зад. 1 a) $z^4 = 8$

$z^4 = 8 + 0i$

$r = \sqrt{8^2 + 0^2} = \sqrt{64} = 8$

$z^4 = 8(\cos 0 + i\sin 0)$

$$z = \sqrt[4]{8}\left(\cos\frac{0+2k\pi}{4} + i\sin\frac{0+2k\pi}{4}\right) = \sqrt[4]{8}\left(\cos\frac{2k\pi}{4} + i\sin\frac{2k\pi}{4}\right),$$

$k = 0, 1, 2, 3$

при $k = 0$:

$$z_0 = \sqrt[4]{8}(\cos 0 + i\sin 0) = \sqrt[4]{8}$$

при $k = 1$:

$$z_1 = \sqrt[4]{8}\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right) = \sqrt[4]{8}\, i$$

при $k = 2$:

$$z_2 = \sqrt[4]{8}(\cos\pi + i\sin\pi) = -\sqrt[4]{8}$$

при $k = 3$:

$$z_3 = \sqrt[4]{8}\left(\cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2}\right) = -\sqrt[4]{8}\, i$$

зад. 1 б) $x^{78} + 9x^{52} + 40x^{26} - 50 = 0$

Пол. $x^{26} = y$

$y^3 + 9y^2 + 40y - 50 = 0$

Схема на Хорнер

	1	9	40	-50
1	1	10	50	0

$\Rightarrow y_1 = 1$

$(y-1)(y^2+10y+50) = 0$

$y^2 + 10y + 50 = 0$

$D = 25 - 50 = -25$

$y_{2,3} = -5 \pm \sqrt{-25} = -5 \pm \sqrt{i^2 25}$

$y_{2,3} = -5 \pm 5i$, връщаме се в полагането

I. $x^{26} = 1 + 0i \qquad r = \sqrt{1^2 + 0^2} = 1$

$x^{26} = 1(\cos 2k\pi + i\sin 2k\pi)$

$x = \left(\cos \frac{2k\pi}{26} + i\sin \frac{2k\pi}{26}\right) = \left(\cos \frac{k\pi}{13} + i\sin\frac{k\pi}{13}\right) \quad k \in [0, 25]$

II. $x^{26} = -5 + 5i \qquad r = \sqrt{(-5)^2 + 5^2} = \sqrt{50}$

$\cos\varphi = \frac{-5}{5\sqrt{2}} = -\frac{\sqrt{2}}{2}$

$\sin\varphi = \frac{5}{5\sqrt{2}} = \frac{\sqrt{2}}{2}$

$\varphi = \frac{3\pi}{4} + 2k\pi$

$x^{26} = \sqrt{50}\left(\cos\left(\frac{3\pi}{4} + 2k\pi\right) + i\sin\left(\frac{3\pi}{4} + 2k\pi\right)\right)$

$x^{26} = \sqrt{50}\left(\cos\left(\frac{3\pi + 8k\pi}{4}\right) + i\sin\left(\frac{3\pi + 8k\pi}{4}\right)\right)$

$x = \sqrt[52]{50}\left(\cos\left(\frac{3\pi + 8k\pi}{104}\right) + i\sin\left(\frac{3\pi + 8k\pi}{104}\right)\right), \quad k \in [0, 25]$

III $x^{26} = -5-5i$ $\qquad r = \sqrt{(-5)^2+(-5)^2} = \sqrt{50}$

$$\cos\varphi = \frac{-5}{5\sqrt{2}} = -\frac{\sqrt{2}}{2} \qquad \sin\varphi = \frac{-5}{5\sqrt{2}} = -\frac{\sqrt{2}}{2}$$

$$\varphi = \frac{5\pi}{4} + 2k\pi$$

$$x^{26} = \sqrt{50}\left(\cos\left(\frac{5\pi}{4}+2k\pi\right) + i\sin\left(\frac{5\pi}{4}+2k\pi\right)\right)$$

$$x^{26} = \sqrt{50}\left(\cos\left(\frac{5\pi+8k\pi}{4}\right) + i\sin\left(\frac{5\pi+8k\pi}{4}\right)\right)$$

$$x = \sqrt[52]{50}\left(\cos\left(\frac{5\pi+8k\pi}{104}\right) + i\sin\left(\frac{5\pi+8k\pi}{104}\right)\right), \quad k \in [0, 25]$$

зад. 1 б) $\dfrac{(\sqrt{3}+15i)^{181}}{(156+96i\sqrt{3})^{90}} = \dfrac{(\sqrt{3}+15i)^{180}}{(156+96i\sqrt{3})^{90}} \cdot (\sqrt{3}+15i) =$

$$= \underbrace{\left(\frac{(\sqrt{3}+15i)^2}{156+96i\sqrt{3}}\right)^{90}}_{\overset{\shortparallel}{Q}} \cdot (\sqrt{3}+15i) \quad , \quad Q = \frac{(\sqrt{3}+15i)^2}{156+96i\sqrt{3}}$$

$$Q = \frac{3+30\sqrt{3}i+225i^2}{156+96i\sqrt{3}} = \frac{-222+30\sqrt{3}i}{156+96i\sqrt{3}} = \frac{\cancel{6}^{1}}{\cancel{12}_{2}} \cdot \frac{(-37+5\sqrt{3}i)}{(13+8\sqrt{3}i)} =$$

$$= \frac{1}{2} \cdot \frac{-37+5\sqrt{3}i}{13+8\sqrt{3}i} \cdot \frac{13-8\sqrt{3}i}{13-8\sqrt{3}i} = \frac{-481+296\sqrt{3}i+65\sqrt{3}i-120i^2}{2(169-192i^2)} =$$

$$= \frac{-361+361\sqrt{3}i}{2 \cdot 361} = \frac{\cancel{361}(-1+\sqrt{3}i)}{2 \cdot \cancel{361}} = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$$

$$\cos\varphi = -\frac{1}{2} \quad \sin\varphi = \frac{\sqrt{3}}{2} \quad \Rightarrow \varphi = \frac{2\pi}{3}$$

$$r = \sqrt{\frac{1}{4}+\frac{3}{4}} = 1$$

$$\Rightarrow -\frac{1}{2} + \frac{\sqrt{3}}{2}i = 1\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right) = Q$$

Връщаме се и имаме, че:

$$\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)^{90} \cdot (\sqrt{3}+15i) =$$

$$= \left(\cos\frac{2\pi \cdot \cancel{90}^{30}}{\cancel{3}} + i\sin\frac{2\pi \cdot \cancel{90}^{30}}{\cancel{3}}\right) \cdot (\sqrt{3}+15i) =$$

$$= (\cos 60\pi + i\sin 60\pi) \cdot (\sqrt{3}+15i) = \quad \left| \begin{array}{l} \cos 60\pi = \cos 2k\pi \\ \sin 60\pi = \sin 2k\pi \end{array} \right.$$

$$= (1+i0) \cdot (\sqrt{3}+15i) = \sqrt{3}+15i$$

зад. 2 $\begin{cases} 4x_1 - 19x_2 - 4x_3 - (5-\mu)x_4 = 1-\lambda \\ x_1 - 2x_2 - 2x_3 - x_4 = 2 \\ 2x_1 - 9x_2 - 2x_3 - 2x_4 = -2 \\ 2x_1 - 6x_2 - 3x_3 - 2x_4 = \lambda \end{cases}$

$$\left(\begin{array}{cccc|c} 4 & -19 & -4 & -5+\mu & 1-\lambda \\ 1 & -2 & -2 & -1 & 2 \\ 2 & -9 & -2 & -2 & -2 \\ 2 & -6 & -3 & -2 & \lambda \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & -2 & -2 & -1 & 2 \\ 2 & -9 & -2 & -2 & -2 \\ 2 & -6 & -3 & -2 & \lambda \\ 4 & -19 & -4 & -5+\mu & 1-\lambda \end{array}\right)$$

$$\sim \left(\begin{array}{cccc|c} 1 & -2 & -2 & -1 & 2 \\ 0 & -5 & 2 & 0 & -6 \\ 0 & -2 & 1 & 0 & \lambda-4 \\ 0 & -11 & 4 & \mu-1 & -\lambda-7 \end{array}\right) \cdot\left(-\frac{2}{5}\right) \quad \cdot\left(-\frac{11}{5}\right) \sim$$

$$\sim \left(\begin{array}{cccc|c} 1 & -2 & -2 & -1 & 2 \\ 0 & -5 & 2 & 0 & -6 \\ 0 & 0 & 1/5 & 0 & \frac{5\lambda-8}{5} \\ 0 & 0 & -\frac{2}{5} & \mu-1 & \frac{31-5\lambda}{5} \end{array}\right) \cdot 2 \sim \left(\begin{array}{cccc|c} 1 & -2 & -2 & -1 & 2 \\ 0 & -5 & 2 & 0 & -6 \\ 0 & 0 & \frac{1}{5} & 0 & \frac{5\lambda-8}{5} \\ 0 & 0 & 0 & \mu-1 & \lambda+3 \end{array}\right)$$

1) при $\mu = 1$ и $\lambda \neq -3$ системата е несъвместима

$$0x_1 + 0x_2 + 0x_3 + 0x_4 = \lambda + 3$$

2) при $\mu = 1$ и $\lambda = -3$ системата е неопределена

$x_4 = p$, $x_3 = 5.(-3) - 8 = -23$

$$x_2 = \frac{-6 - 2(-23)}{-5} = \frac{-6+46}{-5} = \frac{40}{-5} = -8$$

$$x_1 = -16 - 23.2 + p + 2 = p - 60$$

Решение на системата е:

$$=\{(p-69, -8, -23, p) / p \in F\}$$

3) при $\mu \neq 1$ и $\lambda \neq -3$ системата е определена

$$x_4 = \frac{\lambda+3}{\mu-1}, \quad x_3 = 5\lambda - 8$$

$$x_2 = \frac{-6-2(5\lambda-8)}{-5} = \frac{-10\lambda+10}{-5} = \frac{\cancel{5}(-2\lambda+2)}{\cancel{-5}} = 2\lambda - 2$$

$$x_1 = 2(2\lambda-2) + 2(5\lambda-8) + \frac{\lambda+3}{\mu-1} + 2$$

$$x_1 = 4\lambda - 4 + 10\lambda - 16 + \frac{\lambda+3}{\mu-1} + 2$$

$$x_1 = 14\lambda - 18 + \frac{\lambda+3}{\mu-1}$$

$$x_1 = \frac{14\lambda(\mu-1) - 18(\mu-1) + \lambda + 3}{\mu-1}$$

$$x_1 = \frac{14\lambda\mu - 14\lambda - 18\mu + 18 + \lambda + 3}{\mu-1} = \frac{14\lambda\mu - 13\lambda - 18\mu + 21}{\mu-1}$$

Решението на системата е:

$$= \left(\frac{14\lambda\mu - 13\lambda - 18\mu + 21}{\mu-1}, 2\lambda - 2, 5\lambda - 8, \frac{\lambda+3}{\mu-1}\right)$$

зад. 3 а)

$$\begin{array}{c}\begin{pmatrix} -1 & \lambda-3 & 7 & -2 \\ -1 & 5 & 1 & 1 \\ 2 & -8 & -4 & -1 \\ -2 & \mu-1 & \mu+7 & -2 \end{pmatrix}\end{array}\begin{array}{l} \cdot(-1) \\ \\ \cdot 2 \\ \cdot(-2) \end{array} \sim \begin{pmatrix} 0 & \lambda-8 & 6 & -3 \\ -1 & 5 & 1 & 1 \\ 0 & 2 & -2 & 1 \\ 0 & \mu-11 & \mu+5 & -4 \end{pmatrix}\begin{array}{l} \cdot 3 \\ \cdot(-1) \\ \\ \cdot 4 \end{array}$$

$$\sim \begin{pmatrix} 0 & \lambda-2 & 0 & 0 \\ -1 & 3 & 3 & 0 \\ 0 & 2 & -2 & 1 \\ 0 & \mu-3 & \mu-3 & 0 \end{pmatrix}$$

При $\lambda \neq 2$ и $\mu \neq 3$ рангът е равен на 4

б)

$$\left(\begin{array}{ccc|c} -1 & -1 & 2 & -2 \\ \lambda-3 & 5 & -8 & \mu-1 \\ 7 & 1 & -4 & \mu+7 \\ -2 & 1 & -1 & -2 \end{array}\right) \sim \left(\begin{array}{ccc|c} -5 & 1 & 0 & -6 \\ \lambda+13 & -3 & 0 & \mu+15 \\ 15 & -3 & 0 & \mu+15 \\ -2 & 1 & -1 & -2 \end{array}\right) \sim$$

$$\sim \left(\begin{array}{ccc|c} -5 & 1 & 0 & -6 \\ \lambda-2 & 0 & 0 & \mu-3 \\ 0 & 0 & 0 & \mu-3 \\ 3 & 0 & -1 & 4 \end{array}\right)$$

При $\lambda \neq 2$ и $\mu \neq 3$ има неопределеност

$\alpha_1 = p$, $\alpha_2 = 5p-6$, $\alpha_3 = 3p-4$

$v = pa_1 + (5p-6)a_2 + (3p-4)a_3$, $p \in F$

при $p=1$: $v = a_1 - a_2 - a_3$

при $p=2$: $v = 2a_1 + 4a_2 + 2a_3$

при $p=-1$: $v = -a_1 - 11a_2 - 7a_3$

зад. 4

$$A=\begin{pmatrix} -g_i & g_i & g_i & g_i & g_i & -\lambda+g_i \\ -g_i & g_i & g_i & g_i & -\lambda+g_i & g_i \\ -g_i & g_i & g_i & -\lambda+g_i & g_i & g_i \\ -g_i & g_i & -\lambda+g_i & g_i & g_i & g_i \\ -g_i & -\lambda+g_i & g_i & g_i & g_i & g_i \\ -\lambda+g_i & -g_i & -g_i & -g_i & -g_i & -g_i \end{pmatrix} \sim$$

$+ \quad - \quad - \quad - \quad - $

$$\sim\begin{pmatrix} +\lambda & +\lambda & \lambda & \lambda & \lambda & -\lambda+g_i \\ 0 & 0 & 0 & 0 & -\lambda & g_i \\ 0 & 0 & 0 & -\lambda & 0 & g_i \\ 0 & 0 & -\lambda & 0 & 0 & g_i \\ 0 & -\lambda & 0 & 0 & 0 & g_i \\ -\lambda & 0 & 0 & 0 & 0 & -g_i \end{pmatrix} \sim$$

$\cdot\frac{1}{\lambda} \quad \cdot\frac{1}{\lambda} \quad \cdot\frac{1}{\lambda} \quad \cdot\frac{1}{\lambda} \quad \cdot\frac{1}{\lambda}$ $\qquad \lambda\neq 0$

$$\sim\begin{pmatrix} -1 & 1 & 1 & 1 & 1 & -\lambda+g_i \\ 0 & 0 & 0 & 0 & -1 & g_i \\ 0 & 0 & 0 & -1 & 0 & g_i \\ 0 & 0 & -1 & 0 & 0 & g_i \\ 0 & -1 & 0 & 0 & 0 & g_i \\ -1 & 0 & 0 & 0 & 0 & -g_i \end{pmatrix} \sim$$

$g_i \quad g_i \quad \cdot g_i \quad \cdot g_i \quad \cdot(-g_i)$

$$\sim \begin{pmatrix} -1 & 1 & 1 & 1 & 1 & \lambda+54i \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \sim$$

$\cancel{\frac{1}{54i-\lambda}}$

$\cdot \frac{1}{\lambda-54i}$ $\qquad \lambda-54i \neq 0$

$$\sim \begin{pmatrix} -1 & 1 & 1 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

при $\lambda \neq 0$ и $\lambda - 54i \neq 0$ $\quad r = 6$

при $\lambda = 0$: $r = 1$

при $\lambda - 54i = 0$ $\quad r = 5$

зад. 5 $V=\left\{\begin{pmatrix} a_{11} & 0 & a_{13} \\ 0 & a_{22} & 0 \\ a_{31} & 0 & a_{31} \end{pmatrix} \middle| a_{11}, a_{13}, a_{22}, a_{31}, a_{33} \in \mathbb{R}\right\}$

$U=\{A\in V \mid a_{31}=a_{11}\}$ и $W=\{A\in V \mid a_{13}+a_{22}+a_{31}=0\}$

а) Затвореност относно събиране:

$$A=\begin{pmatrix} a_{11} & 0 & a_{13} \\ 0 & a_{22} & 0 \\ a_{31} & 0 & a_{33} \end{pmatrix}\in V,\ B=\begin{pmatrix} b_{11} & 0 & b_{13} \\ 0 & b_{22} & 0 \\ b_{31} & 0 & b_{33} \end{pmatrix}\in V$$

$$A+B=\begin{pmatrix} a_{11}+b_{11} & 0 & a_{13}+b_{13} \\ 0 & a_{22}+b_{22} & 0 \\ a_{31}+b_{31} & 0 & a_{31}+b_{31} \end{pmatrix}$$

$a_{11}+b_{11},\ a_{13}+b_{13},\ a_{22}+b_{22},\ a_{31}+b_{31},\ a_{31}+b_{31}\in\mathbb{R}$

$\Rightarrow V$ е затворено относно събирането

Затвореност относно умножение с число:

Нека $A\in V$ и $\lambda\in\mathbb{R}$

$$\lambda A=\lambda\begin{pmatrix} a_{11} & 0 & a_{13} \\ 0 & a_{22} & 0 \\ a_{31} & 0 & a_{33} \end{pmatrix}=\begin{pmatrix} \lambda a_{11} & 0 & \lambda a_{13} \\ 0 & \lambda a_{22} & 0 \\ \lambda a_{31} & 0 & \lambda a_{33} \end{pmatrix}$$

$\lambda a_{11}, \lambda a_{13}, \lambda a_{22}, \lambda a_{31}, \lambda a_{33}\in\mathbb{R}$

$\Rightarrow V$ е затворено относно умножението с число

Базис и размерност на V

$A=a_{11}.E_{11}+a_{13}E_{13}+a_{22}.E_{22}+a_{31}E_{31}+a_{33}E_{33},$

където:

$$E_{11}=\begin{pmatrix}1&0&0\\0&0&0\\0&0&0\end{pmatrix},\ E_{13}=\begin{pmatrix}0&0&1\\0&0&0\\0&0&0\end{pmatrix},\ E_{22}=\begin{pmatrix}0&0&0\\0&1&0\\0&0&0\end{pmatrix},$$

$$E_{31}=\begin{pmatrix}0&0&0\\0&0&0\\1&0&0\end{pmatrix} \text{ и } E_{33}=\begin{pmatrix}0&0&0\\0&0&0\\0&0&1\end{pmatrix}$$

$E_{11}, E_{13}, E_{22}, E_{31}, E_{33}$ са базис на V

$V=\ell(E_{11}, E_{13}, E_{22}, E_{31}, E_{33})$

и $\dim V = 5$

б) Нека $A_1=\begin{pmatrix}a_{11}&0&a_{13}\\0&a_{22}&0\\a_{11}&0&a_{33}\end{pmatrix}\in U$, тогава:

$A_1=a_{11}\cdot E_{11}+a_{13}\cdot E_{13}+a_{22}\cdot E_{22}+a_{11}\cdot E_{31}+a_{33}\cdot E_{33}$

$A_1=a_{11}(E_{11}+E_{31})+a_{22}E_{22}+a_{13}\cdot E_{13}+a_{33}\cdot E_{33}$

Сл. $U=\ell(E_{11}+E_{31}, E_{13}, E_{22}, E_{33})$

$\Rightarrow U<V$ и $\dim U=4$

Нека $A_2=\begin{pmatrix}a_{11}&0&-a_{31}-a_{22}\\0&a_{22}&0\\a_{31}&0&a_{33}\end{pmatrix}\in W$ тогава:

$A_2 = a_{11} \cdot E_{11} + (-a_{31} - a_{22}) E_{13} + a_{22} \cdot E_{22} + a_{31} \cdot E_{31} + a_{33} \cdot E_{33}$

$A_2 = a_{11} \cdot E_{11} - a_{31} E_{13} - a_{22} E_{13} + a_{22} E_{22} + a_{31} \cdot E_{31} + a_{33} \cdot E_{33}$

$A_2 = a_{11} \cdot E_{11} + a_{31} (E_{31} - E_{13}) + a_{22} (E_{22} - E_{13}) + a_{33} \cdot E_{33}$

Сл. $W = \ell(E_{11}, E_{31} - E_{13}, E_{22} - E_{13}, E_{33})$

$\Rightarrow W \subset V$ и $\dim W = 4$

в) $U \cap W$: Трябва условията за U и W да бъдат изпълнени.

$$\left.\begin{array}{l} \text{за } U: a_{31} = a_{11} \\ \text{за } W: a_{13} + a_{22} + a_{31} = 0 \end{array}\right\} \Rightarrow a_{13} = -a_{22} - a_{11}$$

$$A = \begin{pmatrix} a_{11} & 0 & -a_{22} - a_{11} \\ 0 & a_{22} & 0 \\ a_{11} & 0 & a_{33} \end{pmatrix} \in (U \cap W)$$

$A = a_{11} \cdot E_{11} + (-a_{22} - a_{11}) E_{13} + a_{22} \cdot E_{22} + a_{11} \cdot E_{31} + a_{33} \cdot E_{33}$

$A = a_{11} (E_{11} + E_{31} - E_{13}) + a_{22} (E_{22} - E_{13}) + a_{33} \cdot E_{33}$

Сл. $(U \cap W) = \ell(E_{11} + E_{31} - E_{13}, E_{22} - E_{13}, E_{33})$

и $\dim (U \cap W) = 3$

Ще използваме формулата за размерност на сумата:

$$\dim(U+W) = \dim U + \dim W - \dim(U \cap W)$$

Взимаме стойностите на размерностите на U и W от б) и заместваме

$$\dim(U+W) = 4+4-3 = 5$$

$$\dim(U+W) = 5$$