

# **COMP0147 Discrete Mathematics for Computer Scientists Notes**

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Notes adapted from:

- Lecture notes by Max Kanovich and Robin Hirsch [1].
- *A First Course in Abstract Algebra* by Joseph J. Rotman [2].



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# 1 Set Theory

## 1.1 Set Notations

- Set definition:  $A = \{a, b, c\}$
- Set membership (element-of):  $a \in A$
- Set builder notation:  $\{x \mid x \in \mathbb{R} \wedge x^2 = x\}$
- Empty set:  $\emptyset$

## 1.2 Properties

- No structure
- No order
- No copies

For example,  $a, b, c$  are references to actual objects in

$$\{a, b, c\} \Leftrightarrow \{c, a, b\} \Leftrightarrow \{a, b, c, b\}$$

## 1.3 Set Equality

**Definition 1.3.1** (Set Equality). Set  $A = B$  iff:

1.  $A \subseteq B \implies \forall x(x \in A \rightarrow x \in B)$
2.  $B \subseteq A \implies \forall y(y \in B \rightarrow y \in A)$

**Remark.**  $A = B \Leftrightarrow A \subseteq B \wedge B \subseteq A$

## 1.4 Set Operations

- *Union:*  $A \cup B := \{x \mid x \in A \vee x \in B\}$
- *Intersection:*  $A \cap B := \{x \mid x \in A \wedge x \in B\}$
- *Relative Complement:*  $A \setminus B := \{x \mid x \in A \wedge x \notin B\}$
- *Absolute Complement:*  $A^c := U \setminus A := \{x \mid x \in U \wedge x \notin A\}$
- *Symmetric Difference:*  $A \Delta B := (A \setminus B) \cup (B \setminus A) := (A \cup B) \setminus (A \cap B)$
- *Cartesian Product:*  $A \times B := \{(x, y) \mid x \in A \wedge y \in B\}$

## 1.5 Boolean Algebra

**Definition 1.5.1** (De Morgan's Laws).

$$\neg(p \vee q) := \neg p \wedge \neg q \quad (1.1)$$

$$\neg(p \wedge q) := \neg p \vee \neg q \quad (1.2)$$

**Definition 1.5.2** (Idempotent Laws).

$$p \vee p := p \quad (1.3)$$

$$p \wedge p := p \quad (1.4)$$

**Definition 1.5.3** (Commutative Laws).

$$p \vee q := q \vee p \quad (1.5)$$

$$p \wedge q := q \wedge p \quad (1.6)$$

**Definition 1.5.4** (Associative Laws).

$$p \vee (q \vee r) := (p \vee q) \vee r \quad (1.7)$$

$$p \wedge (q \wedge r) := (p \wedge q) \wedge r \quad (1.8)$$

**Definition 1.5.5** (Distributive Laws).

$$p \wedge (q \vee r) := (p \wedge q) \vee (p \wedge r) \quad (1.9)$$

$$p \vee (q \wedge r) := (p \vee q) \wedge (p \vee r) \quad (1.10)$$

**Definition 1.5.6** (Identity Laws).

$$p \vee F := p \quad (1.11)$$

$$p \vee T := T \quad (1.12)$$

$$p \wedge T := p \quad (1.13)$$

$$p \wedge F := F \quad (1.14)$$

**Definition 1.5.7** (Absorption Laws).

$$p \vee (p \wedge q) := p \quad (1.15)$$

$$p \wedge (p \vee q) := p \quad (1.16)$$

**Definition 1.5.8** (Implication and Negation Laws).

- *Identity:*  $p \rightarrow q := \neg p \vee q$
- *Counter-example:*  $\neg(p \rightarrow q) := p \wedge \neg q$
- *Equivalences:*  $p \rightarrow q \rightarrow r := (p \wedge q) \rightarrow r := q \text{ to } (p \rightarrow r)$



- *Absorption:*  
 $p \rightarrow T := T$   
 $p \rightarrow F := \neg p$   
 $T \rightarrow p := p$   
 $F \rightarrow p := T$
- *Contrapositive:*  $p \rightarrow q := \neg q \rightarrow \neg p$
- *Law of Excluded Middle:*  
 $p \vee \neg p := T$   
 $p \wedge \neg p := F$
- *Double Negation:*  $\neg\neg p := p$
- *Reduction to Absurdity:*  $\neg p \rightarrow F := p$

## 1.6 Set Algebra

**Definition 1.6.1** (De Morgan's Laws).

$$(A \cup B)^c := A^c \cap B^c \quad (1.17)$$

$$(A \cap B)^c := A^c \cup B^c \quad (1.18)$$

**Definition 1.6.2** (Idempotent Laws).

$$A \cup A := A \quad (1.19)$$

$$A \cap A := A \quad (1.20)$$

**Definition 1.6.3** (Commutative Laws).

$$A \cup B := B \cup A \quad (1.21)$$

$$A \cap B := B \cap A \quad (1.22)$$

**Definition 1.6.4** (Associativity Laws).

$$A \cup (B \cup C) := (A \cup B) \cup C \quad (1.23)$$

$$A \cap (B \cap C) := (A \cap B) \cap C \quad (1.24)$$

**Definition 1.6.5** (Distributive Laws).

$$A \cap (B \cup C) := (A \cap B) \cup (A \cap C) \quad (1.25)$$

$$A \cup (B \cap C) := (A \cup B) \cap (A \cup C) \quad (1.26)$$

**Definition 1.6.6** (Identity Laws).

$$A \cup \emptyset := A \quad (1.27)$$

$$A \cap \emptyset := \emptyset \quad (1.28)$$

$$A \cap U := A \quad (1.29)$$

$$A \cup U := U \quad (1.30)$$

**Definition 1.6.7** (Absorption Laws).

$$A \cup (A \cap B) := A \quad (1.31)$$

$$A \cap (A \cup B) := A \quad (1.32)$$

**Definition 1.6.8** (Difference Identity Laws).

$$C \setminus (A \cup B) := (C \setminus A) \cap (C \setminus B) \quad (1.33)$$

$$C \setminus (A \cap B) := (C \setminus A) \cup (C \setminus B) \quad (1.34)$$

**Definition 1.6.9** (Complement-Difference Identity Law).

$$C \setminus D := C \cap D^c \quad (1.35)$$

**Definition 1.6.10** (Double Complement Law).

$$(D^c)^c := D \quad (1.36)$$

**Definition 1.6.11** (Contraposition).

$$C \subseteq D \Leftrightarrow D^c \subseteq C^c \quad (1.37)$$

$$C = D \Leftrightarrow C^c = D^c \quad (1.38)$$

**Definition 1.6.12** (Arbitrary Union).

Given sets  $A_1, A_2, \dots, A_n$  where  $I = \{1, 2, \dots, n\}$

$$A_1 \cup A_2 \cup \dots \cup A_n := \bigcup_{i \in I} A_i \quad (1.39)$$

Then

$$x \in \bigcup_{i \in I} A_i \Leftrightarrow \exists i \in I: x \in A_i \quad (1.40)$$

**Definition 1.6.13** (Arbitrary Intersection).

Given sets  $A_1, A_2, \dots, A_n$  where  $I = \{1, 2, \dots, n\}$

$$A_1 \cap A_2 \cap \dots \cap A_n := \bigcap_{i \in I} A_i \quad (1.41)$$

Then

$$x \in \bigcap_{i \in I} A_i \Leftrightarrow \forall i \in I: x \in A_i \quad (1.42)$$

## 2 Functions

### 2.1 Function Basics

**Definition 2.1.1** (Function). A function  $f$  is a mapping from  $X$  to  $Y$

$$f: X \mapsto Y \quad (2.1)$$

- $\text{domain}(f) = X$
- $\text{image}(f) = f(X)$

**Definition 2.1.2** (Total Function). A function is *total* if

$$\text{domain}(f) = X \quad (2.2)$$

**Definition 2.1.3** (Partial Function). A function is *partial* if

$$\text{domain}(f) \subseteq X \quad (2.3)$$

**Definition 2.1.4** (Surjection). A function  $f: X \mapsto Y$  is *surjective* iff

$$f(X) = Y \Leftrightarrow \forall y \in Y: \exists x \in X: f(x) = y \quad (2.4)$$

Namely each  $y \in Y$  has a corresponding  $x \in X$ .

**Definition 2.1.5** (Injection (Encodings, One-to-one)). A function  $f: X \mapsto Y$  is *injective* iff

$$\forall x_1, x_2 \in X: x_1 \neq x_2 \rightarrow f(x_1) \neq f(x_2) \quad (2.5)$$

$$\Leftrightarrow \forall x_1, x_2 \in X: f(x_1) = f(x_2) \rightarrow x_1 = x_2 \quad (2.6)$$

Namely each distinct element  $x \in X$  maps to a different element in  $Y$ .

**Definition 2.1.6** (Bijection). A function  $f: X \mapsto Y$  is *bijective* iff  $f$  is both *injective* and *surjective*.

$$\text{Bijective}(f) := \text{Injective}(f) \wedge \text{Surjective}(f) \quad (2.7)$$

The *inverse bijection*  $f^{-1}: Y \mapsto X$  does exist.

## 2.2 Composition of Injections

**Proposition 2.2.1** (Composition of Injection). Given *injections*  $f: X \mapsto Y$  and  $g: Y \mapsto Z$ , then their *composition*  $h: X \mapsto Z$  is given by

$$h(x) = g(f(x)) \quad (2.8)$$

Then  $h$  is also an *injective* function. Namely  $h = g \circ f$  where  $h$  is composed from  $g$  and  $f$  with  $f$  applied first.

*Proof.* Given any  $x_1, x_2 \in X$  where  $x_1 \neq x_2$ , then

$$f(x_1) \neq f(x_2) \quad (2.9)$$

as  $f$  is *injective*, and thus

$$h(x_1) = g(f(x_1)) \neq g(f(x_2)) = h(x_2) \quad (2.10)$$

$h$  is *injective* consequently. ■

## 2.3 Composition of Surjection

**Proposition 2.3.1** (Composition of Surjection). Given *surjections*  $f: X \mapsto Y$  and  $g: Y \mapsto Z$ , then their *composition*  $h: X \mapsto Z$  is given by

$$h(x) = g(f(x)) \quad (2.11)$$

Then  $h$  is also a *surjective* function.

*Proof.* To prove  $h: X \mapsto Z$  is *injective*, it is required to prove that

$$\forall z \in Z: \exists x \in X: h(x) = z \quad (2.12)$$

Where  $h(x) \Leftrightarrow (g \circ f)(x) \Leftrightarrow g(f(x))$ .

Given any element  $z \in Z$  ( $\forall z \in Z$ ):

1. That  $g: Y \mapsto Z$  is *surjective* by definition, then  $\exists y \in Y: g(y) = z$ .
2. That  $f: X \mapsto Y$  is *surjective* by definition, then  $\exists x \in X: f(x) = y$ .

Then  $\forall z \in Z: \exists x \in X: h(x) = (g \circ f)(x) = g(f(x)) = g(y) = z$  holds true. ■

## 2.4 Composition of Bijection

**Proposition 2.4.1** (Composition of Bijection). Given *bijections*  $f: X \mapsto Y$  and  $g: Y \mapsto Z$ , then their composition  $h: X \mapsto Z$  is given by

$$h(x) = g(f(x)) \quad (2.13)$$

Then  $h$  is also a *bijective* function; an *inverse bijection*  $h^{-1}: Z \mapsto X$  also exists.

## 2.5 Cardinality of Sets

**Definition 2.5.1** (Cardinality). The number of elements in a set  $X$  is denoted  $|X|$ .

**Definition 2.5.2** (Equal Cardinality and Bijection).

$$|X| = |Y| \quad (2.14)$$

Holds true if there exists a *bijection*  $h: X \mapsto Y$  (one-to-one correspondence between  $X$  and  $Y$ ).

Namely,  $X$  and  $Y$  have the same number of distinct elements, and each distinct element  $x \in X$  corresponds to exactly one distinct element  $y \in Y$ .

**Theorem 2.5.1** (Cantor-Bernstein). Given

1. *injective* function  $f: X \mapsto Y$

2. *injective* function  $g: Y \mapsto X$

Then there exists a *bijection* function  $h: X \mapsto Y$ .

Equivalently,

$$(|X| \leq |Y|) \wedge (|Y| \leq |X|) \rightarrow (|X| = |Y|) \quad (2.15)$$

**Remark.** Examples include countable sets, enumerable sets

$$|\mathbb{Q}| = |\mathbb{Z}| = |\mathbb{N}| = \aleph_0 \quad (2.16)$$

Where the cardinality of countable sets such as the *rational numbers*, *integers* and the *natural numbers* is denoted as "alpeh-zero" ( $\aleph_0$ ).

On the other hand, continuum such as the *real numbers* are not countable and as such

$$|\mathbb{R}| > \aleph_0 \quad (2.17)$$



# 3 Permutations

## 3.1 Permutation Basics

**Definition 3.1.1** (Permutation). The bijection – *permutation* – of

$$\begin{array}{ccccccccc} 1 & 2 & 3 & \cdots & n \\ \downarrow & \downarrow & \downarrow & \cdots & \downarrow \\ 1 & 2 & 3 & \cdots & n \end{array} \quad (3.1)$$

Is denoted as

$$\left( \begin{array}{ccccccccc} 1 & 2 & 3 & \cdots & n \\ 1 & 2 & 3 & \cdots & n \end{array} \right) \quad (3.2)$$

Where  $\sigma: \{1, \dots, n\} \rightarrow \{1, \dots, n\}$  is the *permutation* bijection.

**Definition 3.1.2** (Counting Permutations).

$$|S_n| := n! \quad (3.3)$$

Which is the number of different ways to permute  $n$  elements  $\{1, 2, \dots, n\} \subset \mathbb{Z}$ . Together, the different permutations for  $n$  distinct elements is the *symmetric group*  $S_n$ .

**Remark.** For example, with  $S_3 = \{1, 2, 3\}$ , there are  $3! = 6$  different ways to arrange the three distinct elements

$$\begin{array}{ccc} \left( \begin{array}{ccc} 1 & 2 & 3 \\ 1 & 2 & 3 \end{array} \right) & \left( \begin{array}{ccc} 1 & 2 & 3 \\ 1 & 3 & 2 \end{array} \right) & \left( \begin{array}{ccc} 1 & 2 & 3 \\ 2 & 1 & 3 \end{array} \right) \\ \left( \begin{array}{ccc} 1 & 2 & 3 \\ 2 & 3 & 1 \end{array} \right) & \left( \begin{array}{ccc} 1 & 2 & 3 \\ 3 & 1 & 2 \end{array} \right) & \left( \begin{array}{ccc} 1 & 2 & 3 \\ 3 & 2 & 1 \end{array} \right) \end{array} \quad (3.4)$$

**Definition 3.1.3** (Order of Permutation). The *order* of a permutation  $\sigma$  is the smallest  $k \in \mathbb{Z}^+$  such that

$$\sigma^k = \epsilon \quad (3.5)$$

Where  $\epsilon$  is the *identity permutation*

$$\epsilon(x) = x \quad (3.6)$$

**Definition 3.1.4** (Sign of Permutation). The *sign* of a permutation  $\text{sgn } \sigma: \sigma \rightarrow \{-1, +1\}$  where  $\sigma \in S_n$  is defined as

$$\text{sgn}(\sigma) = (-1)^k \quad (3.7)$$

Where  $k$  is the number of *disorders* within  $\sigma$ , the number of pairs  $(x, y)$  such that  $x > y \rightarrow \sigma(x) < \sigma(y)$  or the converse  $x < y \rightarrow \sigma(x) > \sigma(y)$ . Additionally,

$$\text{sgn}(\sigma) = \begin{cases} +1 & \text{if } k \text{ is even} \\ -1 & \text{if } k \text{ is odd} \end{cases} \quad (3.8)$$

**Remark.** For example, in

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$$

$1 < 2$  but  $\sigma(1) = 2 > \sigma(2) = 1$ , hence a disorder.

For each  $i \in \{1, \dots, n\}$ , starting from  $i = 1$ , compare  $\sigma(i)$  with  $\sigma(i + 1), \dots, \sigma(n)$  and add the number of disordered pairs, then move on to  $i + 1$  and compare  $\sigma(i + 1)$  with  $\sigma(i + 2), \dots, \sigma(n)$  and so on.

**Theorem 3.1.1** (Composition of Permutation).

$$\text{sgn}(\sigma_1 \sigma_2) := \text{sgn}(\sigma_1) \cdot \text{sgn}(\sigma_2) \quad (3.9)$$

Where

$\circ$	even	odd
even	even	odd
odd	odd	even

Table 3.1: Sign Changes on Composition



## 4 Binary Relations

**Definition 4.0.1** (Binary Relation). A binary relation  $R(x, y)$  describes some relationship between  $x$  and  $y$  where  $R: X \rightarrow Y$ ,  $R \subseteq X \times Y$ ,  $x \in X$  and  $y \in Y$ . This relation can be expressed in infix notation as  $xRy$ .

### 4.1 Equivalence Relations

**Definition 4.1.1** (Equivalence Relation). A binary relation  $E(x, y)$  is an *equivalence relation* on  $X$  iff it satisfies all three conditions:

1. **Reflexivity**

$$\forall x \in X: E(x, x)$$

2. **Symmetry**

$$\forall x, y \in X: E(x, y) \rightarrow E(y, x)$$

3. **Transitivity**

$$\forall x, y, z \in X: E(x, y) \wedge E(y, z) \rightarrow E(x, z)$$

### 4.2 Equivalence Classes

**Definition 4.2.1** (Equivalence Class). If  $a \in X$ , the *equivalence class*  $[a]$  is

$$[a] := \{x \in X: E(x, a)\} \subseteq X \quad (4.1)$$

**Definition 4.2.2** (Congruence and Equivalence Class of mod  $m$  on  $\mathbb{Z}$ ). For *congruence mod  $m$*  on  $\mathbb{Z}$ , if  $a \in \mathbb{Z}$  then the *congruence class* of  $a$  is

$$[a]_m := \{x \in \mathbb{Z}: x = a + km\} \quad (4.2)$$

Where  $k \in \mathbb{Z}$ . Since  $x = a + km \Leftrightarrow x \equiv a \pmod{m}$ , then the *equivalence class* of  $a$  is also the *congruence class*.

$$\Leftrightarrow [a]_m := \{x \in \mathbb{Z}: x \equiv a \pmod{m}\} \quad (4.3)$$

**Definition 4.2.3** (Set of Remainders). Over  $\mathbb{Z}$ , the *remainder*  $r$  from the integer division  $k \div m$  is

$$r \bmod m \equiv k \bmod m \quad (4.4)$$

Then the set of remainders  $G_m$  from the integer division  $k \div m$  is defined by

$$G_m := \{0, 1, 2, \dots, m-2, m-1\} \quad (4.5)$$

### 4.3 Quotient Groups

**Definition 4.3.1** (Quotient Group). A *quotient group* is a group constructed via congruence mod  $m$ .

**Definition 4.3.2** (Congruence Class). If  $m \leq 2$  and  $a \in \mathbb{Z}$  then the *congruence class* of  $a \bmod m$  is  $[a] \subseteq \mathbb{Z}$

$$[a] := \{b \in \mathbb{Z} : b \equiv a \bmod m\} \quad (4.6)$$

$$\Leftrightarrow \{a + km : k \in \mathbb{Z}\} \quad (4.7)$$

$$\Leftrightarrow \{\dots, a - 2m, a - m, a, a + m, a + 2m, \dots\} \quad (4.8)$$

**Remark.** Let  $E(x, y) := "x - y \equiv 0 \bmod 2"$ , that is,  $x - y$  is divisible by 2. Then,

$$[k]_2 := \{y : E(k, y)\} \quad (4.9)$$

Where  $[k]_2$  is the congruence class of integers modulo 2.

Computing  $[0]_2$  and  $[1]_2$  yields

- $[0]_2 = \{0, 2, -2, 4, -4, \dots, 2n, -2n, \dots\}$
- $[1]_2 = \{1, -1, 3, -3, \dots, 2n + 1, \dots\}$

Observe that

$$[1]_2 \oplus [1]_2 \Leftrightarrow [2]_2 \Leftrightarrow [0]_2 \quad (4.10)$$

It can be deduced that  $[0]_2$  and  $[1]_2$  are two congruence (and equivalence) classes which partition the integers  $\mathbb{Z}$  into two disjoint subsets – integers which are odd, and integers which are even. This may be denoted as

$$\mathbb{Z}/E \equiv \{\text{EVEN}, \text{ODD}\} \quad (4.11)$$

**Definition 4.3.3** (Congruence Modular Arithmetic (mod  $m$ ) on  $\mathbb{Z}$ ).

$$[a]_m \oplus [b]_m \equiv [a + b]_m \quad (4.12)$$

$$[a]_m \otimes [b]_m \equiv [a \cdot b]_m \quad (4.13)$$

If  $a_1 \equiv a_2 \bmod m$  and  $b_1 \equiv b_2 \bmod m$  then

$$a_1 + b_1 \equiv a_2 + b_2 \bmod m \quad (4.14)$$

$$a_1 \cdot b_1 \equiv a_2 \cdot b_2 \bmod m \quad (4.15)$$

$$(4.16)$$

**Remark.** We may introduce addition (+) and multiplication (\*) over the remainders  $G_m$  previously defined as

$$G_m := \{0, 1, 2, \dots, m - 2, m - 1\} \quad (4.17)$$

For example, given  $m = 3$ , then the multiplication and addition table of  $+$  (mod 3) and  $*$  (mod 3) over  $G_3$  can be computed:

$+$ (mod 3)	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

$*$ (mod 3)	0	1	2
0	0	0	0
1	0	1	2
2	0	2	1

Table 4.1: Multiplication and Addition Table of  $G_3$



# 5 Groups

## 5.1 Group Basics

A *group* is an abstract collection consisting of:

- A *nonempty set*  $G$ .
- A *binary operation*  $\star: G \times G \rightarrow G$ .

It has the following properties:

1. **Closure**

$$\forall x, y: x \in G \wedge y \in G \rightarrow x \star y \in G \quad (5.1)$$

2. **Associativity**

$$\forall x, y, z \in G: (x \star y) \star z \equiv x \star (y \star z) \quad (5.2)$$

3. **Neutral Element**

$$\exists \epsilon \in G: \forall x \in G: x \star \epsilon \equiv \epsilon \star x \equiv x \quad (5.3)$$

That there exists a unique *neutral* element  $\epsilon \in G$ .

4. **Invertibility**

$$\forall x \in G: \exists y \in G: x \star y \equiv y \star x \equiv \epsilon \quad (5.4)$$

That there exists a unique *inverse* element  $y := x^{-1} \in G$  where  $x^{-1}$  denotes the *inverse* element of  $x$ .

**Definition 5.1.1** (Commutative Group). An *commutative group* (or *abelian group*) is a *group* for which its operation  $\star: G \times G \rightarrow G$  satisfies the additional *commutative* property:

- **Commutativity**

$$\forall x, y \in G: x \star y \equiv y \star x \quad (5.5)$$

## 5.2 Multiplicative Group

**Proposition 5.2.1** (Multiplicative Group). A *multiplicative group* is a *group*  $(G, *)$  which has the binary operation  $\ast: G \times G \rightarrow G$ :

- **Closure, Associativity.** The multiplication operation  $\ast: G \times G \rightarrow G$  is closed and is left associative.
- **Neutral Element.** The neutral element  $\epsilon$  is unique.
- **Invertibility.** The inverse element  $x^{-1}$  is unique.
- For all  $a, b \in G$  the equation

$$a \ast x = b \quad (5.6)$$

Has the unique solution

$$x = a^{-1} \ast b \quad (5.7)$$

Since

$$a * x = b \Leftrightarrow a^{-1} * (a * x) = a^{-1} * b \quad (\text{Multiply by inverse element}) \quad (5.8)$$

$$\Leftrightarrow (a^{-1} * a) * x = a^{-1} * b \quad (\text{Associativity}) \quad (5.9)$$

$$\Leftrightarrow \epsilon * x = a^{-1} * b \quad (\text{Invertibility}) \quad (5.10)$$

$$\Leftrightarrow x = a^{-1} * b \quad (\text{Neutral Element}) \quad (5.11)$$

**Remark.** An example of a multiplicative group is permutations under composition, namely  $S_n$  is a group  $(G, \circ)$  where  $\circ: G \times G \rightarrow G$ .

For example, let  $G$  be the set of permutations

$$\epsilon = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} \quad \sigma_1 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} \quad \sigma_2 = \sigma_1^2 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} \quad (5.12)$$

To verify that  $G$  does form a group with composition  $\circ$ , one may draw the multiplication table for the group. Note that

$$\sigma_2 \sigma_2 = \sigma_1^4 = \sigma_1^3 \sigma_1 = \epsilon \sigma_1 = \sigma_1 \quad (5.13)$$

$\circ$	$\epsilon$	$\sigma_1$	$\sigma_2$
$\epsilon$	$\epsilon$	$\sigma_1$	$\sigma_2$
$\sigma_1$	$\sigma_1$	$\sigma_2$	$\epsilon$
$\sigma_2$	$\sigma_2$	$\epsilon$	$\sigma_1$

Table 5.1: Multiplication Table of Composition  $\circ$  over  $G$

### 5.3 Additive Group

**Definition 5.3.1** (Additive Group). An *additive group* is a group  $(G, +)$  with the binary operation  $+: G \times G \rightarrow G$ . It has the same properties of a general group.

1. **Closure**

$$\forall x, y: x \in G \wedge y \in G \rightarrow x + y \in G \quad (5.14)$$

2. **Associativity**

$$\forall x, y, z \in G: (x + y) + z \equiv x + (y + z) \quad (5.15)$$

3. **Neutral Element**

$$\exists \epsilon \in G: \forall x \in G: x + \epsilon \equiv \epsilon + x \equiv x \quad (5.16)$$

That there exists an unique *neutral* element  $0_G \in G$  (usually denoted simply as 0).

4. **Invertibility**

$$\forall x \in G: \exists y \in G: x + y \equiv y + x \equiv 0 \quad (5.17)$$

That there exists an unique *inverse* element  $y := -x \in G$  where  $-x$  denotes the *inverse* element of  $x$ .

**Remark.** An example of an additive group is  $(\mathbb{Z}, +)$  (i.e. addition over the integers). Then for any of such *commutative group*  $(G, +)$

- *Neutral element* 0 is unique.
- *Inverse element*  $-x$  is unique.
- For any  $a, b \in G$  the equation

$$a + x = b \quad (5.18)$$

Has a unique solution

$$x = b + (-a) = b - a \quad (5.19)$$

## 5.4 Associativity of Sequential Composition of Functions

**Definition 5.4.1** (Sequential Composition of Functions). Let  $f * g$  denote the sequential composition of functions  $f * X \rightarrow Y$  and  $g: Y \rightarrow Z$  such that  $f * g: X \rightarrow Z$  where  $f$  is applied first then  $g$ , i.e.  $\forall x \in X: (f * g)(x) := g(f(x))$ .

**Proposition 5.4.1** (Associativity of Sequential Composition of Functions). Given sets  $X, Y$  and  $Z$  and

- *Injection*  $f: A \rightarrow B$
- *Injection*  $g: B \rightarrow C$
- *Injection*  $h: C \rightarrow D$

Then their composition is associative:

$$(f * g) * h \equiv f * (g * h) \quad (5.20)$$

*Proof.*

Let  $s = (f * g)$  and  $t = (s * h)$ , then  $t(x) = h(s(x)) = h(g(f(x)))$ .

Let  $u = (g * h)$  and  $v = (f * u)$ , then  $v(x) = u(f(x)) = h(g(f(x)))$ .

Together they yield the desired equality  $t(x) = v(x)$ . ■

## 5.5 Subgroups

**Definition 5.5.1** (Subgroup). Given a *group*  $(G, *)$ , then the subset  $H \subseteq G$  is a *subgroup* of  $G$  if it fulfills the properties:

1. **Closure**

$$\forall x, y: x \in H \wedge y \in H \rightarrow x * y \in H \quad (5.21)$$

2. **Neutral Element**

$$\epsilon \in H \quad (5.22)$$

That is, the *neutral* element  $\epsilon$  from  $G$  is contained within the subset  $H \subseteq G$ .

3. **Invertibility**

$$\forall x \in H: x^{-1} \in H \quad (5.23)$$

## 5.6 Lagrange's Theorem

**Theorem 5.6.1** (Lagrange's Theorem). Given a finite group of order  $n$   $(G, *)$  where

$$G := \{g_1, g_2, \dots, g_n\} \quad (5.24)$$

And its subgroup  $(H, *)$  of order  $k \leq n$

$$H := \{h_1, h_2, \dots, h_k\} \quad (5.25)$$

Then  $k|n$  ( $k$  divides  $n$ ).

$G$  can be partitioned into  $\ell$  disjoint subsets of the same size  $k$  such that

$$n = k\ell \quad (5.26)$$

**Definition 5.6.1** (Left Coset). Given  $(G, *)$  is a group,  $(H, *)$  is a subgroup of  $(G, *)$  and  $g \in G$  then the left coset  $gH$  of  $H$  in  $G$  with respect to  $g$  is defined as

$$gH := \{g * h : h \in H\} \quad (5.27)$$

**Remark.** Visually,

$$G \equiv \left. \begin{array}{c} \boxed{g_1 H} \\ \boxed{g_2 H} \\ \vdots \\ \boxed{g_\ell H} \end{array} \right\} \ell \text{ disjoint subsets} \quad (5.28)$$

To verify that the left cosets together do in fact reconstruct  $G$ , check the multiplication table

$*$	$h_1$	$h_2$	$\dots$	$h_k$
$g_1 H$	$g_1 * h_1$	$g_1 * h_2$	$\dots$	$g_1 * h_k$
$g_2 H$	$g_2 * h_1$	$g_2 * h_2$	$\dots$	$g_2 * h_k$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
$g_\ell H$	$g_\ell * h_1$	$g_\ell * h_2$	$\dots$	$g_\ell * h_k$

Table 5.2: Multiplication Table from  $\ell$  Left Cosets, Each of Size  $|H| = k$

**Proposition 5.6.1.** For any  $a, b \in G$  from  $(G, *)$

$$(a * b)^{-1} \equiv b^{-1} * a^{-1} \quad (5.29)$$



*Proof.*

$$(a * b)^{-1} \Leftrightarrow (a * b)^{-1} * \epsilon \quad (\text{Neutral element}) \quad (5.30)$$

$$\Leftrightarrow (a * b)^{-1} * (a * a^{-1}) \quad (\text{Invertibility}) \quad (5.31)$$

$$\Leftrightarrow (a * b)^{-1} * ((a * \epsilon) * a^{-1}) \quad (\text{Neutral element}) \quad (5.32)$$

$$\Leftrightarrow (a * b)^{-1} * [(a * (b * b^{-1})) * a^{-1}] \quad (\text{Invertibility}) \quad (5.33)$$

$$\Leftrightarrow (a * b)^{-1} * [(a * b) * (b^{-1} * a^{-1})] \quad (\text{Associativity}) \quad (5.34)$$

$$\Leftrightarrow [(a * b)^{-1} * (a * b)] * (b^{-1} * a^{-1}) \quad (\text{Associativity}) \quad (5.35)$$

$$\Leftrightarrow \epsilon * (b^{-1} * a^{-1}) \quad (\text{Invertibility}) \quad (5.36)$$

$$\Leftrightarrow b^{-1} * a^{-1} \quad (\text{Neutral Element}) \quad (5.37)$$

■

*Proof.* For a constructive proof of Lagrange's Theorem:

Let the binary relation  $E(x, y)$  be defined on the group  $(G, *)$ , with its subgroup  $(H, *)$

$$E(x, y) := x^{-1} * y \in H \quad (5.38)$$

For the equivalence

$$x = y \Leftrightarrow x^{-1} * y = 1 \quad (5.39)$$

Then for each of the required properties:

- **Neutral Element** from *Reflexivity* of  $E(x, y)$

$$\forall x \in G: E(x, x) \quad (5.40)$$

Since

$$E(x, x) \equiv x^{-1} * x \in H \equiv \epsilon \in H \quad (5.41)$$

Then this satisfies the *reflexivity* requirement for *equivalence relations*, and proves the *neutral element* requirement for *subgroups*.

- **Invertibility** from *Symmetry* of  $E(x, y)$

$$\forall x, y \in G: E(x, y) \rightarrow E(y, x) \quad (5.42)$$

Let for some  $h \in H$ ,  $x^{-1} * y = h$ , then by proposition 5.6.1

$$y^{-1} * x \equiv (x^{-1} * y)^{-1} \equiv h^{-1} \in H \quad (5.43)$$

Which satisfies the *symmetry* requirement for *equivalence relations*, and proves the *invertibility* requirement for *subgroups*.

- **Closure** from *Transitivity* of  $E(x, y)$

$$\forall x, y, z \in G: E(x, y) \wedge E(y, z) \rightarrow E(x, z) \quad (5.44)$$

Let for some  $h_1, h_2 \in H$ ,  $(x^{-1} * y = h_1) \wedge (y^{-1} * z = h_2)$ , then

$$x^{-1} * z \Leftrightarrow x^{-1} * \epsilon * z \quad (5.45)$$

$$\Leftrightarrow (x^{-1} * y) * (y^{-1} * z) \quad (5.46)$$

$$\Leftrightarrow h_1 * h_2 \in H \quad (5.47)$$

Which satisfies the *transitivity* requirement for *equivalence relations*, and proves the *closure* requirement for *subgroups*. ■

**Remark.** To demonstrate Lagrange's Theorem, let the *group* be constructed from  $x * y \pmod{10}$ .

Let  $(G, *)$  be a finite *group* of order  $n = 4$  where

$$G = \{1, 3, 7, 9\} \quad (5.48)$$

And  $(H, *)$  be its *subgroup* of order  $k = 2$ .

Constructing the multiplication table yields

$* \pmod{10}$	1	9
$1 * H$	1	9
$3 * H$	3	7
$7 * H$	7	3
$9 * H$	9	1

Table 5.3: Multiplication Table for  $(G, *)$

There are only  $\ell = 2$  disjoint subsets (unique cosets)  $gH$ ;  $G$  can be partitioned into  $\ell$  disjoint subsets, each of size  $|H| = 2$  such that  $4 = n = k\ell = 2 \cdot 2$ .

Visually,

$$G = \left. \begin{array}{l} 1 * H = 9 * H = \{1, 9\} \\ 3 * H = 7 * H = \{3, 7\} \end{array} \right\} \ell = 2 \quad (5.49)$$

### 5.6.1 Equivalence Classes

**Definition 5.6.2** (Equivalence Class). Given *group*  $(G, *)$  and its *subgroup*  $(H, *)$ , then the *equivalence class*  $[g]$  is defined as

$$[g] := \{y \in G \mid g^{-1} * y \in H\} \quad (5.50)$$

Then

$$\forall h \in H: g^{-1} * y = h \Leftrightarrow y = g * h \quad (5.51)$$

Which yields the equivalence

$$\{y \in G \mid g^{-1} * y \in H\} \equiv \{y \in G \mid y \in gH\} \quad (5.52)$$

Hence

$$[g] \equiv gH \quad (5.53)$$

That the *equivalence class*  $[g]$  is exactly the *left coset*  $gH$ .

Let  $\ell$  be the number of disjoint equivalence class  $[g]$ , then  $G$  can be partitioned into  $\ell$  disjoint subsets where visually,

$$G = \left. \begin{array}{c} [g_1] \equiv g_1 H \\ [g_2] \equiv g_2 H \\ \vdots \\ [g_\ell] \equiv g_\ell H \end{array} \right\} \ell \text{ disjoint subsets} \quad (5.54)$$

**Proposition 5.6.2.**

$$\forall g \in G: |gH| \equiv |H| \equiv k \quad (5.55)$$

*Proof.* Let  $I$  be the set of indices  $I := \{1, \dots, k\}$

$$\forall i, j \in I: (h_i = h_j) \Leftrightarrow (g * h_i = g * h_j) \quad (5.56)$$

$$\Leftrightarrow \forall i, j \in I: (h_i \neq h_j) \Leftrightarrow (g * h_i \neq g * h_j) \quad (5.57)$$

■



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