

# **COMP0147 Discrete Mathematics for Computer Scientists Notes**

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Notes adapted from lecture notes by Max Kanovich and Robin Hirsch [1].

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# 1 Foundations

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## 1.1 Set Theory

### 1.1.1 Set Notations

- Set definition:  $A = \{a, b, c\}$
- Set membership (element-of):  $a \in A$
- Set builder notation:  $\{x \mid x \in \mathbb{R} \wedge x^2 = x\}$
- Empty set:  $\emptyset$

### 1.1.2 Properties

- No structure
- No order
- No copies

For example,  $a, b, c$  are references to actual objects in

$$\{a, b, c\} \Leftrightarrow \{c, a, b\} \Leftrightarrow \{a, b, c, b\}$$

### 1.1.3 Set Equality

**Definition 1.1.1** (Set Equality). Set  $A = B$  iff:

1.  $A \subseteq B \implies \forall x(x \in A \rightarrow x \in B)$
2.  $B \subseteq A \implies \forall y(y \in B \rightarrow y \in A)$

**Remark.**  $A = B \Leftrightarrow A \subseteq B \wedge B \subseteq A$

### 1.1.4 Set Operations

- *Union:*  $A \cup B \equiv \{x \mid x \in A \vee x \in B\}$
- *Intersection:*  $A \cap B \equiv \{x \mid x \in A \wedge x \in B\}$
- *Relative Complement:*  $A \setminus B \equiv \{x \mid x \in A \wedge x \notin B\}$
- *Absolute Complement:*  $A^c \equiv U \setminus A \equiv \{x \mid x \in U \wedge x \notin A\}$
- *Symmetric Difference:*  $A \Delta B \equiv (A \setminus B) \cup (B \setminus A) \equiv (A \cup B) \setminus (A \cap B)$
- *Cartesian Product:*  $A \times B \equiv \{(x, y) \mid x \in A \wedge y \in B\}$

### 1.1.5 Boolean Algebra

**Definition 1.1.2** (De Morgan's Laws).

$$\neg(p \vee q) \equiv \neg p \wedge \neg q \quad (1.1)$$

$$\neg(p \wedge q) \equiv \neg p \vee \neg q \quad (1.2)$$

**Definition 1.1.3** (Idempotent Laws).

$$p \vee p \equiv p \quad (1.3)$$

$$p \wedge p \equiv p \quad (1.4)$$

**Definition 1.1.4** (Commutative Laws).

$$p \vee q \equiv q \vee p \quad (1.5)$$

$$p \wedge q \equiv q \wedge p \quad (1.6)$$

**Definition 1.1.5** (Associative Laws).

$$p \vee (q \vee r) \equiv (p \vee q) \vee r \quad (1.7)$$

$$p \wedge (q \wedge r) \equiv (p \wedge q) \wedge r \quad (1.8)$$

**Definition 1.1.6** (Distributive Laws).

$$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r) \quad (1.9)$$

$$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r) \quad (1.10)$$

**Definition 1.1.7** (Identity Laws).

$$p \vee F \equiv p \quad (1.11)$$

$$p \vee T \equiv T \quad (1.12)$$

$$p \wedge T \equiv p \quad (1.13)$$

$$p \wedge F \equiv F \quad (1.14)$$

**Definition 1.1.8** (Absorption Laws).

$$p \vee (p \wedge q) \equiv p \quad (1.15)$$

$$p \wedge (p \vee q) \equiv p \quad (1.16)$$

**Definition 1.1.9** (Implication and Negation Laws).

- *Identity:*  $p \rightarrow q \equiv \neg p \vee q$
- *Counter-example:*  $\neg(p \rightarrow q) \equiv p \wedge \neg q$
- *Equivalences:*  $p \rightarrow q \rightarrow r \equiv (p \wedge q) \rightarrow r \equiv q \rightarrow (p \rightarrow r)$
- *Absorption:*
  - $p \rightarrow T \equiv T$
  - $p \rightarrow F \equiv \neg p$
  - $T \rightarrow p \equiv p$
  - $F \rightarrow p \equiv T$
- *Contrapositive:*  $p \rightarrow q \equiv \neg q \rightarrow \neg p$
- *Law of Excluded Middle:*
  - $p \vee \neg p \equiv T$
  - $p \wedge \neg p \equiv F$
- *Double Negation:*  $\neg\neg p \equiv p$
- *Reduction to Absurdity:*  $\neg p \rightarrow F \equiv p$

### 1.1.6 Set Algebra

**Definition 1.1.10** (De Morgan's Laws).

$$(A \cup B)^c \equiv A^c \cap B^c \quad (1.17)$$

$$(A \cap B)^c \equiv A^c \cup B^c \quad (1.18)$$

**Definition 1.1.11** (Idempotent Laws).

$$A \cup A \equiv A \quad (1.19)$$

$$A \cap A \equiv A \quad (1.20)$$

**Definition 1.1.12** (Commutative Laws).

$$A \cup B \equiv B \cup A \quad (1.21)$$

$$A \cap B \equiv B \cap A \quad (1.22)$$

**Definition 1.1.13** (Associativity Laws).

$$A \cup (B \cup C) \equiv (A \cup B) \cup C \quad (1.23)$$

$$A \cap (B \cap C) \equiv (A \cap B) \cap C \quad (1.24)$$

**Definition 1.1.14** (Distributive Laws).

$$A \cap (B \cup C) \equiv (A \cap B) \cup (A \cap C) \quad (1.25)$$

$$A \cup (B \cap C) \equiv (A \cup B) \cap (A \cup C) \quad (1.26)$$

**Definition 1.1.15** (Identity Laws).

$$A \cup \emptyset \equiv A \quad (1.27)$$

$$A \cap \emptyset \equiv \emptyset \quad (1.28)$$

$$A \cap U \equiv A \quad (1.29)$$

$$A \cup U \equiv U \quad (1.30)$$

**Definition 1.1.16** (Absorption Laws).

$$A \cup (A \cap B) \equiv A \quad (1.31)$$

$$A \cap (A \cup B) \equiv A \quad (1.32)$$

**Definition 1.1.17** (Difference Identity Laws).

$$C \setminus (A \cup B) \equiv (C \setminus A) \cap (C \setminus B) \quad (1.33)$$

$$C \setminus (A \cap B) \equiv (C \setminus A) \cup (C \setminus B) \quad (1.34)$$

**Definition 1.1.18** (Complement-Difference Identity Law).

$$C \setminus D \equiv C \cap D^c \quad (1.35)$$

**Definition 1.1.19** (Double Complement Law).

$$(D^c)^c \equiv D \quad (1.36)$$

**Definition 1.1.20** (Contraposition).

$$C \subseteq D \Leftrightarrow D^c \subseteq C^c \quad (1.37)$$

$$C = D \Leftrightarrow C^c = D^c \quad (1.38)$$



**Definition 1.1.21** (Arbitrary Union).

Given sets  $A_1, A_2, \dots, A_n$  where  $I = \{1, 2, \dots, n\}$

$$A_1 \cup A_2 \cup \dots \cup A_n \equiv \bigcup_{i \in I} A_i \quad (1.39)$$

Then

$$x \in \bigcup_{i \in I} A_i \Leftrightarrow \exists i \in I: x \in A_i \quad (1.40)$$

**Definition 1.1.22** (Arbitrary Intersection).

Given sets  $A_1, A_2, \dots, A_n$  where  $I = \{1, 2, \dots, n\}$

$$A_1 \cap A_2 \cap \dots \cap A_n \equiv \bigcap_{i \in I} A_i \quad (1.41)$$

Then

$$x \in \bigcap_{i \in I} A_i \Leftrightarrow \forall i \in I: x \in A_i \quad (1.42)$$

## 1.2 Functions

**Definition 1.2.1** (Function). A function  $f$  is a mapping from  $X$  to  $Y$

$$f: X \mapsto Y \quad (1.43)$$

- $\text{domain}(f) = X$
- $\text{image}(f) = f(X)$

**Definition 1.2.2** (Total Function). A function is *total* if

$$\text{domain}(f) = X \quad (1.44)$$

**Definition 1.2.3** (Partial Function). A function is *partial* if

$$\text{domain}(f) \subseteq X \quad (1.45)$$

**Definition 1.2.4** (Surjection). A function  $f: X \mapsto Y$  is *surjective* iff

$$f(X) = Y \Leftrightarrow \forall y \in Y: \exists x \in X: f(x) = y \quad (1.46)$$

Namely each  $y \in Y$  has a corresponding  $x \in X$ .

**Definition 1.2.5** (Injection (Encodings, One-to-one)). A function  $f: X \mapsto Y$  is *injective* iff

$$\forall x_1, x_2 \in X: x_1 \neq x_2 \rightarrow f(x_1) \neq f(x_2) \quad (1.47)$$

$$\Leftrightarrow \forall x_1, x_2 \in X: f(x_1) = f(x_2) \rightarrow x_1 = x_2 \quad (1.48)$$

Namely each distinct element  $x \in X$  maps to a different element in  $Y$ .

**Definition 1.2.6** (Bijection). A function  $f: X \mapsto Y$  is *bijection* iff  $f$  is both *injective* and *surjective*.

$$\text{Bijection}(f) \equiv \text{Injective}(f) \wedge \text{Surjective}(f) \quad (1.49)$$

The *inverse bijection*  $f^{-1}: Y \mapsto X$  does exist.

### 1.2.1 Composition of Injections

**Proposition 1.2.1** (Composition of Injection). Given *injections*  $f: X \mapsto Y$  and  $g: Y \mapsto Z$ , then their *composition*  $h: X \mapsto Z$  is given by

$$h(x) = g(f(x)) \quad (1.50)$$

Then  $h$  is also an *injective* function. Namely  $h = g \circ f$  where  $h$  is composed from  $g$  and  $f$  with  $f$  applied first.

*Proof.* Given any  $x_1, x_2 \in X$  where  $x_1 \neq x_2$ , then

$$f(x_1) \neq f(x_2) \quad (1.51)$$

as  $f$  is *injective*, and thus

$$h(x_1) = g(f(x_1)) \neq g(f(x_2)) = h(x_2) \quad (1.52)$$

$h$  is *injective* consequently. ■

### 1.2.2 Composition of Surjection

**Proposition 1.2.2** (Composition of Surjection). Given *surjections*  $f: X \mapsto Y$  and  $g: Y \mapsto Z$ , then their *composition*  $h: X \mapsto Z$  is given by

$$h(x) = g(f(x)) \quad (1.53)$$

Then  $h$  is also a *surjective* function.

*Proof.* To prove  $h: X \mapsto Z$  is *injective*, it is required to prove that

$$\forall z \in Z: \exists x \in X: h(x) = z \quad (1.54)$$

Where  $h(x) \Leftrightarrow (g \circ f)(x) \Leftrightarrow g(f(x))$ .

Given any element  $z \in Z$  ( $\forall z \in Z$ ):

1. That  $g: Y \mapsto Z$  is *surjective* by definition, then  $\exists y \in Y: g(y) = z$ .
2. That  $f: X \mapsto Y$  is *surjective* by definition, then  $\exists x \in X: f(x) = y$ .

Then  $\forall z \in Z: \exists x \in X: h(x) = (g \circ f)(x) = g(f(x)) = g(y) = z$  holds true. ■

### 1.2.3 Composition of Bijection

**Proposition 1.2.3** (Composition of Bijection). Given *bijections*  $f: X \mapsto Y$  and  $g: Y \mapsto Z$ , then their composition  $h: X \mapsto Z$  is given by

$$h(x) = g(f(x)) \quad (1.55)$$

Then  $h$  is also a *bijective* function; an *inverse bijection*  $h^{-1}: Z \mapsto X$  also exists.

### 1.2.4 Cardinality of Sets

**Definition 1.2.7** (Cardinality). The number of elements in a set  $X$  is denoted  $|X|$ .

**Definition 1.2.8** (Equal Cardinality and Bijection).

$$|X| = |Y| \quad (1.56)$$

Holds true if there exists a *bijection*  $h: X \mapsto Y$  (one-to-one correspondence between  $X$  and  $Y$ ).

Namely,  $X$  and  $Y$  have the same number of distinct elements, and each distinct element  $x \in X$  corresponds to exactly one distinct element  $y \in Y$ .

**Theorem 1.2.1** (Cantor-Bernstein). Given

1. *injective* function  $f: X \mapsto Y$
2. *injective* function  $g: Y \mapsto X$

Then there exists a *bijective* function  $h: X \mapsto Y$ .

Equivalently,

$$(|X| \leq |Y|) \wedge (|Y| \leq |X|) \rightarrow (|X| = |Y|) \quad (1.57)$$

**Remark.** Examples include countable sets, enumerable sets

$$|\mathbb{Q}| = |\mathbb{Z}| = |\mathbb{N}| = \aleph_0 \quad (1.58)$$

Where the cardinality of countable sets such as the *rational numbers*, *integers* and the *natural numbers* is denoted as "aleph-zero" ( $\aleph_0$ ).

On the other hand, continuum such as the *real numbers* are not countable and as such

$$|\mathbb{R}| > \aleph_0 \quad (1.59)$$



# Bibliography

- [1] Max Kanovich and Robin Hirsch.  
“Lecture Notes on Discrete Mathematics for Computer Scientists”.  
URL: [http://www.cs.ucl.ac.uk/1819/a4u/t2/comp0147\\_discrete\\_mathematics\\_for\\_computer\\_scientists/](http://www.cs.ucl.ac.uk/1819/a4u/t2/comp0147_discrete_mathematics_for_computer_scientists/).