$$w_{2z} = \dot\theta_1 + \dot\theta_2 , \qquad \dot w_{2z} = \ddot\theta_1 + \ddot\theta_2$$

$$\dot v_2 = \dot w_2 \times P_2^* + w_2 \times (w_2 \times P_2^*) + \dot v_1$$

$$= \begin{bmatrix} 0 \\ 0 \\ \ddot\theta_1 + \ddot\theta_2 \end{bmatrix} \begin{bmatrix} l_2 C_{12} \\ l_2 S_{12} \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \dot\theta_1 + \dot\theta_2 \end{bmatrix} \times \left(\begin{bmatrix} 0 \\ 0 \\ \dot\theta_1 + \dot\theta_2 \end{bmatrix} \begin{bmatrix} l_2 C_{12} \\ l_2 S_{12} \\ 0 \end{bmatrix} \right) + \dot v_1$$

$$= \begin{bmatrix} -l_2 S_{12}(\ddot\theta_1 + \ddot\theta_2) - l_2 C_{12}(\dot\theta_1 + \dot\theta_2)^2 - l_1 S_1 \ddot\theta_1 - l_1 C_1 \dot\theta_1^2 \\ l_2 C_{12}(\ddot\theta_1 + \ddot\theta_2) - l_2 S_{12}(\dot\theta_1 + \dot\theta_2)^2 + l_1 C_1 \ddot\theta_1 - l_1 S_1 \dot\theta_1^2 + g \\ 0 \end{bmatrix}$$

$$\dot v_{c2} = \dot w_2 \times S_2 + w_2 \times (w_2 \times S_2) + \dot v_2 , \quad \text{where } S_2 = -\frac{1}{2} P_2^*$$

$$= \begin{bmatrix} -\frac{1}{2} l_2 S_{12}(\ddot\theta_1 + \ddot\theta_2) - \frac{1}{2} l_2 C_{12}(\dot\theta_1 + \dot\theta_2)^2 - l_1 S_1 \ddot\theta_1 - l_1 C_1 \dot\theta_1^2 \\ \frac{1}{2} l_2 C_{12}(\ddot\theta_1 + \ddot\theta_2) - \frac{1}{2} l_2 S_{12}(\dot\theta_1 + \dot\theta_2)^2 + l_1 C_1 \ddot\theta_1 - l_1 S_1 \dot\theta_1^2 + g \\ 0 \end{bmatrix}$$

$$F_2 = m_2 \dot v_{c2} , \quad J_2 = \begin{bmatrix} \frac{1}{4} m_2 l_2^2 & 0 & 0 \\ 0 & \frac{1}{4} m_2 l_2^2 & 0 \\ 0 & 0 & \frac{1}{4} m_2 l_2^2 \end{bmatrix} , \quad N_2 = J_2 \dot w_2 + w_2 \times (J_2 w_2)$$

$$f_2 = F_2 + f_3 , \quad \text{where } f_3 = \begin{bmatrix} 0 \\ mg \\ 0 \end{bmatrix}$$

$$\rightarrow n_2 = n_3 + P_2^* \times f_3 + (P_2^* + S_2) \times F_2 + N_2 , \quad n_3 = 0$$

$$= \begin{bmatrix} l_2 C_{12} \\ l_2 S_{12} \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ mg \\ 0 \end{bmatrix} + \begin{bmatrix} \frac{1}{2} l_2 C_{12} \\ \frac{1}{2} l_2 S_{12} \\ 0 \end{bmatrix} \times m_2 \dot v_{c2} + N_2$$

due to N_2,

$$\rightarrow \tau_2 = n_2^t \vec z_1 , \quad \vec z_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$