Problem Chosen	2022	Team Control Number
A	**MCM/ICM** **Summary Sheet**	XJ162

test

Summary

Keywords: 123456

Contents

1   Introduction

1.1   Problem Restatement

Finless porpoise is the only freshwater mammal in the Yangtze River at present, which is distributed in the middle and lower reaches of the Yangtze River, Dongting Lake and Poyang Lake, and its population has decreased dramatically in the past 20 years. According to the statistics, the number of finless porpoises in the Yangtze River was more than 2,700 in 1991. However, in the year of 2006, there were fewer than 1,800 finless porpoises surviving in the area. In 2011, there were probably just over 1,000 of them, and in 2018 there were about 1,012.

In fact, since the 1980s, the ecologists along with the government had explored and developed three conservation strategies: in situ conservation, ex situ conservation and artificial breeding. Among them, ex situ protection, that is, selecting some waters with similar ecological environment to the Yangtze River to establish ex situ protection, is the most direct and effective measure to protect the Yangtze finless porpoise.

China has set up five ex-situ protected sites until now, in which more than 150 Yangtze finless porpoises are conserved. On September 18, 2021, CCTV reported that the population of the Yangtze finless porpoise is growing steadily. The population decline of the Yangtze finless porpoise has been curbed, but its critically endangered status remains unchanged.

Based on what has been discussed above, please address the following problems:

1   (a) Establish a mathematical model to predict the population number of finless porpoises in five ex situ protected areas after 20 years.

  (b) Explain how the sex ratio of 150 finless porpoises in ex situ protected areas affects the population development of finless porpoises.

2 Will the Yangtze finless porpoise become functionally extinct without ex situ conservation strategies?

3 Based on your analysis, please submit no more than 2 pages of recommendations for the protection of finless porpoises to the relevant authorities.

1.2   Overview of Our Work
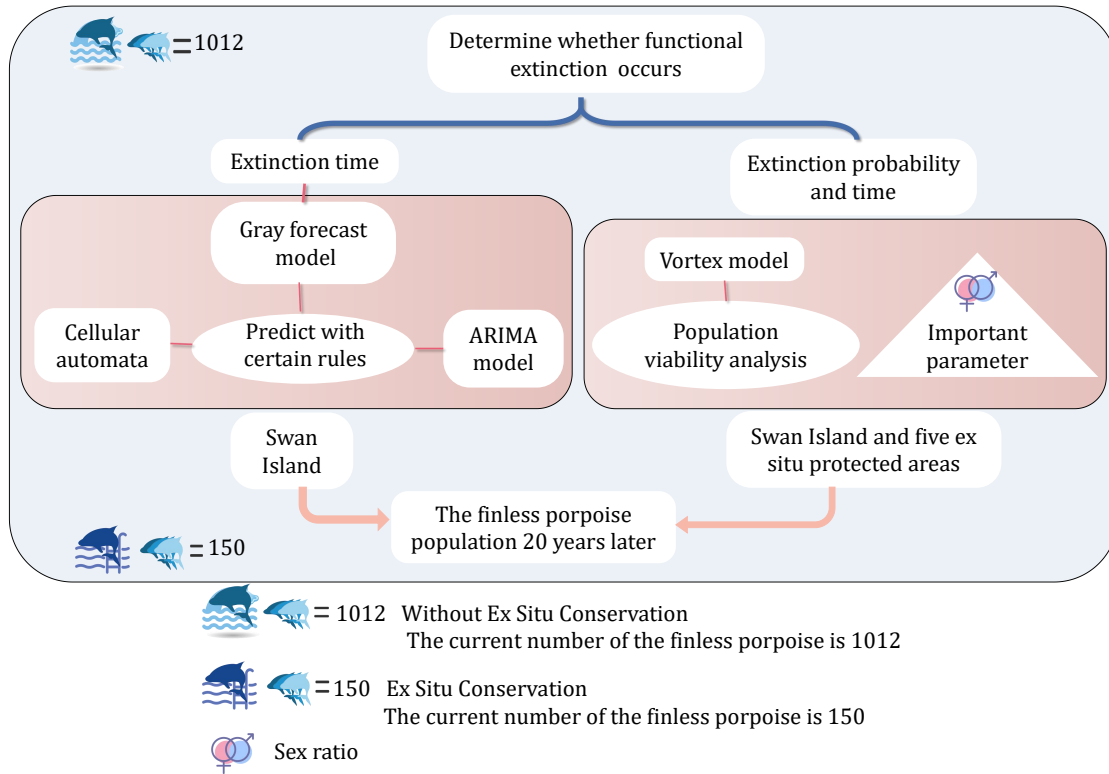
2   Assumptions and Justifications

These are necessary assumptions for simplifying the model.

1. The carrying capacity per unit area of each ex-situ conservation layout is constant.

2.

3   Notations

Table 3.1: Notation Descriptions

Symbol	Definition
K	Carrying capacity
N_t	Size of finless porpoise population in the year of $1991 + t$

4   Introduction and Results of Models on Problem 1(a)

Note that because of the deficiency of the statistics about the other four ex-situ conservations, the size of finless porpoise population per unit area is considered the same as that in Swan Oxbow of the Yangtse River.

Considering tremendous cost on massive finless porpoise population census, merely six years of data was collected in Swan Oxbow of the Yangtse River during the three decades since 1992. (Zhigang, 2020) Thus, we've applied **Lagrange interpolation** to obtain other years' data in Swan Oxbow.

4.1 Relation between Population Size and Time based on Lagrange Interpolation

Given n distinct real values $x_1, x_2, \cdots, x_n$ and n real values $y_1, y_2, \cdots, y_n$ (not necessarily distinct), there is a unique polynomial P with real coefficients satisfying $P(x_i) = y_i$ for $i \in \{1, 2, \cdots, n\}$, such that deg($P$)<n.

The polynomial $P(x)$ is defined as follows:

$$P(x) = \sum_{k=1}^{n} y_k p_k(x), \quad p_k(x) = \frac{(x - x_1) \cdots (x - x_{k-1})(x - x_{k+1}) \cdots (x - x_n)}{(x_k - x_1) \cdots (x_k - x_{k-1})(x_k - x_{k+1}) \cdots (x_k - x_n)}$$

After substituted the number in 1992, 2002, 2005, 2007, 2015 and 2021, the figure of the polynomial is as follows:
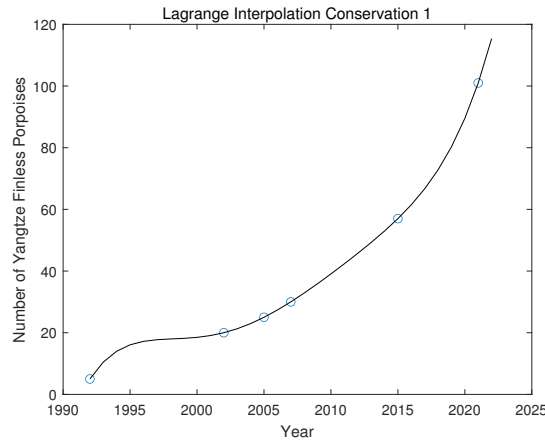
Figure 4.1: Langrange Interpolation Conservation

According to Lagrange interpolation and the definition of N_t :

$$N_t = P(t) \quad t = 1, \cdots, 30$$

and the exact numbers are listed below:

4.2 Model I: Vortex model based finless porpoise analysis

4.3 Model II: Auto Regressive Integrated Moving Average(ARIMA) model

We apply ARIMA model in order to predict the size of finless porpoise population in five ex-situ conservation areas after 20 years, whose **algorithm block diagram** is shown as Figure 4.2.

Algorithm 1 Vortex model based Finless Porpoise Population Size Prediction Algorithm

---

Input: features of finless porpoises described in table 6.1

 1: **for** scenario $\leftarrow 1$ to m **do**
 2:     **if** NumberOfPopulations ≥ 1 **then**
 3:         READ POPULATION and MIGRATION
 4:         **for** each population, pSource $\leftarrow 1$ to n **do**
 5:             $BreedEV[P] = \sqrt{BreedEV[P]^2 - BreedEV[P] \times EVConcordance}$
 6:             Calculate the parameters $\leftarrow$ LOCAL_EV, CATASTROPHES, MIGRATE, BREED, MORTALITY(p)
 7:         **end for**
 8:         **for** each Year $\leftarrow 1$ to l **do**
 9:             $SE = \sqrt{PE \times (1 - PE)}$
10:             **if** population extincted **then**
11:                 $YearRecolonized[P] = CurrentYear$
12:             **else**
13:                 $TimeToRecolonization[p] = CurrentYear - YearExtinct[p]$
14:             **end if**
15:             **if** not extinct $\&\&PopulationSize[p], N > CarryingCapacity[p]$ **then**
16:                 **for** each indiividual **do**
17:                     **if** $RAND() > \frac{K}{N}$ **then**
18:                         Extinct
19:                     **else**
20:                         **if** $N > K$ **then**
21:                             $YearExtinct[p] = CurrentYear$
22:                         **end if**
23:                         **if** others **then**
24:                             $TimeToReextinction[p] = CurrrentYear - Recolonization[p]$
25:                         **end if**
26:                     **end if**
27:                 **end for**
28:             **end if**
29:         **end for**
30:     **end if**
31: **end for**

Output: The Number of iterated population in each year

Table 4.1: Estimated size of population on year basis from 1992 to 2022

Year	Number	Year	Number	Year	Number
1992	5	2002	20	2012	45.7364
1993	10.4864	2003	21.2919	2013	49.2555
1994	13.9987	2004	22.9647	2014	52.9737
1995	16.0839	2005	25	2015	57
1996	17.2014	2006	27.3612	2016	61.4941
1997	17.7296	2007	30	2017	66.6734
1998	17.9732	2008	32.8636	2018	72.82
1999	18.1701	2009	35.9015	2019	80.2876
2000	18.4981	2010	39.0724	2020	89.5084
2001	19.0819	2011	42.3514	2021	101

n is the capacity of sample;$\hat{\sigma_\epsilon^2}$ is the estimation of σ_ϵ^2 relating to p and q. Suppose $p = \hat{p}, q = \hat{q}$, such that equation(4.1) reaches the minimum, than we deem the series is $\mathbf{ARMA}(\hat{p}, \hat{q})$.

Suppose $\mathbf{ARMA}(p, q)$ series has an unknown average parameter μ, the model becomes

$$\phi(B)(X_y - \mu) = \theta(B)\epsilon_t,$$

meanwhile, the number of unknown parameters is $k = p + q + 2$, the AIC is: choose p, q such that

$$\min \mathbf{AIC} = n \ln \hat{\sigma_\epsilon^2} + 2(p + q + 2). \tag{4.2}$$

In fact, equations(4.1)and(4.2)have the same minimum point $\hat{p}, \hat{q}$. After that, we usually choose $p = 1, q = 1$ to make parameter estimation over ARMA model.

It's demonstrated that the differential operation can stabilize certain class of non-stationary series. And It's emphasized that stationary test must be conducted previously. Stationary test can be applied by calculating sample autocorrelation function and sample coefficient of partial function.

If the functions are truncated or trending to 0 (meaning being controlled by negative index), than the series belongs to ARMA model.

If at least one of the functions above is not truncated or trending to 0, than it's not stationary.

Suppose the series is non-stationary, which can be transformed to a stationary series by d -degree differential operation, denoted as $\mathbf{ARIMA}(p, q, d)$ series,than differentiate the sample by d -degree:

$$W_t = \nabla^d X_t, \quad t = d + 1, \ldots, n$$

After that, apply stationary test on W_t and repeat steps above until it becomes a stationary series, Than W_t(which is denoted as X_t) complies ARMA model.

Under regular circumstances, the time series we obtain in the real world has tendency, seasonality and non-stationarity. Thus, it's vital for us to transfer the non-stationary time series to stationary time series and make an assumption that the time series is an Auto Regressive Moving Average (ARMA) series to predict the future data. ARMA series is defined as follows.

$$X_t - \phi_1 X_{t-1} - \cdots - \phi_p X_{t-p}$$
$$= \epsilon_t - \theta_1 \epsilon_{t-1} - \cdots - \theta_q \epsilon_{t-q}$$

ϵ_1 is a stationary white noise whose average is zero and deviation is σ_ϵ^2; X_t is an ARMA series with p and q degree, recorded briefly as $\mathbf{ARMA}(p, q)$ series. Akaike Information Criterion(AIC) is one of the most commonly used criterion to determine the degree of $\mathbf{ARMA}(p, q)$: choose p, q such that

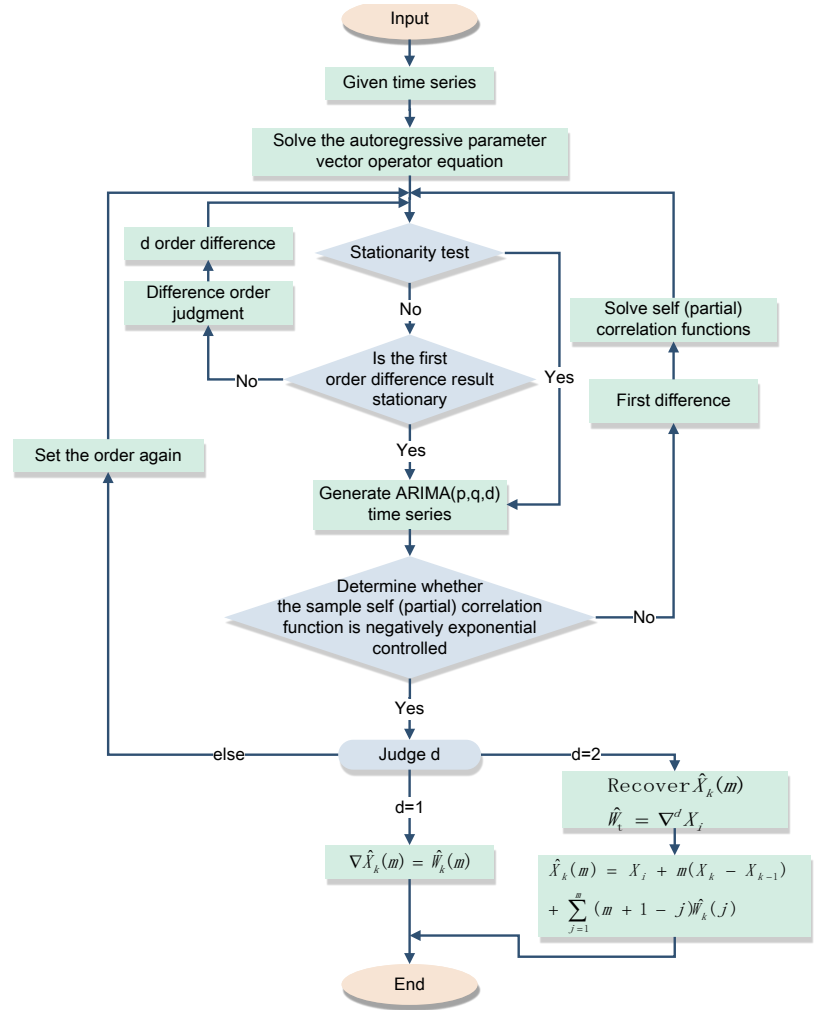$$\min \mathbf{AIC} = n \ln \hat{\sigma}_\epsilon^2 + 2(p + q + 1) \quad (4.1)$$

Figure 4.2: Algorithm Block of ARIMA

The figures below describe the result of ARIMA model on the time series of the size of finless porpoise N_t, $t = 1 \cdots 30$, in which the Figure(4.3(c)) clearly shows that the number will decline and settle around 62 in the future 20 years.

4.4 Model III: Cellular Automata based Population Size Prediction

4.5 Model IV: Gray Forecast model

The very heart of Gray Forecast is Gray model, which is used to model and forecast the approximate exponential law after summing up the original data. It is suitable for the prediction scenario with less data. While **Vehulst** model is primarily used to describe procedures with saturation state, that is S-shape procedure, which can be applied in the sphere of biological growth and reproductive prediction. The fundamental is as follows.

Suppose the sample is $x^{(0)} = \{x^{(0)}(1), x^{(0)}(2) \cdots x^{(0)}(n)\}$, and it is accumulated once to generate

a (1-AGO) sequence, which is:

$$x^{(1)} = (x^{(1)}(1), x^{(1)}(2), \cdots, x^{(1)}(n))$$

$z^{(1)}$ is the mean-generating sequence of $X^{(1)}$, which is:

$$z^{(1)} = (z^{(1)}(2), z^{(1)}(3), \cdots, z^{(1)}(n)).$$

Then, we call the following equations Gray Verhulst model with parameters a and b:

$$x^{(0)} + az^{(1)} = b(x^{(1)})^2$$

, and we call the following equations the winterization equations of of Gray Verhulst model with t denoting time:

$$\frac{\mathbf{d}x^{(1)}}{\mathbf{d}t} + ax^{(1)} = b(x^{(1)})^2$$

Theorem 1. Suppose the Gray Verhulst model is depicted as above, if $\boldsymbol{u} = [a, b]^T$ is the parameter vector,and

$$\boldsymbol{B} = \begin{bmatrix} -z^{(1)}(2) & (z^{(1)}(2))^2 \\ -z^{(1)}(3) & (z^{(1)}(3))^2 \\ \vdots & \vdots \\ -z^{(1)}(n) & (z^{(1)}(n))^2 \end{bmatrix}, \boldsymbol{Y} = \begin{pmatrix} x^{(0)}(2) \\ x^{(0)}(3) \\ \vdots \\ x^{(0)}(n) \end{pmatrix},$$

then the least square estimation of parameter $\boldsymbol{u}$ satisfies

$$\hat{\boldsymbol{u}} = [\hat{a}, \hat{b}]^T = (\boldsymbol{B}^T\boldsymbol{B})^{-1}\boldsymbol{B}^T\boldsymbol{Y}$$

Theorem 2 Suppose Gray Verhulst model is defined as above, then solution to the winterization equations is

$$x^{(1)}(t) = \frac{\hat{a}x^{(0)}(1)}{\hat{b}x^{(0)}(1) + [\hat{a} - \hat{b}x^{(0)}(1)]e^{\hat{a}t}},$$

and the time response sequence of Gray Verhulst model is

$$\hat{x}^{(1)}(k+1) = \frac{\hat{a}x^{(0)}(1)}{\hat{b}x^{(0)}(1) + [\hat{a} - \hat{b}x^{(0)}(1)]e^{\hat{a}t}},$$

and the reductive formula is

$$\hat{x}^{(0)}(k+1) = \hat{x}^{(1)}(k+1) - \hat{x}^{(1)}(k)$$

# 5   Solution to Problem 1(b) based on Vortex model

Considering the abundant parameter settings in Vortex software among which sex ratio is easy to edit and is of great significance, we decide to further apply Vortex model in order to address the problem 1(b) concerning the effects of sex ratio in five ex-situ conservation zones.
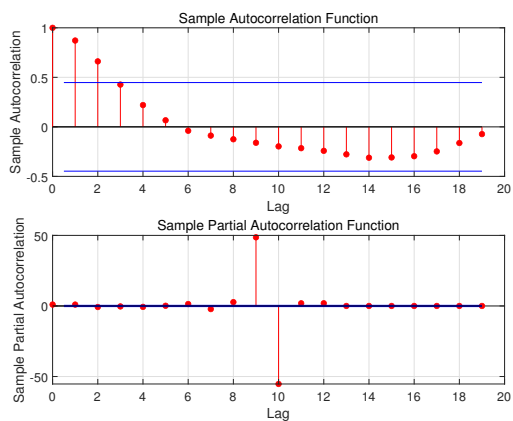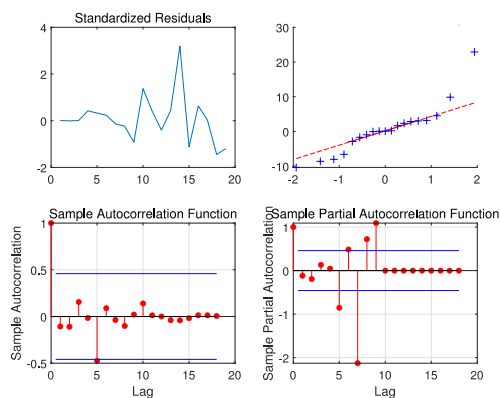
(a) Sample Autocorrelation Function and Sample Partial Autocorrelation Function
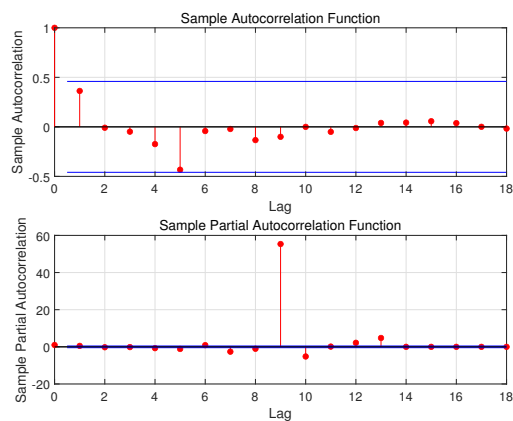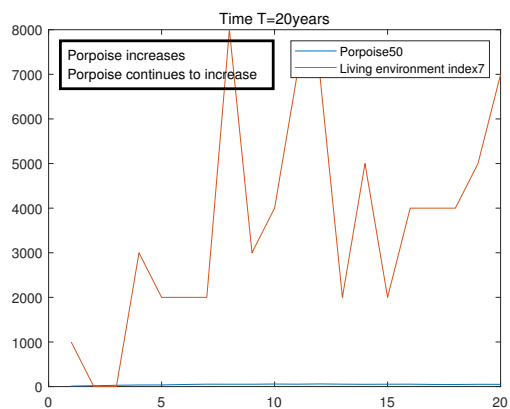

(b) Sample Autocorrelation Function and Sample Partial Autocorrelation Function
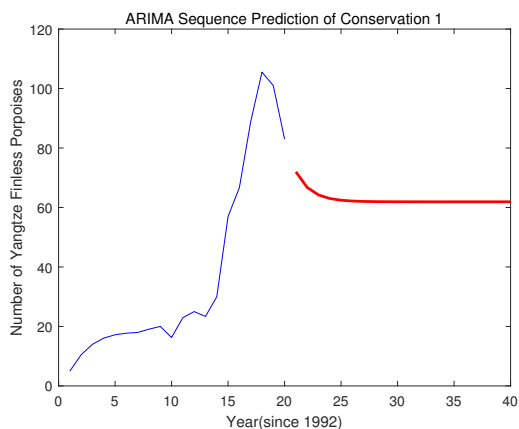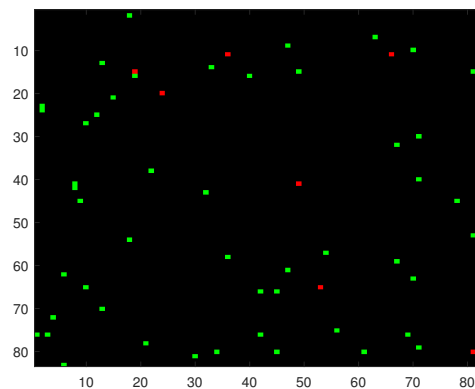

(c) ARIMA Sequence Prediction of Conservation

(d) Standardaized Residuals and QQ figure

(e) Living Environment Index and Population

(f) Cell Distribution

6 Vortex model and Gray Forecast model based Population Prediction without Ex-Situ Conservation

The known **existing wild finless porpoise population is 1012** (Gang et al., 2021). According to previous researches, female finless porpoises are more vulnerable than the male ones in their early ages, so the ratio of male finless porpoises to female ones which are 0 - 6 years old is $3.7 : 1$, and the ratio of those older than 6 is $2 : 1$. For simplicity, we perceive that **the ratio of male finless porpoises to female ones in all ages** is **$2.85 : 1$**, which is the median of the above two ratios. What's more, we assume that **the carrying capacity** K of finless porpoises in the whole Yangtze River is **3000**.

In order to thoroughly discuss the risk of finless porpoise extinction, we define two types of **functional extinction** and one type of **species extinction**.

The first type of functional extinction (**Functional Extinction I**) is that there only exists one sex, which means this population is no longer able to reproduce and bond to be extinct in the future. The second type of functional extinction (**Functional Extinction I**) is that the number of remaining individuals in the population is unable to prevent inbreeding depression which will cause the fading and extinction of the population. We define the number of population that reaches the second functional extinction is **200**.

The **Species Extinction** is defined as no living individual of this population found in this area.

6.1 Result of Vortex model

The result of Vortex model simulation based on previous conditions is shown as Table 6.1, in which the meaning of these parameters is shown in Table 6.2.

In a nut shell, from the **meanTE** value in Table 6.1 we can draw a reasonable conclusion that without ex-situ conservation actions, the finless porpoises in Yangtze River will suffer **Functional Extinction II after 55.1 years, Functional Extinction II after 71.3 years and Species Extinction after 95.5 years.**

What's shown in Figure 6.1 describes the variation trends in these scenarios.

6.2 Result of Gray Forecast model

Given the fact that the Gray Forecast model is unable to predict the date of Functional Extinction I, we deploy the model to measure Functional Extinction II.

The Figure 6.2 has explicitly shown that it will take the finless porpoise population 55 to 60 years to reach Functional Extinction II, which is unbelievably similar to the conclusion of Vortex model from section 6.1.

---

[1]Modifiable when discussing different scenarios.

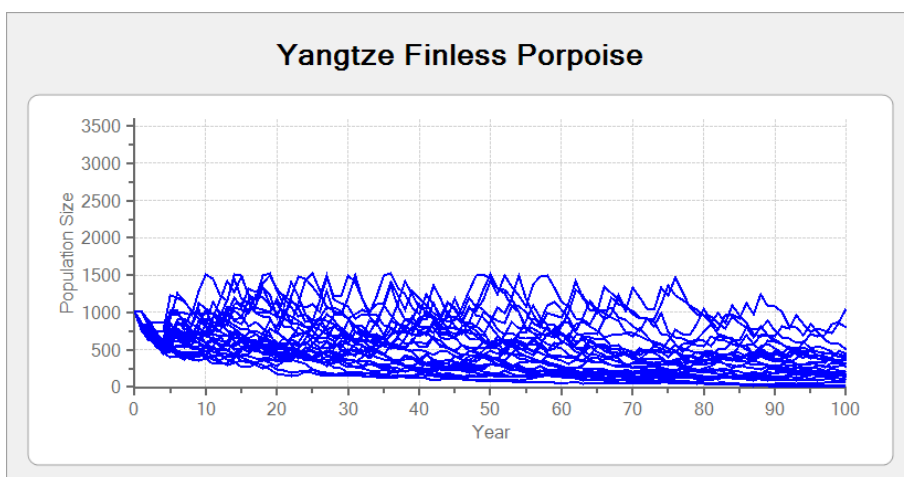[2]$P(N)$ is the percent of females the breed when the population size is N, which can be dfined as $P(N) = P(0) - [P(0) - P(K)(\frac{N}{K})^B]\frac{N}{N+A}$
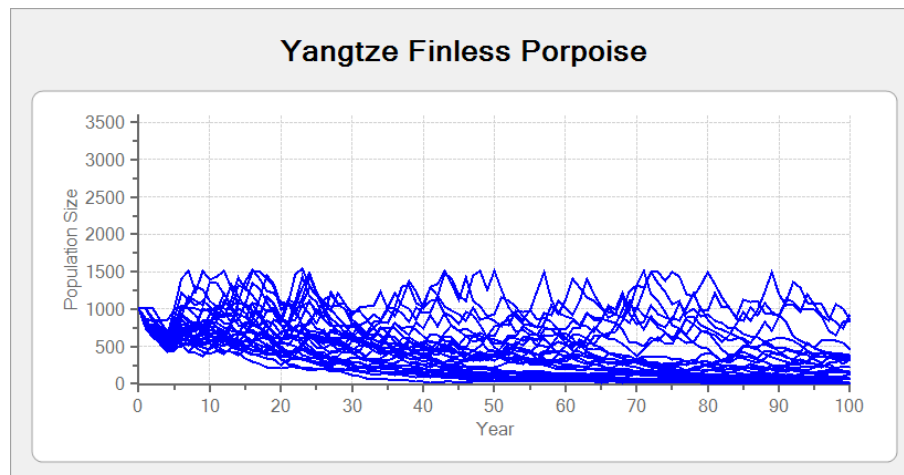
(a) Variation trend in terms of Species Extinction
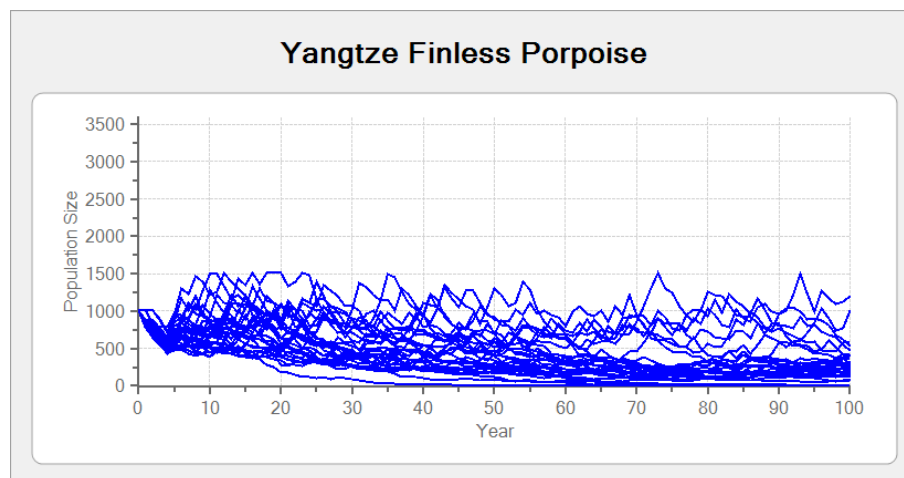
(b) Variation trend in terms of Functional Extinction I

(c) Variation trend in terms of Functional Extinction II

Figure 6.1: Variation Trends in Different Scenarios

| Scenario	Species Extinction	Functional Extinction I	Functional Extinction II
nRuns	30	30	30
stoch-r	-0.0239	-0.0165	-0.0120
SD(r)	0.1399	0.1340	0.1359
PE	0.0667	0.1000	0.4667
N-extant	256.00	340.00	427.25
SD(N-ext)	259.57	257.31	214.72
N-all	238.93	306.03	268.37
SD(N-all)	258.75	264.76	236.37
GeneDiv	0.8852	0.9300	0.9621
SD(GD)	0.1257	0.0321	0.0132
nAlleles	31.71	35.59	51.56
SD(nA)	20.83	19.06	16.43
medianTE	0	0	80
meanTE	95.5	71.3	55.1
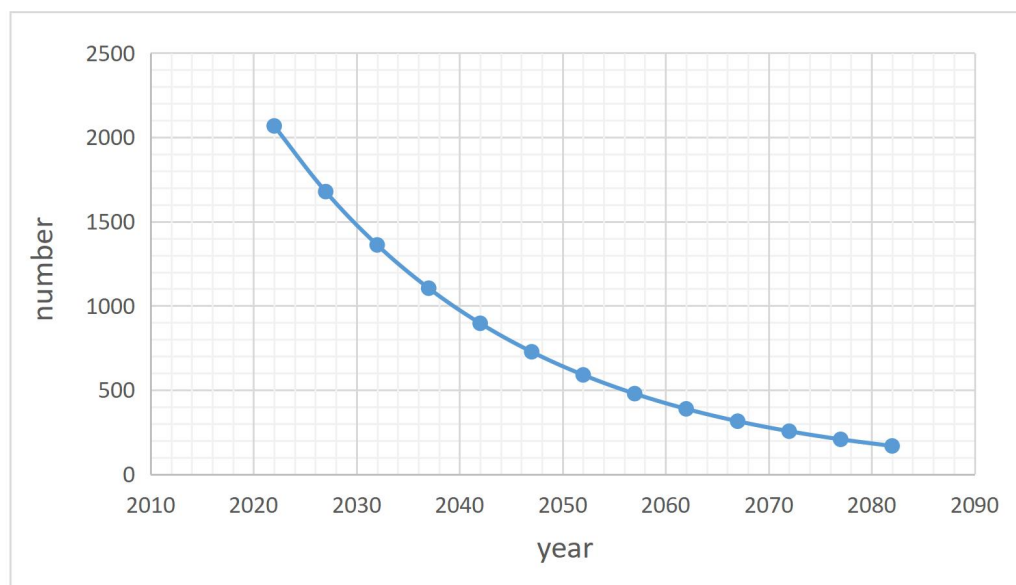
Figure 6.2: Predicted Population Size based on Gray Forecast

Table 6.1: Values inputted into Vortex

Times simulated	1000times
Years simulated	100a
Reporting interval	10a
Populations simulated	1
Inbreeding depression (Y/N)	Y
Heterosis or Lethal	H
Lethal equivalents	3.14
EV correlation between reproduction and survival	0.5
EV correlation among populations	0.5
Types of catastrophe	2
Monogamous, Polygynous or Hermaphroditic	P
Female breeding age	4a
Male breeding age	5a
Maximum breeding age	15a
Sex ratio (proportion males) at birth[1]	0.5
Maximum litter size	1
Density dependent breeding[2] (Y/N)?	Y
$P(0)$ (the percent of adult females that breed at low densities when there is no Allee effect)	70%
$P(K)$ (the percent that breed when the population is at carrying capacity)	25%
B	2
A	2
Percent litter size 1	100%
Moralities in different ages	see Table
Catastrophes and influence	see Table
Percent males in breeding pool	70%
Start at stable age distribution (Y/N)?	Y
Trend in K (Y/N) ?	Y
Years of trend	5a
Percent age in K	-10%
Harvest (Y/N)?	N
Supplement (Y/N)?	N

Table 6.2: Important output of Vortex(Lacy et al., 2021)

Symbol	Definition
r	Innate rate of increase
Stoch r	The mean population growth rate experienced in the simulations, averaged across all years in which the population was extant.
N-extant	Average extant population size
N-all	Average population size
PE	Probability of Extinction
GeneDiv	Genetic Diversity
TE	Time of Extinction(year)
medianTE	If at least 50% of the iterations went extinct, the median time to extinction
SD	Standard Deviation
nAlleles	The mean number of alleles remaining within extant populations (from an original number equal to twice the number of founder individuals)
K	Carrying capacity
N_t	Size of finless porpoise population in the year of $1991 + t$

7   Sensitivity Test

8   Evaluation of Model

9   Conclusions

REPORT

To: 123

From: 123

Date: January 15, 2022

References

Gang, H., Bin, W., Weiping, W., & Haihua, W. (2021). Analysis of population viability analysis of yangtze finless porpoise in different simulated scenarios. *Progress In Fishery Sciences*, *2*(42), 28-35.

Lacy, R.C., & Pollak, J. (2021). Vortex: A stochastic simulation of the extinction process. version 10.5.5. *IUCN SSC Conservation Planning Specialist Group and Chicago Zoological Society*, *1*(1), 22-76.

Zhigang, L. (2020). *The changes of micro-ecological of diseased yangtze finless and research on its protection under ex-situ* (Unpublished doctoral dissertation). Huazhong Agricultural University.

Appendices

Input python source:

```python
import numpy as np
import math


def predict(data):
    x1 = data.cumsum()
    z = (x1[:len(x1) - 1] + x1[1:]) / 2.0
    B = np.array([-z, z*z]).T
    Y = data[1:]
    u = np.dot(np.dot(np.linalg.inv(np.dot(B.T, B)), B.T), Y)
    a, b = u[0], u[1]
    return [a*data[0]/(b*data[0]+(a-b*data[0])*math.exp(a*i)) for i in range(len(data))]
    # Gray Forecast Model Function


if __name__ == '__main__':
    raw_data = np.loadtxt('conservation_1.txt')
    data = np.array(raw_data)
    # [5.0000, 12.4864, 13.9987,  16.0839, 17.201,
    # 17.7296, 17.9732, 19.0819, 20.0000,  21.2919,
    # 22.9647,  25.0000,  28.3612, 30.0000,  57.0000,
    # 66.6734,  88.8200,  105.5084, 101.0000, 52]
    predict_data = predict(data)  # Prediction
    result = np.ediff1d(predict_data)  # Diminishing
    print('Original result: ', data[1:])
    print('Prediction result: ', result)
    print('Relative error: ', (np.array(result[:len(data)])
                    - np.array(data[1:len(data)])) / np.array(data[1:len(data)]))
```

Input matlab source:

```matlab
clc;
clear;
x = [1992, 2002, 2005, 2007, 2015, 2021]    % Year
y = [5, 20, 25, 30, 57, 101]     % Number of Yangtze Finless Porpoises
xi = 1992:1:2022    % Prediction
yi = lagrange(x, y, xi)     % Lagrange Interpolation
plot(x, y, 'o', xi, yi, 'k')
title('Lagrange Interpolation Conservation 1')
xlabel('Year')
ylabel('Number of Yangtze Finless Porpoises')


function yy=lagrange(x,y,xx)     % Lagrange Function
m = length(x);
n = length(y);
if m~= n, error('Length of vector x and y should be the same');
end
s = 0;
for i = 1:n
```

```matlab
    t = ones(1,length(xx));
    for j = 1:n
        if j~=i,
            t = t.*(xx - x(j))/(x(i) - x(j))    %Data (x, y) at interpolation point xx
        end
    end
    s = s + t * y(i)
end
yy = s
```

```matlab
clc;
clear;
data = textread('conservation_1.txt');
data=nonzeros(data');
% Remove the zero elements in the order of the original data
r11=autocorr(data);
% Calculate the self correlation coefficient
r12=parcorr(data);
% Calculate partial correlation coefficient
figure
subplot(211),autocorr(data);
subplot(212),parcorr(data);
% The autocorrelation and partial autocorrelation
%of the original data are drawn on a graph
diff=diff(data);
% R11 is positive, not controlled by a negative exponential,
%so calculate the first order difference
r21=autocorr(diff);
% Calculate the self correlation coefficient
r22=parcorr(diff);
% Calculate the partial correlation coefficient
adf=adftest(diff);
% If adf == 1, stable time sequence
figure
subplot(211),autocorr(diff);
subplot(212),parcorr(diff);
% Plot on the same figure, stable time sequence
n=length(diff);
% Caculate the differencial data
k=0;
  for i = 0:3
    for j = 0:3
        if i == 0 & j == 0
            continue
        elseif i == 0
            ToEstMd = arima('MALags', 1:j, 'Constant', 0);
        elseif j == 0
            ToEstMd = arima('ARLags', 1:i, 'Constant', 0);
        else
            ToEstMd = arima('ARLags', 1:i, 'MALags', 1:j, 'Constant', 0);
            % Model structure
        end
        k = k + 1;
        R(k) = i;
```

```matlab
        M(k) = j;
        [EstMd,EstParamCov,LogL,info] = estimate(ToEstMd,diff);
        % Fitness, and estimate model parameter
        numParams = sum(any(EstParamCov));
        [aic(k), bic(k)] = aicbic(LogL,numParams,n);
    end
end
fprintf('R, M, AIC, BIC value: \n%f');
check  = [R',M',aic',bic'];
res=infer(EstMd,diff);
% Verification
figure
subplot(2,2,1)
plot(res./sqrt(EstMd.Variance))
% Standardlized residual
title('Standardized Residuals')
subplot(2,2,2),qqplot(res)
% Fit the hypothesis of normality
subplot(2,2,3),autocorr(res)
subplot(2,2,4),parcorr(res)


% The autocorrelation coefficient rapidly decreases to 0 after 1 order lag,
% and the partial correlation coefficient
%is the same as the self correlation coefficient,
% so p = 1, q = 1
p=input('p = ');
q=input('q = ');
ToEstMd=arima('ARLags', 1:p, 'MALags', 1:q, 'Constant', 0);
 [EstMd, EstParamCov, LogL, info] = estimate(ToEstMd, diff);
 dx_forest = forecast(EstMd, 20, 'Y0', diff);        % 20 years prediction
 x_forest = data(end)+cumsum(dx_forest)
figure
h4 = plot(data, 'b');
hold on
h5 = plot(length(data)+1: length(data)+20, x_forest, 'r', 'LineWidth', 2);
title('ARIMA Sequence Prediction of Conservation 1')
xlabel('Year(since 1992)')
ylabel('Number of Yangtze Finless Porpoises')
hold off
```