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Social Influence Diffusion and Coordinated Decision Making on Networks

Wynn C. Stirling
Department of Electrical and Computer Engineering
Brigham Young University
Provo, Utah, 84602
Email: wynn_stirling@byu.edu

Luca Tummolini
Institute of Cognitive Sciences and Technologies
Italian National Research Council
Rome, Italy
Email: luca.tummolini@istc.cnr.it

***Abstract*—Essential properties of a social influence network are a) the mechanism by which social influence is diffused throughout the network and b) the mechanism by which social influence is combined to create emergent interrelationships that enable coordinated decision making. Conditional game theory addresses these issues by a) enabling individuals to invoke conditional preferences to dynamically modulate their preferences in response to the intentions of those who influence them, and b) aggregating the conditional preferences to generate a comprehensive social model that enables agents to coordinate their individual choices. This paper develops an operational definition of coordination and establishes a formal relationship between group level coordination and individual level performance for both acyclical and cyclical social influence networks.**

I. Introduction

An influence network is a collective of decision-making entities, or agents, who are connected by some form of communication or control that induce them to respond to the social influence exerted by others. An essential attribute of each individual agent is that its behavior conform to a **principle of rationality**: an agent will act in its own best interest according to what he believes (cf. [1]–[3]). The rationality principle that is often invoked assumes a **narrow view of self-interest**—the doctrine that self-interest amounts to being committed to maximizing its own welfare without overt regard for the welfare of others. Under that rationality scenario, exploiting or even damaging others is justified if doing so increases one's welfare, which is typically defined in terms of material payoffs or other individual notions of satisfaction. Thus, self-interest so defined entails selfishness. Since the actions of others will generally effect the agent's payoff, each agent must define its decision rule as a function or the set of possible joint actions of all agents. A prime example is the familiar payoff array of noncooperative game theory.

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This narrow view of self-interest, however, is an inadequate rationality principle to govern the behavior of members of a network who are responsive to social influence. In addition to concern for their own welfare, such individuals are also disposed to consider how their preferences are influenced by the *preferences, not just the actions, of others.* To make such assessments rationally, the agent must adopt a principle of rationality that goes beyond the narrow view of self-interest.

As influence diffuses throughout the network, the result is an emergent social conglomeration that may exhibit some form of systematic behavior (which could range from anarchy to a well-designed organization). A conglomeration may be cooperative, such as the formation of a business partnership, it could be conflictive, such as a competitive sports match like tennis; it could be mixed, such as a family, where some members are cooperative and some are conflictive; or it could fail to exhibit any discernable systematic behavior.

Although even the narrow view of self-interest would motivate the members of such a conglomeration to maximize their individual material benefit, it may be actually more appropriate also in light of their own individual interests for them to respond in accordance with the emergent social structure. It is more appropriate for a business partnership to settle on a productive division of labor than for each partner to maximize individual control. It is more appropriate to aim to win a match than to try to score as many points as possible. In other words, in a network setting, the operational concept of performance is the effectiveness of an individual as part of a conglomeration, rather than focusing exclusively on maximizing individual benefit (although such focus may very well be the emergent role for the agent). In fact, as Arrow has observed, reliance on a narrow view of self-interest as a principle of rationality has a specific and limited application.

> Rationality in application is not merely a property of the individual. Its useful and powerful implications derive from the conjunction of individual rationality and other basic concepts of neoclassical theory—equilibrium, competition, and completeness of markets. . . . When these assumptions fail, the very concept of rationality becomes threatened, because perceptions of others and, in particular, their ra-

tionality become part of one's own rationality [4, p. 203].

A concept of individually rational behavior that extends beyond the narrow view of self-interest and fits well with the concept of individuals functioning in their emergent role as part of a conglomeration is the notion of coordination. As defined by the Oxford English Dictionary, *to coordinate* is "to place or arrange (things) in proper position relative to each other and to the system of which they form parts; to bring into proper combined order as parts of a whole" [5].

The thesis of this paper is that a notion of **coordinated self-interest** is a principle of rationality that applies to social influence networks. Self-interest, in this view, does not necessarily amount to selfishness but only that an agent can be motivated only by his own goals. Generally speaking, coordinated self-interest is a form of rational behavior that comports with Arrow's assertion that the one's rationality is influenced by the rationality of others. To be precise, however, coordinated self-interest must be explicitly operationalized.

Conditional game theory, developed by Stirling [6], is an extension of conventional noncooperative game theory that is designed to accommodate complex models of social behavior, including models involving the diffusion of social influence and the resulting impact on agent behavior as the agents interact. The key features of this approach are a) a mechanism to enable the diffusion of social influence throughout the network, and b) a mechanism to aggregate the emergent social relationships to create a comprensive social model that enables the creation of solution concepts that explicitly actualize coordinated self-interest.

II. Previous Research

Social psychologists and mathematicians have studied social influence network theory since the 1950s, with much of the research focusing on the organizational structure of so-called *small groups*, defined as loosely coupled collectives of mutually interacting individuals [7]. Specifically, much of the emphasis has been placed on the structure of such organizations [8]–[12]. The basic model is that an individual's socially adjusted preference payoff is a convex combination of its own payoff and a weighted sum of the payoffs of those agents who influence it. Game theory has been widely used as a framework within which to study coordination [13]–[28]). All of these approaches, however, share a common feature: *they separate the way preferences are specified from the way solutions are defined.* They follow the dictum expressed by Friedman that "economic theory proceeds largely to take wants as fixed. This is primarily a case of division of labor. The economist has little to say about the formation of wants; this is the province of the psychologist. The economist's task is to trace the consequences of any given set of wants" [29, p. 13]. The conventional way agents express their preferences is with payoff functions that are fixed, immutable, and unconditional—they are *categorical*. Each agent must incorporate all material and social considerations regarding each outcome into a single preference ordering. These payoffs are then juxtaposed in a payoff array, to which a solution concept, such as Nash equilibrium, can be applied.

The assumption that one's preferences can be adequately expressed by a single ordering has long been criticized. As Sen observed, "The *purely* economic man is indeed close to being a social moron. Economic theory has been much preoccupied with this rational fool decked in the glory of his *one* all-purpose ordering. To make room for the different concepts related to his behavior **we need a more elaborate structure** [italic emphasis in original, bold emphasis added]" [30, pp. 335 - 336]. A single preference ordering may be an appropriate mechanism for one whose concept of rational behavior is limited to the narrow view of self-interest, but for environments involving social influence, a preference mechanism that provides a more elaborate structure is required.

III. Socially Rational Preference Concepts

A **game scenario** comprises the following elements:

- a set $\mathbf{X} = \{X_1, \ldots, X_n\}$ of $n \geq 2$ **agents**, each of which possesses a finite **action set** $\mathcal{A}_i = \{x_{i1}, \ldots x_{iN_i}\}$;
- an **outcome set** $\boldsymbol{\mathcal{A}} = \mathcal{A}_1 \times \cdots \times \mathcal{A}_n$, comprising all possible combinations of actions or **profiles** of the form $\mathbf{a} = (a_1, \ldots, a_n)$ where $a_i \in \mathcal{A}_i$, $i = 1, \ldots, n$;
- a **preference concept** for each X_i that defines the criteria and mechanism for its evaluation of the elements of $\boldsymbol{\mathcal{A}}$; and
- a **solution concept** for each X_i that defines the criteria and mechanism for evaluating the elements of its action set $\mathcal{A}_i$.

The distinction between the preference concept and the solution concept is critical. The former involves the evaluation of group-level outcomes, and the latter involves the evaluation of individual-level actions. This is the inescapable dichotomy of game theory: *how to reconcile preferences defined over group-level outcomes with solutions defined over individual-level actions.* For the game to be well-formed, it is essential that the preference and solution concepts must be compatible. A game is **rationally congruent** if the preference and solution concepts are governed by the same principle of rationality.

Since the preferences for a noncooperative game are defined by categorical payoffs, the governing rationality principle is narrow self-interest. For the game to be rationally congruent, the solution concept must also be governed by narrow self-interest, which is indeed the case for solution concepts such as Nash equilibria. However, if the solution concept invokes notions of sociality that go beyond narrow self-interest, then congruence will be violated. A good example is the Prisoner's Dilemma. Mutual defection is the unique Nash equilibrium, and invoking that decision ensures that the game is congruent. Empirical studies reveal, however, that people often take advantage of opportunity to cooperate, which implies that they are invoking a principle of rationality that is different from narrow self-interest, thereby violating congruence.

One standard way to address this dichotomy is by modifying the payoffs. An example is Fehr and Schmidt's approach of modeling "fairness" as "inequity aversion," which they define as "self-centered if people do not care per se about inequity that exists among other people but are only interested in the fairness of their own material payoff relative to the payoff of others" [31, p. 819]. This definition only simulates the intrinsically social concept of fairness but not really having it. Although adjusting one's categorical preferences in an attempt to account for the social as well as the material interests of the agent certainly changes one's preferences, *it does not change one's preference concept.* Changing the preference concept requires, as Sen suggests, a more elaborate structure that ensures that the preference concept is governed by the same rationality principle that governs the solution concept, namely, coordinated self-interest.

The mechanism used by conditional game theory to diffuse influence is *conditionalization*—the concept of defining preferences in terms of hypothetical propositions. Graph theory provides a convenient framework within which to develop this concept. An influence network is expressed as a directed graph whose vertices are agents and whose edges define the influence relationships between parents and children. If X_j influences X_i, then there is a directed edge, denoted $X_j \longrightarrow X_i$, from X_j to X_i. For each X_i, its **parent set** is $\mathrm{pa}\,(X_i) = \{X_{i_1}, \ldots X_{i_{q_i}}\}$, where $X_{i_k} \to X_i$. If $\mathrm{pa}\,(X_i) = \varnothing$ then X_i is a **root vertex**.

An element $a_{ij} \in \mathcal{A}_j$ is a **conjecture** by X_i for X_j, denoted $X_i \vDash a_{ij}$, meaning that X_i hypothesizes that X_j intends to actualize a_{ij}. The element a_{ii} is a **self-conjecture** by X_i, denoted $X_i \vDash a_{ii}$. A **conjecture profile** by X_i is the profile $\mathbf{a}_i = (a_{i1}, \ldots, a_{in}) \in \boldsymbol{\mathcal{A}}$, denoted $X_i \vDash \mathbf{a}_i$. A **joint conjecture profile** is a set of conjecture profiles for the collective, denoted $\boldsymbol{\alpha}_{1:n} = (\mathbf{a}_1, \ldots, \mathbf{a}_n) \in \boldsymbol{\mathcal{A}}^n$.

Given the parent set $\mathrm{pa}\,(X_i) = \{X_{i_1}, \ldots, X_{i_{q_i}}\}$, a **conditioning conjecture profile** by X_i asserted for X_{i_k}, denoted $X_{i_k} \vDash \mathbf{a}_{i_k}$, is a hypothesis by X_i that X_{i_k} conjectures the profile $\mathbf{a}_{i_k} = (a_{i_k 1}, \ldots, a_{i_k n})$. A **conditioning conjecture set** by X_i for $\mathrm{pa}\,(X_i)$ is the set of conditioning conjecture profiles $\boldsymbol{\alpha}_{\mathrm{pa}(i)} = (\mathbf{a}_{i_1}, \ldots, \mathbf{a}_{i_{q_i}}) \in \boldsymbol{\mathcal{A}}^{q_i}$, denoted $\mathrm{pa}\,(X_i) \vDash \boldsymbol{\alpha}_{\mathrm{pa}(i)}$.

A **conjecture hypothetical proposition**, denoted

$$H(\mathbf{a}_i|\boldsymbol{\alpha}_{\mathrm{pa}(i)})\colon\ \mathrm{pa}\,(X_i) \vDash \boldsymbol{\alpha}_{\mathrm{pa}(i)} \Rightarrow X_i \vDash \mathbf{a}_i\,, \tag{1}$$

is that, if $\mathrm{pa}\,(X_i) \vDash \boldsymbol{\alpha}_{\mathrm{pa}(i)}$ (the antecedent), then $X_i \vDash \mathbf{a}_i$ (the consequent). The conditioning symbol "|" separates the consequent on the left from the antecedent on the right. For each $\boldsymbol{\alpha}_{\mathrm{pa}(i)} \in \boldsymbol{\mathcal{A}}^{q_i}$, let

$$\mathcal{H}(\boldsymbol{\alpha}_{\mathrm{pa}(i)}) = \{H(\mathbf{a}_i|\boldsymbol{\alpha}_{\mathrm{pa}(i)}),\ \mathbf{a}_i \in \boldsymbol{\mathcal{A}}\} \tag{2}$$

denote the set of all hypothetical propositions with respect to $\boldsymbol{\alpha}_{\mathrm{pa}(i)}$ as $\mathbf{a}_i$ ranges over $\boldsymbol{\mathcal{A}}$, and let $\succsim_{i|\mathrm{pa}(i)}$ denote a linear ordering over $\mathcal{H}(\boldsymbol{\alpha}_{\mathrm{pa}(i)})$. A **conditional payoff** $u_{i|\mathrm{pa}(i)}(\cdot|\boldsymbol{\alpha}_{\mathrm{pa}(i)})\colon \boldsymbol{\mathcal{A}} \to \mathbb{R}$ is a function such that

$$H(\mathbf{a}_i|\boldsymbol{\alpha}_{\mathrm{pa}(i)}) \succsim_{i|\mathrm{pa}(i)} H(\mathbf{a}'_i|\boldsymbol{\alpha}_{\mathrm{pa}(i)}) \tag{3}$$

if, and only if,

$$u_{i|\mathrm{pa}(i)}(\mathbf{a}_i|\boldsymbol{\alpha}_{\mathrm{pa}(i)}) \geq u_{i|\mathrm{pa}(i)}(\mathbf{a}'_i|\boldsymbol{\alpha}_{\mathrm{pa}(i)})\,, \tag{4}$$

which means that, given the common antecedent $\mathrm{pa}\,(X_i) \vDash \boldsymbol{\alpha}_{\mathrm{pa}(i)}$, the consequent $X_i \vDash \mathbf{a}_i$ is either strictly preferred to the consequent $X_i \vDash \mathbf{a}'_i$ or is indifferent. If $\mathrm{pa}\,(X_i) = \varnothing$, then $u_{i|\mathrm{pa}(i)}(\mathbf{a}_i|\boldsymbol{\alpha}_{\mathrm{pa}(i)}) = u_i(\mathbf{a}_i)$, a categorical payoff.

An example of a three-agent network is

$$\begin{tikzcd} & X_1 \arrow[dl, "u_{2|1}"'] \arrow[dr] & \\ X_2 \arrow[rr, "u_{3|12}"] & & X_3 \end{tikzcd} \tag{5}$$

where X_1 is a root vertex and thus must have a categorical payoff u_1, $\mathrm{pa}\,(X_2) = \{X_1\}$ with conditional payoff $u_{2|1}$, and $\mathrm{pa}\,(X_3) = \{X_1, X_2\}$ with conditional payoff $u_{3|12}$.

Conditional payoffs enable agents to modulate their preferences in response to the social influence exerted by their parents. This mechanism allows agents to expand their individual rationality by incorporating the interests of others into their own rationality without surrendering their own intrinsic individual volition.

A **conditional game** is a triple $\{\mathbf{X}, \boldsymbol{\mathcal{A}}, \boldsymbol{\mathcal{U}}\}$, where $\mathbf{X} = \{X_1, \ldots, X_n\}$, $\boldsymbol{\mathcal{A}} = \mathcal{A}_1 \times \cdots \times \mathcal{A}_n$, and

$$\boldsymbol{\mathcal{U}} = \{u_{i|\mathrm{pa}(i)}(\cdot|\boldsymbol{\alpha}_{\mathrm{pa}(i)})\ \forall \boldsymbol{\alpha}_{\mathrm{pa}(i)} \in \boldsymbol{\mathcal{A}}^{q_i},\ i = 1, \ldots n\}\,. \tag{6}$$

A conditional game degenerates to a conventional noncooperative normal-form game when $\boldsymbol{\mathcal{U}} = \{u_i,\ i = 1, \ldots, n\}$. Thus, conditional game theory is an an extension of conventional noncooperative game theory.

IV. Socially Rational Solution Concepts

A. Coordination

The existence of social influence relationships among the members of a network invites the development of notions of group-level behavior. The concept of a "group value" or "group payoff," however, is generally eschewed by noncooperative game theory. Shubik [32] warns against the "anthropomorphic trap" of ascribing preference to a group: "It may be meaningful, in a given setting, to say that group 'chooses' or 'decides' something. It is rather less likely to be meaningful to say that the group 'wants' or 'prefers' something" (p. 124). Ascribing material value to a group is inherently problematic, since game theory requires that preferences be defined by individuals. Nevertheless, as the members of a group interrelate, a notion of shared intentionality may emerge. Bratman [33] introduces the concept of "augmented individualism," which asserts that there is no discontinuity between individual and joint intentionality: "shared intention consists primarily of interrelated attitudes (especially intentions) . . . and that the contents of the attitudes that are constitutive of basic cases of shared intention need not in general essentially involve the very idea of shared intention (though on occasion they may)" (p. 12). According to this thesis, shared intentionality is an emergent phenomenon that arises endogenously as the agents interact. In the context of social influence networks, shared intentionality is essentially a synonym for coordination.

In terms of network functionality, it is often the case that the propensity of a group to coordinate is more relevant than the propensity of the individuals to optimize. Focusing on performance without considering coordination is an incomplete characterization of group behavior. Similarly, focusing on coordination without considering performance is an incomplete characterization of individual behavior. Coordination without performance is unproductive, and performance without coordination is equivocal. A full understanding of the functionality of a group requires the assessment of both attributes.

Game theory requires that preferences be defined operationally (i.e., preference orderings), rather than merely descriptively. Coordination, however, is typically defined descriptively using rationality arguments to justify socially oriented behavior for various game scenarios (cf. [13]–[16]). Although rationality based definitions characterize the behavioral attributes of coordination, they do not explain the mechanisms that generate the behavior. In other words, they do not define the concept *operationally*. An operational definition of coordination must be expressed in mathematical terms to give precision to descriptive interpretations.

As the conditional preferences diffuse throughout the network, nascent social interrelationships are established between its members. The aggregation of these interrelationships creates a comprehensive social model of the network that incorporates all social influence. Given a conditional game $\{\mathbf{X}, \boldsymbol{\mathcal{A}}, \boldsymbol{\mathcal{U}}\}$, an **aggregation functional** as a mapping $F\colon \boldsymbol{\mathcal{U}} \to [0,1]$ that generates a **social model** $u_{1:n}\colon \boldsymbol{\mathcal{A}}^n \to [0,1]$ of the form

$$u_{1:n}(\mathbf{a}_1, \dots, \mathbf{a}_n) = F[u_{i|\mathrm{pa}(i)}(\mathbf{a}_i|\boldsymbol{\alpha}_{\mathrm{pa}(i)}),\ i = 1, \dots n]\,. \quad (7)$$

To ensure that the aggregation is well defined, it is necessary to require that pathological situations that threaten the autonomy and independence of all agents cannot occur. In particular, it is essential that no agent be **subjugated**. To define this condition, let $\mathbf{a}_{-i} = (\mathbf{a}_1, \dots, \mathbf{a}_{i-1}, \mathbf{a}_{i+1}, \dots \mathbf{a}_n) \in \boldsymbol{\mathcal{A}}^{n-1}$ denote the subset of the joint conjecture profile that excludes $\mathbf{a}_i$. If, for any $\mathbf{a} \in \boldsymbol{\mathcal{A}}$ the inequality

$$u_i(\mathbf{a}) > u_i(\mathbf{a}')\ \forall \mathbf{a}' \neq \mathbf{a} \quad (8)$$

were to hold for some X_i , then the social model would satisfy the inequality

$$u_{1:n}(\mathbf{a}, \mathbf{a}_{-i}) < u_{1:n}(\mathbf{a}', \mathbf{a}_{-i})\ \forall \mathbf{a}' \neq \mathbf{a},\ \forall \mathbf{a}_{-i} \in \boldsymbol{\mathcal{A}}^{n-1} \quad (9)$$

If X_i is subjugated, then, no matter what conjecture profile it most prefers, all joint conjecture profiless that include its most preferred conjecture profile are dominated by all other joint conjecture profiles. In other words, the fact that X_i most prefers *any* outcome distorts the social model. Notice that if the inequality in (9) were reversed, then X_i would become a **subverter**—one who unilaterally controls the ordering of the aggregated behavior of the group. Both subjugation and subversion are pathological social attributes that produce dysfunctional behavior. Thus, a minimal concept of a well defined network is that subjugation and its dual, subversion, must be impossible.

The key result of this section is that necessary and sufficient conditions to avoid subjugation are isomorphic to necessary and sufficient conditions to avoid a sure loss—a gambling scenario, termed a Dutch book, such that no matter what the outcome of the wager, the entry fee will exceed the payoff.

Theorem 1 *Subjugation is isomorphic to sure loss.*

Proof 1 *Let $\succeq_b$ and $\succeq_p$ denote linear belief and preference orderings over $\boldsymbol{\mathcal{A}}$. Since both orderings are complete, there exists a permutation π: $\boldsymbol{\mathcal{A}} \to \boldsymbol{\mathcal{A}}$ such that, for any pair $(\mathbf{a}, \mathbf{a}') \in \boldsymbol{\mathcal{A}} \times \boldsymbol{\mathcal{A}}$, $\mathbf{a} \succeq_b \mathbf{a}'$ if, and only if, $\pi(\mathbf{a}) \succeq_p \pi(\mathbf{a}')$. Since π is bijective, it is an order isomorphism.*

Let u_i denote a categorial payoff defined over the product action set $\boldsymbol{\mathcal{A}}$. By the order isomorphism, u_i may also be interpreted as a belief function. Also, the social model $u_{1:n}$ may be interpreted as a belief function defined over the product event set $\boldsymbol{\mathcal{A}}^n$. Suppose a gambler enters a lottery to win \$1 if $\mathbf{a}$ is realized. On the basis of (8), a fair entry fee is \$$p > 1/2$. Suppose, also, that on the basis of (9), the gambler enters a lottery to win \$1 if $\mathbf{a}$ is not realized, with a fair entry fee of \$$q > 1/2$. The gambler is sure to win exactly \$1 by entering both lotteries, but a sure loss results, since the total entry fee is \$$(p + q) > 1$.

The Dutch book theorem and its converse establish that a sure loss is impossible if, and only if, the gambler's beliefs are consistent with the axioms of probability theory. Thus, by the order isomorphism, subjugation is impossible if, and only if, payoffs are expressed and aggregated according to the syntax of probability theory. In accord with the term *coherence* as introduced by de Finetti [34], a group is **socially coherent** if subjugation (and subversion) is impossible.

Since payoffs are unique up to positive affine transformations, it may be assumed without loss of generality that all payoffs are nonnegative and sum to unity and thus become mass functions with the same syntax as probability mass functions. Consequently, an acyclical conditional game network is isomorphic to a Bayesian network.[1] The key difference between the two interpretations is that, with a Bayesian network, vertices are random variables with beliefs governed by probability mass functions, whereas the vertices of a condition game network are volitional agents with preferences governed by conditional payoff mass functions.

The isomorphic relationship between a conditional game network and a Bayesian network makes it possible to apply a fundamental theorem of Bayesian networks, the *Markov condition*: nondescendent nonparents of a vertex have no influence on the vertex, given the state of its parent vertices [35]. Consequently, the joint probability mass function of all vertices is uniquely given by the product of all conditional

[1]To be precise, the vertices of a conditional game network are isomorphic to random vectors, but Bayesian network theory easily extends to that case.

and categorical probability mass functions [36]–[38]. By the isomorphism, the unique socially coherent social model is the product of the categorical and conditional payoffs. This establishes the following result.

Theorem 2 *For an acyclical conditional game to be socially coherent, the social model must be of the form*

$$u_{1:n}(\mathbf{a}_1, \ldots, \mathbf{a}_n) = \prod_{i=1}^{n} u_{i|\mathrm{pa}(i)}(\mathbf{a}_i|\boldsymbol{\alpha}_{\mathrm{pa}(i)})\,. \tag{10}$$

The social model is isomorphic to a joint probability mass function of a set of random variables. Analogous to the way that a joint probability mass function serves as a comprehensive model of all statistical relationships among random variables, the social model serves as a comprehensive model of all social interrelationships among agents.

Although the social model may conceivably be used to define rational behavior from a *group* perspective, it is not immediately obvious how it may be used to define rational behavior from an *individual* perspective. It does, however, provide a social model from which individually rational solution concepts may be conceived.

B. Marginalization

Since the mathematical structure of the social model is analogous to the joint probability mass function of a collective of random vectors, probability-based operations may be applied. Of particular importance is marginalization. The **ex post payoff** for X_i is the i-th marginal of the social model, that is,

$$v_i(\mathbf{a}_i) = \sum_{-\mathbf{a}_i} u_{1:n}(\mathbf{a}_1, \ldots, \mathbf{a}_n)\,, \tag{11}$$

where the notation $\sum_{-\mathbf{a}_i}$ means that the summation is taken over all elements except $\mathbf{a}_i$. The ex post payoffs given by (11) define each individual's emergent preferences over the outcomes after consideration of social influence. Since these payoffs are unconditional, they may be juxtaposed in a payoff array and conventional noncooperative game-theoretic solution concepts, such as Nash equilibria, may be applied.

A profile $\mathbf{a} = (a_1, \ldots, a_n) \in \boldsymbol{\mathcal{A}}$ is an **ex post Nash equilibrium** if, for $\mathbf{a}' = (a_1, \ldots, a_i', \ldots, a_n)$ (that is, $\mathbf{a}'$ differs from $\mathbf{a}$ in the ith position only), then $v_i(\mathbf{a}) \geq v_i(\mathbf{a}')$ for all $a_i' \neq a_i$ and for all $i = 1, \ldots, n$. Even though the ex post marginals take social influence into account, a Nash equilibrium is a solution based on narrow self-interest, and thus generates an incongruity, since the preference concept is based on social influence. If this were the end of the story, then conditional game theory would serve as a pre-game preference-processing mechanism that would culminate in the application of classical game theory using the ex post utilities. Fortunately, however, there is more to be said.

C. Coordinatability

The ordering provided by the social model is with respect to joint conjecture profiles $\boldsymbol{\alpha}_{1:n} = (\mathbf{a}_1, \ldots, \mathbf{a}_n)$, with each conjecture profile of the form $\mathbf{a}_i = (a_{i1}, \ldots, a_{in})$, where $a_{ii} \in \mathcal{A}_i$ is a conjecture by X_i and $a_{ij} \in \mathcal{A}_j$ is a conjecture for X_j by X_i. What is most relevant to coordination, however, is how the self-conjectures a_{ii}, $i = 1, \ldots, n$ fit together to generate a coordinated outcome. Since each X_i has direct control over only a_{ii}, its own self-conjecture, it is necessary to develop an expression that accounts for the way these self-conjectures combine to form a notion of coordination.

Given a joint conjecture profile $\boldsymbol{\alpha}_{1:n} = (\mathbf{a}_1, \ldots, \mathbf{a}_n)$, the **coordination conjecture profile**, denoted $\mathbf{a} = (a_{11}, \ldots, a_{nn})$, comprises the i-th element of each X_i's conjecture profile, $i = 1, \ldots, n$. The **coordination function**, denoted $w_{1:n}$, for $\{X_1, \ldots, X_n\}$, is obtained by summing the social model over all elements of each $\mathbf{a}_i$ except the i-th elements, yielding

$$\begin{aligned} &w_{1:n}(a_{11}, \ldots, a_{nn}) \\ &= \sum_{-a_{11}} \cdots \sum_{-a_{nn}} u_{1:n}[(a_{11}, \ldots, a_{1n}), \ldots, (a_{n1} \ldots, a_{nn})]\,. \end{aligned} \tag{12}$$

The coordination function is a mass function that assignes a portion of its mass to each of the possible coordination conjectures as a measure of the joint intentionality of the group. Unlike a group payoff, which would be a measure of material benefit to the group, the coordination function is a measure of group sociality that may or may not correspond to material benefit.

The **individual coordinated payoff** for X_i is the i-th marginal of $w_{1:n}$, that is,

$$w_i(a_{ii}) = \sum_{-a_{ii}} w_{1:n}(a_{11}, \ldots, a_{1n})\,. \tag{13}$$

As an example, consider the conditional game $\{\{X_1, X_2\}, \mathcal{A}_1 \times \mathcal{A}_2, \{u_{X_1}, u_{X_2|X_1}\}\}$ with network

$$X_1 \xrightarrow[u_{2|1}]{} X_2\,. \tag{14}$$

The social model is

$$\begin{aligned} &u_{12}[(a_{11}, a_{12}), (a_{21}, a_{22})] = \\ &\qquad u_1(a_{11}, a_{12}) u_{2|1}(a_{21}, a_{22}|a_{11}, a_{12})\,, \end{aligned} \tag{15}$$

yielding the coordination function

$$w_{12}(a_{11}, a_{22}) = \sum_{a_{12}a_{21}} u_{12}[(a_{11}, a_{12}), (a_{21}, a_{22})]\,, \tag{16}$$

and the coordinated individual payoffs are

$$w_i(a_{ii}) = \sum_{-a_{ii}} w_{12}(a_{11}, a_{22}),\ i = 1, 2\,. \tag{17}$$

Coordination is a principle of behavior on a parallel with, but different from, performance. Performance deals with the efficiency and effectiveness of individual behavior in terms of material payoffs. Coordination, on the other hand, is an organizational property that characterizes the way the individuals fit together to form a coherent network. To put it succinctly, *individuals perform; groups coordinate*.

It is important to stress that w_i serves as the operational measure of individual behavior in terms of performance, and $w_{1:n}$ serves as the operational measure of group behavior in terms of coordination. It may seem superfluous to offer different interpretations of these two measures. If payoffs were aggregated summatively (e.g., utilitarianism), then a notion of performance for the group may be meaningful—at least the payoffs would all be expressed in the same units. But the concept of aggregation used by conditional game theory is multiplicative, which raises the prospect of creating a concept of value different from the individual concepts. To illustrate, consider the network (14), and suppose X_1 represents electrical current and X_2 represents voltage. The product of current and voltage is electric power—an emergent phenomenon that is inherently "social" in the abstract sense in that it involves interrelationships between two distinct concepts. In a very real operational sense, electric power is the aggregation of current and voltage that produces a new phenomenon that could not be anticipated before their aggregation.

Coordination is an *emergent* property of a network. The players do not come to the social encounter with ex ante intentions of group-level behavior. Rather, such group-level intentions emerge endogenously as their conditional preferences combine and thereby define a nascent notion of coordination. Coordination may be positive if the social influence is such that the agents engage in cooperation (e.g., teams). Coordination may be negative if the social influence is such that the agents engage in conflict (e.g., athletic contests and military engagements)

V. Reciprocal Influence

Thus far, considerations of social influence have been unidirectional—from parent to child. With this model, the child is able to modulate its preferences according to the hypothesized intentions of the parent, but there is no reciprocal way for the child to influence the parent. In other words there is no opportunity for dialogue—a dynamic process whereby individuals may work together with the hope of converging to a result that is rational, fair, and legitimate. The assumption of noncooperative game theory is that the agents do not communicate. However, as argued by Misyak, et al., [39], the agents may certainly engage in *virtual* communication by independently reasoning how the other might be reasoning. Such scenarios are examples of Schelling's concept of *focal points*, also termed *salience*. "People *can* often concert their intentions or expectations with others if each knows that the other is trying to do the same. Many situations—perhaps every situation for people who are practiced at this kind of game—provide some clue for coordinating behavior, some focal point for each person's expectations of what the other expect him to expect to be expected to do" [13, p. 57].

Consider the two-agent scenario

$$\begin{tikzcd} X_1 \arrow[r, bend left, "u_{2|1}"] & X_2 \arrow[l, bend left, "u_{1|2}"] \end{tikzcd}, \tag{18}$$

where X_1 influences X_2 who in turn influences X_1 and so forth. Such a cyclic influence relationship could result in an infinite regression with no ultimate resolution. There is a natural and powerful way, however, to resolve this problem. The key observation is to recognize that a cyclic network can be expressed dynamically as a time sequence of acyclic networks. From this perspective, (18) may be viewed as

$$X_1 \xrightarrow[u_{2|1},\ s=0]{} X_2 \tag{19}$$

at time $s = 0$, followed by

$$X_2 \xrightarrow[u_{1|2},\ s=\delta]{} X_1 \tag{20}$$

at time $s = \delta$, followed by

$$X_1 \xrightarrow[u_{2|1},\ s=2\delta]{} X_2 \tag{21}$$

for $s = 2\delta$, and so on.

Let $v_1(\mathbf{a}_1, 0)$ denote X_1's marginal payoff mass function at time $s = 0$. At $s = \delta$, the social model is, following (10),

$$u_{12}(\mathbf{a}_1, \mathbf{a}_2, \delta) = u_{2|1}(\mathbf{a}_2|\mathbf{a}_1)v_1\mathbf{a}_1, 0)\,, \tag{22}$$

and X_2's marginal is computed using (11):

$$v_2(\mathbf{a}_2, \delta) = \sum_{\mathbf{a}_1} u_{12}(\mathbf{a}_1, \mathbf{a}_2, \delta)\,. \tag{23}$$

At $s = 2\delta$ the social model is

$$u_{21}(\mathbf{a}_2, \mathbf{a}_1, 2\delta) = u_{1|2}(\mathbf{a}_1|\mathbf{a}_2)v_2(\mathbf{a}_2, \delta)\,, \tag{24}$$

and X_1's marginal is

$$v_1(\mathbf{a}_1, 2\delta) = \sum_{\mathbf{a}_2} u_{21}(\mathbf{a}_2, \mathbf{a}_1, 2\delta)\,. \tag{25}$$

This process may be continued for $s = 3\delta$, $s = 4\delta$, etc. Given this dynamic structure, the issue devolves around whether this time sequence of updates converges. To develop this theory, it is convenient to introduce notation that is amenable to the dynamic nature of this representation.

A. Matrix Form Dynamics Model

Let

$$\boldsymbol{\mathcal{A}} = \mathcal{A}_1 \times \cdots \times \mathcal{A}_n = \{\mathbf{x}_1, \ldots, \mathbf{x}_N\}\,, \tag{26}$$

with $N = \prod_{i=1}^{n} N_i$, where each profile $\mathbf{x}_k$ is of the form

$$\mathbf{x}_k = (x_{1k_1}, \ldots, x_{nk_n})\,, \tag{27}$$

where x_{ik_i} is the k_ith element of $\mathcal{A}_i$, and define the **payoff mass vector**

$$\mathbf{v}_i(s) = \begin{bmatrix} v_i(\mathbf{x}_1, s) \\ \vdots \\ v_i(\mathbf{x}_N, s) \end{bmatrix}. \tag{28}$$

Next, define the **agent-to-agent transition matrix**

$$T_{i+1|i} = \begin{bmatrix} u_{i+1|i}(\mathbf{x}_1|\mathbf{x}_1) & \cdots & u_{i+1|i}(\mathbf{x}_1|\mathbf{x}_N) \\ \vdots & \vdots & \vdots \\ u_{i+1|i}(\mathbf{x}_N|\mathbf{x}_1) & \cdots & u_{i+1|i}(\mathbf{x}_N|\mathbf{x}_N) \end{bmatrix} . \tag{29}$$

This notation enables combining the operations defined by (22) and (23) into the single expression

$$\mathbf{v}_2(\delta) = T_{2|1}\mathbf{v}_1(0) , \tag{30}$$

and replacing (24) and (25) with

$$\mathbf{v}_1(2\delta) = T_{1|2}\mathbf{v}_2(\delta) . \tag{31}$$

More generally, A **path** of length k from vertex X_{i_1} to vertex X_{i_k}, denoted $X_{i+1} \mapsto X_{i_k}$, is a sequence $\{X_{i_1}, \ldots, X_{i_k}\}$ of distinct vertices such that an edge exists between X_{i_j} and $X_{i_{j+1}}$, $j = 1 \ldots, k-1$. A k cycle is a path $X_j = X_{k_1} \mapsto X_{i_k} = X_j$.

Tracing the path from X_i around the cycle back to X_i updates the marginal mass vector $\mathbf{v}_i$ k times, where the indices are incremented $\bmod\ k$:

$$\begin{aligned} \mathbf{v}_{i+1}(\delta) &= T_{i+1|i}\mathbf{v}_i(0) \\ \mathbf{v}_{i+2}(2\delta) &= T_{i+2|i+1}T_{i+1|i}\mathbf{v}_i(0) \\ &\ \vdots \\ \mathbf{v}_{i+k-1}[(k-1)\delta] &= T_{i+k-1|i+k-2}\cdots T_{i+2|i+1}T_{i+1|i}\mathbf{v}_i(0) . \end{aligned}$$

The loop is closed with the final update of the cycle, yielding

$$\mathbf{v}_{i+k}(k\delta) = T_{i+k|i+k-1}T_{i+k-1|i+k-2}\cdots T_{i+2|i+1}T_{i+1|i}\mathbf{v}_i(0) \tag{32}$$

or, since all indices are incremented $\bmod\ k$,

$$\mathbf{v}_i(k\delta) = T_{i|i+k-1}T_{i+k-1|i+k-2}\cdots T_{i+2|i+1}T_{i+1|i}\mathbf{v}_i(0) . \tag{33}$$

Now define the **closed-loop transition matrix**

$$T_i = T_{i|i+k-1}T_{i+k-1|i+k-2}\cdots T_{i+2|i+1}T_{i+1|i} . \tag{34}$$

Also, it is convenient to express time in units equal to the interval $k\delta$. Thus, (33) becomes

$$\mathbf{v}_i(1) = T_i\mathbf{v}_i(0) \tag{35}$$

for $i = 1, \ldots, k$. The closed-loop transition matrices for the cycle are as follows.

$$\begin{aligned} T_1 &= T_{1|k}T_{k|k-1}\cdots T_{3|2}T_{2|1} \\ T_2 &= T_{2|1}T_{1|k}\cdots T_{4|3}T_{3|2} \\ &\ \vdots \\ T_k &= T_{k|k-1}T_{k-1|k-2}\cdots T_{3|2}T_{2|1}T_{1|k} . \end{aligned} \tag{36}$$

After t cycles,

$$\begin{aligned} \mathbf{v}_i(t) &= T_i\mathbf{v}_i(t-1) \\ &= T_iT_i\mathbf{v}_i(t-2) \\ &\ \vdots \\ &= T_i\cdots T_i\mathbf{v}_i(0) \\ &= T_i^t\mathbf{v}_i(0) . \end{aligned} \tag{37}$$

The key issue devolves around the convergence properties of T_i^t as $t \to \infty$.

B. Convergence of Closed-Loop Transition Matrices

Theorem 3 (Markov Convergence) *Let T be a square matrix with nonnegative entries such that each column sums to unity and there exists an integer m such that all elements of T^m are strictly positive. Then T possesses a unity eigenvalue of algebraic and geometric multiplicity one such that the corresponding eigenvector $\overline{\mathbf{v}}$ of T satisfies*

- $T\overline{\mathbf{v}} = \overline{\mathbf{v}}$
- $\overline{T} = \lim_{t\to\infty} T^t = \begin{bmatrix}\overline{\mathbf{v}} & \cdots & \overline{\mathbf{v}}\end{bmatrix}$
- $\overline{\mathbf{v}} = \overline{T}\mathbf{v}(0)$ *for every initial mass vector* $\mathbf{v}(0)$

For a proof, see Luenberger [40].

Thus, a group whose dynamic behavior is governed by transition matrices that meet the conditions of Theorem 3 will converge to a group where the agents possess constant marginal payoff mass functions, that is,

$$\lim_{t\to\infty} \mathbf{v}_i(t) = \overline{\mathbf{v}}_i = \begin{bmatrix} \overline{v}_i(x_{i1}) \\ \vdots \\ \overline{v}_i(x_{iN}) \end{bmatrix} . \tag{38}$$

This valuaton vector, termed the **steady-state payoff**, is the eigenvector corresponding to the unique unity eigenvalue of T.

VI. Coordinated Interactive Decisions on Communication Networks

Communication networks are essential for telecommunications, computer networks, and human organizational networks. A network topology that frequently arises is a linear chain of the form

$$X_1 \rightleftarrows X_2 \rightleftarrows X_3 , \tag{39}$$

where X_1 communicates with X_2, who also communicates with X_3, but X_1 and X_3 do not communicate with each other. The influence that X_2 exerts on X_1 and X_3 is expressed by the transition relations

$$\begin{aligned} \mathbf{v}_1 &= T_{1|2}\mathbf{v}_2 \\ \mathbf{v}_3 &= T_{3|2}\mathbf{v}_2 , \end{aligned} \tag{40}$$

where the agent-to-agent transforms $T_{1|2}$ and $T_{3|2}$ are defined by the conditional utilities $u_{1|2}$ and $u_{3|2}$ according to (29).

Since X_2 is influenced by both X_1 and X_3, the influence from both of those entities must be taken into consideration via a transform of the form

$$\mathbf{v}_2 = T_{2|13}\mathbf{v}_{13}\,. \tag{41}$$

It remains, however, to develop an expresion for $\mathbf{v}_{13}$. Considering only the linkage

$$X_1 \xrightarrow{u_{2|1}} X_2 \xleftarrow[u_{2|3}]{} X_3\,, \tag{42}$$

let u_{123} denote the corresponding social model, which, following (10), may be expressed as

$$u_{123}(\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3) = u_{13|2}(\mathbf{a}_1, \mathbf{a}_3|\mathbf{a}_2)u_2(\mathbf{a}_2)\,. \tag{43}$$

Since there are no direct linkages between X_1 and X_3, X_1 and X_2 are socially independent, given X_2; thus, $u_{13|2}(\mathbf{a}_1, \mathbf{a}_3|\mathbf{a}_2) = u_{1|2}(\mathbf{a}_1|\mathbf{a}_2)u_{3|2}(\mathbf{a}_3|\mathbf{a}_2)$.

$$\begin{aligned} v_{13}(\mathbf{a}_1, \mathbf{a}_3) &= \sum_{\mathbf{a}_2} u_{123}(\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3) \\ &= \sum_{\mathbf{a}_2} u_{1|2}(\mathbf{a}_1|\mathbf{a}_2)u_{3|2}(\mathbf{a}_3|\mathbf{a}_2)v_2(\mathbf{a}_2) \end{aligned} \tag{44}$$

or, in matrix form,

$$\mathbf{v}_{13} = T_{13|2}\mathbf{v}_2\,. \tag{45}$$

Thus,

$$\begin{aligned} \mathbf{v}_2 &= T_{2|13}\mathbf{v}_{13} \\ \mathbf{v}_{13|2} &= T_{13|2}\mathbf{v}_2 \end{aligned}\,, \tag{46}$$

and the loop is closed by

$$\mathbf{v}_2 = T_{2|13}\mathbf{v}_{13} = T_{2|13}T_{13|2}\mathbf{v}_2 = T_2\mathbf{v}_2\,, \tag{47}$$

where $T_2 = T_{2|13}T_{13|2}$.

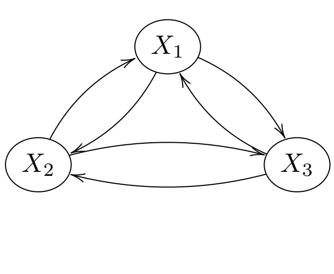
In steady state, X_2's payoff converges to $\overline{\mathbf{v}}_2$, and the steady-state utilities for X_1 and X_3 become

$$\begin{aligned} \overline{\mathbf{v}}_1 &= T_{1|2}\overline{\mathbf{v}}_2 \\ \overline{\mathbf{v}}_3 &= T_{3|2}\overline{\mathbf{v}}_2\,. \end{aligned} \tag{48}$$

The resulting steady-state network may be expressed as

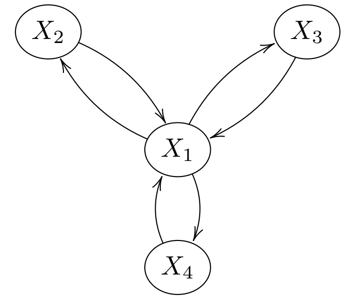
$$X_1 \xleftarrow[T_{1|2}]{} X_2 \xrightarrow[T_{3|2}]{} X_3 \tag{49}$$

In addition to linear chains, other useful topologies are ring networks of the form



(50)

and star networks of the form



, (51)

both of which admit a similar analysis.

VII. Conclusion

A well-formed game must be rationally congruent in that the preference and solution concepts are motivated by the same principle of rationality. Congruence is violated, however, if the solution concept explicitly accounts for social as well as individual considerations, but the preference concept account only for individual categorical preferences. Conditional game theory resolves this issue by providing a mechanism to expand rationality beyond narrow self-interest to accommodate the interests of others as well as the self.

The isomorphic relationship between subjugation and sure loss ensures that social coherence can be maintained (via the Dutch book theorem) by requiring conditional utilities to conform to the syntax of probability theory, thereby rendering subjugation to be impossible.

This approach establishes parallel measures of individual and group behavior by expressing individual behavior in terms of performance via utilities and group behavior in terms of coordination value. Measures of both individual performance and group coordination are necessary for a complete model of a decision-making influence network.

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