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Numbers come in a lot of different forms. We all know that 2 is a number. Many of us have seen π in some capacity and recognize that it is a number. In a similar manner, you may know that $\sqrt{9001}$ and $3i + 17$ are numbers. But without understanding what these numbers mean, they are worthless. The first step in our journey towards establishing meaning for numbers starts with classification of these numbers.

> **Definition 0.1** (Number Set)
> A number set is a collection of numbers.

First definition down. This is a small, but significant definition as well. Number sets will be a recurring character throughout this book.

> **Definition 0.2** (Natural Number)
> A number is called natural if we can use it to count. We denote the set of natural numbers as $\mathbb{N}$.
> $1, 2, 3, 4$, and 5 are examples natural numbers.

We use natural numbers every single day. However, there are some quantities that we cannot describe using natural numbers. Enter the integers.

> **Definition 0.3** (Integer)
> A number is an integer if it has no fractional part. An integer can be either positive or negative. We denote the set of integers as $\mathbb{Z}$.
> $-4, -2, 0, 3, 17, 9002$ are all examples of integers.

Definition overload.

> **Theorem 0.4** (First Theorem)
> This is our first theorem.

bangus

Hello. This is a fucking box.

Problem 0.5 (Dang)
This is a fucking boxed problem.

Problem 0.6 (The Brangle Effect)
Test?

Definition 0.7 (Farster)
Test drangulus. A farster is a powerful, yet extraordinarily weak, contradiction. It is always true, except every single situation.

Definition 0.8
Second definition. Joseph is an amateur.

yecdahiluzhnvurhnuanhurnhruinaufio c ur aiuor gaiu gyioa gria gri sr ya riia irg iayro gaior giagroiua iuor hauio riua ruio hdsui qgyud uqgeyui aiuyeg uyagdui ygruyi ag riy gy gyu y agfyug a u vgika uii gyia y gvaky gkadkya gygfy ar tsh sths rt sts ts gs tbhfhfhfhhfhfhfhfhffhhffhhhfhfhfhfhhfhfhfhfhfhf fhs agid gad ggfd ahfgdk adgfjk agkfh gakjdfh akjgsf kagfagyagykrua gsfhj agky

> This is the killer box that should be on the side of the text.

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Example 0.9
If we take an egg and throw it at someones face, we get:

$$4 + 7 + 2 = 2$$

where 2 is not 2.

Intersections as a Concept

An important idea in math is the idea of a set. Sets are collections of objects. We can talk about the set of flowers, the set of things that are red, or whatever other category of thing we can think of. If something belongs to both sets, we say that it is in the intersection of those sets.

> **Problem 1.1**
>
> What's something that's in the intersection of the set of flowers and the set of things that are red? What would the intersection of the solution sets of two linear equations be like?

We've briefly mentioned sets before.

Recall: What is the solution set for the line $y = -x$? What about the line $y = 2x + 1$?

> **Definition 1.2** (Linear System)
>
> A linear system is made up of multiple linear equations. The solution set for a linear system is the intersection of the solution sets of the linear functions.

Over the course of the past few sections, we've been reinforcing this idea that plotting a linear equation as a line makes some features easier to see, such as other solutions, and overall trends of those solutions. Furthermore, finding the equation that goes with a line that has been plotted gives us some extra power, such as the ability to find solutions that are outside of the plot, or express the facts of the situation in words so that we can communicate these ideas more easily.

Problem 1.3

Find some solutions to the equation $y = 3 - x$. Find some solutions to the equation $y = 2x$.
   How would you find the intersection of their solution sets?

Graphing Linear Systems

We've mentioned before that a coordinate system can be created with any scale we want. The x and y axes don't even need to have the same scale. That idea requires some refinement now: if we put two lines on the same coordinate system, we have to use them same scale for each line. When we do this, if we're plotting the line from a set of points, we need to make sure we finish plotting the first line before moving on to the second.
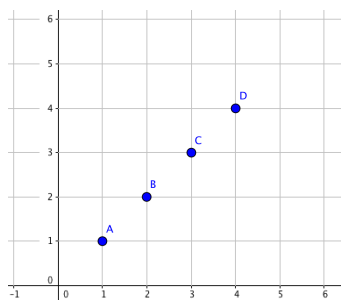
Figure 1.1: Plotting points for the line $y = x$.
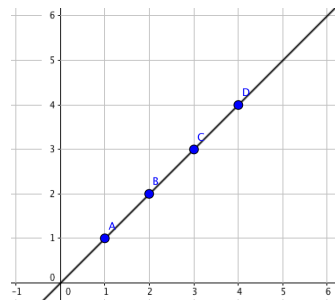
Figure 1.2: The line $y = x$.
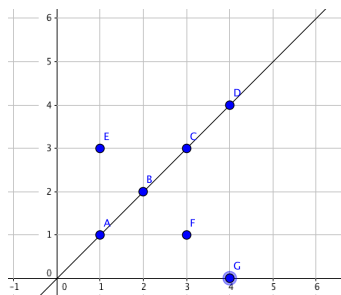
Figure 1.3: Plotting points for a new line.
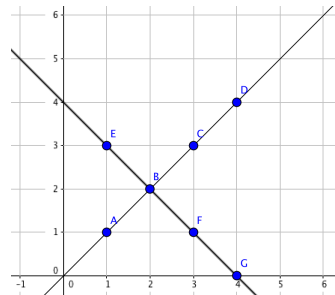
Figure 1.4: The lines $y = x$ and $y = -x + 4$.

It's no coincidence that we call the objects in two sets the "intersection". A single solution in the solution set of an equation is a point on the line that corresponds to that equation. If we have two lines on a graph, the point where they intersect is a point that's on both of the lines at the same time. That means that the point is going to be in the solution set of both equations.

Example 1.4

Find the intersection for the following system of linear equations.

$$y - 4x = -4$$

$$2y + x = 10$$

Using this strategy, we want to plot both equations. To do this, we want them in a form that we can use to plot them, so we'll put them in slope-intercept form. For the first line:

$$\begin{aligned} y - 4x &= -4 \\ y &= 4x - 4 \end{aligned}$$

And for the second line:

$$\begin{aligned} 2y + x &= 10 \\ 2y &= 10 - x \\ y &= 5 - \tfrac{x}{2} \\ y &= -\tfrac{1}{2}x + 5 \end{aligned}$$

Plotting both lines then shows that they have an intersection.
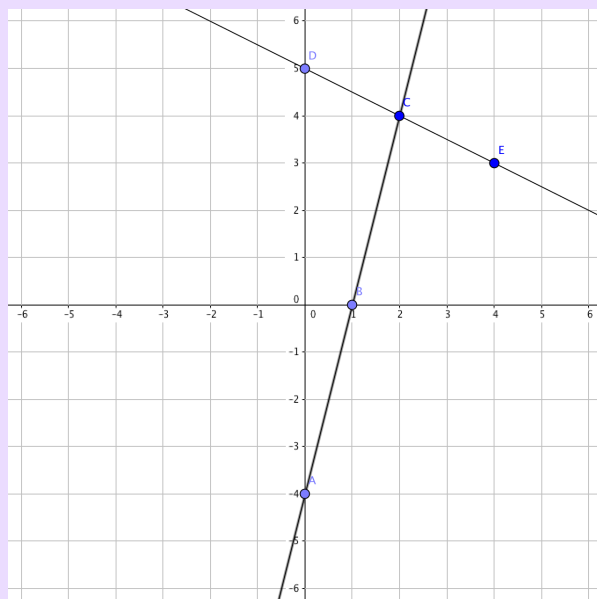
We can then see that the lines intersect at the point $(2,4)$. This means that $(2,4)$ is in the solution set to both equations, and is thereby the solution to this system of equations.

This is a useful tool for finding solutions to systems of linear equations as long as being close is good enough. Unfortunately, some systems of linear equations have solutions that aren't on the grid-lines for a particular scale.

Problem 1.5

Consider the following system of linear equations.

$$y = -\frac{1}{2}x + 3$$

$$y = 2x$$

What do you think the intersection is? What could you do to convince someone who didn't agree?

There are some lines that do not intersect.

[Parallel Lines] We call two lines with the same slope parallel lines. If two lines are parallel, they will never intersect. Likewise, if two lines intersect, then they are not parallel.

Substitution

Although most people aren't accustomed to the idea, there is more than one way to do most things in math. This is a good thing! If the only way to find the solution to a system of equations was to graph them and point at it, it would be pretty difficult to solve a lot of these systems. This is where substitution comes in. Consider two equations in slope-intercept form.

$$y = 2x + 4$$

$$y = 3x + 2$$

If we were to find a solution to this system of equations, there would only be one y-value, and it would apply for both of the equations. In other words, we know that y will have the same value for each equation. Let's say we know that the solution has a y-value of 8. We would then have,

$$8 = 2x + 4$$

$$8 = 3x + 2$$

Well, if the right side of the first equation is equal to 8, and the right side of the second equation is equal to 8, they must be equal to each other! This is the driving force of substitution. If two things have the same value, they're equal to each other.

$$2x + 4 = 3x + 2$$

And this is single linear equation like we had in the second unit! We can solve this.

$$
\begin{aligned}
2x + 4 &= 3x + 2 \\
4 &= x + 2 \\
2 &= x
\end{aligned}
$$

So now we know the solution has an *x*-value of 2. Nice. So what about that *y*-value? Is it really 8? We can plug the *x*-value we just found into one of those equations and find out. We'll choose the first equation.

$$
\begin{aligned}
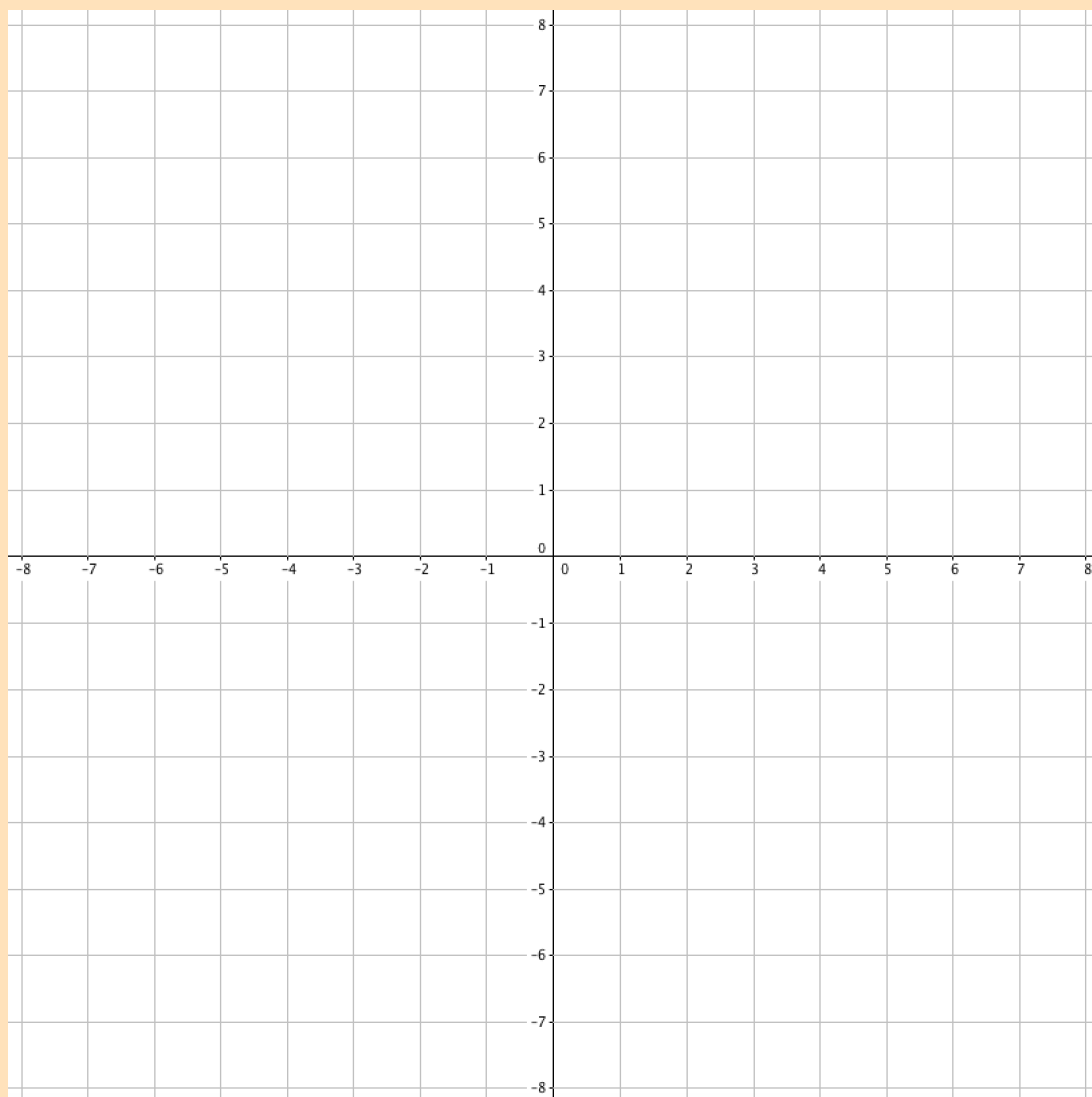y &= 2 \times 2 + 4 \\
y &= 4 + 4 \\
y &= 8
\end{aligned}
$$

Awesome. Now we think the solution has an *x*-value of 2, and a *y*-value of 8. If this is true, then $(2, 8)$ must be a solution to the second equation as well. We can check that.

$$
\begin{aligned}
y &= 3x + 2 \\
8 &= 3 \times 2 + 2 \\
8 &= 6 + 2
\end{aligned}
$$

Everything checks out. We've found a solution to this linear system.

Problem 1.6
Try using the graphing strategy on this linear system. Compare the results.

For this last problem, the solution had nice, clean integer values for the solution. The equations were also given in slope-intercept form. This won't always be the case.

Example 1.7

Solve the system of linear equations.

$$\begin{aligned} 4y - 16x &= 25 \\ 2y + x &= -1 \end{aligned}$$

To use the substitution strategy the way we did last time, we want to solve each equation for either y or x. We had the equations solved for y last time, and solving for y should be natural to us by now, so we'll do that for now.

For the first equation:

$$\begin{aligned} 4y - 16x &= 25 \\ 4y &= 16x + 25 \\ y &= 4x + \tfrac{25}{4} \end{aligned}$$

And for the second equation:

$$\begin{aligned} 2y + x &= -1 \\ 2y &= -x - 1 \\ y &= -\tfrac{1}{2}x - \tfrac{1}{2} \end{aligned}$$

Now we're in a similar position as we were at the beginning of the last problem. We realize that both $-\tfrac{1}{2}x - \tfrac{1}{2}$ and $4x + \tfrac{25}{4}$ are equal to y, and so we set them equal to each other.

$$\begin{aligned} -\tfrac{1}{2}x - \tfrac{1}{2} &= 4x + \tfrac{25}{4} \\[4pt] -\tfrac{1}{2} &= 4\tfrac{1}{2}x + \tfrac{25}{4} \\[4pt] -\tfrac{1}{2} - \tfrac{25}{4} &= 4\tfrac{1}{2}x \\[4pt] -\tfrac{2}{4} - \tfrac{25}{4} &= \tfrac{9}{2}x \\[4pt] -\tfrac{27}{4} &= \tfrac{9}{2}x \\[4pt] -\tfrac{27}{4} \div \tfrac{9}{2} &= x \\[4pt] -\tfrac{3}{2} &= x \end{aligned}$$

So we've figured out that the solution has an x-value of $-\tfrac{3}{2}$. Now we need to figure out the y-value. Again, we plug our x-value into one of the equations, and the y-value should come out. The great part is that, for the whole sequence where we solved for y in the first equation, each of those equations is equivalent. We can choose the new version of the first equation to find our y-value.

$$\begin{aligned} y &= 4 \times \left(-\tfrac{3}{2}\right) + \tfrac{25}{4} \\ y &= -\tfrac{12}{2} + \tfrac{25}{4} \\ y &= -\tfrac{24}{4} + \tfrac{25}{4} \\ y &= \tfrac{1}{4} \end{aligned}$$

This tells us that the point in the first equation where the x-value is $-\frac{3}{2}$ has a y-value of $\frac{1}{4}$. Then, the solution is the point, $\left(-\frac{3}{2}, \frac{1}{4}\right)$.

Let's try a tougher one.

Example 1.8

Solve the system of linear equations.

$$2y - 3x = 9$$

$$5y + 2x = -10$$

For the first equation:

$$\begin{aligned} 2y - 3x &= 9 \\ 2y &= 3x + 9 \\ y &= \tfrac{3}{2}x + \tfrac{9}{2} \end{aligned}$$

And for the second equation:

$$\begin{aligned} 5y + 2x &= -10 \\ 5y &= -10 - 2x \\ y &= -\tfrac{2}{5}x - 2 \end{aligned}$$

Now, setting the right side of each equation equal,

$$\tfrac{3}{2}x + \tfrac{9}{2} = -\tfrac{2}{5}x - 2$$

$$\tfrac{3}{2}x + \tfrac{2}{5}x + \tfrac{9}{2} = -2$$

$$\tfrac{3}{2}x + \tfrac{2}{5}x = -\tfrac{9}{2} - 2$$

$$\tfrac{15}{10}x + \tfrac{4}{10}x = -\tfrac{9}{2} - \tfrac{4}{2}$$

$$\tfrac{19}{10}x = -\tfrac{13}{2}$$

$$x = -\tfrac{13}{2} \div \tfrac{19}{10}$$

$$x = -\tfrac{65}{19}$$

Special Cases

There are some special cases that we may come across when solving linear systems. This leads us to our first Theorem.

Theorem 1.9 (Solutions of a Linear System)

There is either a single solution, no solution, or infinite solutions to a linear system with two equations.

Evidence: Consider the possibilities for two lines in a linear system. The first case is they intersect. To see an example, flip back to Example 1.4. Since lines remain straight, they will never intersect again. There is our first case (a single solution). If they don't, then they must be parallel.

Exercises

Ex. 1 –   Find the solution for the following linear system:

$$\begin{aligned} x &= 4 \\ y &= 2x+1 \end{aligned}$$

Is there one method for solving this system that is easier than the others? Give evidence for your answer.

Ex. 2 –   Find a linear system that has the solution $(2,5)$.

Ex. 3 –   Your philosophy teacher wants you to compare apples and oranges for an assignment. Your friend picks up a total of 12 bundles of fruit and paid $74.00. The apples weighed 5 lbs per bundle and the oranges weighed 4 lbs per bundle. If the bundles of apples cost $5 a piece and the bundles of oranges cost $7 a piece, how many bundles of apples did your friend buy?

Ex. 4 –   Consider the linear system:

$$\begin{aligned} y &= x \\ y &= -x \end{aligned}$$

with respective slopes 1 and -1. If we change the slopes of the two lines to 2 and -2, without changing the y-intercept, does it change the solution? If so, what is the new solution. Provide evidence for your claim.

Ex. 5 –   Solve the following system of equations using whatever method you choose:

$$2x - y = 3$$

$$-3y = 4 + x$$

Ex. 6 –   In what type of situations is it more difficult to use graphing than substitution or elimination to solve a linear system? Give an example of a system in which this is true.

Ex. 7 –   Find the solution for the system:

$$\begin{aligned} 7x + 4y &= 3 \\ x + \tfrac{1}{7}y &= 3 \end{aligned}$$

Ex. 8 –   Solve the following system in two ways (using substitution and elimination) and compare your answers:

$$\frac{1}{2}x + y = 6$$

$$4x = y$$

Should the solutions be the same or different. Provide evidence for your answer.

Ex. 9 –   It costs $5.00 for a student and $10.00 for non-students to see a movie at the Cinema 12. If the theater had 75 people go to a movie one night and made $500 dollars from ticket sales, how many students saw a movie that night?

Ex. 10 –   Find the solution for the linear system:

$$2x + y = 17$$

$$\frac{1}{2}y = -x + 1$$

Ex. 11 –   Use whatever method you choose to find the solution for the system:

$$1002x + 404y \;=\; 2016$$

$$4x + 6y \;=\; 36$$

Ex. 12 –   Solve the following linear system:

$$2x + y = 7$$

$$\frac{1}{3}y = -\frac{2}{3}x + \frac{7}{3}$$