COMP0147 Discrete Mathematics for Computer Scientists Notes

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Notes adapted from lecture notes by Max Kanovich and Robin Hirsch [1].

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1 Foundations

Contents

1.1 Set Theory

1.1.1 Set Notations

- Set definition: $A = \{a, b, c\}$
- Set membership (element-of): $a \in A$
- Set builder notation: $\{x \mid x \in \mathbb{R} \wedge x^2 = x\}$
- Empty set: $\emptyset$

1.1.2 Properties

- No structure
- No order
- No copies

For example, a, b, c are references to actual objects in

$$\{a, b, c\} \Leftrightarrow \{c, a, b\} \Leftrightarrow \{a, b, c, b\}$$

1.1.3 Set Equality

Definition 1.1.1 (Set Equality). Set $A = B$ iff:

1. $A \subseteq B \implies \forall\, x(x \in A \rightarrow x \in B)$
2. $B \subseteq A \implies \forall\, y(y \in B \rightarrow y \in A)$

Remark. $A = B \Leftrightarrow A \subseteq B \wedge B \subseteq A$

1.1.4 Set Operations

- *Union*: $A \cup B \equiv \{x \mid x \in A \vee x \in B\}$
- *Intersection*: $A \cap B \equiv \{x \mid x \in A \wedge x \in B\}$
- *Relative Complement*: $A \setminus B \equiv \{x \mid x \in A \wedge x \notin B\}$
- *Absolute Complement*: $A^c \equiv U \setminus A \equiv \{x \mid x \in U \wedge x \notin A\}$
- *Symmetric Difference*: $A \Delta B \equiv (A \setminus B) \cup (B \setminus A) \equiv (A \cup B) \setminus (A \cap B)$
- *Cartesian Product*: $A \times B \equiv \{(x, y) \mid x \in A \wedge y \in B\}$

1.1.5 Boolean Algebra

Definition 1.1.2 (De Morgan's Laws).

$$\neg(p \vee q) \equiv \neg p \wedge \neg q \tag{1.1}$$
$$\neg(p \wedge q) \equiv \neg p \vee \neg q \tag{1.2}$$

Definition 1.1.3 (Idempotent Laws).

$$p \vee p \equiv p \tag{1.3}$$
$$p \wedge p \equiv p \tag{1.4}$$

Definition 1.1.4 (Commutative Laws).

$$p \vee q \equiv q \vee p \tag{1.5}$$
$$p \wedge q \equiv q \wedge p \tag{1.6}$$

Definition 1.1.5 (Associative Laws).

$$p \vee (q \vee r) \equiv (p \vee q) \vee r \tag{1.7}$$
$$p \wedge (q \wedge r) \equiv (p \wedge q) \wedge r \tag{1.8}$$

Definition 1.1.6 (Distributive Laws).

$$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r) \tag{1.9}$$
$$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r) \tag{1.10}$$

Definition 1.1.7 (Identity Laws).

$$p \lor \mathrm{F} \equiv p \tag{1.11}$$
$$p \lor \mathrm{T} \equiv \mathrm{T} \tag{1.12}$$
$$p \land \mathrm{T} \equiv p \tag{1.13}$$
$$p \land \mathrm{F} \equiv \mathrm{F} \tag{1.14}$$

Definition 1.1.8 (Absorption Laws).

$$p \lor (p \land q) \equiv p \tag{1.15}$$
$$p \land (p \lor q) \equiv p \tag{1.16}$$

Definition 1.1.9 (Implication and Negation Laws).

- *Identity*: $p \to q \equiv \neg p \lor q$
- *Counter-example*: $\neg(p \to q) \equiv p \land \neg q$
- *Equivalences*: $p \to q \to r \equiv (p \land q) \to r \equiv q\, to (p \to r)$
- *Absorption*:
  $p \to \mathrm{T} \equiv T$
  $p \to \mathrm{F} \equiv \neg p$
  $\mathrm{T} \to p \equiv p$
  $\mathrm{F} \to p \equiv T$
- *Contrapositive*: $p \to q \equiv \neg q \to \neg p$
- *Law of Excluded Middle*:
  $p \lor \neg p \equiv \mathrm{T}$
  $p \land \neg p \equiv \mathrm{F}$
- *Double Negation*: $\neg\neg p \equiv p$
- *Reduction to Absurdity*: $\neg p \to \mathrm{F} \equiv p$

1.1.6 Set Algebra

Definition 1.1.10 (De Morgan's Laws).

$$(A \cup B)^c \equiv A^c \cap B^c \tag{1.17}$$
$$(A \cap B)^c \equiv A^c \cup B^c \tag{1.18}$$

Definition 1.1.11 (Idempotent Laws).

$$A \cup A \equiv A \tag{1.19}$$
$$A \cap A \equiv A \tag{1.20}$$

Definition 1.1.12 (Commutative Laws).

$$A \cup B \equiv B \cup A \tag{1.21}$$
$$A \cap B \equiv B \cap A \tag{1.22}$$

Definition 1.1.13 (Associativity Laws).

$$A \cup (B \cup C) \equiv (A \cup B) \cup C \tag{1.23}$$
$$A \cap (B \cap C) \equiv (A \cap B) \cap C \tag{1.24}$$

Definition 1.1.14 (Distributive Laws).

$$A \cap (B \cup C) \equiv (A \cap B) \cup (B \cap C) \tag{1.25}$$
$$A \cup (B \cap C) \equiv (A \cup B) \cap (B \cup C) \tag{1.26}$$

Definition 1.1.15 (Identity Laws).

$$A \cup \emptyset \equiv A \tag{1.27}$$
$$A \cap \emptyset \equiv \emptyset \tag{1.28}$$
$$A \cap U \equiv A \tag{1.29}$$
$$A \cup U \equiv U \tag{1.30}$$

Definition 1.1.16 (Absorption Laws).

$$A \cup (A \cap B) \equiv A \tag{1.31}$$
$$A \cap (A \cup B) \equiv A \tag{1.32}$$

Definition 1.1.17 (Difference Identity Laws).

$$C \setminus (A \cup B) \equiv (C \setminus A) \cap (C \setminus B) \tag{1.33}$$
$$C \setminus (A \cap B) \equiv (C \setminus A) \cup (C \setminus B) \tag{1.34}$$

Definition 1.1.18 (Complement-Difference Identity Law).

$$C \setminus D \equiv C \cap D^c \tag{1.35}$$

Definition 1.1.19 (Double Complement Law).

$$(D^c)^c \equiv D \tag{1.36}$$

Definition 1.1.20 (Contraposition).

$$C \subseteq D \Leftrightarrow D^c \subseteq C^c \tag{1.37}$$
$$C = D \Leftrightarrow C^c = D^c \tag{1.38}$$

Definition 1.1.21 (Arbitrary Union).
Given sets $A_1, A_2, \dots, A_n$ where $I = \{1, 2, \dots, n\}$

$$A_1 \cup A_2 \cup \dots \cup A_n \equiv \bigcup_{i \in I} A_i \tag{1.39}$$

Then

$$x \in \bigcup_{i \in I} A_i \Leftrightarrow \exists\, i \in I : x \in A_i \tag{1.40}$$

Definition 1.1.22 (Arbitrary Intersection).
Given sets $A_1, A_2, \dots, A_n$ where $I = \{1, 2, \dots, n\}$

$$A_1 \cap A_2 \cap \cdots \cap A_n \equiv \bigcap_{i \in I} A_i \tag{1.41}$$

Then

$$x \in \bigcap_{i \in I} A_i \Leftrightarrow \forall\, i \in I\colon x \in A_i \tag{1.42}$$

1.2 Functions

Definition 1.2.1 (Function). A function f is a mapping from X to Y

$$f\colon X \mapsto Y \tag{1.43}$$

- $\text{domain}(f) = X$
- $\text{image}(f) = f(X)$

Definition 1.2.2 (Total Function). A function is *total* if

$$\text{domain}(f) = X \tag{1.44}$$

Definition 1.2.3 (Partial Function). A function is *partial* if

$$\text{domain}(f) \subseteq X \tag{1.45}$$

Definition 1.2.4 (Surjection). A function $f\colon X \mapsto Y$ is *surjective* iff

$$f(X) = Y \Leftrightarrow \forall\, y \in Y\colon\ \exists x \in X\colon\ f(x) = y \tag{1.46}$$

Namely each $y \in Y$ has a corresponding $x \in X$.

Definition 1.2.5 (Injection (Encodings, One-to-one)). A function $f\colon X \mapsto Y$ is *injective* iff

$$\forall\, x_1, x_2 \in X\colon x_1 \neq x_2 \to f(x_1) \neq f(x_2) \tag{1.47}$$

$$\Leftrightarrow \forall\, x_1, x_2 \in X\colon f(x_1) = f(x_2) \to x_1 = x_2 \tag{1.48}$$

Namely each distinct element $x \in X$ maps to a different element in Y.

Definition 1.2.6 (Bijection). A function $f\colon X \mapsto Y$ is *bijective* iff f is both *injective* and *surjective*.

$$\text{Bijective}(f) \equiv \text{Injective}(f) \wedge \text{Surjective}(f) \tag{1.49}$$

The *inverse bijection* $f^{-1}\colon Y \mapsto X$ does exist.

1.2.1 Composition of Injections

Proposition 1.2.1 (Composition of Injection)**.** Given *injections* $f\colon X \mapsto Y$ and $g\colon Y \mapsto Z$, then their *composition* $h\colon X \mapsto Z$ is given by

$$h(x) = g(f(x)) \tag{1.50}$$

Then h is also an *injective* function. Namely $h = g \circ f$ where h is composed from g and f with f applied first.

Proof. Given any $x_1, x_2 \in X$ where $x_1 \neq x_2$, then

$$f(x_1) \neq f(x_2) \tag{1.51}$$

as f is *injective*, and thus

$$h(x_1) = g(f(x_1)) \neq g(f(x_2)) = h(x_2) \tag{1.52}$$

h is *injective* consequently. ■

1.2.2 Composition of Surjection

Proposition 1.2.2 (Composition of Surjection)**.** Given *surjections* $f\colon X \mapsto Y$ and $g\colon Y \mapsto Z$, then their *composition* $h\colon X \mapsto Z$ is given by

$$h(x) = g(f(x)) \tag{1.53}$$

Then h is also a *surjective* function.

Proof. To prove $h\colon X \mapsto Z$ is *injective*, it is required to prove that

$$\forall\, z \in Z\colon\ \exists\, x \in X\colon\ h(x) = z \tag{1.54}$$

Where $h(x) \Leftrightarrow (g \circ f)(x) \Leftrightarrow g(f(x))$.

Given any element $z \in Z$ ($\forall\, z \in Z$):

1. That $g\colon Y \mapsto Z$ is *surjective* by definition, then $\exists\, y \in Y\colon\ g(y) = z$.
2. That $f\colon X \mapsto Y$ is *surjective* by definition, then $\exists\, x \in X\colon\ f(x) = y$.

Then $\forall\, z \in Z\colon\ \exists\, x \in X\colon\ h(x) = (g \circ f)(x) = g(f(x)) = g(y) = z$ holds true. ■

1.2.3 Composition of Bijection

Proposition 1.2.3 (Composition of Bijection)**.** Given *bijections* $f\colon X \mapsto Y$ and $g\colon Y \mapsto Z$, then their composition $h\colon X \mapsto Z$ is given by

$$h(x) = g(f(x)) \tag{1.55}$$

Then h is also a *bijective* function; an *inverse bijection* $h^{-1}\colon Z \mapsto X$ also exists.

1.2.4 Cardinality of Sets

Definition 1.2.7 (Cardinality)**.** The number of elements in a set X is denoted $|X|$.

Definition 1.2.8 (Equal Cardinality and Bijection)**.**

$$|X| = |Y| \tag{1.56}$$

Holds true if there exists a *bijection* $h\colon X \mapsto Y$ (one-to-one correspondence between X and Y).

Namely, X and Y have the same number of distinct elements, and each distinct element $x \in X$ corresponds to exactly one distinct element $y \in Y$.

Theorem 1.2.1 (Cantor-Bernstein)**.** Given

1. *injective* function $f\colon X \mapsto Y$
2. *injective* function $g\colon Y \mapsto X$

Then there exists a *bijective* function $h\colon X \mapsto Y$.

Equivalently,

$$(|X| \leq |Y|) \wedge (|Y| \leq |X|) \rightarrow (|X| = |Y|) \tag{1.57}$$

Remark. Examples include countable sets, enumerable sets

$$|\mathbb{Q}| = |\mathbb{Z}| = |\mathbb{N}| = \aleph_0 \tag{1.58}$$

Where the cardinality of countable sets such as the *rational numbers*, *integers* and the *natural numbers* is denoted as "alpeh-zero" ($\aleph_0$).

On the other hand, continuum such as the *real numbers* are not countable and as such

$$|\mathbb{R}| > \aleph_0 \tag{1.59}$$

1.3 Permutations

Definition 1.3.1 (Permutation)**.** The bijection – *permutation* – of

$$\begin{array}{ccccc} 1 & 2 & 3 & \cdots & n \\ \downarrow & \downarrow & \downarrow & \cdots & \downarrow \\ 1 & 2 & 3 & \cdots & n \end{array} \tag{1.60}$$

Is denoted as

$$\begin{pmatrix} 1 & 2 & 3 & \cdots & n \\ 1 & 2 & 3 & \cdots & n \end{pmatrix} \tag{1.61}$$

Where $\sigma\colon \{1, \dots, n\} \rightarrow \{1, \dots, n\}$ is the *permutation* bijection.

Definition 1.3.2 (Counting Permutations)**.**

$$|S_n| \equiv n! \tag{1.62}$$

Which is the number of different ways to permutate n elements $\{1, 2, \dots, n\} \subset \mathbb{Z}$. Together, the different permutations for n distinct elements is the *symmetric group* S_n.

Remark. For example, with $S_3 = \{1, 2, 3\}$, there are $3! = 6$ different ways to arrange the three distinct elements

$$\begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} \quad \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} \quad \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$$
$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} \quad \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} \tag{1.63}$$

Definition 1.3.3 (Order of Permutation)**.** The *order* of a permutation σ is the smallest $k \in \mathbb{Z}^+$ such that

$$\sigma^k = \epsilon \tag{1.64}$$

Where ϵ is the *identity permutation*

$$\epsilon(x) = x \tag{1.65}$$

Definition 1.3.4 (Sign of Permutation)**.** The *sign* of a permutation $\operatorname{sgn} \sigma \colon \sigma \to \{-1, +1\}$ where $\sigma \in S_n$ is defined as

$$\operatorname{sgn}(x) = (-1)^k \tag{1.66}$$

Where k is the number of *disorders* within σ, the number of pairs (x, y) such that $x > y \to \sigma(x) < \sigma(y)$ or the converse $x < y \to \sigma(x) > \sigma(y)$. Additionally,

$$\operatorname{sgn}(x) = \begin{cases} +1 & \text{if k is even} \\ -1 & \text{if k is odd} \end{cases} \tag{1.67}$$

Remark. For example, in

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$$

$1 < 2$ but $\sigma(1) = 2 > \sigma(2) = 1$, hence a disorder.

For each $i \in \{1, \dots, n\}$, starting from $i = 1$, compare $\sigma(i)$ with $\sigma(i+1), \dots, \sigma(n)$ and add the number of disordered pairs, then move on to $i + 1$ and compare $\sigma(i + 1)$ with $\sigma(i + 2), \dots, \sigma(n)$ and so on.

Theorem 1.3.1 (Composition of Permutation)**.**

$$\operatorname{sgn}(\sigma_1 \sigma_2) \equiv \operatorname{sgn}(\sigma_1) \cdot \operatorname{sgn}(\sigma_2) \tag{1.68}$$

Where

- even $\circ$ even = even
- even $\circ$ odd = odd
- odd $\circ$ even = odd
- odd $\circ$ odd = even

1.4 Binary Relations

Definition 1.4.1 (Binary Relation)**.** A binary relation $R(x, y)$ describes some relationship between x and y where $R \colon X \to Y$, $R \subseteq X \times Y$, $x \in X$ and $y \in Y$. This relation can be expressed in infix notation as xRy.

1.4.1 Equivalence Relations

Definition 1.4.2 (Equivalence Relation). A binary relation $E(x, y)$ is an *equivalence relation* on X iff it satisfies all three conditions:

1. **Reflexivity**
   $\forall\, x \in X\colon E(x, x)$
2. **Symmetry**
   $\forall\, x, y \in X\colon E(x, y) \to E(y, x)$
3. **Transitivity**
   $\forall\, x, y, z \in X\colon E(x, y) \wedge E(y, z) \to E(x, z)$

1.4.2 Equivalence Classes

Definition 1.4.3 (Equivalence Class). If $a \in X$, the *equivalence class* $[a]$ is

$$[a] \equiv \{x \in X\colon E(x, a)\} \subseteq X \tag{1.69}$$

Definition 1.4.4 (Congruence Class). For *congruence mod* m on $\mathbb{Z}$, if $a \in \mathbb{Z}$ then the *congruence class* of a is

$$[a]_m \equiv \{x \in \mathbb{Z}\colon x = a + km\} \tag{1.70}$$

Where $k \in \mathbb{Z}$.

Bibliography

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