Lab 1.Spectral Analysis of Deterministic Signals

Abstract

The objective is to use discrete Fourier transform and its implementation with the help of matrix multiplication

1. Input Signal

Let us first define a "complex-valued signal" $x[k]$ of a certain block length N ranging from $0 \leq k \leq N-1$.

We will now perform an DFT of $x[k]$ since we are interested in the frequency spectrum of it.

2. DFT Definition

The discrete Fourier transform pair for a discrete-time signal $x[k]$ with sample index k and the corresponding DFT spectrum $X[\mu]$ with frequency index μ is given as

$$\text{DFT} : X[\mu] = \sum_{k=0}^{N-1} x[k] \cdot \mathrm{e}^{-\mathrm{j}\frac{2\pi}{N}k\mu} \tag{1}$$

$$\text{IDFT} : x[k] = \frac{1}{N} \sum_{\mu=0}^{N-1} X[\mu] \cdot \mathrm{e}^{+\mathrm{j}\frac{2\pi}{N}k\mu} \tag{2}$$

Note the sign reversal in the exp()-function and the $1/N$ normalization in the IDFT. This convention is used by the majority of DSP text books and also in Python's ‘numpy.fft.fft()‘, ‘numpy.fft.ifft()‘ and Matlab's ‘fft()‘, ‘ifft()‘ routines.

3. DFT and IDFT with Matrix Multiplication

Now we do a little better: We should think of the DFT/IDFT in terms of a matrix operation setting up a set of linear equations.

For that we define a column vector containing the samples of the discrete-time signal $x[k]$

$$\mathbf{x}_k = (x[k=0], x[k=1], x[k=2], \dots, x[k=N-1])^{\mathrm{T}} \tag{3}$$

and a column vector containing the DFT coefficients $X[\mu]$

$$\mathbf{x}_\mu = (X[\mu=0], X[\mu=1], X[\mu=2], \dots, X[\mu=N-1])^{\mathrm{T}} \tag{4}$$

Then, the matrix operations

$$\text{DFT: } \mathbf{x}_\mu = \mathbf{W}^* \mathbf{x}_k \tag{5}$$

$$\text{IDFT: } \mathbf{x}_k = \frac{1}{N} \mathbf{W} \mathbf{x}_\mu \tag{6}$$

hold.
$()^{\mathrm{T}}$ is the transpose, $()^*$ is the conjugate complex.
The $N \times N$ Fourier matrix is defined as (element-wise operation $\odot$)

$$\mathbf{W} = \mathrm{e}^{+\mathrm{j}\frac{2\pi}{N} \odot \mathbf{K}} \tag{7}$$

using the so called twiddle factor (note that the sign in the exp() is our convention)

$$W_N = \mathrm{e}^{+\mathrm{j}\frac{2\pi}{N}} \tag{8}$$

and the outer product

$$\mathbf{K} = \begin{bmatrix} 0 \\ 1 \\ 2 \\ \vdots \\ N-1 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 & 2 & \cdots & N-1 \end{bmatrix} \tag{9}$$

containing all possible products $k\,\mu$ in a suitable arrangement.
For the simple case $N = 4$ these matrices are

$$\mathbf{K} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 3 \\ 0 & 2 & 4 & 6 \\ 0 & 3 & 6 & 9 \end{bmatrix} \rightarrow \mathbf{W} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & +\mathrm{j} & -1 & -\mathrm{j} \\ 1 & -1 & 1 & -1 \\ 1 & -\mathrm{j} & -1 & +\mathrm{j} \end{bmatrix} \tag{10}$$

4. Fourier Matrix Properties

The DFT and IDFT basically solve two sets of linear equations, that are linked as forward and inverse problem.

This is revealed with the important property of the Fourier matrix

$$\mathbf{W}^{-1} = \frac{\mathbf{W}^{\mathrm{H}}}{N} = \frac{\mathbf{W}^{*}}{N}, \tag{11}$$

the latter holds since the matrix is symmetric.

Thus, we see that by our convention, the DFT is the inverse problem (signal analysis) and the IDFT is the forward problem (signal synthesis)

$$\text{DFT: } \mathbf{x}_\mu = \mathbf{W}^{*}\mathbf{x}_k \rightarrow \mathbf{x}_\mu = N\mathbf{W}^{-1}\,\mathbf{x}_k \tag{12}$$

$$\text{IDFT: } \mathbf{x}_k = \frac{1}{N}\mathbf{W}\mathbf{x}_\mu. \tag{13}$$

The occurrence of the N, $1/N$ factor is due to the prevailing convention in signal processing literature.

If the matrix is normalised as $\frac{\mathbf{W}}{\sqrt{N}}$, a so called unitary matrix results, for which the important property

$$(\frac{\mathbf{W}}{\sqrt{N}})^{\mathrm{H}}\,(\frac{\mathbf{W}}{\sqrt{N}}) = \mathbf{I} = (\frac{\mathbf{W}}{\sqrt{N}})^{-1}\,(\frac{\mathbf{W}}{\sqrt{N}}) \tag{14}$$

holds, i.e. the complex-conjugate, transpose is equal to the inverse $(\frac{\mathbf{W}}{\sqrt{N}})^{\mathrm{H}} = (\frac{\mathbf{W}}{\sqrt{N}})^{-1}$ and due to the matrix symmetry also $(\frac{\mathbf{W}}{\sqrt{N}})^{*} = (\frac{\mathbf{W}}{\sqrt{N}})^{-1}$ is valid.

This tells that the matrix $\frac{\mathbf{W}}{\sqrt{N}}$ is **orthonormal**, i.e. the matrix spans a orthonormal vector basis (the best what we can get in linear algebra world to work with) of N normalized DFT eigensignals.

So, DFT and IDFT is transforming vectors into other vectors using the vector basis of the Fourier matrix.

5. Check DFT Eigensignals and -Frequencies

The columns of the Fourier matrix $\mathbf{W}$ contain the eigensignals of the DFT. These are

$$w_\mu[k] = \cos(\frac{2\pi}{N}k\mu) + \mathrm{j}\sin(\frac{2\pi}{N}k\mu) \tag{15}$$

since we have intentionally set up the matrix this way.

The eigensignals for $0 \leq \mu \leq N-1$ therefore exhibit a certain digital frequency, the so called DFT eigenfrequencies.

The nice thing about the chosen eigenfrequencies, is that the eigensignals are "orthogonal".

This choice of the vector basis is on purpose and one of the most important ones in linear algebra and signal processing.

We might for example check orthogonality with the "complex" inner product of some matrix columns.

6. Initial Example: IDFT Signal Synthesis for N=8

Let us synthesize a discrete-time signal by using the IDFT in matrix notation for $N = 8$.

The signal should contain a DC value, the first and second eigenfrequency with different amplitudes, such as

$$\mathbf{x}_\mu = [8, 2, 4, 0, 0, 0, 0, 0]^{\mathrm{T}} \tag{16}$$

7. Tasks

Synthesize a discrete-time signal by using the IDFT in matrix notation for different values of N. Show the matrices W and K. Plot the signal synthesized.

Variants

1. $$\mathbf{x}_\mu = [6, 2, 4, 3, 4, 5, 0, 0, 0, 0]^{\mathrm{T}} \tag{17}$$

2. $$\mathbf{x}_\mu = [10, 5, 6, 6, 2, 4, 3, 4, 5, 0, 0, 0, 0]^{\mathrm{T}} \tag{18}$$

3. $$\mathbf{x}_\mu = [6, 2, 4, 3, 4, 5, 0, 0, 0, 0]^{\mathrm{T}} \tag{19}$$

4. $$\mathbf{x}_\mu = [6, 2, 4, 3, 4, 5, 0, 0, 0]^{\mathrm{T}} \tag{20}$$

5. $$\mathbf{x}_\mu = [6, 4, 4, 5, 3, 4, 5, 0, 0, 0, 0]^{\mathrm{T}} \tag{21}$$

6.
$$\mathbf{x}_\mu = [7, 2, 4, 3, 4, 5, 0, 0, 0, 0]^{\mathrm{T}} \tag{22}$$

7.
$$\mathbf{x}_\mu = [6, 8, 2, 4, 3, 4, 5, 0, 0, 0]^{\mathrm{T}} \tag{23}$$

8.
$$\mathbf{x}_\mu = [6, 2, 4, 4, 4, 5, 0, 0, 0, 0]^{\mathrm{T}} \tag{24}$$

9.
$$\mathbf{x}_\mu = [6, 5, 4, 3, 4, 5, 0, 0, 0, 0]^{\mathrm{T}} \tag{25}$$

10.
$$\mathbf{x}_\mu = [6, 2, 4, 3, 4, 4, 0, 0, 0, 0]^{\mathrm{T}} \tag{26}$$

11.
$$\mathbf{x}_\mu = [6, 2, 4, 3, 4, 5, 0, 0, 0, 0]^{\mathrm{T}} \tag{27}$$

12.
$$\mathbf{x}_\mu = [10, 5, 6, 6, 2, 4, 3, 4, 5, 0, 0, 0, 0]^{\mathrm{T}} \tag{28}$$

13.
$$\mathbf{x}_\mu = [6, 2, 4, 3, 4, 5, 0, 0, 0, 0]^{\mathrm{T}} \tag{29}$$

14.
$$\mathbf{x}_\mu = [6, 2, 4, 3, 4, 5, 0, 0, 0]^{\mathrm{T}} \tag{30}$$

15.
$$\mathbf{x}_\mu = [6, 4, 4, 5, 3, 4, 5, 0, 0, 0, 0]^{\mathrm{T}} \tag{31}$$

Reports in the form:

1. Report (file .pdf)

2. file .ipynb

3. pdf-export the file .ipynb

upload to the remote repozitorium (e.g. Github) and link save in the report. Upload the report to eLearning.ubb.edu.pl.

References

References

[pandasUG] Pandas User's Guide https://pandas.pydata.org/pandas-docs/stable/user_guide/index.html

[DA2016] Data Analysis with Python and pandas using Jupyter Notebook https://dev.socrata.com/blog/2016/02/01/pandas-and-jupyter-notebook.html

[MIT] https://ocw.mit.edu/courses/res-6-008-digital-signal-processing-spring-2011/pages/study-materials/