2

General Relativity

This chapter is a brief introduction to some of the basic concepts of general relativity. It is not meant to provide a thorough introduction to general relativity and we refer the reader to the many excellent textbooks written on that subject. The main purpose of this chapter is to derive the results of general relativity necessary for our discussion of the properties of compact stars and its astrophysical applications in the later chapters. A separate chapter is devoted to the subject of gravitational waves (see Chapter 10).

2.1 Gravity and the Equivalence Principle

Let us introduce the concept of general relativity by looking at Newton's law and Newton's expression for the gravitational force. According to Newton's law a massive particle feels a kinetic force

$$F_{\text{kin}} = m_i \cdot a \tag{2.1}$$

that is proportional to the acceleration a with the coefficient m_i. We will denote the coefficient m_i as the inertial mass of the particle. According to Newton, the classical gravitational force

$$F_g = -m_g \cdot \nabla\phi \tag{2.2}$$

is proportional to the gradient of the gravitational potential ϕ with the coefficient m_g. We will denote the coefficient m_g as the gravitational mass of the particle. In experiments first performed by Galileo Galilei, it was found out that every object falls with the same acceleration in a gravitational field so that

$$a = -\nabla\phi \tag{2.3}$$

independent of its inertial and gravitational mass. Hence, the inertial and the gravitational mass have to be the same

$$m_g = m_i. \tag{2.4}$$

Eq. (2.4) constitutes the weak equivalence principle (or WEP).

Weak equivalence principle: All particles experience the same acceleration in a gravitational field irrespective of their masses.

WEP has been tested to a high accuracy in many different experiments. Modern torsion balance experiments have tested the WEP down to a level of 10^{-13} (Wagner et al., 2012). The satellite experiment MICROSCOPE has tested the weak equivalence principle at a level of 10^{-15} (Touboul et al., 2017). There are equivalent formulations of the WEP. For example, WEP implies that there is a preferred trajectory of particles through spacetime on which particles move that is determined just by gravity.

An extension of the WEP is the Einstein equivalence principle (EEP), which includes any form of matter, not only particles. Consider an observer in a sealed box performing experiments. From the equivalence of the inertial and the gravitational mass, an observer cannot distinguish whether an object is accelerated by the acceleration of the box or whether it is accelerated by the presence of a gravitational field. It is impossible to detect the existence of a gravitational field by local experiments. It is important to add the condition of locality to the experiments as gradients of the gravitational field will lead to tidal forces on larger scales that would be measurable. The observer will measure locally only the laws of physics as they would be in a spacetime without gravity, so that the kinematics are governed by special relativity. The EEP can be stated as follows (see e.g., Schutz, 2009):

Einstein equivalence principle: In small enough regions of spacetime, any physical experiment not involving gravity will have the same result if performed in a freely falling inertial frame as if performed in the flat spacetime of gravity.

We will heavily use EEP in the following sections. It allows the transfer of equations valid in special relativity to its general form in general relativity as both equations have to be of the same form in a small enough region of spacetime.

EEP has important implications for the way gravity couples to matter. As all physical laws, except gravity, are included in defining the EEP, it implies that gravity couples to the other forces, of nature in a special way. Let us be more specific here. An atom consists of electrons and a nucleus bound by electromagnetic forces, which is stabilized by quantum mechanics. The mass of an atom is not the sum of the mass of electrons and the nucleus but slightly less. The difference in mass

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is dubbed the mass defect or simply the binding energy of the atom. Seemingly, gravity couples not only to the masses of the constituents of the atom but it is also sensitive to the mass defect generated by the quantum electromagnetic forces. Hence, gravity couples to electromagnetic energy. The same story goes for the nucleus of an atom. The nucleus consists of protons and neutrons. The mass of the nucleus is not the sum of the masses of the protons and neutrons in the nucleus but somewhat less, which is just the mass defect or the binding energy of the nucleus. In the nucleus, the nuclear forces or the strong interactions are at work. So gravity couples to energy generated by strong interactions. The same reasoning should also apply for the other fundamental force of the standard model left, weak interactions. The masses of the elementary particle, quarks and leptons, are generated by the Higgs mechanism. So gravity couples also to energy (mass) generated by the weak interactions. The reader is invited to fancy at a system consisting of leptons only that is bound by weak interactions only, as, for example, a hypothetical neutrino ball, and perform the same reasoning for nuclei and strong interactions.

The equivalence of performing experiments in an accelerating system and in one at rest being exposed to a gravitational field can be exploited in Einstein's famous way via thought experiments. Hereby one imagines certain experimental situations to arrive at statements about physical laws. Let us imagine a rocket that is constantly accelerating in free space that shall be equal to the gravitational acceleration on Earth. According to EEP, experiments performed inside the rocket are equivalent to the ones performed in a rocket standing on Earth as the systems are exposed to the same acceleration. From this setting one can immediately derive two important consequences of EEP that form the basis of general relativity: the gravitational redshift of light and the gravitational bending of light in a gravitational field.

Imagine an electromagnetic wave with a fixed frequency (e.g., from a laser) being emitted from the bottom of the rocket and being measured at the top of the rocket. If the rocket is continuously accelerating, the speed of the rocket at the time of the detection will be correspondingly larger compared to the one at the time of emission for each crest (or trough) of the wave. The time of arrival of the wave crests of the light at the detector will be correspondingly larger than the one at constant speed. This effect leads to a decrease of the frequency of the wave being measured at the top of the rocket compared to the original frequency, that is, a redshift of the electromagnetic wave. Switching the emitter with the detector, so that the wave travels from the top of the rocket to its bottom, the electromagnetic wave will be blueshifted when measured at the bottom of the rocket. EEP now states that the experimental situation is equivalent to the one in a gravitational field with the same (gravitational) acceleration. Hence, an electromagnetic wave being emitted within a gravitational potential to the outside will be measured to have a lower frequency compared to original frequency by an outside observer. This is

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the famous gravitational redshift effect predicted by general relativity. In fact, as we have seen, the gravitational redshift is more fundamental as it is a consequence of EEP.

We can derive an approximate expression for the gravitational redshift by using the Doppler shift formula for nonrelativistic velocities. The Doppler shift depends on the velocity difference of the emitting source and the receiver Δv. The relative change in the wavelength $\Delta\lambda$ measured by the receiver in comparison to the original wavelength of the emitter λ is given by

$$\frac{\Delta\lambda}{\lambda} = \frac{\Delta v}{c}, \tag{2.5}$$

where we put in the velocity of light c as the velocity of the wave. For the thought experiment of the rocket, the difference in the velocity comes from the constant acceleration a of the rocket within the time span the light is traveling from the bottom of the rocket to the top $\Delta t = \Delta x/c$, where Δx is the distance between the emitter and the detector. The difference in the arrival of wave crests corresponds now to

$$\frac{\Delta\lambda}{\lambda} = \frac{a \cdot \Delta t}{c} = a \cdot \frac{\Delta x}{c^2}. \tag{2.6}$$

The equivalence principle now demands that the shift in wavelength is exactly the same as for a gravitational field with the same (gravitational) acceleration due to the gradient of the gravitational potential $\nabla\phi$:

$$\frac{\Delta\lambda}{\lambda} = \nabla\phi \frac{\Delta x}{c^2} = \frac{\Delta\phi \cdot \Delta x}{\Delta x \cdot c^2} = \frac{\Delta\phi}{c^2}. \tag{2.7}$$

There is a change in the sign from Eq. (2.3) as the gravitational acceleration is pointing inward and the wave is traveling outward. In the last step, we assumed a gravitational potential with a constant gradient. However, the expression holds also for the general case. Note that $\Delta\phi$ for the case studied here is a positive quantity so that the observed wavelength will be longer than it should be. Specifying the wavelength for the emitting source by λ_e and the one observed at the detector by λ_d one finds

$$\frac{\lambda_d}{\lambda_e} = 1 + \frac{\Delta\phi}{c^2}. \tag{2.8}$$

As the dispersion relation for light reads

$$c = \lambda \cdot \nu, \tag{2.9}$$

with the frequency ν, the corresponding expression for the ratio of the frequencies is

$$\frac{\nu_d}{\nu_e} = \left(1 + \frac{\Delta\phi}{c^2}\right)^{-1} \approx 1 - \frac{\Delta\phi}{c^2} \tag{2.10}$$

to first order in $\Delta\phi/c^2$. The redshift factor z is defined as

$$1 + z = \frac{\lambda_d}{\lambda_e} \tag{2.11}$$

so that the gravitational redshift corresponds to a redshift factor of

$$z = \frac{\Delta\phi}{c^2}. \tag{2.12}$$

For light emitted from the surface of a massive body with mass M and a radius R, the gravitational potential is

$$\phi = \frac{GM}{R}. \tag{2.13}$$

For an outside observer at infinity, the observed redshift factor is then

$$z = \frac{GM}{Rc^2}, \tag{2.14}$$

where R is the radius of the massive body. We notice that the redshift factor is equal to the compactness of an object, see Eq. (1.4). Be aware that the expression equation (2.14) only holds for weak gravitational fields.

The second prediction of EEP is the bending of light in a gravitational potential. Imagine a laser beam being emitted from one side of the rocket to a detector at the other side of the rocket. If the rocket is moving with a constant velocity, the path of the laser beam will be a straight line. If the rocket is accelerating, there will be a difference in the velocity between the time of the emission and the time of detection. The detector needs to be moved down toward the bottom of the rocket to measure the laser beam compared to the nonaccelerating case. The path of the laser beam will be no longer a straight one, instead the laser beam is bent due to the acceleration of the rocket. According to EEP, the same experimental situation will be present for a gravitational field with the same (gravitational) acceleration. Hence, a gravitational field will bend the path of light.

The strong equivalence principle (SEP) is an extension of EEP. While the EEP explicitly dismisses experiments involving gravity, SEP explicitly includes all experiments, including those involving gravity.

Strong equivalence principle: In small enough regions of spacetime, any physical experiment *including gravity* will have the same result if performed in a freely falling inertial frame as if performed in the flat spacetime of gravity.

SEP not only includes self-gravitating systems, where the gravitational binding energy is important, but also gravitational systems as a gravitational two-body system moving in the presence of a gravitational field from a third body. Extensions

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of general relativity can be compatible with EEP but can violate SEP, for example, by the presence of a fifth force.

Tests of SEP have been performed within the solar system by lunar laser ranging experiments. However, the gravitational fields involved are rather weak. A more stringent test of SEP would be provided by the study of three compact stars orbiting each other. In fact, such a system has been observed. The pulsar PSR J0337+1715 is part of a triple system together with two white dwarfs that are gravitationally bound to each other (Ransom et al., 2014). The universality of free fall has been tested for the white dwarf and the pulsar in the gravitational potential of the second white dwarf, providing a test of SEP at the level of 2.6×10^{-6} (Archibald et al., 2018).

2.2 Special Relativity and the Metric

The basic building principle of special relativity is the constancy of the speed of light c, independent of the coordinate system. For a light ray, the relative difference in time $\mathrm{d}t$ and the relative differences in three-dimensional space $\mathrm{d}x$, $\mathrm{d}y$, and $\mathrm{d}z$ are related such that the difference of the squared quantities

$$\mathrm{d}s^2 = -c^2 \cdot \mathrm{d}t^2 + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2 \tag{2.15}$$

is zero. This condition has to be fulfilled in any coordinate system to ensure the constancy of the speed of light. Hence, the quantity $\mathrm{d}s^2$ is an invariant quantity. Moreover, it connects time with space coordinates so that one has to consider coordinates in the combined spacetime that is now four-dimensional. Any measurement, where $\mathrm{d}s^2 < 0$, is allowed as the corresponding velocity would be lower than the speed of light. Those distances in spacetime are called timelike and are causal in character. The path of particles through spacetime will be timelike to ensure causality. Measurements with $\mathrm{d}s^2 >$ are not allowed as it would involve velocities larger than the speed of light. Such distances are called spacelike and are acausal in character. In summary, distances in the four-dimensional spacetime can be classified as

$$\mathrm{d}s^2 \begin{cases} = 0 & \text{lightlike or null} \\ < 0 & \text{timelike} \\ > 0 & \text{spacelike} \end{cases} . \tag{2.16}$$

We note that the sign of Eq. (2.15) can be chosen arbitrarily. While textbooks in general relativity prefer the same sign as chosen here, textbooks in particle physics usually adopt the other sign. While this would change the conditions for timelike and spacelike distances, it would not change the classification itself. Eq. (2.15) can be written in matrix form as

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$$ds^2 = \sum_{\mu,\nu} \eta_{\mu\nu} dx^\mu dx^\nu = \eta_{\mu\nu} dx^\mu dx^\nu, \tag{2.17}$$

where $\mu = 1, \ldots 4$ and $\nu = 1, \ldots 4$. Here we introduced the Einstein's summation convention where one has to sum over repeated upper and lower indices. The matrix $\eta_{\mu\nu}$ stands for the metric of the spacetime of special relativity. The quantity x^μ is a four-vector in spacetime, where the 0-component corresponds to the time coordinate and the other three components to the space coordinates. For Cartesian coordinates

$$\eta_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{2.18}$$

For general coordinates, the metric $\eta_{\mu\nu}$ will be different. For spherical coordinates, for example, one finds

$$ds^2 = -c^2 dt^2 + dr^2 + r^2 \cdot \left(d\theta^2 + \sin^2\theta d\phi^2\right) \tag{2.19}$$

$$= -c^2 dt^2 + dr^2 + r^2 \cdot d\Omega^2, \tag{2.20}$$

where $d\Omega$ stands for the differential solid angle. In a general curved spacetime of general relativity, the four-dimensional distance is defined as

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu, \tag{2.21}$$

where $g_{\mu\nu}$ is the metric defining curved spacetime distances and thereby spacetime itself. The metric depends on the spacetime coordinates and is a tensor of rank two. As the metric is symmetric in the indices $g_{\mu\nu} = g_{\nu\mu}$, it has ten independent components in four dimensions ($D(D+1)/2$ components for D dimensions).

Let us look at a four-vector in spacetime A^μ and define the following quantity:

$$A_\mu = g_{\mu\nu} A^\nu \tag{2.22}$$

with a lower index. The components of A_μ are the ones of a so-called one-form. A one-form maps a vector to a real number as

$$A_\mu A^\mu = g_{\mu\nu} A^\nu A^\mu. \tag{2.23}$$

If we choose $A^\mu = x^\mu$, we know from the definition of the metric that the expression in Eq. (2.23) is a scalar quantity independent of the choice of the coordinate system. This should be true then for any choice of the vector A^μ. Let us denote g as the determinant of the metric $g_{\mu\nu}$. The determinant g must be nonvanishing to ensure a well-defined coordinate system, where the coordinates are providing independent directions in spacetime. Then one can define the inverse relation

$$A^\mu = g^{\mu\nu} A_\nu. \tag{2.24}$$

The metric can then be used to lower or raise indices. Combining Eqs. (2.22) and (2.24) one finds

$$A^\mu = g^{\mu\nu} g_{\nu\rho} A^\rho. \tag{2.25}$$

As this holds for any vector, the metric has to fulfill the condition that

$$g^{\mu\nu} g_{\nu\rho} = \delta^\mu_\rho, \tag{2.26}$$

where δ^μ_ρ is the Kronecker symbol

$$\delta^\mu_\nu = \begin{cases} 1 & \text{for } \mu = \nu \\ 0 & \text{for } \mu \neq \nu. \end{cases} \tag{2.27}$$

The metric is its own inverse and can be used to lower or raise its own indices.

For an observer at rest, the time measured must be a coordinate invariant quantity. The metric for an observer at rest reads

$$\mathrm{d}s^2 = g_{00} \cdot c^2 \cdot \mathrm{d}t^2. \tag{2.28}$$

One can define a coordinate independent scalar quantity, the proper time τ of an observer at rest by

$$c^2 \cdot \mathrm{d}\tau^2 = -\mathrm{d}s^2 = -g_{00} \cdot c^2 \cdot \mathrm{d}t^2. \tag{2.29}$$

Eigentime differences are then related to the 00-component of the metric tensor

$$\Delta\tau = \sqrt{-g_{00}} \cdot \Delta t. \tag{2.30}$$

Note, that t is a time coordinate but not the physical time being measured. Only the proper time has a physical meaning.

For weak gravitational fields, the metric has to recover the Newtonian limit. For this we take a look at the gravitational redshift discussed in connection with the equivalence principle. Recall that there is a light beam emitted from the bottom of a rocket to the top of it within a gravitational field characterized by the gravitational potential ϕ. The observed frequency ν_d will be shifted compared to the original frequency of the emitter ν_e by the difference in the gravitational potential, see Eq. (2.10):

$$\frac{\nu_d}{\nu_e} \approx 1 - \frac{\Delta\phi}{c^2}. \tag{2.31}$$

The frequency of the wave is the inverse of the time difference between two consecutive crests of the wave. Hence, the ratio of the frequencies is given by the inverse proper time difference at the location of the emitting source and the detector

$$\frac{\nu_d}{\nu_e} = \frac{\Delta\tau_e}{\Delta\tau_d} = \frac{\left(-g_{00}^{(e)}\right)^{1/2}}{\left(-g_{00}^{(d)}\right)^{1/2}}, \tag{2.32}$$

where we have chosen to have the same time coordinates within the rocket. Hence, the relative change in the frequencies is solely determined by the square root of the 00-component of the metric tensor. Comparing Eq. (2.31) with Eq. (2.32), it is apparent that for weak gravitational fields

$$g_{00} = -(1+\phi)^2 \approx -(1+2\phi)\,, \tag{2.33}$$

which gives correctly

$$\frac{\nu_d}{\nu_e} = \frac{1+\phi_e/c^2}{1+\phi_d/c^2} \approx 1 + \frac{\phi_e - \phi_d}{c^2} = 1 - \frac{\Delta\phi}{c^2}. \tag{2.34}$$

As general relativity has to recover the correct limit for weak gravitational fields, Eq. (2.33) can be used to fix undetermined constants of the full metric tensor.

2.3 Einstein's Equation

Here we give heuristic arguments for the form of the Einstein equations. From here on we will use natural units by setting formally $c = 1$. Newton's gravitational law can be written in a form to describe a distribution of mass characterized by a mass density distribution $\rho(r)$. The gravitational potential is given by the Laplace equation:

$$\nabla^2\phi(r) = 4\pi G \cdot \rho(r), \tag{2.35}$$

where G is the gravitational constant. From the equivalence principle it is known that space tells matter how to move. The characteristic quantity to describe spacetime is the metric of the spacetime $g_{\mu\nu}$, which has to replace the gravitational potential ϕ. The differential equation of general relativity has to involve then the metric tensor and its derivatives up to second order on the left-hand side.

In a general curved spacetime, the partial derivative has to be replaced by the covariant derivative, which takes into account the curvature of spacetime. The covariant derivative of a vector is defined as

$$\nabla_\mu V^\nu = \partial_\mu V^\nu + \Gamma^\nu_{\mu\lambda} V^\lambda, \tag{2.36}$$

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where $\Gamma^{\lambda}_{\mu\nu}$ stands for the Christoffel symbols that are a measure of the curvature of spacetime. Here ∂_μ stands for the shorthand notation

$$\partial_\mu = \frac{\partial}{\partial x^\mu}. \tag{2.37}$$

In the following, we will also denote the partial derivative with respect to some index with a comma followed by the index:

$$\partial_\mu V^\nu = V^\nu{}_{,\mu} \tag{2.38}$$

and for the covariant derivative we use a semicolon instead:

$$\nabla_\mu V^\nu = V^\nu{}_{;\mu} = V^\nu{}_{,\mu} + \Gamma^{\nu}_{\mu\lambda} V^\lambda. \tag{2.39}$$

For a product of vectors, we demand the Leibniz rule to hold:

$$\nabla_\kappa \left(V^\mu \cdot W^\nu\right) = \left(\nabla_\kappa V^\mu\right) \cdot W^\nu + V^\mu \cdot \left(\nabla_\kappa W^\nu\right). \tag{2.40}$$

The Christoffel symbols should be given in terms of the metric and its first derivatives. They can be fixed by demanding that the covariant derivative of the metric vanishes

$$\nabla_\kappa g_{\mu\nu} = 0, \tag{2.41}$$

which is equivalent to saying that the covariant derivative is metric compatible. It ensures that one can lower and raise indices of the argument of a covariant derivative. Hence,

$$\nabla_\kappa g^{\mu\nu} = 0. \tag{2.42}$$

The metric can be considered as a map of a product of two vectors to the real numbers from its definition (Eq. (2.21)). Considering the metric tensor as a product of two vectors, the Leibniz rule, Eq. (2.40), inquires that the covariant derivative has to act on both indices, so that it involves two Christoffel symbols:

$$\nabla_\kappa g^{\mu\nu} = \partial_\kappa g^{\mu\nu} + \Gamma^{\mu}_{\kappa\lambda} g^{\lambda\nu} + \Gamma^{\nu}_{\kappa\lambda} g^{\mu\lambda}. \tag{2.43}$$

Using the condition of metric compatibility, Eq. (2.41), and the definition of the covariant derivative, Eq. (2.36), one arrives at an expression for the Christoffel symbols:

$$\Gamma^{\lambda}_{\mu\nu} = \frac{1}{2} g^{\lambda\kappa} \left(g_{\kappa\nu,\mu} + g_{\kappa\mu,\nu} - g_{\mu\nu,\kappa}\right), \tag{2.44}$$

where

$$g_{\mu\nu,\kappa} = \frac{\partial g_{\mu\nu}}{\partial x^\kappa} \tag{2.45}$$

are the partial derivative of the metric tensor in the comma notation. We note that the Christoffel symbols are symmetric in the lower two indices (which is attributed as being torsion-free).

The quantity to describe a general curved spacetime should be a tensor that describes the curvature of spacetime. Hence, the tensor shall depend on the derivatives of the metric tensor up to second order. This can be achieved by either using the derivative of the Christoffel symbol or by the Christoffel symbol squared. Let us have a look at the second covariant derivative of a vector. If we look at the antisymmetrized version of the second covariant derivative, the partial derivatives will cancel out. The result shall then be proportional to the vector again

$$\nabla_\mu \nabla_\lambda V^\kappa - \nabla_\lambda \nabla_\mu V^\kappa = R^\kappa{}_{\mu\lambda\nu} V^\nu, \tag{2.46}$$

where we introduced a new tensor of rank four $R^\kappa{}_{\mu\nu\lambda}$, which is called the Riemann curvature tensor. It will depend only on the derivative of the Christoffel symbols and the Christoffel symbols squared. However, it has more independent components than we need to fix the ten components of the metric tensor $g_{\mu\nu}$. In fact, the Riemann curvature tensor has 20 independent components. We can transform the Riemann curvature tensor from a rank four tensor to a tensor of rank two by summing over one upper and one lower index. This is called a contraction of a tensor. The contraction of the Riemann curvature tensor gives

$$R_{\mu\nu} = R^\lambda{}_{\mu\lambda\nu} \tag{2.47}$$

and the resulting tensor is called the Ricci tensor. The antisymmetrized second covariant derivative of a vector involving the Ricci tensor is given by contracting the corresponding lower and upper index of Eq. (2.46)

$$\nabla_\mu \nabla_\lambda V^\lambda - \nabla_\lambda \nabla_\mu V^\lambda = R_{\mu\nu} V^\nu. \tag{2.48}$$

The Ricci tensor is symmetric in its indices and given by the derivative of the Christoffel symbols and the Christoffel symbol squared. Indeed, it has the right properties describing the curvature of spacetime. Using the definition of the covariant derivative, Eq. (2.36), one finds the expression of the Ricci tensor in terms of the Christoffel symbols and its derivative as

$$R_{\mu\nu} = \Gamma^\alpha_{\mu\alpha,\nu} - \Gamma^\alpha_{\mu\nu,\alpha} - \Gamma^\alpha_{\mu\nu}\Gamma^\beta_{\alpha\beta} + \Gamma^\alpha_{\mu\beta}\Gamma^\beta_{\nu\alpha}. \tag{2.49}$$

The comma in the subscript stands again for the partial derivative with respect to the index following the comma

$$\Gamma^\alpha_{\mu\nu,\kappa} = \frac{\partial}{\partial x^\kappa}\Gamma^\alpha_{\mu\nu}. \tag{2.50}$$

In vacuum, the Einstein equations read then simply

$$R_{\mu\nu} = 0. \tag{2.51}$$

These are 10 coupled nonlinear differential equations as the Ricci tensor is symmetric in its lower indices. Due to the nonlinear character of the differential equation (2.51), there are nontrivial solutions to the Einstein equations even in vacuum. The most famous one is the Schwarzschild metric, which will be discussed in the following section.

2.4 The Schwarzschild Metric

Shortly after Einstein has written down the differential equations of general relativity (Einstein, 1916a), Schwarzschild found an analytic solution for the Einstein equations in vacuum in 1916, the famous Schwarzschild metric (Schwarzschild, 1916b). As already discussed, the Einstein equations in vacuum read

$$R_{\mu\nu} = 0. \tag{2.52}$$

The starting point to solve the Einstein equation in vacuum, Eq. (2.52), is the ansatz for the metric, which is chosen to be static and spherical symmetric. Extending the metric for spherical coordinates to the general case, there are in principle three different metric functions possible:

$$\mathrm{d}s^2 = -A(r) \cdot \mathrm{d}t^2 + B(r) \cdot \mathrm{d}r^2 + C(r) \cdot r^2 \mathrm{d}\Omega^2. \tag{2.53}$$

We have the freedom to choose the coordinates, the physics must be invariant under coordinate transformations. The angular coordinates are fixed by assuming spherical symmetry. One can define a new radial coordinate by absorbing the function $C(r)$ via

$$\bar{r} = C(r) \cdot r \tag{2.54}$$

so that

$$\mathrm{d}s^2 = -A(\bar{r}) \cdot \mathrm{d}t^2 + \bar{B}(\bar{r}) \cdot \mathrm{d}\bar{r}^2 + \bar{r}^2 \mathrm{d}\Omega^2. \tag{2.55}$$

The radial coordinate can be relabeled back to $\bar{r} \to r$. Hence, there are only two metric functions left that need to be determined from the Einstein equations. For later convenience, we write those metric functions in exponential form:

$$\mathrm{d}s^2 = -e^{2\alpha(r)}\mathrm{d}t^2 + e^{2\beta(r)}\mathrm{d}r^2 + r^2 \cdot \left(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2\right). \tag{2.56}$$

The components of the metric are then given by

$$g_{00} = -e^{2\alpha(r)} \tag{2.57}$$
$$g_{11} = e^{2\beta(r)} \tag{2.58}$$
$$g_{22} = r^2 \tag{2.59}$$
$$g_{33} = r^2 \sin^2\theta \tag{2.60}$$

and all other components are zero. The components of the inverse metric are just given by inverse of the components, as there are only diagonal components:

$$g^{00} = -e^{-2\alpha(r)} \tag{2.61}$$
$$g^{11} = e^{-2\beta(r)} \tag{2.62}$$
$$g^{22} = r^{-2} \tag{2.63}$$
$$g^{33} = \left(r^2 \sin^2\theta\right)^{-1} \tag{2.64}$$

and all other components are zero. Here, we need to demand that $r \neq 0$. The procedure is now to calculate the components of the Ricci tensor from the Christoffel symbols. The Christoffel symbols as defined in Eq. (2.44) have the following nonzero components

$$\begin{aligned}
&\Gamma^1_{00} = \alpha' \cdot e^{2(\alpha-\beta)} && \Gamma^0_{10} = \alpha' \\
&\Gamma^1_{11} = \beta' && \Gamma^2_{12} = \Gamma^3_{13} = r^{-1} \\
&\Gamma^1_{22} = -r \cdot e^{-2\beta} && \Gamma^3_{23} = \cot\theta \\
&\Gamma^1_{33} = -r \cdot \sin^2\theta \cdot e^{-2\beta} && \Gamma^2_{33} = -\sin\theta\cos\theta.
\end{aligned} \tag{2.65}$$

The prime indicates the partial derivative with respect to r:

$$\alpha' = \frac{\partial\alpha}{\partial r} \qquad \beta' = \frac{\partial\beta}{\partial r}. \tag{2.66}$$

Note that the Christoffel symbols are symmetric in the lower indices, giving other nonzero components that we have not written down explicitly here. Now one can compute the components of the Ricci tensor. Only the diagonal components are nonzero:

$$R_{00} = e^{2(\alpha-\beta)} \cdot \left(-\alpha'' + \alpha' \cdot \beta' - \alpha'^2 - \frac{2}{r} \cdot \alpha'\right) \tag{2.67}$$
$$R_{11} = \alpha'' - \alpha' \cdot \beta' + \alpha'^2 - \frac{2}{r} \cdot \beta' \tag{2.68}$$
$$R_{22} = e^{-2\beta} \cdot \left(1 + r \cdot \alpha' - r \cdot \beta'\right) - 1 \tag{2.69}$$
$$R_{33} = \sin^2\theta \cdot R_{22}, \tag{2.70}$$

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where double primes indicate the second derivative with respect to r:

$$\alpha'' = \frac{\partial^2 \alpha}{\partial r^2}. \tag{2.71}$$

In vacuum, all components are vanishing according to the Einstein equations (Eq. (2.51)). Inspection of Eqs. (2.67) and (2.68) shows that similar terms are present in those equations. One can take a look at the combination

$$e^{2(\alpha+\beta)} \cdot R_{00} + R_{11} = \frac{2}{r}\left(\alpha' + \beta'\right) = 0, \tag{2.72}$$

which leads to

$$\alpha' + \beta' = 0. \tag{2.73}$$

So up to a constant and a sign the functions $\alpha(r)$ and $\beta(r)$ are equal. The constant can be absorbed either by redefining the time coordinate or by setting it to zero by demanding that for $r \to \infty$ one has to recover flat spacetime. In any case, we can set

$$\alpha(r) = -\beta(r). \tag{2.74}$$

Let us look now at the Einstein equation for the 22-component of the Ricci tensor (Eq. (2.69)):

$$\begin{aligned} R_{22} &= e^{-2\beta} \cdot \left(1 + r \cdot \alpha' - r \cdot \beta'\right) - 1 \\ &= e^{2\alpha} \cdot \left(1 + 2r \cdot \alpha'\right) - 1 \\ &= \frac{\mathrm{d}}{\mathrm{d}r}\left(r \cdot e^{2\alpha}\right) - 1 = 0, \end{aligned} \tag{2.75}$$

where we used Eq. (2.73). The differential equation can be readily solved to lead to the solution

$$e^{2\alpha} = 1 - \frac{R_s}{r} \tag{2.76}$$

with an integration constant R_s, which is called the Schwarzschild radius. Finally, the metric for a static, spherical symmetric spacetime reads

$$\mathrm{d}s^2 = -\left(1 - \frac{R_s}{r}\right)\mathrm{d}t^2 + \left(1 - \frac{R_s}{r}\right)^{-1}\mathrm{d}r^2 + r^2 \cdot \mathrm{d}\Omega^2 \tag{2.77}$$

and is called the Schwarzschild metric. One can convince oneself that the Schwarzschild metric is also a solution for the other components of the Einstein equation in vacuum individually ($R_{00} = 0$ and $R_{11} = 0$) not only of the combination used in Eq. (2.72).

The integration constant R_s can be fixed by looking at the weak field limit.

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For weak gravitational fields, the 00-component of the metric is given by the Newtonian gravitational potential ϕ:

$$g_{00} = -(1 + 2\phi) = -\left(1 - \frac{2GM}{r}\right). \tag{2.78}$$

One sees that one can interpret the Schwarzschild radius as

$$R_s = 2GM. \tag{2.79}$$

Note that Eq. (2.79) is the definition of the quantity M, which can be interpreted as a conventional Newtonian mass for an observer at asymptotically flat spacetime. We recall from the equivalence principle that the concept of a mass is absent in general relativity. In the limit of vanishing mass M, one recovers flat spacetime. In the limit $r \to \infty$, one recovers asymptotically flat spacetime. The Schwarzschild metric has two singularities. The one for $r = 0$ is a true singularity. The one for $r = R_s$, however, is a coordinate dependent singularity. One can find different coordinates, where this singularity is absent. However, $r = R_s$ marks a special place in spacetime. As one can read off from the Schwarzschild metric, for both radii below and above $r = R_s$, the timelike and the spacelike components of the metric change signs. So the Schwarzschild time coordinate will be the new radial coordinate inside the event horizon and vice versa. This condition marks the event horizon as the point of no return for anything going to $r < R_s$. A timelike trajectory will turn beyond the event horizon to one which inadvertently has to go to the singularity at $r \to 0$. The coordinate r switches to the timelike coordinate within the event horizon and, as time just goes on and on, the radial coordinate r will go to zero, inadvertedly reaching the singularity at $r = 0$.

2.5 Energy-Momentum Tensor

In the following, we discuss the properties of matter in a four-dimensional spacetime. From special relativity, one knows that a massive particle moves along a timelike path through spacetime, as every particle can not move faster than the speed of light. A massless particle, as light, moves along lightlike (null) curves. This path is called a worldline. A worldline can be considered as a map of real numbers λ onto the curved spacetime: the worldline is a parameterized curve given by $x^\mu(\lambda)$. For massive particles, one can use the proper time as parameter. We define the four-velocity as a tangent vector of the wordline

$$u^\mu = \frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}. \tag{2.80}$$

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The proper time τ is defined by

$$d\tau^2 = -g_{\mu\nu} \cdot dx^\mu dx^\nu. \tag{2.81}$$

Division by the proper time difference squared results in

$$g_{\mu\nu} \cdot u^\mu u^\nu = -1. \tag{2.82}$$

This is the definition of u^μ being a timelike vector. For a null vector, the right-hand side would be zero, for a spacelike vector, the right-hand side would be $+1$. In the rest frame of the particle, only the time component is nonvanishing. Hence, the components of the four-velocity in the rest frame are given by

$$u^\mu = \left((-g_{00})^{-1/2}, 0, 0, 0\right), \tag{2.83}$$

where we used Eq. (2.82). The corresponding one-form for the four-velocity is then

$$u_\mu = \left(-(-g_{00})^{1/2}, 0, 0, 0\right). \tag{2.84}$$

For flat spacetime, the expressions are particular simple. Replacing $g_{\mu\nu}$ with $\eta_{\mu\nu}$, Eq. (2.82) reads

$$\eta_{\mu\nu} \cdot u^\mu u^\nu = -1 \tag{2.85}$$

and the four-velocity is given by

$$u^\mu = (1, 0, 0, 0) \tag{2.86}$$

as $\eta_{00} = -1$ in flat space. The corresponding components of the one-form are

$$u_\mu = (-1, 0, 0, 0). \tag{2.87}$$

A general four-velocity in flat spacetime can be written as

$$u^\mu = \gamma\,(1, \boldsymbol{v}), \tag{2.88}$$

where $\boldsymbol{v}$ is the three-dimensional velocity. The components of the one-form are

$$u_\mu = \gamma\,(-1, \boldsymbol{v}). \tag{2.89}$$

The prefactor γ can be determined from the requirement that the four-velocity is a timelike vector (Eq. (2.85)):

$$\gamma = \frac{1}{\sqrt{1 - v^2}}, \tag{2.90}$$

where $v = |\boldsymbol{v}|$. One recovers the standard gamma-factor of special relativity for a relativistic boost.

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We define now the four-momentum by

$$p^\mu = m \cdot u^\mu = (E, \boldsymbol{p}), \tag{2.91}$$

where m is the rest mass of the particle. The energy of the particle E appears as the 0-component of the four-momentum, the three-momentum $\boldsymbol{p}$ as the spatial components of the four-momentum. Using Eq. (2.82), the norm of the four-momentum squared is

$$g_{\mu\nu} \cdot p^\mu p^\nu = p_\mu \cdot p^\mu = -m^2, \tag{2.92}$$

which is a scalar quantity and thereby coordinate independent. For a flat spacetime, one recovers the relativistic energy-momentum equation from Eq. (2.92) when plugging in the components of the four-momentum (Eq. (2.91))

$$E = \sqrt{p^2 + m^2}, \tag{2.93}$$

where $p = |\boldsymbol{p}|$. One can make also a connection to the components of a boosted four-velocity of Eq. (2.88). The boost factor γ and the three-velocity v in terms of the components of the four-momentum are given by

$$\gamma = \frac{E}{m} \qquad \boldsymbol{v} = \frac{\boldsymbol{p}}{E}, \tag{2.94}$$

which can be derived from the definition of the four-momentum equation (2.91) and the properties of the four-velocity.

For many particles we want to describe the system as a fluid with macroscopic quantities, which are the energy density ε, the pressure P, and the flow of the bulk matter u^μ. We stress that thermodynamic quantities, such as the energy density and the pressure, are only well defined in the rest frame of the bulk matter. However, to couple matter to gravity, we need to extend the description of the properties of matter to spacetime quantities, that is, to vectors and tensors. In addition to the bulk properties, we need to define the flux of energy and momentum in a certain direction. In four dimensions, this corresponds to define the flux of the four-momentum p^μ along a four-dimensional direction. The spacetime quantity to describe matter must therefore be a product of the four-momentum and the four-velocity, that is, a tensor. Indeed, this is the energy-momentum tensor $T^{\mu\nu}$ that in symbolic notation we can write as

$$T^{\mu\nu} \propto p^\mu \cdot p^\nu, \tag{2.95}$$

where we replaced the four-velocity with the four-momentum to arrive at the correct dimensionality of the energy-momentum tensor. For a distribution of particles, we need to sum up over all four-momenta. The integral needs to be weighted by a function $f(p)$, which describes the distribution of the energy and momentum of

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the particles. The distribution function $f(p)$ shall be a scalar quantity. Energy and momentum of the particles are connected by the energy-momentum relation (Eq. (2.93)) for locally flat spacetimes, which can be ensured by a δ-distribution. The energy-momentum tensor can be defined then by the following integral over the flat four-dimensional space of four-momentum:

$$T^{\mu\nu} = \frac{g}{(2\pi)^4} \int d^4p \; p^\mu p^\nu \cdot f(p) \cdot \delta\left(p^0 - \sqrt{p^2 + m^2}\right) \cdot \theta(p^0), \tag{2.96}$$

where g stands for the degeneracy factor of the particles. The θ-function ensures that the energy of the particles has to be positive. Integration over p^0 with $p^0 = E(p)$ results in

$$T^{\mu\nu} = \frac{g}{(2\pi)^3} \int d^3p \; \frac{p^\mu p^\nu}{E(p)} \cdot f(p), \tag{2.97}$$

which is still a well-defined tensorial quantity. Let us look at the diagonal components of the energy-momentum tensor, assuming an isotropic distribution of the three-momentum p. The 00-component is

$$T^{00} = \frac{g}{(2\pi)^3} \int d^3p \; \frac{p^0 p^0}{E} \cdot f(p) \tag{2.98}$$

$$= \frac{g}{(2\pi)^3} \int d^3p \; E(p) \cdot f(p). \tag{2.99}$$

In fact, it corresponds to the integral of all single particle energies $E(p)$. We can interpret the 00-component of the energy-momentum tensor as the energy density ε. The diagonal ii-components are equal for an isotropic momentum distribution and read

$$T^{11} = T^{22} = T^{33} \tag{2.100}$$

$$= \frac{g}{(2\pi)^3} \int d^3p \; \frac{p^i p^i}{E(p)} \cdot f(p) \tag{2.101}$$

$$= \frac{g}{(2\pi)^3} \int d^3p \; \frac{p^2}{E(p)} \cdot f(p), \tag{2.102}$$

as $p^1 = p^2 = p^3 = p$. Note that the repeated spatial indices are not meant to be summed up. The ii-components of the energy-momentum tensor can be interpreted as a pressure P, as we will show now. In fact, the integral as defined in Eq. (2.102) is known as the pressure integral. The first part of the integrand can be interpreted as the product of the three-momentum times the relativistic three-velocity of the particle v:

$$\frac{p^2}{E(p)} = p \cdot \frac{p}{E(p)} = p \cdot v, \tag{2.103}$$

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see Eq. (2.94). In Newtonian physics, the pressure is the force per unit area, that is, the change of momentum per unit area. Imagine a unit area that is hit by particles with a momentum p per unit time resulting in a force on that unit area. The number of particles hitting the unit area per unit time are within a volume given by the unit area times the path length the particle can travel within a unit time. The path length per unit time is nothing else than the velocity of the particles. Hence, the pressure is proportional to the momentum times the velocity of the particle summed over all particles. This is exactly what Eq. (2.102) is expressing.

We construct now the energy-momentum tensor by looking at bulk matter at rest. Then the 00-component is the energy density ε ('flux of energy along time'). The mixed $0i$-components T^{0i} are describing the three-dimensional flow of energy, which we will set to zero in the rest frame of the bulk matter. The diagonal spatial components must be the pressure along the three different spatial directions of the flow ('flux of three-momentum along space'). The remaining off-diagonal components of T^{ij} with $i \neq j$ describe a flux of momentum with a three-dimensional flow in a perpendicular direction, that is, they stand for shear forces of matter. In summary, we have the following components of the energy-momentum tensor

$$
\begin{array}{ll}
T^{00} = \varepsilon & \text{energy density} \\
T^{0i} = T^{i0} & \text{flux of energy density} \\
T^{ii} = P^{i} & \text{pressure in direction } i \text{ (no summation)} \\
T^{ij} = T^{ji} & \text{off-diagonal } (i \neq j) \text{ momentum flux (shear).}
\end{array}
\tag{2.104}
$$

We consider in the following a so-called ideal fluid, which has no shear forces and no heat flux, so we set all off-diagonal elements to zero. Also, matter is considered to be isotropic so that the spatial diagonal elements in the rest frame are equal. This is the equivalent of having equal pressure in all three-dimensional directions. The energy-momentum tensor for an ideal fluid in the rest frame as constructed in this way is then given by

$$
T^{\mu\nu}_{\text{rest}} = \begin{pmatrix} \varepsilon & 0 & 0 & 0 \\ 0 & P & 0 & 0 \\ 0 & 0 & P & 0 \\ 0 & 0 & 0 & P \end{pmatrix}. \tag{2.105}
$$

We need now to cast the expression of the energy-momentum at rest in a covariant form. Let us start with flat space. The covariant tools at hand are, besides the four-velocity of bulk matter u^μ, just the metric tensor $\eta_{\mu\nu}$. A combination of these covariant quantities is

$$
T^{\mu\nu} = (\varepsilon + P) \cdot u^\mu u^\nu + P \cdot \eta^{\mu\nu}, \tag{2.106}
$$

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which gives indeed in the rest frame of the bulk matter the correct answer as $u^\mu = \delta_0^\mu$ and

$$(\varepsilon + P)\delta_0^\mu \delta_0^\nu + P\eta^{\mu\nu} = \begin{pmatrix} \varepsilon + P & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} -P & 0 & 0 & 0 \\ 0 & P & 0 & 0 \\ 0 & 0 & P & 0 \\ 0 & 0 & 0 & P \end{pmatrix}.$$

The expression of the energy-momentum tensor in Eq. (2.106) is in a covariant form. Hence, it is valid in any reference frame and constitutes the correct covariant expression for the energy-momentum tensor of an ideal fluid in special relativity. Energy-momentum conservation is ensured by the vanishing derivative of the energy-momentum tensor:

$$\partial_\mu T^{\mu\nu} = 0 \tag{2.107}$$

For a boosted four-velocity, the 0-component of Eq. (2.107) gives

$$\partial_t \varepsilon + \nabla(\varepsilon \cdot \boldsymbol{v}) = 0, \tag{2.108}$$

which is the continuity equation. The spatial components of Eq. (2.107) give

$$\varepsilon \cdot [\partial_t \boldsymbol{v} + (\boldsymbol{v} \cdot \nabla)\boldsymbol{v}] = -\nabla P, \tag{2.109}$$

which is the Euler equation of hydrodynamics. In fact, Eq. (2.107) is the basic equation defining relativistic hydrodynamics.

The extension of relativistic hydrodynamics to hydrodynamics in curved spacetime is now straight forward. The general recipe is

(1) write down a covariant expression using special relativity
(2) replace the metric of flat spacetime $\eta_{\mu\nu}$ with the general one $g_{\mu\nu}$
(3) replace all partial derivatives ∂_μ with the covariant derivative ∇_μ.

Let us apply this recipe to the energy-momentum tensor. The first step is done with Eq. (2.106). The second step results in

$$T^{\mu\nu} = (\varepsilon + P) \cdot u^\mu u^\nu + P \cdot g^{\mu\nu} \tag{2.110}$$

and the third step in the energy-momentum conservation

$$\nabla_\mu T^{\mu\nu} = 0. \tag{2.111}$$

Indeed, Eqs. (2.110) and (2.111) are the correct expressions for the energy-momentum tensor and for energy-momentum conservation in a general curved spacetime.

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2.6 The Full Einstein Equation with Matter

For including matter to the Einstein equations, we know from the equivalence principle that gravity couples to all forms of energy. The characteristic quantity to describe the energy of matter is the symmetric energy-momentum tensor $T^{\mu\nu}$, which replaces the mass density distribution ρ on the right-hand side of the full equations of general relativity that we are looking for. In addition, energy-momentum conservation is ensured in flat space by the condition

$$\partial_\mu T^{\mu\nu} = T^{\mu\nu}{}_{,\mu} = 0. \tag{2.112}$$

For curved spacetime, we have to use the covariant derivative. In Eq. (2.36), the covariant derivative of vector was written as

$$\nabla_\mu V^\nu = \partial_\mu V^\nu + \Gamma^\nu_{\mu\lambda} V^\lambda. \tag{2.113}$$

A tensor can be considered as a product of two vectors. Actually, this is how we defined the energy-momentum tensor in Eq. (2.96), as a product of two four-momenta. Hence, we need to take the covariant derivative with respect to both vectors. This implies that for a tensor one has to repeat the formula for the covariant derivative for each index of the tensor. Then, the energy-momentum conservation in a curved spacetime reads

$$\nabla_\mu T^{\mu\nu} = \partial_\mu T^{\mu\nu} + \Gamma^\mu_{\mu\lambda} T^{\lambda\nu} + \Gamma^\nu_{\mu\lambda} T^{\mu\lambda}. \tag{2.114}$$

We can denote the covariant derivative with a colon instead of a comma

$$T^{\mu\nu}{}_{;\mu} = \nabla_\mu T^{\mu\nu}. \tag{2.115}$$

However, the covariant derivative of the Ricci tensor does not vanish as

$$\nabla_\mu R^{\mu\nu} = \frac{1}{2}\nabla_\mu \left(g^{\mu\nu} R\right) = \frac{1}{2} g^{\mu\nu} \nabla_\mu R \tag{2.116}$$

with the Ricci scalar or curvature scalar R defined as

$$R = g_{\mu\nu} R^{\mu\nu} = R^\mu{}_\mu. \tag{2.117}$$

The second equality of Eq. (2.116) follows from the property of the covariant derivative that the covariant derivative of the metric vanishes.

The relation equation (2.116) is also called the twice-contracted Bianchi identity. It is clear from Eq. (2.116) that a combination of the Ricci tensor and the curvature scalar R with the metric tensor will do the job. The resulting tensor describing the curvature of spacetime with a vanishing derivative is the Einstein tensor

$$G^{\mu\nu} = R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R \tag{2.118}$$

for which indeed

$$\nabla_\mu G^{\mu\nu} = \nabla_\mu \left(R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R \right) = 0. \tag{2.119}$$

The measure of spacetime curvature on the left-hand side of the differential equation we are looking for should be the Einstein tensor. The measure of the energy content of matter is the energy-momentum tensor on the right-hand side of the differential equation. The constant of proportionality has to be the gravitational constant. The Einstein equations are then given by

$$G^{\mu\nu} = 8\pi G \cdot T^{\mu\nu}. \tag{2.120}$$

where $G_{\mu\nu}$ is the Einstein tensor, which depends on the metric and its derivatives up to second order, and $T^{\mu\nu}$ is the energy-momentum tensor that describes the coupling to matter. The factor 8π instead of the factor 4π comes from the condition to arrive at the correct Newtonian limit, that is, the Newtonian law for weak gravitational fields.

We close by showing that the full Einstein equations in vacuum

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 0 \tag{2.121}$$

can be cast in the form

$$R_{\mu\nu} = 0 \tag{2.122}$$

used to derive the Schwarzschild metric. One just needs to show that the curvature scalar R vanishes in vacuum. This can be seen immediately by contracting Eq. (2.121) with the metric

$$\begin{aligned} g^{\mu\nu} G_{\mu\nu} &= g^{\mu\nu} R_{\mu\nu} - \frac{1}{2} g^{\mu\nu} g_{\mu\nu} R \\ &= R - \frac{1}{2} \delta^\mu_\mu R = R - 2R = -R = 0, \end{aligned} \tag{2.123}$$

where we used Eq. (2.26) for the inverse metric. Note that the indices of the Kronecker delta needs to be summed according to the summation convention.

2.7 Tolman–Oppenheimer–Volkoff Equation

We consider now static spherically symmetric objects (a star) consisting of matter. The full Einstein equations need to be solved now:

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G \cdot T_{\mu\nu}. \tag{2.124}$$

The star shall be a sphere with radius R, which has to be larger than the Schwarzschild radius $R_s = 2GM$. The solution outside the star for $r > R$ has no matter and has to be the solution in vacuum. The Birkhoff theorem states that the outside solution has to be the Schwarzschild metric.

Birkhoff's theorem: Any spherically symmetric solution of the Einstein equations in vacuum must be static and asymptotically flat and is given by the Schwarzschild metric.

So the interior solution of the full Einstein equation has to be matched to the Schwarzschild metric as the exterior solution at the radius R of the star. For the interior solution, we are looking for static, spherically symmetric solutions of the full Einstein equation. For the metric, we choose the ansatz

$$\mathrm{d}s^2 = -e^{2\alpha(r)} \cdot \mathrm{d}t^2 + e^{2\beta(r)} \cdot \mathrm{d}r^2 + r^2 \cdot \mathrm{d}\Omega^2 \tag{2.125}$$

according to our discussion for the static, spherical symmetric solution of the vacuum Einstein equation. For the full Einstein equations, we need to determine now the components of the Einstein tensor $G_{\mu\nu}$. The Ricci scalar R can be calculated from the components of the Ricci tensor, Eqs. (2.67)–(2.70):

$$R = 2e^{-2\beta}\left[-\alpha'' - (\alpha')^2 + \alpha'\beta' + \frac{2}{r}\left(\beta' - \alpha'\right) - \frac{1}{r^2}\right] + \frac{2}{r^2}, \tag{2.126}$$

where primes indicate the derivative with respect to r. The components of the Einstein tensor are then given by

$$G_{00} = \frac{1}{r^2} e^{2(\alpha-\beta)} \cdot \left(2r\beta' - 1 + e^{2\beta}\right) \tag{2.127}$$

$$G_{11} = \frac{1}{r^2} \cdot \left(2r\alpha' + 1 - e^{2\beta}\right) \tag{2.128}$$

$$G_{22} = r^2 e^{-2\beta} \cdot \left[\alpha'' + (\alpha')^2 - \alpha'\beta' + \frac{1}{r}\left(\alpha' - \beta'\right)\right] \tag{2.129}$$

$$G_{33} = \sin^2\theta \cdot G_{22}. \tag{2.130}$$

For the right-hand side of the Einstein equation, we need to specify now the energy-momentum tensor $T_{\mu\nu}$. We model the matter inside the star as a perfect fluid:

$$T_{\mu\nu} = (\varepsilon + P)\, u_\mu u_\nu + P \cdot g_{\mu\nu} \tag{2.131}$$

and fix the four-vector to be $u^\mu = (1, 0, 0, 0)$, that is, at rest with respect to the matter. Note that the energy-momentum tensor is written down with lower indices

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in Eq. (2.131), so that there appears the metric coefficients squared for the explicit expression of its components:

$$T_{\mu\nu} = \begin{pmatrix} e^{2\alpha} \cdot \varepsilon & 0 & 0 & 0 \\ 0 & e^{2\beta} \cdot P & 0 & 0 \\ 0 & 0 & r^2 \cdot P & 0 \\ 0 & 0 & 0 & r^2 \sin^2\theta \cdot P \end{pmatrix}. \tag{2.132}$$

Putting in the expressions for the Einstein tensor and the energy-momentum tensor into the full Einstein equations, three independent differential equations appear:

$$\frac{1}{r^2} e^{-2\beta} \cdot \left(2r\beta' - 1 + e^{2\beta}\right) = 8\pi G \cdot \varepsilon \tag{2.133}$$

$$\frac{1}{r^2} e^{-2\beta} \cdot \left(2r\alpha' + 1 - e^{2\beta}\right) = 8\pi G \cdot P \tag{2.134}$$

$$e^{-2\beta} \cdot \left[\alpha'' + (\alpha')^2 - \alpha'\beta' + \frac{1}{r}\left(\alpha' - \beta'\right)\right] = 8\pi G \cdot P. \tag{2.135}$$

We can rewrite the metric function $\beta(r)$ in such a form that we can easily recover the exterior Schwarzschild metric by introducing a mass function $m_r(r)$ via

$$e^{2\beta(r)} = \left[1 - \frac{2Gm_r(r)}{r}\right]^{-1}. \tag{2.136}$$

We will denote the expression on the right-hand side also as the Schwarzschild factor. At the radius of the star $r = R$, the metric component equals the one of the Schwarzschild metric with the total mass of the star $m_r(r = R) = M$. Replacing $\beta(r)$ in Eq. (2.133) results in the differential equation

$$\frac{\mathrm{d}m_r(r)}{\mathrm{d}r} = 4\pi r^2 \varepsilon(r) \tag{2.137}$$

or in integral form

$$m_r(r) = 4\pi \int_0^r \mathrm{d}r'\, r'^2 \cdot \varepsilon(r'), \tag{2.138}$$

which can be interpreted as mass conservation. The total mass of the star is then given by

$$M = m_r(r = R) = 4\pi \int_0^R \mathrm{d}r'\, r'^2 \cdot \varepsilon(r'). \tag{2.139}$$

Eq. (2.138) looks like an integral over the energy density within the radius r, that is, the mass within a sphere of radius r. However, the expression of the integral on the right-hand side does not involve the proper spatial integration measure. In fact, the integration measure is not an invariant quantity.

For finding the proper volume element for an integral in curved spacetime, let us have a look at the static, spherical symmetric expression for the metric ansatz, Eq. (2.125), and set $\mathrm{d}t = 0$:

$$\mathrm{d}s^2 = g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = e^{2\beta}\mathrm{d}r^2 + r^2\mathrm{d}\theta^2 + r^2\sin^2\theta\mathrm{d}\phi^2. \tag{2.140}$$

Let us take the determinant of this expression that should be an invariant quantity

$$\det(g_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu) = \det(g_{\mu\nu})\cdot\det(\mathrm{d}x^\mu\mathrm{d}x^\nu) \tag{2.141}$$

$$= \left(\sqrt{-g}\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi\right)^2 \tag{2.142}$$

$$= \left(e^\beta\, r^2\sin\theta\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi\right)^2, \tag{2.143}$$

where we introduced the determinant of the metric g:

$$g = \det(g_{\mu\nu}). \tag{2.144}$$

The final expression looks like the familiar integral measure for spherical coordinates squared. However, there appears an additional factor from the metric function $\beta(r)$, which makes the expression a coordinate invariant quantity. Hence, the corresponding integral of the energy density of a star in curved spacetime reads

$$\overline{M} = 4\pi\int_0^R \mathrm{d}r'\, r'^2\, e^{\beta(r)}\cdot\varepsilon(r') = 4\pi\int_0^R \mathrm{d}r'\, r'^2\left(1-\frac{2Gm_r(r')}{r'}\right)^{-1/2}\cdot\varepsilon(r'). \tag{2.145}$$

The difference of the mass of the star M and the integral of the energy density $\overline{M}$ is only given by the square-root of the Schwarzschild factor, which gives a different weight of the integral. Note that always $\overline{M} \geq M$ as the radius is larger than the Schwarzschild radius of matter contained within the sphere of radius r: $r > 2Gm_r(r)$. The difference between M and $\overline{M}$ is due to the gravitational binding energy. Therefore, the mass of the star M is also called the gravitational mass of the star. It is the mass of the star measured by an exterior observer at asymptotically flat spacetime distance.

We continue now to solve the Einstein equation and take a look at the 11-component of the Einstein equation, see Eq. (2.134). We can rewrite that equation by using the expression for the mass function $m_r(r)$:

$$\frac{\mathrm{d}\alpha(r)}{\mathrm{d}r} = \frac{Gm_r(r)}{r^2}\left(1+\frac{4\pi r^3P(r)}{m_r(r)}\right)\left(1-\frac{2Gm_r(r)}{r}\right)^{-1}, \tag{2.146}$$

which fixes the metric function $\alpha(r)$ now in terms of the mass function $m_r(r)$ and the pressure $P(r)$. One could now tackle the last equation, Eq. (2.135), but it turns out to be easier to use the equation of energy-momentum conservation instead

$$\nabla_\mu T^{\mu\nu} = 0. \tag{2.147}$$

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For our purpose, only the $\mu = \nu = 1$ component is relevant, which leads to the differential equation for the pressure

$$\frac{\mathrm{d}P(r)}{\mathrm{d}r} = -\left(\varepsilon(r) + P(r)\right)\frac{\mathrm{d}\alpha(r)}{\mathrm{d}r}. \tag{2.148}$$

The two equations, Eqs. (2.146) and (2.148), can be combined to give

$$\frac{\mathrm{d}P(r)}{\mathrm{d}r} = -\frac{Gm_r(r)\varepsilon(r)}{r^2}\left(1 + \frac{P(r)}{\varepsilon(r)}\right)\left(1 + \frac{4\pi r^3 P(r)}{m_r(r)}\right)\left(1 - \frac{2Gm_r(r)}{r}\right)^{-1}, \tag{2.149}$$

which is called the Tolman–Oppenheimer–Volkoff equation or TOV equation in short. The first term on the right-hand side is the Newtonian term from hydrostatic equilibrium:

$$\frac{dP(r)}{\mathrm{d}r} = -\frac{Gm_r(r)\rho(r)}{r^2}, \tag{2.150}$$

where $\rho(r)$ is the mass density. The three additional terms of the TOV equation are corrections from general relativity. The first correction term modifies the mass density $\rho(r)$ and takes into account that gravity couples to the energy density $\varepsilon(r)$ and the pressure $P(r)$ of matter. The second correction term modifies the mass function $m_r(r)$ and adds another correction term from the pressure of matter. The third correction term modifies the radius and takes into account the warpage of spacetime that is described by the Schwarzschild factor.

The Einstein equations provide only three independent equations, Eqs. (2.137), (2.146), and (2.148). It can be shown that the 22-component of the Einstein equation do not provide an additional constraint once the condition of energy-momentum conservations is used instead. As there are four independent quantities in the equations, the two metric functions $\alpha(r)$ and $\beta(r)$ (or $m_r(r)$), the energy density $\varepsilon(r)$ and the pressure $P(r)$, an additional equation is needed to close the system of equations. The missing equation is the relation between the pressure and energy density of matter

$$P = P(\varepsilon), \tag{2.151}$$

which fixes the properties of matter of which the star is made. This relation is called the equation of state (or EOS). The EOS describes the physics beyond gravity, the quantum physics of matter. We note that the TOV equation is not sensitive to any other detail from the properties of matter than the bulk quantities energy density and pressure. The number density or the composition of matter is not relevant for the properties of static spherical symmetric stars in general relativity. The EOS, Eq. (2.151), together with the TOV equation, Eq. (2.149), and the equation of mass conservation, Eq. (2.137), fixes all quantities $m_r(r)$, $\varepsilon(r)$, and $P(r)$ to determine the bulk properties of compact stars.

Let us close with a brief historical outline of the TOV equation. Tolman wrote down the equations for static spheres of matter in his textbook in 1934 (Tolman, 1934). Motivated by the work of Landau on neutron stars (Landau, 1932), Oppenheimer started to work out the maximum mass of neutron stars with his graduate student Volkoff. Oppenheimer was in close contact with his colleague Tolman at Caltech, so they published back-to-back two papers in 1939 referring to each other. In the first paper, Tolman worked out analytic solutions for static spheres of fluid for various forms of the metric functions that relate to certain forms of EOS (Tolman, 1939). In the second paper, Oppenheimer and Volkoff set up the equation in the form of Eq. (2.149) and solved it numerically for an EOS of a free gas of neutrons (Oppenheimer and Volkoff, 1939). However, neither Tolman nor Oppenheimer and Volkoff were the first to find a solution to the static Einstein equation for a sphere of fluid; that was Schwarzschild in (1916a).

2.8 The Schwarzschild Solution for a Sphere of Fluid

For solving the TOV equation, one needs to define the EOS. Schwarzschild considered a particular simple ansatz for the EOS. He used as an EOS one of an incompressible fluid, that is, one where the energy density is constant throughout the sphere with radius R:

$$\varepsilon(r) = \begin{cases} \varepsilon_*, & r \le R \\ 0, & r > R. \end{cases} \tag{2.152}$$

The integration of the mass function, Eq. (2.138), is easily solved to give

$$m_r(r) = \begin{cases} \frac{4\pi}{3} r^3 \varepsilon_*, & r \le R \\ M, & r > R. \end{cases} \tag{2.153}$$

where M is the total mass of the sphere

$$M = \frac{4\pi}{3} R^3 \varepsilon_*. \tag{2.154}$$

Also, the TOV equation can now be solved analytically to give the pressure as a function of r:

$$P(r) = \frac{R\sqrt{R-2GM} - \sqrt{R^3 - 2GMr^2}}{\sqrt{R^3 - 2GMr^2} - 3R\sqrt{R-2GM}} \cdot \varepsilon_*. \tag{2.155}$$

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The pressure increases continuously toward the center of the sphere $r \to 0$ and the maximum pressure is given by

$$P(0) = \frac{R\sqrt{R-2GM} - \sqrt{R^3}}{\sqrt{R^3} - 3R\sqrt{R-2GM}} \cdot \varepsilon_*. \tag{2.156}$$

The pressure will diverge at the center of the star when the denominator is becoming zero which relates to the following condition for the critical mass and radius of the star:

$$M_c = \frac{4}{9G} R_c \tag{2.157}$$

or in terms of the Schwarzschild radius R_s:

$$R_c = \frac{9}{8} R_s. \tag{2.158}$$

The critical radius is just slightly larger than the radius of a nonrotating black hole. In order to have well-behaved pressure throughout the star, the radius of the star has to be above the critical radius

$$R > \frac{9}{8} R_s, \tag{2.159}$$

which has been noted by Schwarzschild in his original paper (Schwarzschild, 1916a). Buchdahl showed several years later that this limit on the radius of a star holds for any EOS where the energy density is not increasing with the radius (Buchdahl, 1959). Hence, Eq. (2.159) holds for any reasonable EOS and is known as the Buchdahl limit:

Buchdahl limit: All static fluid spheres where the energy density is not increasing outward cannot be more compact than

$$C_{\text{max}} = \frac{4}{9}.$$

The limit can be slightly strengthened by arguing that the pressure cannot exceed the energy density, so that the maximum pressure allowed at the center corresponds to $p_{\text{max}} = \varepsilon_*$. As we will see in the chapter on compact stars, Chapter 4, this represents the causal limit above which the speed of sound in the medium is larger than the speed of light. In this case, the maximum allowed compactness relates to

$$C_{\text{max, causal}} = \frac{3}{8} \tag{2.160}$$

as you are invited to show in the exercises (see also Buchdahl [1959] for a general discussion).

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Exercises

(2.1) Show that the equation for the gravitational shift of the wavelength, Eq. (2.7), holds also for a general gravitational potential with an arbitrary dependence on the distance (we assumed a constant gradient in the gravitational potential in the derivation).

(2.2) Show that the Schwarzschild metric, Eq. (2.77), is a solution of $R_{00} = 0$ and $R_{11} = 0$.

(2.3) Derive the relations for the relativistic boost factors, Eq. (2.94), from the definition of the four-momentum.

(2.4) Derive the continuity equation, Eq. (2.108), and the Euler equation, Eq. (2.109), from the conservation of the energy-momentum tensor, Eq. (2.147), for the nonrelativistic limit. Use the nonrelativistic expression for the four-vector $u^\mu = (1, \boldsymbol{v})$ for velocities much smaller than the velocity of light c. Which terms have to be neglected to result in the continuity and the Euler equations, and can you find arguments for neglecting them?

(2.5) Show that the condition of metric compatibility of the metric tensor, Eq. (2.41), results in the expression of the Christoffel symbols in terms of the partial derivatives of the metric tensor, Eq. (2.44).

(2.6) Derive the differential equation for the pressure in terms of the metric function $\alpha(r)$, Eq. (2.148), from the equation of energy-momentum conservation, Eq. (2.147).

(2.7) For a general metric in spherical symmetry, the metric functions could also depend on the time coordinate. Birkhoff's theorem states that the time dependence drops out in the equation so that the general solution for a spherical symmetry is the Schwarzschild solution. Prove Birkhoff's theorem by starting with the general ansatz for the metric in the form

$$\mathrm{d}s^2 = -A(r,t)\mathrm{d}t^2 + B(r,t)\mathrm{d}r^2 + C(r,t)r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2\right) \quad (2.161)$$

and choose proper coordinates to end up in the form

$$\mathrm{d}s^2 = -e^{2\alpha(r,t)}\mathrm{d}t^2 + e^{2\beta(r,t)}\mathrm{d}r^2 + r^2 \cdot \left(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2\right), \quad (2.162)$$

where the metric functions are assumed to depend on time also. Derive the time-dependent Christoffel symbols and the Ricci tensor (note: there are off-diagonal terms). Show that Einstein's equations result in a metric that is time independent.

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(2.8) Solve the TOV equation for an incompressible fluid of the form

$$\varepsilon(r) = \begin{cases} \varepsilon_*, & r \leq R \\ 0, & r > R, \end{cases} \tag{2.163}$$

where R is the radius of the star. Show that the solution is given by

$$P(r) = \frac{R\sqrt{R - 2GM} - \sqrt{R^3 - 2GMr^2}}{\sqrt{R^3 - 2GMr^2} - 3R\sqrt{R - 2GM}} \cdot \varepsilon_*, \tag{2.164}$$

which is called the Schwarzschild solution.

(2.9) Derive the Buchdahl limit from Eq. (2.156), which states that the maximum compactness of the star is given by

$$C_{\text{max}} = \frac{4}{9}$$

and corresponds to the case of infinite pressure of the Schwarzschild solution.

(2.10) Derive the condition for the maximum compactness of a sphere $C_{\text{max}} = 3/8$ from the Schwarzschild solution, Eq. (2.156), for an incompressible causal fluid where the pressure is not allowed to violate the causal limit $P = \varepsilon$.

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