4

Compact Stars

4.1 Spheres in Hydrostatic Equilibrium

Let us consider a homogeneous sphere containing the total mass M within the total radius R. The mass density is denoted by $\rho(r)$. Integrating the mass density over a sphere with a radius r gives the mass $m_r(r)$ contained within the sphere of radius r:

$$m_r(r) = \int \mathrm{d}r'^3 \rho(r') = 4\pi \int_0^r \mathrm{d}r' \; r'^2 \rho'(r), \tag{4.1}$$

which is expressing the conservation of mass. Note, that $m_r(r)$ is the mass of the entire sphere of radius r and should not be confused with the mass at the radius r. The conservation of mass reads in differential form:

$$\frac{\mathrm{d}m_r(r)}{\mathrm{d}r} = 4\pi r^2 \rho(r). \tag{4.2}$$

The boundary conditions can be fixed by demanding that the mass at the center vanishes $m_r(r = 0) = 0$, while the mass at the surface is just given by the total mass $m_r(r = R) = M$ by definition.

In the following we consider the hydrostatic equilibrium within a homogeneous sphere. Hydrostatic equilibrium means that the pressure of matter onto an area A balances the gravitational force. The net force on matter vanishes locally. Let us assume that the sphere is in hydrostatic equilibrium up to radius r. Then imagine that one adds a small amount (or a shell) of matter of thickness dr within an arbitrary area A on top of the sphere. This adds an additional mass of

$$\mathrm{d}m_r(r) = A \cdot \rho(r)\mathrm{d}r \tag{4.3}$$

according to mass conservation. The additional pull of the gravitational force is given by

$$\mathrm{d}F_G(r) = -G\frac{m_r(r) \cdot \mathrm{d}m_r(r)}{r^2} = -G\frac{m_r(r) \cdot A \cdot \rho(r)\mathrm{d}r}{r^2}. \tag{4.4}$$

On the other hand, the pressure of the added matter over the area A also produces a force

$$dF_P(r) = A \cdot dP(r). \tag{4.5}$$

The additional pressure force has to balance the additional gravitational force to keep the hydrostatic equilibrium

$$dF_P(r) = A \cdot dP(r) = -G\frac{m_r(r) \cdot A \cdot \rho(r)dr}{r^2} = dF_G(r). \tag{4.6}$$

The area A cancels out so that one can consider hydrostatic equilibrium for an arbitrary area, as for example, a cylinder or the entire shell. It follows the equation of hydrostatic equilibrium

$$\frac{dP(r)}{dr} = -G\frac{m_r(r) \cdot \rho(r)}{r^2} \tag{4.7}$$

for the Newtonian case. The boundary conditions are that the pressure at the center is given by $P(0) = P_c$ and that the pressure vanishes at the surface $P(R) = 0$. Note, that the pressure has to vanish at the surface to ensure hydrostatic equilibrium.

4.1.1 Relativistic Hydrostatic Equilibrium

The Newtonian equation of hydrostatic equilibrium can be extended to general relativity. We have all necessary equations in Chapter 2 on general relativity. However, for the reader's convenience, we will summarize the condition of hydrostatic equilibrium in general relativity using partly a heuristic derivation.

In general relativity, the equation of hydrostatic equilibrium follows from the generalized relativistic form of the conservation of the energy–momentum tensor. The form of the energy–momentum tensor is assumed to be of the form of an ideal fluid, which means that there are no effects from dissipation, heat transport, or shear or bulk viscosity. The only quantities that enter the expression for the energy–momentum tensor are the pressure P and the energy density ε of the matter. From general relativity, only the metric tensor $g^{\mu\nu}$ and the four velocity u^μ can be additional ingredients for the expression of the energy–momentum tensor. Locally, the energy–momentum tensor has to be of the form

$$T^{\mu\nu} = \begin{pmatrix} \varepsilon & & & \\ & P & & \\ & & P & \\ & & & P \end{pmatrix}, \tag{4.8}$$

which is what we used to have in the Newtonian limit. Consider first the extension of this expression for its Lorentz-invariant form in flat space. For the relativistic

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form, the only matrices at hand are $\eta^{\mu\nu}$ and $u^\mu u^\nu$. It is then straightforward to see that the form of the energy–momentum tensor for an ideal fluid shall be assumed to be

$$T^{\mu\nu} = (\varepsilon + P) \cdot u^\mu u^\nu + P \cdot \eta^{\mu\nu} \tag{4.9}$$

in its general Lorentz-invariant form. In the local restframe, the four velocity is given by $u^\mu = (1, 0, 0, 0)$ and one recovers Eq. (4.8). The relativistic form of the conservation of energy and momentum of the fluid is ensured by a vanishing covariant derivative

$$\partial_\mu T^{\mu\nu} = 0 \tag{4.10}$$

in flat space.

The extension to general relativity is now easy. One simply needs to replace the flat space metric $\eta^{\mu\nu}$ by the general metric $g^{\mu\nu}$ for curved spacetime

$$T^{\mu\nu} = (\varepsilon + P) \cdot u^\mu u^\nu + P \cdot g^{\mu\nu}. \tag{4.11}$$

The conservation of the energy–momentum tensor is now given by the corresponding covariant derivative form

$$\nabla_\mu T^{\mu\nu} = 0 \tag{4.12}$$

for general curved spacetime.

The effects from relativity and from the extension to curved spacetime give correction factors to hydrostatic equilibrium for each entity m_r, ρ, and radius r. The corrections have different origins. First, in general relativity, gravity couples to the total energy density not only the rest mass density. So, the mass density has to be replaced by the energy density. Second, gravity couples to the energy–momentum tensor, not only to the energy density. So, the pressure of matter will also enter as an additional correction. The pressure will appear as a correction to the mass density ρ and to the mass m_r contained within the sphere of radius r. Third, in curved spacetime there will be a correction from the metric tensor. For the static case in spherical symmetry, the corresponding metric is the Schwarzschild metric and the corresponding correction factor will be given by the Schwarzschild factor of the metric. One expects that the gravitational pull on matter will be stronger when including corrections from general relativity and the Schwarzschild factor shall enter, correspondingly modifying the radius r in the denominator. The final expression from general relativity for hydrostatic equilibrium for a static spherical spacetime, that is, for the Schwarzschild metric, is given by the Tolman–Oppenheimer–Volkoff (TOV) equation

$$\frac{dP}{dr} = -G\frac{m_r(r)\varepsilon(r)}{r^2}\left(1+\frac{P(r)}{\varepsilon(r)}\right)\left(1+\frac{4\pi r^3 P(r)}{m_r(r)}\right)\left(1-\frac{2Gm_r(r)}{r}\right)^{-1}, \tag{4.13}$$

see Eq. (2.149). In addition to replacing the mass density ρ with the energy density ε, there are three correction factors, for m_r, ε, and the radius r, in line with our arguments given earlier. The Newtonian limit can be easily recovered by setting $\varepsilon \to \rho$, $P = 0$, and ignoring the Schwarzschild factor in the denominator. Note that the three correction factors from general relativity all enhance the gravitational effects so that a larger pressure gradient is needed to ensure hydrostatic equilibrium compared to the Newtonian case.

The equation of the conservation of the mass in general relativity turns out to be

$$\frac{dm_r}{dr} = 4\pi r^2 \varepsilon(r), \tag{4.14}$$

see Eq. (2.137). One notes that replacing the mass density by the energy density seems to be sufficient to go from the Newtonian limit to the full expression in general relativity. Surprisingly, the Schwarzschild factor does not enter the expression. From the integral version of mass conservation, one expects that the Schwarzschild factor will enter via the integral measure of the curved spacetime of the Schwarzschild metric. In fact, it does not, so that m_r and thereby also the total mass of the sphere M is *smaller* than the covariant integral of the energy density over the whole sphere. This feature originates from the additional binding energy of matter in the presence of a gravitational field so that the mass seen from an outside observer is smaller compared to flat space.

4.1.2 Compact Stars of Noninteracting Neutrons

The TOV equations were solved numerically for the first time by Oppenheimer and Volkoff in their classic paper (Oppenheimer and Volkoff, 1939) on matter consisting of a free gas of neutrons, that is, noninteracting neutrons. The work of Oppenheimer and Volkoff was motivated by the work of Landau of 1932, who discussed for the first time the possible existence of dense stars looking like one giant nucleus (Landau, 1932). The interesting history surrounding the paper of Landau has been nicely described in Yakovlev et al. (2013).

For solving the TOV equation numerically, one needs to define the initial conditions of the two first-order differential equations, Eqs. (4.13) and (4.14). Usually one adopts initial values for the pressure and the mass at $r = 0$:

$$P(r=0) = P_c \qquad m_r(r=0) = 0. \tag{4.15}$$

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For a given equation of state (EOS) of the form $P = P(\varepsilon)$, the two differential equations are then integrated from $r = 0$ until the pressure vanishes at the radius R of the sphere. This radius defines the radius of the compact star. The integrated mass at that radius R is denoted as the gravitational mass of the star $M = m_r(r = R)$. This procedure is the repeated for several values of the central pressure P_c, thereby generating a family of solutions of the TOV equations for a given fixed EOS.

There are a few general comments in order. First, the matter contained within radius r has to fulfill the condition that $r > 2Gm_r(r)$, as the Schwarzschild factor diverges for $r = 2Gm_r(r)$. This condition implies that the interior solution does not allow for the presence of a Schwarzschild black hole at any radius r. Second, it is clear from Eq. (4.13) that the pressure gradient has to be always negative for realistic forms of matter with $\varepsilon > 0$ and $P > 0$. A negative pressure gradient implies that the pressure is a continuously decreasing function of radius r, that is, the pressure cannot jump as a function of r. Physically, this is a consequence of hydrostatic equilibrium, of course. It also means, that if there exists a range in the EOS where the pressure does not change with energy density, the corresponding size of the shell of matter within that range of energy density will shrink to zero, as demanded by hydrostatic equilibrium. Hence, the energy density can jump as a function of radius r while the corresponding pressures before and after the jump in energy density are the same. Third, a compact star configuration with a well-defined total radius R implies that the pressure falls sufficiently fast as a function of energy density so that eventually the condition of a vanishing pressure at the surface of the compact star can be met. This condition hinges on the form of the EOS and it can happen that the pressure never vanishes for any radius r. Actually, there are known examples, also realistic ones, where this happens, in particular for matter at nonzero temperature. If the integration of the mass reaches a well-defined asymptotic limit, it implies that there is a halo of matter surrounding the compact star configuration. A radius R of that compact star configuration can then be defined at a nonvanishing minimum pressure delineating the regime of the dense core and the dilute halo. In other instances this might not be fulfilled. A well-known example is the spherical distribution of dark matter around galaxies where the mass increases linearly with radius r. Such a kind of solution will be discussed in connection with the Lane–Emden equation when discussing the properties of white dwarfs.

The EOS for a free gas of neutron matter generates compact star solutions with a well-defined radius R and a gravitational mass M. The numerical solution of the TOV equations for the EOS of a free gas of neutrons is depicted in Figure 4.1, which shows the gravitational mass M and the radius R of the neutron star as a function of the central energy density ε_c in units of the energy density of nuclear

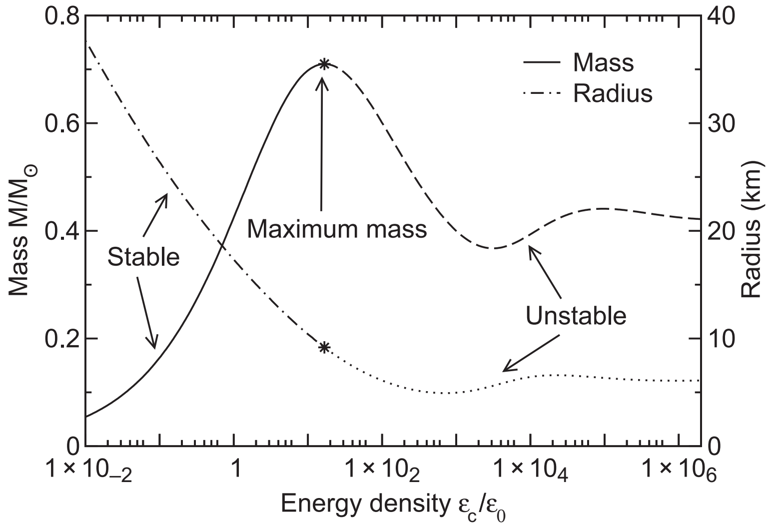

Figure 4.1 The mass and radius as a function of the central pressure for a compact star consisting of free neutron matter.

matter $\varepsilon_0 = 140\,\text{MeV}\,\text{fm}^{-3}$ (see e.g., Section 7.3.1). The radius R decreases with increasing central energy density, reaches a minimum, and starts to oscillate around an asymptotic limit. The gravitational mass M increases with increasing central energy density, reaches a maximum, then a minimum, and starts to oscillate around an asymptotic limit. The first maximum in the gravitational mass is at a lower central energy density than the first minimum in the radius.

Figure 4.2 shows the gravitational mass M versus the radius R of a neutron star consisting of noninteracting neutrons. The sequence of solutions starts at small masses and large radii. For small central energy densities, the mass scales with the radius as $M \cdot R^3 =$ constant. With increasing central pressure, the mass increases while the radius decreases until a maximum in the mass is reached. Afterwards, the mass–radius relation spirals around a limiting value for asymptotically large central energy densities. This behavior for the radius and the mass is quite generic for compact stars. Most notably, there exists an upper bound on the mass of a compact star. For noninteracting neutron matter, the maximum mass is $M_{\text{max}} = 0.71 M_\odot$, with a radius of $R = 9.1$ km.

It turns out that in the case studied here the solutions up to the maximum mass configuration are stable and that all other solutions beyond the maximum mass configuration are unstable. There are exceptions, though, under special conditions to be discussed in Chapter 9 on hybrid stars.

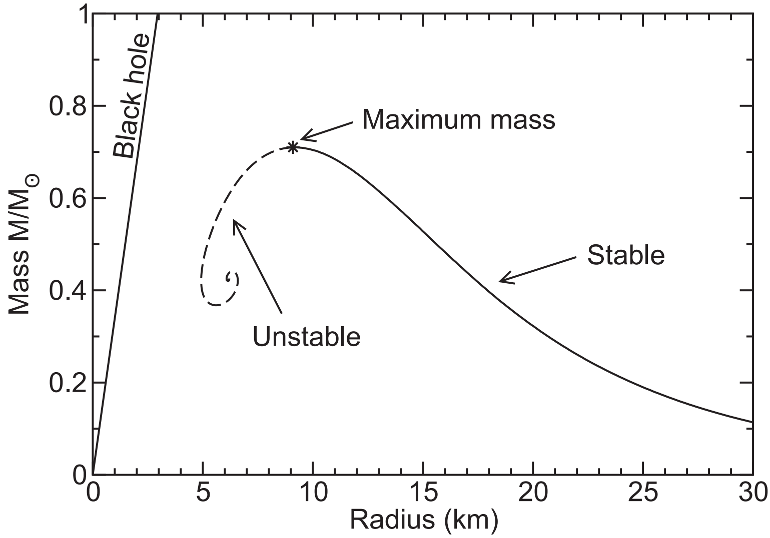

Figure 4.2 The mass–radius relation for a compact star consisting of free neutron matter. The dashed line denotes unstable configurations. The line labelled 'Black hole' shows the Schwarzschild radius.

4.2 Maximum Masses of Compact Stars

The existence of a maximum mass for neutron stars can be understood within Landau's argument for a maximum mass of compact stars (see Landau, 1932; Shapiro and Teukolsky, 1983). The argument can be extended to white dwarfs and to compact stars in general.

4.2.1 Landau's Argument for a Maximum Mass

Let us consider a static homogenous sphere of matter with a total mass M and a radius R consisting of free fermions of the mass m. Classically, the total energy $E(R)$ of a particle for radius R is the sum of the gravitational energy and the kinetic energy of the particle

$$E(R) = E_G + E_{kin} = -G\frac{Mm}{R} + E_{kin}. \tag{4.16}$$

In general, the energy of the particle is given by the relativistic energy–momentum relation:

$$E(k) = \sqrt{k^2 + m^2}. \tag{4.17}$$

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For a fluid of free fermions, the characteristic momentum is given by the Fermi momentum k_F, so we set $k = k_F$ in the following. The Fermi momentum is related to the number density as

$$n = \frac{g}{6\pi^2} k_f^3, \tag{4.18}$$

where g is the degeneracy factor. For neutrons, we have $g = (2s + 1) = 2$ from the spin degree of freedom. For simplicity, we set the number density equal to the average density $\bar{n}$ of the sphere, which is defined by

$$\bar{n} = \frac{N}{V} = \frac{3N}{4\pi R^3}, \tag{4.19}$$

where N stands for the number of fermions in the sphere. This gives the following relation for the Fermi momentum:

$$k_F = \left(\frac{6\pi^2}{g}\right)^{1/3} n^{1/3} = \left(\frac{9\pi}{2g}\right)^{1/3} \frac{N^{1/3}}{R}. \tag{4.20}$$

Assuming that the total mass of the sphere is just given by the sum of the masses of the fermions, one has

$$M = N \cdot m. \tag{4.21}$$

Let us consider now the ultra-relativistic limit for the kinetic energy of the fermion. Then, $E_{\text{kin}} = k_F$ and the total energy of the particle reads

$$E(R) = -G\frac{Mm}{R} + k_F. \tag{4.22}$$

Inserting the approximate expressions for the total mass M and the Fermi momentum one arrives at

$$E(R) = -G\frac{Nm^2}{R} + \left(\frac{9\pi}{2g}\right)^{1/3} \frac{N^{1/3}}{R}. \tag{4.23}$$

One notices that both terms scale as $1/R$. The total energy depends now only on the number of fermions in the sphere N for a fixed fermion mass m. For large values of N, the first term, the gravitational energy, dominates, for small values of N, the second term, the kinetic energy, dominates. In principle, there are three cases to be considered depending on the number of fermions N in the star:

$E(R) < 0$**:** the system is unstable as the energy gets smaller and smaller for $R \to 0$, so that the system collapses eventually to a black hole

$E(R) > 0$**:** the energy can be minimized by increasing the radius until the number density and correspondingly the Fermi momentum reaches the nonrelativistic region for a Fermi fluid ($k_F < m$). Then, $E_{kin} \propto k_F^2/m_f \propto R^{-2}$ and a stable minimum exists for a finite value of R

$E(R) = 0$ **:** the system is marginally stable.

As the gravitational potential dominates for large values of N, giving a negative sign for the total energy, stable solutions exist only up to a maximum number of fermions $N_{\max}$ given by the marginally stable case. Hence, the limit of stability $E(R) = 0$ implies that

$$G N_{\max} m^2 = \left(\frac{9\pi}{2g}\right)^{1/3} N_{\max}^{1/3} \tag{4.24}$$

or

$$N_{\max} = \left(\frac{9\pi}{2g}\right)^{1/2} \left(\frac{m_{\mathrm{P}}}{m}\right)^3, \tag{4.25}$$

where we have set $G = 1/m_{\mathrm{P}}^2$ in natural units with m_{P} being the Planck mass. The maximum mass is then given by

$$M_{\max} = N_{\max} \cdot m = \left(\frac{9\pi}{2g}\right)^{1/2} \frac{m_{\mathrm{P}}^3}{m^2}. \tag{4.26}$$

The corresponding radius can be estimated by setting $k_F = m$, as this defines the onset of the ultra-relativistic limit for the fermions. Using Eq. (4.20), we get

$$R_{\mathrm{crit}} = \left(\frac{9\pi}{2g}\right)^{1/3} \frac{N_{\max}^{1/3}}{m} = \left(\frac{9\pi}{2g}\right)^{1/2} \frac{m_{\mathrm{P}}}{m^2}, \tag{4.27}$$

where we inserted $N_{\max}$ from Eq. (4.25). We define now the Landau mass M_{L} and the corresponding Landau radius R_{L} by

$$M_{\mathrm{L}} = \frac{m_{\mathrm{P}}^3}{m^2} \qquad \text{and} \qquad R_{\mathrm{L}} = \frac{m_{\mathrm{P}}}{m^2}. \tag{4.28}$$

So far we have not fixed the fermion mass and the expressions apply to any fermion star consisting of fermions of a mass m. For a neutron star with a fermion mass of the neutron, $m = m_n$, the actual values of the Landau mass and the Landau radius are

$$M_{\mathrm{L}} = \frac{m_{\mathrm{P}}^3}{m_n^2} = 1.848 M_\odot \qquad R_{\mathrm{L}} = \frac{m_{\mathrm{P}}}{m_n^2} = 2.729\ \mathrm{km}. \tag{4.29}$$

We note that the values are not too far from the numerically exact ones for a neutron star consisting of free neutrons, which are $M_{\max} = 0.71 M_\odot$ and $R = 9.1$ km. The number of neutrons for the maximum mass configuration is about

$$N_{\max} \sim \frac{m_{\mathrm{P}}^3}{m_n^3} = \frac{M_{\mathrm{L}}}{m_n} \sim 2 \times 10^{57}. \tag{4.30}$$

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Note that as the Landau mass is close to the mass of our Sun, the number of fermions in a neutron star is about the same as in the Sun.

4.2.2 Maximum Mass for White Dwarfs

Landau's argument can be readily extended to the case of white dwarfs. White dwarf material consists of two charged components, electrons and nuclei, which balance each other such that the total charge of the white dwarf vanishes. For each nucleus with a charge Z and mass number A, there are Z electrons around due to the charge neutrality condition. The number density of electrons has to be then, with n_A being the number density of nuclei with mass number A:

$$n_e = Z \cdot n_A = n_p, \tag{4.31}$$

which is just the number density of protons in the white dwarf. While electrons have a higher momentum than the nuclei due to their smaller mass, the more massive nuclei are responsible for the total mass of the white dwarf. The mass of nuclei is to good approximation given by $m_A \approx A \cdot m_p$. So the total energy for each nucleus reads

$$E(R) = -G\frac{Mm_A}{R} + Z \cdot k_{F,e}, \tag{4.32}$$

where $k_{F,e}$ is the Fermi momentum of the electrons. The total mass of the white dwarf is given by the sum of the masses of the nuclei or by the sum of the masses of the nucleons

$$M = N_N \cdot m_p = \frac{N_N}{A} \cdot m_A, \tag{4.33}$$

where we introduced the number of nucleons in the star N_N. Note that the number of protons balancing the charge of the electrons is given by $N_p = (Z/A) \cdot N_N = N_e$. The total energy for each nucleus now reads

$$E(R) = -G\frac{N_N A m_p^2}{R} + Z \cdot \left(\frac{9\pi}{2g_e}\right)^{1/3} \frac{N_e^{1/3}}{R}, \tag{4.34}$$

where we replaced the Fermi momentum of the electrons in terms of the number of electrons N_e with g_e being the degeneracy factor of the electrons. In terms of the number of nucleons, one finds

$$E(R) = -G\frac{N_N A m_p^2}{R} + Z \cdot \left(\frac{9\pi}{2g_e}\right)^{1/3} \frac{N_N^{1/3}}{R} \left(\frac{Z}{A}\right)^{1/3}. \tag{4.35}$$

The arguments for the stability of the white dwarf can be repeated analogously to the case of neutron stars. The critical condition for stability $E(R) = 0$ determines

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the maximum mass of white dwarfs and results in the condition for the maximum number of nucleons $N_{N,\max}$:

$$G N_{N,\max} A m_p^2 = Z \cdot \left(\frac{9\pi}{2g}\right)^{1/3} \left(\frac{Z}{A}\right)^{1/3} N_{N,\max}^{1/3}. \tag{4.36}$$

The maximum number of nucleons is then given by

$$N_{N,\max} = \left(\frac{9\pi}{2g_e}\right)^{1/2} \left(\frac{m_{\mathrm{P}}}{m_p}\right)^3 \left(\frac{Z}{A}\right)^2. \tag{4.37}$$

There appears an additional factor $(Z/A)^2$ compared to the case of neutron stars, otherwise the expression is the same as the degeneracy factors are the same $g_e = g_n$. The corresponding maximum mass for white dwarfs is

$$M_{\max} = \left(\frac{9\pi}{2g_e}\right)^{1/2} \frac{m_{\mathrm{P}}^3}{m_p^2} \left(\frac{Z}{A}\right)^2 = 2.66 \cdot \frac{m_{\mathrm{P}}^3}{m_p^2} \left(\frac{Z}{A}\right)^2. \tag{4.38}$$

Compare this expression to the numerically exact one, the Chandrasekhar mass, in natural units:

$$M_{\mathrm{Chandra}} = \left(\frac{2Z}{A}\right)^2 1.44 M_\odot = 3.12 \cdot \frac{m_{\mathrm{P}}^3}{m_p^2} \left(\frac{Z}{A}\right)^2. \tag{4.39}$$

Except for the numerical prefactor, which is of the order of 1, the expressions are the same. Most interestingly, one recovers with Landau's argument the dependence of the maximum mass of white dwarfs on the charge-to-mass ratio (Z/A) of the nuclei and on the nucleon mass. Note that the electron mass does not enter the expression for the maximum mass of white dwarfs. Most strikingly, the expressions for the maximum mass of neutron stars and white dwarfs are similar and just differ by the charge-to-mass ratio squared. In summary

$$M_{\mathrm{wd}} \sim M_{\mathrm{ns}} \sim \frac{m_{\mathrm{P}}^3}{m_N^2} \sim 1 M_\odot, \tag{4.40}$$

as the maximum mass is determined in both cases by the nucleon mass. The critical radius for white dwarfs is given by $k_{F,e} = m_e$ so that

$$R_{\mathrm{crit}} = \left(\frac{9\pi}{2g_e}\right)^{1/3} \frac{N_{e,\max}^{1/3}}{m_e} \tag{4.41}$$

$$= \left(\frac{9\pi}{2g_e}\right)^{1/3} \frac{N_{N,\max}^{1/3}}{m_e} \left(\frac{Z}{A}\right)^{1/3}. \tag{4.42}$$

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We realize that the electron mass m_e is entering one of our expressions for the first time. Using the expression for the maximum number of nucleons one arrives at

$$R_{\text{crit}} = \left(\frac{9\pi}{2g_e}\right)^{1/2} \frac{m_{\text{P}}}{m_p \cdot m_e} \left(\frac{Z}{A}\right), \tag{4.43}$$

so that the radius of a white dwarf is characteristically about

$$R_{\text{wd}} \sim \frac{m_{\text{P}}}{m_p \cdot m_e} \sim 5,000\,\text{km}, \tag{4.44}$$

which is about the size of Earth. Note, that the radius corresponding to the maximum mass of a white dwarf is related to the electron mass, or more generically to the mass of the fermion that is providing the pressure. The smaller the mass of this fermion, the larger the compact star. Comparing the expressions for the radius of the white dwarf with the one of a neutron star, see Eq. (4.27), one sees that they scale with the ratio of the proton to the electron mass. Modulo a factor of Z/A, the radius of a white dwarf is larger by the ratio of the proton to electron mass

$$\frac{R_{\text{wd}}}{R_{\text{ns}}} \sim \frac{m_p}{m_e} \sim 2,000, \tag{4.45}$$

that is, by about three orders of magnitude. Indeed, this is what one finds in more refined models of neutron stars and white dwarfs and from observations. Characteristically, the radius of a neutron star is about 10 km and that of white dwarfs is about 10,000 km.

4.3 Scaling Solutions for Compact Stars

Landau's argument for a maximum mass and the corresponding radius has been derived for arbitrary fermion masses. One notices that both the Landau mass and the Landau radius scale as the inverse fermion mass squared. In fact, this features originates from the scaling behavior of the TOV equations. Although the arguments in deriving the expressions for the Landau mass and radius seem to be simple minded, the result for the Landau's scaling solution are exactly the same for the full general relativistic treatment. The deeper reason behind this is the nearly magical power of scaling arguments, which are just based on dimensional reasoning. The scaling result can only depend on the dimensionful parameters of the problem in certain combinations that are allowed by the equations studied. In principle, Landau's derivation of the maximum mass and the radius are just that, looking for a reasonable combination of dimensionful parameters of the problem at hand to arrive at a scaling solution. We will delineate the procedure in the following in more detail and extend it to more general classes of EOSs.

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We note in passing that scaling solutions are well known and studied in stellar evolution. A particular kind of scaling solution of the equations of hydrostatic equilibrium for stellar configurations will be adopted, for example, for the study of the Lane–Emden equations for white dwarfs, see Chapter 5.

4.3.1 Scaling of the TOV Equations

First let us introduce dimensionless quantities for the EOS:

$$P = \varepsilon_0 \cdot P' \qquad \varepsilon = \varepsilon_0 \cdot \varepsilon' \tag{4.46}$$

which we denote by a prime. Here ε_0 stands for a constant with the dimension of an energy density (which is in natural units equivalent to the dimension of a pressure). The aim is to look for a solution of the TOV equations in terms of that energy density ε_0 alone. We rescale the other dimensionful quantities of the TOV equations, which are the radial coordinate r and the mass m_r with radius r such that the TOV equations are scale-free:

$$r = a \cdot r' \qquad m_r = b \cdot m'_r, \tag{4.47}$$

with dimensionful coefficients a and b to be determined. Again, the primed quantities denote the scale invariant quantities. The rescaled TOV equation in terms of the dimensionless primed quantities reads

$$\frac{\varepsilon_0 \cdot dP'}{a \cdot dr'} = -G\frac{bm'_r\varepsilon_0\varepsilon'}{a^2 \cdot r'^2}\left(1 + \frac{\varepsilon_0 \cdot P'}{\varepsilon_0 \cdot \varepsilon'}\right)\left(1 + \frac{4\pi a^3 \cdot r'^3\varepsilon_0 P'}{b \cdot m'_r}\right)\left(1 - \frac{2Gb \cdot m'_r}{a \cdot r'}\right)^{-1}. \tag{4.48}$$

Three remaining dimensionful quantities are present, the coefficients a and b and the gravitational constant G. One sees that the TOV equations are becoming independent from those remaining dimensionful quantities under the conditions that

$$G \cdot b = a \qquad a^3 \cdot \varepsilon_0 = b. \tag{4.49}$$

Note that this holds true also for the correction terms from general relativity, including the Schwarzschild factor of the metric. The conditions from Eq. (4.49) can be recast to determine the unknown coefficients

$$a = (G \cdot \varepsilon_0)^{-1/2} \qquad b = (G^3 \cdot \varepsilon_0)^{-1/2}. \tag{4.50}$$

Then the dimensionless TOV equation reads

$$\frac{dP'}{dr'} = -\frac{m'_r\varepsilon'}{r'^2}\left(1 + \frac{P'}{\varepsilon'}\right)\left(1 + \frac{4\pi r'^3 P'}{m'_r}\right)\left(1 - \frac{2m'_r}{r'}\right)^{-1} \tag{4.51}$$

without any dimensionful quantity appearing in the expression. One can convince oneself that the same scaling conditions for the radius and the mass in Eq. (4.50) transform the equation of mass conservation to a fully dimensionless form:

$$\frac{dm'_r}{dr'} = 4\pi r'^2 \varepsilon'. \tag{4.52}$$

For a given EOS in the form $P' = P'(\varepsilon')$, the dimensionless TOV equations can be solved numerically in terms of a dimensionless mass–radius relation, energy density profile, and so on. The physical quantities are recovered by fixing the scaling energy density ε_0. That means that each quantity of the solution scales with the factor ε_0 to some power. In particular, any mass and radius rescales as given by Eqs. (4.47) and (4.50):

$$m_r = \frac{m'_r}{(G^3\varepsilon_0)^{1/2}} \qquad r = \frac{r'}{(G\varepsilon_0)^{1/2}}, \tag{4.53}$$

which includes also the maximum mass $M_{\max}$ and its radius R_{crit}, which have to scale as $1/\sqrt{\varepsilon_0}$ also

$$M_{\max} = \frac{M'}{(G^3\varepsilon_0)^{1/2}} \qquad R_{\text{crit}} = \frac{R'}{(G\varepsilon_0)^{1/2}}, \tag{4.54}$$

or in terms of the Planck mass m_{P}:

$$M_{\max} = M' \cdot \frac{m_{\text{P}}^3}{\varepsilon_0^{1/2}} \qquad R_{\text{crit}} = R' \cdot \frac{m_{\text{P}}}{\varepsilon_0^{1/2}}. \tag{4.55}$$

The entire mass–radius relation has to scale accordingly. Recall that the primed quantities are dimensionless numbers to be determined numerically by solving the TOV equations.

The scaling relations of Eq. (4.55) have several important applications. The recipe is as follows: solve the dimensionless TOV equations for a given type of dimensionless EOS and then rescale the results with $\sqrt{\varepsilon_0}$ to arrive at physical values. Let us give some examples in the following.

4.3.2 EOS for a Free (Massive) Fermi Gas

For a free gas of fermions, the only scale that appears is the fermion mass. We rescale the EOS by the fermion mass simply by $P' = P/m_f^4$ and $\varepsilon' = \varepsilon/m_f^4$,

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which implies that the scaling constant is $\varepsilon_0 = m_f^4$. Then, the maximum mass and the corresponding radius from Eq. (4.55) are given by

$$M_{\rm max} = M' \cdot (G^3 \cdot m_f^4)^{-1/2} = M' \cdot \frac{m_{\rm P}^3}{m_f^2} \tag{4.56}$$

$$R_{\rm crit} = R' \cdot (G \cdot m_f^4)^{-1/2} = R' \cdot \frac{m_{\rm P}}{m_f^2}, \tag{4.57}$$

where we recognize the Landau mass and Landau radius as solutions. So, we have shown that the Landau mass and radius are solutions of the full general relativistic equations for compact stars, the TOV equations, and not just for the classical limit. The dimensionless prefactors M' and R' can be computed either by solving the dimensionless TOV equations with the EOS for a free Fermi gas or by using the known numerical values for a free gas of neutrons with an appropriate rescaling. Numerically one finds that these prefactors are given by $M' = 0.384$ and $R' = 3.367$. Let us look at the compactness of a star, which we defined as the ratio of the gravitational mass to its radius

$$C = \frac{GM}{R}. \tag{4.58}$$

Note that the compactness of the star is dimensionless and measures how close the star's radius is to that of a black hole of the same mass. A compact star of free fermions has the maximum compactness at the maximum mass and is given by the ratios of the dimensionless prefactors of Eqs. (4.56) and (4.57)

$$C_{\rm max} = \frac{M'}{R'} = 0.11, \tag{4.59}$$

which is independent on the fermion mass.

4.3.3 EOS for a Relativistic Fermi Gas with a Vacuum Term

The EOS of relativistic particles

$$P = \frac{1}{3}\varepsilon \tag{4.60}$$

is similar to a polytrope with a power of $\gamma = 1$. As we will see, this EOS produces unstable configurations in the Newtonian case, that is, for the Lane–Emden equation, see Chapter 5. Moreover, the EOS does not contain any dimensionful quantity so there is nothing to define a dimensionful scaling constant ε_0. However, the EOS

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can be modified to be able to arrive at stable solutions for the TOV equations. The trick is to introduce a dimensionful constant to the EOS in the form

$$P = \frac{1}{3}(\varepsilon - \varepsilon_0), \tag{4.61}$$

where ε_0 is an offset of the energy density. The EOS has now the special property that the pressure vanishes at a nonvanishing energy density ε_0. This feature is absent in the EOSs studied so far. The EOS of fermion gases or of neutron matter have a vanishing pressure at a vanishing energy density, even for the case of interacting fermions that is to be studied later.

In general, microscopic theories of matter based on a quantum field theoretical description have a vanishing energy density in the vacuum by definition. More precisely, one can add a constant to the Lagrangian density and it would not change the equations of motion of the quantum fields so that one can fix the energy density of the vacuum to be 0. For gravity an additional constant in the Lagrangian density counts and has an impact on the equations of motion. In fact, for gravity this additional constant is nothing less than what is known as the cosmological constant.

There are known cases of microscopic models, where the form of equation equation (4.61) appears. One of the most famous ones is that of the MIT bag model, which we will discuss in more detail in Chapter 8 on quark stars. The MIT bag model considers a gas of free relativistic quarks that are confined within a bag stabilized by some pressure B from the outside. The quantity B is assumed to be a constant and usually dubbed the MIT bag constant. The total pressure in the bag is then

$$P = P_{\text{free}} - B. \tag{4.62}$$

Thermodynamic consistency demands that the energy density has to be

$$\varepsilon = \varepsilon_{\text{free}} + B, \tag{4.63}$$

which the reader is invited to check as an exercise. Note that the pressure from the outside is the pressure of the vacuum, that is, it corresponds to nonvanishing vacuum contributions. Also, as seen from Eq. (4.63), the vacuum pressure corresponds to an energy density with a negative sign. The nonvanishing vacuum energy density can be attributed to nonvanishing vacuum expectation values of the quantum chromodynamics (QCD) vacuum, in particular to the quark and gluon condensates. We note in passing that similar reasoning would also apply to the electroweak theory and the nonvanishing vacuum expectation value of the Higgs field.

The EOS of the MIT bag model can be recast to the form of Eq. (4.61). As we know that for a relativistic massless gas of particles,

$$P_{\text{free}} = \frac{1}{3}\varepsilon_{\text{free}} \tag{4.64}$$

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the equations can be combined to the final form

$$P = \frac{1}{3}(\varepsilon - 4B), \tag{4.65}$$

which is of the form of Eq. (4.61) with $\varepsilon_0 = 4B$. From our scaling analysis, we can write down immediately how the maximum mass and the corresponding radius depend on the value of the MIT bag constant B:

$$M_{\mathrm{max}} = M' \cdot (G^3 \cdot \varepsilon_0)^{-1/2} = 2.01\, M_\odot \cdot \frac{(145\,\mathrm{MeV})^2}{\sqrt{B}} \tag{4.66}$$

$$R_{\mathrm{crit}} = R' \cdot (G \cdot \varepsilon_0)^{-1/2} = 10.9\,\mathrm{km} \cdot \frac{(145\,\mathrm{MeV})^2}{\sqrt{B}}, \tag{4.67}$$

where $B^{1/4} = 145\,\mathrm{MeV}$ is the standard MIT bag model value and the numerical prefactors are taken from the known solution for that value (see e.g., Baym and Chin, 1976; Witten, 1984). The mass–radius relation of the kind of EOSs that have a vanishing pressure at a nonvanishing energy density are special and correspond to the class of selfbound stars. Selfbound stars have a vanishing pressure at the surface of the star by the properties of matter and do not need gravity to be stabilized – hence, the name selfbound. A maximum mass configuration exists, though, when the gravitational energy reaches the scale given by ε_0, destabilizing the compact star. The maximum compactness for the EOS of the form of Eq. (4.65) is given by

$$C_{\mathrm{max}} = \frac{GM_{\mathrm{max}}}{R_{\mathrm{crit}}} = \frac{M'}{R'} = 0.271 \tag{4.68}$$

and is independent of the choice of the MIT bag constant B or the scaling energy density ε_0. In comparison to Eq. (4.59), these selfbound stars are considerably more compact than compact stars made of free fermions.

4.3.4 Limiting EOS from Causality

Zel'dovich has shown (Zel'dovich, 1961) that the stiffest possible EOS is not of the form of an ultra-relativistic gas of particles, see Eq. (4.60), but of the form

$$P = \varepsilon. \tag{4.69}$$

Indeed, by looking at the speed of sound, one realizes that for this EOS, the speed of sound squared is

$$c_s^2 = \frac{\partial P}{\partial \varepsilon} = 1, \tag{4.70}$$

so that the speed of sound is equal to the speed of light, the maximum value allowed by causality. Therefore, one can call the Zel'dovich EOS, Eq. (4.69), also the limiting causal EOS, as it gives the maximum pressure for a given energy density allowed by causality. The Zel'dovich EOS can be realized in microscopic models, if the dominant term of the energy density scales as

$$\varepsilon = c \cdot n^2, \tag{4.71}$$

where n stands for the number density and c is a constant with dimensions $(1/\text{mass})^2$. For neutron stars, n would be the baryon number density. Thermodynamic consistency then fixes the pressure to be

$$P = c \cdot n^2, \tag{4.72}$$

with the same prefactor so that one recovers the Zel'dovich equation, Eq. (4.69). We will encounter the Zel'dovich EOS as the high-density EOS for an interacting gas of fermions in Section 4.4.

If we look at the Zel'dovich EOS, Eq. (4.69), we realize that no mass scale appears and, moreover, that the EOS will lead to unstable solutions for the TOV equations, as the polytropic index is like the case for the ultra-relativistic gas discussed in Section 4.3.3. We know now how to arrive at stable solutions for the TOV equation, we simply introduce a constant to the EOS. It is convenient to introduce the following extended form of the Zel'dovich EOS

$$P = P_f + (\varepsilon - \varepsilon_f), \tag{4.73}$$

with a nonvanishing pressure P_f for an energy density ε_f. Note that for $P_f = 0$ we would recover an EOS for selfbound stars. Here we have now a different application in mind. Assume that we know the low-density EOS up to some pressure P_f with the corresponding energy density $\varepsilon_f = \varepsilon_f(P_f)$ for neutron stars. Then, the stiffest possible EOS for higher densities has to be the Zel'dovich EOS of the form given in Eq. (4.73). As the EOS provides the maximum pressure allowed by causality, the corresponding maximum mass is the highest possible mass and gives an upper limit to the maximum mass allowed by causality.

The neutron matter EOS is known to about the saturation density of normal nuclear matter with an energy density of $\varepsilon_{\text{nm}} = 140\,\text{MeV}\,\text{fm}^{-3}$. The pressure at that energy density is about two orders of magnitude smaller, so it can be safely ignored. Then, the limiting EOS contains only one dimensionful quantity and we can proceed with our scaling analysis. The maximum mass then has to scale with the fiducial energy density ε_f as

$$M_{\max} = M' \cdot (G^3 \cdot \varepsilon_f)^{-1/2} = 4.2 M_\odot \cdot \sqrt{\frac{\varepsilon_{\text{nm}}}{\varepsilon_f}}, \tag{4.74}$$

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where ε_{nm} is the ground-state energy density of nuclear matter. This expression for the limiting maximum mass of a neutron star was derived by Rhoades and Ruffini (1974) and is called the Rhoades–Ruffini mass limit.

It turns out that the numerical factor is rather insensitive to the choice of the low-density nuclear EOSs as the pressure of the neutron star matter EOS is so small. However, in their original paper, Rhoades and Ruffini chose twice the saturation density as their fiducial energy density setting $\varepsilon_f = 2\varepsilon_{\text{nm}}$. Looking at Eq. (4.74), one arrives at a limiting maximum mass of a neutron star of $3M_\odot$, a value often generically quoted in the literature as the maximum mass of a neutron star allowed by causality. As we do not know at present the neutron star matter EOS at $2\varepsilon_{\text{nm}}$, that value of the maximum mass should be taken with corresponding care. The maximum mass of a neutron star allowed by causality alone with our present knowledge of the neutron star matter EOS up to the energy density at saturation density is $4.2M_\odot$.

4.3.5 Selfbound Linear EOS

The Zel'dovich EOS in the form of Eq. (4.73) with vanishing fiducial pressure $P_f = 0$ and the MIT bag model EOS, Eq. (4.65), can be combined to a special class of EOSs

$$P = s \cdot (\varepsilon - \varepsilon_0)\,, \tag{4.75}$$

with a dimensionless constant s. As s is equivalent to the speed of sound, $s = c_s^2$, in the medium

$$c_s^2 = \frac{\partial P}{\partial \varepsilon} = s, \tag{4.76}$$

it is bounded by $0 < s \leq 1$. For $s = 1/3$, we recover the MIT bag model, for $s = 1$, the Zel'dovich EOS as the limiting causal EOS. EOSs of the type of Eq. (4.75) are the simplest type of EOS, having a nonvanishing energy density at vanishing pressure. The solutions of the TOV equations will constitute classes of so-called selfbound stars, which are bound by themselves as the pressure vanishes at the surface of the star, even when gravity is absent. Note that ordinary stars including compact stars are stabilized by the nonvanishing pressure gradient of matter being counterbalanced by the gravitational force.

In all cases for the selfbound EOS of Eq. (4.75), the solution of the TOV equation will exhibit that the maximum mass and its radius will scale with $1/\varepsilon_0^{1/2}$, as shown earlier. The numerical prefactors, however, will now depend on the parameter s. For $s = 1/3$, one can use Eq. (4.66) and rewrite it in the form

$$M_{\max} = 2.57\, M_{\odot} \cdot \left(\frac{\varepsilon_{\mathrm{nm}}}{\varepsilon_0}\right)^{1/2} \qquad R_{\mathrm{crit}} = 14.0\,\mathrm{km} \cdot \left(\frac{\varepsilon_{\mathrm{nm}}}{\varepsilon_0}\right)^{1/2}, \tag{4.77}$$

for $s = 2/3$, one finds numerically

$$M_{\max} = 3.64\, M_{\odot} \cdot \left(\frac{\varepsilon_{\mathrm{nm}}}{\varepsilon_0}\right)^{1/2} \qquad R_{\mathrm{crit}} = 16.4\,\mathrm{km} \cdot \left(\frac{\varepsilon_{\mathrm{nm}}}{\varepsilon_0}\right)^{1/2}, \tag{4.78}$$

and for the limiting causal EOS $s = 1$

$$M_{\max} = 4.23\, M_{\odot} \cdot \left(\frac{\varepsilon_{\mathrm{nm}}}{\varepsilon_0}\right)^{1/2} \qquad R_{\mathrm{crit}} = 17.6\,\mathrm{km} \cdot \left(\frac{\varepsilon_{\mathrm{nm}}}{\varepsilon_0}\right)^{1/2}. \tag{4.79}$$

Here we have computed the dimensionful numerical prefactors for the energy density of saturated nuclear matter $\varepsilon_{\mathrm{nm}} = 140\,\mathrm{MeV\,fm}^{-3}$. We see that the equation for the maximum mass for the case $s = 1$ coincides with the one for the Rhoades–Ruffini mass limit presented in Eq. (4.74). We stress that the relative changes of the numerical prefactors for the maximum mass and the corresponding radius are different and do not scale with the parameter s. In fact, the mass–radius relations for all three cases are different and cannot be mapped into each other by rescaling.

Note that the compactness, the ratio of the maximum mass, and the radius are independent of the chosen value of the scale, ε_0, which drops out. Hence, the dimensionless form of the EOS alone defines the compactness of the whole mass–radius relation, including the maximum mass configuration. In most cases, this defines also a maximum compactness for all stable solutions to the TOV equations, as the maximum compactness usually coincides with the maximum mass. For the cases considered here, one finds that the maximum mass configuration gives the maximum compactness of

$$C_{\max} = \frac{GM_{\max}}{R_{\mathrm{crit}}} = \frac{M'}{R'} = \begin{cases} 0.271 & \text{for } s = 1/3 \\ 0.328 & \text{for } s = 2/3 \\ 0.354 & \text{for } s = 1 \end{cases}, \tag{4.80}$$

where M' and R' are the dimensionless numerical prefactors for the mass and radius, as defined in Eq. (4.55). The compactness is related to the redshift factor $1 + z$ via

$$1 + z = \frac{1}{\sqrt{1 - \frac{2GM}{R}}} \tag{4.81}$$

and the maximum redshift can be determined to be

$$z_{\max} = \frac{1}{\sqrt{1 - \frac{2GM_{\max}}{R_{\mathrm{crit}}}}} - 1 = \frac{1}{\sqrt{1 - \frac{2M'}{R'}}} - 1. \tag{4.82}$$

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The numerical values are as follows

$$z_{\text{max}} = \begin{cases} 0.478 & \text{for } s = 1/3 \\ 0.705 & \text{for } s = 2/3 \\ 0.851 & \text{for } s = 1 \end{cases}, \tag{4.83}$$

so that the maximum redshift stays in all cases below 1 and is at maximum $z_{\text{max}} = 0.851$. As the case $s = 1$ constitutes the stiffest possible EOS, it leads to the most compact star configurations allowed by causality. It was shown by Lindblom that this maximum redshift applies also for the case of neutron stars (Lindblom, 1984).

The central energy density for the maximum mass configuration is the maximum central energy density possible for the given EOS. For the cases considered here, one gets the following numerical values:

$$\frac{\varepsilon_{\text{max}}}{\varepsilon_0} = \begin{cases} 4.81 & \text{for } s = 1/3 \\ 3.54 & \text{for } s = 2/3 \\ 3.03 & \text{for } s = 1 \end{cases}. \tag{4.84}$$

Note that those numbers just give the relative ratio of the central energy density to the surface energy density for the maximum mass configuration. It is instructive to quote the maximum central energy densities for the maximum mass of $M_{\text{max}} = 2M_\odot$, which are

$$\frac{\varepsilon_{\text{max}}}{\varepsilon_{\text{nm}}} = \begin{cases} 7.91 & \text{for } s = 1/3 \\ 11.8 & \text{for } s = 2/3 \\ 13.5 & \text{for } s = 1 \end{cases} \tag{4.85}$$

in terms of the energy density of nuclear matter ε_{nm}. So, for a maximum mass of $M_{\text{max}} = 2M_\odot$ and the causal EOS with $c_s^2 = 1$, the maximum possible energy density is $\varepsilon_{\text{max}} = 13.5\varepsilon_{\text{nm}}$, which constitutes the ultimate energy density of observable cold matter (Lattimer and Prakash, 2005). Note that the maximum energy density changes inversely with the maximum mass. So a higher maximum mass results in a lower ultimate energy density. In fact, the maximum mass scales with $\varepsilon_0^{-1/2}$ and, obviously, the maximum energy density with ε_0. So the product of the maximum mass squared times the maximum energy density is independent of ε_0, stated first explicitly by Lattimer and Prakash (2011). In other terms, the maximum energy density scales inversely with the maximum mass squared and to embarass them I will denote the scaling relation for the ultimate energy density as the Lattimer–Prakash relation:

$$\varepsilon_{\text{ultimate}} = 13.5\varepsilon_{\text{nm}} \left(\frac{2M_\odot}{M_{\text{max}}} \right)^2 . \tag{4.86}$$

If one finds that the maximum mass is $M_{\text{max}} = 3M_\odot$, for example, the ultimate energy density from causality will be only $\varepsilon_{\text{ultimate}} = 6\varepsilon_{\text{nm}}$.

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4.4 Interacting Fermions

So far we considered free ideal gases of fermions. Interactions between the fermions have been neglected for considering the maximum masses of compact star configurations and their corresponding radius. In the following, we discuss the possible impact of interactions on the mass–radius relation and the maximum mass of compact stars.

We assume that the contribution from interactions is proportional to the number density of fermions n to the power of γ. Motivated by this, we add an interaction term to the EOS of the form

$$\varepsilon_{\text{int}} = c \cdot n^{\gamma}, \tag{4.87}$$

with c being a constant that determines the strength of the interaction. The corresponding contribution to the pressure has to be then of the form

$$P_{int} = -\left.\frac{\partial E_{\text{int}}}{\partial V}\right|_{N,T=0} = n^2 \frac{(\partial \varepsilon_{\text{int}}/n)}{\partial n} = (\gamma - 1) \cdot c \cdot n^{\gamma} = (\gamma - 1) \cdot \varepsilon_{\text{int}}. \tag{4.88}$$

The speed of sound is then given by

$$c_s^2 = \frac{\partial P}{\partial \varepsilon} = \gamma - 1 \tag{4.89}$$

if the EOS is dominated by the interaction terms. Then, the overall EOS will be acausal for $\gamma > 2$. For two-body interactions between fermions, $\gamma = 2$ seems to be a reasonable value. In fact, such a kind of interaction term can be motivated from relativistic mean-field models where the interaction between fermions is mediated by the exchange of vector mesons (see e.g., Glendenning, 2000).

Let us consider now an EOS for interacting fermions, including the kinetic terms for the pressure and energy density. First have a look at the nonrelativistic limit:

$$\varepsilon^{(\text{nr})} = m \cdot n + c \cdot n^2 \qquad p^{(\text{nr})} = \frac{1}{5m^{8/3}} \left(\frac{6\pi^2}{g}\right)^{2/3} \cdot n^{5/3} + c \cdot n^2. \tag{4.90}$$

For low densities, the first terms dominate

$$\varepsilon^{(\text{nr})} \approx m \cdot n \qquad p^{(\text{nr})} \approx \frac{1}{5m^{8/3}} \left(\frac{6\pi^2}{g}\right)^{2/3} \cdot n^{5/3} \propto \varepsilon^{5/3} \tag{4.91}$$

and we recover the low-density limit of a noninteracting Fermi gas, a polytrope with a power of $5/3$. At some intermediate density the terms from the interactions will start to be dominant. With increasing density, the pressure will be dominated by the interaction term, as the kinetic term of the nonrelativistic pressure is much

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smaller compared to the mass term of the energy density. The energy density can still be dominated by the mass term for some intermediate density so that

$$\varepsilon \approx m \cdot n \qquad p \approx c \cdot n^2 \propto \varepsilon^2 \tag{4.92}$$

and we recover a polytrope with a power of $\Gamma = 2$. Finally, in the ultra-relativistic limit, the interaction terms dominate the pressure and the energy density. Hence,

$$\varepsilon \approx c \cdot n^2 \qquad p \approx c \cdot n^2 \approx \varepsilon \tag{4.93}$$

and the EOS approaches the limiting causal EOS with a power of one in terms of the energy density ε.

From the discussion of the general solution of compact stars for polytropic EOSs, see Section 5.2, we will see later that a polytrope with a power of $\Gamma = 5/3$ gives a mass–radius relation of $M \cdot R^3 = \text{const.}$, a polytrope with a power of $\Gamma = 2$ gives a mass–radius relation with a constant radius, and a polytrope with a power of $\Gamma = 1$ gives unstable solutions. Pretty generically, we conclude that a compact star consisting of interacting fermions will have a mass–radius relation that looks like that of noninteracting fermions at large radii, that is, at low densities, that is independent of the mass for low radii, that is, at moderate densities, and that reaches a maximum mass at high densities. Beyond the density corresponding to the maximum mass, the solutions are unstable.

It is even possible to make some generic statements about the maximum mass for the case of interacting fermions. First, let us introduce a dimensionless measure of the interaction strength. By dimensionless analysis, the coefficient c of the interaction term has the units of mass^{-2}. Let us denote that mass scale as the interaction mass scale m_I and the mass of the fermion as m_f. We can rewrite the contribution from the interaction to the pressure by

$$P_{\text{int}} = c \cdot n^2 = \frac{1}{m_I^2} \cdot n^2. \tag{4.94}$$

By rescaling all the terms with the Fermi mass, one arrives at

$$\frac{P_{\text{int}}}{m_f^4} = \frac{m_f^2}{m_I^2} \cdot \left(\frac{n}{m_f^3}\right)^2 = y^2 \cdot \left(\frac{n}{m_f^3}\right)^2, \tag{4.95}$$

with the dimensionless interaction strength $y = m_f/m_I$.

Figure 4.3 shows the mass–radius relation of compact stars with interacting fermions for different interaction strengths y. Note that the plot is shown in units of the Landau mass and the Landau radius, so that all quantities are dimensionless and valid for arbitrary fermion masses. One sees that for small values of the interaction strength $y < 1$, the mass–radius relation scales as $M \propto R^{-3}$ for large radii and looks like the one for the noninteracting case. For large values of the interaction

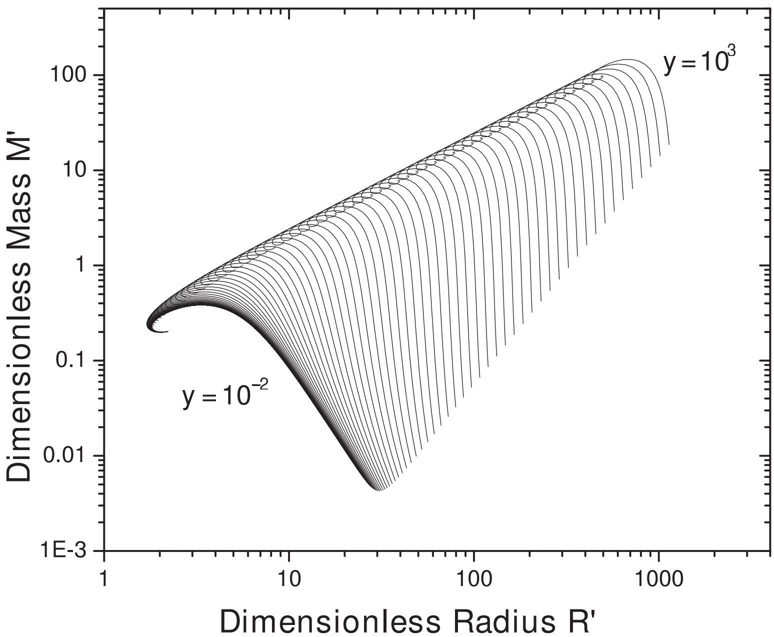

Figure 4.3 The mass–radius relation for compact stars consisting of interacting fermions for different interaction strengths y (in 20 equal steps for each decade) given in units of the Landau mass and Landau radius. Reprinted with permission from Narain et al. (2006). Copyright (2006) by the American Physical Society

strength $y > 1$, the mass–radius relation becomes independent of the mass with a constant radius. The maximum mass and the radius are growing linearly with the interaction strength y. In summary, for the weakly interacting case, ($y < 1$): $M \cdot R^3 = \text{const.}$ and $M_{\max} = \text{const.}$, while for the strongly interacting case, ($y > 1$): $R \approx \text{const.}$ and $M_{\max} \propto y$.

The different behavior of the mass–radius relation for the weakly and the strongly interacting cases can be understood in simple terms. For the weakly interacting case, the fermions are becoming relativistic before the interaction terms start to dominate the pressure, so that the maximum mass is reached before interactions are important. One recovers the case for noninteracting fermions. For the strongly interacting case, the interaction terms are dominating the pressure and energy density before the fermions are becoming relativistic. The maximum mass is then reached when the interaction terms are starting to become dominant for ε, that is, when the contribution from the mass term and the interaction term for the energy density are about equal

$$m_f \cdot n_c \approx c \cdot n_c^2 \tag{4.96}$$

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and the corresponding critical density n_c can be estimated to scale with the interaction strength y as

$$n_c \sim m_f/c = m_f \cdot m_I^2 = \frac{m_f^3}{y^2}. \tag{4.97}$$

For comparison, the critical density for the free case corresponds to $n_c \sim m_f^3$, as $n \sim k_F^3$ and the critical Fermi momentum is just given by the onset of having relativistic fermions $k_{F,c} = m_f$. We realize now that for $y < 1$, the instability stemming from relativistic fermions is happening at a critical density, which is lower than the one stemming from the interaction terms. Hence, the mass–radius relation looks like that of a compact star with noninteracting fermions. For $y > 1$, however, the instability stemming from the interaction terms is happening at a density lower than the one stemming from relativistic fermions.

The critical values for the pressure and the energy density for $y > 1$ are scaling as

$$\varepsilon_c \sim m_f \cdot n_c \sim m_f^2 \cdot m_I^2 \sim \frac{m_f^4}{y^2} \qquad P_c \sim \frac{1}{m_I^2} \cdot n_c^2 \sim \frac{1}{m_I^2} \cdot \frac{m_f^6}{y^4} \sim \frac{m_f^4}{y^2}. \tag{4.98}$$

We know that for the noninteracting case, the critical energy density and the pressure scales as m_f^4 and the maximum mass and the corresponding radius scales as given by the Landau mass and radius, that is, proportional to $1/m_f^2$. As the critical energy density and pressure for the strongly interacting case scales now as m_f^4/y^2, in analogy the maximum mass and the corresponding radius have to scale as

$$M_L^{\text{int}} = y \cdot \frac{m_{\text{P}}^3}{m_f^2} = y \cdot M_L \qquad R_L^{\text{int}} = y \cdot \frac{m_{\text{P}}}{m_f^2} = y \cdot R_L \qquad (y > 1), \tag{4.99}$$

hence, linear in the interaction strength y. Numerically one finds the following expression for arbitrary values of y:

$$M_{\text{max, int}} = (0.384 + 0.165 \cdot y) \cdot M_L \tag{4.100}$$

and

$$R_{\text{crit, int}} = (3.367 + 0.797 \cdot y) \cdot R_L. \tag{4.101}$$

The compactness of a star is given by the ratio of the radius to its Schwarzschild radius. The maximum compactness is given by the compactness of the maximum mass configuration. For weakly interacting fermions $y \ll 1$, one recovers the maximum compactness of a free fermion star. Fermion stars with strongly interaction fermions can be more compact. In summary,

$$C_{\max} = \frac{GM_{\max}}{R_{\text{crit}}} = \begin{cases} 0.11 & \text{for } y \ll 1 \\ 0.21 & \text{for } y \gg 1 \end{cases} \tag{4.102}$$

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for our choice of the interaction terms in the EOS. We see that interactions can make a neutron star more compact by a factor of two compared the noninteracting case. Of course, a different ansatz for the interaction between the fermions could result in an even more compact configurations.

Finally, let us put the interaction strength y in perspective with the known forces. In weak interactions of the standard model, the interaction strength is given by Fermi's constant, which is in natural units given by

$$G_F = 1.1663787(6) \times 10^{-5}\,\mathrm{GeV}^{-2} = (292.806\,\mathrm{GeV})^{-2} \tag{4.103}$$

and the corresponding interaction mass scale would be $m_I \sim 300\,\mathrm{GeV}$. For strong interactions of the standard model (QCD), the typical mass scale is given by $\Lambda_{\mathrm{QCD}} \sim 200\,\mathrm{MeV}$, as known from perturbative QCD and the running of the strong coupling constant α_s (see Chapter 8). In the nonperturbative regime, QCD can be described by chiral effective field theory, where the interaction strength is controlled by the pion decay constant $f_\pi = 92\,\mathrm{MeV}$. In fact, interaction terms to first order in the chiral expansion for the pion–nucleon interaction depend on $1/f_\pi^2$. So, for the strong interaction, the typical mass scale would be roughly $m_I \sim 100\,\mathrm{MeV}$. For a neutron star, where the mass of the fermion (the neutron) is about 1 GeV, one would expect sizable corrections from interactions, as $y \sim m_f/m_I = m_n/f_\pi \sim 10$. The maximum mass of a neutron star, including effects from strong interactions would then be much larger than the one for the free case. In fact, this is indeed seen in refined models of neutron stars, see Chapter 7. The numerical expression for the maximum mass, Eq. (4.100), gives then a maximum mass of a neutron star of about $3.8M_\odot$ compared to $0.71M_\odot$ for the free case.

One can even include gravity in these considerations. The corresponding mass scale is, of course, Newton's constant or, equivalently, the Planck mass, which is in natural units:

$$G = \frac{1}{m_{\mathrm{P}}^2} = (1.220890(13) \times 10^{19}\,\mathrm{GeV})^{-2}. \tag{4.104}$$

As any reasonable fermion mass is well below the Planck scale, one would recover in any reasonable case the weakly interacting case $y < 1$ for any fermion star where the fermions are only interacting on scales of the gravitational interaction, which is equivalent to the noninteracting case for compact star configurations.

4.5 Boson Stars

So far we have considered compact stars made of fermions where the Fermi pressure counterbalances the gravitational pull of matter. In this section we are discussing the case when compact stars consist of bosons instead of fermions.

Boson stars are normally considered as a solution to the combined Einstein equations with the Klein–Gordon equation of a scalar boson. As we will see, the essential features of these solutions can be well understood in terms the of TOV equations and the EOS of bosonic matter, which we will pursue in the following.

Bosons at zero temperature are in a Bose condensate, where all particles are occupying the lowest energy state. For a free gas of bosons, the lowest energy state has zero momentum, so there is no pressure generated by the bosonic particles. Naturally, one can ask oneselves if boson stars can be stable, if there is no pressure at all. Here comes quantum mechanics to the rescue. The Heisenberg uncertainty principle demands that there is an uncertainty in the momentum of particles, which is also valid for bosons. A massive particle has a Compton wavelength, which corresponds to a nonvanishing momentum of the particle, even if it is in the lowest energy state.

We can use Landau's argument for a maximum mass also for the case of bosons. Let us consider a sphere of bosons and have a look at a relativistic boson sitting at the radius R of that sphere. The total energy of that boson is the sum of gravitational energy and kinetic energy

$$E(R) = E_g + E_{\rm kin} = -G\frac{m_b M}{R} + E_{\rm kin}, \tag{4.105}$$

with the total mass of the boson star $M = N \cdot m_b$, where N is the number of bosons in the compact star and m_b the mass of the boson. For a relativistic boson, the energy is equal to the momentum. The uncertainty principle relates the momentum of the boson to the size of the quantum system so that

$$E_{\rm kin} = k_b \sim \frac{1}{R}, \tag{4.106}$$

where R is the radius of the boson star. Note that this implies that the boson star is something like a macroscopic quantum ball. So, the kinetic energy and the gravitational energy of the boson scale, with $1/R$ as in the case for fermions:

$$E(R) = -G\frac{N \cdot m_b^2}{R} + \frac{1}{R}. \tag{4.107}$$

In the nonrelativistic limit, the kinetic energy of the bosons is proportional to $E_{\rm kin} = k_b^2/m_b \sim 1/R^2$, as in the case for fermions. One can repeat the stability analysis for bosons in an analogy to the one of fermions. For a positive total energy, the compact star will expand to minimize the total energy. Eventually, the boson becomes nonrelativistic, the kinetic energy scales as $1/R^2$, generating a stable minimum at a nonvanishing value of the radius R. For a negative total energy, the compact star will shrink to minimize the total energy, eventually collapsing to a black hole. Hence, there exists a maximum number of bosons $N_{\max}$, which is given by the limiting case of stability $E(R) = 0$. This condition translates to

$$G \cdot N_{\max} \cdot m_b^2 = 1, \tag{4.108}$$

so that the maximum number of bosons is given by

$$N_{\max} = \frac{1}{G \cdot m_b^2} = \frac{m_{\mathrm{P}}^2}{m_b^2}. \tag{4.109}$$

The maximum mass of a compact star of free bosons is then

$$M_{\max} = N_{\max} \cdot m_b = \frac{m_{\mathrm{P}}^2}{m_b}, \tag{4.110}$$

which differs by a factor m_{P}/m_b compared to the Landau maximum mass for free fermions, Eq. (4.28). Usually, $m_{\mathrm{P}} \gg m_b$, so that the maximum mass of a boson star is orders of magnitude smaller compared to the maximum mass of a fermion star with the same particle mass. The critical radius can be estimated from the onset of relativity for the boson's kinetic energy from Eq. (4.106) to be

$$R_{\mathrm{crit}} = \frac{1}{m_b}, \tag{4.111}$$

which is just the Compton wavelength of the boson. The critical radius differs by the same factor m_{P}/m_b from the case for free fermions. Hence, boson stars are orders of magnitude smaller compared to fermion stars that consist of particles with the same particle mass. As the critical radius of a boson star is determined by the Compton wavelength, which is a quantum mechanical quantity, the radius is microscopically small and boson stars can be considered to be quantum balls.

Interactions between the boson will change the picture drastically. Let us proceed by introducing an interaction term to the energy density, which scales as the number density n squared, similar to the case we studied before for fermions:

$$\varepsilon = m_b \cdot n + c \cdot n^2, \tag{4.112}$$

where c is a constant of dimension $(1/\text{mass})^2$, specifying the strength of the interaction. We ignore the effect from the quantum zero-point energy, which we consider to be negligible compared to the interaction energy. A posteriori we will see that this is indeed justified. The pressure can be calculated by using Eq. (3.69) to be

$$P = c \cdot n^2. \tag{4.113}$$

For low number densities, the mass term in the energy density dominates and one recovers an EOS of the form $P \sim \varepsilon^2$, which is a polytrope with a power law with the power of $\gamma = 2$. We know that this type of EOS leads to stable compact star configurations with a constant radius in the mass–radius diagram. At high densities, the energy density is dominated by the interaction term and we recover an EOS of the form $P = \varepsilon$, which is a power law with $\gamma = 1$. We have seen that this EOS leads to unstable compact star configurations. The limiting case is where the mass term and the interaction term in the expression of the energy density are of equal magnitude, giving us a criteria for determining the maximum mass of a compact star with interacting bosons. The critical number density is accordingly determined via

$$m_b \cdot n_c = c \cdot n_c^2 \rightarrow n_c = \frac{m_b}{c}, \tag{4.114}$$

giving a critical energy density of about

$$\varepsilon_c \sim m_b \cdot n_c = \frac{m_b^2}{c} = m_b^2 \cdot m_I^2 = \frac{m_b^4}{y^2}. \tag{4.115}$$

Here we introduced the interaction mass scale m_I defined via the relation $c = 1/m_I^2$ and the interaction strength $y = m_b/m_I$ in an analogy with the case of interacting fermions. Now we are in the position to use the scaling properties of the TOV equations, Eq. (4.55). The maximum mass and the corresponding radius of a compact star with interacting bosons is then given by choosing the scaling energy density to be $\varepsilon_0 = m_b^4/y^2$, which results in

$$M_{\max} = M' \cdot y \cdot \frac{m_{\mathrm{P}}^3}{m_b^2} = M' \cdot y \cdot M_L \tag{4.116}$$

and

$$R_{\mathrm{crit}} = R' \cdot y \cdot \frac{m_{\mathrm{P}}}{m_b^2} = R' \cdot y \cdot R_L. \tag{4.117}$$

The numerical values for the scaling coefficients are $M' = 0.164$ and $R' = 0.763$. Surprisingly, we recover the Landau mass and radius for boson stars for an interaction strength of $y = 1$. This means that boson stars have a similar maximum mass and a radius as its fermionic counterparts with the same particle mass if interactions are included in the EOS. The dependence of the maximum mass and the corresponding radius on the interaction strength y is linear, as in the case for fermion stars with an interaction strength of $y > 1$, see Eqs. (4.100) and (4.101). Also, the numerical prefactors are very close to the fermionic case. Interestingly, if one sets the interaction strength by choosing the interaction mass scale to the Planck scale, $m_I = m_{\mathrm{P}}$, one arrives at the expression for the maximum mass and the corresponding radius for the noninteracting case, Eqs. (4.110) and (4.111), derived by using Landau's argument.

Figure 4.4 shows the mass–radius relation for boson stars for different interaction strengths y. The mass and the radius are given in dimensionless units, that is, in

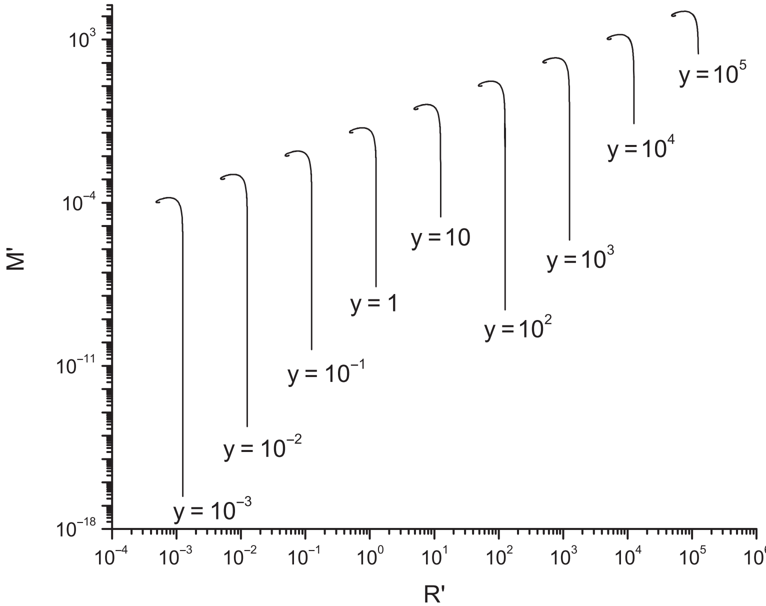

Figure 4.4 The mass–radius relation for compact stars consisting of interacting bosons for different interaction strengths y given in units of the Landau mass and Landau radius. Reprinted with permission from Agnihotri et al. (2009). Copyright (2009) by the American Physical Society

units of the Landau mass M_L and the Landau radius R_L. One sees that the mass–radius relation exhibits a constant radius over a large range of compact star masses. The maximum mass as well as the characteristic radius scale linearly with the interaction scale y. Note that the plot has a double logarithmic scale.

The maximum compactness of boson stars is given by the compactness of the maximum mass configuration

$$C_{\max} = \frac{M'}{R'} = 0.21 \tag{4.118}$$

and is independent of the boson mass and the interaction strength y. Hence, this value would be also applicable for noninteracting boson stars by choosing the interaction mass scale to be the Planck scale, as discussed earlier. The maximum compactness of boson stars is about the same compared to the case of strongly interacting fermion stars, see Eq. (4.102).

The history of boson stars is quite interesting. Wheeler was the first to study self-gravitating spheres for the combined Einstein–Maxwell equations, that is, compact stars of photons, which he dubbed geons (Wheeler, 1955) but the solutions turned out to be unstable. Derrick showed that there exists no stable time-independent solution of a nonlinear equation of motion of a scalar field (Derrick, 1964), also known as Derrick's theorem. Later, Kaup followed by a work of Ruffini and Bonazzola found stable solutions of the Einstein equation together with the Klein–Gordon equations for a fundamental scalar field (Kaup, 1968; Ruffini and Bonazzola, 1969). Derrick's theorem does not apply, as time-dependent solutions were considered for the scalar field. The resulting boson stars were of microscopic sizes and prevented from collapse to a black hole just by Heisenberg's uncertainty principle. This is the case we considered for noninteracting bosons earlier and our estimates for the maximum mass and the radius of the boson star do actually apply. Colpi, Shapiro, and Wasserman introduced a scalar field with self-interactions for the study of boson stars (Colpi et al., 1986). They found that the mass as well as the radius of a boson star with scalar selfinteractions increases drastically. The resulting relations of the maximum mass and the corresponding radius scale with the Landau mass and radius times the square-root of interaction strength parameter, which they showed to be consistent with a scaling analysis, in an exact analogy to the scaling analysis we discussed earlier. Another class of boson stars were introduced by Friedberg, Lee, and Pang for nontopological soliton fields (Friedberg et al., 1987), where the property of the soliton ensured the stability of the boson star.

In the standard model of particle physics, there exists no known stable massive boson. Bosons in the standard model are unstable on the timescale of electroweak interactions, such as the W, Z, and Higgs boson. The only stable bosons, the gluons and the photons, have a vanishing mass and likely form an unstable boson star. There are massive bosons known in strong interaction physics, as the scalar and vector mesons. These mesons are unstable, too, and most of them decay on the timescale of strong interactions. The most stable mesons are pseudoscalar mesons, which decay on the timescale of weak interactions. The charged pions have a mass of $m_\pi \sim 140\,\text{MeV}$ and a lifetime of $\tau \sim 3 \times 10^{-8}\,\text{s}$ for the K_L^0. The kaons have a mass of about $m_K \sim 500\,\text{MeV}$ and lifetimes of up to $\tau \sim 5 \times 10^{-8}\,\text{s}$. These timescale are still orders of magnitudes too small to be of astrophysical interest for boson stars. There might be stable bosons beyond the standard model, as for example, the hypothetical axion, a pseudoscalar particle that is considered a candidate for cold dark matter or other unknown fundamental scalar fields.

Exercises

(4.1) Estimate the maximum mass and the corresponding radius for hypothetical compact stars made of the following fermions via the Landau mass $M_L = m_\mathrm{P}^3/m_f^2$ and the Landau radius $R_L = m_\mathrm{P}/m_f^2$:

(a) for a neutrino star with $m_\nu = 1\,\text{eV}$

(b) for a neutralino star with $m_{\nu_s} = 100\,\text{GeV}$.

How does the maximum mass and the corresponding radius change when including interactions? Use the interaction mass scale from QCD and from weak interactions.

(4.2) Show that the mass–radius relation for compact stars made of a free Fermi gas with the mass m and the degeneracy factor g scales as $1/(\sqrt{g}m^2)$. What would be the maximum mass of gas of free nucleons, if protons are uncharged, that is, for a degeneracy factor of $g = 4$ instead of $g = 2$ for the case of free neutrons?

(4.3) Look at an interaction-dominated gas of nucleons of the form $\varepsilon = n^2/m_I^2$. Show that the pressure has the form $p = \varepsilon = n^2/m_I^2$ by using thermodynamic relations.

(4.4) Show that a pressure of the form $P(\mu) = a \cdot \mu^2 - c$, where a and c are constants, results in an EOS of the form $P = \varepsilon - 2c$ independent of the value of a.

(4.5) Derive the thermodynamic relation for the speed of sound squared

$$c_s^2 = \frac{\mathrm{d}P}{\mathrm{d}\varepsilon} = \frac{\mathrm{d}\ln\mu}{\mathrm{d}\ln n} \tag{4.119}$$

using the chain rule and thermodynamic relations for vanishing temperature. Solve the differential equation for a constant speed of sound $s = c_s^2$ and show that in general

$$P(\mu) = a \cdot \mu^{(s+1)/s} - c, \tag{4.120}$$

where a and c are constants. Check that the solution results in an EOS of the form $P = s \cdot (\varepsilon - \varepsilon_{\mathrm{vac}})$, with a vacuum energy $\varepsilon_{\mathrm{vac}} = (1 + 1/s) \cdot c$. Conclude how the corresponding mass–radius relation has to scale with parameters a and c.