibilities of derived functors in algebraic geometry. For the rigorous approach to dualities and derived categories, see either [Con00] or [LH09]. For a thorough survey of Fourier-Mukai transforms in algebraic geometry, see [Huy06].

[2]This generalization of the full faithfullness theorem to stacks will be the subject of Jack's talk next week.

3. Derived Equivalences of K3 Surfaces

We now study the rich theory of derived equivalences of K3 surfaces. Recall the definition of a K3 surface:

Definition 3.1. A K3 surface is a compact complex surface X with trivial canonical bundle

$$\omega_X \simeq \mathcal{O}_X \text{ and } H^1(X, O_X) = 0.$$

Remark 3.2. Every K3 surface is Kahler.

Remark 3.3. Algebraic K3 surfaces are dense in the moduli space of all K3 surfaces.

Question 1. *When are two K3 surfaces equivalent?*

Answer:

Theorem 3.4 ([PSS71])**.** *(Global Torelli Theorem) Two complex K3 surface X and Y are isomorphic if and only if there exists a Hodge isometry*

$$H^2(X;\mathbb{Z}) \to H^2(Y;\mathbb{Z}).$$

Remark 3.5. Originally proven by Pyatetski-Shapiro–Shafarevich in 1971 in the paper [PSS71] for algebraic K3 Surfaces. This result was then extended to all K3 surfaces by numerous authors, most notably in the work of Gritsenko–Hulek–Sankaran [GHS07].

It is natural to then formulate the derived analogue of question 1:

Question 2. *When are two K3 surfaces derived equivalent?*

Let X be a K3 surface then

$$H^*(X;\mathbb{Z}) = H^0(X;\mathbb{Z}) \oplus H^2(X;\mathbb{Z}) \oplus H^4(X;\mathbb{Z}).$$

The Mukai pairing $\langle -, - \rangle : H^*(X;\mathbb{Z}) \times H^*(X;\mathbb{Z}) \to H^4(X;\mathbb{Z}) \xrightarrow{\sim} \mathbb{Z}$ is defined as the pairing

$$\langle \alpha, \beta \rangle := \alpha_1\beta_1 - \alpha_0\beta_2 - \alpha_2\beta_0 \in H^4(X;\mathbb{Z}) \xrightarrow{\sim} \mathbb{Z},$$

where $\alpha = (\alpha_0, \alpha_1, \alpha_2)$ and $\beta = (\beta_0, \beta_1, \beta_2)$ with $\alpha_i, \beta_i \in H^{2i}(X;\mathbb{Z})$.

Definition 3.6. A Mukai lattice for X a K3 surface denoted $\hat{H}(X;\mathbb{Z})$ is the integral cohomology ring $H^*(X;\mathbb{Z})$ along with the Mukai pairing $\langle -, - \rangle$ defined as above.

A K3 surface admits a weight two Hodge decomposition

$$\begin{aligned} \hat{H}^{2,0}(X) &\simeq H^{2,0}(X) \\ \hat{H}^{1,1}(X) &\simeq H^{1,1}(X) \oplus H^0(X) \oplus H^4(X) \\ \hat{H}^{0,2}(X) &\simeq H^{0,2}(X). \end{aligned}$$

Theorem 3.7 ([Muk84], [Orl97])**.** *(Derived Torelli Theorem) Let X and Y be K3 surfaces with Mukai lattices $\hat{H}(X;\mathbb{Z})$ and $\hat{H}(Y;\mathbb{Z})$ respectively, then there is a derived equivalence*

$$\mathrm{D}(X) \simeq \mathrm{D}(Y),$$

if and only if there exists a Hodge isometry between Mukai lattices $\hat{H}(X;\mathbb{Z}) \to \hat{H}(Y;\mathbb{Z})$.

Remark 3.8. Mukai initially proved the forward direction in [Muk84], later Orlov proved the converse in [Orl97].

Let X be a K3 surface we define the set of Fourier-Mukai partners

$$\mathrm{FM}(X) = \{Y | \mathrm{D}(X) \simeq \mathrm{D}(Y)\}/\simeq$$

where $\simeq$ quotient by autoequivalences.

Remark 3.9. Since $X \in \mathrm{FM}(X)$ the set of Fourier-Mukai partners for X is never empty.

A famous example of a K3 surface is a Kummer surface X. Kummer surfaces are obtained from resolving an abelian surface $A/\sim$ where we have taken a quotient by the involution corresponding to the inversion map.

Example 3.10. Let X be the Kummer surface associated to an abelian surface A, that is $X = \mathrm{Km}(A)$ then

$$\mathrm{FM}(X) = \{X\}.$$

Therefore given any two Kummer surfaces $X = \mathrm{Km}(A)$ and $Y = \mathrm{Km}(B)$ with A and B abelian surfaces then $\mathrm{D}(A) \simeq \mathrm{D}(B)$ if and only if $X \simeq Y$.

Now suppose k is an algebraically closed field with positive characteristic.

Theorem 3.11 (Theorem 6.1, [LO11])**.** *(Derived Torelli Theorem in Positive Characteristic)*
Let X and Y be K3 surfaces over k. If there exists a kernel $\mathcal{P} \in \mathrm{D}(X \times Y)$ such that we have a filtered equivalence $\mathrm{D}(X) \to \mathrm{D}(Y)$ then $X \simeq Y$.

Remark 3.12. See paragraph 2.11 in Leiblich–Olsson [LO11] for the precise definition of filtered equivalence.

Theorem 3.13 (Theorem 4.1, [LO11])**.** *If X and Y are K3 surfaces over a finite field $\mathbb{F}$ and derived equivalent, (i,e $\mathrm{D}(X) \simeq \mathrm{D}(Y)$) then X and Y have the same zeta-function and the same number of points over $\mathbb{F}$.*

$$\#X(\mathbb{F}) = \#Y(\mathbb{F}).$$

Appendix A. Triangulated Categories

Triangulated Categories as we will define were first introduced and developed in 1962-1963 by Dieter Puppe and Jean-Louis Verdier respectively. Puppe was an algebraic topologist primarily interested in triangulated categories as an abstraction of the stable homotopy category whilst Verdier, who at the time was a PhD student of Grothendieck, was an algebraic geometer interested in how the derived category admits this triangulated structure. Both Puppe and Verdier gave similar definitions of a triangulated category at around the same time with the only notable difference being Verdier's inclusion of the octahedral axiom (TR4)[3]

Definition A.1. Let $\mathcal{T}$ be a category equipped with an automorphism $\Sigma : \mathcal{T} \to \mathcal{T}$. A triangle (A, B, C) where $A, B, C \in \mathcal{T}$ is an ordered triple (u, v, w) of morphisms where $u : A \to B$, $v : B \to C$ and $w : C \to \Sigma A$. That is a triangle

$$A \xrightarrow{u} B \xrightarrow{v} C \xrightarrow{w} \Sigma A.$$

Definition A.2. A morphism of triangles is a triple (f, g, h) forming a commutative diagram in $\mathcal{T}$:

$$\begin{tikzcd}
A \arrow[r, "u"] \arrow[d, "f"] & B \arrow[r, "v"] \arrow[d, "g"] & C \arrow[r, "w"] \arrow[d, "h"] & \Sigma A \arrow[d, "\Sigma f"] \\
A \arrow[r, "u'"] & B \arrow[r, "v'"] & C \arrow[r, "w'"] & \Sigma A'
\end{tikzcd}$$

An isomorphism of triangles is then defined in the natural way.

Definition A.3. Let $\mathcal{T}$ be an additive category. Then $\mathcal{T}$ is a **triangulated category** if $\mathcal{T}$ admits an additive equivalence $\Sigma : \mathcal{T} \to \mathcal{T}$, called the translation or shift functor, and a collection of distinguished (or exact) triangles in $\mathcal{T}$, which are triangles (u, v, w) that satisfy the following four axioms:

(TR1) i) Every morphism $u : A \to B$ can be embedded in an exact triangle (u, v, w).
ii)Any triangle of the form

$$A \xrightarrow{id} A \longrightarrow 0 \longrightarrow \Sigma A,$$

is exact.
iii) if (u, v, w) is a triangle on (A, B, C), isomorphic to an exact triangle (u', v', w') on (A', B', C') then (u, v, w) is also exact.

$$\begin{tikzcd}
A \arrow[r, "u"] \arrow[d, "\cong"] & B \arrow[r, "v"] \arrow[d, "\cong"] & C \arrow[r, "w"] \arrow[d, "\cong"] & \Sigma A \arrow[d, "\cong"] \\
A \arrow[r, "u'"] & B \arrow[r, "v'"] & C \arrow[r, "w'"] & \Sigma A'
\end{tikzcd}$$

(TR2) (Rotation). If (u, v, w) is an exact triangle on (A, B, C)

$$\begin{tikzcd}
& C \arrow[dl, "w"'] & \\
A \arrow[rr, "u"] & & B \arrow[ul, "v"']
\end{tikzcd}$$

then both its "rotates" $(-\Sigma^{-1}w, u, v)$ and $(v, w, -\Sigma u)$ are exact triangles on $(-\Sigma^{-1}w, u, v)$ and $(B, C, \Sigma A)$

$$\begin{tikzcd}
& \Sigma A \arrow[dl, "-\Sigma u"'] & \\
B \arrow[rr, "v"] & & C \arrow[ul, "w"']
\end{tikzcd}
\qquad
\begin{tikzcd}
& B \arrow[dl, "v"'] & \\
\Sigma^{-1} C \arrow[rr, "-\Sigma^{-1} w"] & & A \arrow[ul, "u"']
\end{tikzcd}$$

(TR3) Suppose there exists a commutative diagram of distinguished triangles (A, B, C) and (A', B', C') with vertical arrows $f : A \to A'$ and $g : B \to B'$: Then the diagram can be completed to a morphism of triangles, by a non-unique morphism $h : C \to C'$.

$$\begin{tikzcd}
A \arrow[r, "u"] \arrow[d, "f"] & B \arrow[r, "v"] \arrow[d, "g"] & C \arrow[r, "w"] \arrow[d, dashed, "\exists h"] & \Sigma A \arrow[d, "\Sigma f"] \\
A' \arrow[r, "u'"] & B' \arrow[r, "v'"] & C' \arrow[r, "w'"] & \Sigma A'
\end{tikzcd}$$

[3] hence why (TR4) is sometimes referred to as the Verdier axiom.

(TR4)(**Octahedral axiom/Verdier axiom**) Given objects $A, B, C, A', B', C' \in \mathcal{T}$, if there are three exact triangles: (u, j, ∂) on (A, B, C'), (v, x, i) on (B, C, A') and (vu, y, δ) on (A, C, B').

$$\begin{tikzcd} & C \arrow[dl, "\partial"'] & \\ A \arrow[rr, "u"] & & B \arrow[ul, "j"'] \end{tikzcd} \qquad \begin{tikzcd} & C' \arrow[dl, "i"'] & \\ A \arrow[rr, "v"] & & B' \arrow[ul, "x"'] \end{tikzcd} \qquad \begin{tikzcd} & B' \arrow[dl, "\delta"'] & \\ A \arrow[rr, "vu"] & & C \arrow[ul, "y"'] \end{tikzcd}$$

Then there is a fourth exact triangle $(f, g, (\Sigma j)i)$ on (C', B', A') such that in the following octahedron the four exact triangles form four of the faces and the remaining faces commute.

$$\begin{tikzcd} & & & C \arrow[dl, "x"'] \arrow[dddr, "y"] & \\ B \arrow[urrr, "v"] \arrow[ddrr, "j"'] & & A' \arrow[ll, "i"'] \arrow[dd, "(\Sigma j)i"] & & \\ & & & A' \arrow[ulll, dashed, "u"] \arrow[ul] \arrow[uu, "vu"'] \arrow[dr] & \\ & & C' \arrow[ur, dashed, "\partial"] \arrow[rr, "\exists f"] & & B' \arrow[ul, dashed, "\exists g\ \ \delta"'] \end{tikzcd}$$

Remark A.4. There is some uncertainty as to whether TR4 is the "right" axiom for the definition of a triangulated category, and whether instead it is more natural to impose that the morphisms of distinguished triangles should admit mapping cones which also form distinguished triangles. In [Nee01] Neeman calls such a condition (TR4') and in particular shows that (TR4') implies the octahedral axiom (TR4), with the converse implication proven in [Nee91]. For the reader interested in further such discussion see [Nee91] and the relevant sections of [May01].

Definition A.5. An additive functor $F : \mathcal{T} \to \mathcal{T}'$ between triangulated categories $\mathcal{T}$ and $\mathcal{T}'$ is called exact if the following conditions are satisfied

(1) There exists a functorial isomoprhism $F \circ T_{\mathcal{T}} \xrightarrow{\sim} T_{\mathcal{T}'} \circ F$.
(2) Any distinguished triangle

$$A \longrightarrow B \longrightarrow C \longrightarrow A[1]$$

in $\mathcal{T}$ is mapped to a distinguished triangle

$$F(A) \longrightarrow F(B) \longrightarrow F(C) \longrightarrow F(A)[1]$$

in $\mathcal{T}'$ where $F(A[1])$ is identified with $F(A)[1]$ via the functor isomorphism in 1).

Definition A.6. A subcategory $\mathcal{T}' \subset \mathcal{T}$ of a triangulated category is a triangulated subcategory if $\mathcal{T}'$ admits the structure of a triangulated category such that the inclusion $i : \mathcal{T}' \to \mathcal{T}$ is exact.

Proposition A.7. *Let $\mathcal{T}' \subset \mathcal{T}$ be a full subcategory. $\mathcal{T}$ is a triangulated subcategory if and only if $\mathcal{T}'$ is invariant under the shift functor $\Sigma : \mathcal{T} \to \mathcal{T}$ and for any distinguished triangle*

$$A \longrightarrow B \longrightarrow C \longrightarrow A[1]$$

we have that $C \cong D$ for $D \in \mathcal{T}$.

Definition A.8. Two triangulated categories $\mathcal{T}$ and $\mathcal{T}'$ are equivalent if there exists an exact equivalence $F : \mathcal{T} \to \mathcal{T}'$.

Definition A.9. A triangulated category $\mathcal{T}$ is decomposable into triangulated subcategories $\mathcal{A} \subset \mathcal{T}$ and $\mathcal{B} \subset \mathcal{T}$ if the following three conditions are satisfied:

(1) The categories $\mathcal{A}$ and $\mathcal{B}$ contain objects non-isomorphic to 0.
(2) For every object $F \in \mathcal{T}$, there exists a distinguished triangle

$$A \longrightarrow F \longrightarrow B \longrightarrow A[1]$$

where $A \in \mathcal{A}$ and $B \in \mathcal{B}$.
(3) For every pair of objects $B_1 \in \mathcal{T}_1$ and $B_2 \in \mathcal{T}_2$, there exist no morphisms in $\mathcal{T}$ between them, i.e.,

$$\mathrm{Hom}(B_1, B_2) = \mathrm{Hom}(B_2, B_1) = 0.$$

Appendix B. Properties of Triangulated Categories

Definition B.1. Let $\mathcal{A}$ be a k-linear category. A Serre functor $S : \mathcal{A} \to \mathcal{A}$ is an additive functor that is also an autoequivalence such that for any two objects $A, B \in \mathcal{T}$ there exists an isomorphism

$$\eta_{A,B} : \mathrm{Hom}(A, B) \to \mathrm{Hom}(B, S(A))^*.$$

Proposition B.2 ([BK90])**.** *Any Serre functor on a triangulated category over a field k is exact.*

Definition B.3. Let $\mathcal{T}$ be a triangulated category. A subclass $\Omega \subset \mathcal{T}$ of the objects of $\mathcal{T}$ is called a spanning class of $\mathcal{T}$ if for any object $B \in \mathcal{T}$:

(1) $\mathrm{Hom}_{\mathcal{T}}(A, B[i]) = 0$ for all $A \in \Omega$ and all $i \in \mathbb{Z}$, then $B \cong 0$.
(2) $\mathrm{Hom}_{\mathcal{T}}(B[i], A) = 0$ for all $A \in \Omega$ and all $i \in \mathbb{Z}$, then $B \cong 0$.

Proposition B.4 (Corollary. 3.19, [Huy06])**.** *If X is a smooth projective variety and L is an amble line bundle on X, then the powers $L^i, i \in \mathbb{Z}$, form a spanning class in $\mathrm{D}(X)$.*

Example B.5. Let $\mathcal{E}^\bullet \in \mathrm{D}(X)$ be any object and

$$\mathcal{E}^{\bullet\perp} := \{\mathcal{F}^\bullet \in \mathrm{D}(X) \mid \mathrm{Hom}(\mathcal{E}^\bullet, \mathcal{F}^\bullet[i]) = 0 \text{ for all } i \in \mathbb{Z}\}.$$

Then $\Omega = \{\mathcal{E}^\bullet\} \cup \mathcal{E}^{\bullet\perp} \subset \mathrm{D}(X)$ is a spanning class.

Remark B.6. If the triangulated category admits a Serre functor, conditions (1) and (2) above are equivalent.

Proposition B.7 ([Orl97])**.** *Let $F : \mathcal{T} \to \mathcal{T}'$ be an exact functor between triangulated categories with left and right adjoints: $G \dashv F \dashv H$. Suppose Ω is the spanning class of $\mathcal{T}$ such that for all objects $A, B \in \Omega$ and all $i \in \mathbb{Z}$ the natural homomorphisms*

$$F : \mathrm{Hom}(A, B[i]) \to \mathrm{Hom}(F(A), F(B[i]))$$

are bijective. Then F is fully faithful.

Proposition B.8 (Corollary, 1.56, [Huy06])**.** *Let $F : \mathcal{T} \to \mathcal{T}'$ be an exact functor between triangulated categories $\mathcal{T}$ and $\mathcal{T}'$ with left adjoint $F \dashv H$. Furthermore assume that Ω is a spanning class of $\mathcal{T}$ satisfying the following conditions*

(1) *For all $A, B \in \Omega$ the natural morphisms*

$$\mathrm{Hom}(A, B[i]) \to \mathrm{Hom}(F(A), F(B)[i]),$$

are bijective for all $i \in \mathbb{Z}$.

(2) *The categories $\mathcal{T}$ and $\mathcal{T}'$ admit Serre functors $S_{\mathcal{T}}$ and respectively $S_{\mathcal{T}'}$ such that for all $A \in \Omega$,*

$$F(S_{\mathcal{T}}(A)) = S_{\mathcal{T}'}(F(A)).$$

(3) *The category $\mathcal{T}'$ is indecomposable and $\mathcal{T}$ is non-trivial.*

Then F is an equivalence.

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