Jürgen Schaffner-Bielich: Compact Stars

4.3 Scaling Solutions for Compact Stars

Basic Idea

- scaling arguments = reasonable combination of dimensionful parameters

1. make TOV dimensionless
2. solve dimensionless TOV
3. rescale with $\sqrt{\varepsilon_0}$ $\rightarrow$ physical solution

Making TOV Dimensionless

- $$\frac{dP}{dr} = -G\frac{m_r}{r^2}\varepsilon\left(1+\frac{P}{\varepsilon}\right)\left(1+\frac{4\pi r^3 P}{m_r}\right)\left(1-\frac{2Gm_r}{r}\right)^{-1}$$

$P = \varepsilon_0 P'$	$\varepsilon = \varepsilon_0 \varepsilon'$
$r = ar'$	$m_r = bm_r'$

- $$\frac{\varepsilon_0 dP'}{adr'} = -G\frac{bm_r'}{(ar')^2}\varepsilon_0\varepsilon'\left(1+\frac{P'}{\varepsilon'}\right)\left(1+\frac{4\pi(ar')^3\varepsilon_0 P'}{bm_r'}\right)\left(1-\frac{2Gbm_r'}{ar'}\right)^{-1}$$

$Gb = a$	$a^3\varepsilon_0 = b$
$a = (G\varepsilon_0)^{-\frac{1}{2}}$	$\mathrm{b} = (G^3\varepsilon_0)^{-\frac{1}{2}}$

$$\frac{dP'}{dr'} = -G\frac{m_r'}{r'^2}\varepsilon'\left(1+\frac{P'}{\varepsilon'}\right)\left(1+\frac{4\pi r'^3 P'}{m_r'}\right)\left(1-\frac{2m_r'}{r'}\right)^{-1}$$

$$\boldsymbol{m_r = \frac{m_r'}{\sqrt{G^3\varepsilon_0}} \sim \frac{1}{\sqrt{\varepsilon_0}}} \qquad \boldsymbol{r = \frac{r}{\sqrt{G\varepsilon_0}} \sim \frac{1}{\sqrt{\varepsilon_0}}}$$

Making TOV Dimensionless

$Gb = a$	$a^3\varepsilon_0 = b$
$a = (G\varepsilon_0)^{-\frac{1}{2}}$	$b = (G^3\varepsilon_0)^{-\frac{1}{2}}$

$$\frac{dP'}{dr'} = -G\frac{m'_r}{r'^2}\varepsilon'\left(1+\frac{P'}{\varepsilon'}\right)\left(1+\frac{4\pi r'^3P'}{m'_r}\right)\left(1-\frac{2m'_r}{r'}\right)^{-1}$$

$$\boldsymbol{m_r = \frac{m'_r}{\sqrt{G^3\varepsilon_0}} \sim \frac{1}{\sqrt{\varepsilon_0}}} \qquad \boldsymbol{r = \frac{r}{\sqrt{G\varepsilon_0}} \sim \frac{1}{\sqrt{\varepsilon_0}}}$$

$$\boldsymbol{M_{max} = \frac{M'}{\sqrt{G^3\varepsilon_0}} = M'\frac{m_p^3}{\sqrt{\varepsilon_0}}} \qquad \boldsymbol{R_{crit} = \frac{R'}{\sqrt{G\varepsilon_0}} = R'\frac{m_p}{\sqrt{\varepsilon_0}}}$$

Example: Free Massive Fermi Gas

- dimensionful quantity: fermion mass

$P' = \frac{P'}{\varepsilon_0} = \frac{P'}{\mathrm{m_f^4}}$	$\varepsilon' = \frac{\varepsilon}{\varepsilon_0} = \frac{\varepsilon'}{\mathrm{m_f^4}}$

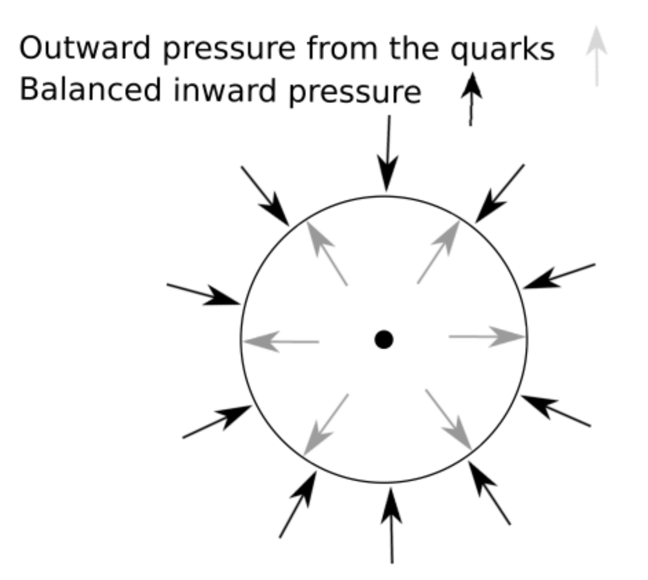
- $\rightarrow \varepsilon_0 = m_f^4$

$\boldsymbol{M_{max} = M' \frac{m_p^3}{\sqrt{\varepsilon_0}} = M' \frac{m_p^3}{m_f^2}}$	$\boldsymbol{R_{crit} = R' \frac{m_p}{\sqrt{\varepsilon_0}} = R' \frac{m_p}{m_f^2}}$

- Compactness $C = \frac{GM}{R}$
- $\mathrm{C_{max}} = \frac{M'}{R'} = \frac{0.384}{3.367} = 0.11$

Example: Relativistic Fermi Gas with Vacuum Term

- relativistic particles: $P = \frac{1}{3}\varepsilon \quad \rightarrow \quad \boxed{P = \frac{1}{3}(\varepsilon - \varepsilon_0)}$
- vanishing P at nonvanishing ε
- MIT bag model:
  - $P = P_{free} - B \quad \leftrightarrow \quad \varepsilon = \varepsilon_{free} + B$
  - relativistic massless gas of particles: $P_{free} = \frac{1}{3}\varepsilon_{free}$
  - $P = \frac{1}{3}\varepsilon_{free} - B = \frac{1}{3}(\varepsilon - B) - B = \boxed{\frac{1}{3}(\varepsilon - 4B)}$
    $\rightarrow \varepsilon_0 = 4B$
  - $C = 0.271 \quad (C_{free\ Fermion} = 0.11)$



Limiting EOS from causality

- stiffest possible EOS (Zel'dovich): $P = \varepsilon$
  - $c_{sound}^2 = \frac{\partial P}{\partial \varepsilon} = 1$
  - maximum P for given ε
  - unstable TOV solutions
- know low-density EOS for $0 < P < P_f$
- stiffest possible EOS for higher densities must be Zel'dovich EOS
  - maximum $P \quad \leftrightarrow \quad$ maximum M
- Rhoades-Ruffini mass limit: $M_{max} = M' \frac{1}{\sqrt{G^3 \varepsilon_f}} = 4.2\ M_{solar} \sqrt{\frac{\varepsilon_{nm}}{\varepsilon_f}}$

$$\boxed{P = P_f + (\varepsilon - \varepsilon_f)}$$

Selfbound Linear EOS

- selfbound stars:
  - vanishing P at nonvanishing ε
  - vanishing P at surface
  - do not need gravity to be stabilized
- $P = s(\varepsilon - \varepsilon_0)$
  - $c_{sound}^2 = \frac{\partial P}{\partial \varepsilon} = s$
- reedshift factor: $1 + z = \frac{1}{\sqrt{\frac{1}{1-2C}}}$

Selfbound Linear EOS

s=1/3 MIT bag model	$M_{max} = 2.57\ M_{solar}\sqrt{\frac{\varepsilon_{nm}}{\varepsilon_0}}$	$R_{crit} = 14\ km\sqrt{\frac{\varepsilon_{nm}}{\varepsilon_0}}$	$C_{max} = 0.271$	$z_{max} = 0.478$	$\frac{\varepsilon_{max}}{\varepsilon_0} = 4.81$	$\frac{\varepsilon_{max}(2M_{solar})}{\varepsilon_{nm}} = 7.91$
s=2/3	$M_{max} = 3.64\ M_{solar}\sqrt{\frac{\varepsilon_{nm}}{\varepsilon_0}}$	$R_{crit} = 16.4\ km\sqrt{\frac{\varepsilon_{nm}}{\varepsilon_0}}$	$C_{max} = 0.328$	$z_{max} = 0.705$	$\frac{\varepsilon_{max}}{\varepsilon_0} = 3.54$	$\frac{\varepsilon_{max}(2M_{solar})}{\varepsilon_{nm}} = 11.8$
s=3/3 Fel'dovich	$M_{max} = 4.23\ M_{solar}\sqrt{\frac{\varepsilon_{nm}}{\varepsilon_0}}$	$R_{crit} = 17.6\ km\sqrt{\frac{\varepsilon_{nm}}{\varepsilon_0}}$	$C_{max} = 0.354$	$z_{max} = 0.851$	$\frac{\varepsilon_{max}}{\varepsilon_0} = 3.03$	$\frac{\varepsilon_{max}(2M_{solar})}{\varepsilon_{nm}} = 13.5$

Thank you for listening

NOTES 1

- free massive fermi gas
  - P=?
  - rescale with fermi mass^4
  - C=0,11
- relativistic fermi gas with vacuum term
  - P=1/3 e -> polytrope 1
  - rescale with offset P=1/3(e-e0) -> pressure vanishes at nonvanishing e
  - MIT bag model
    - gas of free relativistic quarks in bag stabilizes by outside pressure B -> P_tot = P_free – B and e = e_free + B (nonvanishing vacuum energy density contributions due to quark gluon condensates)
    - relativistic massless gas of particles: P_free = 1/3 e_free -> P = 1/3 e_free – B = 1/3 (e-B) –B=1/3 (e-4B)
    - M_max, R_crit -> C=0.271 -> more compact than free fermion star

NOTES 2

- limiting EOS from causality
  - stiffest possible EOS (Zel'dovich): $P = \varepsilon$
    - $c_{sound}^2 = \frac{\partial P}{\partial \varepsilon} = 1$
    - maximum P for given ε
    - unstable TOV solutions
- know low-density EOS for $0 < P < P_f$
- stiffest possible EOS for higher densities must be Zel'dovich EOS
  - maximum $P \quad \leftrightarrow \quad$ maximum M

$$\boxed{P = P_f + (\varepsilon - \varepsilon_f)}$$

- Rhoades-Ruffini mass limit: $M_{max} = M' \frac{1}{\sqrt{G^3 \varepsilon_f}} = 4.2\ M_{solar} \sqrt{\frac{\varepsilon_{nm}}{\varepsilon_f}}$
  - would give M_max = 3 M_s for e_f=e_nm but we know 4.2 M_s is true

NOTES 3

- selfbound stars:
  - vanishing P at nonvanishing ε
  - vanishing P at surface
  - do not need gravity to be stabilized
- selfbound linear eos -> see table
- reedshift factor: $1+z=\frac{1}{\sqrt{\frac{1}{1-2C}}}$

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Selfbound Linear EOS- whole table

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