COMP0147 Discrete Mathematics for Computer Scientists Notes

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Notes adapted from lecture notes by Max Kanovich and Robin Hirsch [1].

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1 Foundations

Contents

1.1 Set Theory

1.1.1 Set Notations

- Set definition: $A = \{a, b, c\}$
- Set membership (element-of): $a \in A$
- Set builder notation: $\{x \mid x \in \mathbb{R} \land x^2 = x\}$
- Empty set: $\emptyset$

1.1.2 Properties

- No structure
- No order
- No copies

For example, a, b, c are references to actual objects in

$$\{a, b, c\} \Leftrightarrow \{c, a, b\} \Leftrightarrow \{a, b, c, b\}$$

1.1.3 Set Equality

Definition 1.1.1 (Set Equality). Set $A = B$ iff:
1. $A \subseteq B \implies \forall\, x(x \in A \to x \in B)$
2. $B \subseteq A \implies \forall\, y(y \in B \to y \in A)$

Remark. $A = B \Leftrightarrow A \subseteq B \land B \subseteq A$

1.1.4 Set Operations

- *Union*: $A \cup B \equiv \{x \mid x \in A \lor x \in B\}$
- *Intersection*: $A \cap B \equiv \{x \mid x \in A \land x \in B\}$
- *Relative Complement*: $A \setminus B \equiv \{x \mid x \in A \land x \notin B\}$
- *Absolute Complement*: $A^c \equiv U \setminus A \equiv \{x \mid x \in U \land x \notin A\}$
- *Symmetric Difference*: $A \Delta B \equiv (A \setminus B) \cup (B \setminus A) \equiv (A \cup B) \setminus (A \cap B)$
- *Cartesian Product*: $A \times B \equiv \{(x, y) \mid x \in A \land y \in B\}$

1.1.5 Boolean Algebra

Definition 1.1.2 (De Morgan's Laws).

$$\neg(p \lor q) \equiv \neg p \land \neg q \tag{1.1}$$

$$\neg(p \land q) \equiv \neg p \lor \neg q \tag{1.2}$$

Definition 1.1.3 (Idempotent Laws).

$$p \lor p \equiv p \tag{1.3}$$

$$p \land p \equiv p \tag{1.4}$$

Definition 1.1.4 (Commutative Laws).

$$p \lor q \equiv q \lor p \tag{1.5}$$

$$p \land q \equiv q \land p \tag{1.6}$$

Definition 1.1.5 (Associative Laws).

$$p \lor (q \lor r) \equiv (p \lor q) \lor r \tag{1.7}$$

$$p \land (q \land r) \equiv (p \land q) \land r \tag{1.8}$$

Definition 1.1.6 (Distributive Laws).

$$p \land (q \lor r) \equiv (p \land q) \lor (p \land r) \tag{1.9}$$

$$p \lor (q \land r) \equiv (p \lor q) \land (p \lor r) \tag{1.10}$$

Definition 1.1.7 (Identity Laws)**.**

$$p \lor \text{F} \equiv p \tag{1.11}$$
$$p \lor \text{T} \equiv \text{T} \tag{1.12}$$
$$p \land \text{T} \equiv p \tag{1.13}$$
$$p \land \text{F} \equiv \text{F} \tag{1.14}$$

Definition 1.1.8 (Absorption Laws)**.**

$$p \lor (p \land q) \equiv p \tag{1.15}$$
$$p \land (p \lor q) \equiv p \tag{1.16}$$

Definition 1.1.9 (Implication and Negation Laws)**.**

- *Identity*: $p \to q \equiv \neg p \lor q$
- *Counter-example*: $\neg(p \to q) \equiv p \land \neg q$
- *Equivalences*: $p \to q \to r \equiv (p \land q) \to r \equiv q \, to(p \to r)$
- *Absorption*:
    $p \to \text{T} \equiv T$
    $p \to \text{F} \equiv \neg p$
    $\text{T} \to p \equiv p$
    $\text{F} \to p \equiv T$
- *Contrapositive*: $p \to q \equiv \neg q \to \neg p$
- *Law of Excluded Middle*:
    $p \lor \neg p \equiv \text{T}$
    $p \land \neg p \equiv \text{F}$
- *Double Negation*: $\neg\neg p \equiv p$
- *Reduction to Absurdity*: $\neg p \to \text{F} \equiv p$

1.1.6 Set Algebra

Definition 1.1.10 (De Morgan's Laws)**.**

$$(A \cup B)^c \equiv A^c \cap B^c \tag{1.17}$$
$$(A \cap B)^c \equiv A^c \cup B^c \tag{1.18}$$

Definition 1.1.11 (Idempotent Laws)**.**

$$A \cup A \equiv A \tag{1.19}$$
$$A \cap A \equiv A \tag{1.20}$$

Definition 1.1.12 (Commutative Laws).

$$A \cup B \equiv B \cup A \tag{1.21}$$
$$A \cap B \equiv B \cap A \tag{1.22}$$

Definition 1.1.13 (Associativity Laws).

$$A \cup (B \cup C) \equiv (A \cup B) \cup C \tag{1.23}$$
$$A \cap (B \cap C) \equiv (A \cap B) \cap C \tag{1.24}$$

Definition 1.1.14 (Distributive Laws).

$$A \cap (B \cup C) \equiv (A \cap B) \cup (B \cap C) \tag{1.25}$$
$$A \cup (B \cap C) \equiv (A \cup B) \cap (B \cup C) \tag{1.26}$$

Definition 1.1.15 (Identity Laws).

$$A \cup \emptyset \equiv A \tag{1.27}$$
$$A \cap \emptyset \equiv \emptyset \tag{1.28}$$
$$A \cap U \equiv A \tag{1.29}$$
$$A \cup U \equiv U \tag{1.30}$$

Definition 1.1.16 (Absorption Laws).

$$A \cup (A \cap B) \equiv A \tag{1.31}$$
$$A \cap (A \cup B) \equiv A \tag{1.32}$$

Definition 1.1.17 (Difference Identity Laws).

$$C \setminus (A \cup B) \equiv (C \setminus A) \cap (C \setminus B) \tag{1.33}$$
$$C \setminus (A \cap B) \equiv (C \setminus A) \cup (C \setminus B) \tag{1.34}$$

Definition 1.1.18 (Complement-Difference Identity Law).

$$C \setminus D \equiv C \cap D^c \tag{1.35}$$

Definition 1.1.19 (Double Complement Law).

$$(D^c)^c \equiv D \tag{1.36}$$

Definition 1.1.20 (Contraposition).

$$C \subseteq D \Leftrightarrow D^c \subseteq C^c \tag{1.37}$$
$$C = D \Leftrightarrow C^c = D^c \tag{1.38}$$

Definition 1.1.21 (Arbitrary Union). Given sets $A_1, A_2, \dots, A_n$ where $I = \{1, 2, \dots, n\}$

$$A_1 \cup A_2 \cup \dots \cup A_n \equiv \bigcup_{i \in I} A_i \tag{1.39}$$

Then

$$x \in \bigcup_{i \in I} A_i \Leftrightarrow \exists\, i \in I : x \in A_i \tag{1.40}$$

Definition 1.1.22 (Arbitrary Intersection). Given sets $A_1, A_2, \dots, A_n$ where $I = \{1, 2, \dots, n\}$

$$A_1 \cap A_2 \cap \dots \cap A_n \equiv \bigcap_{i \in I} A_i \tag{1.41}$$

Then

$$x \in \bigcap_{i \in I} A_i \Leftrightarrow \forall\, i \in I : x \in A_i \tag{1.42}$$

1.2 Functions

Definition 1.2.1 (Function). A function f is a mapping from X to Y

$$f : X \mapsto Y \tag{1.43}$$

- $\mathrm{domain}(f) = X$
- $\mathrm{image}(f) = f(X)$

Definition 1.2.2 (Total Function). A function is *total* if

$$\mathrm{domain}(f) = X \tag{1.44}$$

Definition 1.2.3 (Partial Function). A function is *partial* if

$$\mathrm{domain}(f) \subseteq X \tag{1.45}$$

Definition 1.2.4 (Surjection). A function $f : X \mapsto Y$ is *surjective* iff

$$f(X) = Y \Leftrightarrow \forall\, y \in Y : \exists\, x \in X : f(x) = y \tag{1.46}$$

Namely each $y \in Y$ has a corresponding $x \in X$.

Definition 1.2.5 (Injection (Encodings, One-to-one)). A function $f : X \mapsto Y$ is *injective* iff

$$\forall\, x_1, x_2 \in X : x_1 \neq x_2 \rightarrow f(x_1) \neq f(x_2) \tag{1.47}$$
$$\Leftrightarrow \forall\, x_1, x_2 \in X : f(x_1) = f(x_2) \rightarrow x_1 = x_2 \tag{1.48}$$

Namely each distinct element $x \in X$ maps to a different element in Y.

Definition 1.2.6 (Bijection). A function $f : X \mapsto Y$ is *bijective* iff f is both *injective* and *surjective*.

$$\mathrm{Bijective}(f) \equiv \mathrm{Injective}(f) \wedge \mathrm{Surjective}(f) \tag{1.49}$$

The *inverse bijection* $f^{-1} : Y \mapsto X$ does exist.

1.2.1 Composition of Injections

Proposition 1.2.1 (Composition of Injection). Given *injections* $f \colon X \mapsto Y$ and $g \colon Y \mapsto Z$, then their *composition* $h \colon X \mapsto Z$ is given by

$$h(x) = g(f(x)) \tag{1.50}$$

Then h is also an *injective* function. Namely $h = g \circ f$ where h is composed from g and f with f applied first.

Proof. Given any $x_1, x_2 \in X$ where $x_1 \neq x_2$, then

$$f(x_1) \neq f(x_2) \tag{1.51}$$

as f is *injective*, and thus

$$h(x_1) = g(f(x_1)) \neq g(f(x_2)) = h(x_2) \tag{1.52}$$

h is *injective* consequently. ∎

1.2.2 Composition of Surjection

Proposition 1.2.2 (Composition of Surjection). Given *surjections* $f \colon X \mapsto Y$ and $g \colon Y \mapsto Z$, then their *composition* $h \colon X \mapsto Z$ is given by

$$h(x) = g(f(x)) \tag{1.53}$$

Then h is also a *surjective* function.

Proof. To prove $h \colon X \mapsto Z$ is *injective*, it is required to prove that

$$\forall z \in Z \colon \exists x \in X \colon h(x) = z \tag{1.54}$$

Where $h(x) \Leftrightarrow (g \circ f)(x) \Leftrightarrow g(f(x))$.

Given any element $z \in Z$ ($\forall z \in Z$):
1. That $g \colon Y \mapsto Z$ is *surjective* by definition, then $\exists y \in Y \colon g(y) = z$.
2. That $f \colon X \mapsto Y$ is *surjective* by definition, then $\exists x \in X \colon f(x) = y$.

Then $\forall z \in Z \colon \exists x \in X \colon h(x) = (g \circ f)(x) = g(f(x)) = g(y) = z$ holds true. ∎

1.2.3 Composition of Bijection

Proposition 1.2.3 (Composition of Bijection). Given *bijections* $f \colon X \mapsto Y$ and $g \colon Y \mapsto Z$, then their composition $h \colon X \mapsto Z$ is given by

$$h(x) = g(f(x)) \tag{1.55}$$

Then h is also a *bijective* function; an *inverse bijection* $h^{-1} \colon Z \mapsto X$ also exists.

Bibliography

[1] Max Kanovich and Robin Hirsch.
    "Lecture Notes on Discrete Mathematics for Computer Scientists".
    URL: `http://www.cs.ucl.ac.uk/1819/a4u/t2/comp0147_discrete_mathematics_for_computer_scientists/`.