1 The effective potential

The main ingredient used for the analysis of the orbits in static spherically symmetric spacetimes is the effective potential, from which the orbit equations can be derived. This makes raytracing associated to the photon sphere, black hole shadows, and perihelion possible, and provides insight into the stability of orbits.

The metric of a static spherically symmetric and asymptotically flat spacetime can be written in the form

$$ds^2 = -f(r)dt^2 + g(r)dr^2 + h(r)\left(d\theta^2 + \sin^2(\theta)\, d\phi^2\right) \tag{1}$$

In this spacetime, the general relativistic point particle Lagrangian density becomes

$$\mathcal{L} = g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu = -f(r)\dot{t}^2 + \mathfrak{g}(r)\dot{r}^2 + h(r)\left(\dot{\theta}^2 + \sin^2(\theta)\, \dot{\phi}^2\right), \tag{2}$$

where dot denotes derivative with respect to the proper time τ for timelike geodesics, and the affine parameter λ for null geodesics. Due to spherical symmetry of the metric (in particular conservation of the direction of the angular momentum), we can always fix θ to, e.g, its value on the equatorial plane $\theta(\tau) = \pi/2$. This, after integrating once, reduces the Euler-Lagrange equations and constants of motion associated to the above $\mathcal{L}$, to

$$\dot{t} = \frac{E}{f(r)}, \tag{3}$$

$$\dot{\phi} = \frac{L}{h(r)}, \tag{4}$$

$$\dot{r}^2 = \frac{1}{\mathfrak{g}(r)}\left[\frac{1}{f(r)}E^2 + \left(\sigma - \frac{L^2}{h(r)}\right)\right], \tag{5}$$

where E is the conserved energy of the orbiting object, and L is the magnitude of its conserved angular momentum, L, which are

$$E = -\frac{1}{2}\frac{\partial \mathcal{L}}{\partial \dot{t}}, \tag{6}$$

$$L = \frac{1}{2}\frac{\partial \mathcal{L}}{\partial \dot{\phi}}, \tag{7}$$

The parameter σ, which is the conserved magnitude of the four-velocity (or momentum in case of null geodesics) of the geodesics, is 0 for massless particles and -1 for massive particles. Rearranging Eq. (5) such that the terms E^2 is isolated yields an equation mimicking the equation of motion of a single particle moving in one dimension under an effective potential V_{eff}, namely,

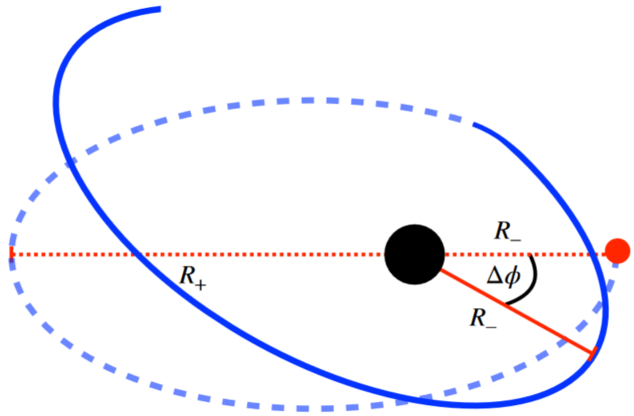
$$f(r)\mathfrak{g}(r)\,\dot{r}^2 - f(r)\left(\sigma - \frac{L^2}{h(r)}\right) = E^2. \tag{8}$$

From this, we can read-off the effective potential as

$$V_{\text{eff}}(r) = f(r)\left(\sigma - \frac{L^2}{h(r)}\right). \tag{9}$$

2 Perihelion shift

To derive the perihelion shift of a massive object on an orbit around a central stationary spherically symmetric gravitating object we consider its trajectory given as function $r(\phi)$.



Exaggerated schematic of the perihelion shift.

It can be determined from the effective potential and the angular momentum as

$$\frac{dr}{d\phi} = \frac{\dot{r}}{\dot{\phi}} = \frac{h(r)}{L}\sqrt{\frac{1}{f(r)g(r)}(E^2 + V_{\text{eff}}(r,L))}\,. \tag{10}$$

where the effective potential of the black hole is

$$V_{\text{eff}}(r) = f(r)\left(\sigma - \frac{L^2}{h(r)}\right) \tag{11}$$

At the perihelion R_- (the point of closest encounter to the central mass, the minimum of $r(\phi)$) and the aphelion R_+ (the point of farthest distance to the central mass, the maximum of $r(\phi)$) the following must hold

$$\frac{dr}{d\phi}(R_-) = 0 \text{ and } \frac{dr}{d\phi}(R_+) = 0\,. \tag{12}$$

The two equations can be solved for the constants of motion E and L as functions of R_+ or R_-. The perihelion shift is then obtained from [S. Weinberg, *Gravitation and Cosmology*, 1972]

$$2\left|\int_{R_-}^{R_+}\frac{d\phi}{dr}\,dr\right| = \int_0^{2\pi+\Delta\phi} d\phi = \Delta\phi + 2\pi \tag{13}$$

After some algebra we obtain

$$\frac{d\phi}{dr} = \frac{L}{h(r)}\frac{1}{\sqrt{\frac{1}{f(r)\mathfrak{g}(r)}(E^2 + V_{\text{eff}}(r,L))}} \tag{14}$$

$$= \frac{\frac{1}{r^2}\frac{1}{\sqrt{f(r)\mathfrak{g}(r)}}\,\sqrt{f(R_-) - f(R_+)}R_-R_+}{\sqrt{f(R_-)f(R_+)(R_-^2 - R_+^2) + \frac{f(r)}{r^2}\left(f(R_-)R_+^2(r^2 - R_-^2) + f(R_+)R_-^2(-r^2 + R_+^2)\right)}}\,. \tag{15}$$

In general the integral to obtain the perihelion shift is difficult to evaluate. To investigate the influence of the LQG modification as good as possible analytically, we assume the LQG parameters and the Schwarzschild radius r_s, hence the mass $r_s = 2GM$, are small against all other appearing scales and evaluate the integral to leading order in these parameters.

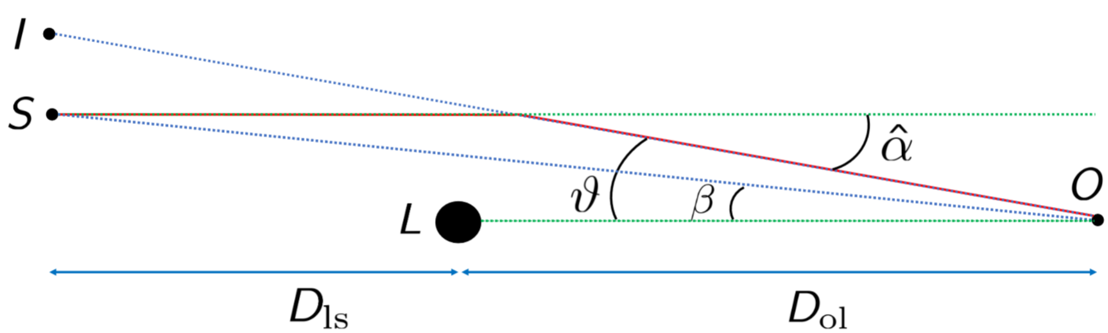
For different models discussed in this paper, we find the following results, which we present by $R_\pm = (1 \pm e)a$ in terms of the eccentricity e and the semimajor axis a of the rotating elliptical orbit.

3 Light deflection

We assume that the observer, black hole, and the source lie on the equatorial plane $\theta = \pi/2$. The constant of motion for null geodesics will be $\sigma = 0$. At the radius of closest distance of approach $r = r_0$, we have $\frac{dr}{d\phi} = 0$, so from Eq. (5), we find $E^2/L^2 = f(r_0)/h(r_0)$, where $f(r_0)$ and $h(r_0)$ are the value of the metric function at $r = r_0$. Then, we can write Eq. (5) as

$$\frac{d\phi}{dr} = \frac{1}{\sqrt{\frac{h(r)}{g(r)}\left(\frac{h(r)}{h(r_0)}\frac{f(r_0)}{f(r)} - 1\right)}}. \tag{16}$$

A schematic diagram illustrating the lensing effect is provided in Fig. below. To calculate the deflection angle $\hat{\alpha}$, we only require an initial and a final radial coordinates of the source and the observer, where the origin of the coordinate system in on the lens.



lensing configuration.

By using equation (16), we can determine the deflection rotation of the trajectory, $\Delta\phi$. Assuming r_{ol} and r_{ls} are the radial distance of the lens from the observer and of lens from the source, then the total change in ϕ will be

$$\Delta\phi(r_0; r_{ls}, r_{ol}) = \int_{r_{ls}}^{r_0} \frac{dr}{\sqrt{\frac{h(r)}{\mathfrak{g}(r)}\left(\frac{h(r)}{h(r_0)}\frac{f(r_0)}{f(r)} - 1\right)}} + \int_{r_0}^{r_{ol}} \frac{dr}{\sqrt{\frac{h(r)}{\mathfrak{g}(r)}\left(\frac{h(r)}{h(r_0)}\frac{f(r_0)}{f(r)} - 1\right)}}, \tag{17}$$

We assume that the distance of the lens from the observer, $D_{\rm ol}$, and the distance of the lens from the source, $D_{\rm ls}$, are much larger than the closest distance of approach $D_{\rm ol}, D_{\rm ls} \gg r_0$. Consequently, we can express the effective deflection angle as follows:

$$\hat{\alpha}(r_0) = 2\int_{r_0}^{\infty} \frac{dr}{\sqrt{\frac{h(r)}{\mathfrak{g}(r)}\left(\frac{h(r)}{h(r_0)}\frac{f(r_0)}{f(r)} - 1\right)}} - n\pi. \tag{18}$$

Here, $n = 3, 5, \ldots$ controls the number of rotations of rays around the lens, in this case once or twice. Since we are interested in light trajectories that complete at least one full rotation around the lens before reaching the observer, we introduce the term $n\pi$ in equation (18). We refer to this as the effective deflection angle because we replace r_{ol} and r_{ls} with ∞. Although this introduces a very small discrepancy in the numerical calculations, it does not invalidate our results.

We use the lens equation introduced in [K. S. Virbhadra and G. F. Ellis, *Schwarzschild black hole lensing*, Physical Review D 62 (2000) 084003; K. Virbhadra, Relativistic images of schwarzschild black hole lensing, Physical Review D 79 (2009) 083004] for large and small deflections, $\hat{\alpha}$,

$$\tan(\beta) = \tan(\vartheta) - \frac{D_{\rm ls}}{D_{\rm os}} \left[\tan(\vartheta) + \tan(\hat{\alpha} - \vartheta)\right]. \tag{19}$$

where $D_{\rm os} = D_{\rm ol} + D_{\rm ls}$, and ϑ can be calculated from

$$b = \sqrt{\frac{h(r_0)}{f(r_0)}} = D_{\rm ol} \sin(\vartheta). \tag{20}$$

in which b is the impact parameter. To derive equation (20), we employ the flat trigonometric relations inherent to geometric optics. These relations are also applicable in non-asymptotically flat spacetimes. This is because they pertain to an observer's line of sight, but inherently incorporates the effects of curved spacetime. These effects are implicitly included in the integral (17), thus allowing the lens equation to be relevant not only in asymptotic regions but also in regions? that are non-flat.

For a specified source position β, we refer to Fig. above, where we numerically solve the lens equation Eq. (19) to obtain the minimum distance of approach, denoted by r_0, and the corresponding image position ϑ.

4 Time delay

The difference between the time for the light rays to travel from the source to the observer in a curved spacetime compared to the time taken in a flat spacetime, is referred to as the time delay. To compute that, we rewrite Eq. (5) for the null geodesic in the form

$$\frac{dt}{dr} = \frac{1}{\sqrt{\frac{f(r)}{g(r)}\left(1 - \frac{f(r)}{f(r_0)}\frac{h(r_0)}{h(r)}\right)}} \tag{21}$$

where we have used the fact that at $r = r_0$ we have $\frac{dr}{dt} = 0$ and from Eq. (5) we get $L^2/E^2 = h(r_0)/f(r_0)$. Then the time delay for source position β in an isotropic spherically symmetric metric can be expressed as

$$\tau(r_0; r_{\rm ls}, r_{\rm ol}) = \left[\int_{r_0}^{r_{\rm ls}} dr + \int_{r_0}^{r_{\rm ol}} dr\right] \frac{1}{\sqrt{\frac{f(r)}{g(r)}\left(1 - \frac{f(r)}{f(r_0)}\frac{h(r_0)}{h(r)}\right)}} - D_{\rm os}\sec(\beta), \tag{22}$$

where $D_{\rm os} = D_{\rm ol} + D_{\rm ls}$ is the distance from observer to the source, $r_{\rm ls} = \sqrt{D_{\rm ls}^2 + D_{\rm os}^2 \tan^2(\beta)}$, and $r_{\rm ol} = D_{\rm ol}$. The first two terms in Eq. (22) represent the light travel time from the source to the closest approach point, and from there to the observer, respectively. The final term accounts for the light travel time directly from the source to the observer in the absence of a gravitational curvature. In addition to the familiar primary and secondary images formed by weak field gravitational lenses –those arising from rays that do not complete a full rotation around the lens– we are intrigued by a more exotic phenomenon related to the light rays that complete at least one full rotation around the lens. These rays give rise to a remarkable infinite sequence of relativistic images, which manifest as demagnified images on both sides of the optical axis. Unlike the standard images formed by non-rotating rays, these relativistic images trace paths that encircle the lens. Their behavior is governed by the strong gravitational field near massive objects. The infinite sequence of relativistic images emerges due to multiple rotations of the light rays. As we move away from the optical axis (measured with respect to the lens), the magnification of relativistic images decreases rapidly. The angular position of the light source plays a crucial role in determining the magnification properties of these relativistic images. Observing these relativistic images presents significant challenges. Their faintness, intricate patterns, and the need for ultra-high resolution observations make their detection complicated.

For circularly symmetric gravitational lensing, magnification is given by

$$\mu = \left(\frac{\sin(\beta)}{\sin(\theta)} \frac{d\beta}{d\theta} \right)^{-1}. \tag{23}$$

To find the magnification, we need the first derivative of the deflection angle with respect to the angular coordinate θ: $\frac{d\hat{\alpha}}{d\theta} = \frac{d\hat{\alpha}}{dr_0}\frac{dr_0}{d\theta}$. The term $\frac{dr_0}{d\theta}$ can be obtained using Eq. (20), while calculating $\frac{d\hat{\alpha}}{dr_0}$ can be intricate, and its details are discussed in [M. B. J. Poshteh and R. B. Mann, *Gravitational lensing by black holes in Einsteinian cubic gravity*, Physical Review D 99 (2019) 024035]. When deriving the final expression of magnification –particularly for numerical calculations– we often rely on asymptotic flatness.