Bloch Groups and Dilogarithms over Finite Fields

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Introduction: What is the Pre-Bloch Group?

The pre-Bloch group:

- Connected to Polylogarithms, and Geometry
- Studied with Algebraic K-theory.

One existing result from Hutchinson is:

- The pre-Bloch group over a finite field is cyclic, and thus finite.

This project sought to examine the following question:

- Can it be shown that the pre-Bloch group over a finite field is finite using only linear algebra?

Why are we interested in this:

- It would give us information on the underlying structure of the pre-Bloch group.

Relations and Presentation Matrices

Consider the $\mathbb{Z}$-module generated by $G = \{x_1, x_2, x_3\}$ which is subject to the following system of equations:

$$r_1(x) = x_1 + 17x_2 - x_3 = 0$$

$$r_2(x) = x_1 - 6x_2 - x_3 = 0$$

$$r_3(x) = -x_1 - x_2 + x_3 = 0$$

We call each equation a "relation". We can represent this information with a matrix M, which we call a presentation matrix. In this example:

$$M = \begin{pmatrix} 1 & 1 & -1 \\ 17 & -6 & -1 \\ -1 & -1 & 1 \end{pmatrix}$$

Our module is then the quotient $\mathbb{Z}[G]/M\mathbb{Z}[G]$ (the cokernel of M). We say this module is 'presented' by M. This module is finite if and only if the rank of M is maximal.

The Pre-Bloch Group over a Field

We define the Pre-Bloch Group over a Field using this same idea.

Definition: The Five-Term Relation, $R(x, y)$

Let $R(x, y)$, such that $x, y \in \mathbb{F}\backslash\{0, 1\}$ and $x \neq y$, be an element in $\mathbb{Z}[\mathbb{F}\backslash\{0, 1\}]$ such that:

$$R(x, y) = [x] - [y] + [\frac{y}{x}] - [\frac{1 - x^{-1}}{1 - y^{-1}}] + [\frac{1 - x}{1 - y}] \quad (1)$$

We will call $R(x, y)$ the "five term relation".

Definition: The Pre-Bloch Group over a field, $\mathcal{P}(\mathbb{F})$

The pre-Bloch group $\mathcal{P}(\mathbb{F})$ is the abelian group generated by the symbols $[x]$, with $x \in \mathbb{F}\backslash\{0, 1\}$, subject to the relations $R(x, y) = 0$

The Pre-Bloch Group - Example

Example: What is a matrix which presents $\mathcal{P}(\mathbb{F}_5)$?

- We recall $\mathbb{F}_5 = \{0, 1, 2, 3, 4\}$ and thus $\mathbb{Z}[\mathbb{F}_5 \backslash \{0, 1\}] = \mathbb{Z}[\{2, 3, 4\}]$
- We then determine all of our relations:

$$R(x, y) = [x] - [y] + [\frac{y}{x}] - [\frac{1 - x^{-1}}{1 - y^{-1}}] + [\frac{1 - x}{1 - y}]$$

$$R(2, 3) = [2] - [3] + [\frac{3}{2}] - [\frac{1 - 2^{-1}}{1 - 3^{-1}}] + [\frac{1 - 2}{1 - 3}] = 0$$

$$R(2, 3) = [2] - [3] + [3 \cdot 3] - [\frac{1 - 3}{1 - 2}] + [\frac{-1}{-2}] = 0$$

$$R(2, 3) = [2] - [3] + [4] - [2] + [3] = 0[2] + 0[3] + 1[4] = 0$$

The Pre-Bloch Group - Example $\mathcal{P}(\mathbb{F}_5)$

Example (cont.): What is a matrix which presents $\mathcal{P}(\mathbb{F}_5)$?

$$R(2,4) = [2] - [4] + [\frac{4}{2}] - [\frac{1-2^{-1}}{1-4^{-1}}] + [\frac{1-2}{1-4}] = 3[2] + 0[3] - 2[4] = 0$$

$$R(3,2) = [3] - [2] + [\frac{2}{3}] - [\frac{1-3^{-1}}{1-2^{-1}}] + [\frac{1-3}{1-2}] = 0[2] + 0[3] + 1[4] = 0$$

$$R(3,4) = [3] - [4] + [\frac{4}{3}] - [\frac{1-3^{-1}}{1-4^{-1}}] + [\frac{1-3}{1-4}] = -1[2] + 2[3] + 0[4] = 0$$

$$R(4,2) = [4] - [2] + [\frac{2}{4}] - [\frac{1-4^{-1}}{1-2^{-1}}] + [\frac{1-4}{1-2}] = -1[2] + 2[3] + 0[4] = 0$$

$$R(4,3) = [4] - [3] + [\frac{3}{4}] - [\frac{1-4^{-1}}{1-3^{-1}}] + [\frac{1-4}{1-3}] = 1[2] - 2[3] + 2[4] = 0$$

Example (cont.): What is a matrix which presents $\mathcal{P}(\mathbb{F}_5)$?

- Thus a presentation matrix of $\mathcal{P}(\mathbb{F}_5)$ is:

$$M = \begin{pmatrix} 0 & 3 & 0 & -1 & -1 & 1 \\ 0 & 0 & 0 & 2 & 2 & -2 \\ 1 & -2 & 1 & 0 & 0 & 2 \end{pmatrix}$$

- And finally $\mathcal{P}(\mathbb{F}_5)$ is the group $\mathbb{Z}[\mathbb{F}_5 \backslash \{0,1\}]/M \cdot \mathbb{Z}[\mathbb{F}_5 \backslash \{0,1\}]$

Note that there are $(q-2)\cdot(q-3)$ relations for each $\mathbb{F}_q$.

- We have $q-2$ generators from which to choose x.
- We then have $q-3$ remaining choices for y (as $y \neq x$)

Goal, Challenges, and Solutions

Goal:

1. $\mathcal{P}(\mathbb{F}_q)$ is finite
2. $\mathrm{rank}(P_q) = q - 2$
3. A consistent set of $q - 2$ relations which are linearly independent.

- Challenge 1: Time consuming
- Solution: Created Mathematica Program
- Challenge 2: Tough to find $q - 2$ consistent linearly independent relations. Can't simply take the first $q - 2$ relations.
- Solution: Three approaches were taken to resolve this problem
  - Found a consistent set of $\frac{q-3}{2}$ linearly independent relations.
  - Found a suspected set of $q - 3$ linearly independent relations.
  - Found a different approach with interesting observations. $(\star)$

Presentation Matrix has Constant Row and Column Sums

To begin our path towards proving the rank is maximal, we first note we can rewrite our five term relation using powers of a primitive root as:

$$R(r^i, r^j) = [r^i] - [r^j] + [r^{j-i}] - [\frac{1-r^{-i}}{1-r^{-j}}] + [\frac{1-r^i}{1-r^j}]$$

We then use this to order the Presentation Matrix (which we call M_q when ordered in this manner) as:

$$M_q := \begin{pmatrix} \uparrow & \dots & \uparrow & \uparrow & \dots & \uparrow & \dots & \uparrow & \dots & \uparrow \\ R(r^1, r^2) & \dots & R(r^1, r^{q-2}) & R(r^2, r^1) & \dots & R(r^2, r^{q-2}) & \dots & R(r^{q-2}, r^1) & \dots & R(r^{q-2}, r^{q-3}) \\ \downarrow & \dots & \downarrow & \downarrow & \dots & \downarrow & \dots & \downarrow & \dots & \downarrow \end{pmatrix}$$

We then will look at some examples to motivate an identity which will be the key to this approach.

$$M_4 = \begin{pmatrix} 3 & -2 \\ -2 & 3 \end{pmatrix} M_5 = \begin{pmatrix} 3 & 0 & -1 & 1 & 0 & -1 \\ -2 & 1 & 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & -2 & 0 & 2 \end{pmatrix}$$

Presentation Matrix has Constant Row and Column Sums

Theorem: The sum of all coefficients of $[x]$ is constant

Let F_q be a field. Then:

$$\sum_{x\in\mathbb{F}_q\setminus\{0,1\}} \Big(\sum_{y\in\mathbb{F}_q\setminus\{0,1,x\}} R(x,y)\Big) = \sum_{x\in\mathbb{F}_q\setminus\{0,1\}} (q-3)[x] \tag{2}$$

Proof:

$$R(x,y) = [x] - [y] + [\frac{y}{x}] - [\frac{1-x^{-1}}{1-y^{-1}}] + [\frac{1-x}{1-y}]$$

If we fix x and sum over all y we get

$$\sum_{y\in\mathbb{F}_q\setminus\{0,1,x\}} R(x,y) = \sum_{y\in\mathbb{F}_q\setminus\{0,1,x\}} ([x] - [y] + [\frac{y}{x}] - [\frac{1-x^{-1}}{1-y^{-1}}] + [\frac{1-x}{1-y}])$$

Presentation Matrix has Constant Row and Column Sums

Note that x is invariant here, and we've $q-3$ sums so:

$$\sum_{y\in\mathbb{F}_q\setminus\{0,1,x\}} R(x,y) = (q-3)[x] + \sum_{y\in\mathbb{F}_q\setminus\{0,1,x\}} (-[y]+[\frac{y}{x}]-[\frac{1-x^{-1}}{1-y^{-1}}]+[\frac{1-x}{1-y}])$$

If we sum over all x we get $\sum_{x\in\mathbb{F}_q\setminus\{0,1\}} (\sum_{y\in\mathbb{F}_q\setminus\{0,1,x\}} R(x,y))$

$$= \sum_{x\in\mathbb{F}_q\setminus\{0,1\}} (q-3)[x] + \sum_{x\in\mathbb{F}_q\setminus\{0,1\}} (\sum_{y\in\mathbb{F}_q\setminus\{0,1,x\}} (-[y]+[\frac{y}{x}]-[\frac{1-x^{-1}}{1-y^{-1}}]+[\frac{1-x}{1-y}]))$$

$$\implies \sum_{x\in\mathbb{F}_q\setminus\{0,1\}} (\sum_{y\in\mathbb{F}_q\setminus\{0,1,x\}} R(x,y)) = \sum_{x\in\mathbb{F}_q\setminus\{0,1\}} (q-3)[x]$$

The Matrix MT$_q$

Prompted by the difficulty in finding $(q-2)$ linearly independent columns, we looked at another matrix using the following result:

$$\mathrm{rank}(A) = \mathrm{rank}(A \cdot A^T)$$

Using this we turn our attention to the $(q-2) \times (q-2)$ matrix:

$$MT_q := (M_q) \cdot (M_q)^T$$

If we show this has maximum rank we have proven our achieved. Explicitly, with $a_{i,j}$ in M_q and $m_{i,j}$ in MT_q, we have:

$$MT_q = \begin{pmatrix} \sum_{j=1}^{v}(a_{1,j})^2 & \sum_{j=1}^{v} a_{1,j}a_{2,j} & \cdots & \sum_{j=1}^{v} a_{1,j}a_{q-2,j} \\ \sum_{j=1}^{v} a_{2,j}a_{1,j} & \sum_{j=1}^{v}(a_{2,j})^2 & \cdots & \sum_{j=1}^{v} a_{2,j}a_{q-2,j} \\ \vdots & \vdots & \ddots & \vdots \\ \sum_{j=1}^{v} a_{q-2,j}a_{1,j} & \sum_{j=1}^{v} a_{q-2,j}a_{2,j} & \cdots & \sum_{j=1}^{v}(a_{q-2,j})^2 \end{pmatrix}$$

The Matrix MT_q

$$\begin{pmatrix}
74 & -4 & -6 & -4 & -4 & -6 & -4 & -4 & -4 & -2 & -4 & -4 & -2 & -4 & -8 \\
-4 & 72 & -4 & -4 & -4 & -4 & -4 & -4 & -4 & -4 & -4 & -4 & -4 & -6 & -4 \\
-6 & -4 & 74 & -4 & -4 & -2 & -4 & -4 & -4 & -6 & -4 & -4 & -8 & -4 & -2 \\
-4 & -4 & -4 & 74 & -6 & -4 & -6 & -4 & -2 & -4 & -2 & -8 & -4 & -4 & -4 \\
-4 & -4 & -4 & -6 & 74 & -4 & -2 & -4 & -6 & -4 & -8 & -2 & -4 & -4 & -4 \\
-6 & -4 & -2 & -4 & -4 & 74 & -4 & -4 & -4 & -8 & -4 & -4 & -6 & -4 & -2 \\
-4 & -4 & -4 & -6 & -2 & -4 & 74 & -4 & -8 & -4 & -6 & -2 & -4 & -4 & -4 \\
-4 & -4 & -4 & -4 & -4 & -4 & -4 & 70 & -4 & -4 & -4 & -4 & -4 & -4 & -4 \\
-4 & -4 & -4 & -2 & -6 & -4 & -8 & -4 & 74 & -4 & -2 & -6 & -4 & -4 & -4 \\
-2 & -4 & -6 & -4 & -4 & -8 & -4 & -4 & -4 & 74 & -4 & -4 & -2 & -4 & -6 \\
-4 & -4 & -4 & -2 & -8 & -4 & -6 & -4 & -2 & -4 & 74 & -6 & -4 & -4 & -4 \\
-4 & -4 & -4 & -8 & -2 & -4 & -2 & -4 & -6 & -4 & -6 & 74 & -4 & -4 & -4 \\
-2 & -4 & -8 & -4 & -4 & -6 & -4 & -4 & -4 & -2 & -4 & -4 & 74 & -4 & -6 \\
-4 & -6 & -4 & -4 & -4 & -4 & -4 & -4 & -4 & -4 & -4 & -4 & -4 & 72 & -4 \\
-8 & -4 & -2 & -4 & -4 & -2 & -4 & -4 & -4 & -6 & -4 & -4 & -6 & -4 & 74
\end{pmatrix}$$

Figure: MT_{17}

The Matrix MT_q

We have proven:

- $\sum_{i=1}^{q-2} m_{i,j} = \sum_{j=1}^{q-2} m_{i,j} = \sum_{i=1}^{q-2}(\sum_{j=1}^{v} a_{s,j}a_{i,j}) = q-3$ for a fixed s.

By using Hutchinson's result that $\mathrm{rank}(M_q) = q-2$ we have proven:

- (MT_q) is invertible.
- $x^T(MT_q)x > 0 \ \forall x \in \mathbb{R}^{q-3}$

It remains to be determined if the following are true for MT_q:

- For $E_q := \{\lambda : \lambda \text{ is an eigenvalue of } MT_q\}$, $\min(E_q) = q-3$
- $2|\sum_{j=1}^{v}(a_{s,j})^2| > \sum_{i=1}^{q-2}|(\sum_{j=1}^{v} a_{s,j}a_{i,j})|$
- $m_{i,j} = \sum_{p=1}^{v} a_{i,p}a_{j,p} \begin{cases} > 0 \text{ if } i = j \\ < 0 \text{ if } i \neq j \end{cases} \qquad (\star)$
- MT_q is bisymmetric

Conclusion:

So to summarise:

1. Want: $\mathcal{P}(\mathbb{F}_q)$ is finite.
2. Found: $\frac{q-3}{2}$ linearly independent relations.
3. Suspect: Set of size $q-3$ is linearly independent, but the proof remains elusive.
4. Found another approach focusing on an interesting matrix, MT_q
5. We have the identity that the sum of the row is $q-3$.
6. If you can show the non-diagonals are negative, you have proven $\mathcal{P}(\mathbb{F}_q)$ is finite, and that MT_q is strictly diagonally dominant, positive definite, and invertible.

Questions:

Thank you for your attention, I would welcome any questions you may have.