Part3- Extensive Form Games

Based on slides by David Sarne and Jackson, Leyton-Brown & Shoham

The Notion of Sequence

- The notion of sequence, or time, is an important ingredient of a game.

- Normal form game does not involve any explicit representation of sequence

- Extensive form game is an alternative representation for that purpose

Perfect Information Extensive Form Game

- A (finite) perfect-information game (in extensive form) is defined by the tuple (N ; A ; H ; Z ; χ ; ρ ; σ ; u), where:
  - Players: N
  - Actions: A
  - Choice nodes and labels for these nodes:
    - Choice nodes: H
    - Action function: $\chi : H \rightarrow 2^A$ assigns to each choice node a set of possible actions
    - Player function: $\rho : H \rightarrow N$ assigns to each non-terminal node h a player $i \epsilon N$ who chooses an action at h

Perfect Information Extensive Form Game

- A (finite) perfect-information game (in extensive form) is defined by the tuple (N ; A ; H ; Z ; χ ; ρ ; σ ; u), where:
  - Terminal nodes: Z is a set of terminal nodes, disjoint from H
  - Successor function: $\sigma : H\boldsymbol{x}A \rightarrow H$ **U** Z maps a choice node and an action to a new choice node or terminal node such that for all $h_1, h_2 \in H$ and $a_1, a_2 \in A$, if $\sigma(h_1, a_1) = \sigma(h_2, a_2)$ then $h_1 = h_2$ and $a_1 = a_2$
    - Choice nodes form a tree: nodes encode history
  - Utility function: $u = (u_1, ..., u_n)$; $u_i : Z \rightarrow R$ is a utility function for player i on the terminal nodes Z

Example- The Sharing Game

- A brother and a sister needs to divide 2 indivisible and identical presents from their parents.
- The brother is the first to suggest a split.
- If the sister doesn't agree, they both get nothing.
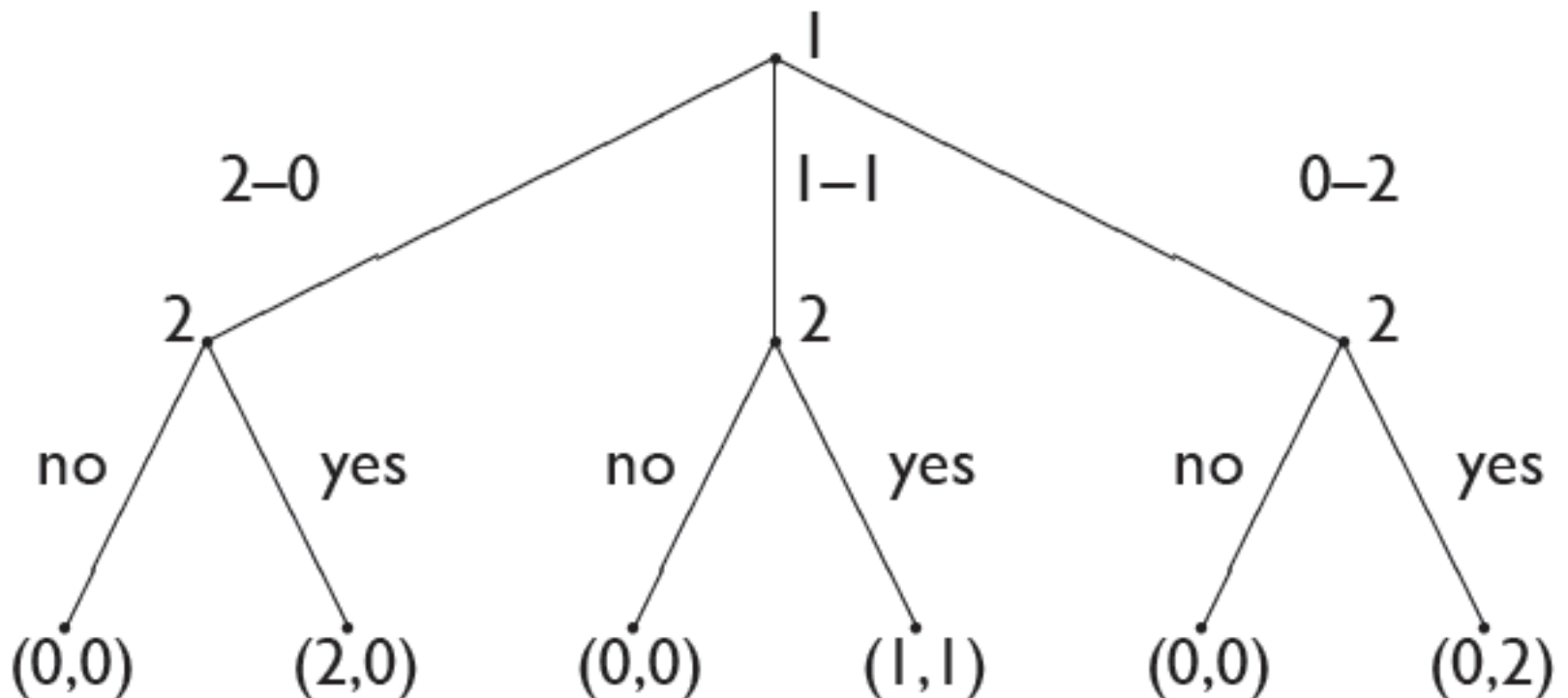
Strategies
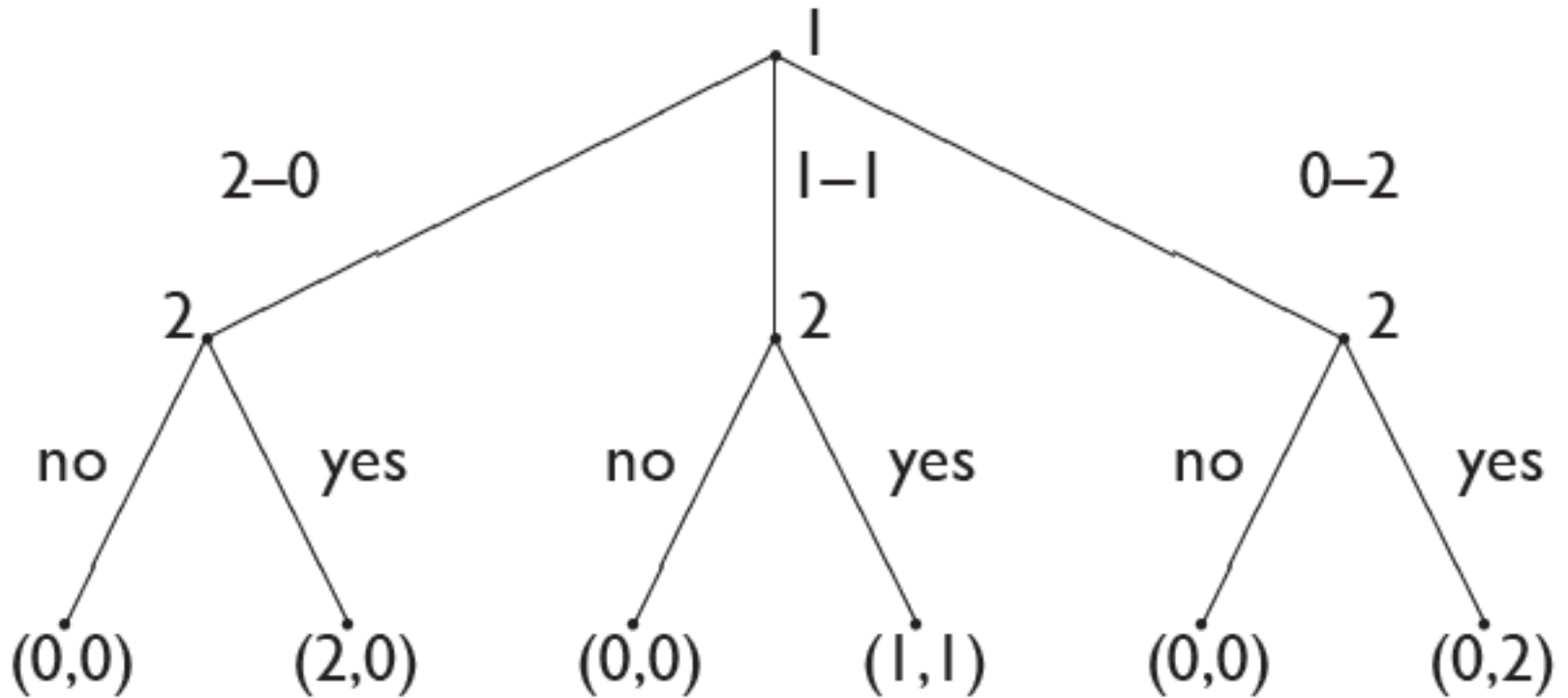
- How many pure strategies are there?
  – Player 1 has 3 strategies (2-0,1-1,0-2)
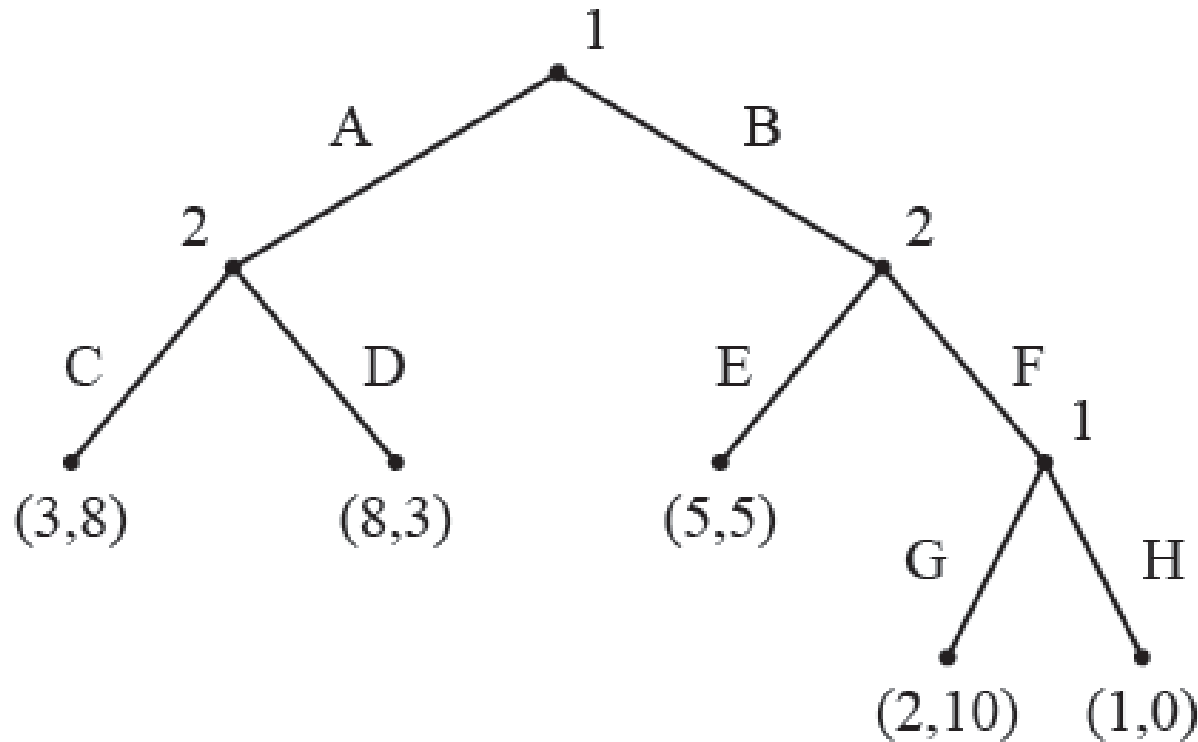 – Player 2 has 8 strategies

Pure Strategies

A pure strategy for a player in a perfect-information game is a complete specification of which deterministic action to take at every node belonging to that player.

Definition (pure strategies)

Let $G = (N, A, H, Z, \chi, \rho, \sigma, u)$ be a perfect-information extensive-form game. Then the **pure strategies** of player i consist of the cross product

$$\prod_{h \in H, \rho(h) = i} \chi(h)$$

Example

- What are the pure strategies of player2
  – $S2 = \{(C,E),(C,F),(D,E),(D,F)\}$
- What are the pure strategies of player 1?
  – $S1 = \{(B,G),(B,H),(A,G),(A,H)\}$

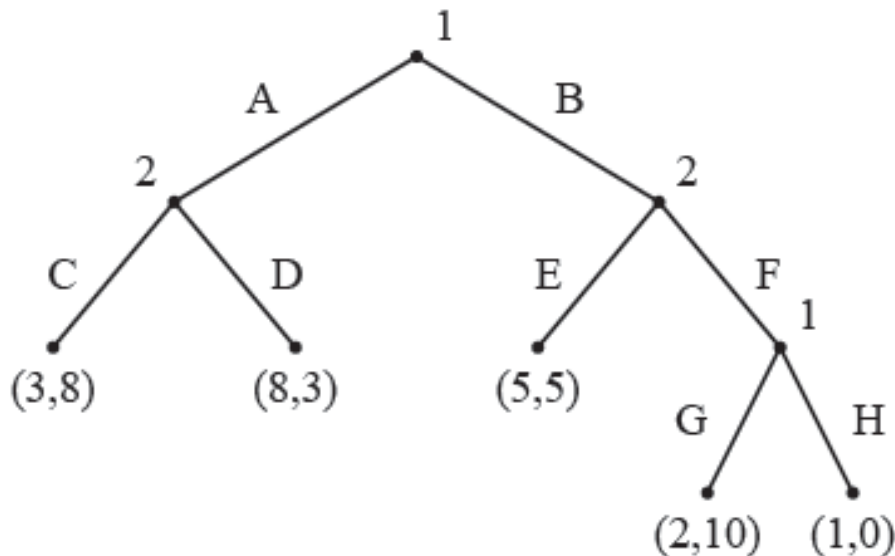We will never reach to H, but still, (A,G) is different from (A,H)

Nash Equilibrium

Using the basic definition of pure strategy it is easy to redefine:

- Mixed strategy
- Best response
- Nash Equilibrium

Converting to a Normal-form Game

For every perfect-information game there exists a corresponding normal-form game with the same strategy space.

	CE	CF	DE	DF
AG	3, 8	3, 8	8, 3	8, 3
AH	3, 8	3, 8	8, 3	8, 3
BG	5, 5	2, 10	5, 5	2, 10
BH	5, 5	1, 0	5, 5	1, 0

Note:
Every result/proof that is true for normal form games is also true for extensive form games

The Advantages of Extensive Form Game

- An explicit representation of time
- A compact representation
  - The temporal structure of the extensive-form representation can result in a certain redundancy within the normal form.
  - E.g., in the previous example we had 16 outcomes in the normal form game versus only 5 with the extensive form game.

- But…
  - the reverse transformation— from the normal form to the perfect-information extensive form—does not always exist
  - E.g. matching coins

What about Mixed Strategies?

Theorem

Every perfect information game in extensive form has a *PSNE*

- Why?

  - Since the game is sequential

- Since players take turns, and everyone gets to see everything that happened thus far before making a move, it is never necessary to introduce randomness into action selection.

- Note: this is true only for perfect information extensive form game.

"Let's Burn the Boats"

- What does it mean?
  - You give away alternative B
  - So the other player won't have a choice other than trying to achieve alternative A, which is better for you.

- **Paradox of commitment**
  - It is by limiting my own options that I can manage to influence the rival's course of actions in my interest.

Hernan Cortes and the Aztecs

- From a good Spanish family.

  - Failed at being a European soldier.
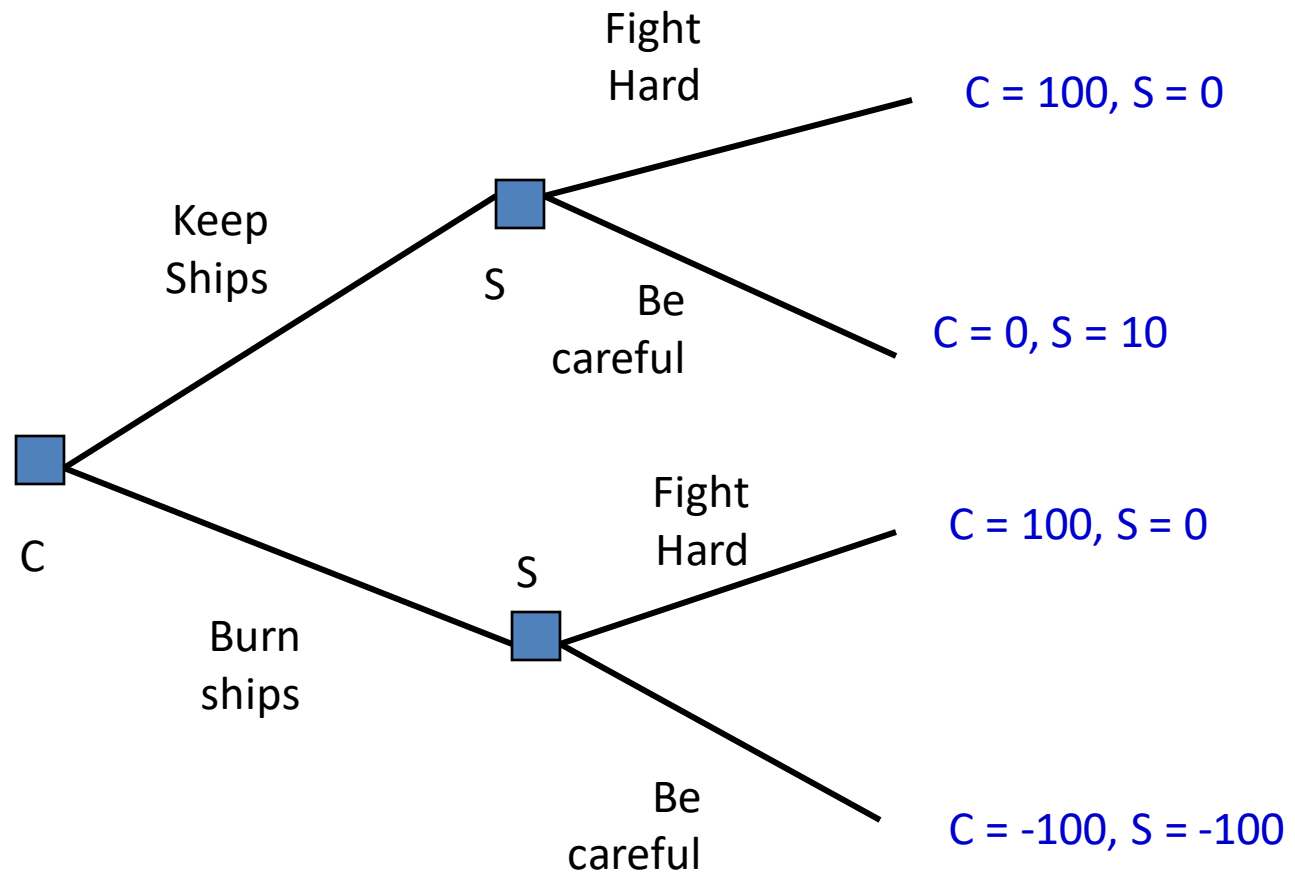
  - Failed at becoming a lawyer.

  Cortes was hungry for wealth and recognition.
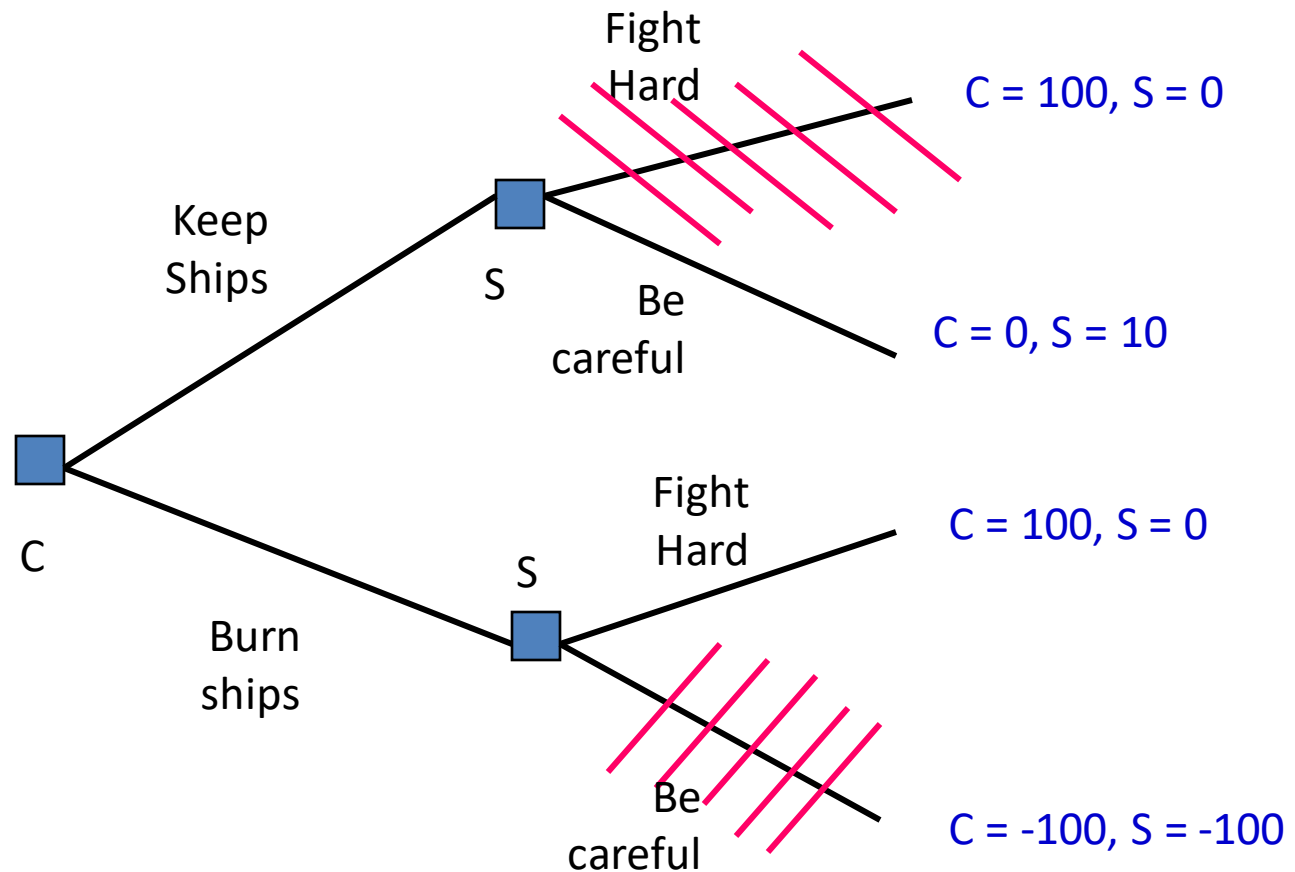
Cortes

- 1519 – landed on the coast of Mexico with :
  - 600 men, 16 horses, and a few cannon.
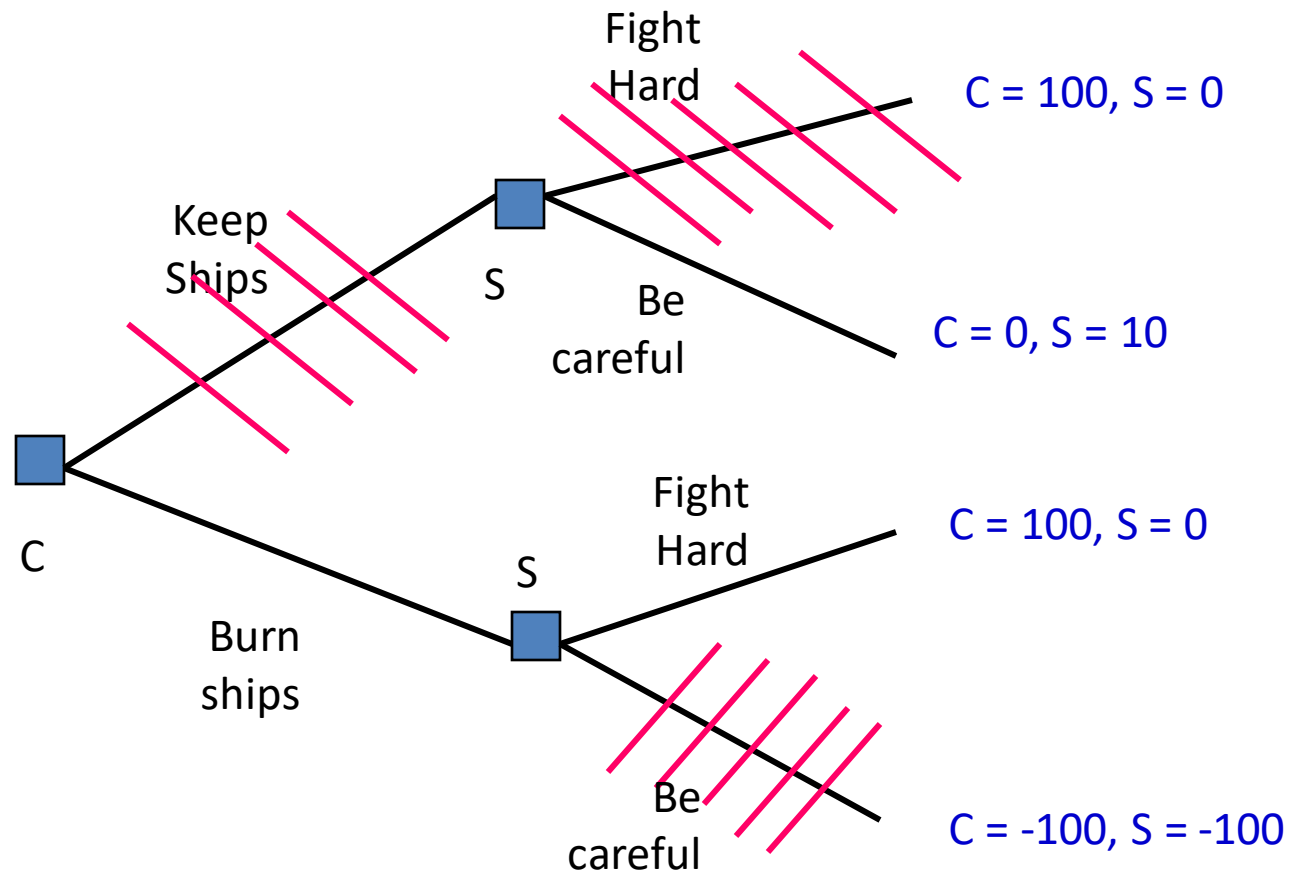- WHY WOULD CORTES BURN HIS SHIPS WHEN HE GOT TO THE NEW WORLD?

Think of Cortes trying to motivate his own soldiers

Fight Hard → C = 100, S = 0

Keep Ships → S

Be careful → C = 0, S = 10

C

Burn ships → S

Fight Hard → C = 100, S = 0

Be careful → C = -100, S = -100

If no retreat possible, will fight hard or die. But if retreat is possible, may fight less hard and 'run away'

Fight Hard

C = 100, S = 0

Keep Ships

S

Be careful

C = 0, S = 10

C

Fight Hard

C = 100, S = 0

Burn ships

S

Be careful

C = -100, S = -100

So Cortes wants to burn his ships. It is a credible commitment not to retreat – and this alters how his own troops behave.

Fight Hard

C = 100, S = 0

Keep Ships

S

Be careful

C = 0, S = 10

C

Fight Hard

C = 100, S = 0

S

Burn ships

Be careful

C = -100, S = -100

Where else did we see it?

- Nick from Golden Balls
- Chicken game

- Business guru, Tom Peters swears by Cortés' destructive strategy by going so far as to sug-gest that every company hire a CDO – a Chief Destructive Officer.

Subgame-perfect equilibrium

- Find all the Nash equilibria in the game
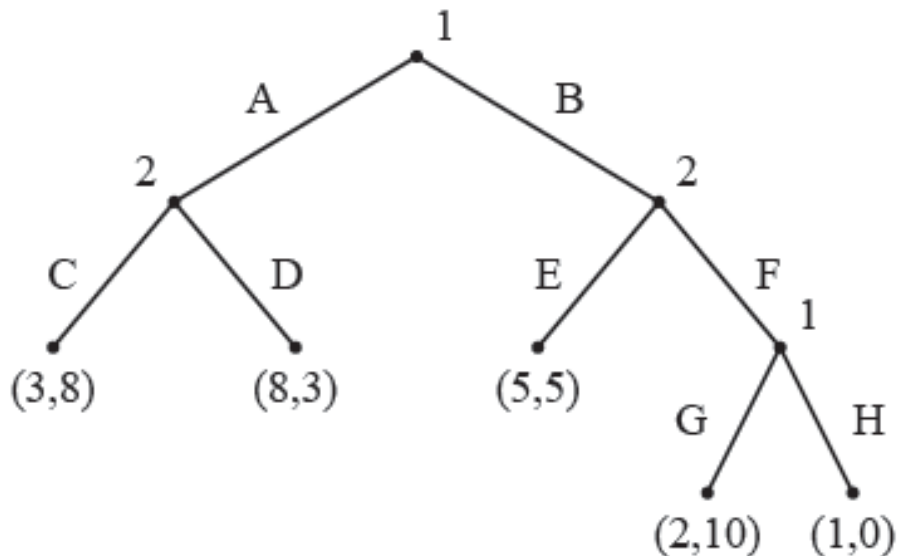
	CE	CF	DE	DF
AG	3, 8	3, 8	8, 3	8, 3
AH	3, 8	3, 8	8, 3	8, 3
BG	5, 5	2, 10	5, 5	2, 10
BH	5, 5	1, 0	5, 5	1, 0

- Is (AG,CF) a reasonable equilibrium?

Subgame-perfect equilibrium
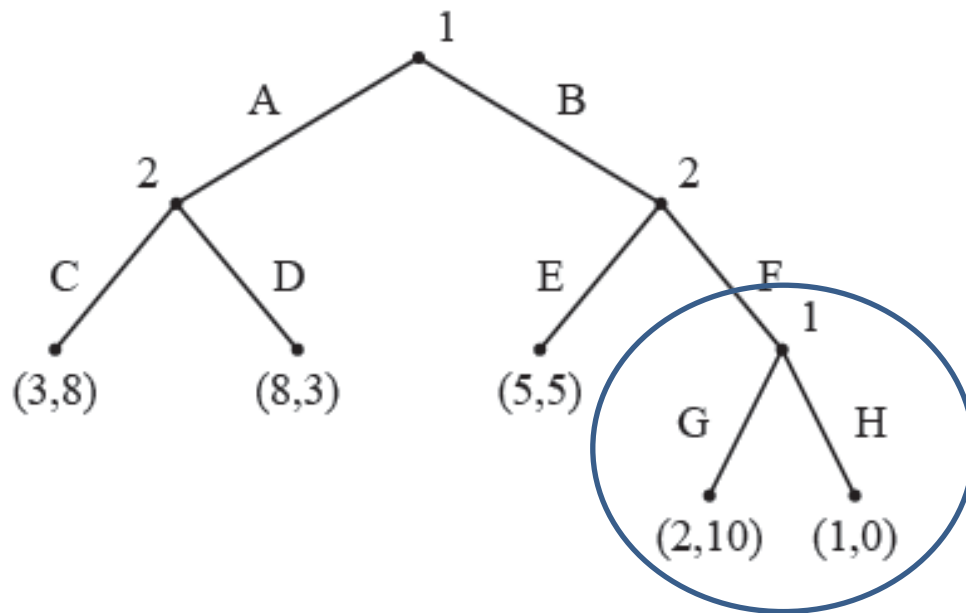
	CE	CF	DE	DF
AG	3, 8	3, 8	8, 3	8, 3
AH	3, 8	3, 8	8, 3	8, 3
BG	5, 5	2, 10	5, 5	2, 10
BH	5, 5	1, 0	5, 5	1, 0

- Is (BH,CE) a reasonable equilibrium?
  - Why player 1 would play H if he can play G?
  - Since the choice in H is a threat that keeps the equilibrium. Player 2 know that if he chooses F player 1 will choose H and therefore player 2 prefers to keep choosing E.
  - However, is this a credible threat? Will player 1 actually play H if we reach to that point?

Formal Definition

Definition (subgame of G rooted at h)

The subgame of G rooted at h is the restriction of G to the descendents of h.

Definition (subgames of G)

The set of subgames of G is defined by the subgames of G rooted at each of the nodes in G.

- *The subgame-perfect equilibria (SPE) of a game G are all strategy profiles s such that for any subgame G' of G, the restriction of s to G' is a Nash equilibrium of G'.*

Recall our example
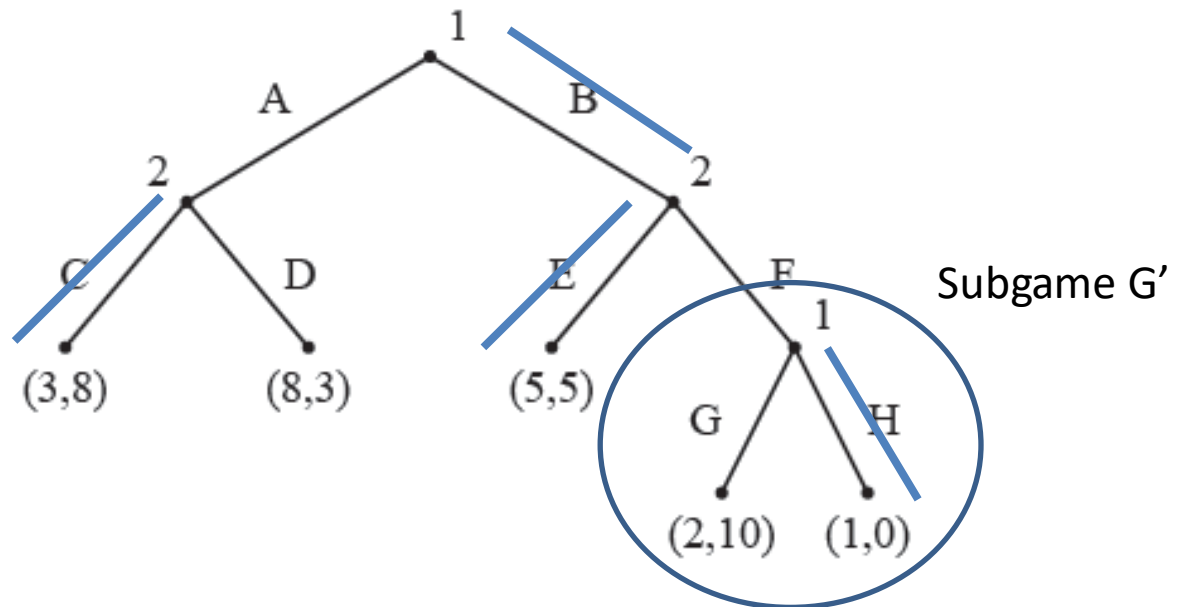
Subgame G'

In this subgame, H is not part of a N.E. Thus, (BH,CE) is not subgame-perfect equilibrium

Additional Notes

- The definition of subgame-perfect rules out "noncredible threats".

- Since G is its own subgame, every SPE is also a Nash equilibrium.

- Every perfect-information extensive-form game has at least one subgame-perfect equilibrium (proof is by induction on the height of the game tree).
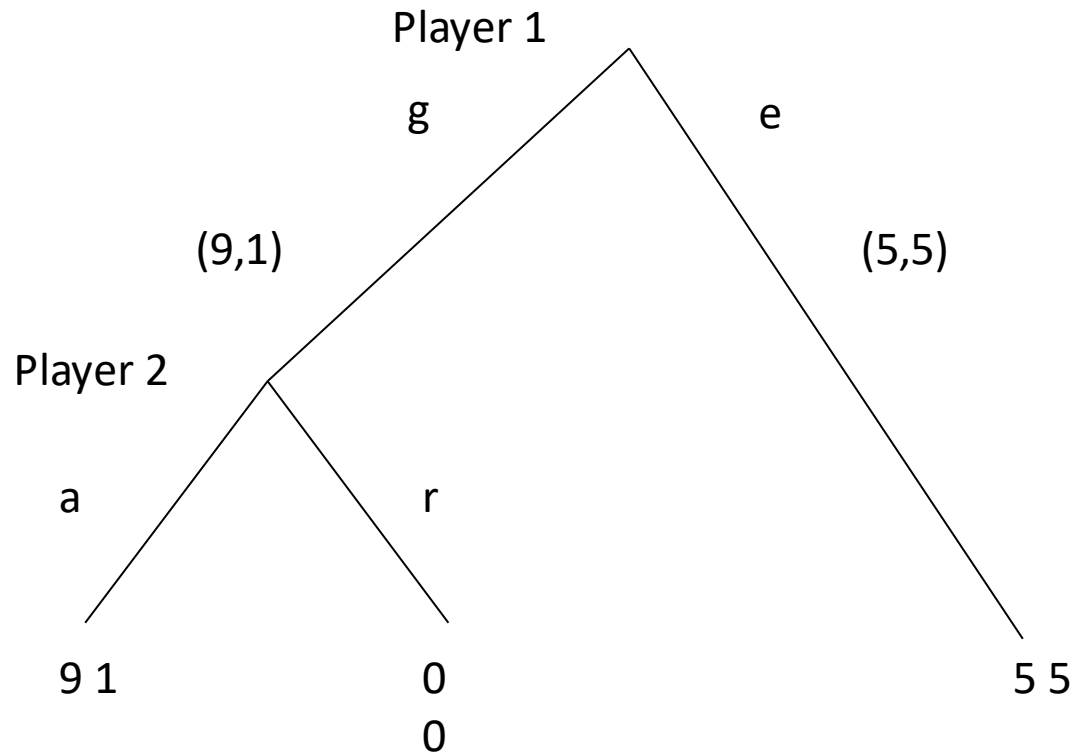
What is the Subgame-perfect Equilibrium?
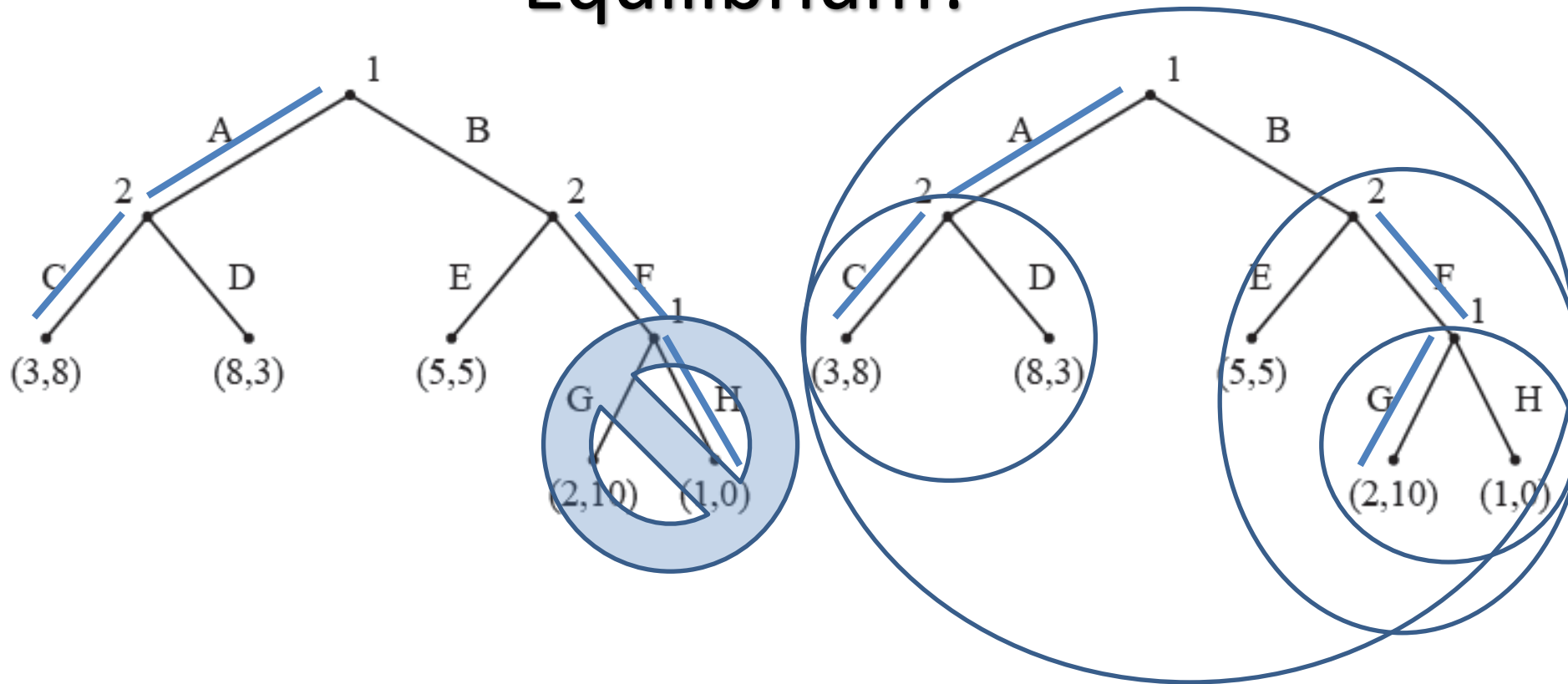
The strategy H in (AH,CF) is called off path

Example: Mini Ultimatum Game

- Proposer (Player 1) can suggest one of two splits of £10: (5,5) and (9,1).

- Responder (Player 2) can decide whether to accept or reject (9,1), but has to accept (5,5). Reject leads to 0 for both
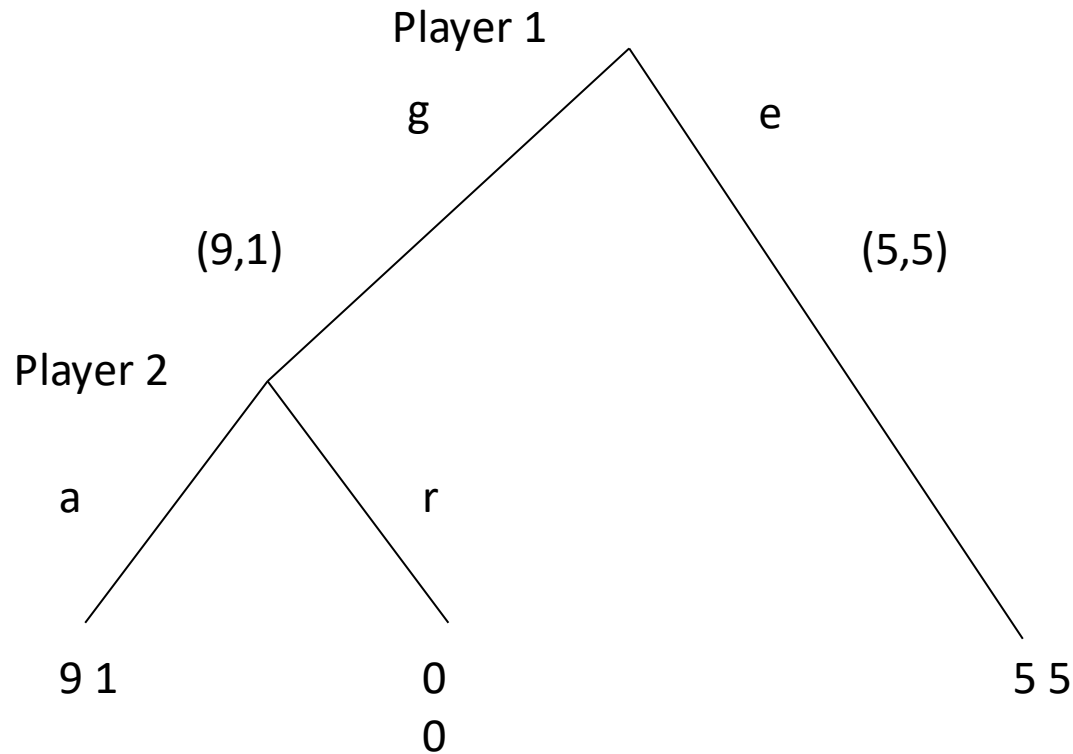
Player 1

g

e

(9,1)

(5,5)

Player 2

a

r

9 1

0
0

5 5

Mini Ultimatum Game in Strategic Form

		Player	2
		accept (9,1)	reject (9,1)
Player 1	propose (5,5)	5,5	5,5
	propose (9,1)	9,1	0,0

- There are **two** equilibria:
  1. (propose (9,1), accept (9,1))
  2. (propose (5,5), reject (9,1)).
- Equilibrium 2 is in **weakly dominated** strategies (reject (9,1) is weakly dominated)

Example: Mini Ultimatum Game
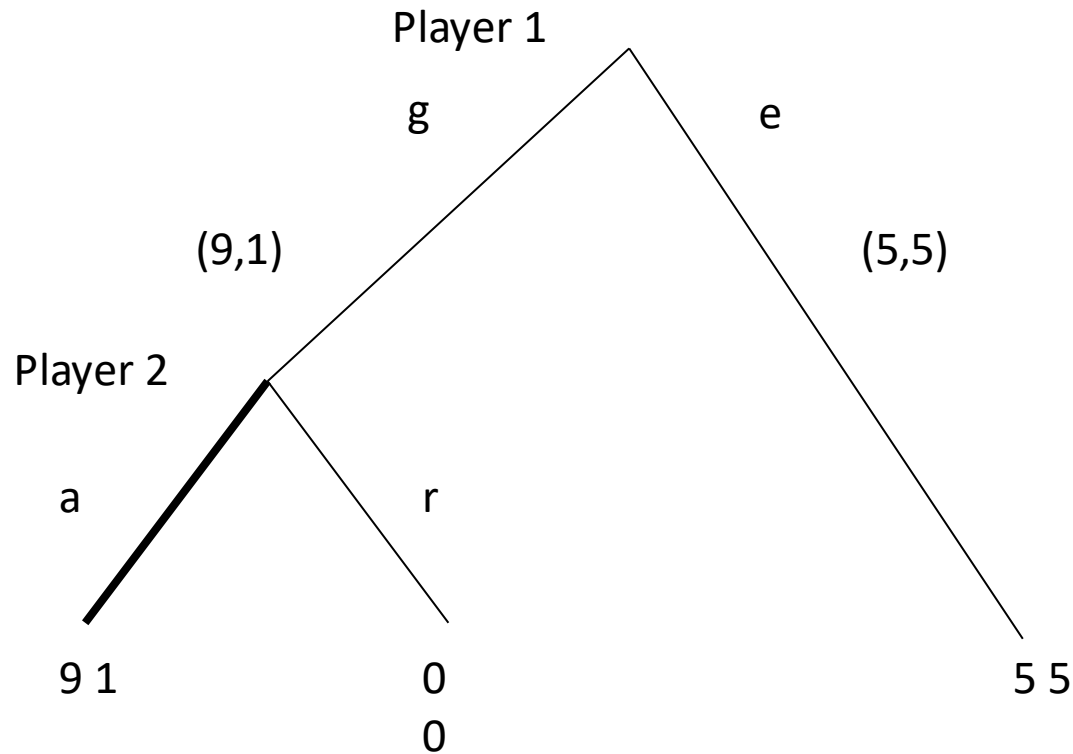
- There is one subgame of length 1, following (9,1)

Player 1

g   e

(9,1)   (5,5)

Player 2

a   r

9 1   0   5 5

0

# Example: Mini Ultimatum Game

- There is one subgame of length 1, following (9,1)
- The optimal action is accept

Player 1

g   e

(9,1)   (5,5)

Player 2

a   r

9 1   0   5 5
                     0

Example: Mini Ultimatum Game

- There is one subgame of length 1, following (9,1)
- The optimal action is accept
- There is one subgame of length 2, the whole game

Player 1

g   e

(9,1)   (5,5)

Player 2

a   r

9 1   0   5 5
           0
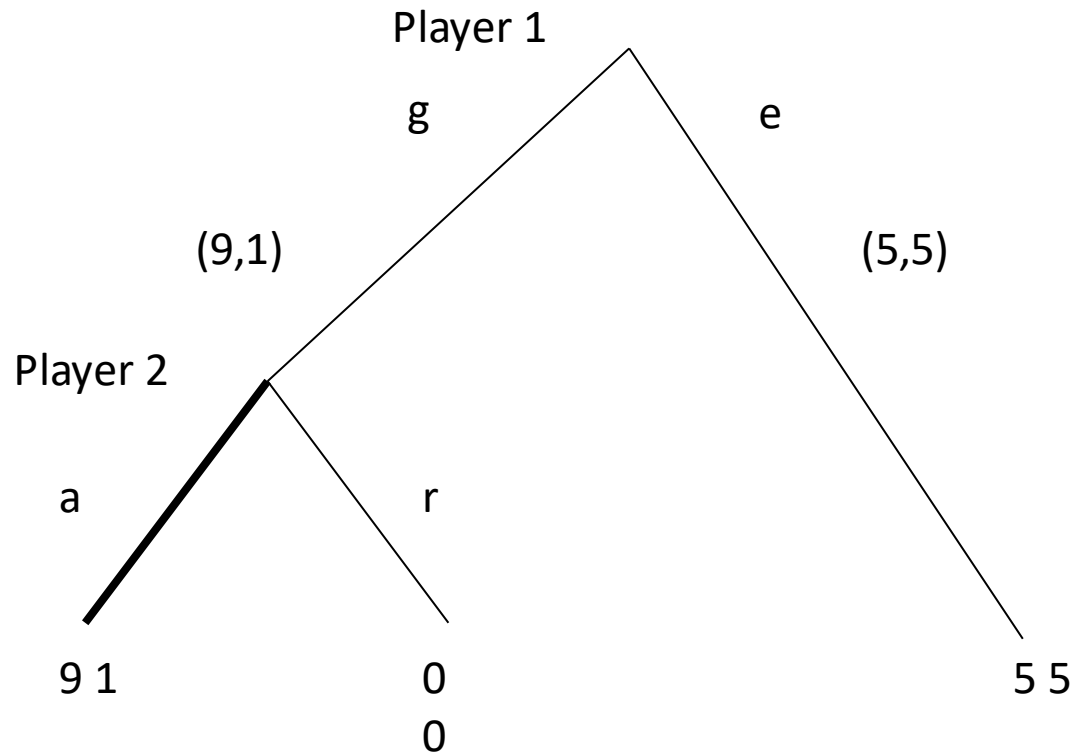
Example: Mini Ultimatum Game

- There is one subgame of length 1, following (9,1)
- The optimal action is accept
- There is one subgame of length 2, the whole game
- Taking "accept" in the subgame of length 1 as given, we see that (9,1) is optimal

Player 1

g   e

(9,1)   (5,5)

Player 2

a   r

9 1   0   5 5
           0
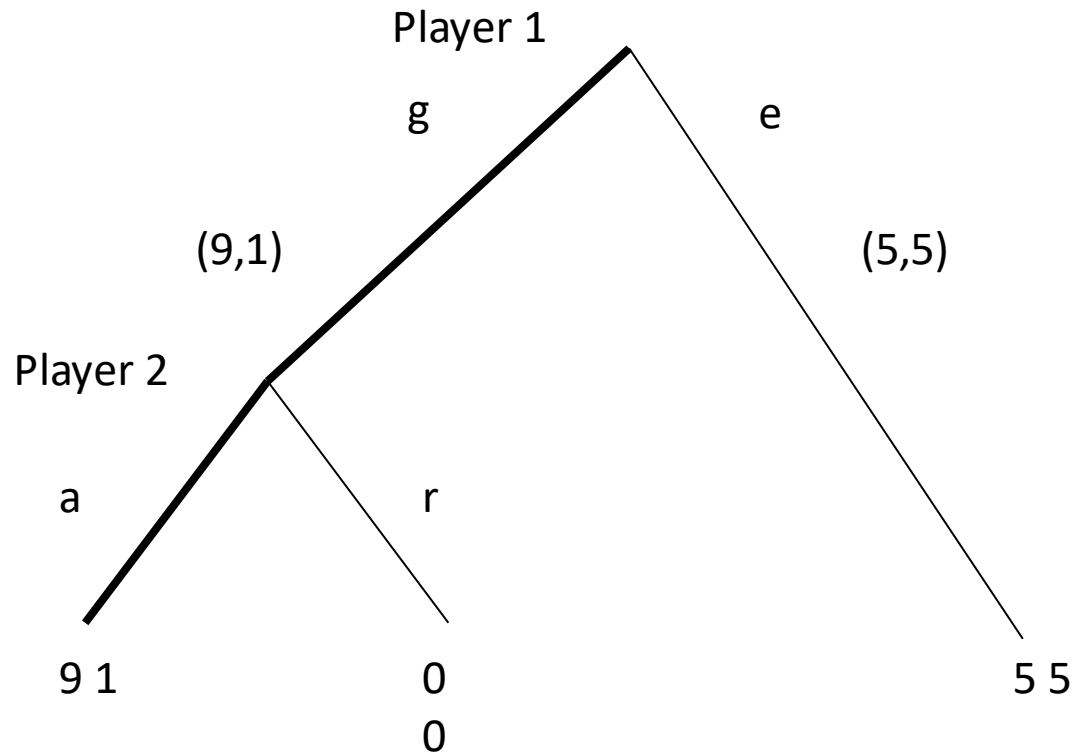
Computing one SPE - Backward Induction

all leaf nodes

set of all actions available to player at that node

function BACKWARDINDUCTION (node h) **returns** $u(h)$
if $h \in Z$ **then**
$\quad$ ⌊ **return** $u(h)$
$best\_util \leftarrow -\infty$
forall $a \in \chi(h)$ **do**
$\quad$ $util\_at\_child \leftarrow$ BACKWARDINDUCTION$(\sigma(h, a))$
$\quad$ **if** $util\_at\_child_{\rho(h)} > best\_util_{\rho(h)}$ **then**
$\quad\quad$ ⌊ $best\_util \leftarrow util\_at\_child$
return $best\_util$

vector denoting the utility for each player

Computing one SPE - Backward Induction

- This procedure does not return an equilibrium strategy for each of the n players, but rather describes how to label each node with a vector of n real numbers.

- This labeling can be seen as an extension of the game's utility function to the nonterminal nodes H.

- The players' equilibrium strategies follow straightforwardly from this extended utility function: every time a given player i has the opportunity to act at a given node $h \in H$ (i.e., $\rho(h) = i$), that player will choose an action $a_i \in \chi(h)$ that solves argmax $a_i \in \chi(h)$ $u_i (\sigma(a_i,h))$.

Example: Voting for a Payraise

Three legislators are voting on whether to give themselves a pay raise.
All three want the pay raise; however,
Each face a small cost in voter resentment c>0.
The benefit for the raise is greater than cost: b>c
They vote in the order 1-2-3. Simple majority rule
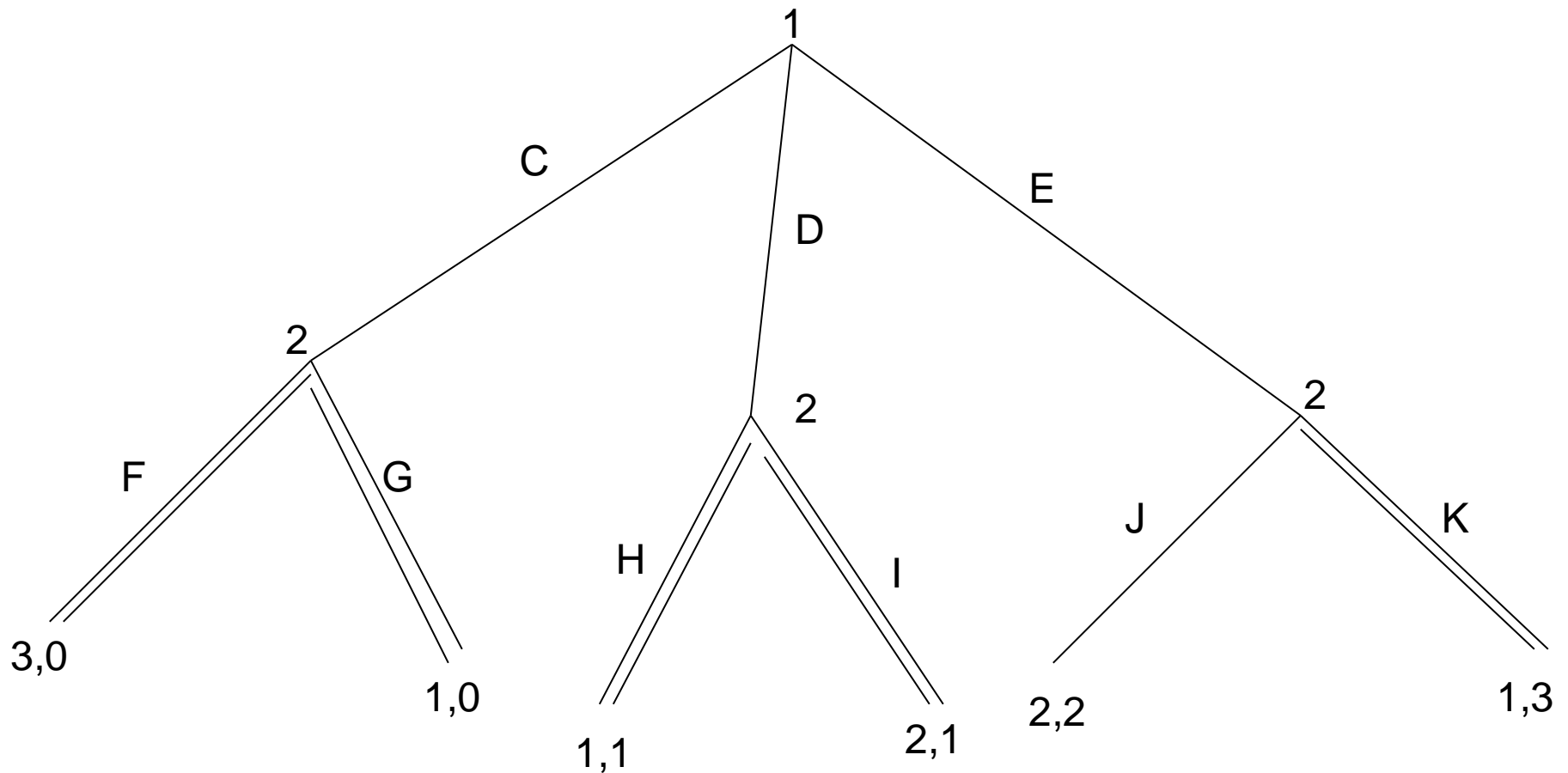What is the outcome obtained by backward induction?
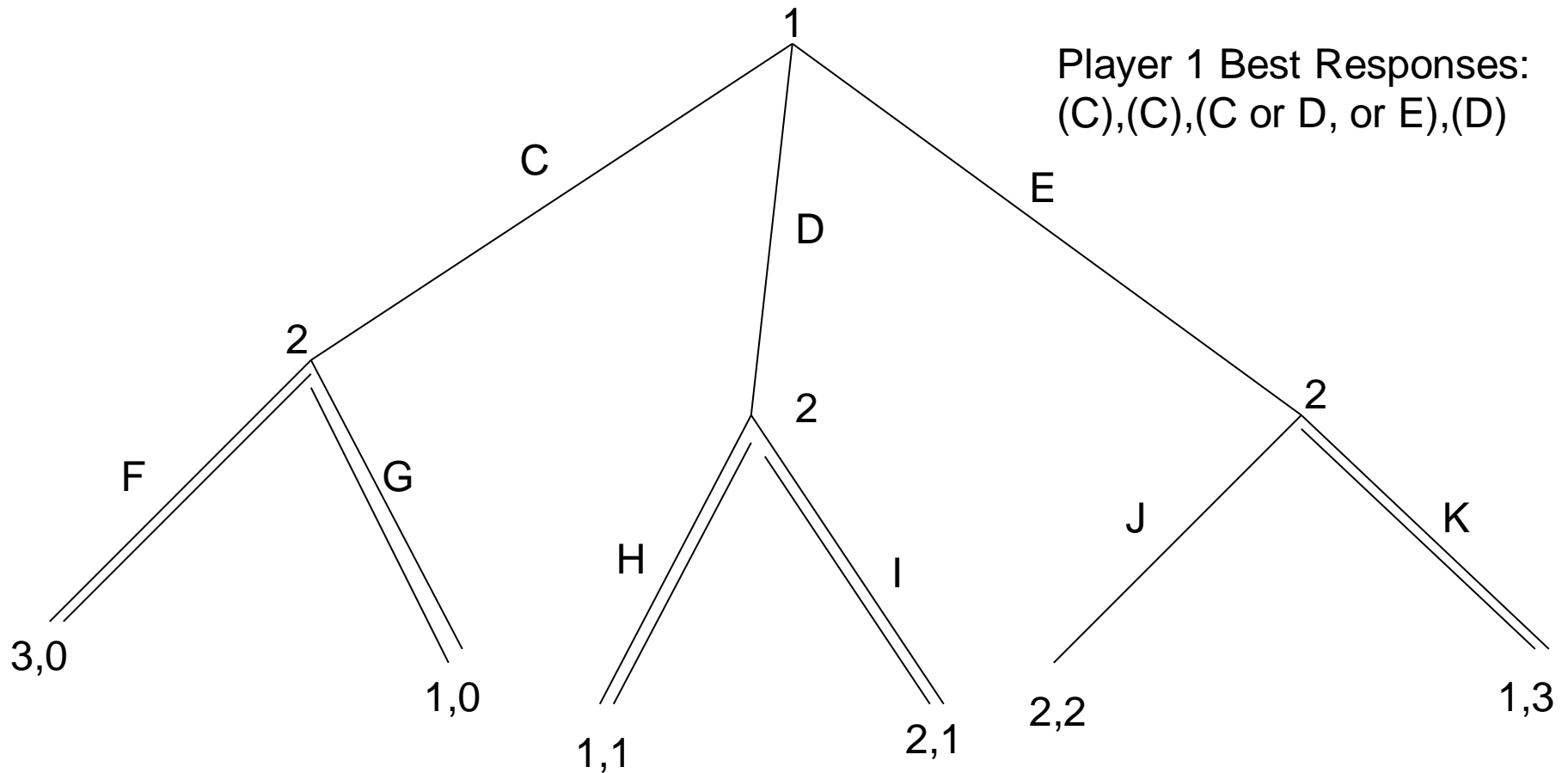   (N; NY; nyyn)
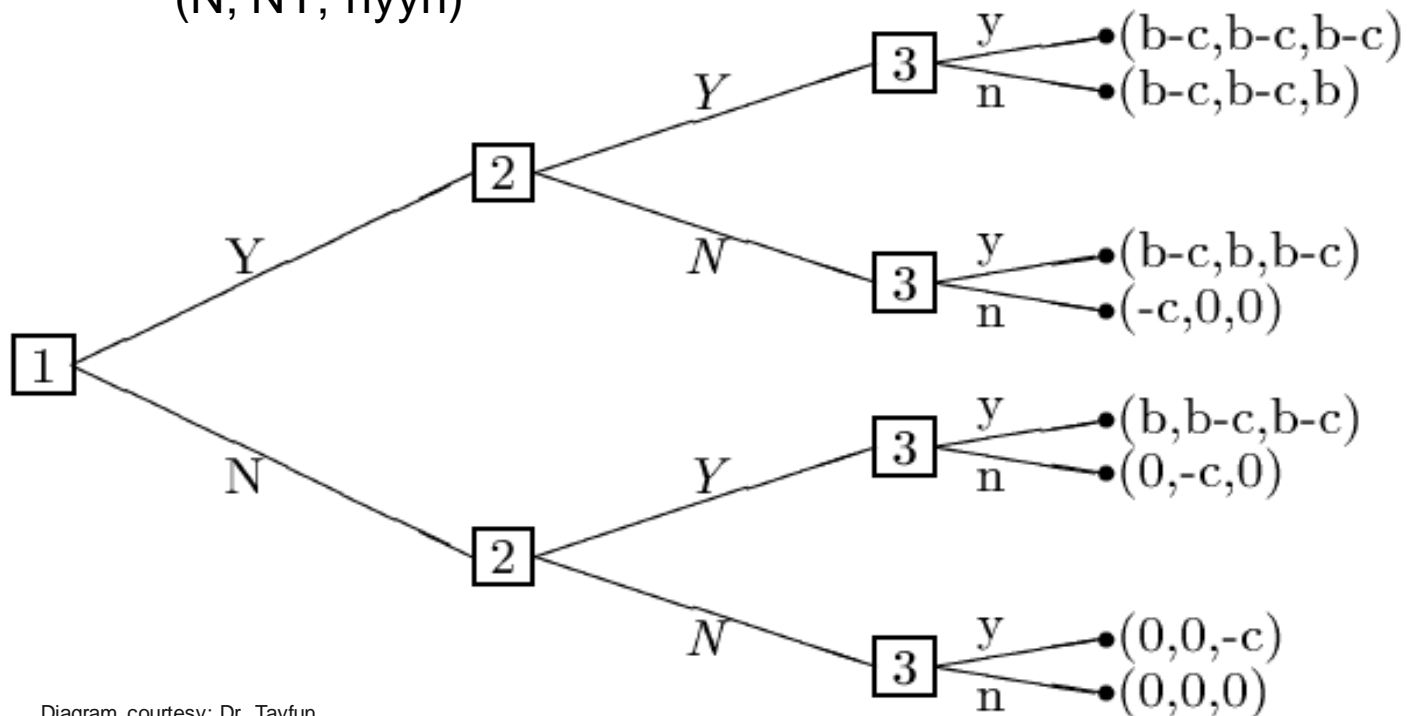
Multiple SPE
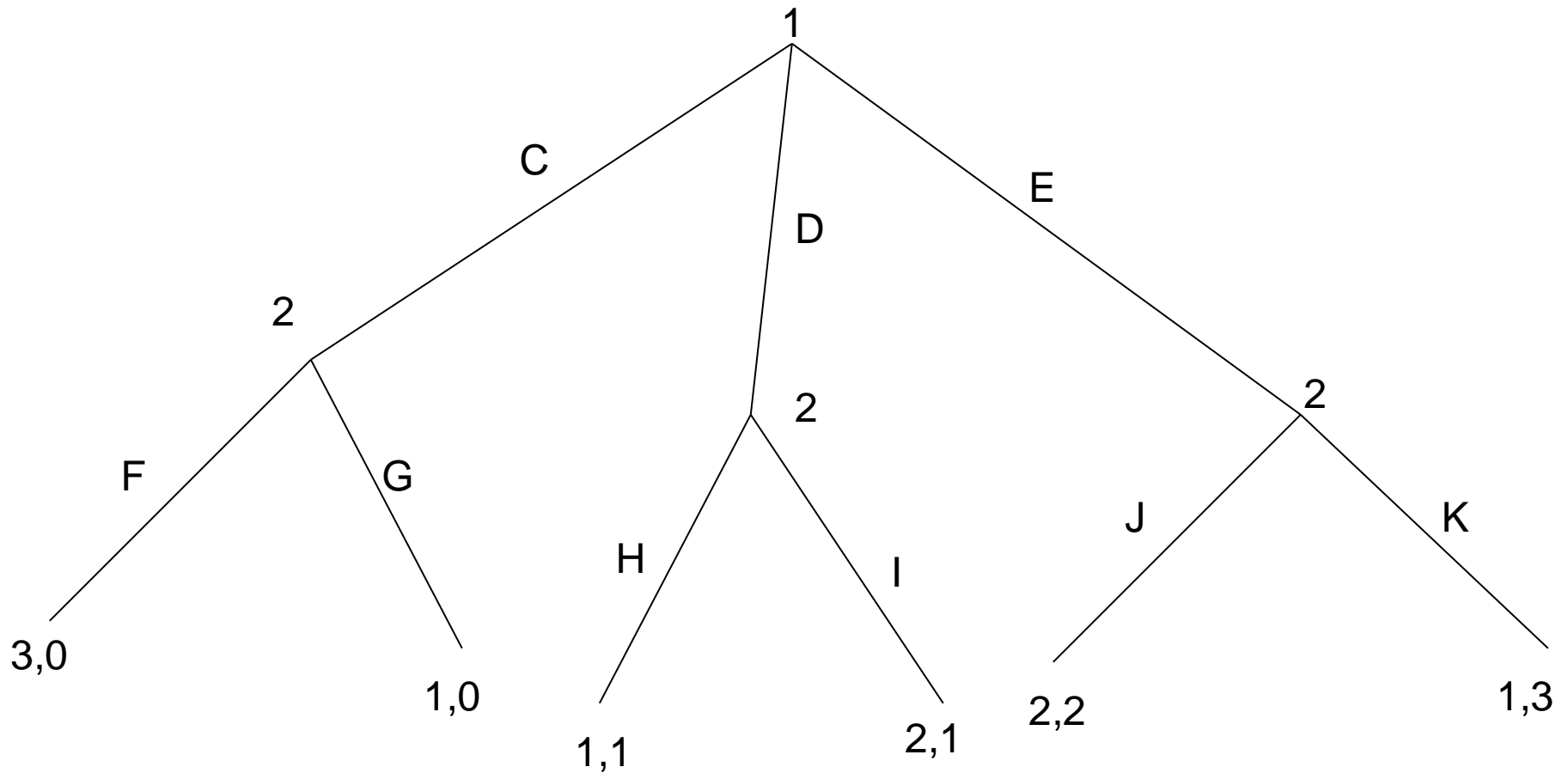
Multiple SPE

Player 2 Optimal Strategies:
(FHK),(FIK),(GHK),(GIK)

Multiple SPE

Player 2 Optimal Strategies:
(FHK),(FIK),(GHK),(GIK)

Player 1 Best Responses:
(C),(C),(C or D, or E),(D)

Multiple SPE

All SPE

Player 2 Optimal Strategies:
(FHK),(FIK),(GHK),(GIK)
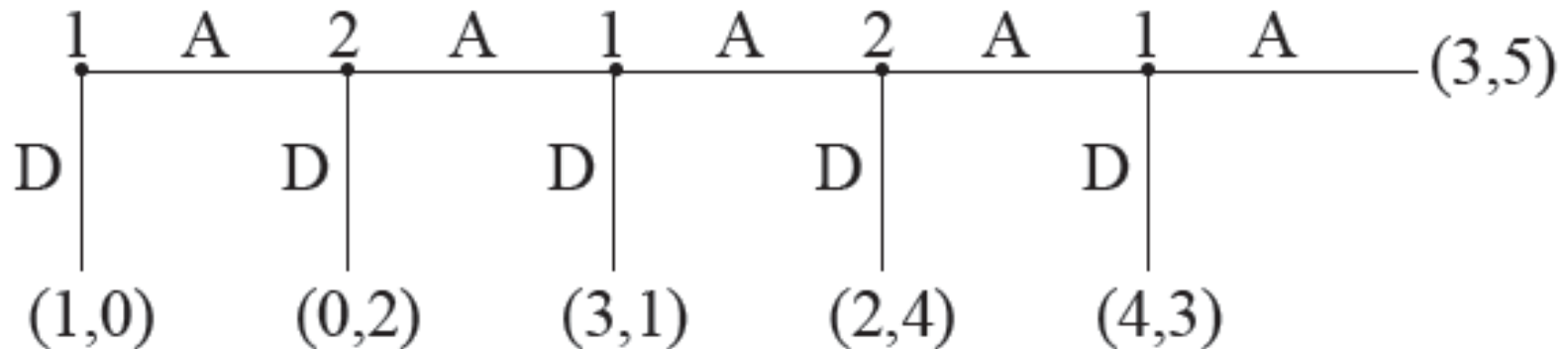
Player 1 Best Responses:
(C),(C),(C or D, or E),(D)

Strategy Pairs:
(C,FHK), (C,FIK), (C,GHK),
(D,GHK), (E,GHK), (D,GIK)

1

C

D

E

2

F

G

2

H

I

2

J

K

3,0

1,0

1,1

2,1

2,2

1,3
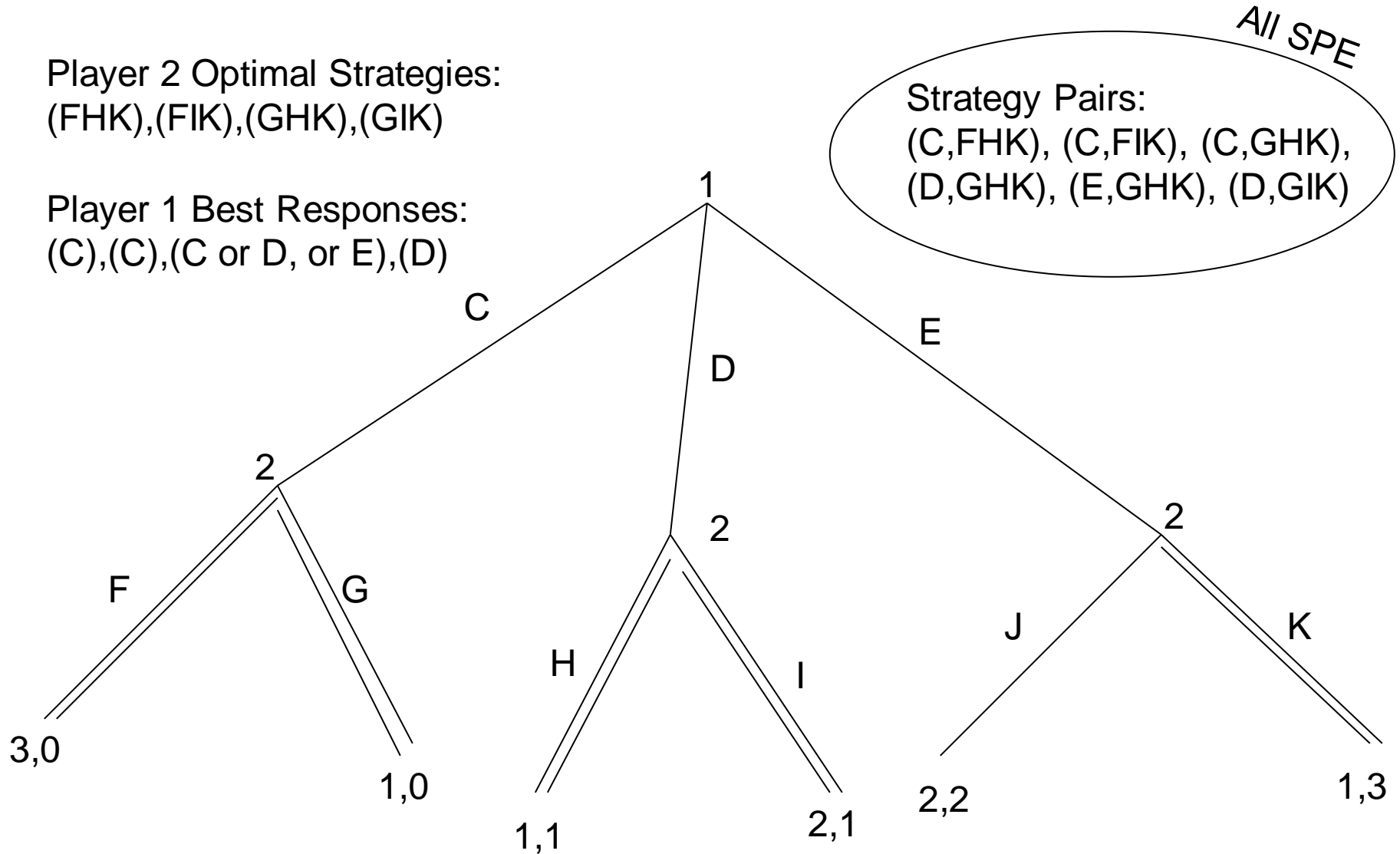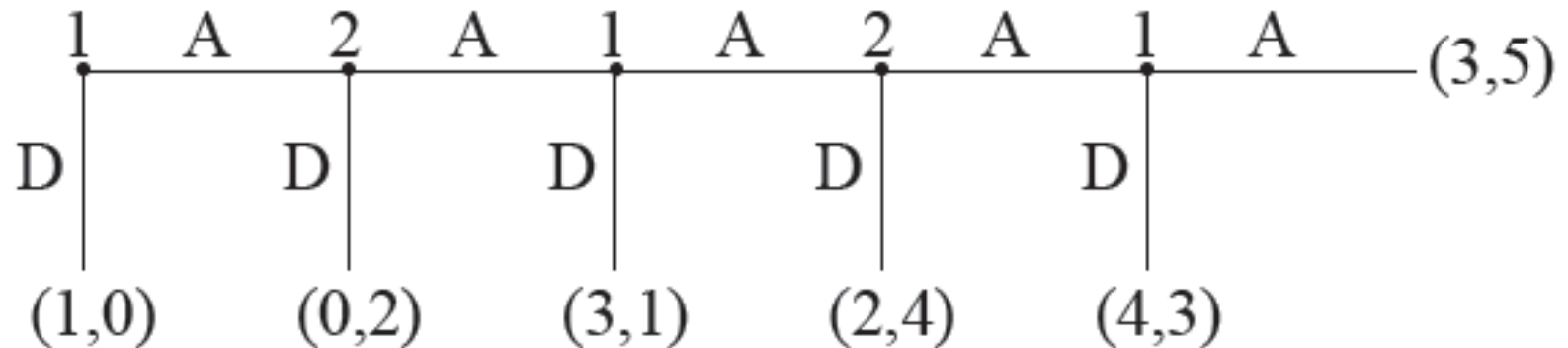
Centipede game

- What is the subgame-perfect equilibrium?
  - Player 1 chooses D at the first round
- Note that this outcome is Pareto-dominated by almost all the other outcomes.

Centipede game

- Will people play according to this SPE?
- What should player 2 do if player 1 chose A at the first round?
  - According to the SPE, he should choose D.
  - But player 1 was also supposed to choose D according to this rational.
  - In other words, you have reached a state to which your analysis has given a probability of zero.

Constant-Sum Centipede game
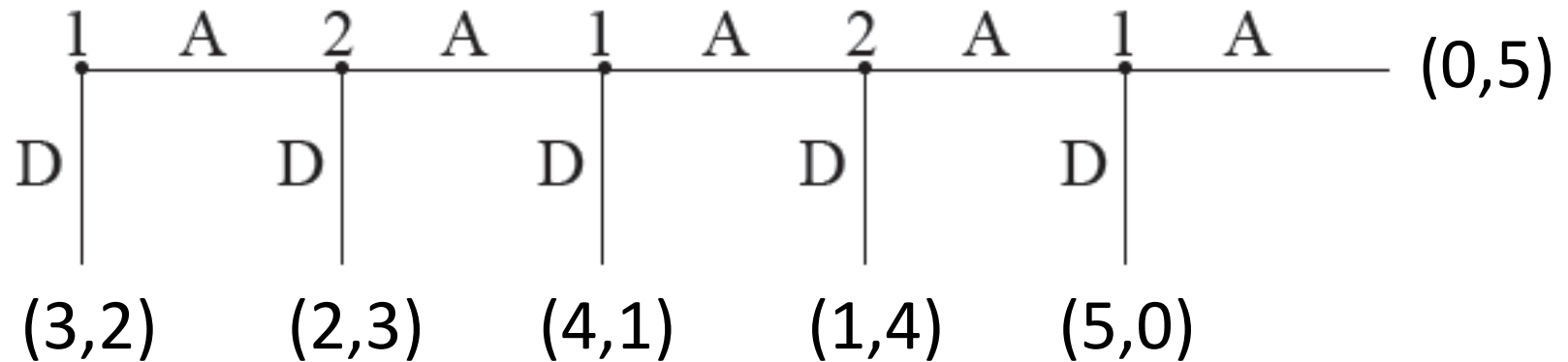
- Instead of having increasing payoffs for both players, the sum of their payoffs is always the same.

- In this case, backward induction gives much more accurate results.

Computing SPE, 2-player Zero-sum Games

- Backward Induction has another name in the two-player, zero-sum context: the minimax algorithm.

- Recall that in such games, only a single payoff number is required to characterize any outcome:
  - Player 1 wants to maximize this number, while player 2 wants to minimize it.
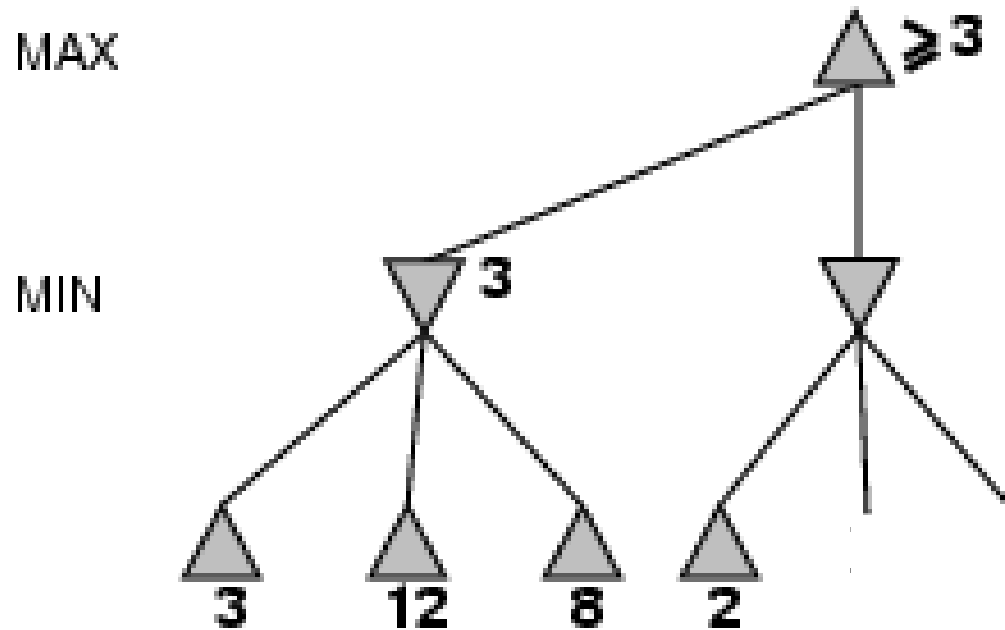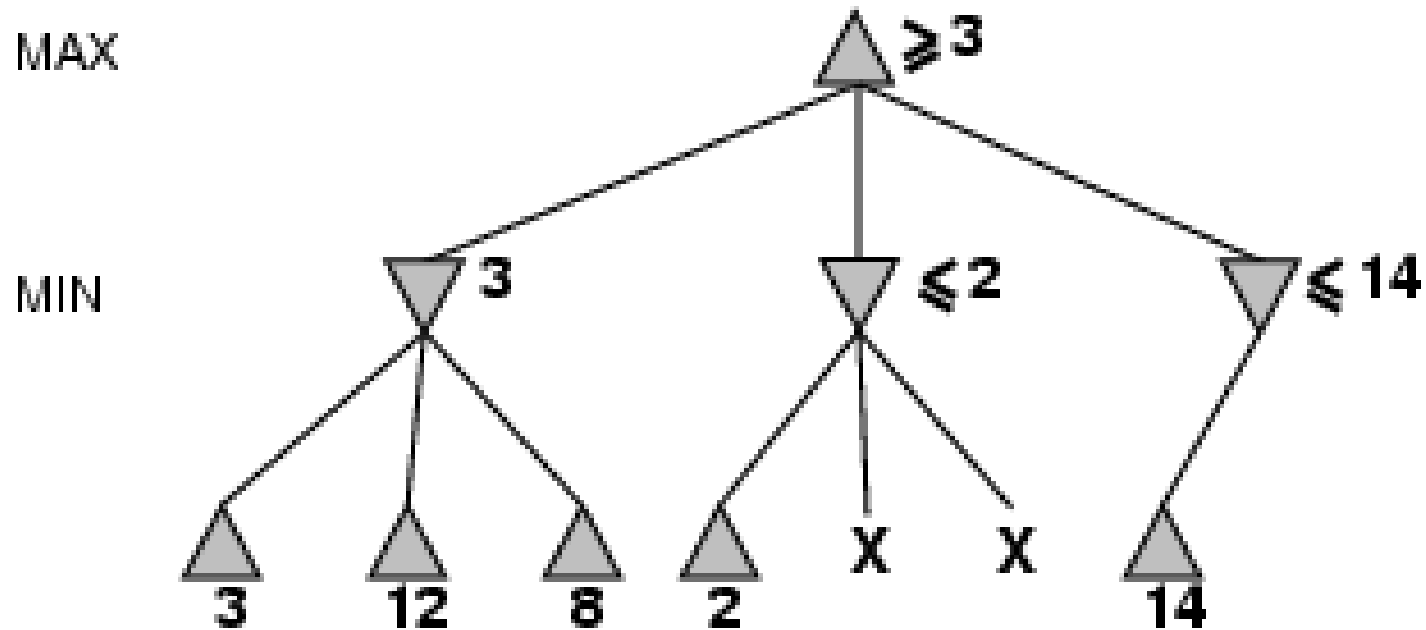 - We call player 1 the MAX player, and player 2 the MIN player.
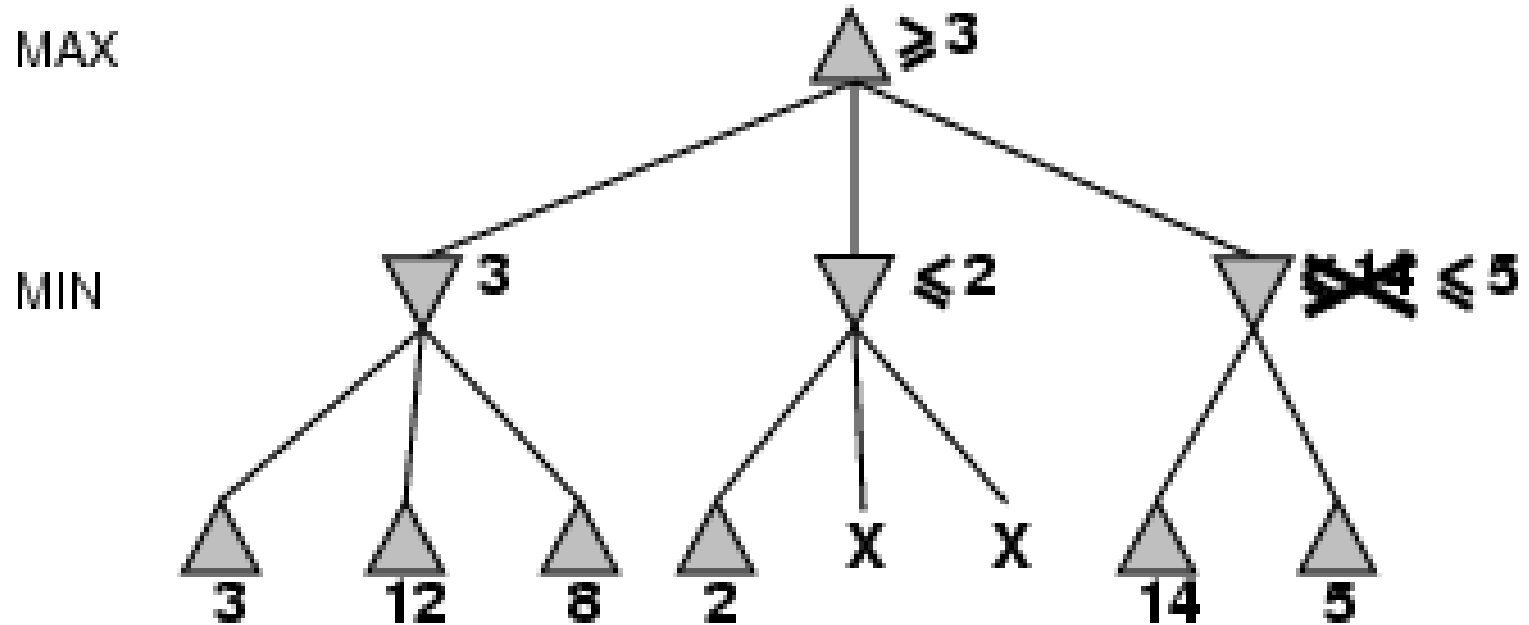
The Minimax Algorithm

α-β pruning example

MAX
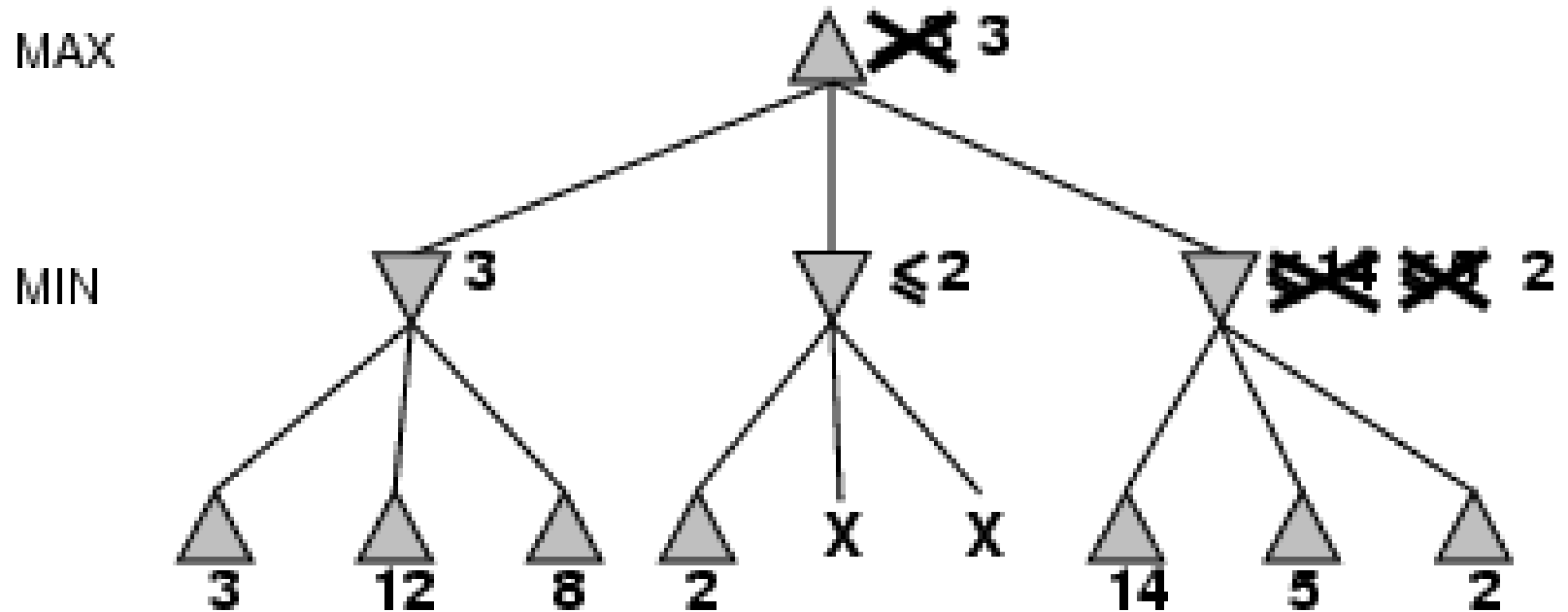
MIN

3

3   12   8
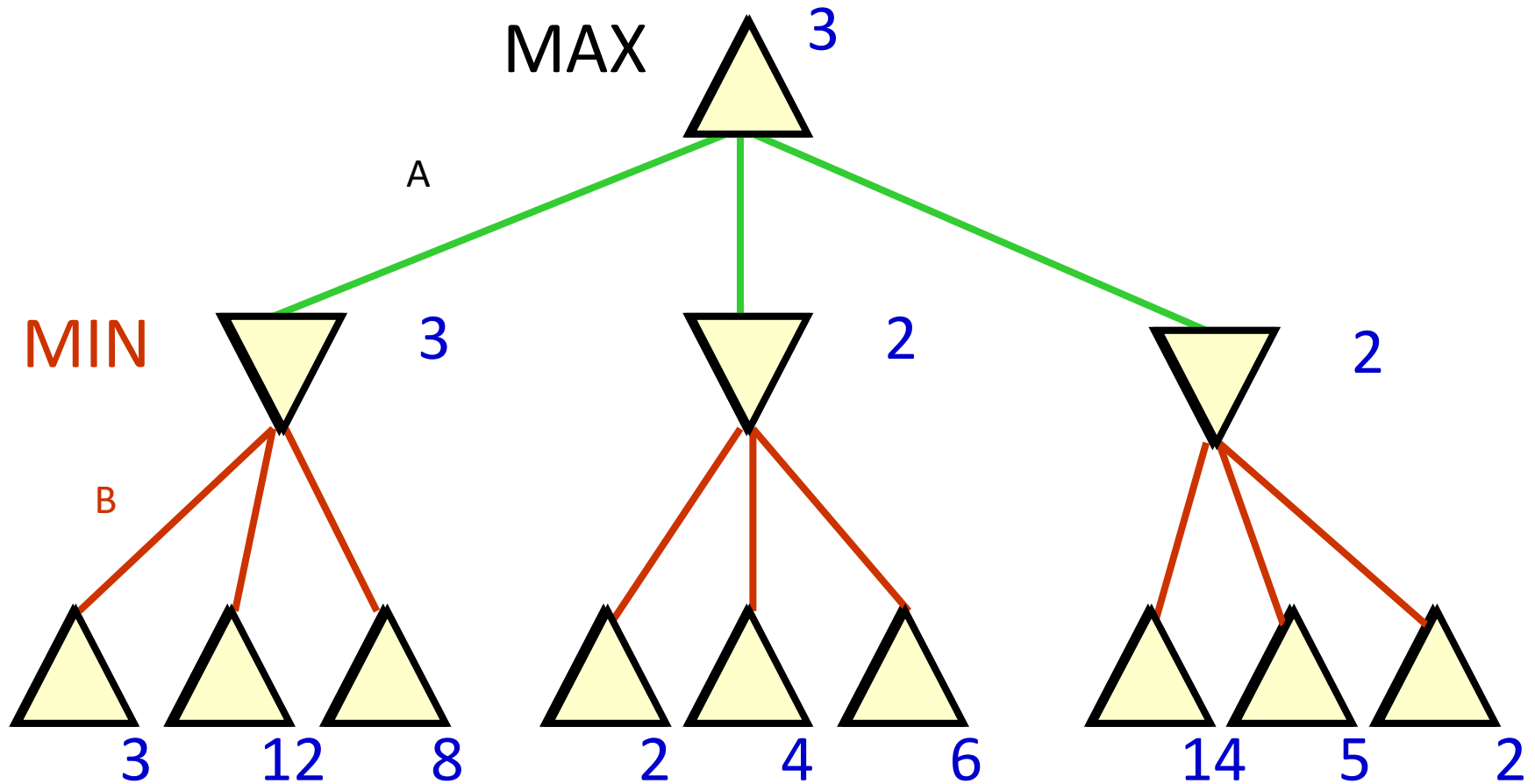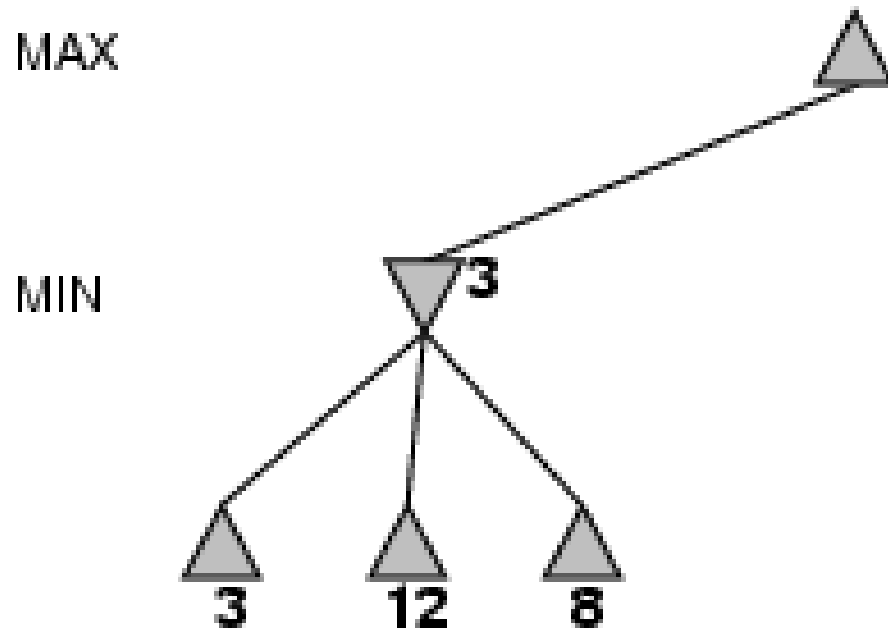
α-β pruning example

α-β pruning example

α-β pruning example

α-β pruning example

Why is it called α-β?

- α is the value of the best (i.e., highest-value) choice found so far at any choice point along the path for *max*

- If *v* is worse than α, *max* will avoid it
  →prune that branch
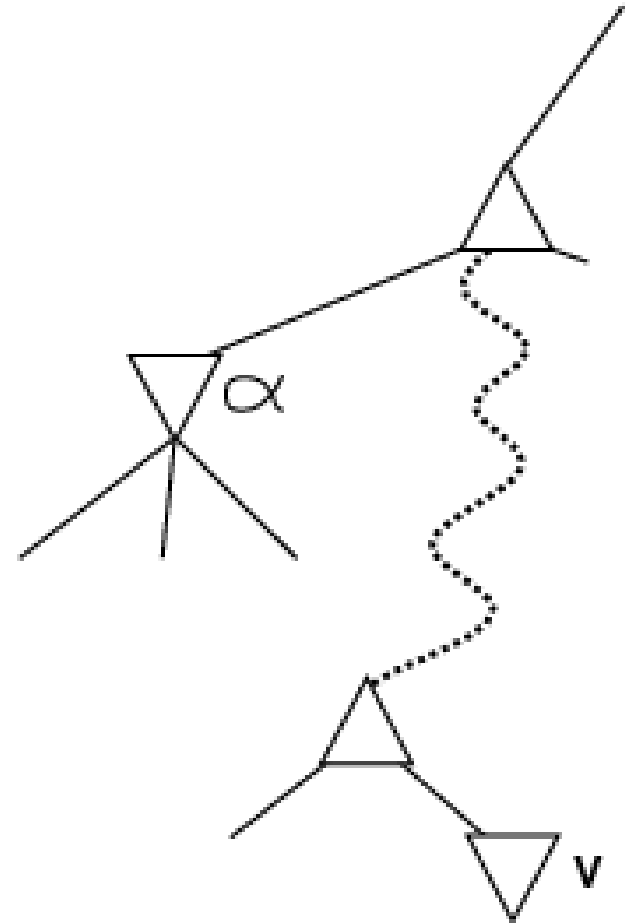
- Define β similarly for *min*
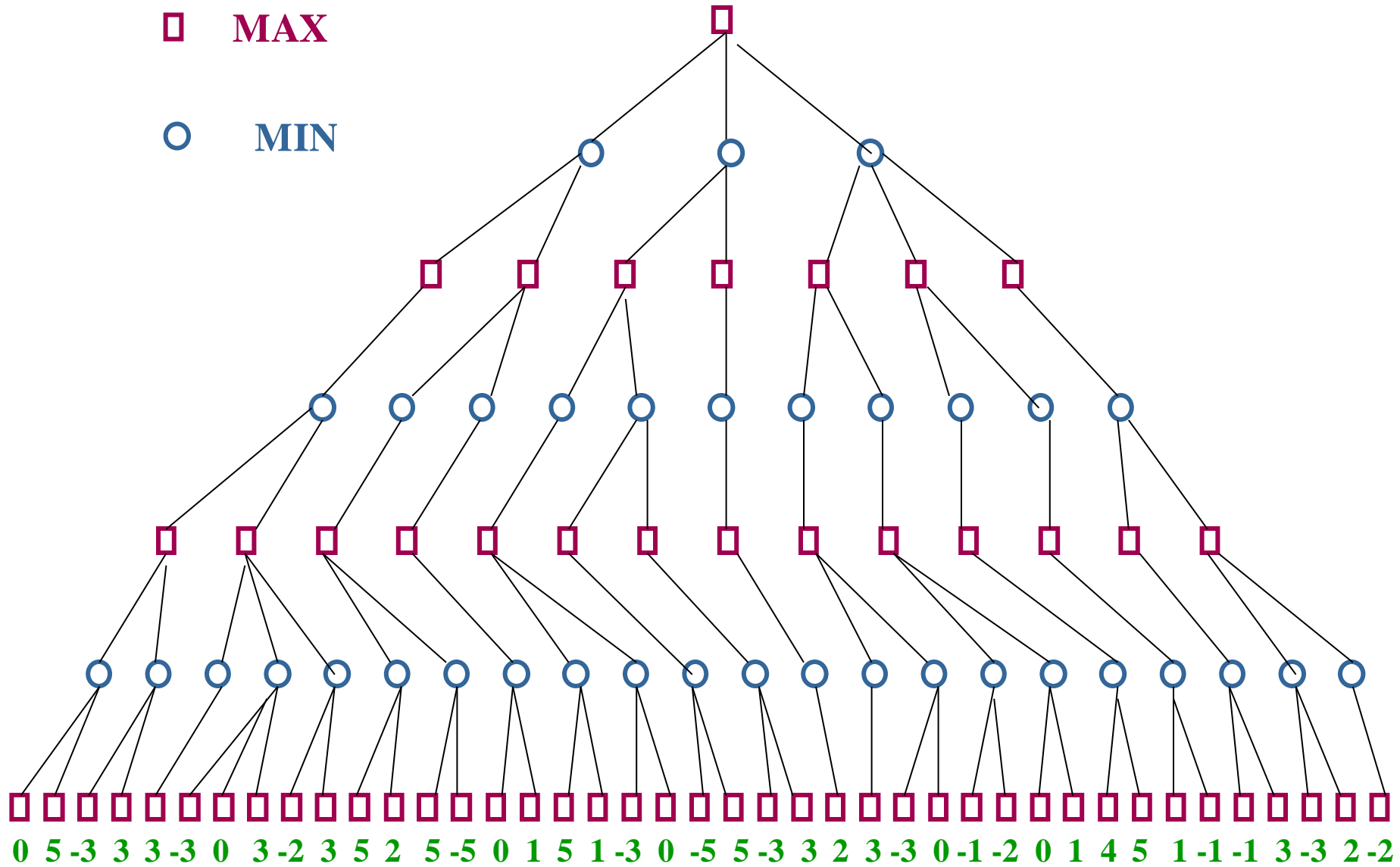
MAX

MIN

..
..
..

MAX

MIN

α

v

The α-β algorithm

function ALPHABETAPRUNING (node h, real α, real β) **returns** $u_1(h)$

if $h \in Z$ **then**

    return $u_1(h)$ // h is a terminal node

$best\_util \leftarrow (2\rho(h) - 3) \times \infty$ // $-\infty$ for player 1; ∞ for player 2

forall $a \in \chi(h)$ **do**

    if $\rho(h) = 1$ **then**

        $best\_util \leftarrow \max(best\_util, \text{ALPHABETAPRUNING}(\sigma(h,a), \alpha, \beta))$

        if $best\_util \geq \beta$ **then**

            return $best\_util$

        $\alpha \leftarrow \max(\alpha, best\_util)$

    else

        $best\_util \leftarrow \min(best\_util, \text{ALPHABETAPRUNING}(\sigma(h,a), \alpha, \beta))$

        if $best\_util \leq \alpha$ **then**

            return $best\_util$

        $\beta \leftarrow \min(\beta, best\_util)$

return $best\_util$

α- β pruning

MAX

MIN

0 5 -3 3 3 -3 0 3 -2 3 5 2 5 -5 0 1 5 1 -3 0 -5 5 -3 3 2 3 -3 0 -1 -2 0 1 4 5 1 -1 -1 3 -3 2 -2

α - βpruning

MAX

MIN

0 5 -3 3 3 -3 0 3 -2 3 5 2 5 -5 0 1 5 1 -3 0 -5 5 -3 3 2 3 -3 0 -1 -2 0 1 4 5 1 -1 -1 3 -3 2 -2