Renormalizing a Scalar-Fermion Yukawa Model

Adam Monteleone

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This is a calculation I had to do from a second course in quantum field theory PHYC90057. It beautifully demonstrates a lot of the machinery used in perturbative quantum field theory[1] and as far I know is nowhere on the internet. The exposition is verbose throughout, with the hope that other graduate students find it helpful.

1 Introduction

Consider the bare lagrangian describing a two-component massless spinor ψ and two complex scalar fields ϕ and S

$$\mathcal{L} = (\partial_\mu \phi)^\dagger \partial^\mu \phi + \psi^\dagger i\overline{\sigma}^\mu \partial_\mu \psi + S^* S + [\lambda_1 S \phi^2 + i\lambda_2 \psi^T \sigma^2 \psi \phi + \mathbf{h.c.}] - m^2|\phi|^2,$$

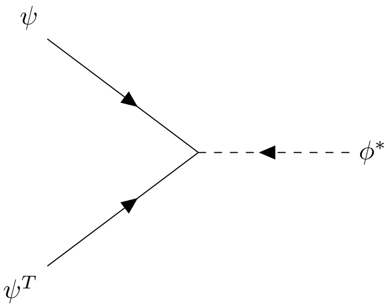
where $(\overline{\sigma})^\mu := (I, -\sigma^i)$ with λ_1 and λ_2 real parameters. Note for two-component spinors the σ^μ matrices are analogues of the Dirac matrices γ^μ. We begin by determining from $\mathcal{L}$ the equations of motion for S and S^*

$$0 = \frac{\partial \mathcal{L}}{\partial S^*} = S + \lambda_1(\phi^*)^2 \text{ and } 0 = \frac{\partial \mathcal{L}}{\partial S} = S^* + \lambda_1 \phi^2.$$

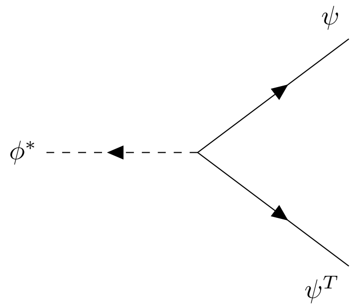
Solving the equations for S and S^*, we find the on-shell solutions are given by $S = -\lambda_1(\phi^*)^2$ and $S^* = -\lambda_1\phi^2$. The complex scalar fields S and S^* can then be integrated out of $\mathcal{L}$ by substituting our on-shell expressions back into the Lagrangian

$$\mathcal{L} = |\partial\phi|^2 + \psi^\dagger(i\overline{\sigma}\cdot\partial)\psi + i\lambda_2\psi^T\sigma^2\psi\phi - i\lambda_2\psi^\dagger\sigma^2\psi^*\phi^* - m^2|\phi|^2 - \lambda_1^2|\phi|^4.$$

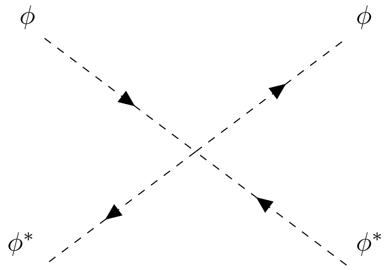
The above is an effective Lagrangian in the fields ϕ and ψ with now an additional quartic self-interaction term for ϕ. From $\mathcal{L}$ we can determine the interaction vertices for our theory:



$= -2\lambda_2\sigma^2,$



$= 2\lambda_2\sigma^2$



$= -4i\lambda_1^2,$

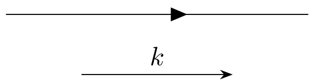
where we have taken into account the symmetry factor of the quartic self-interaction coming from the connected four point scalar function. Observe at first order there are 4 possible contractions of $|i\rangle$ and $\langle f|$

$$\langle p,\tilde{p}|T\phi\phi^*\phi\phi^*|k,\tilde{k}\rangle = \langle p,\tilde{p}|\phi\phi^*\phi\phi^*|k,\tilde{k}\rangle + \langle p,\tilde{p}|\phi\phi^*\phi\phi^*|k,\tilde{k}\rangle + \langle p,\tilde{p}|\phi\phi^*\phi\phi^*|k,\tilde{k}\rangle + \langle p,\tilde{p}|\phi\phi^*\phi\phi^*|k,\tilde{k}\rangle.$$

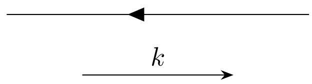
[1] whilst also suppressing superficial complexities like spinor indices. The indices can be done away with as long as the orientations of the Feynman diagrams given are adhered to. This is why our complex scalar field has a directed propagator.

2 The One-Loop Correction of the Two-Point Function for ϕ

We begin by introducing the momentum-space representation for the fermion propagator, given by



$= \frac{ik_\mu \sigma^\mu}{k^2}$ and



$= \frac{ik_\mu \overline{\sigma}^\mu}{k^2}$.

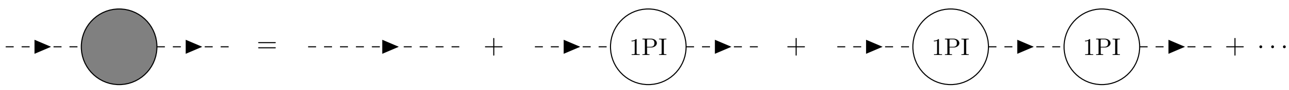
Before computing corrections to the propagator at one loop, we recount some basic theory. Recall that a 1-particle irreducible (1PI) diagram is by definition any diagram that cannot be split in two by removing a single line. Diagrammatically, the circle with 1PI in the centre represents the sum over all 1PI two-point diagrams. Algebraically we denote it by the expression $-i\Pi_\phi(p^2)$ called the 1PI *amplitude* of ϕ where $\Pi(p^2)$ is the *self energy of the scalar field* ϕ. The 2-point Greens function in momentum space denoted $D_F(p)$ is given by summing over all connected two point diagrams, however a connected two point Feynman diagram can be decomposed recursively into a bare propagator, plus a bare propagator concatenated with a 1PI diagram connected to an arbitrary connected two point diagram. Algebraically this recursion is stated as

$$D_F(p) = D_0(p) + D_0(p)[-i\Pi(p^2)]D_F(p),$$

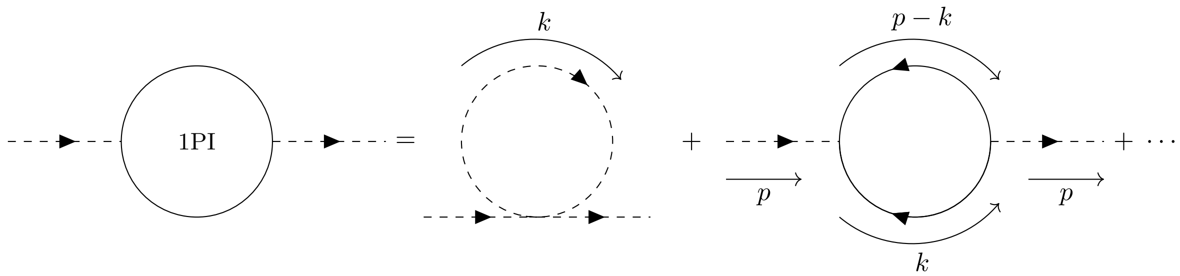
where $D_0(p)$ is the *bare propagator.* This recursion gives a geometric series which can be re-summed to yield

$$D_F(p) = \frac{i}{p^2 - (m^2 + \Pi_r(p^2)) + i\varepsilon},$$

where $\Pi_r(p^2)$ is the *renormalized self-energy* of the scalar field ϕ.[2]The recursion above also implies that $D_F(p)$ can be expressed as the following sum of diagrams:



Using our interaction vertices from the previous page, we can deduce the form fo the diagrams that appear in the expansion of the 1PI amplitude of ϕ to order $O(\lambda_1^2, \lambda_2^2)$, they are



Now applying the standard Feynman rules to the above diagrams respectively gives

$$\begin{aligned}
-i\Pi_\phi &= -4i\lambda_1^2 \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2 - m^2} \\
&= 4\lambda_1^2 \int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2 - m^2}, \\
-i\Pi_\psi(p^2) &= \frac{1}{2}(-2\lambda_2)(2\lambda_2) \int \frac{d^4k}{(2\pi)^4} \operatorname{Tr}\left(\frac{\sigma^2 i(p-k)\cdot\overline{\sigma}^T}{(p-k)^2} \sigma^2 \frac{i(k\cdot\overline{\sigma}}{k^2} \right) \\
&= 2\lambda_2^2 \int \frac{d^4k}{(2\pi)^4} \frac{(p-k)_\mu k_\nu}{(p-k)^2 k^2} \operatorname{Tr}\left(\sigma^2 (\overline{\sigma}^T)^\mu \sigma^2 \overline{\sigma}^\nu\right).
\end{aligned}$$

[2]It is worth pointing out that from the pole in the denominator (when compared with the pole in the formula of $D_F(p)$ given by the Källen-Lehmann spectral representation) relates physical mass squared of ϕ to the bare mass by $m^2_{\text{phys}} = m^2 + \Pi_r(m^2)$.

From $(\overline{\sigma}^T)^\mu = (I, -\sigma^i)$ it can be show that the identity $\sigma^2(\overline{\sigma}^T)^\mu\sigma^2 = \sigma^\mu$ holds. Combining this with a standard QFT trace identity we find

$$\mathrm{Tr}(\sigma^2(\overline{\sigma}^T)^\mu)\sigma^2\overline{\sigma}^\nu) = \mathrm{Tr}(\sigma^\mu\sigma^\nu) = 2g^{\mu\nu}.$$

By definition of the Minkowski product $(k-p)\cdot k = (k-p)_\mu k_\nu g^{\mu\nu}$, then from the identity above we have

$$-i\Pi_\psi(p^2) = -4\lambda_2^2 \int \frac{d^4k}{(2\pi)^4} \frac{(k-p)\cdot k}{(p-k)^2k^2}.$$

Adding the contribution of each diagrams gives the 1PI amplitude of ϕ, with $-i\Pi(p^2) = -i\Pi_\phi - i\Pi_\psi(p^2)$ to be

$$-i\Pi(p^2) = 4\lambda_1^2 \int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2-m^2} - 4\lambda_2^2 \int \frac{d^4k}{(2\pi)^4} \frac{(k-p)\cdot k}{(p-k)^2k^2}.$$

Note from the above equation it can be seen that as the loop 4-momentum k goes to the UV there is a quadratic divergence

$$-i\Pi(p^2) \to 4(\lambda_1^2 - \lambda_2^2) \int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2},$$

unless the couplings are equal in which case there is a UV fixed point. To calculate the one loop correction we make use the following three standard techniques in sequence, they are: Feynman parameters, Wick rotation and dimensional regularization. To begin, we introduce the technique of Feynman parameters, which involves exploiting integral identities of the form (or similar to)

$$\frac{1}{A_1A_2} = \int_0^1 dx \frac{1}{(A_1 + (A_2 - A_1)x)^2}.$$

The product of factors in the denominator can then be combined in a way where they form a single quadratic expression raised to the second power. Then after completing the square and changing variables the integral over the momentum variable can now be done however there will also be auxiliary variables that can be integrated out. Therefore for the current integral at hand we apply the integral identity above to $-i\Pi_\psi(p^2)$ and complete the square to obtain

$$\begin{aligned}
\frac{1}{k^2(p-k)^2} &= \int_0^1 dx \frac{1}{(k^2 + ((p-k)^2 - k^2)x)^2} \\
&= \int_0^1 dx \frac{1}{(k^2 + p^2x - 2pkx)^2} \\
&= \int_0^1 dx \frac{1}{((k-px)^2 - p^2x(x-1))^2}.
\end{aligned}$$

Define $\Delta := p^2x(x-1)$ and $\ell := k - px$ then $d^4\ell = d^4k$, the numerator simplifies as

$$\begin{aligned}
(k-p)\cdot k &= (\ell + p(x-1))\cdot(\ell + px) \\
&= \ell^2 + \Delta + O(\ell).
\end{aligned}$$

Noting that odd integrals vanish our full expression simplifies to

$$\begin{aligned}
-i\Pi_\psi(p^2) &= -4\lambda_2^2 \int \frac{d^4k}{(2\pi)^4} \int_0^1 dx \frac{(k-p)\cdot k}{((k-px)^2 - p^2x(x-1))^2} \\
&= -4\lambda_2^2 \int_0^1 dx \int \frac{d^4\ell}{(2\pi)^4} \left[\frac{\ell^2}{(\ell^2-\Delta)^2} + \frac{\Delta}{(\ell^2-\Delta)^2}\right],
\end{aligned}$$

where we are left with a quadratic divergence and logarithmic divergence respectively. Currently the loop integrals above are written with contractions using a Lorentz signature. However if they were in Euclidean space we could do them in spherical polar coordinates in a relatively straight-forward way. To progress further we introduce our second technique, *Wick rotation.* A Wick rotation will transform the Lorentz invariance into rotational invariance. To perform a Wick rotation we define Euclidean 4-momentum variables $\ell_E^0 := -i\ell^0$ and $\ell_E^i = \ell^i$ this implies $\ell^2 = -\ell_E^2$ and $d^4\ell = id^4\ell_E$. Here we have analytically continued one of the variables into the complex plane. Wick rotating, our expression becomes

$$-i\Pi_\psi(p^2) = -4i\lambda_2^2 \int_0^1 dx \int \frac{d^4\ell_E}{(2\pi)^4}\left[\frac{-\ell_E^2}{(\ell_E^2+\Delta)^2} + \frac{\Delta}{(\ell_E^2+\Delta)^2}\right].$$

Unfortunately, this integral is still divergent in the UV. One way to proceed from here would be to introduce a cut-off, that way we could still obtain some approximate answer however it turns out such an answer would violate the Ward identity and introduce a mass that is proportional to the cut-off. To study the behaviour of the loop integral in a way which respects the Ward identity we use *dimensional regularisation*. The idea is to work not in dimension 4 but dimension d, and interpret $d = 4$ as a divergence. There is a way to then extract the leading order terms for the divergent integral at $d = 4$, this is done by doing perturbation theory in dimension $d = 4 - \varepsilon$.

Consider now our loop integral in d-dimensions (where $d \neq 4$) the d-dimensional closed form of both loop integrals of this form are a standard identity in QFT and hence we quote the results. The identity that corresponds to the first loop integral is

$$\int \frac{d^d\ell_E}{(2\pi)^d}\frac{\ell_E^2}{(\ell_E^2+\Delta)^n} = \frac{1}{(4\pi)^{\frac{d}{2}}}\frac{d}{2}\frac{\Gamma(n-\frac{d}{2}-1)}{\Gamma(n)}\left(\frac{1}{\Delta}\right)^{n-\frac{d}{2}-1},$$

and the identity for the second loop integral is given by

$$\int \frac{d^d\ell_E}{(2\pi)^d}\frac{1}{(\ell_E^2+\Delta)^2} = \frac{1}{(4\pi)^{\frac{d}{2}}}\frac{\Gamma(2-\frac{d}{2})}{\Gamma(2)}\left(\frac{1}{\Delta}\right)^{2-\frac{d}{2}}.$$

There is a subtlety when working in an arbitrary integer dimension, namely our couplings are no longer dimensionless. It follows from dimensional analysis on our Lagrangian that $\lambda_2 \mapsto \mu^{\frac{4-d}{2}}\lambda_2$ where μ is an arbitrary parameter of mass dimension one that we introduce. Therefore for $d \neq 4$ our 1PI amplitude has the form

$$-i\Pi_\psi(p^2) = -4i\lambda_2^2\mu^{4-d}\int_0^1 dx\left[\frac{-1}{(4\pi)^d}\frac{d}{2}\frac{\Gamma(1-\frac{d}{2})}{\Delta^{1-\frac{d}{2}}} + \frac{1}{(4\pi)^{\frac{d}{2}}}\frac{\Gamma(2-\frac{d}{2})}{\Delta^{2-\frac{d}{2}}}\right].$$

Substituting $d = 4 - \varepsilon$, and applying some basic complex analysis we obtain the expansion for each term:

- $\Gamma(1-\frac{d}{2}) = \Gamma(-1+\frac{\varepsilon}{2}) \approx -\frac{2}{\varepsilon} + \gamma_E - 1$;
- $\Gamma(2-\frac{d}{2}) = \Gamma(\frac{\varepsilon}{2}) \approx \frac{2}{\varepsilon} - \gamma_E$;
- $\mu^{4-d} = \mu^\varepsilon = (\mu^2)^{\frac{\varepsilon}{2}} \approx 1 + \frac{\varepsilon}{2}\log(\mu^2)$;
- $(4\pi)^{-\frac{d}{2}} = (4\pi)^{-2+\frac{\varepsilon}{2}} \approx \frac{1}{(4\pi)^2}\left(1+\frac{\varepsilon}{2}\log(4\pi)\right)$;
- $\Delta^{\frac{d}{2}-1} = \Delta^{1-\frac{\varepsilon}{2}} \approx \Delta(1-\frac{\epsilon}{2}\log(\Delta))$;
- $\Delta^{\frac{d}{2}-2} = \Delta^{-\frac{\varepsilon}{2}} \approx 1 - \frac{\varepsilon}{2}\log(\Delta)$.

Define the constant $\tilde{\mu}^2 := 4\pi e^{-\gamma_E}\mu^2$, expanding both expressions up to order $O(\varepsilon)$ yields

$$\begin{aligned}
\frac{1}{(4\pi)^{\frac{d}{2}}}\frac{\Gamma(1-\frac{d}{2})}{\Delta^{1-\frac{d}{2}}} &= \frac{\Delta}{16\pi^2}\left(-\frac{2}{\varepsilon}+\gamma_E+-1\right)\left(1-\frac{\varepsilon}{2}\log(\Delta)\right)\left(1+\frac{\varepsilon}{2}\log(\mu^2)\right)\left(1+\frac{\varepsilon}{2}\log(4\pi)\right) \\
&= \frac{\Delta}{16\pi^2}\left[-\frac{2}{\varepsilon}+\gamma_E-1+\log(\Delta)-\log(\mu^2)-\log(4\pi)\right] \\
&= -\frac{\Delta}{16\pi^2}\left[\frac{2}{\varepsilon}+\log\left(\frac{\tilde{\mu}^2}{\Delta}\right)+1\right],
\end{aligned}$$

and

$$
\begin{aligned}
\frac{1}{(4\pi)^{\frac{d}{2}}}\frac{\Gamma(2-\frac{d}{2})}{\Delta^{2-\frac{d}{2}}} &= \frac{1}{16\pi^2}\left(1-\frac{\varepsilon}{2}\log(\Delta)\right)\left(\frac{2}{\varepsilon}-\gamma_E\right)\left(1+\frac{\varepsilon}{2}\log(4\pi)\right)\left(1+\frac{\varepsilon}{2}\log(\mu^2)\right) \\
&= \frac{1}{16\pi^2}\left(\frac{2}{\varepsilon}-\gamma_E-\log(\Delta)\right)\left(1+\frac{\varepsilon}{2}\log(4\pi)+\frac{\varepsilon}{2}\log(\mu^2)\right) \\
&= \frac{1}{16\pi^2}\left(\frac{2}{\varepsilon}+\log(4\pi)+\log(\mu^2)-\gamma_E-\log(\Delta)\right) \\
&= \frac{\Delta}{16\pi^2}\left(\frac{2}{\varepsilon}+\log\left(\frac{\tilde{\mu}^2}{\Delta}\right)\right).
\end{aligned}
$$

Combining everything together up to order ε yields our 1-loop contribution from the second Feynamn diagram:

$$
\begin{aligned}
-i\Pi_\psi(p^2) &= -4i\lambda_2^2\int_0^1 dx\left[-\frac{\Delta}{16\pi^2}\left(-2+\frac{\varepsilon}{2}\right)\left(\frac{2}{\varepsilon}+\log(\frac{\tilde{\mu}^2}{\Delta})+1\right)+\frac{\Delta}{16\pi^2}\left(\frac{2}{\varepsilon}+\log\left(\frac{\tilde{\mu}^2}{\Delta}\right)\right)\right] \\
&= -\frac{3i\lambda_2^2}{4\pi^2}\int_0^1 dx\Delta\left[\frac{2}{\varepsilon}+\log\left(\frac{\tilde{\mu}^2}{\Delta}\right)+\frac{1}{3}+O(\varepsilon)\right].
\end{aligned}
$$

Recall that we also had to evaluate the loop integral associated to correction from the first Feynman diagram

$$
-i\Pi_\phi^2 = 4\lambda_1^2\int\frac{d^4k}{(2\pi)^4}\frac{1}{k^2-m^2}.
$$

Performing a Wick rotation, we set $k_E^0 := -ik^0$ and $k^2 = -k_E^2$ which yields

$$
-i\Pi_\phi^2 = -4i\lambda_1^2\int\frac{d^4k_E}{(2\pi)^4}\frac{1}{k_E^2+m^2}.
$$

Dimensional regularizing, and making use of d-dimensional loop integral identity, gives

$$
\int\frac{d^dk}{(2\pi)^d}\frac{1}{(k_E^2+m^2)} = \frac{1}{(4\pi)^{\frac{d}{2}}}\left(\frac{\Gamma(1-\frac{d}{2})}{(m^2)^{1-\frac{d}{2}}}\right).
$$

Using the expansion derived before we expand each term out to first order

$$
\begin{aligned}
-i\Pi_\phi &= -\frac{4i\lambda_1^2\mu^{4-d}}{(4\pi)^{\frac{d}{2}}}\left(\frac{\Gamma(1-\frac{d}{2})}{(m^2)^{1-\frac{d}{2}}}\right) \\
&= -4i\lambda_1^2\left(m^2(1-\frac{\varepsilon}{2}\log(m^2))\right)\left(-\frac{2}{\varepsilon}+\gamma_E+1\right)\frac{1}{(4\pi)^2}\left(1+\frac{\varepsilon}{2}\log(4\pi)\right)\left(1+\frac{\varepsilon}{2}\log(\mu^2)\right) \\
&= -\frac{i\lambda_1^2m^2}{4\pi^2}\left(1-\frac{\varepsilon}{2}\log(m^2)\right)\left(-\frac{2}{\varepsilon}+\gamma_E-1\right)\left(1+\frac{\varepsilon}{2}\log(4\pi)\right)\left(1+\frac{\varepsilon}{2}\log(\mu^2)\right) \\
&= -\frac{i\lambda_1^2m^2}{4\pi^2}\left[-\frac{2}{\varepsilon}+\log(m^2)+\gamma_E-1-\log(4\pi)-\log(\mu^2)\right] \\
&= \frac{i\lambda_1^2m^2}{4\pi^2}\left[\frac{2}{\varepsilon}+\log\left(\frac{\tilde{\mu}^2}{m^2}\right)+1\right].
\end{aligned}
$$

Putting everything together we find that the one-loop correction to the two point function for ϕ is given by

$$
i\Pi(p^2) = \frac{i\lambda_1^2m^2}{4\pi^2}\left[\frac{2}{\varepsilon}+\log\left(\frac{\tilde{\mu}^2}{m^2}\right)+1\right]-\frac{3i\lambda_2^2}{4\pi^2}\int_0^1 dx\Delta\left[\frac{2}{\varepsilon}+\log\left(\frac{\tilde{\mu}^2}{\Delta}\right)+\frac{1}{3}\right]+O(\varepsilon).
$$

3 A Renormalized Perturbation Theory for $\mathcal{L}$ in the $\overline{\text{MS}}$ Scheme

In this section we give a full account of how to construct a renormalized perturbation theory for $\mathcal{L}$ in the *modified minimal subtraction* scheme $\overline{\text{MS}}$. Recall that in the MS scheme the finite parts of the counterterms are chosen to be zero and in the $\overline{\text{MS}}$ scheme the universal constants the appear in the regularization (which are $\log(4\pi) + \gamma_E$) are subtracted off[3].

From now we denote the bare fields and masses by ψ_0, ϕ_0 and m_0 respectively. Recall the bare Lagrangian after integrating out S is given by

$$\mathcal{L}_0 = |\partial\phi_0|^2 + \psi_0^\dagger(i\overline{\sigma}\cdot\partial)\psi_0 + i\lambda_2\psi_0^T\sigma^2\psi_0\phi_0 - i\lambda_2\psi_0^\dagger\sigma^2\psi_0^*\phi_0^* - m_0^2|\phi_0|^2 - \lambda_1^2|\phi_0|^4.$$

We denote the renormalized fields and bare masses by ψ, ϕ and m respectively. We introduce the following wavefunction renormalizations

$$\psi_0 := Z_\psi^{\frac{1}{2}}\psi \text{ and } \phi_0 := Z_\phi^{\frac{1}{2}}\phi$$

Our renormalized the infinities appearing in our bare theory we will introduce a renormalized Lagrangian $\mathcal{L} = \mathcal{L}_0 - \mathcal{L}_{\text{ct}}$ where $\mathcal{L}_{\text{ct}}$ is the counterterm lagrangian which will be carefully deduced and which will cancel with the divergences. Substituting the definition of the bare quantities into our bare Lagrangian $\mathcal{L}_0 = \mathcal{L} + \mathcal{L}_{\text{c.t.}}$ we have

$$\begin{aligned}\mathcal{L}_0 &= |\partial\phi_0|^2 + \psi_0^\dagger(i\overline{\sigma}\cdot\partial)\psi_0 + i\lambda_{2,0}\psi_0^T\sigma^2\psi_0\phi_0 - i\lambda_{2,0}\psi_0^\dagger\sigma^2\psi_0^*\phi_0^* - m_0^2|\phi_0|^2 - \lambda_{1,0}^2|\phi_0|^4 \\ &= Z_\phi|\partial\phi|^2 + Z_\psi\psi^\dagger(i\overline{\sigma}\cdot\partial)\psi + [i\lambda_{2,0}Z_\phi^{\frac{1}{2}}Z_\psi\psi^T\sigma^2\psi\phi + \textbf{h.c.}] - m_0^2Z_\phi|\phi_0|^2 - \lambda_{1,0}^2Z_\phi^2|\phi_0|^4.\end{aligned}$$

To deduce the additional counterterms, first observe that $\mathcal{L}_0 = \mathcal{L} + \mathcal{L}_{\text{c.t.}}$, this decomposition therefore suggests the relations

$$Z_\phi = 1 + \delta_\phi, \quad Z_\psi = 1 + \delta_\psi,$$

$$\lambda_{1,0}^2Z_\phi^2 = \lambda_1^2 + \delta_1, \quad \lambda_{2,0}Z_\phi^{\frac{1}{2}}Z_\psi = \lambda_2 + \delta \quad \text{and} \quad m_0^2Z_\phi = m^2 + \delta_m.$$

Since $\mathcal{L}_{\text{c.t.}} = \mathcal{L}_0 - \mathcal{L}$ we can rearrange to deduce the respective definitions of the counterterms:

$$\delta_\phi := Z_\phi - 1, \quad \delta_\psi := Z_\psi - 1,$$

$$\delta_1 := \lambda_1^2 - \lambda_{1,0}^2Z_\phi^2 = \lambda_1^2 - \lambda_{1,0}^2Z_1, \quad \delta_2 := \lambda_{2,0}Z_\phi^{\frac{1}{2}}Z_\psi - \lambda_2\delta = \lambda_{2,0}Z_2 - \lambda_2 \quad \text{and} \quad \delta_m := m_0^2Z_\phi - m^2,$$

with $Z_1 := Z_\phi^2$ and $Z_2 := Z_\phi^{\frac{1}{2}}Z_\psi$. The above relations can then be substituted into the bare Lagrangian which gives

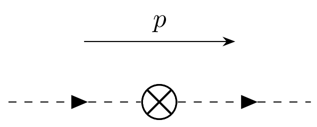
$$\begin{aligned}\mathcal{L}_0 &= (1+\delta_\phi)|\partial\phi|^2 + (1+\delta_\psi)\psi^\dagger(i\overline{\sigma}\cdot\partial)\psi + [i(\lambda_2+\delta_2)\psi^\dagger\sigma^2\psi\phi + \textbf{h.c.}] - (m^2+\delta_m)|\phi|^2 - (\lambda_1^2+\delta_1)|\phi|^4 \\ &= \mathcal{L} + \delta_\phi|\partial\phi|^2 + \delta_\psi\psi^\dagger(i\overline{\sigma}\cdot\partial)\psi + [i\delta_2\psi^\dagger\sigma^2\psi\phi + \textbf{h.c.}] - \delta_m|\phi|^2 - \delta_1|\phi|^4.\end{aligned}$$

Therefore the counterterm Lagrangian is given by

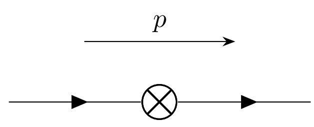
$$\mathcal{L}_{\text{c.t.}} = \delta_\phi|\partial\phi|^2 + \delta_\psi\psi^\dagger(i\overline{\sigma}\cdot\partial)\psi + [i\delta_2\psi^\dagger\sigma^2\psi\phi + \textbf{h.c.}] - \delta_m|\phi|^2 - \delta_1|\phi|^4.$$

From the counterterm Lagrangian the Feynman diagram and rules for the counterterm insertions can be read off as

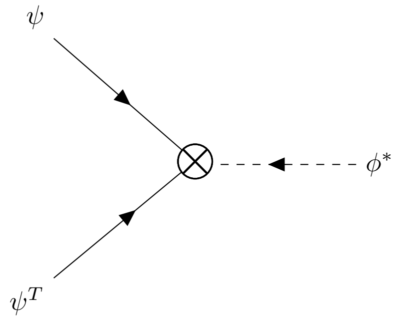
[3] The advantage of using the $\overline{\text{MS}}$ scheme is that it is makes it easier to deduce the counterterms, additionally the 1-loop expressions also have a simpler form. The disadvantage is that it is not strictly true that the renormalized mass is the physical mass. An on-shell subtraction scheme can be used alternatively, however this scheme has the opposite advantages and disadvantages of the former.



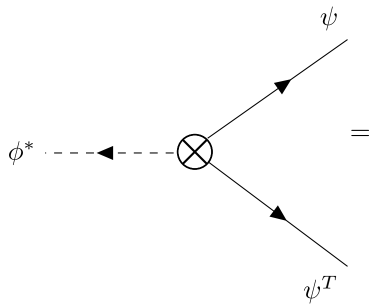
$= i(p^2\delta_\phi - \delta_m)$, and



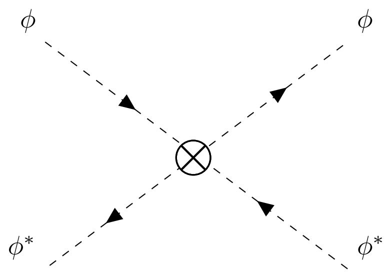
$= i\delta_\psi \sigma \cdot p.$



$= -2\delta_2\sigma^2,$



$= 2\delta_2\sigma^2$



$= -4i\delta_1$

We need to compute the radiative corrections to each of the amputated vertices adding the counterterm insertion.

The three-point function δ_2 is zero as it's clear that neither $\psi\psi^T \mapsto \phi^*$ or $\phi^* \mapsto \psi\psi^T$ don't have radiative corrections at one-loop. This is because loop cannot be formed in such a way with the diagrams that keeps the orientations of the edges consistent.

Next we derive counterterms for the scalar 2-point function. In the previous section we computed the bosonic self energy however now we must now also take into account the Feynman diagram associated to the δ_ϕ and δ_m counterterm insertions. Therefore

$$
\begin{aligned}
-i\Pi(p^2) &= -i\Pi_\phi - i\Pi_\psi(p^2) + i(p^2\delta_\phi - \delta_m) \\
&= \frac{i\lambda_1^2 m^2}{4\pi^2}\left[\frac{2}{\varepsilon} + \log\left(\frac{\tilde{\mu}^2}{m^2}\right) + 1\right] - \frac{3i\lambda_2^2}{4\pi^2}\int_0^1 dx\Delta\left[\frac{2}{\varepsilon} + \log\left(\frac{\tilde{\mu}^2}{\Delta}\right) + \frac{1}{3}\right] + i(p^2\delta_\phi - \delta_m),
\end{aligned}
$$

since our counterterms should cancel these loop corrections, we set the respective counterterms equal to

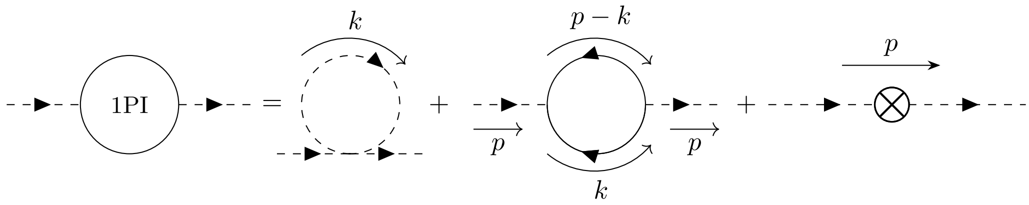
$$
-i\delta_m = i\Pi_\phi \quad \text{and} \quad -i\delta_\phi = i\lim_{p^2 \longrightarrow 0}\frac{\partial \Pi_\psi}{\partial p^2}.
$$

Hence

$$
\delta_m = \Pi_\phi = \frac{\lambda_1^2 m^2}{4\pi^2}\left[\frac{2}{\varepsilon} + \log\left(\frac{\tilde{\mu}^2}{m^2}\right) + 1\right],
$$

applying the $\overline{\text{MS}}$ scheme the counterterm δ_m is redefined using the above equation but with finite parts dropped and with universal constants included

$$
\delta_m := \frac{\lambda_1^2 m^2}{2\pi^2\varepsilon} + \log(4\pi e^{-\gamma_E}).
$$



Similarly,

$$\begin{aligned}
\delta_\phi &= \lim_{p^2\to 0}\frac{\partial}{\partial p^2}\left(\frac{-3\lambda_2^2 i}{4\pi^2}\int_0^1 dx\ \Delta\left[\frac{2}{\varepsilon}+\log(\tilde{\mu}^2)+\frac{1}{3}-\log(\Delta)\right]\right)\\
&= \frac{-3\lambda_2^2 i}{4\pi^2}\int_0^1 dx\ x(x-1)\left[\frac{2}{\varepsilon}+\log(\tilde{\mu}^2)-1-\lim_{p^2\to 0}\frac{\partial}{\partial p^2}\left(p^2\log(p^2x(x-1))\right)\right]\\
&= \frac{-3\lambda_2^2 i}{4\pi^2}\int_0^1 dx\ x(x-1)\left[\frac{2}{\varepsilon}+\log(\tilde{\mu}^2)-\frac{2}{3}-\lim_{p^2\to 0}\log(\Delta)\right].
\end{aligned}$$

As $p^2\to 0$, the term $\log(\Delta)\to-\infty$ and so there is an IR divergence. To regulate this we now introduce a fictitious fermion mass term M_ψ such that $M_\psi^2 << \mu^2$. The Δ above can then be replaced with the IR regulated Δ given by

$$\Delta := \Delta_{\text{IR}} = p^2x(x-1)+M_\psi^2.$$

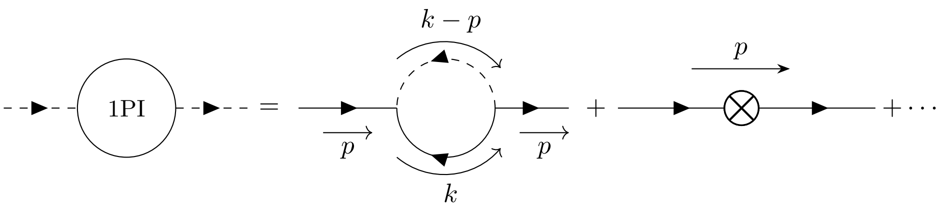
Now we can take $p^2\to 0$ to obtain a finite expression that isolates the ultraviolet pole and keeps the IR regulator

$$\begin{aligned}
\delta_\phi^{\text{reg}} &= \frac{-3\lambda_2^2}{4\pi^2}\int_0^1 dx\ x(x-1)\left[\frac{2}{\varepsilon}+\log(\tilde{\mu}^2)-\frac{2}{3}-\log(M_\psi^2)\right]\\
&= \frac{\lambda_2^2}{8\pi^2}\left[\frac{2}{\varepsilon}+\log(\tilde{\mu}^2)-\frac{2}{3}-\log(M_\psi^2)\right],\ \text{as}\ \int_0^1 x(x-1)dx=\frac{1}{6}.
\end{aligned}$$

Applying the $\overline{\text{MS}}$ scheme the corresponding counterterm δ_ϕ above is redefined dropping the finite part and adding in the universal constants

$$\delta_\phi := \frac{\lambda_2^2}{4\pi^2\varepsilon}+\log(4\pi e^{-\gamma_E}).$$

Next to determine is the δ_ψ counterterm. The fermion self energy at 1-loop is given by which mathematically



corresponds to $-i\Sigma(p)=-i\Sigma_1(p)+i\delta_\psi\sigma\cdot p$. In order to determine δ_ψ, we must compute the contribution to the 1-loop amplitude given by $-i\Sigma_1(p)$. Using the Feynman rules and noting that the sign coming from the fermion interchange in $\psi^\dagger\sigma^2\psi^*\psi^T\sigma^2\psi$, yields

$$\begin{aligned}
-i\Sigma_1(p) &= 4\lambda_2^2\int\frac{d^4k}{(2\pi)^4}\frac{(i)^2k_\mu\sigma^2\overline{\sigma}^\mu\sigma^2}{k^2[(k-p)^2-m^2]}\\
&= -4\lambda_2^2\int\frac{d^4k}{(2\pi)^4}\frac{k_\mu\sigma^\mu}{k^2[(k-p)^2-m^2]}\quad\text{since}\ \sigma^2\overline{\sigma}^\mu\sigma^2=\sigma^\mu.
\end{aligned}$$

Using Feynman parameters where $A = k^2$ and $B = (k-p)^2 - m^2$ we obtain

$$\begin{aligned} A + (B-A)x &= k^2 + ((k-p)^2 - k^2 - m^2)x \\ &= k^2 + xp^2 - 2xkp - m^2 x \\ &= (k - xp)^2 - x^2p^2 + xp^2 - m^2 x. \end{aligned}$$

Let $\Delta := x^2 p(1-x) + m^2 x$, and let $\ell = k - xp$ then $d^4\ell = d^4 k$, substituting this into our expression gives

$$-i\Sigma_1(p) = -4\lambda_2^2 \int_0^1 dx \int \frac{d^4\ell}{(2\pi)^4} \frac{(\ell + xp)_\mu \sigma^\mu}{(\ell^2 - \Delta)^2} = -4\lambda_2^2 (\sigma \cdot p) \int_0^1 dx \int \frac{d^4\ell}{(2\pi)^4} \frac{x}{(\ell^2 - \Delta)^2},$$

where the integral with ℓ to the odd power is odd and hence vanishes. Performing a Wick rotation: set $\ell_E^0 := -i\ell^0$ and set $\ell_E^i := \ell^i$ then $d^4\ell = id^4\ell_E$ and $\ell_E^2 = -\ell^2$, so

$$-i\Sigma_1(p) = -4i\lambda_2^2 (\sigma \cdot p) \int_0^1 dx\, x \int \frac{d^4\ell_E}{(2\pi)^4} \frac{1}{(\ell_E^2 + \Delta)^2}.$$

Applying dimensional regularization, where $d = 4 - \varepsilon$ and using the standard identities, expanding to order ε gives

$$-i\Sigma_1(p) = \frac{-i\lambda_2^2(\sigma \cdot p)}{4\pi^2} \int_0^1 dx\, x \left[\frac{2}{\varepsilon} + \log\left(\frac{\tilde{\mu}}{\Delta}\right) + O(\varepsilon)\right] = \frac{-i\lambda_2^2(\sigma \cdot p)}{8\pi^2} \left[\frac{2}{\varepsilon} + \log\left(\frac{\tilde{\mu}}{\Delta}\right) + O(\varepsilon)\right],$$

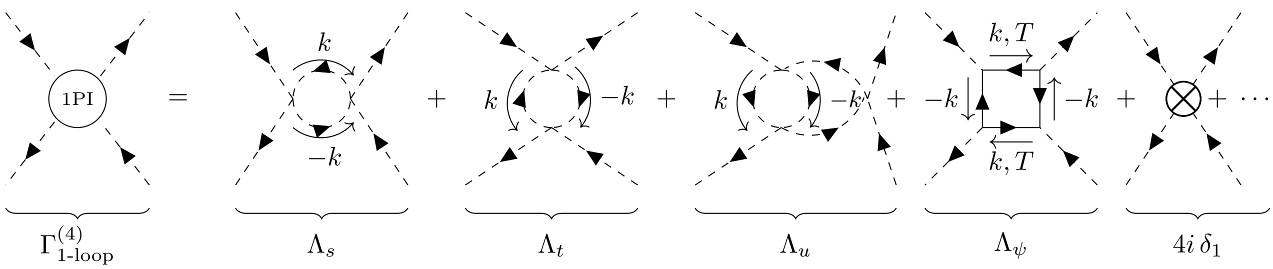
hence our counterterm for ψ should be equal to

$$\delta_\psi = \frac{\lambda_2^2}{8\pi^2} \left[\frac{2}{\varepsilon} + \log\left(\frac{\tilde{\mu}}{\Delta}\right) + O(\varepsilon)\right].$$

Again, applying the $\overline{\text{MS}}$ scheme the corresponding counterterm δ_ψ above is redefined dropping the finite part and adding in the universal constants

$$\delta_\psi := \frac{\lambda_2^2}{4\pi^2 \varepsilon} + \log(4\pi e^{-\gamma_E}).$$

Next we compute the δ_1 counterterm by considering a 1-loop correction to the 4-point scalar interaction. Setting the external momenta to zero, we have at 1-loop that:



$$i\Gamma^{(4)}_{\text{1-loop}} = i\Lambda_s + i\Lambda_t + i\Lambda_u + i\Lambda_\psi + 4i\delta_1.$$

Computing the $i\Lambda_\psi$ contribution using Feynman rules (and taking into account the sign from fermion interchange) yields

$$\begin{aligned} i\Lambda_\psi &= (-1)(-2\lambda_2)^2(2\lambda_2)^2 \int \frac{d^4k}{(2\pi)^4} i^4 \, \mathrm{Tr}\big(\sigma^2[\overline{\sigma}^\mu]^T \sigma^2 \overline{\sigma}^\nu \sigma^2 [\overline{\sigma}^\rho]^T \sigma^\lambda\big) \frac{k_\mu(-k_\nu)\, k_\rho(-k_\lambda)}{k^8} \\ &= -16\lambda_2^2 \int \frac{d^4k}{(2\pi)^4} \mathrm{Tr}\big(\sigma^2[\overline{\sigma}^\mu]^T \sigma^2 \overline{\sigma}^\nu \sigma^2 [\overline{\sigma}^\rho]^T \sigma^\lambda\big) \frac{k_\mu k_\nu k_\rho k_\lambda}{k^8}. \end{aligned}$$

The expression using the trace simplifies using the fact that $\sigma^2[\overline{\sigma}^\mu]^T\sigma^2 = \sigma^\mu$ together with an identity for the trace

$$\begin{aligned}\operatorname{Tr}\!\big(\sigma^2[\overline{\sigma}^\mu]^T\sigma^2\overline{\sigma}^\nu\sigma^2[\overline{\sigma}^\rho]^T\sigma^\lambda\big) &= \operatorname{Tr}\!\big(\sigma^\mu\overline{\sigma}^\nu\sigma^\rho\overline{\sigma}^\lambda\big)\\ &= 2\big[g^{\mu\nu}g^{\lambda\rho} + g^{\mu\rho}g^{\nu\lambda} - g^{\mu\lambda}g^{\nu\rho} - i\,\epsilon^{\mu\nu\lambda\rho}\big]\,.\end{aligned}$$

Now, using the fact that $k_\mu g^{\mu\nu}k_\nu = k\cdot k = k^2$, we simplify

$$\begin{aligned}i\Lambda_\psi &= -32\lambda_2^2\int\frac{d^4k}{(2\pi)^4}\big(g^{\mu\nu}g^{\lambda\rho} + g^{\mu\rho}g^{\nu\lambda} - g^{\mu\lambda}g^{\nu\rho} - i\,\epsilon^{\mu\nu\lambda\rho}\big)\frac{k_\mu k_\nu k_\rho k_\lambda}{k^8}\\ &= -32\lambda_2^2\int\frac{d^4k}{(2\pi)^4}\frac{1}{k^4},\end{aligned}$$

Next we introduce a mass regulator M, such that $M^2 << \mu^2$ to regulate the IR divergence

$$i\Lambda_\psi = -32\lambda_2^2\int\frac{d^4k}{(2\pi)^4}\frac{1}{(k^2-M^2)^2} = -32\lambda_2^2 i\int\frac{d^4k_E}{(2\pi)^4}\frac{1}{(k_E^2+M^2)^2},$$

where we Wick rotated, setting $k_E^0 := -ik^0$, $k_E^i := k^i$ so $id^4k_E = d^4k$ and $k^2 = -k_E^2$. Now, together with our together with our standard d-dimensional integral identities we dimensionally regularise this loop integral. Setting $d = 4-\varepsilon$ and expanding up to first order gives

$$i\Lambda_\psi = -32i\lambda_2^4(\mu^2)^{4-d}\int\frac{d^dk_E}{(2\pi)^d}\frac{1}{(k^2+M^2)^2} = -\frac{2i\lambda_2^4}{\pi^2}\left[\frac{2}{\varepsilon} + \log\left(\frac{\tilde{\mu}^2}{M^2}\right) + O(\varepsilon)\right].$$

Similarly, to compute the $i\Lambda_s$ we apply the Feynman rules to the s-channel diagram to get our loop expression. Performing the calculation just as we did above and expanding to first order we obtain

$$\begin{aligned}i\Lambda_s &= (-4i\lambda_1)^2\int\frac{d^4k}{(2\pi)^4}\frac{(i^2)}{(k^2-m^2)^2}\\ &= \frac{i\lambda_1^4}{\pi^2}\left[\frac{2}{\varepsilon} + \log\left(\frac{\tilde{\mu}^2}{m^2}\right) + O(\varepsilon)\right].\end{aligned}$$

Again, applying the Feynman rules to the t and u-channel diagrams, gives expression of the form

$$i\Lambda_t = i\Lambda_s \quad \text{and} \quad i\Lambda_u = \frac{1}{2}i\Lambda_s.$$

Putting everything together, the 1-loop amplitude for the 4-point function is found to be

$$\begin{aligned}i\Gamma^{(4)}_{\text{1-loop}} &= i\Lambda_s + i\Lambda_t + i\Lambda_u + i\Lambda_\psi + 4i\delta_1\\ &= \frac{5}{2}i\Lambda_s + i\Lambda_\psi + 4i\delta_1.\end{aligned}$$

Hence, our counterterm must be of the form

$$\begin{aligned}\delta_1 &= -\frac{1}{4}\left[\frac{5}{2}\Lambda_s + \Lambda_\psi\right]\\ &= -\frac{1}{4}\left[\frac{5\lambda_1^4}{2\pi^2}\left(\frac{2}{\varepsilon} + \log\left(\frac{\tilde{\mu}}{m^2}\right)\right) - \frac{2\lambda_2^4}{\pi^2}\left(\frac{2}{\varepsilon} + \log\left(\frac{\tilde{\mu}^2}{M^2}\right)\right)\right]\end{aligned}$$

Since we are in the $\overline{\text{MS}}$ scheme the counterterm is then redefined, dropping finite terms, and adding universal constants. Therefore our final counterterm in the renormalized PT theory is given by

$$\delta_1 := -\frac{1}{4\pi^2\varepsilon}\left(5\lambda_1^4 - 4\lambda_2^4\right) + \log(4\pi e^{-\gamma_E}).$$

4 One-Loop β-Functions for $\mathcal{L}$ and RG Flow

To compute the β functions for the parameters in our theory we use the Callman-Symanzik equation,

$$\beta_\lambda = \mu\frac{\partial}{\partial\mu}\left(-\delta_\lambda + \frac{1}{2}\lambda\sum_i \delta_{Z_i}\right) = -2B - \lambda\sum_i A_i$$

where $\delta_\lambda = B\frac{2}{\varepsilon}$ and $\delta_{Z_i} = -A_i\frac{2}{\varepsilon}$. Recall the counterterms we deduced for $\mathcal{L}$ in the previous section:

$$\delta_\psi = \frac{\lambda_2^2}{4\pi^2\varepsilon} + \log(4\pi e^{-\gamma_E}), \quad \delta_m = \frac{\lambda_1^2 m^2}{2\pi^2\varepsilon} + \log(4\pi e^{-\gamma_E}), \quad \delta_\phi = \frac{\lambda_2^2}{4\pi^2\varepsilon} + \log(4\pi e^{-\gamma_E}),$$

$$\delta_1 = -\frac{1}{4\pi^2\varepsilon}\left(5\lambda_1^4 - 4\lambda_2^4\right) + \log(4\pi e^{-\gamma_E}) \quad \text{and} \quad \delta_2 = 0.$$

The beta function for λ_1^2 is then found by using the formula above with $B = \frac{-1}{8\pi^2}(5\lambda_1^4 - 4\lambda_2^4)$, where we sum over four external legs in the self interaction vertex each contributing $A_\phi = -\frac{\lambda_2^2}{8\pi^2}$ that is

$$\begin{aligned}
\beta_{\lambda_1^2}(\lambda_1^2, \lambda_2^2) &= -2\left(\frac{-1}{8\pi^2}(5\lambda_1^4 - 4\lambda_2^4)\right) - \lambda_1^2\sum_{i=1}^4 \frac{\lambda_2^2}{8\pi^2} \\
&= \frac{1}{4\pi^2}(5\lambda_1^4 - 4\lambda_2^4) + 4\lambda_1^2\left(\frac{\lambda_2^2}{8\pi^2}\right) \\
&= \frac{1}{4\pi^2}(\lambda_1^4 + 4\lambda_1^2\lambda_2^2 - 4(\lambda_1^2 - \lambda_2^2)).
\end{aligned}$$

To compute the beta function for λ_2 we again use the same formula however this time $B = 0$ as $\delta_2 = 0$,

$$\beta_{\lambda_2}(\lambda_1^2, \lambda_2^2) = -2B - \lambda_2^2\sum_{i=1}^3 A_i = -\lambda_2^2\left[2\left(-\frac{\lambda_2^2}{8\pi^2}\right) - \frac{\lambda_2^2}{8\pi^2}\right] = \frac{3\lambda_2^3}{8\pi^2}.$$

Having obtained β_{λ_2} for the linear coupling, we now use the chain rule to translate it into the flow of its square

$$\beta_{\lambda_2^2} = \mu\frac{\partial}{\partial\mu}(\lambda_2^2(\mu)) = 2\lambda_2\mu\frac{\partial}{\partial\mu}(\lambda_2(\mu)) = 2\lambda_2\beta_{\lambda_2} = \frac{3\lambda_2^4}{4\pi^2}.$$

Next we analyse the renormalization group flow of the beta functions, in particular we consider whether the condition $\lambda_1^2 = \lambda_2^2$ is stable under renormalization group flow. Let

$$\Delta(\mu) := \lambda_1^2(\mu) - \lambda_2^2(\mu),$$

then if $\Delta = 0$ stays zero for all scales, the condition is said to be preserved. Let $t = \log(\mu)$ the RG equation gives

$$\begin{aligned}
\frac{d\Delta}{dt} &= \frac{d}{dt}(\lambda_1^2 - \lambda_2^2) \\
&= \beta_{\lambda_1^2}(\lambda_1^2, \lambda_2^2) - \beta_{\lambda_2^2}(\lambda_1^2, \lambda_2^2) \\
&= \frac{1}{4\pi^2}(\lambda_1^4 + 4\lambda_1^2\lambda_2^2 + 4(\lambda_1^2 - \lambda_2^2)) - \frac{3\lambda_2^4}{4\pi^2} \\
&= \frac{1}{4\pi^2}((\lambda_1^2 - \lambda_2^2)(\lambda_1^2 + 3\lambda_2^2) + 4(\lambda_1^2 - \lambda_2^2)(\lambda_1^2 + \lambda_2^2)) \\
&= \frac{(5\lambda_1^2 + 7\lambda_2^2)}{4\pi^2}\Delta.
\end{aligned}$$

Since $\frac{5\lambda_1^2+7\lambda_2^2}{4\pi^2} > 0$, by the above $\frac{d\Delta}{dt} = 0$ if $\Delta = 0$ for all t, and hence all scales μ. Therefore the condition $\lambda_1 = \lambda_2$ is preserved under renormalization group evolution. Moreover we see that $|\Delta|$ deceases in the IR and increases in the UV.