Investigation of the moment of inertia

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1 Introduction

The purpose of this experiment was to explore how the moment of inertia depends on the mass distribution of a rotating object. Specifically, it involved determining the moment of inertia and verifying Steiner's theorem using a torsion pendulum. This was achieved by measuring the moments of inertia of a metal rod with movable weights and a circular disk with several possible mounting positions.

Theoretical Background

For a single point mass with mass m rotating around an axis at a distance r, we can obtain the following relationship for the moment of inertia:

$$I = mr^2. \tag{1}$$

Steiner's theorem states that the moment of inertia (I) of a rigid body about any axis is equal to the moment of inertia about a parallel axis through the center of mass (I_0) plus the product of the body's total mass (m) and the square of the distance (d) between the two parallel axes.

$$I = I_0 + md^2, \tag{2}$$

For standard bodies, the theoretical moments of inertia are as follows:

$$I_{\text{rod}} = \frac{1}{12}ml^2, \tag{3}$$

$$I_{\text{disk}} = \frac{1}{2}MR^2, \tag{4}$$

where m is the mass of the rod, l its length, M is the mass of the disk, and R its radius.

2 Apparatus and Measurement System

A torsion pendulum apparatus was used to measure the rotational inertia, or moment of inertia, of different objects. The core of the setup included a fixed base, a vertical shaft with a coiled spring providing the restoring torque, and a platform for securing the objects to be tested.

As the pendulum swings, it moves with simple harmonic motion. The time it takes to complete one full oscillation (its period) is governed by the spring's torsional stiffness and the object's moment of inertia. To ensure accurate readings, the mechanism used low-friction bearings to minimize resistance. For each measurement, the duration of five complete oscillations was recorded using a stopwatch, with the first oscillation ignored to account for any initial instability.

1. a steel rod with adjustable weights mounted symmetrically at various distances from the rotation axis,
2. a circular disk with multiple mounting positions used to verify Steiner's theorem.

Element	**Mass [g]**	**Dimensions [mm]**
Steel rod	136	Length: 620
Weight (red)	260	—
Weight (blue)	260	—
Circular plate	436	Diameter: 320 (*Radius*: 160)

Measurements of rod with weights

d [cm]	$5T_1$ [s]	$5T_2$ [s]	T_{avg}	$T[s]$
0 (without weights)	12.73	12.54	12.64	2.53
5	14.36	14.27	14.32	2.86
10	18.36	18.17	18.27	3.65
15	23.87	23.64	23.76	4.72
20	29.80	29.81	29.81	5.96
25	36.95	36.74	36.85	7.37
30	42.34	42.55	42.45	8.49

Measurement of disk

d [cm]	$5T_1$ **[s]**	$5T_2$ **[s]**	$5T_3$ **[s]**	$5T_4$ **[s]**
0	14.23	14.17	14.13	14.15
2	14.13	14.30	14.14	14.63
4	14.66	14.56		
6	15.64	15.74		
8	16.64	16.58		
10	18.83	18.68		
12	19.74	19.63		
14	22.27	22.17		

3 Calculations and data processing

Processing rod data

Firstly, we need to process the data for further calculations, we will need T^2 and the same with distance of weight to the center of mass (r^2).

d [cm]	T^2 [s^2]	r [m]	r^2 [m^2]
0	6.391	0.000	0.00000
5	8.180	0.05	0.0025
10	13.323	0.10	0.0100
15	22.278	0.15	0.0225
20	35.522	0.20	0.0400
25	54.317	0.25	0.0625
30	72.080	0.30	0.0900

Table 1: Completed data table based on measurements.

Determination of restoring torque of spring

To determine restoring torque we need to use formula:

$$I = D\left(\frac{T}{2\pi}\right)^2 \tag{5}$$

$$I = I_B + m_w r^2 \tag{6}$$

Where I_B is our moment of inertia of rod and m_w mass of our weights, then after substituting to our formula (Figure 6.) we have:

$$D\left(\frac{T}{2\pi}\right)^2 = 2m_w r^2 + D\left(\frac{T_B}{2\pi}\right)^2 \tag{7}$$

after simplification we have got:

$$T^2 = \frac{8\pi^2 m_w}{D} r^2 + T_B \tag{8}$$

Now, we need to transform our above formula to the form of $y = ax + b$ (linear function) which we can you by substituting $y = T^2$, $x = r^2$, $a = \frac{8\pi^2 m_w}{D}$ and finally $b = T_B$. To find the D we need to calculate a which we can do by applying linear regression to relationship between T^2 and r^2 and put the values to the following equation:

$$D = \frac{8\pi^2 m_w}{a_{reg}} \tag{9}$$

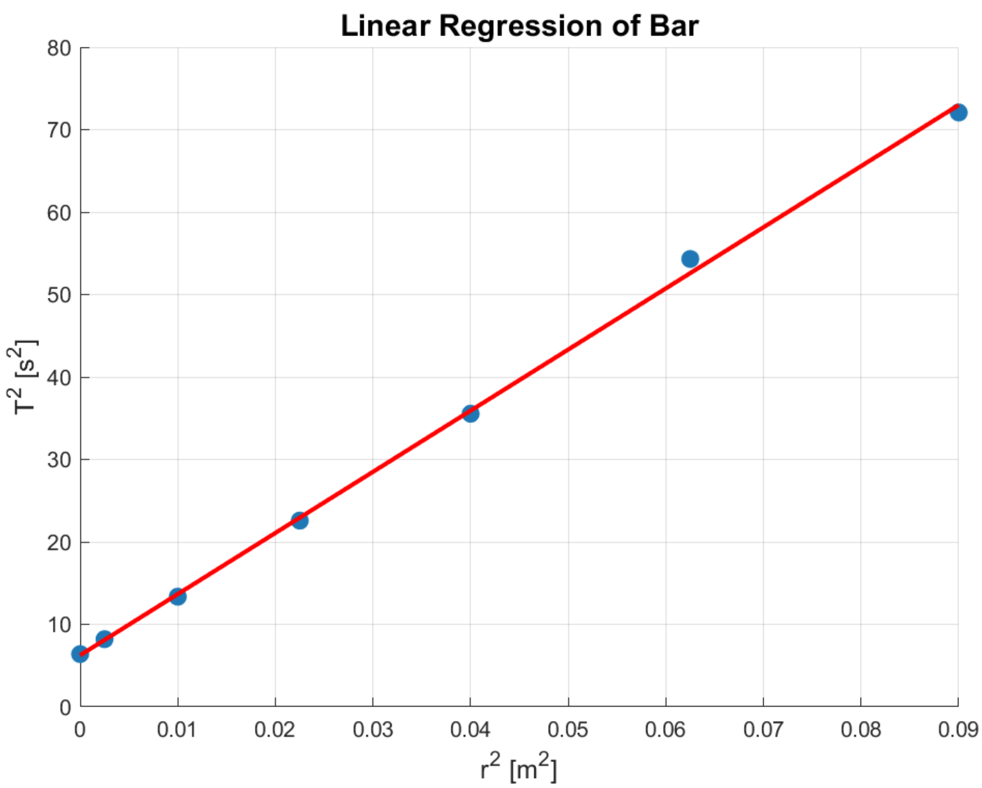

Figure 1: Relation between $\mathrm{T}^2 and r^2$

From our linear regression we obtain:

$$a = 741.49 s^2 m^{-2},\ b = 6.25 s^2$$

After substituting the values we have:

$$D = \frac{8\pi^2 \times 0.260}{741.49} = 0.027869[N \cdot m \cdot rad^{-1}]$$

Finally we have:

$$D = (2.8 \times 10^{-2}) N \cdot m \cdot rad^{-1}$$

Processing disk data

For our disk the table will look like this

d [cm]	$5T_{avg}$	T
0	14.17	2.834
2	14.30	2.860
4	14.61	2.922
6	15.69	3.138
8	16.61	3.322
10	18.76	3.752
12	19.69	3.938
14	22.22	4.444

Table 2: Average time of 5 oscillations (5T) and calculated period for one oscillations (T)

Moments of inertia for the rod and disk

At the moment constants we have are:

$$D = 0.028 N\ m\ rad^{-1},\ m_r = 0.136 kg,\ l_r = 0.62 m,\ m_d = 0.436 kg,\ r = 0.160 m$$

We use equations for theoretical and experimental torsion pendulum which are:

$$I_{exp} = D \left(\frac{T}{2\pi}\right)^2$$

$$I_{rod,theor} = \frac{1}{12} m_r l_r^2,\ I_{disk,theor} = \frac{1}{2} m_d r^2$$

Calculations for rod (for $d = 0$)

$$I_{exp,rod} = 0.028 \left(\frac{2.834}{2\pi}\right)^2 = 4.5398 \times 10^{-3} kg \cdot m^2$$

$$I_{theor,rod} = \frac{1}{12} \times 0.136 \times 0.620^2 = 4.3565 \times 10^{-3} kg\ m^2$$

Calculations for disk (for $d = 0$)

$$I_{exp,disk} = 0.028 \left(\frac{2.834}{2\pi}\right)^2 = 5.6964 \times 10^{-3} kg\ m^2$$

$$I_{theor,disk} = \frac{1}{2} \times 0.436 \times 0.160^2 = 5.5580 \times 10^{-3} kg\ m^2$$

Object	$T[s]$	$I_{exp}[kgm^2]$	$I_{theor}[kgm^2]$
Steel rod	2.530	4.5398×10^{-3}	4.3565×10^{-3}
Disk	2.834	5.6964×10^{-3}	5.5580×10^{-3}

Table 3: Values for theoretical and experimental moment of inertia over symmetry axis

Comparison between experimental and theoretical moments of inertia

The difference between experimental and theoretical values can be expressed as:

$$\Delta I[\%] = 100 \times \frac{|I_{exp} - I_{theor}|}{I_{theor}} \tag{10}$$

If we substitute values for the rod we have

$$\Delta I_{rod}[\%] = 100 \times \frac{|4.5398 \times 10^{-3} - 4.3565 \times 10^{-3}|}{4.3565 \times 10^{-3}} = 4.21\%$$

and for the disk:

$$\Delta I_{disk}[\%] = 100 \times \frac{|5.6964 \times 10^{-3} - 5.5580 \times 10^{-3}|}{5.5580 \times 10^{-3}} = 2.49\%.$$

Object	$I_{exp}[kgm^2]$	$I_{theor}[kgm^2]$	$\Delta I[\%]$
Steel rod	4.5398×10^{-3}	4.3565×10^{-3}	4.21
Disk	5.6964×10^{-3}	5.5580×10^{-3}	2.49

Table 4: Difference between experimental and theoretical moments of inertia

Experimental confirmation of Steiner's Theorem

To check Steiner's theorem, a disk was attached to a torsion pendulum at several different points. The distance (d) between the disk's center and the pendulum's rotation axis was varied from 0 cm to 14 cm in 2 cm steps (0, 2, 4,... 14 cm). For each distance d, the pendulum's oscillation period (T) was measured. This measurement allowed the experimental moment of inertia (I_{exp}) to be calculated using the formula

$$I_{\text{exp}} = D \left(\frac{T}{2\pi}\right)^2.$$

This experimental result was then compared to the theoretical value from Steiner's theorem, which is given by

$$I = \frac{1}{2} m_d r^2 + m d^2. \tag{11}$$

This equation can also be written as

$$I = I_0 + m_d d^2, \ I_0 = \frac{1}{2} m_d r^2,$$

where m_d is the disk's mass, R is its radius, d is the offset distance, and is the moment of inertia about the disk's own center.

Our values are: $D = 0.028 N \cdot m \cdot rad^{-1}$, $m_d = 0.436 kg$, $r = 0.160 m$ and we take in example $d = 0.02m$

So after substituting our values for theoretical formula (Figure 11) we have:

$$I_0 = \frac{1}{2} \times 0.436 \times 0.160^2 = 5.5580 \times 10^{-3} kgm^2,$$

$$I_{theor} = 5.5580 \times 10^{-3} + 0.436 \times 0.02^2 = 5.7552 \times 10^{-3} kgm^2$$

and after experimental (Figure 5):

$$I_{exp} = 0.028 \left(\frac{2.860}{2\pi} \right)^2 = 5.8014 \times 10^{-3} kgm^2$$

And the difference is:

$$\Delta I[\%] = 100 \times \frac{|I_{exp} - I_{theor}|}{I_{theor}} = 100 \times \frac{|5.8014 \times 10^{-3} - 5.7552 \times 10^{-3}|}{5.7552 \times 10^{-3}} = 1.20\%$$

d [m]	d^2 [m]	T [s]	I_{theor} (10^{-3} kg·m^2)	I_{exp} (10^{-3} kg·m^2)	ΔI (%)
0.00	0.00	2.834	5.5580	5.6960	2.48
0.02	0.0004	2.860	5.7324	5.8014	1.20
0.04	0.0016	2.922	6.2556	6.0544	3.22
0.06	0.0036	3.138	7.1276	6.9836	2.02
0.08	0.0064	3.322	8.3484	7.8272	6.24
0.10	0.0100	3.752	9.9180	9.9829	0.65
0.12	0.0144	3.938	11.8364	10.9986	7.08
0.14	0.0196	4.444	14.1036	14.0039	0.71

Table 5: experimental and theoretical moments of inertia

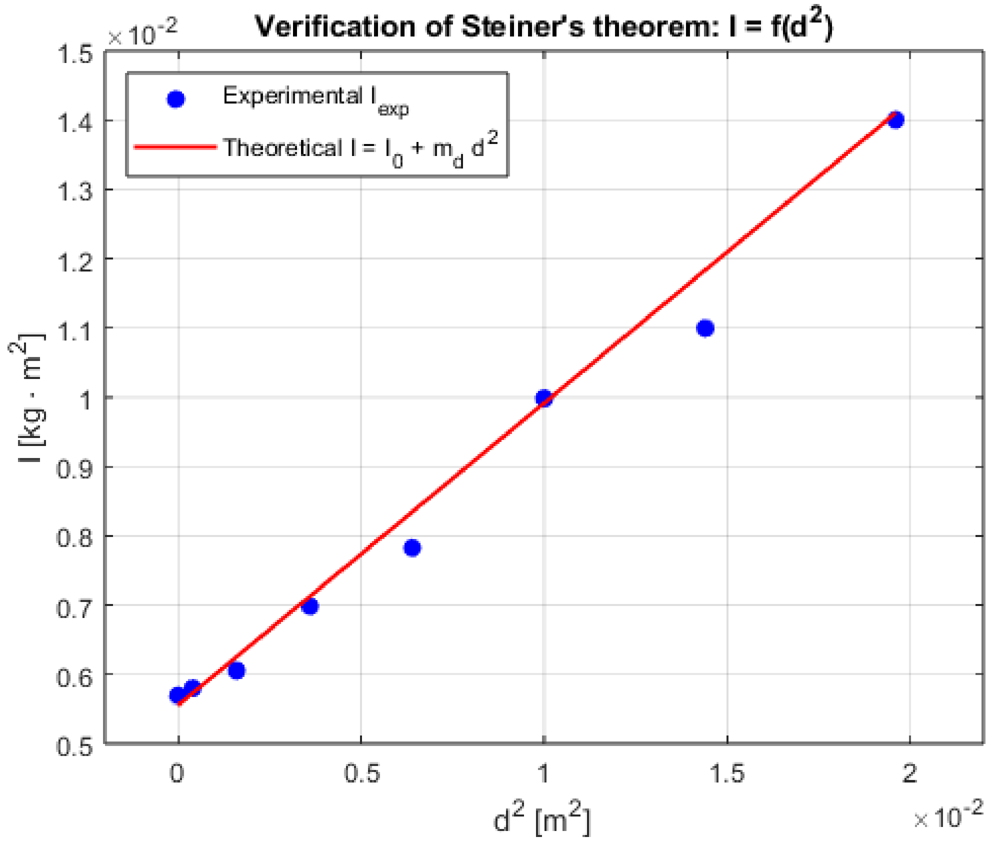

Figure 2: Relation between T^2 and r^2

4 Error estimation

Uncertainty of the torsional constant D from the regression equation:

$$T^2 = a_{reg}r^2 + b$$

and the slope coefficient is:

$$a_{reg} = 741.49s^2m^{-2}$$

Determination of the uncertainty of the slope coefficient a_{reg}

The uncertainty of a_{reg} can be determined using the least squares method:

$$a_{reg} = \frac{\sum(x_i - \overline{x})(y_i - \overline{y})}{\sum(x_i - \overline{x})^2}), b = \overline{y} - a_{reg}\overline{x}, \tag{12}$$

$$u(a_{reg}) = \sqrt{\frac{1}{n-2} \cdot \frac{\sum(y_i - a_{reg}x_i - b)^2}{\sum(x_i - \overline{x})^2}}$$

for our data ($x_i = r_i^2$, $y_i = T_i^2$):

$$x_i = \{0.00000, 0.0025, 0.0100, 0.0225, 0.0400, 0.0625, 0.0900\}[m^2],$$

$$y_i = \{6.391, 8.180, 13.323, 22.278, 35.522, 54.317, 72.080\}[s].$$

After substitution:

$$\overline{x} = 0.325,\ \overline{y} = 30.2987,\ S_{xx} = 0.006825$$

Our parameters for linear regression were:

$$a_{reg} = 741.49 s^2 m^{-2},\ b = 6.25 s^2.$$

Residual sum of squares:

$$RSS = \sum(y_i - a_{reg}x_i - b)^2 = 4.5117$$

From this we can tell:

$$s^2 = \frac{RSS}{n-2} = \frac{4.5117}{5} = 0.902344,$$

so the uncertinity is:

$$u(a_{reg}) = \sqrt{\frac{s^2}{S_{xx}}} = \sqrt{\frac{0.902344}{0.006825}} = 11.50 s^2 m^{-2}$$

Final result is:

$$a_{reg} = (741.5 \pm 11.5) s^2 m^{-2}$$

Uncertainty of the torsional constant D

From

$$D = \frac{8\pi^2 m_w}{a_{reg}} \tag{13}$$

So the uncertainty formula is:

$$\frac{u(D)}{D} = \sqrt{\left(\frac{u(a_{reg})}{a_{reg}}\right)^2 + \left(\frac{u(m_w)}{m_w}\right)^2}.$$

So the components are:

$$\frac{u(a_{reg})}{a_{reg}} = \frac{11.50}{741.5} = 0.01551,\ \frac{u(m_w)}{m_w} = \frac{0.0003}{0.260} = 0.001154.$$

Relative uncertainty is:

$$\frac{u(D)}{D} = \sqrt{(0.1551)^2 + (0.001154)^2} = 0.01555,$$

$$D = \frac{8\pi^2 * 0.260}{741.5} = 0.02769 N \cdot m \cdot rad^{-1}$$

From this we can calculate absolute uncertainty,

$$u(D) = 0.01555 \times 0.02769 = 4.305 \times 10^{-4} N \cdot m \cdot rad^{-1} \tag{14}$$

So finally we have:

$$D = (0.028 \pm 0.00043)\ N \cdot m \cdot rad^{-1}$$

5 Final results

Linear regression

$$a_{reg} = (741.5 \pm 11.5) s^2 m^{-2}$$

Torsional constant

$$D = (0.028 \pm 0.00043)\ N \cdot m \cdot rad^{-1}$$

Moment of inertia

Results for steel rod:

$$I_{theor,rod} = 4.437 \times 10^{-3} kg\ m^2,\ I_{exp,rod} = 4.540 \times 10^{-3} kg\ m^2$$

$$\Delta I_{rod} = 4.21\%$$

Results for disk:

$$I_{theor,disk} = 5.558 \times 10^{-3} kg\ m^2,\ I_{exp,disk} = 5.696 \times 10^{-3} kg\ m^2$$

$$\Delta I_{disk} = 2.49\%$$

Confirmation of Steiner's Theorem

$$\Delta I \text{ within } (1\text{–}7)\%$$

6 Conclusions

The experiment demonstrated the relationship between the moment of inertia and the mass distribution of a rigid body using a torsion pendulum. From the regression analysis, the torsional constant of the spring was determined to be

$$D = (0.028 \pm 0.00043)\ N \cdot m \cdot rad^{-1}$$

indicating a high level of precision in the measurement. The experimentally obtained moments of inertia for the steel rod and the circular disk were found to closely match the theoretical values calculated from geometric formulas. For the rod, the relative difference between experimental and theoretical values was

$$\Delta I_{\mathrm{rod}} = 4.21\%$$

and for the disk,

$$\Delta I_{\text{disk}} = 2.49\%$$

These deviations can be explained by common experimental inaccuracies such as human reaction time when measuring oscillation periods, small miss alignments in the mounting system, and energy losses caused by air resistance and internal damping.

The verification of Steiner's theorem confirmed the expected linear dependence of the moment of inertia on the square of the distance from the rotation axis:

$$I(d) = I_0 + Md^2$$

The experimentally measured moments of inertia for the disk agreed with the theoretical predictions within a deviation of 1–7%, confirming the validity of Steiner's theorem within the limits of experimental accuracy. Overall, the results verify the theoretical relationship between mass distribution and rotational inertia, demonstrating that the torsion pendulum is a reliable method for determining the moment of inertia of rigid bodies.