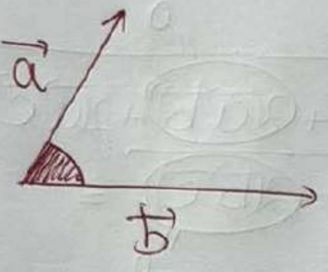
Упражнение №3
(част 2)
Скаларно произведение

I Скаларно произведение на два вектора

$\vec{a} \neq \vec{0}, \vec{b} \neq \vec{0}$

деф.: $\langle \vec{a}, \vec{b} \rangle = (\vec{a}.\vec{b}) = \underbrace{|\vec{a}|.|\vec{b}|.\cos \sphericalangle(\vec{a},\vec{b})_e}_{\text{число}}$

$\sphericalangle(\vec{a},\vec{b})_e$ - елементарногеометричен ъгъл м/у векторите $\vec{a}, \vec{b}$



- неориентиран
- $\sphericalangle(\vec{a},\vec{b})_e = \sphericalangle(\vec{b},\vec{a})_e$
- $\sphericalangle(\vec{a},\vec{b}) \in [0;\pi]$
- няма периодичност, т.е. $\cos x = \lambda \Rightarrow \exists!\, x_0 \in [0;\pi] : \cos x_0 = \lambda$

св-ства:

1) $(\vec{a}.\vec{b}) = (\vec{b}.\vec{a})$ - комутативност
2) $(\vec{a}+\vec{b}).\vec{c} = (\vec{a}.\vec{c}) + (\vec{b}.\vec{c})$
3) $(\lambda\vec{a}).\vec{b} = \lambda(\vec{a}.\vec{b})$
4) $(\vec{a}.\vec{a}) = |\vec{a}|.|\vec{a}|.\underset{\overset{\|}{1}}{\cos 0} = |\vec{a}|^2 \Rightarrow \underbrace{\vec{a}^2 = |\vec{a}|^2}_{\text{скаларен квадрат}} \Rightarrow |\vec{a}| = \sqrt{\vec{a}^2}$
5) $\cos \sphericalangle(\vec{a},\vec{b})_e = \dfrac{(\vec{a}.\vec{b})}{|\vec{a}|.|\vec{b}|}$!
6) $(\vec{a}.\vec{b}) = 0 \Leftrightarrow \vec{a} \perp \vec{b}$!

1 зад. $\vec{a}, \vec{b}, \vec{c}$ - ЛНЗ

$|\vec{a}| = 1, |\vec{b}| = 2, |\vec{c}| = \sqrt{2}$

$\sphericalangle(\vec{a},\vec{b})_e = \frac{\pi}{2}, \sphericalangle(\vec{b},\vec{c})_e = \frac{\pi}{2},$

$\sphericalangle(\vec{a},\vec{c})_e = \frac{\pi}{4}$

$\vec{p} = \vec{a} + \vec{b} - \vec{c}$

$\vec{q} = 2\vec{a} - 3\vec{b} + \vec{c}$

$\vec{r} = \vec{a} + \lambda\vec{b} - \vec{c}$

а) $|\vec{p}| = ?, |\vec{q}| = ?$

б) $(\vec{p}.\vec{q}) = ?$

в) $\lambda = ?$, така че $\vec{p} \perp \vec{r}$

г) $\lambda = ?$, така че $|\vec{r}| = \sqrt{5}$

реш.: Коефициенти на метриката:

$\vec{a}^2 = 1$, $\vec{a}.\vec{b} = 0$

$\vec{b}^2 = 4$, $\vec{b}.\vec{c} = 0$

$\vec{c}^2 = 2$, $\vec{a}.\vec{c} = |\vec{a}|.|\vec{c}|.\cos\sphericalangle(\vec{a},\vec{c}) = 1.\sqrt{2}.\frac{\sqrt{2}}{2} = 1$

a) $|\vec{p}| = ?$, $|\vec{q}| = ?$

$\vec{p} = \vec{a} + \vec{b} - \vec{c}$

$|\vec{p}| = \sqrt{\vec{p}^2} = \sqrt{(\vec{a}+\vec{b}-\vec{c})(\vec{a}+\vec{b}-\vec{c})} = \sqrt{\vec{a}^2+\vec{b}^2+\vec{c}^2+\underbrace{2(\vec{a}\vec{b})}_{=0}-2(\vec{a}\vec{c})-\underbrace{2(\vec{b}\vec{c})}_{=0}} =$

$= \sqrt{1+4+\cancel{2}-\cancel{2}} = \sqrt{5}$

$|\vec{q}|^2 = \vec{q}^2 = (2\vec{a}-3\vec{b}+\vec{c})^2 = 4\vec{a}^2+9\vec{b}^2+\vec{c}^2-\underbrace{12(\vec{a}\vec{b})}_{=0}+4(\vec{a}\vec{c})-\underbrace{6(\vec{b}\vec{c})}_{=0} =$

$= 4+36+2+4 = 46 \Rightarrow |\vec{q}| = \sqrt{\vec{q}^2} = \sqrt{46}$

б) $(\vec{p}.\vec{q}) = ?$, $\cos\sphericalangle(\vec{p},\vec{q}) = ?$

$(\vec{p}.\vec{q}) = (\vec{a}+\vec{b}-\vec{c})(2\vec{a}-3\vec{b}+\vec{c}) = 2\vec{a}^2 - \underbrace{3\vec{a}\vec{b}}_{=0} + \vec{a}\vec{c} +$

$+ 2\underbrace{\vec{a}\vec{b}}_{=0} - 3\vec{b}^2 + \underbrace{\vec{b}\vec{c}}_{=0} - 2\vec{a}\vec{c} + 3\underbrace{\vec{b}\vec{c}}_{=0} - \vec{c}^2 =$

$= \cancel{2}+1-3.4-\cancel{2}-2 = 1-14 = -13$

$\cos\sphericalangle(\vec{p},\vec{q}) = \frac{\vec{p}.\vec{q}}{|\vec{p}|.|\vec{q}|} = \frac{-13}{\sqrt{46}.\sqrt{5}} = \frac{-13\sqrt{230}}{230}$

$(\vec{p}.\vec{q}) < 0 \Longleftrightarrow \sphericalangle(\vec{p},\vec{q}) \in (\frac{\pi}{2};\pi)$

$\downarrow$

тъп ъгъл!

в) $\lambda = ?$, $\vec{p} \perp \vec{r} \Longleftrightarrow \vec{p}.\vec{r} = 0$

$(\vec{p}.\vec{r}) = (\vec{a}+\vec{b}-\vec{c})(\vec{a}+\lambda\vec{b}-\vec{c}) = \vec{a}^2 + \underbrace{\lambda\vec{a}\vec{b}}_{=0} - \vec{a}\vec{c} + \underbrace{\vec{a}\vec{b}}_{=0} + \lambda\vec{b}$

$-\underbrace{\vec{b}\vec{c}}_{=0} - \vec{a}\vec{c} - \lambda\underbrace{\vec{b}\vec{c}}_{=0} + \vec{c}^2 = \cancel{1} - \cancel{1} + \lambda 4 - 1 + 2$

$(\vec{p}\cdot\vec{r})=0 \Rightarrow 1+4\lambda=0$

$\lambda=-\frac{1}{4}$ //

г) $\lambda=?,\ |\vec{r}|=\sqrt{5}$

$|\vec{r}|^2=\vec{r}^2=(\vec{a}+\lambda\vec{b}-\vec{c})^2=\vec{a}^2+\lambda^2\vec{b}^2+\vec{c}^2+2\lambda\overset{0}{\overset{\shortparallel}{\vec{a}\vec{b}}}-2\vec{a}\vec{c}-2\lambda\underset{0}{\underset{\shortparallel}{\vec{b}\vec{c}}}=$

$=1+\lambda^2\cdot 4+\cancel{2}-\cancel{2}=1+4\lambda^2$

$|\vec{r}|=\sqrt{5} \Rightarrow 5=|\vec{r}|^2=\vec{r}^2=1+4\lambda^2$

$4=4\lambda^2 \Rightarrow \lambda=\pm 1$ //

2 зад. $\vec{a},\vec{b},\vec{c}$ – ЛНЗ

Даден е вектор $\vec{p}$: $\vec{p}\perp\vec{a},\ \vec{p}\perp\vec{b},\ \vec{p}\perp\vec{c}$

Да се док., че $\vec{p}=\vec{0}$.

док.: Нека $\vec{p}=\alpha\vec{a}+\beta\vec{b}+\gamma\vec{c}$ (Всеки четири вектора в геом. пр. са линейно зависими)

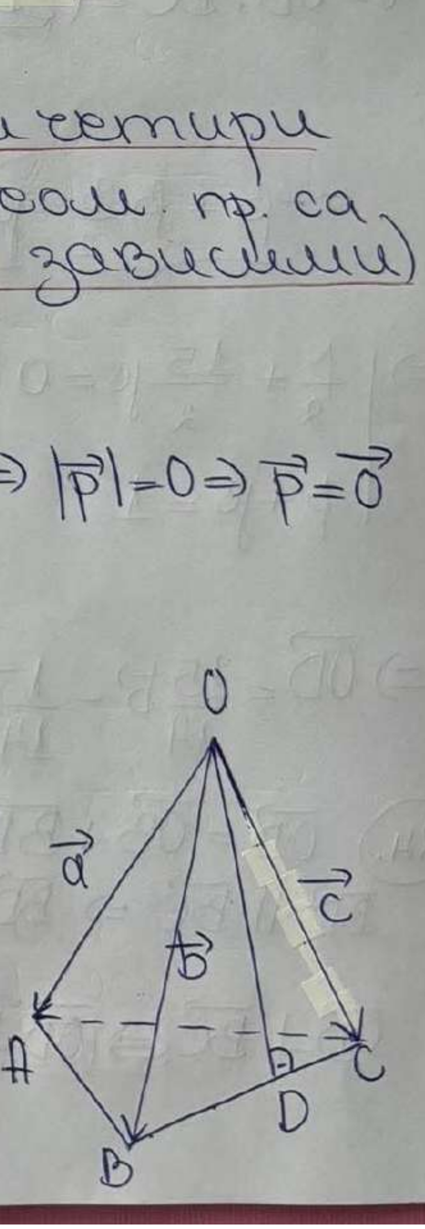
$\vec{p}=\alpha\vec{a}+\beta\vec{b}+\gamma\vec{c}\ /\cdot\vec{p}$

$\vec{p}^2=\alpha\underbrace{(\vec{a}\cdot\vec{p})}_{0}+\beta\underbrace{(\vec{b}\cdot\vec{p})}_{0}+\gamma\underbrace{(\vec{c}\cdot\vec{p})}_{0} \Rightarrow \vec{p}^2=0 \Rightarrow |\vec{p}|=0 \Rightarrow \vec{p}=\vec{0}$

3 зад. $\vec{a},\vec{b},\vec{c}: |\vec{a}|=2,\ |\vec{b}|=1,\ |\vec{c}|=3$

$\sphericalangle(\vec{a},\vec{b})_e=\sphericalangle(\vec{b},\vec{c})_e=\sphericalangle(\vec{c},\vec{a})_e=\frac{\pi}{3}$

Тетраедър $OABC$: $\vec{OA}=\vec{a},\ \vec{OB}=\vec{b},\ \vec{OC}=\vec{c}$



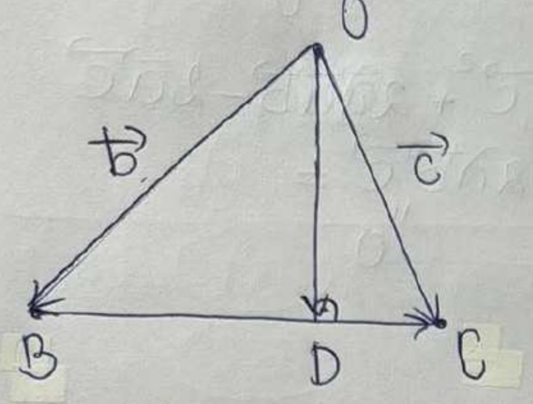
а) Ако т. D $\begin{cases} \in BC \\ OD\perp BC \end{cases}$, да се изрази $\vec{OD}$ чрез $\vec{a},\vec{b},\vec{c}$

коеф. на метриката:

$\vec{a}^2 = 4$, $\vec{a}\cdot\vec{b} = |\vec{a}|.|\vec{b}|.\cos\angle(a,b)e = \cancel{2}.1.\frac{1}{\cancel{2}} = 1$

$\vec{b}^2 = 1$, $\vec{b}\cdot\vec{c} = 1.3.\frac{1}{2} = \frac{3}{2}$

$\vec{c}^2 = 3^2 = 9$, $\vec{a}\cdot\vec{c} = \cancel{2}.3.\frac{1}{\cancel{2}} = 3$



$\vec{OD}$ е от линейната обвивка на $\vec{b}$ и $\vec{c}$, т.е. $D\in(BCO)$ и $\vec{OD}, \vec{b}, \vec{c}$ – компланарни

$\Rightarrow \vec{OD} = \beta\vec{b} + \gamma\vec{c}$

1н. $\begin{cases} \vec{OD} = \beta\vec{b} + \gamma\vec{c} \\ \beta + \gamma = 1 \end{cases}$ (от осн. задача)

$$0 = \vec{OD}\cdot\vec{BC} = \underbrace{(\beta\vec{b} + \gamma\vec{c})}_{=\vec{OD}}\underbrace{(-\vec{b} + \vec{c})}_{=\vec{BC}} = -\beta\vec{b}^2 + \beta\vec{b}\vec{c} - \gamma\vec{b}\vec{c} + \gamma\vec{c}^2 =$$

$$= \underbrace{-\beta + \beta.\frac{3}{2}}\ \underbrace{-\gamma.\frac{3}{2} + \gamma.\overset{2}{9}} = \frac{1}{2}\beta + \frac{15}{2}\gamma$$

$$\Rightarrow \begin{cases} \frac{\beta}{2} + \frac{15}{2}\gamma = 0 \quad /.2 \\ \beta + \gamma = 1 \end{cases} \Rightarrow \begin{cases} \beta + 15\gamma = 0 \\ \beta = 1 - \gamma \end{cases} \Rightarrow 1 - \gamma + 15\gamma = 0$$

$$14\gamma = -1$$

$$\gamma = -\frac{1}{14} \Rightarrow \beta = \frac{15}{14}$$

$$\Rightarrow \vec{OD} = \frac{15}{14}\vec{b} - \frac{1}{14}\vec{c}$$

2н. $\vec{OD} = \vec{OB} + \vec{BD} = \vec{b} + \vec{BD}$

$\vec{BD} \parallel \vec{BC} \Rightarrow \vec{BD} = k.\vec{BC} = k(-\vec{b} + \vec{c}) \Rightarrow \vec{OD} = \vec{b}(1-k) + k\vec{c}$

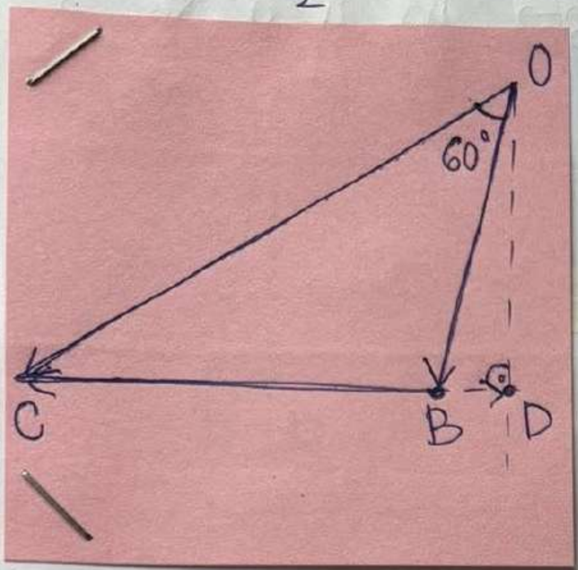
$\vec{BC} = (\vec{BO} + \vec{OC}) = (\vec{OC} - \vec{OB})$

$\vec{OD} \perp \vec{BC} \Leftrightarrow (\vec{OD}\cdot\vec{BC}) = 0 \Rightarrow (\vec{b}(1-k) + k\vec{c}).(-\vec{b} + \vec{c}) = 0$

$\vec{b}^2(k-1) + \vec{b}\vec{c}(1-k) - k\vec{c}\vec{b} + k\vec{c}^2 = 0$

$$k-1+\frac{3}{2}(1-k)+\frac{3}{2}(-k)+k\cdot 9=0$$

$$10k-1+\frac{3}{2}-3k=0 \Rightarrow 7k=-\frac{1}{2}$$



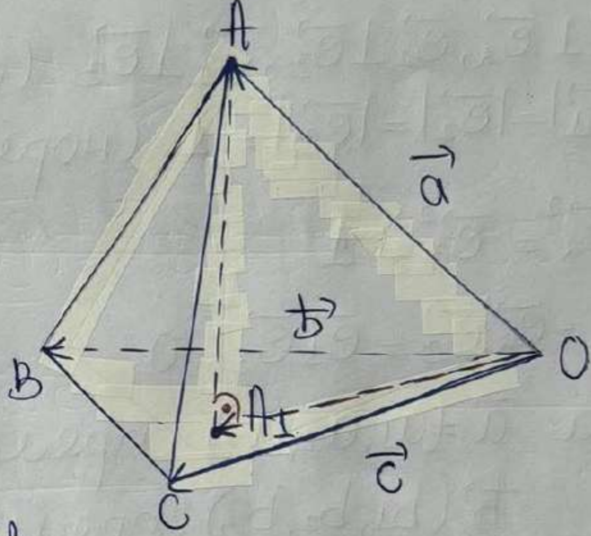
$$k=-\frac{1}{14} \Rightarrow \overrightarrow{BD}=-\frac{1}{14}\overrightarrow{BC} \text{ и}$$

$$\overrightarrow{OD}=\frac{15}{14}\vec{b}-\frac{1}{14}\vec{c}$$

б) т. A_1 $\begin{cases} \in (BOC) \\ AA_1 \perp (BOC) \end{cases}$

Да се изрази $\overrightarrow{OA_1}$ чрез $\vec{a}, \vec{b}, \vec{c}$.

$|\overrightarrow{OA_1}|=?$



реш.:

$\overrightarrow{OA_1}, \vec{b}, \vec{c}$ – компланарни

$\Rightarrow \exists!\,(p, q): \overrightarrow{OA_1}=p\cdot\vec{b}+q\vec{c}$

$\overrightarrow{AA_1}=\overrightarrow{AO}+\overrightarrow{OA_1}=-\vec{a}+p\vec{b}+q\vec{c}$

$AA_1 \perp (BCO) \Rightarrow \begin{cases} \overrightarrow{AA_1}\perp\vec{b} \\ \overrightarrow{AA_1}\perp\vec{c}\end{cases} \Leftrightarrow \begin{cases} \overrightarrow{AA_1}\cdot\vec{b}=0 \\ \overrightarrow{AA_1}\cdot\vec{c}=0\end{cases}$

$$\begin{cases} (-\vec{a}+p\vec{b}+q\vec{c})\vec{b}=0 \\ (-\vec{a}+p\vec{b}+q\vec{c})\vec{c}=0 \end{cases} \Rightarrow \begin{cases} -\vec{a}\vec{b}+p\vec{b}^2+q\vec{b}\vec{c}=0 \\ -\vec{a}\vec{c}+p\vec{b}\vec{c}+q\vec{c}^2=0 \end{cases}$$

$$\begin{cases} -1+p+q\cdot\frac{3}{2}=0 \ /\cdot 2 \\ -3+p\cdot\frac{3}{2}+q\cdot 9=0 \ /\cdot 2 \end{cases} \Rightarrow \begin{cases} -2+2p+3q=0 \ /\cdot(-6) \\ -6+3p+18q=0 \end{cases} + \Rightarrow \begin{cases} -2+2p+3q=0 \\ 6-9p=0 \end{cases}$$

$$p=\frac{2}{3}$$

$$\text{и } q=\frac{2}{9}$$

$$\overrightarrow{OA_1} = \frac{2}{3}\vec{b} + \frac{2}{9}\vec{c}$$

$$|\overrightarrow{OA_1}| = ?$$

$$|\overrightarrow{OA_1}|^2 = \left(\frac{2}{3}\vec{b} + \frac{2}{9}\vec{c}\right)^2 = \frac{4}{9}\vec{b}^2 + \frac{4}{81}\vec{c}^2 + \frac{8}{27}\vec{b}\vec{c} = \frac{4}{9} + \frac{4}{\cancel{81}_{9}} \cdot \cancel{9} + \frac{\cancel{8}^{4}}{\cancel{27}_{9}} \cdot \frac{\cancel{3}}{\cancel{2}}$$

$$= 3 \cdot \frac{4}{9} \Rightarrow |\overrightarrow{OA_1}| = \frac{2}{3}\sqrt{3}$$