# COMP0147 Discrete Mathematics for Computer Scientists Notes

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Notes adapted from:

- Lecture notes by Max Kanovich and Robin Hirsch [1].
- *A First Course in Abstract Algebra* by Joseph J. Rotman [2].

# Contents

# 1 Set Theory

## 1.1 Set Notations

- Set definition: $A = \{a, b, c\}$
- Set membership (element-of): $a \in A$
- Set builder notation: $\{x \mid x \in \mathbb{R} \land x^2 = x\}$
- Empty set: $\emptyset$

## 1.2 Properties

- No structure
- No order
- No copies

For example, $a, b, c$ are references to actual objects in

$$\{a, b, c\} \Leftrightarrow \{c, a, b\} \Leftrightarrow \{a, b, c, b\}$$

## 1.3 Set Equality

**Definition 1.3.1** (Set Equality)**.** Set $A = B$ iff:

1. $A \subseteq B \implies \forall\, x(x \in A \to x \in B)$
2. $B \subseteq A \implies \forall\, y(y \in B \to y \in A)$

**Remark.** $A = B \Leftrightarrow A \subseteq B \land B \subseteq A$

## 1.4 Set Operations

- *Union*: $A \cup B \coloneqq \{x \mid x \in A \lor x \in B\}$
- *Intersection*: $A \cap B \coloneqq \{x \mid x \in A \land x \in B\}$
- *Relative Complement*: $A \setminus B \coloneqq \{x \mid x \in A \land x \notin B\}$
- *Absolute Complement*: $A^c \coloneqq U \setminus A \coloneqq \{x \mid x \in U \land x \notin A\}$
- *Symmetric Difference*: $A \Delta B \coloneqq (A \setminus B) \cup (B \setminus A) \coloneqq (A \cup B) \setminus (A \cap B)$
- *Cartesian Product*: $A \times B \coloneqq \{(x, y) \mid x \in A \land y \in B\}$

## 1.5 Boolean Algebra

**Definition 1.5.1** (De Morgan's Laws).

$$\neg(p \lor q) \equiv \neg p \land \neg q \tag{1.1}$$
$$\neg(p \land q) \equiv \neg p \lor \neg q \tag{1.2}$$

**Definition 1.5.2** (Idempotent Laws).

$$p \lor p \equiv p \tag{1.3}$$
$$p \land p \equiv p \tag{1.4}$$

**Definition 1.5.3** (Commutative Laws).

$$p \lor q \equiv q \lor p \tag{1.5}$$
$$p \land q \equiv q \land p \tag{1.6}$$

**Definition 1.5.4** (Associative Laws).

$$p \lor (q \lor r) \equiv (p \lor q) \lor r \tag{1.7}$$
$$p \land (q \land r) \equiv (p \land q) \land r \tag{1.8}$$

**Definition 1.5.5** (Distributive Laws).

$$p \land (q \lor r) \equiv (p \land q) \lor (p \land r) \tag{1.9}$$
$$p \lor (q \land r) \equiv (p \lor q) \land (p \lor r) \tag{1.10}$$

**Definition 1.5.6** (Identity Laws).

$$p \lor \mathrm{F} \equiv p \tag{1.11}$$
$$p \lor \mathrm{T} \equiv \mathrm{T} \tag{1.12}$$
$$p \land \mathrm{T} \equiv p \tag{1.13}$$
$$p \land \mathrm{F} \equiv \mathrm{F} \tag{1.14}$$

**Definition 1.5.7** (Absorption Laws).

$$p \lor (p \land q) \equiv p \tag{1.15}$$
$$p \land (p \lor q) \equiv p \tag{1.16}$$

**Definition 1.5.8** (Implication and Negation Laws).

- *Identity*: $p \to q \equiv \neg p \lor q$
- *Counter-example*: $\neg(p \to q) \equiv p \land \neg q$
- *Equivalences*: $p \to q \to r \equiv (p \land q) \to r \equiv q\ to(p \to r)$

- *Absorption*:
  $p \to \mathrm{T} \equiv T$
  $p \to \mathrm{F} \equiv \neg p$
  $\mathrm{T} \to p \equiv p$
  $\mathrm{F} \to p \equiv T$
- *Contrapositive*: $p \to q \equiv \neg q \to \neg p$
- *Law of Excluded Middle*:
  $p \vee \neg p \equiv \mathrm{T}$
  $p \wedge \neg p \equiv \mathrm{F}$
- *Double Negation*: $\neg\neg p \equiv p$
- *Reduction to Absurdity*: $\neg p \to \mathrm{F} \equiv p$

## 1.6 Set Algebra

**Definition 1.6.1** (De Morgan's Laws).

$$(A \cup B)^c \equiv A^c \cap B^c \tag{1.17}$$
$$(A \cap B)^c \equiv A^c \cup B^c \tag{1.18}$$

**Definition 1.6.2** (Idempotent Laws).

$$A \cup A \equiv A \tag{1.19}$$
$$A \cap A \equiv A \tag{1.20}$$

**Definition 1.6.3** (Commutative Laws).

$$A \cup B \equiv B \cup A \tag{1.21}$$
$$A \cap B \equiv B \cap A \tag{1.22}$$

**Definition 1.6.4** (Associativity Laws).

$$A \cup (B \cup C) \equiv (A \cup B) \cup C \tag{1.23}$$
$$A \cap (B \cap C) \equiv (A \cap B) \cap C \tag{1.24}$$

**Definition 1.6.5** (Distributive Laws).

$$A \cap (B \cup C) \equiv (A \cap B) \cup (B \cap C) \tag{1.25}$$
$$A \cup (B \cap C) \equiv (A \cup B) \cap (B \cup C) \tag{1.26}$$

**Definition 1.6.6** (Identity Laws).

$$A \cup \emptyset \equiv A \tag{1.27}$$
$$A \cap \emptyset \equiv \emptyset \tag{1.28}$$
$$A \cap U \equiv A \tag{1.29}$$
$$A \cup U \equiv U \tag{1.30}$$

**Definition 1.6.7** (Absorption Laws).

$$A \cup (A \cap B) \equiv A \tag{1.31}$$
$$A \cap (A \cup B) \equiv A \tag{1.32}$$

**Definition 1.6.8** (Difference Identity Laws).

$$C \setminus (A \cup B) \equiv (C \setminus A) \cap (C \setminus B) \tag{1.33}$$
$$C \setminus (A \cap B) \equiv (C \setminus A) \cup (C \setminus B) \tag{1.34}$$

**Definition 1.6.9** (Complement-Difference Identity Law).

$$C \setminus D \equiv C \cap D^c \tag{1.35}$$

**Definition 1.6.10** (Double Complement Law).

$$(D^c)^c \equiv D \tag{1.36}$$

**Definition 1.6.11** (Contraposition).

$$C \subseteq D \Leftrightarrow D^c \subseteq C^c \tag{1.37}$$
$$C = D \Leftrightarrow C^c = D^c \tag{1.38}$$

**Definition 1.6.12** (Arbitrary Union).
Given sets $A_1, A_2, \dots, A_n$ where $I = \{1, 2, \dots, n\}$

$$A_1 \cup A_2 \cup \dots \cup A_n \coloneqq \bigcup_{i \in I} A_i \tag{1.39}$$

Then

$$x \in \bigcup_{i \in I} A_i \Leftrightarrow \exists\, i \in I : x \in A_i \tag{1.40}$$

**Definition 1.6.13** (Arbitrary Intersection).
Given sets $A_1, A_2, \dots, A_n$ where $I = \{1, 2, \dots, n\}$

$$A_1 \cap A_2 \cap \dots \cap A_n \coloneqq \bigcap_{i \in I} A_i \tag{1.41}$$

Then

$$x \in \bigcap_{i \in I} A_i \Leftrightarrow \forall\, i \in I : x \in A_i \tag{1.42}$$

# 2 Functions

## 2.1 Function Basics

**Definition 2.1.1** (Function)**.** A function $f$ is a mapping from $X$ to $Y$

$$f\colon X \mapsto Y \tag{2.1}$$

- $\text{domain}(f) = X$
- $\text{image}(f) = f(X)$

**Definition 2.1.2** (Total Function)**.** A function is *total* if

$$\text{domain}(f) = X \tag{2.2}$$

**Definition 2.1.3** (Partial Function)**.** A function is *partial* if

$$\text{domain}(f) \subseteq X \tag{2.3}$$

**Definition 2.1.4** (Surjection)**.** A function $f\colon X \mapsto Y$ is *surjective* iff

$$f(X) = Y \Leftrightarrow \forall\, y \in Y\colon\ \exists\, x \in X\colon\ f(x) = y \tag{2.4}$$

Namely each $y \in Y$ has a corresponding $x \in X$.

**Definition 2.1.5** (Injection (Encodings, One-to-one))**.** A function $f\colon X \mapsto Y$ is *injective* iff

$$\forall\, x_1, x_2 \in X\colon\ x_1 \neq x_2 \rightarrow f(x_1) \neq f(x_2) \tag{2.5}$$
$$\Leftrightarrow \forall\, x_1, x_2 \in X\colon\ f(x_1) = f(x_2) \rightarrow x_1 = x_2 \tag{2.6}$$

Namely each distinct element $x \in X$ maps to a different element in $Y$.

**Definition 2.1.6** (Bijection)**.** A function $f\colon X \mapsto Y$ is *bijective* iff $f$ is both *injective* and *surjective.*

$$\text{Bijective}(f) \coloneqq \text{Injective}(f) \wedge \text{Surjective}(f) \tag{2.7}$$

The *inverse bijection* $f^{-1}\colon Y \mapsto X$ does exist.

## 2.2 Composition of Injections

**Proposition 2.2.1** (Composition of Injection)**.** Given *injections* $f\colon X \mapsto Y$ and $g\colon Y \mapsto Z$, then their *composition* $h\colon X \mapsto Z$ is given by

$$h(x) := g(f(x)) \tag{2.8}$$

Then $h$ is also an *injective* function. Namely $h = g \circ f$ where $h$ is composed from $g$ and $f$ with $f$ applied first.

*Proof.* Given any $x_1, x_2 \in X$ where $x_1 \neq x_2$, then

$$f(x_1) \neq f(x_2) \tag{2.9}$$

as $f$ is *injective*, and thus

$$h(x_1) = g(f(x_1)) \neq g(f(x_2)) = h(x_2) \tag{2.10}$$

$h$ is *injective* consequently. ■

## 2.3 Composition of Surjection

**Proposition 2.3.1** (Composition of Surjection)**.** Given *surjections* $f\colon X \mapsto Y$ and $g\colon Y \mapsto Z$, then their *composition* $h\colon X \mapsto Z$ is given by

$$h(x) := g(f(x)) \tag{2.11}$$

Then $h$ is also a *surjective* function.

*Proof.* To prove $h\colon X \mapsto Z$ is *injective*, it is required to prove that

$$\forall\, z \in Z\colon\ \exists\, x \in X\colon h(x) = z \tag{2.12}$$

Where $h(x) \Leftrightarrow (g \circ f)(x) \Leftrightarrow g(f(x))$.

Given any element $z \in Z$ ($\forall\, z \in Z$):

1. That $g\colon Y \mapsto Z$ is *surjective* by definition, then $\exists\, y \in Y\colon g(y) = z$.
2. That $f\colon X \mapsto Y$ is *surjective* by definition, then $\exists\, x \in X\colon f(x) = y$.

Then $\forall\, z \in Z\colon\ \exists\, x \in X\colon h(x) = (g \circ f)(x) = g(f(x)) = g(y) = z$ holds true. ■

## 2.4 Composition of Bijection

**Proposition 2.4.1** (Composition of Bijection)**.** Given *bijections* $f\colon X \mapsto Y$ and $g\colon Y \mapsto Z$, then their composition $h\colon X \mapsto Z$ is given by

$$h(x) := g(f(x)) \tag{2.13}$$

Then $h$ is also a *bijective* function; an *inverse bijection* $h^{-1}\colon Z \mapsto X$ also exists.

## 2.5 Cardinality of Sets

**Definition 2.5.1** (Cardinality)**.** The number of elements in a set $X$ is denoted $|X|$.

**Definition 2.5.2** (Equal Cardinality and Bijection)**.**

$$|X| = |Y| \tag{2.14}$$

Holds true if there exists a *bijection* $h \colon X \mapsto Y$ (one-to-one correspondence between $X$ and $Y$).

Namely, $X$ and $Y$ have the same number of distinct elements, and each distinct element $x \in X$ corresponds to exactly one distinct element $y \in Y$.

**Theorem 2.5.1** (Cantor-Bernstein)**.** Given

1. *injective* function $f \colon X \mapsto Y$
2. *injective* function $g \colon Y \mapsto X$

Then there exists a *bijective* function $h \colon X \mapsto Y$.

Equivalently,

$$(|X| \leq |Y|) \wedge (|Y| \leq |X|) \to (|X| = |Y|) \tag{2.15}$$

**Remark.** Examples include countable sets, enumerable sets

$$|\mathbb{Q}| = |\mathbb{Z}| = |\mathbb{N}| = \aleph_0 \tag{2.16}$$

Where the cardinality of countable sets such as the *rational numbers*, *integers* and the *natural numbers* is denoted as ”alpeh-zero” ($\aleph_0$).

On the other hand, continuum such as the *real numbers* are not countable and as such

$$|\mathbb{R}| > \aleph_0 \tag{2.17}$$

# 3 Permutations

## 3.1 Permutation Basics

**Definition 3.1.1** (Permutation)**.** The bijection – *permutation* – of

$$\begin{array}{ccccc} 1 & 2 & 3 & \cdots & n \\ \downarrow & \downarrow & \downarrow & \cdots & \downarrow \\ \sigma(1) & \sigma(2) & \sigma(3) & \cdots & \sigma(n) \end{array} \tag{3.1}$$

Is denoted as

$$\begin{pmatrix} 1 & 2 & 3 & \cdots & n \\ \sigma(1) & \sigma(2) & \sigma(3) & \cdots & \sigma(n) \end{pmatrix} \tag{3.2}$$

Where $\sigma\colon \{1, \ldots, n\} \to \{1, \ldots, n\}$ is the *permutation* bijection.

**Definition 3.1.2** (Counting Permutations)**.**

$$|S_n| := n! \tag{3.3}$$

Which is the number of different ways to permutate $n$ elements $\{1, 2, \ldots, n\} \subset \mathbb{Z}$. Together, the different permutations for $n$ distinct elements is the *symmetric group* $S_n$.

**Remark.** For example, with $S_3 = \{1, 2, 3\}$, there are $3! = 6$ different ways to arrange the three distinct elements

$$\begin{array}{ccc} \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} & \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} & \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} \\ \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} & \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} & \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} \end{array} \tag{3.4}$$

**Definition 3.1.3** (Order of Permutation)**.** The *order* of a permutation $\sigma$ is the smallest $k \in \mathbb{Z}^+$ such that

$$\sigma^k = \epsilon \tag{3.5}$$

Where $\epsilon$ is the *identity permutation*

$$\epsilon(x) = x \tag{3.6}$$

**Definition 3.1.4** (Sign of Permutation)**.** The *sign* of a permutation $\operatorname{sgn}\sigma\colon \sigma \to \{-1, +1\}$ where $\sigma \in S_n$ is defined as

$$\operatorname{sgn}(\sigma) = (-1)^k \tag{3.7}$$

Where $k$ is the number of *disorders* within $\sigma$, the number of pairs $(x, y)$ such that $x > y \to \sigma(x) < \sigma(y)$ or the converse $x < y \to \sigma(x) > \sigma(y)$. Additionally,

$$\operatorname{sgn}(\sigma) = \begin{cases} +1 & \text{if k is even} \\ -1 & \text{if k is odd} \end{cases} \tag{3.8}$$

**Remark.** For example, in

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$$

$1 < 2$ but $\sigma(1) = 2 > \sigma(2) = 1$, hence a disorder.

For each $i \in \{1, \dots, n\}$, starting from $i = 1$, compare $\sigma(i)$ with $\sigma(i+1), \dots, \sigma(n)$ and add the number of disordered pairs, then move on to $i + 1$ and compare $\sigma(i+1)$ with $\sigma(i+2), \dots, \sigma(n)$ and so on.

**Theorem 3.1.1** (Composition of Permutation)**.**

$$\operatorname{sgn}(\sigma_1\sigma_2) := \operatorname{sgn}(\sigma_1) \cdot \operatorname{sgn}(\sigma_2) \tag{3.9}$$

Where

| $\circ$ | even | odd |
|---|---|---|
| even | even | odd |
| odd | odd | even |

Table 3.1: Sign Changes on Composition

# 4 Binary Relations

**Definition 4.0.1** (Binary Relation)**.** A binary relation $R(x, y)$ describes some relationship between $x$ and $y$ where $R\colon X \to Y$, $R \subseteq X \times Y$, $x \in X$ and $y \in Y$. This relation can be expressed in infix notation as $xRy$.

## 4.1 Equivalence Relations

**Definition 4.1.1** (Equivalence Relation)**.** A binary relation $E(x, y)$ is an *equivalence relation* on $X$ iff it satisfies all three conditions:

1. **Reflexivity**
   $\forall\, x \in X\colon E(x, x)$
2. **Symmetry**
   $\forall\, x, y \in X\colon E(x, y) \to E(y, x)$
3. **Transitivity**
   $\forall\, x, y, z \in X\colon E(x, y) \wedge E(y, z) \to E(x, z)$

## 4.2 Equivalence Classes

**Definition 4.2.1** (Equivalence Class)**.** If $a \in X$, the *equivalence class* $[a]$ is

$$[a] \coloneqq \{x \in X\colon E(x, a)\} \subseteq X \tag{4.1}$$

**Definition 4.2.2** (Congruence and Equivalence Class of mod $m$ on $\mathbb{Z}$)**.** For *congruence mod* $m$ on $\mathbb{Z}$, if $a \in \mathbb{Z}$ then the *congruence class* of $a$ is

$$[a]_m \coloneqq \{x \in \mathbb{Z}\colon x = a + km\} \tag{4.2}$$

Where $k \in \mathbb{Z}$. Since $x = a + km \Leftrightarrow x \equiv a \bmod m$, then the *equivalence class* of $a$ is also the *congruence class*.

$$\Leftrightarrow [a]_m \coloneqq \{x \in \mathbb{Z}\colon x \equiv a \bmod m\} \tag{4.3}$$

**Definition 4.2.3** (Set of Remainders)**.** Over $\mathbb{Z}$, the *remainder* $r$ from the integer division $k \div m$ is

$$r \bmod m \equiv k \bmod m \tag{4.4}$$

Then the set of remainders $G_m$ from the integer division $k \div m$ is defined by

$$G_m \coloneqq \{0, 1, 2, \dots, m-2, m-1\} \tag{4.5}$$

## 4.3 Quotient Groups

**Definition 4.3.1** (Quotient Group)**.** A *quotient group* is a group constructed via congruence mod $m$.

**Definition 4.3.2** (Congruence Class)**.** If $m \leq 2$ and $a \in \mathbb{Z}$ then the *congruence class* of $a$ mod $m$ is $[a] \subseteq \mathbb{Z}$

$$[a] \coloneqq \{b \in \mathbb{Z} \colon b \equiv a \bmod m\} \tag{4.6}$$
$$\Leftrightarrow \{a + km \colon k \in \mathbb{Z}\} \tag{4.7}$$
$$\Leftrightarrow \{\dots, a - 2m, a - m, a, a + m, a + 2m, \dots\} \tag{4.8}$$

**Remark.** Let $E(x, y) \coloneqq$ "$x - y \equiv 0 \bmod 2$", that is, $x - y$ is divisible by 2. Then,

$$[k]_2 \coloneqq \{y \colon E(k, y)\} \tag{4.9}$$

Where $[k]_2$ is the congruence class of integers modulo 2.
Computing $[0]_2$ and $[1]_2$ yields
- $[0]_2 = \{0, 2, -2, 4, -4, \dots, 2n, -2n, \dots\}$
- $[1]_2 = \{1, -1, 3, -3, \dots, 2n + 1, \dots\}$

Observe that

$$[1]_2 \oplus [1]_2 \Leftrightarrow [2]_2 \Leftrightarrow [0]_2 \tag{4.10}$$

It can be deduced that $[0]_2$ and $[1]_2$ are two congruence (and equivalence) classes which partition the integers $\mathbb{Z}$ into two disjoint subsets – integers which are odd, and integers which are even. This may be denoted as

$$\mathbb{Z}/E \equiv \{\text{EVEN}, \text{ODD}\} \tag{4.11}$$

**Definition 4.3.3** (Congruence Modular Arithmetic $\pmod m$ on $\mathbb{Z}$)**.**

$$[a]_m \oplus [b]_m \equiv [a + b]_m \tag{4.12}$$
$$[a]_m \otimes [b]_m \equiv [a \cdot b]_m \tag{4.13}$$

If $a_1 \equiv a_2 \bmod m$ and $b_1 \equiv b_2 \bmod m$ then

$$a_1 + b_1 \equiv a_2 + b_2 \bmod m \tag{4.14}$$
$$a_1 \cdot b_1 \equiv a_2 \cdot b_2 \bmod m \tag{4.15}$$
$$\tag{4.16}$$

**Remark.** We may introduce addition $(+)$ and multiplication $(*)$ over the remainders $G_m$ previously defined as

$$G_m \coloneqq \{0, 1, 2, \dots, m - 2, m - 1\} \tag{4.17}$$

For example, given $m = 3$, then the multiplication and addition table of $+ \pmod 3$ and $* \pmod 3$ over $G_3$ can be computed:

| + (mod 3) | 0 | 1 | 2 |
|---|---|---|---|
| 0 | 0 | 1 | 2 |
| 1 | 1 | 2 | 0 |
| 2 | 2 | 0 | 1 |

| $*$ (mod 3) | 0 | 1 | 2 |
|---|---|---|---|
| 0 | 0 | 0 | 0 |
| 1 | 0 | 1 | 2 |
| 2 | 0 | 2 | 1 |

Table 4.1: Multiplication and Addition Table of $G_3$

# 5 Groups

## 5.1 Group Basics

A *group* is an abstract collection consisting of:

- A *nonempty set* $G$.
- A *binary operation* $\star\colon G \times G \to G$.

It has the following properties:

1. **Closure**
$$\forall\, x, y\colon x \in G \wedge y \in G \to x \star y \in G \tag{5.1}$$
2. **Associativity**
$$\forall\, x, y, z \in G\colon (x \star y) \star z \equiv x \star (y \star z) \tag{5.2}$$
3. **Neutral Element**
$$\exists\, \epsilon \in G\colon\ \forall\, x \in G\colon x \star \epsilon \equiv \epsilon \star x \equiv x \tag{5.3}$$
That there exists an unique *neutral* element $\epsilon \in G$.
4. **Invertibility**
$$\forall\, x \in G\colon\ \exists\, y \in G\colon x \star y \equiv y \star x \equiv \epsilon \tag{5.4}$$
That there exists an unique *inverse* element $y \coloneqq x^{-1} \in G$ where $x^{-1}$ denotes the *inverse* element of $x$.

**Definition 5.1.1** (Commutative Group)**.** An *commutative group* (or *abelian group*) is a *group* for which its operation $\star\colon G \times G \to G$ satisfies the additional *commutative* property:

- **Commutativity**
$$\forall\, x, y \in G\colon x \star y \equiv y \star x \tag{5.5}$$

## 5.2 Multiplicative Group

**Proposition 5.2.1** (Multiplicative Group)**.** A *multiplicative group* is a *group* $(G, *)$ which has the binary operation $*\colon G \times G \to G$:

- **Closure, Associativity**. The multiplication operation $*\colon G \times G \to G$ is closed and is left associative.
- **Neutral Element**. The neutral element $\epsilon$ is unique.
- **Invertibility**. The inverse element $x^{-1}$ is unique.

- For all $a, b \in G$ the equation
$$a * x = b \tag{5.6}$$
Has the unique solution
$$x = a^{-1} * b \tag{5.7}$$

Since

$$\begin{aligned}
a * x = b &\Leftrightarrow a^{-1} * (a * x) = a^{-1} * b && \text{(Multiply by inverse element)} && (5.8)\\
&\Leftrightarrow (a^{-1} * a) * x = a^{-1} * b && \text{(Associativity)} && (5.9)\\
&\Leftrightarrow \epsilon * x = a^{-1} * b && \text{(Invertibility)} && (5.10)\\
&\Leftrightarrow x = a^{-1} * b && \text{(Neutral Element)} && (5.11)
\end{aligned}$$

**Remark.** An example of a multiplicative group is permutations under composition, namely $S_n$ is a group $(G, \circ)$ where $\circ\colon G \times G \to G$.

For example, let $G$ be the set of permutations

$$\epsilon = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} \quad \sigma_1 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} \quad \sigma_2 = \sigma_1^2 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} \tag{5.12}$$

To verify that $G$ does form a group with composition $\circ$, one may draw the multiplication table for the group. Note that

$$\sigma_2\sigma_2 = \sigma_1^4 = \sigma_1^3\sigma_1 = \epsilon\sigma_1 = \sigma_1 \tag{5.13}$$

| $\circ$ | $\epsilon$ | $\sigma_1$ | $\sigma_2$ |
|---|---|---|---|
| $\epsilon$ | $\epsilon$ | $\sigma_1$ | $\sigma_2$ |
| $\sigma_1$ | $\sigma_1$ | $\sigma_2$ | $\epsilon$ |
| $\sigma_2$ | $\sigma_2$ | $\epsilon$ | $\sigma_1$ |

Table 5.1: Multiplication Table of Composition $\circ$ over $G$

## 5.3 Additive Group

**Definition 5.3.1** (Additive Group)**.** An *additive group* is a *group* $(G, +)$ with the binary operation $+\colon G \times G \to G$. It has the same properties of a general *group*.

1. **Closure**
$$\forall\, x, y\colon x \in G \wedge y \in G \to x + y \in G \tag{5.14}$$
2. **Associativity**
$$\forall\, x, y, z \in G\colon (x + y) + z \equiv x + (y + z) \tag{5.15}$$
3. **Neutral Element**
$$\exists\, \epsilon \in G\colon\ \forall\, x \in G\colon x + \epsilon \equiv \epsilon + x \equiv x \tag{5.16}$$
That there exists an unique *neutral* element $0_G \in G$ (usually denoted simply as 0).

4. **Invertibility**
$$\forall\, x \in G\colon\ \exists\, y \in G\colon x + y \equiv y + x \equiv 0 \tag{5.17}$$
That there exists an unique *inverse* element $y := -x \in G$ where $-x$ denotes the *inverse* element of $x$.

**Remark.** An example of an additive group is $(\mathbb{Z}, +)$ (i.e. addition over the integers).
Then for any of such *commutative group* $(G, +)$
- *Neutral element* 0 is unique.
- *Inverse element* $-x$ is unique.
- For any $a, b \in G$ the equation
$$a + x = b \tag{5.18}$$
Has a unique solution
$$x = b + (-a) = b - a \tag{5.19}$$

## 5.4 Associativity of Sequential Composition of Functions

**Definition 5.4.1** (Sequential Composition of Functions)**.** Let $f * g$ denote the sequential composition of functions $f * X \to Y$ and $g\colon Y \to Z$ such that $f * g\colon X \to Z$ where $f$ is applied first then $g$, i.e. $\forall\, x \in X\colon (f * g)(x) := g(f(x))$.

**Proposition 5.4.1** (Associativity of Sequential Composition of Functions)**.** Given sets $X, Y$ and $Z$ and
- *Injection* $f\colon A \to B$
- *Injection* $g\colon B \to C$
- *Injection* $h\colon C \to D$

Then their composition is associative:
$$(f * g) * h \equiv f * (g * h) \tag{5.20}$$

*Proof.*
Let $s = (f * g)$ and $t = (s * h)$, then $t(x) = h(s(x)) = h(g(f(x)))$.
Let $u = (g * h)$ and $v = (f * u)$, then $v(x) = u(f(x)) = h(g(f(x)))$.
Together they yield the desired equality $t(x) = v(x)$. ■

## 5.5 Subgroups

**Definition 5.5.1** (Subgroup)**.** Given a *group* $(G, *)$, then the subset $H \subseteq G$ is a *subgroup* of $G$ if it fulfills the properties:
1. **Closure**
$$\forall\, x, y\colon x \in H \land y \in H \to x * y \in H \tag{5.21}$$
2. **Neutral Element**
$$\epsilon \in H \tag{5.22}$$
That is, the *neutral* element $\epsilon$ from $G$ is contained within the subset $H \subseteq G$.

3. **Invertibility**

$$\forall\, x \in H\colon x^{-1} \in H \tag{5.23}$$

## 5.6 Lagrange's Theorem

**Theorem 5.6.1** (Lagrange's Theorem)**.** Given a finite *group* of order $n$ $(G, *)$ where

$$G \coloneqq \{g_1, g_2, \dots, g_n\} \tag{5.24}$$

And its *subgroup* $(H, *)$ of order $k \leq n$

$$H \coloneqq \{h1_, h_2, \dots, h_k\} \tag{5.25}$$

Then $k|n$ ($k$ divides $n$).

$G$ can be *partitioned* into $\ell$ disjoint subsets of the same size $k$ such that

$$n = k\ell \tag{5.26}$$

**Definition 5.6.1** (Left Coset)**.** Given $(G, *)$ is a *group*, $(H, *)$ is a *subgroup* of $(G, *)$ and $g \in G$ then the *left coset* $gH$ of $H$ in $G$ with respect to $g$ is defined as

$$gH \coloneqq \{g * h\colon h \in H\} \tag{5.27}$$

**Remark.** Visually,

$$G \equiv \left.\begin{array}{c} \boxed{g_1 H} \\ \boxed{g_2 H} \\ \vdots \\ \boxed{g_\ell H} \end{array}\right\} \ell \text{ disjoint subsets} \tag{5.28}$$

To verify that the *left cosets* together do in fact reconstruct $G$, check the multiplication table

| $*$ | $h_1$ | $h_2$ | $\cdots$ | $h_k$ |
|---|---|---|---|---|
| $g_1 H$ | $g_1 * h_1$ | $g_1 * h_2$ | $\cdots$ | $g_1 * h_k$ |
| $g_2 H$ | $g_2 * h_1$ | $g_2 * h_2$ | $\cdots$ | $g_2 * h_k$ |
| $\vdots$ | $\vdots$ | $\vdots$ | $\ddots$ | $\vdots$ |
| $g_\ell H$ | $g_\ell * h_1$ | $g_\ell * h_2$ | $\cdots$ | $g_\ell * h_k$ |

Table 5.2: Multiplication Table from $\ell$ Left Cosets, Each of Size $|H| = k$

**Proposition 5.6.1.** For any $a, b \in G$ from $(G, *)$

$$(a * b)^{-1} \equiv b^{-1} * a^{-1} \tag{5.29}$$

*Proof.*

$$
\begin{aligned}
(a * b)^{-1} &\Leftrightarrow (a * b)^{-1} * \epsilon && \text{(Neutral element)} && (5.30)\\
&\Leftrightarrow (a * b)^{-1} * (a * a^{-1}) && \text{(Invertibility)} && (5.31)\\
&\Leftrightarrow (a * b)^{-1} * ((a * \epsilon) * a^{-1}) && \text{(Neutral element)} && (5.32)\\
&\Leftrightarrow (a * b)^{-1} * \left[(a * (b * b^{-1})) * a^{-1}\right] && \text{(Invertibility)} && (5.33)\\
&\Leftrightarrow (a * b)^{-1} * \left[(a * b) * (b^{-1} * a^{-1})\right] && \text{(Associativity)} && (5.34)\\
&\Leftrightarrow \left[(a * b)^{-1} * (a * b)\right] * (b^{-1} * a^{-1}) && \text{(Associativity)} && (5.35)\\
&\Leftrightarrow \epsilon * (b^{-1} * a^{-1}) && \text{(Invertibility)} && (5.36)\\
&\Leftrightarrow b^{-1} * a^{-1} && \text{(Neutral Element)} && (5.37)
\end{aligned}
$$

■

*Proof.* For a constructive proof of Lagrange's Theorem:
Let the binary relation $E(x, y)$ be defined on the *group* $(G, *)$, with its *subgroup* $(H, *)$

$$E(x, y) \coloneqq x^{-1} * y \in H \tag{5.38}$$

For the equivalence

$$x = y \Leftrightarrow x^{-1} * y = 1 \tag{5.39}$$

Then for each of the required properties:

- **Neutral Element** from *Reflexivity* of $E(x, y)$

  $$\forall\, x \in G\colon E(x, x) \tag{5.40}$$

  Since

  $$E(x, x) \equiv x^{-1} * x \in H \equiv \epsilon \in H \tag{5.41}$$

  Then this satisfies the *reflexivity* requirement for *equivalence relations*, and proves the *neutral element* requirement for *subgroups*.
- **Invertibility** from *Symmetry* of $E(x, y)$

  $$\forall\, x, y \in G\colon E(x, y) \to E(y, x) \tag{5.42}$$

  Let for some $h \in H$, $x^{-1} * y = h$, then by proposition 5.6.1

  $$y^{-1} * x \equiv (x^{-1} * y)^{-1} \equiv h^{-1} \in H \tag{5.43}$$

  Which satisfies the *symmetry* requirement for *equivalence relations*, and proves the *invertibility* requirement for *subgroups*.

- **Closure** from *Transitivity* of $E(x, y)$

$$\forall\, x, y, z \in G\colon E(x, y) \wedge E(y, z) \to E(x, z) \tag{5.44}$$

Let for some $h_1, h_2 \in H$, $(x^{-1} * y = h_1) \wedge (y^{-1} * z = h_2)$, then

$$\begin{aligned} x^{-1} * z &\Leftrightarrow x^{-1} * \epsilon * z && (5.45)\\ &\Leftrightarrow (x^{-1} * y) * (y^{-1} * z) && (5.46)\\ &\Leftrightarrow h_1 * h_2 \in H && (5.47) \end{aligned}$$

Which satisfies the *transitivity* requirement for *equivalence relations*, and proves the *closure* requirement for *subgroups*.

■

**Remark.** To demonstrate Lagrange's Theorem, let the *group* be constructed from $x * y \pmod{10}$.

Let $(G, *)$ be a finite *group* of order $n = 4$ where

$$G = \{1, 3, 7, 9\} \tag{5.48}$$

And $(H, *)$ be its *subgroup* of order $k = 2$.

Constructing the multiplication table yields

| $* \pmod{10}$ | 1 | 9 |
|---|---|---|
| $1 * H$ | 1 | 9 |
| $3 * H$ | 3 | 7 |
| $7 * H$ | 7 | 3 |
| $9 * H$ | 9 | 1 |

Table 5.3: Multiplication Table for $(G, *)$

There are only $\ell = 2$ disjoint subsets (unique cosets) $gH$; $G$ can be partitioned into $\ell$ disjoint subsets, each of size $|H| = 2$ such that $4 = n = k\ell = 2 \cdot 2$.

Visually,

$$G = \left.\begin{array}{|c|}\hline 1 * H = 9 * H = \{1, 9\} \\ \hline 3 * H = 7 * H = \{3, 7\} \\ \hline \end{array}\right\}\ell = 2 \tag{5.49}$$

### 5.6.1 Equivalence Classes

**Definition 5.6.2** (Equivalence Class)**.** Given *group* $(G, *)$ and its *subgroup* $(H, *)$, then the *equivalence class* $[g]$ is defined as

$$[g] \coloneqq \{y \in G \mid g^{-1} * y \in H\} \tag{5.50}$$

Then

$$\forall\, h \in H\colon g^{-1} * y = h \Leftrightarrow y = g * h \tag{5.51}$$

Which yields the equivalence

$$\{y \in G \mid g^{-1} * y \in H\} \equiv \{y \in G \mid y \in gH\} \tag{5.52}$$

Hence

$$[g] \equiv gH \tag{5.53}$$

That the *equivalence class* $[g]$ is exactly the *left coset* $gH$.

Let $\ell$ be the number of disjoint equivalence class $[g]$, then $G$ can be partitioned into $\ell$ disjoint subsets where visually,

$$G = \left.\begin{array}{c} \boxed{[g_1] \equiv g_1 H} \\ \boxed{[g_2] \equiv g_1 H} \\ \vdots \\ \boxed{[g_\ell] \equiv g_\ell H} \end{array}\right\} \ell \text{ disjoint subsets} \tag{5.54}$$

**Proposition 5.6.2.**

$$\forall\, g \in G\colon |gH| \equiv |H| \equiv k \tag{5.55}$$

*Proof.* Let $I$ be the set of indices $I \coloneqq \{1, \dots, k\}$

$$\forall\, i, j \in I\colon (h_i = h_j) \leftrightarrow (g * h_i = g * h_j) \tag{5.56}$$

$$\Leftrightarrow \forall\, i, j \in I\colon (h_1 \neq h_j) \leftrightarrow (g * h_i \neq g * h_j) \tag{5.57}$$

■

**Remark.** Let $A_n$ be the set of all *even permutations* and $B_n$ be the set of all *odd permutations.*

Given the *group* $(S_n, *)$, then $(A_n, *)$ is a *subgroup* of $S_n$.

With the multiplication table

| $*$ | $A_n$ |
|---|---|
| $\epsilon * A_n$ | $A_n$ |
| $\epsilon * A_n$ | $B_n$ |

Table 5.4: Multiplication Table for Group $S_n$

Since

$$\sigma * A_n \equiv \begin{cases} A_n & \text{if } \sigma \text{ is even} \\ B_n & \text{if } \sigma \text{ is even} \end{cases} \tag{5.58}$$

Hence,

$$|A_n| \equiv \frac{1}{2} \cdot |S_n| \equiv \frac{1}{2} \cdot n! \tag{5.59}$$

### 5.6.2 Order of an Element in Lagrange's Theorem

**Definition 5.6.3** (Order of an Element)**.** Given a *group* $(G, *)$ and element $a \in G$ then the *order* of the element $a$ is the smallest $k \in \mathbb{Z}^+$ such that

$$a^k = \epsilon \tag{5.60}$$

**Proposition 5.6.3.** Given a *group* $(G, *)$ with *order* $n$, then for any $a \in G$, should its *order* $k$ exist, then $k|n$ ($k$ divides $n$).

**Proposition 5.6.4.** Given *group* $(G, *)$,

$$\forall\, a \in G\colon a^{|G|} \equiv 1 \tag{5.61}$$

*Proof.* With the *cyclic subgroup* generated by $a \in G$

$$\{a^m \mid m \in \mathbb{Z}\} = \{\epsilon, a, a^2, ...\} \tag{5.62}$$

■

**Remark.** This may be used to calculate the modulo of integers raised to large exponents. For example, for $2^{20} \pmod{15}$. To compute this, let the *multiplicative group* $(G, *)$ be defined over $G$ of *order* 8 where

$$G = \{1, 2, 4, 7, 8, 11, 13, 14\} \tag{5.63}$$

And the *binary operation* $x * y := x * y \pmod{15}$.

Note that $2^{-1} = 8 \pmod{15}$ and $4^{-1} = 4 \pmod{15}$.

Since $|G| = 8$,

$$2^8 = 1 \pmod{15} \tag{5.64}$$

Then $2^{20} \pmod{15}$ can be calculated by decomposing its exponent:

$$2^{20} = 2^{2\cdot 8+4} = (2^8)^2 * 2^4 = 1 * 16 = 1 \pmod{15} \tag{5.65}$$

# 6 Euclidean Algorithm

## 6.1 Euclidean Algorithm Basics

**Definition 6.1.1** (Euclidean Algorithm)**.** The *Euclidean Algorithm* can be used to compute the *greatest common divisor* of two integers $a, b \in \mathbb{Z}$, denoted $\gcd(a, b)$.

Its process, given $a \geq b$ is

$$\begin{aligned}
a &= q_0 \cdot b + r_1 && (6.1)\\
b &= q_1 \cdot r_1 + r_2 && (6.2)\\
r_1 &= q_2 \cdot r_2 + r_3 && (6.3)\\
&\vdots\\
r_{k-1} &= q_k \cdot r_k + r_{k+1} && (6.4)\\
r_k &= q_{k+1} \cdot r_{k+1} + r_{k+2} && (6.5)\\
&\vdots\\
r_{n-1} &= q_n \cdot r_n + r_{n+1} && (6.6)\\
r_n &= q_{n+1} \cdot r_{n+1} + 0 && (6.7)
\end{aligned}$$

Such that $\gcd(a, b) := r_{n+1}$.

## 6.2 gcd(a, b) as a Linear Combination of a and b

**Proposition 6.2.1.** Given $a, b \in \mathbb{Z}$, then for some $k_1, k_2 \in \mathbb{Z}$, and some $d \in \mathbb{Z}$,

$$d = \gcd(a, b) = k_1 a + k_2 b \tag{6.8}$$

**Remark.** To solve the congruence $4 * x = 1 \pmod{17}$ for $x$, find $x$ in the form of $x = 4^{-1} \pmod{17}$.

For instance, to find $\gcd(34, 13)$ as a linear combination $k_1 a + k_2 b$, then first use the Euclidean algorithm to find $\gcd(34, 13)$:

$$\begin{array}{r|r}
34 = 2 \cdot 13 + 8 & a = 2 \cdot b + r_1 \\
13 = 8 + 5 & b = r_1 + r_2 \\
8 = 5 + 3 & r_1 = r_2 + r_3 \\
5 = 3 + 2 & r_2 = r_3 + r_4 \\
3 = 2 + \boxed{1} & r_3 = r_4 + \boxed{r_5} \\
2 = 2 \cdot 1 + 0 & r_4 = 2 \cdot r_5 + 0
\end{array} \tag{6.9}$$

Note that

$$
\begin{aligned}
a &= 2 \cdot b + r_1 \\
b &= r_1 + r_2 \\
r_1 &= r_2 + r_3 \\
r_2 &= r_3 + r_4 \\
r_3 &= r_4 + \boxed{r_5} \\
r_4 &= 2 \cdot r_5 + 0
\end{aligned}
\quad \Leftrightarrow \quad
\begin{aligned}
r_1 &= a - 2b \\
r_2 &= b - r_1 \\
r_3 &= r_1 - r_2 \\
r_4 &= r_2 + r_3 \\
\boxed{r_5} &= r_3 - r_4
\end{aligned}
\tag{6.10}
$$

It is now possible to *collect* $k_1$ and $k_2$ in a bottom-up manner:

$$\boxed{r_5} = r_3 - r_4 \tag{6.11}$$
$$= r_3 - (r_2 - r_3) \tag{6.12}$$
$$= -r_2 + 2r_3 \tag{6.13}$$
$$= -r_2 - 2(r_1 - r_2) \tag{6.14}$$
$$= 2r_1 - 3r_2 \tag{6.15}$$
$$= 2r_1 - 3(b - r_1) \tag{6.16}$$
$$= -3b + 5r_1 \tag{6.17}$$
$$= -3b + 5(a - 2b) \tag{6.18}$$
$$= 5a - 13b \tag{6.19}$$

Hence $\gcd(34, 13) = gcd(a, b) = 5a - 13b$ for some $a, b \in \mathbb{Z}$. One may verify this by checking that

$$5 \cdot 34 - 13 \cdot 13 = 170 - 169 = 1 \tag{6.20}$$

## 6.3 Problems for Integers Modulo m

- $\boxed{a * x = b \pmod{m} \Leftrightarrow x = a^{-1} * b \pmod{m}}$
  For $\mathbb{R}^+$, given some $a, b, m \in \mathbb{Z}$

$$a * x = b \pmod{m} \tag{6.21}$$
$$\Leftrightarrow a^{-1} * a * x = a^{-1} * b \pmod{m} \tag{6.22}$$
$$\Leftrightarrow x = a^{-1} * b \pmod{m} \tag{6.23}$$

- $\boxed{a^n \pmod{m} \Leftrightarrow (a \cdot a^2 \cdot a^4 \cdot a^8, \dots) \pmod{m}}$
  That is, to decompose the exponent into smaller equivalences.
- $\boxed{x^a = b \pmod{m} \Leftrightarrow x = b^{a^{-1}} \pmod{m}}$

For $\mathbb{R}^+$, given some $a, b, m \in \mathbb{Z}$

$$x^a = b \pmod m \tag{6.24}$$
$$x = \sqrt[a]{b} \pmod m \tag{6.25}$$
$$x = b^{\frac{1}{a}} \pmod m \tag{6.26}$$
$$x = b^{a^{-1}} \pmod m \tag{6.27}$$

- For the discrete logarithm: $\boxed{a^x = b \pmod m \Leftrightarrow x = \log_a b \pmod m}$

## 6.4 Multiplicative Group of Integers Modulo m

**Definition 6.4.1** (Relatively Prime, Coprime)**.** Two integers $a, b \in \mathbb{Z}$ are *relatively prime* (or *coprime*) if

$$\gcd(a, b) = 1 \tag{6.28}$$

**Definition 6.4.2** (Multiplicative Group of mod $m$)**.** Given $m \in \mathbb{Z}$, then

$$G_m^\times := \{a \in \mathbb{Z} \mid (1 \le a < m) \wedge (\gcd(a, b) = 1)\} \tag{6.29}$$

Forms a group $(G_m^\times, * \pmod m)$ under *multiplicative modulo $m$*.

1. **Closure**

$$\forall\, a, b, m \in G_m^\times\colon (\gcd(a, m) = 1) \wedge (\gcd(b, m) = 1) \to (\gcd(a * b, m) = 1) \tag{6.30}$$

2. **Associativity**

   Given by multiplication on integers modulo $m$.

3. **Neutral Element**

$$\forall\, m \in G_m^\times\colon\ \gcd(1, m) = 1 \tag{6.31}$$

4. **Invertibility**

$$\forall\, a \in G_m^\times\colon\ \exists\, y \in G_m^\times\colon a * y = 1 \pmod m \tag{6.32}$$

   For which the inverse element $y$ is denoted $a^{-1}$, giving

$$\forall\, a \in G_m^\times\colon a * a^{-1} = 1 \pmod m \tag{6.33}$$

**Theorem 6.4.1** (Euler Totient Function)**.** *Given the multiplicative modulo group $G_m^\times$, then*

$$\phi(m) := |G_m^\times| \tag{6.34}$$

**Theorem 6.4.2.** *If $p$ is prime then*

$$\phi(p) \equiv p - 1 \tag{6.35}$$

**Theorem 6.4.3.** *If $p$ is prime and $k \ge 1$ then*

$$\phi(p^k) \equiv p^{k-1}(p - 1) \tag{6.36}$$

**Theorem 6.4.4.** If $a, b \in \mathbb{Z}$ and $a, b$ are *relatively prime* (i.e. $\gcd(a, b) = 1$) then

$$\phi(ab) \equiv \phi(a)\phi(b) \tag{6.37}$$

**Theorem 6.4.5.** If $a, m \in \mathbb{Z}$ are *relatively prime* (i.e. $\gcd(a, m) = 1$) then

$$a^{\phi(m)} = 1 \pmod{m} \tag{6.38}$$

**Theorem 6.4.6** (Fermat's Little Theorem)**.** Given $p$ is a prime number, then for any $a \in \mathbb{Z}$

$$a^p \equiv a \pmod{p} \tag{6.39}$$

Additionally, if $a, p \in \mathbb{Z}$ are *relatively prime*, $\gcd(a, p) = 1$,

$$a^{p-1} \equiv 1 \pmod{p} \tag{6.40}$$

**Remark.** Given $a \in G_m^\times$, to find $x$ such that

$$a * x = b \pmod{m} \tag{6.41}$$

Find $a^{-1} \pmod{m}$.

For example, for

$$13 * x = 6 \pmod{34} \tag{6.42}$$

Since

$$x = 13^{-1} * 6 \pmod{34} \tag{6.43}$$

Find $13^{-1} \pmod{34}$ via the *Euclidean algorithm* which gives

$$13^{-1} = 21 \pmod{34} \tag{6.44}$$

Then

$$x = 21 * 6 \pmod{34} \tag{6.45}$$
$$= 126 - 3 * 34 \pmod{34} \tag{6.46}$$
$$= 24 \pmod{34} \tag{6.47}$$

**Remark.** To compute expressions of the form

$$a^n \pmod{m} \tag{6.48}$$

One should decompose $a^n$ to $a^n = a \cdot a^2 \cdot a^4 \cdot \cdots$, and use Fermat's Little Theorem and Euler Totient Function Identities whenever possible.

**Remark.** For equations of the form

$$x^a = b \pmod{m} \tag{6.49}$$

Then

$$x = b^{a^{-1}} \pmod{m} \tag{6.50}$$

If $\gcd(a, \phi(m)) = 1$ then

$$a * y = 1 \pmod{\phi(m)} \tag{6.51}$$
$$x = b^y \pmod{m} \tag{6.52}$$

if $\gcd(b, m) = 1$, that is if $b, m$ are *relatively prime*

$$\begin{aligned} x^a &= (b^y)^a \pmod{m} && (6.53)\\ &= b^{a*y} \pmod{m} && (6.54)\\ &= b^{1+k\phi(m)} \pmod{m} && (6.55)\\ &= b * (b^{\phi(m)})^k \pmod{m} && (6.56)\\ &= b * 1^k \pmod{m} && (6.57)\\ &= b \pmod{m} && (6.58) \end{aligned}$$

## 6.5 Rivest–Shamir–Adleman (RSA) Cryptography

**Definition 6.5.1** (RSA, Public Keys and Private Keys)**.** Given actors Alice and Bob, the process of RSA is

1. Alice provides *secrete* primes $p$ and $q$.
$$n = p * q \tag{6.59}$$
2. Alice provides two integers $d$ and $e$ such that
$$d * e = 1 \pmod{\phi(p * q)} \tag{6.60}$$
3. Alice distributes the pair $(n, e)$ to everyone.
4. Encryption and Decryption is then
$$\text{encrypt}_{n,e}(m) := m^e \pmod{n} \tag{6.61}$$
$$\text{decrypt}_{n,d}(m) := c^d \pmod{n} \tag{6.62}$$
5. Bob *encrypts* message $m$ as the encrypted message $c$ where
$$c := \text{encrypt}_{n,e}(m) \tag{6.63}$$
And sends $c$ to Alice.
6. Alice *decrypts* $c$ as
$$m' = \text{decrypt}_{n,d}(c) \tag{6.64}$$
Check that $\gcd(m, n) = 1$, that is if $m, n$ are *relatively prime*, then
$$\begin{aligned} m' \pmod{n} &= c^d \pmod{n} && (6.65)\\ &= (m^e)^d \pmod{n} && (6.66)\\ &= m^{d*e} \pmod{n} && (6.67)\\ &= m^{1+k\phi(p*q)} \pmod{n} && (6.68)\\ &= m \pmod{n} && (6.69) \end{aligned}$$
Then *only* Alice can decrypt the encrypted message $c$ in polynomial time.

**Remark.** An example of the RSA process:

1. Alice provides secret primes $p = 3, q = 41$

$$n = 3 * 41 = 123 \tag{6.70}$$

2. Alice provides two integers $d = 27, e = 3$

$$\begin{aligned}
d * e \pmod{\phi(3 * 41)} &= 27 * 3 \pmod{\phi(3 * 41)} && (6.71)\\
&= 81 \pmod{[\phi(3) * \phi(41)]} && (6.72)\\
&= 81 \pmod{[2 * 40]} && (6.73)\\
&= 81 \pmod{80} && (6.74)\\
&= 1 \pmod{80} && (6.75)
\end{aligned}$$

3. Alice distributes $(n, e) = (123, 3)$ to everyone.
4. The encryption and decryption functions are

$$\begin{aligned}
\text{encrypt}_{n,e}(m) &= m^3 \pmod{n} && (6.76)\\
\text{decrypt}_{n,d}(c) &= c^{27} \pmod{n} && (6.77)
\end{aligned}$$

5. Given a message $m = 5$ then Bob sends

$$\begin{aligned}
c &= 5^3 \pmod{123} && (6.78)\\
&= 125 \pmod{123} && (6.79)\\
&= 2 \pmod{123} && (6.80)
\end{aligned}$$

6. Alice receives the encrypted message $c = 2$ and decrypts with the fact that $\gcd(123, 5) = 1$

$$\begin{aligned}
m' \pmod{123} &= 2^{27} \pmod{123} && (6.81)\\
&= 5 \pmod{123} && (6.82)
\end{aligned}$$

# 7 Linear Algebra

## 7.1 Matrix Basics

**Definition 7.1.1** (Matrix)**.** A $(n \times m)$-dimension matrix $A$ has $n$ rows and $m$ columns, and each of its entries $a_{j,k}$, for $1 \leq j \leq n$ and $1 \leq k \leq m$ are denoted as

$$A = \begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,m} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n,1} & a_{n,2} & \cdots & a_{n,m} \end{bmatrix} \tag{7.1}$$

**Definition 7.1.2** (Set of Matrices of Dimension $n \times m$)**.** Let $\mathcal{M}(n, m)$ denote the set of all matrices with dimension $n \times m$, that is, having $n$ rows and $m$ columns.

**Definition 7.1.3** (Square Matrix)**.** A *square matrix* is a matrix with dimension $n \times n$.

**Definition 7.1.4** (Matrix Addition)**.** Let $A, B \in \mathcal{M}(n, m)$ be two matrices of the same dimension $n \times m$. Then the sum matrix $C = A + B$ is defined to have entries

$$c_{j,k} = a_{j,k} + b_{j,k} \tag{7.2}$$

That is,

$$\begin{aligned} &\begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,m} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n,1} & a_{n,2} & \cdots & a_{n,m} \end{bmatrix} + \begin{bmatrix} b_{1,1} & b_{1,2} & \cdots & b_{1,m} \\ b_{2,1} & b_{2,2} & \cdots & b_{2,m} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n,1} & b_{n,2} & \cdots & b_{n,m} \end{bmatrix} \\ := &\begin{bmatrix} a_{1,1} + b_{1,1} & a_{1,2} + b_{1,2} & \cdots & a_{1,m} + b_{1,m} \\ a_{2,1} + b_{2,1} & a_{2,2} + b_{2,2} & \cdots & a_{2,m} + b_{2,m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n,1} + b_{n,1} & a_{n,2} + b_{n,2} & \cdots & a_{n,m} + b_{n,m} \end{bmatrix} \end{aligned} \tag{7.3}$$

**Definition 7.1.5** (Matrix Multiplication)**.** Let $A$ be an $(l \times m)$ matrix and $B$ be an $(m \times n)$ matrix. Then their product $C = A \cdot B$ is the $(l \times n)$ matrix where each entry $c_{j,k}$ is

$$c_{j,k} := \sum_{s=1}^{m} a_{j,s} b_{s,k} \tag{7.4}$$

Note that matrix multiplication is *not commutative*, that is, for most cases $A \cdot B \neq B \cdot A$

**Definition 7.1.6** (Identity Matrix)**.** Let $I_n$ denote the *identity* matrix with dimension $n \times n$

$$I_n := \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} \tag{7.5}$$

Notice that all diagonal entries $i_{j,k}$ with indices $j = k$ is 1, while all other entries are 0.

Alternatively, the *identity* matrix can be defined with entries $\delta_{j,k}$ where $\delta$ is the *Kronecker symbol* such that

$$\delta_{j,k} := \begin{cases} 1 & j = k \\ 0 & j \neq k \end{cases} \tag{7.6}$$

**Definition 7.1.7** (Matrix Multiplication by Scalar $\lambda$)**.** Let $\lambda \in \mathbb{R}$ be a constant, then the multiplication of an $(n \times m)$-dimension matrix $A$ by $\lambda$ is defined as

$$\lambda A := \begin{bmatrix} \lambda a_{1,1} & \lambda a_{1,2} & \cdots & \lambda a_{1,m} \\ \lambda a_{2,1} & \lambda a_{2,2} & \cdots & \lambda a_{2,m} \\ \vdots & \vdots & \ddots & \vdots \\ \lambda a_{n,1} & \lambda a_{n,2} & \cdots & \lambda a_{n,m} \end{bmatrix} \tag{7.7}$$

If the dimension of $A$ is $n \times n$, i.e. $A$ is a *square matrix*, then $\lambda A$ is equivalently

$$\lambda A := \begin{bmatrix} \lambda & 0 & \cdots & 0 \\ 0 & \lambda & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda \end{bmatrix} \begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n,1} & a_{n,2} & \cdots & a_{n,n} \end{bmatrix} \tag{7.8}$$

**Lemma 7.1.1.** If $A$ is a matrix with dimension $n \times n$, $A$ is a *square* matrix, then

$$AI \equiv IA \equiv A \tag{7.9}$$

Where $I$ is the *identity* matrix with dimension $n \times n$.

*Proof.* Let $B = AI$, then

$$b_{j,k} = \sum_{s=1}^{n} a_{j,s}\delta_{s,k} \tag{7.10}$$

Only $\delta_{k,k}$ is non-zero, thus $b_{j,k} = a_{j,k}$. The same is true for $IA$. ∎

### 7.1.1 Matrix Addition and Multiplication Properties

**Proposition 7.1.1** (Associative Matrix Multiplication)**.** Given matrices $A \in \mathcal{M}(n, m)$, $B \in \mathcal{M}(m, p)$ and $C \in \mathcal{M}(p, q)$ then

$$(AB)C \equiv A(BC) \tag{7.11}$$

*Proof.* The entry $t_{j,l}$ of $T = (AB)C$ is

$$t_{j,l} = \sum_{k=1}^{p} \left( \sum_{s=1}^{m} a_{j,s} b_{s,k} \right) c_{k,l} \equiv \sum_{k=1}^{p} a_{j,s} \left( \sum_{s=1}^{m} b_{s,k} c_{k,l} \right) = u_{j,l} \tag{7.12}$$

Where $u_{j,l}$ are entries of the matrix $U = A(BC)$ ∎

**Proposition 7.1.2** (Distributive Matrix Multiplication)**.** Given matrices $A \in \mathcal{M}(n,m)$, $B \in \mathcal{M}(m,p)$ and $C \in \mathcal{M}(p,q)$ then

$$A(B+C) = AB + AC \tag{7.13}$$
$$(A+B)C = AC + BC \tag{7.14}$$

*Proof.* Let $S = A(B+C)$ and $E = AB + AB$, then each entry $s_{j,l}$ from $S$ is

$$s_{j,l} = \sum_{s=1}^{m} a_{j,s}(b_{s,l} + c_{s,l}) \equiv \sum_{s=1}^{m} a_{j,s} b_{s,l} + \sum_{s=1}^{m} a_{j,s} c_{s,l} = e_{j,l} \tag{7.15}$$

Where $e_{j,l}$ are entries from $E$.

Let $T = (A+B)C$ and $F = AC + BC$, then each entry $t_{j,l}$ from $T$ is

$$t_{j,l} = \sum_{s=1}^{m} (a_{j,s} + b_{s,l}) c_{s,l} \equiv \sum_{s=1}^{m} a_{j,s} c_{s,l} + \sum_{s=1}^{m} b_{j,s} c_{s,l} = f_{j,l} \tag{7.16}$$

Where $f_{j,l}$ are entries from $F$. ∎

### 7.1.2 Determinant of a Square Matrix

**Definition 7.1.8** (Determinant of a $2 \times 2$ Matrix)**.** Given a $2 \times 2$ *square* matrix $A \in \mathcal{M}(2,2)$

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \tag{7.17}$$

Then the determinant of $A$, denoted $\det(A)$ or $|A|$ is calculated with

$$\det(A) = \left| \begin{bmatrix} a & b \\ c & d \end{bmatrix} \right| = ad - bc \tag{7.18}$$

**Definition 7.1.9** (Determinant of a $3 \times 3$ Matrix)**.** Given a $3 \times 3$ *square* matrix $A \in \mathcal{M}(3,3)$

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \tag{7.19}$$

Then the determinant of $A$, denoted $\det(A)$ or $|A|$ is calculated with

$$\det(A) = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a \begin{vmatrix} \square & \square & \square \\ \square & e & f \\ \square & h & i \end{vmatrix} - b \begin{vmatrix} \square & \square & \square \\ d & \square & f \\ g & \square & i \end{vmatrix} + c \begin{vmatrix} \square & \square & \square \\ d & e & \square \\ g & h & \square \end{vmatrix} \tag{7.20}$$

$$= a \begin{vmatrix} e & f \\ h & i \end{vmatrix} - b \begin{vmatrix} d & f \\ g & i \end{vmatrix} + c \begin{vmatrix} d & e \\ g & h \end{vmatrix} \tag{7.21}$$

$$= aei - afh + bfg - bdi + cdh - ceg \tag{7.22}$$

**Definition 7.1.10** (Upper Triangular Matrix)**.** An $n \times n$ matrix $A \in \mathcal{M}(n,n)$ is called a *upper triangular* (or *right triangular*) matrix if it has the form

$$A = \begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ & a_{2,2} & \cdots & a_{2,n} \\ & & \ddots & \vdots \\ 0 & & & a_{n,n} \end{bmatrix} \tag{7.23}$$

Where all the lower triangular part are 0s.

**Lemma 7.1.2** (Determinant of an Upper Triangular Matrix)**.** Given an $n \times n$ *upper triangular* matrix $A$, then its *determinant* $\det(A)$ can be calculated as

$$\det(A) = \begin{vmatrix} \gamma_1 & * & * & \cdots & * \\ \vdots & \gamma_2 & * & \iddots & \vdots \\ \vdots & \cdots & \gamma_3 & * & * \\ \vdots & \iddots & \vdots & \ddots & * \\ 0 & \cdots & \cdots & \cdots & \gamma_n \end{vmatrix} = \gamma_1 \gamma_2 \cdots \gamma_n \tag{7.24}$$

Where $*$ represents arbitrary entries.

**Corollary 7.1.2.1.** A specialization of this lemma is the case for $3 \times 3$ *upper triangular* matrix $A$:

$$\det(A) = \begin{vmatrix} \gamma_1 & * & * \\ 0 & a & b \\ 0 & c & d \end{vmatrix} = \begin{vmatrix} \gamma_1 & * & * \\ 0 & a & b \\ 0 & 0 & d - b \cdot \frac{c}{a} \end{vmatrix} = \gamma_1(ad - bc) \tag{7.25}$$

## 7.2 Solving Linear System of Equations

**Definition 7.2.1.** Matrices are useful for solving a *linear system of equations* of the form

$$\begin{cases} a_{1,1}x_1 + a1, 2x_2 + \cdots + a_{1,n}x_n & = b_1 \\ a_{2,1}x_1 + a2, 2x_2 + \cdots + a_{2,n}x_n & = b_2 \\ \vdots & \\ a_{n,1}x_1 + an, 2x_2 + \cdots + a_{n,n}x_n & = b_n \end{cases} \tag{7.26}$$

Then, the matrix of the *coefficients* is denoted as $A$ with dimension $n \times n$ where

$$A = \begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n,1} & a_{n,2} & \cdots & a_{n,n} \end{bmatrix} \tag{7.27}$$

The *unknowns* are denoted as $X$ with dimension $n \times 1$ where

$$X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \tag{7.28}$$

The *constants* are denoted as $B$ with dimension $n \times 1$ where

$$B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} \tag{7.29}$$

Together, they yield the matrix equation

$$A \cdot X = B \tag{7.30}$$

To solve for $X$, one needs to find the *inverse* matrix $A^{-1}$ of $A$ such that

$$A \cdot X = B \tag{7.31}$$
$$A^{-1} \cdot A \cdot X = A^{-1} \cdot B \tag{7.32}$$
$$I \cdot X = A^{-1} \cdot B \tag{7.33}$$
$$X = A^{-1} \cdot B \tag{7.34}$$

Where $I$ is the *identity* matrix.

## 7.3 Gaussian Elimination

**Definition 7.3.1** (Augmented Matrix)**.** Given a system of linear equations

$$\begin{cases} a_{1,1}x_1 + a_{1,2}x_2 + \cdots + a_{1,n}x_n & = b_1 \\ a_{2,1}x_1 + a_{2,2}x_2 + \cdots + a_{2,n}x_n & = b_2 \\ \vdots \\ a_{n,1}x_1 + a_{n,2}x_2 + \cdots + a_{n,n}x_n & = b_n \end{cases} \tag{7.35}$$

Then its *augmented* matrix $A|B$ is

$$\left[\begin{array}{ccccc|c} a_{1,1} & a_{1,2} & \cdots & a_{1,n} & b_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} & b_{2,n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n,1} & a_{n,2} & \cdots & a_{n,n} & b_{n,n} \end{array}\right] \tag{7.36}$$

**Definition 7.3.2** (Row Operations)**.**

1. **Multiply and Add Row**
   Multiply row by scalar $\gamma$ then add the result to another row.

$$\det(A') = \det(A) \tag{7.37}$$

2. **Swap Rows**

$$\det(A') = -\det(A) \tag{7.38}$$

3. **Multiply Row**
   Multiply a row by scalar $\gamma$.

$$\det(A') = \gamma \det(A) \tag{7.39}$$

**Definition 7.3.3** (Gaussian Elimination)**.** Using the *row operations* applied to $A|B$ then one transforms $AX = B$ into an equivalent system

$$A'X = B' \tag{7.40}$$

If it is the case that

$$A' = I \tag{7.41}$$

Then there exists a *solution* $X = B'$ to the system

$$B' = A'X = IX = X \tag{7.42}$$

**Definition 7.3.4** (Inverse Matrix)**.** The *inverse* matrix $A^{-1}$ of $A$ is the matrix for which under multiplication yields the *identity* matrix $I$

$$AA^{-1} \equiv A^{-1}A \equiv I \tag{7.43}$$

With *Gaussian Elimination* applied to $A|I$ then one transforms

$$AA^{-1} = I \Rightarrow A'A^{-1} = B' \tag{7.44}$$

If

$$A' = I \tag{7.45}$$

Then there exists a solution to $A^{-1} = B'$

$$B' = A'A^{-1} = IA^{-1} = A^{-1} \tag{7.46}$$

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