IEOR 4500
Robust Arbitrage Model

Notation:

- K = number of scenarios
- $0, 1, 2, \ldots, n$: indices for assets (cash is 0)
- r = risk-free interest rate
- $\bar{\pi}_{0j}$ = today's price for asset j,
- $\bar{\pi}_{kj}$ = expected value of π_{kj}, the price for asset j in scenario $k \geq 1$ $(0 \leq j \leq K)$
- σ_{kj} = maximum deviation of price π_{kj} away from its expectation.

The robust problem we want to solve is:

$$
\begin{aligned}
W^* \doteq \min \quad & \sum_{j=0}^{n} \pi_{0j} x_j, & (1)\\
s.t. \quad & \\
& \sum_{j=0}^{n} \pi_{kj} x_j \quad \geq \quad 0, \quad \text{for all } k, \text{ and} \\
& \quad \text{for all } \pi_{kj} \text{ such that } \bar{\pi}_{kj} - \sigma_{kj} \leq \pi_{kj} \leq \bar{\pi}_{kj} + \sigma_{kj} & (2)\\
& -1 \leq x_j \leq 1, \quad 0 \leq j \leq n. & (3)
\end{aligned}
$$

Lemma 1. $W^* < 0$ if and only if a type A arbitrage exists.

Lemma 1 casts the problem of testing for a type A arbitrage into a Linear Program with an infinite number of constraints. Next we will render this into a useful formulation. Given a vector x, we say that x *is feasible for scenario* k $(1 \leq k \leq K)$ if

$$\sum_{j=0}^{n} \pi_{kj} x_j \quad \geq \quad 0,$$

for every choice of prices π_{kj} for scenario k such that $\bar{\pi}_{kj} - \sigma_{kj} \leq \pi_{kj} \leq \bar{\pi}_{kj} + \sigma_{kj}$ for every asset j. So the robust problem can be rewritten as

$$
\begin{aligned}
W^* \doteq \min \quad & \sum_{j=0}^{n} \pi_{kj} x_j, & (4)\\
s.t. \quad & \\
& x \text{ is feasible for scenario } k, \text{ for every } 1 \leq k \leq K & (5)\\
& -1 \leq x_j \leq 1, \quad 0 \leq j \leq n. & (6)
\end{aligned}
$$

Constraint (5) describes an infinite number of conditions. We can make this into a compact statement as follows. Consider a given, fixed asset vector $\hat{x}$. Note that **$\hat{x}$ is feasible for scenario k if and only if $V_k^*(\hat{x}) \geq 0$**, where

$$V_k^*(\hat{x}) \doteq \min \quad \sum_{j=0}^{n} \hat{x}_j \pi_{kj}, \tag{7}$$

$$s.t. \quad \bar{\pi}_{kj} - \sigma_{kj} \leq \pi_{kj} \leq \bar{\pi}_{kj} + \sigma_{kj} \quad 0 \leq j \leq n. \tag{8}$$

In this LP, the $\hat{x}$ vector is given data, and the π_{kj} are the variables. We can rewrite this LP as follows:

$$V_k^*(\hat{x}) \doteq \min \quad \sum_{j=0}^{n} \hat{x}_j \pi_{kj}, \tag{9}$$

$$s.t. \tag{10}$$

$$\pi_{kj} \;\geq\; \bar{\pi}_{kj} - \sigma_{kj}, \quad 0 \leq j \leq n, \tag{11}$$

$$-\pi_{kj} \;\geq\; -\bar{\pi}_{kj} - \sigma_{kj}, \quad 0 \leq j \leq n. \tag{12}$$

The value of an LP is equal to that of its dual. The dual of (9)-(12) is a *maximization* problem. Thus, it follows that $V_k^*(\hat{x}) \geq 0$ if and only if there exists a vector of duals feasible for the dual of (9)-(12), of nonnegative value. Denoting the dual of constraint (11) by u_{kj}, and the dual of constraint (12) by v_{kj}, we have that $V_k^*(\hat{x}) \geq 0$ (i.e., vector $\hat{x}$ is feasible for scenario k) if there exist values u_{kj}, v_{kj} $(1 \leq j \leq n)$ such that

$$\sum_{j=0}^{n} (\bar{\pi}_{kj} - \sigma_{kj}) u_{kj} \;+\; (-\bar{\pi}_{kj} - \sigma_{kj}) v_{kj} \;\geq\; 0, \tag{13}$$

$$u_{kj} - v_{kj} \;=\; \hat{x}_j, \quad 0 \leq j \leq n, \tag{14}$$

$$u_{kj} \geq 0,\; v_{kj} \geq 0, \quad 0 \leq j \leq n, \tag{15}$$

In summary, our robust arbitrage testing LP is

$$W^* \doteq \min \quad \sum_{j=0}^{n} \pi_{0j} x_j, \tag{16}$$

$$s.t.$$

$$\sum_{j=0}^{n} (\bar{\pi}_{kj} - \sigma_{kj}) u_{kj} \;+\; (-\bar{\pi}_{kj} - \sigma_{kj}) v_{kj} \;\geq\; 0,\; 1 \leq k \leq K, \tag{17}$$

$$u_{kj} - v_{kj} \;-\; x_j \;=\; 0, \quad 0 \leq j \leq n,\; 1 \leq k \leq K, \tag{18}$$

$$u_{kj} \geq 0,\; v_{kj} \geq 0, \quad 0 \leq j \leq n,\; 1 \leq k \leq K, \tag{19}$$

$$-1 \leq x_j \leq 1, \quad 0 \leq j \leq n. \tag{20}$$

In this linear program, the variables are the x, the u and the v. We can think of this process as a 'game': we pick the vector x and there is an adversary who in each scenario will try to manipulate prices so that the value of the portfolio in that scenario becomes negative. By picking x so that constraints (18) and (19) hold, we make the adversary's task **impossible:** no (legal) choice of the prices will make the portfolio value negative in any scenario.