2-D Wave Equation

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October 16, 2008

The wave equation can be stated in the following form,

$$\frac{\partial^2 p}{\partial p^2} = c^2 \nabla^2 p, \tag{1}$$

where p is a physical measure such as a pressure, or the height of the water in a tub. This equation can be discretized over a 2d grid using a finite differencing scheme. Assume the pressures are located across a rectangular 2-D grid in x and y dimensions. We call the value of the pressure $p^t_{x,y}$ for time t at location (x, y) in the grid. One solution to the differential equation (1) is the following:

$$p^{t+1}_{x,y} + p^{t-1}_{x,y} - 2p^t_{x,y} = c^2 \left(p^t_{x+1,y} + p^t_{x-1,y} + p^t_{x,y+1} + p^t_{x,y-1} - 4p^t_{x,y} \right). \tag{2}$$

The left hand side term $\frac{\partial^2 p}{\partial p^2}$ is represented in terms of the values of p at three consecutive timesteps at grid location (x, y). The right hand side terms are discretized using the p values for the 4 locations adjacent to (x, y) in the grid. One can simply solve for $p^{t+1}_{x,y}$ in equation (2) to produce an update rule,

$$p^{t+1}_{x,y} = c^2 \left(p^t_{x+1,y} + p^t_{x-1,y} + p^t_{x,y+1} + p^t_{x,y-1} - 4p^t_{x,y} \right) - p^{t-1}_{x,y} + 2p^t_{x,y}. \tag{3}$$

This update rule specifies how the p value for each location in the grid should be computed, using p values from the two previous time steps. This update rule is easy to code in an application. The application simply maintains two timesteps' worth of p grids, and using these, the next timestep's p grid can be computed. The c term determines the wave speed. This value must be small enough such that the wave cannot move across more than one grid square in a single timestep. Smaller values for c will make the simulation take longer to propagate a wave by a fixed distance, but larger values for c can introduce dispersive and diffusive errors.