

Fundamental Algorithmic Techniques II

October 3, 2025

Outline

Divide & Conquer

Recurrence Relations

Master Theorem

Multiplying Square Matrices

$$\mathbf{C} = \mathbf{A} \cdot \mathbf{B},$$

$$n \text{ operations } \forall i, j : \quad c_{ij} = \sum_{k=0}^n a_{ik} \cdot b_{kj}.$$

Divide and conquer:

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}, \quad C = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix},$$

$$\begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} = \begin{bmatrix} A_{11}B_{11} + A_{12}B_{21} & A_{11}B_{12} + A_{12}B_{22} \\ A_{21}B_{11} + A_{22}B_{21} & A_{21}B_{12} + A_{22}B_{22} \end{bmatrix}. \quad (4.4)$$

Decomposing with 8 sub-operations: $T(n) = 8T(n/2) + \Theta(1)$,
so $\mathbf{T(n)} = \mathbf{\Theta(n^3)}$ (master theorem $c = 3 = \log_2(8)$).

Strassen Algorithm

$$T(n) = 7T(n/2) + \Theta(1), \text{ so } \mathbf{T(n)} = \mathbf{\Theta(n^{2.81})}$$

$$M_1 = (A_{11} + A_{22})(B_{11} + B_{22})$$

$$M_2 = (A_{21} + A_{22})B_{11}$$

$$M_3 = A_{11}(B_{12} - B_{22})$$

$$M_4 = A_{22}(B_{21} - B_{11})$$

$$M_5 = (A_{11} + A_{12})B_{22}$$

$$M_6 = (A_{21} - A_{11})(B_{11} + B_{12})$$

$$M_7 = (A_{12} - A_{22})(B_{21} + B_{22})$$

Result Blocks:

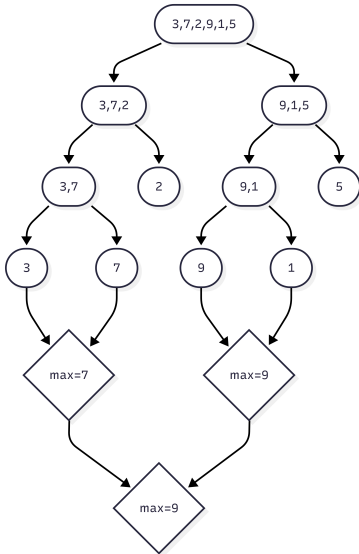
$$C_{11} = M_1 + M_4 - M_5 + M_7$$

$$C_{12} = M_3 + M_5$$

$$C_{21} = M_2 + M_4$$

$$C_{22} = M_1 - M_2 + M_3 + M_6$$

Find Maximum: Divide and Conquer



Problem: Find the maximum element in an array of n numbers.

Approach:

- **Divide:** Split into two halves
- **Conquer:** Recursively find max
- **Combine:** $\max(\text{left}, \text{right})$

Complexity:

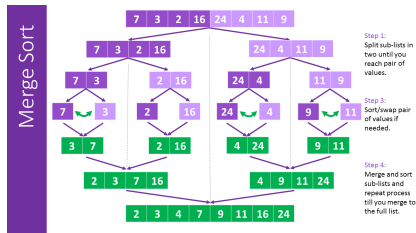
$$T(n) = 2T(n/2) + cn$$
$$\Rightarrow \mathcal{O}(n)$$

Sorting

Insertion sort: $T(n) = an^2 + bn + c$, $a, b, c \in \mathbb{N}$
So $T(N) = \mathcal{O}(n^2)$.

But can we do better?

Yes, actually $\mathcal{O}(n \log(n))$ with MergeSort and QuickSort.



Merge Sort

Recurrence Relation: Mathematical Description

Let $T(n)$ be a recurrence relation defined for $n \geq 1$ by:

$$T(n) = aT\left(\frac{n}{b}\right) + f(n)$$

where:

- $a \geq 1$ is the number of subproblems in the recursion,
- $b > 1$ is the factor by which the input size is reduced in each subproblem,
- $f(n)$ is the cost of dividing the problem and combining the results.

Recursive: Not always good!

Iterative vs Recursive Factorial: Complexity Comparison

	Iterative	Recursive
Time Complexity	$O(n)$	$O(n)$
Space Complexity	$O(1)$	$O(n)$
Stack Overflow?	No	Yes

```
1
2 function factorial_iter(n::Int)
3     result = 1
4     for i in 2:n
5         result *= i
6     end
7     return result
8 end
9
10 function factorial_recu(n::Int)
11     n <= 1 ? 1 : n * factorial(n - 1)
12 end
13
```

Factorial in Julia

Master Theorem

Asymptotic behavior of $T(n) = a \cdot T\left(\frac{n}{b}\right) + f(n)$:

critical exponent: $c_{\text{crit}} = \log_b a$

1 Case 1 (Subproblem Dominated):

If $f(n) = O(n^c)$ where $c < c_{\text{crit}}$, then:

$$T(n) = \Theta(n^{\log_b a})$$

2 Case 2 (Balanced):

If $f(n) = \Theta(n^{c_{\text{crit}}})$, then:

$$T(n) = \Theta(n^{c_{\text{crit}}} \log n) = \Theta(n^{\log_b a} \log n)$$

3 Case 3 (Work Dominated):

If $f(n) = \Omega(n^c)$ where $c > c_{\text{crit}}$, and if the **regularity condition** holds:

$$af\left(\frac{n}{b}\right) \leq kf(n) \quad \text{for constant } k < 1 \text{ and all sufficiently large } n,$$

then:

$$T(n) = \Theta(f(n))$$

Master Theorem : Limitations & Examples

Limitations:

- $T(n)$ not monotone, e.g. $T(n) = \sin(n)$
- $f(n)$ not polynomial, e.g. $f(n) = 2^n$
- a not a constant, e.g. $a = 2n$

Examples: verify!

- 1 $T(n) = 4T(\frac{n}{2}) + n$,
 \Rightarrow Case 1, Subproblem dominated, $T(n) = \Theta(n^2)$
- 2 $T(n) = 2T(\frac{n}{2}) + n$,
 \Rightarrow Case 2, Balanced, $T(n) = \Theta(n \log(n))$
- 3 $T(n) = 3T(\frac{n}{2}) + n^2$,
 \Rightarrow Case 3, Work dominated, $T(n) = \Theta(n^2)$
- 4 $T(n) = 2T(2n) + n \log(n)$,
 \Rightarrow trivially not applicable!
- 5 $T(n) = T(n-1) + 1$,
 \Rightarrow Not applicable! $n-1 \neq n/b$, actually $T(n) = \Theta(n!)$

Master Theorem : Proof with Tree Approach

$$T(n) = aT\left(\frac{n}{b}\right) + f(n)$$

- level 0: work $f(n)$, a subproblems of size n/b

- ...

- level i : work $a^i \cdot f(n/b^i)$, a^i subproblems of size n/b^i

stops when $i = \log_b(n)$, and using $a \cdot \log_b(n) = n \cdot \log_b(a)$:

$$T(n) = \sum_{i=1}^{\log_b n} a^i \cdot f(n/b^i) + \Theta(n^{\log_b a}),$$

with **work** and **leaf decomposition** contributions.

...Study each 3 cases separately for the proof...