

# **COMP0147 Discrete Mathematics for Computer Scientists Notes**

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Notes adapted from:

- Lecture notes by Max Kanovich and Robin Hirsch [1].
- *A First Course in Abstract Algebra* by Joseph J. Rotman [2].



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# 1 Set Theory

## 1.1 Set Notations

- Set definition:  $A = \{a, b, c\}$
- Set membership (element-of):  $a \in A$
- Set builder notation:  $\{x \mid x \in \mathbb{R} \wedge x^2 = x\}$
- Empty set:  $\emptyset$

## 1.2 Properties

- No structure
- No order
- No copies

For example,  $a, b, c$  are references to actual objects in

$$\{a, b, c\} \Leftrightarrow \{c, a, b\} \Leftrightarrow \{a, b, c, b\}$$

## 1.3 Set Equality

**Definition 1.3.1** (Set Equality). Set  $A = B$  iff:

1.  $A \subseteq B \implies \forall x(x \in A \rightarrow x \in B)$
2.  $B \subseteq A \implies \forall y(y \in B \rightarrow y \in A)$

**Remark.**  $A = B \Leftrightarrow A \subseteq B \wedge B \subseteq A$

## 1.4 Set Operations

- *Union:*  $A \cup B := \{x \mid x \in A \vee x \in B\}$
- *Intersection:*  $A \cap B := \{x \mid x \in A \wedge x \in B\}$
- *Relative Complement:*  $A \setminus B := \{x \mid x \in A \wedge x \notin B\}$
- *Absolute Complement:*  $A^c := U \setminus A := \{x \mid x \in U \wedge x \notin A\}$
- *Symmetric Difference:*  $A \Delta B := (A \setminus B) \cup (B \setminus A) := (A \cup B) \setminus (A \cap B)$
- *Cartesian Product:*  $A \times B := \{(x, y) \mid x \in A \wedge y \in B\}$

## 1.5 Boolean Algebra

**Definition 1.5.1** (De Morgan's Laws).

$$\neg(p \vee q) \equiv \neg p \wedge \neg q \quad (1.1)$$

$$\neg(p \wedge q) \equiv \neg p \vee \neg q \quad (1.2)$$

**Definition 1.5.2** (Idempotent Laws).

$$p \vee p \equiv p \quad (1.3)$$

$$p \wedge p \equiv p \quad (1.4)$$

**Definition 1.5.3** (Commutative Laws).

$$p \vee q \equiv q \vee p \quad (1.5)$$

$$p \wedge q \equiv q \wedge p \quad (1.6)$$

**Definition 1.5.4** (Associative Laws).

$$p \vee (q \vee r) \equiv (p \vee q) \vee r \quad (1.7)$$

$$p \wedge (q \wedge r) \equiv (p \wedge q) \wedge r \quad (1.8)$$

**Definition 1.5.5** (Distributive Laws).

$$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r) \quad (1.9)$$

$$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r) \quad (1.10)$$

**Definition 1.5.6** (Identity Laws).

$$p \vee F \equiv p \quad (1.11)$$

$$p \vee T \equiv T \quad (1.12)$$

$$p \wedge T \equiv p \quad (1.13)$$

$$p \wedge F \equiv F \quad (1.14)$$

**Definition 1.5.7** (Absorption Laws).

$$p \vee (p \wedge q) \equiv p \quad (1.15)$$

$$p \wedge (p \vee q) \equiv p \quad (1.16)$$

**Definition 1.5.8** (Implication and Negation Laws).

- *Identity:*  $p \rightarrow q \equiv \neg p \vee q$
- *Counter-example:*  $\neg(p \rightarrow q) \equiv p \wedge \neg q$
- *Equivalences:*  $p \rightarrow q \rightarrow r \equiv (p \wedge q) \rightarrow r \equiv q \rightarrow (p \rightarrow r)$



- *Absorption:*  
 $p \rightarrow T \equiv T$   
 $p \rightarrow F \equiv \neg p$   
 $T \rightarrow p \equiv p$   
 $F \rightarrow p \equiv T$
- *Contrapositive:*  $p \rightarrow q \equiv \neg q \rightarrow \neg p$
- *Law of Excluded Middle:*  
 $p \vee \neg p \equiv T$   
 $p \wedge \neg p \equiv F$
- *Double Negation:*  $\neg\neg p \equiv p$
- *Reduction to Absurdity:*  $\neg p \rightarrow F \equiv p$

## 1.6 Set Algebra

**Definition 1.6.1** (De Morgan's Laws).

$$(A \cup B)^c \equiv A^c \cap B^c \quad (1.17)$$

$$(A \cap B)^c \equiv A^c \cup B^c \quad (1.18)$$

**Definition 1.6.2** (Idempotent Laws).

$$A \cup A \equiv A \quad (1.19)$$

$$A \cap A \equiv A \quad (1.20)$$

**Definition 1.6.3** (Commutative Laws).

$$A \cup B \equiv B \cup A \quad (1.21)$$

$$A \cap B \equiv B \cap A \quad (1.22)$$

**Definition 1.6.4** (Associativity Laws).

$$A \cup (B \cup C) \equiv (A \cup B) \cup C \quad (1.23)$$

$$A \cap (B \cap C) \equiv (A \cap B) \cap C \quad (1.24)$$

**Definition 1.6.5** (Distributive Laws).

$$A \cap (B \cup C) \equiv (A \cap B) \cup (A \cap C) \quad (1.25)$$

$$A \cup (B \cap C) \equiv (A \cup B) \cap (A \cup C) \quad (1.26)$$

**Definition 1.6.6** (Identity Laws).

$$A \cup \emptyset \equiv A \quad (1.27)$$

$$A \cap \emptyset \equiv \emptyset \quad (1.28)$$

$$A \cap U \equiv A \quad (1.29)$$

$$A \cup U \equiv U \quad (1.30)$$

**Definition 1.6.7** (Absorption Laws).

$$A \cup (A \cap B) \equiv A \quad (1.31)$$

$$A \cap (A \cup B) \equiv A \quad (1.32)$$

**Definition 1.6.8** (Difference Identity Laws).

$$C \setminus (A \cup B) \equiv (C \setminus A) \cap (C \setminus B) \quad (1.33)$$

$$C \setminus (A \cap B) \equiv (C \setminus A) \cup (C \setminus B) \quad (1.34)$$

**Definition 1.6.9** (Complement-Difference Identity Law).

$$C \setminus D \equiv C \cap D^c \quad (1.35)$$

**Definition 1.6.10** (Double Complement Law).

$$(D^c)^c \equiv D \quad (1.36)$$

**Definition 1.6.11** (Contraposition).

$$C \subseteq D \Leftrightarrow D^c \subseteq C^c \quad (1.37)$$

$$C = D \Leftrightarrow C^c = D^c \quad (1.38)$$

**Definition 1.6.12** (Arbitrary Union).

Given sets  $A_1, A_2, \dots, A_n$  where  $I = \{1, 2, \dots, n\}$

$$A_1 \cup A_2 \cup \dots \cup A_n := \bigcup_{i \in I} A_i \quad (1.39)$$

Then

$$x \in \bigcup_{i \in I} A_i \Leftrightarrow \exists i \in I: x \in A_i \quad (1.40)$$

**Definition 1.6.13** (Arbitrary Intersection).

Given sets  $A_1, A_2, \dots, A_n$  where  $I = \{1, 2, \dots, n\}$

$$A_1 \cap A_2 \cap \dots \cap A_n := \bigcap_{i \in I} A_i \quad (1.41)$$

Then

$$x \in \bigcap_{i \in I} A_i \Leftrightarrow \forall i \in I: x \in A_i \quad (1.42)$$

## 2 Functions

### 2.1 Function Basics

**Definition 2.1.1** (Function). A function  $f$  is a mapping from  $X$  to  $Y$

$$f: X \mapsto Y \quad (2.1)$$

- $\text{domain}(f) = X$
- $\text{image}(f) = f(X)$

**Definition 2.1.2** (Total Function). A function is *total* if

$$\text{domain}(f) = X \quad (2.2)$$

**Definition 2.1.3** (Partial Function). A function is *partial* if

$$\text{domain}(f) \subseteq X \quad (2.3)$$

**Definition 2.1.4** (Surjection). A function  $f: X \mapsto Y$  is *surjective* iff

$$f(X) = Y \Leftrightarrow \forall y \in Y: \exists x \in X: f(x) = y \quad (2.4)$$

Namely each  $y \in Y$  has a corresponding  $x \in X$ .

**Definition 2.1.5** (Injection (Encodings, One-to-one)). A function  $f: X \mapsto Y$  is *injective* iff

$$\forall x_1, x_2 \in X: x_1 \neq x_2 \rightarrow f(x_1) \neq f(x_2) \quad (2.5)$$

$$\Leftrightarrow \forall x_1, x_2 \in X: f(x_1) = f(x_2) \rightarrow x_1 = x_2 \quad (2.6)$$

Namely each distinct element  $x \in X$  maps to a different element in  $Y$ .

**Definition 2.1.6** (Bijection). A function  $f: X \mapsto Y$  is *bijective* iff  $f$  is both *injective* and *surjective*.

$$\text{Bijective}(f) := \text{Injective}(f) \wedge \text{Surjective}(f) \quad (2.7)$$

The *inverse bijection*  $f^{-1}: Y \mapsto X$  does exist.

## 2.2 Composition of Injections

**Proposition 2.2.1** (Composition of Injection). Given *injections*  $f: X \mapsto Y$  and  $g: Y \mapsto Z$ , then their *composition*  $h: X \mapsto Z$  is given by

$$h(x) := g(f(x)) \quad (2.8)$$

Then  $h$  is also an *injective* function. Namely  $h = g \circ f$  where  $h$  is composed from  $g$  and  $f$  with  $f$  applied first.

*Proof.* Given any  $x_1, x_2 \in X$  where  $x_1 \neq x_2$ , then

$$f(x_1) \neq f(x_2) \quad (2.9)$$

as  $f$  is *injective*, and thus

$$h(x_1) = g(f(x_1)) \neq g(f(x_2)) = h(x_2) \quad (2.10)$$

$h$  is *injective* consequently. ■

## 2.3 Composition of Surjection

**Proposition 2.3.1** (Composition of Surjection). Given *surjections*  $f: X \mapsto Y$  and  $g: Y \mapsto Z$ , then their *composition*  $h: X \mapsto Z$  is given by

$$h(x) := g(f(x)) \quad (2.11)$$

Then  $h$  is also a *surjective* function.

*Proof.* To prove  $h: X \mapsto Z$  is *surjective*, it is required to prove that

$$\forall z \in Z: \exists x \in X: h(x) = z \quad (2.12)$$

Where  $h(x) \Leftrightarrow (g \circ f)(x) \Leftrightarrow g(f(x))$ .

Given any element  $z \in Z$  ( $\forall z \in Z$ ):

1. That  $g: Y \mapsto Z$  is *surjective* by definition, then  $\exists y \in Y: g(y) = z$ .
2. That  $f: X \mapsto Y$  is *surjective* by definition, then  $\exists x \in X: f(x) = y$ .

Then  $\forall z \in Z: \exists x \in X: h(x) = (g \circ f)(x) = g(f(x)) = g(y) = z$  holds true. ■

## 2.4 Composition of Bijection

**Proposition 2.4.1** (Composition of Bijection). Given *bijections*  $f: X \mapsto Y$  and  $g: Y \mapsto Z$ , then their composition  $h: X \mapsto Z$  is given by

$$h(x) := g(f(x)) \quad (2.13)$$

Then  $h$  is also a *bijective* function; an *inverse bijection*  $h^{-1}: Z \mapsto X$  also exists.

## 2.5 Cardinality of Sets

**Definition 2.5.1** (Cardinality). The number of elements in a set  $X$  is denoted  $|X|$ .

**Definition 2.5.2** (Equal Cardinality and Bijection).

$$|X| = |Y| \quad (2.14)$$

Holds true if there exists a *bijection*  $h: X \mapsto Y$  (one-to-one correspondence between  $X$  and  $Y$ ).

Namely,  $X$  and  $Y$  have the same number of distinct elements, and each distinct element  $x \in X$  corresponds to exactly one distinct element  $y \in Y$ .

**Theorem 2.5.1** (Cantor-Bernstein). Given

1. *injective* function  $f: X \mapsto Y$
2. *injective* function  $g: Y \mapsto X$

Then there exists a *bijection* function  $h: X \mapsto Y$ .

Equivalently,

$$(|X| \leq |Y|) \wedge (|Y| \leq |X|) \rightarrow (|X| = |Y|) \quad (2.15)$$

**Remark.** Examples include countable sets, enumerable sets

$$|\mathbb{Q}| = |\mathbb{Z}| = |\mathbb{N}| = \aleph_0 \quad (2.16)$$

Where the cardinality of countable sets such as the *rational numbers*, *integers* and the *natural numbers* is denoted as "aleph-zero" ( $\aleph_0$ ).

On the other hand, continuum such as the *real numbers* are not countable and as such

$$|\mathbb{R}| > \aleph_0 \quad (2.17)$$



# 3 Permutations

## 3.1 Permutation Basics

**Definition 3.1.1** (Permutation). The bijection – *permutation* – of

$$\begin{array}{ccccc} 1 & 2 & 3 & \cdots & n \\ \downarrow & \downarrow & \downarrow & \cdots & \downarrow \\ \sigma(1) & \sigma(2) & \sigma(3) & \cdots & \sigma(n) \end{array} \quad (3.1)$$

Is denoted as

$$\begin{pmatrix} 1 & 2 & 3 & \cdots & n \\ \sigma(1) & \sigma(2) & \sigma(3) & \cdots & \sigma(n) \end{pmatrix} \quad (3.2)$$

Where  $\sigma: \{1, \dots, n\} \rightarrow \{1, \dots, n\}$  is the *permutation* bijection.

**Definition 3.1.2** (Counting Permutations).

$$|S_n| := n! \quad (3.3)$$

Which is the number of different ways to permute  $n$  elements  $\{1, 2, \dots, n\} \subset \mathbb{Z}$ . Together, the different permutations for  $n$  distinct elements is the *symmetric group*  $S_n$ .

**Remark.** For example, with  $S_3 = \{1, 2, 3\}$ , there are  $3! = 6$  different ways to arrange the three distinct elements

$$\begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} \quad \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} \quad \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} \\ \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} \quad \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} \quad (3.4)$$

**Definition 3.1.3** (Order of Permutation). The *order* of a permutation  $\sigma$  is the smallest  $k \in \mathbb{Z}^+$  such that

$$\sigma^k = \epsilon \quad (3.5)$$

Where  $\epsilon$  is the *identity permutation*

$$\epsilon(x) = x \quad (3.6)$$

**Definition 3.1.4** (Sign of Permutation). The *sign* of a permutation  $\text{sgn } \sigma: \sigma \rightarrow \{-1, +1\}$  where  $\sigma \in S_n$  is defined as

$$\text{sgn}(\sigma) = (-1)^k \quad (3.7)$$

Where  $k$  is the number of *disorders* within  $\sigma$ , the number of pairs  $(x, y)$  such that  $x > y \rightarrow \sigma(x) < \sigma(y)$  or the converse  $x < y \rightarrow \sigma(x) > \sigma(y)$ . Additionally,

$$\text{sgn}(\sigma) = \begin{cases} +1 & \text{if } k \text{ is even} \\ -1 & \text{if } k \text{ is odd} \end{cases} \quad (3.8)$$

**Remark.** For example, in

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$$

$1 < 2$  but  $\sigma(1) = 2 > \sigma(2) = 1$ , hence a disorder.

For each  $i \in \{1, \dots, n\}$ , starting from  $i = 1$ , compare  $\sigma(i)$  with  $\sigma(i+1), \dots, \sigma(n)$  and add the number of disordered pairs, then move on to  $i+1$  and compare  $\sigma(i+1)$  with  $\sigma(i+2), \dots, \sigma(n)$  and so on.

**Theorem 3.1.1** (Composition of Permutation).

$$\text{sgn}(\sigma_1 \sigma_2) := \text{sgn}(\sigma_1) \cdot \text{sgn}(\sigma_2) \quad (3.9)$$

Where

| $\circ$ | even | odd  |
|---------|------|------|
| even    | even | odd  |
| odd     | odd  | even |

Table 3.1: Sign Changes on Composition



## 4 Binary Relations

**Definition 4.0.1** (Binary Relation). A binary relation  $R(x, y)$  describes some relationship between  $x$  and  $y$  where  $R: X \rightarrow Y$ ,  $R \subseteq X \times Y$ ,  $x \in X$  and  $y \in Y$ . This relation can be expressed in infix notation as  $xRy$ .

### 4.1 Equivalence Relations

**Definition 4.1.1** (Equivalence Relation). A binary relation  $E(x, y)$  is an *equivalence relation* on  $X$  iff it satisfies all three conditions:

1. **Reflexivity**

$$\forall x \in X: E(x, x)$$

2. **Symmetry**

$$\forall x, y \in X: E(x, y) \rightarrow E(y, x)$$

3. **Transitivity**

$$\forall x, y, z \in X: E(x, y) \wedge E(y, z) \rightarrow E(x, z)$$

### 4.2 Equivalence Classes

**Definition 4.2.1** (Equivalence Class). If  $a \in X$ , the *equivalence class*  $[a]$  is

$$[a] := \{x \in X: E(x, a)\} \subseteq X \quad (4.1)$$

**Definition 4.2.2** (Congruence and Equivalence Class of mod  $m$  on  $\mathbb{Z}$ ). For *congruence mod  $m$*  on  $\mathbb{Z}$ , if  $a \in \mathbb{Z}$  then the *congruence class* of  $a$  is

$$[a]_m := \{x \in \mathbb{Z}: x = a + km\} \quad (4.2)$$

Where  $k \in \mathbb{Z}$ . Since  $x = a + km \Leftrightarrow x \equiv a \pmod{m}$ , then the *equivalence class* of  $a$  is also the *congruence class*.

$$\Leftrightarrow [a]_m := \{x \in \mathbb{Z}: x \equiv a \pmod{m}\} \quad (4.3)$$

**Definition 4.2.3** (Set of Remainders). Over  $\mathbb{Z}$ , the *remainder*  $r$  from the integer division  $k \div m$  is

$$r \bmod m \equiv k \bmod m \quad (4.4)$$

Then the set of remainders  $G_m$  from the integer division  $k \div m$  is defined by

$$G_m := \{0, 1, 2, \dots, m-2, m-1\} \quad (4.5)$$

### 4.3 Quotient Groups

**Definition 4.3.1** (Quotient Group). A *quotient group* is a group constructed via congruence mod  $m$ .

**Definition 4.3.2** (Congruence Class). If  $m \leq 2$  and  $a \in \mathbb{Z}$  then the *congruence class* of  $a$  mod  $m$  is  $[a] \subseteq \mathbb{Z}$

$$[a] := \{b \in \mathbb{Z} : b \equiv a \pmod{m}\} \quad (4.6)$$

$$\Leftrightarrow \{a + km : k \in \mathbb{Z}\} \quad (4.7)$$

$$\Leftrightarrow \{\dots, a - 2m, a - m, a, a + m, a + 2m, \dots\} \quad (4.8)$$

**Remark.** Let  $E(x, y) := "x - y \equiv 0 \pmod{2}"$ , that is,  $x - y$  is divisible by 2. Then,

$$[k]_2 := \{y : E(k, y)\} \quad (4.9)$$

Where  $[k]_2$  is the congruence class of integers modulo 2.

Computing  $[0]_2$  and  $[1]_2$  yields

- $[0]_2 = \{0, 2, -2, 4, -4, \dots, 2n, -2n, \dots\}$
- $[1]_2 = \{1, -1, 3, -3, \dots, 2n + 1, \dots\}$

Observe that

$$[1]_2 \oplus [1]_2 \Leftrightarrow [2]_2 \Leftrightarrow [0]_2 \quad (4.10)$$

It can be deduced that  $[0]_2$  and  $[1]_2$  are two congruence (and equivalence) classes which partition the integers  $\mathbb{Z}$  into two disjoint subsets – integers which are odd, and integers which are even. This may be denoted as

$$\mathbb{Z}/E \equiv \{\text{EVEN}, \text{ODD}\} \quad (4.11)$$

**Definition 4.3.3** (Congruence Modular Arithmetic (mod  $m$ ) on  $\mathbb{Z}$ ).

$$[a]_m \oplus [b]_m \equiv [a + b]_m \quad (4.12)$$

$$[a]_m \otimes [b]_m \equiv [a \cdot b]_m \quad (4.13)$$

If  $a_1 \equiv a_2 \pmod{m}$  and  $b_1 \equiv b_2 \pmod{m}$  then

$$a_1 + b_1 \equiv a_2 + b_2 \pmod{m} \quad (4.14)$$

$$a_1 \cdot b_1 \equiv a_2 \cdot b_2 \pmod{m} \quad (4.15)$$

$$(4.16)$$

**Remark.** We may introduce addition (+) and multiplication (\*) over the remainders  $G_m$  previously defined as

$$G_m := \{0, 1, 2, \dots, m - 2, m - 1\} \quad (4.17)$$

For example, given  $m = 3$ , then the multiplication and addition table of  $+$  (mod 3) and  $*$  (mod 3) over  $G_3$  can be computed:

| $+$ (mod 3)   0 1 2 | $*$ (mod 3)   0 1 2 |
|---------------------|---------------------|
| 0   0 1 2           | 0   0 0 0           |
| 1   1 2 0           | 1   0 1 2           |
| 2   2 0 1           | 2   0 2 1           |

Table 4.1: Multiplication and Addition Table of  $G_3$



# 5 Groups

## 5.1 Group Basics

A *group* is an abstract collection consisting of:

- A *nonempty set*  $G$ .
- A *binary operation*  $\star: G \times G \rightarrow G$ .

It has the following properties:

1. **Closure**

$$\forall x, y: x \in G \wedge y \in G \rightarrow x \star y \in G \quad (5.1)$$

2. **Associativity**

$$\forall x, y, z \in G: (x \star y) \star z \equiv x \star (y \star z) \quad (5.2)$$

3. **Neutral Element**

$$\exists \epsilon \in G: \forall x \in G: x \star \epsilon \equiv \epsilon \star x \equiv x \quad (5.3)$$

That there exists an unique *neutral* element  $\epsilon \in G$ .

4. **Invertibility**

$$\forall x \in G: \exists y \in G: x \star y \equiv y \star x \equiv \epsilon \quad (5.4)$$

That there exists an unique *inverse* element  $y := x^{-1} \in G$  where  $x^{-1}$  denotes the *inverse* element of  $x$ .

**Definition 5.1.1** (Commutative Group). An *commutative group* (or *abelian group*) is a *group* for which its operation  $\star: G \times G \rightarrow G$  satisfies the additional *commutative* property:

- **Commutativity**

$$\forall x, y \in G: x \star y \equiv y \star x \quad (5.5)$$

## 5.2 Multiplicative Group

**Proposition 5.2.1** (Multiplicative Group). A *multiplicative group* is a *group*  $(G, *)$  which has the binary operation  $\ast: G \times G \rightarrow G$ :

- **Closure, Associativity.** The multiplication operation  $\ast: G \times G \rightarrow G$  is closed and is left associative.
- **Neutral Element.** The neutral element  $\epsilon$  is unique.
- **Invertibility.** The inverse element  $x^{-1}$  is unique.

- For all  $a, b \in G$  the equation

$$a * x = b \quad (5.6)$$

Has the unique solution

$$x = a^{-1} * b \quad (5.7)$$

Since

$$a * x = b \Leftrightarrow a^{-1} * (a * x) = a^{-1} * b \quad (\text{Multiply by inverse element}) \quad (5.8)$$

$$\Leftrightarrow (a^{-1} * a) * x = a^{-1} * b \quad (\text{Associativity}) \quad (5.9)$$

$$\Leftrightarrow \epsilon * x = a^{-1} * b \quad (\text{Invertibility}) \quad (5.10)$$

$$\Leftrightarrow x = a^{-1} * b \quad (\text{Neutral Element}) \quad (5.11)$$

**Remark.** An example of a multiplicative group is permutations under composition, namely  $S_n$  is a group  $(G, \circ)$  where  $\circ: G \times G \rightarrow G$ .

For example, let  $G$  be the set of permutations

$$\epsilon = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} \quad \sigma_1 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} \quad \sigma_2 = \sigma_1^2 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} \quad (5.12)$$

To verify that  $G$  does form a group with composition  $\circ$ , one may draw the multiplication table for the group. Note that

$$\sigma_2 \sigma_2 = \sigma_1^4 = \sigma_1^3 \sigma_1 = \epsilon \sigma_1 = \sigma_1 \quad (5.13)$$

| $\circ$    | $\epsilon$ | $\sigma_1$ | $\sigma_2$ |
|------------|------------|------------|------------|
| $\epsilon$ | $\epsilon$ | $\sigma_1$ | $\sigma_2$ |
| $\sigma_1$ | $\sigma_1$ | $\sigma_2$ | $\epsilon$ |
| $\sigma_2$ | $\sigma_2$ | $\epsilon$ | $\sigma_1$ |

Table 5.1: Multiplication Table of Composition  $\circ$  over  $G$

### 5.3 Additive Group

**Definition 5.3.1** (Additive Group). An *additive group* is a *group*  $(G, +)$  with the binary operation  $+: G \times G \rightarrow G$ . It has the same properties of a general *group*.

1. **Closure**

$$\forall x, y: x \in G \wedge y \in G \rightarrow x + y \in G \quad (5.14)$$

2. **Associativity**

$$\forall x, y, z \in G: (x + y) + z \equiv x + (y + z) \quad (5.15)$$

3. **Neutral Element**

$$\exists \epsilon \in G: \forall x \in G: x + \epsilon \equiv \epsilon + x \equiv x \quad (5.16)$$

That there exists an unique *neutral* element  $0_G \in G$  (usually denoted simply as 0).

#### 4. Invertibility

$$\forall x \in G: \exists y \in G: x + y \equiv y + x \equiv 0 \quad (5.17)$$

That there exists an unique *inverse* element  $y := -x \in G$  where  $-x$  denotes the *inverse* element of  $x$ .

**Remark.** An example of an additive group is  $(\mathbb{Z}, +)$  (i.e. addition over the integers). Then for any of such *commutative group*  $(G, +)$

- *Neutral element* 0 is unique.
- *Inverse element*  $-x$  is unique.
- For any  $a, b \in G$  the equation

$$a + x = b \quad (5.18)$$

Has a unique solution

$$x = b + (-a) = b - a \quad (5.19)$$

### 5.4 Associativity of Sequential Composition of Functions

**Definition 5.4.1** (Sequential Composition of Functions). Let  $f * g$  denote the sequential composition of functions  $f * X \rightarrow Y$  and  $g: Y \rightarrow Z$  such that  $f * g: X \rightarrow Z$  where  $f$  is applied first then  $g$ , i.e.  $\forall x \in X: (f * g)(x) := g(f(x))$ .

**Proposition 5.4.1** (Associativity of Sequential Composition of Functions). Given sets  $X, Y$  and  $Z$  and

- *Injection*  $f: A \rightarrow B$
- *Injection*  $g: B \rightarrow C$
- *Injection*  $h: C \rightarrow D$

Then their composition is associative:

$$(f * g) * h \equiv f * (g * h) \quad (5.20)$$

*Proof.*

Let  $s = (f * g)$  and  $t = (s * h)$ , then  $t(x) = h(s(x)) = h(g(f(x)))$ .

Let  $u = (g * h)$  and  $v = (f * u)$ , then  $v(x) = u(f(x)) = h(g(f(x)))$ .

Together they yield the desired equality  $t(x) = v(x)$ . ■

### 5.5 Subgroups

**Definition 5.5.1** (Subgroup). Given a *group*  $(G, *)$ , then the subset  $H \subseteq G$  is a *subgroup* of  $G$  if it fulfills the properties:

#### 1. Closure

$$\forall x, y: x \in H \wedge y \in H \rightarrow x * y \in H \quad (5.21)$$

#### 2. Neutral Element

$$\epsilon \in H \quad (5.22)$$

That is, the *neutral* element  $\epsilon$  from  $G$  is contained within the subset  $H \subseteq G$ .

## 3. Invertibility

$$\forall x \in H: x^{-1} \in H \quad (5.23)$$

## 5.6 Lagrange's Theorem

**Theorem 5.6.1** (Lagrange's Theorem). Given a finite *group* of order  $n$   $(G, *)$  where

$$G := \{g_1, g_2, \dots, g_n\} \quad (5.24)$$

And its *subgroup*  $(H, *)$  of order  $k \leq n$

$$H := \{h_1, h_2, \dots, h_k\} \quad (5.25)$$

Then  $k|n$  ( $k$  divides  $n$ ).

$G$  can be *partitioned* into  $\ell$  disjoint subsets of the same size  $k$  such that

$$n = k\ell \quad (5.26)$$

**Definition 5.6.1** (Left Coset). Given  $(G, *)$  is a *group*,  $(H, *)$  is a *subgroup* of  $(G, *)$  and  $g \in G$  then the *left coset*  $gH$  of  $H$  in  $G$  with respect to  $g$  is defined as

$$gH := \{g * h : h \in H\} \quad (5.27)$$

**Remark.** Visually,

$$G \equiv \left. \begin{array}{c} \boxed{g_1 H} \\ \boxed{g_2 H} \\ \vdots \\ \boxed{g_\ell H} \end{array} \right\} \ell \text{ disjoint subsets} \quad (5.28)$$

To verify that the *left cosets* together do in fact reconstruct  $G$ , check the multiplication table

| $*$        | $h_1$          | $h_2$          | $\dots$  | $h_k$          |
|------------|----------------|----------------|----------|----------------|
| $g_1 H$    | $g_1 * h_1$    | $g_1 * h_2$    | $\dots$  | $g_1 * h_k$    |
| $g_2 H$    | $g_2 * h_1$    | $g_2 * h_2$    | $\dots$  | $g_2 * h_k$    |
| $\vdots$   | $\vdots$       | $\vdots$       | $\ddots$ | $\vdots$       |
| $g_\ell H$ | $g_\ell * h_1$ | $g_\ell * h_2$ | $\dots$  | $g_\ell * h_k$ |

Table 5.2: Multiplication Table from  $\ell$  Left Cosets, Each of Size  $|H| = k$

**Proposition 5.6.1.** For any  $a, b \in G$  from  $(G, *)$

$$(a * b)^{-1} \equiv b^{-1} * a^{-1} \quad (5.29)$$



*Proof.*

$$(a * b)^{-1} \Leftrightarrow (a * b)^{-1} * \epsilon \quad (\text{Neutral element}) \quad (5.30)$$

$$\Leftrightarrow (a * b)^{-1} * (a * a^{-1}) \quad (\text{Invertibility}) \quad (5.31)$$

$$\Leftrightarrow (a * b)^{-1} * ((a * \epsilon) * a^{-1}) \quad (\text{Neutral element}) \quad (5.32)$$

$$\Leftrightarrow (a * b)^{-1} * [(a * (b * b^{-1})) * a^{-1}] \quad (\text{Invertibility}) \quad (5.33)$$

$$\Leftrightarrow (a * b)^{-1} * [(a * b) * (b^{-1} * a^{-1})] \quad (\text{Associativity}) \quad (5.34)$$

$$\Leftrightarrow [(a * b)^{-1} * (a * b)] * (b^{-1} * a^{-1}) \quad (\text{Associativity}) \quad (5.35)$$

$$\Leftrightarrow \epsilon * (b^{-1} * a^{-1}) \quad (\text{Invertibility}) \quad (5.36)$$

$$\Leftrightarrow b^{-1} * a^{-1} \quad (\text{Neutral Element}) \quad (5.37)$$

■

*Proof.* For a constructive proof of Lagrange's Theorem:

Let the binary relation  $E(x, y)$  be defined on the group  $(G, *)$ , with its subgroup  $(H, *)$

$$E(x, y) := x^{-1} * y \in H \quad (5.38)$$

For the equivalence

$$x = y \Leftrightarrow x^{-1} * y = 1 \quad (5.39)$$

Then for each of the required properties:

- **Neutral Element** from *Reflexivity* of  $E(x, y)$

$$\forall x \in G: E(x, x) \quad (5.40)$$

Since

$$E(x, x) \equiv x^{-1} * x \in H \equiv \epsilon \in H \quad (5.41)$$

Then this satisfies the *reflexivity* requirement for *equivalence relations*, and proves the *neutral element* requirement for *subgroups*.

- **Invertibility** from *Symmetry* of  $E(x, y)$

$$\forall x, y \in G: E(x, y) \rightarrow E(y, x) \quad (5.42)$$

Let for some  $h \in H$ ,  $x^{-1} * y = h$ , then by proposition 5.6.1

$$y^{-1} * x \equiv (x^{-1} * y)^{-1} \equiv h^{-1} \in H \quad (5.43)$$

Which satisfies the *symmetry* requirement for *equivalence relations*, and proves the *invertibility* requirement for *subgroups*.

- **Closure** from *Transitivity* of  $E(x, y)$

$$\forall x, y, z \in G: E(x, y) \wedge E(y, z) \rightarrow E(x, z) \quad (5.44)$$

Let for some  $h_1, h_2 \in H$ ,  $(x^{-1} * y = h_1) \wedge (y^{-1} * z = h_2)$ , then

$$x^{-1} * z \Leftrightarrow x^{-1} * \epsilon * z \quad (5.45)$$

$$\Leftrightarrow (x^{-1} * y) * (y^{-1} * z) \quad (5.46)$$

$$\Leftrightarrow h_1 * h_2 \in H \quad (5.47)$$

Which satisfies the *transitivity* requirement for *equivalence relations*, and proves the *closure* requirement for *subgroups*. ■

**Remark.** To demonstrate Lagrange's Theorem, let the *group* be constructed from  $x * y \pmod{10}$ .

Let  $(G, *)$  be a finite *group* of order  $n = 4$  where

$$G = \{1, 3, 7, 9\} \quad (5.48)$$

And  $(H, *)$  be its *subgroup* of order  $k = 2$ .

Constructing the multiplication table yields

| $* \pmod{10}$ | 1 | 9 |
|---------------|---|---|
| $1 * H$       | 1 | 9 |
| $3 * H$       | 3 | 7 |
| $7 * H$       | 7 | 3 |
| $9 * H$       | 9 | 1 |

Table 5.3: Multiplication Table for  $(G, *)$

There are only  $\ell = 2$  disjoint subsets (unique cosets)  $gH$ ;  $G$  can be partitioned into  $\ell$  disjoint subsets, each of size  $|H| = 2$  such that  $4 = n = k\ell = 2 \cdot 2$ .

Visually,

$$G = \left. \begin{array}{l} 1 * H = 9 * H = \{1, 9\} \\ 3 * H = 7 * H = \{3, 7\} \end{array} \right\} \ell = 2 \quad (5.49)$$

### 5.6.1 Equivalence Classes

**Definition 5.6.2** (Equivalence Class). Given *group*  $(G, *)$  and its *subgroup*  $(H, *)$ , then the *equivalence class*  $[g]$  is defined as

$$[g] := \{y \in G \mid g^{-1} * y \in H\} \quad (5.50)$$

Then

$$\forall h \in H: g^{-1} * y = h \Leftrightarrow y = g * h \quad (5.51)$$

Which yields the equivalence

$$\{y \in G \mid g^{-1} * y \in H\} \equiv \{y \in G \mid y \in gH\} \quad (5.52)$$

Hence

$$[g] \equiv gH \quad (5.53)$$

That the *equivalence class*  $[g]$  is exactly the *left coset*  $gH$ .

Let  $\ell$  be the number of disjoint equivalence class  $[g]$ , then  $G$  can be partitioned into  $\ell$  disjoint subsets where visually,

$$G = \left. \begin{array}{c} [g_1] \equiv g_1 H \\ [g_2] \equiv g_2 H \\ \vdots \\ [g_\ell] \equiv g_\ell H \end{array} \right\} \ell \text{ disjoint subsets} \quad (5.54)$$

**Proposition 5.6.2.**

$$\forall g \in G: |gH| \equiv |H| \equiv k \quad (5.55)$$

*Proof.* Let  $I$  be the set of indices  $I := \{1, \dots, k\}$

$$\forall i, j \in I: (h_i = h_j) \leftrightarrow (g * h_i = g * h_j) \quad (5.56)$$

$$\Leftrightarrow \forall i, j \in I: (h_i \neq h_j) \leftrightarrow (g * h_i \neq g * h_j) \quad (5.57)$$

■

**Remark.** Let  $A_n$  be the set of all *even permutations* and  $B_n$  be the set of all *odd permutations*.

Given the *group*  $(S_n, *)$ , then  $(A_n, *)$  is a *subgroup* of  $S_n$ .

With the multiplication table

| *                | $A_n$ |
|------------------|-------|
| $\epsilon * A_n$ | $A_n$ |
| $\epsilon * A_n$ | $B_n$ |

Table 5.4: Multiplication Table for Group  $S_n$

Since

$$\sigma * A_n \equiv \begin{cases} A_n & \text{if } \sigma \text{ is even} \\ B_n & \text{if } \sigma \text{ is even} \end{cases} \quad (5.58)$$

Hence,

$$|A_n| \equiv \frac{1}{2} \cdot |S_n| \equiv \frac{1}{2} \cdot n! \quad (5.59)$$

**5.6.2 Order of an Element in Lagrange's Theorem**

**Definition 5.6.3** (Order of an Element). Given a *group*  $(G, *)$  and element  $a \in G$  then the *order* of the element  $a$  is the smallest  $k \in \mathbb{Z}^+$  such that

$$a^k = \epsilon \quad (5.60)$$

**Proposition 5.6.3.** Given a *group*  $(G, *)$  with *order*  $n$ , then for any  $a \in G$ , should its *order*  $k$  exist, then  $k|n$  ( $k$  divides  $n$ ).

**Proposition 5.6.4.** Given *group*  $(G, *)$ ,

$$\forall a \in G: a^{|G|} \equiv 1 \quad (5.61)$$

*Proof.* With the *cyclic subgroup* generated by  $a \in G$

$$\{a^m \mid m \in \mathbb{Z}\} = \{\epsilon, a, a^2, \dots\} \quad (5.62)$$

■

**Remark.** This may be used to calculate the modulo of integers raised to large exponents. For example, for  $2^{20} \pmod{15}$ . To compute this, let the *multiplicative group*  $(G, *)$  be defined over  $G$  of *order* 8 where

$$G = \{1, 2, 4, 7, 8, 11, 13, 14\} \quad (5.63)$$

And the *binary operation*  $x * y := x * y \pmod{15}$ .

Note that  $2^{-1} = 8 \pmod{15}$  and  $4^{-1} = 4 \pmod{15}$ .

Since  $|G| = 8$ ,

$$2^8 = 1 \pmod{15} \quad (5.64)$$

Then  $2^{20} \pmod{15}$  can be calculated by decomposing its exponent:

$$2^{20} = 2^{2 \cdot 8 + 4} = (2^8)^2 * 2^4 = 1 * 16 = 1 \pmod{15} \quad (5.65)$$

# 6 Euclidean Algorithm

## 6.1 Euclidean Algorithm Basics

**Definition 6.1.1** (Euclidean Algorithm). The *Euclidean Algorithm* can be used to compute the *greatest common divisor* of two integers  $a, b \in \mathbb{Z}$ , denoted  $\gcd(a, b)$ .

Its process, given  $a \geq b$  is

$$a = q_0 \cdot b + r_1 \quad (6.1)$$

$$b = q_1 \cdot r_1 + r_2 \quad (6.2)$$

$$r_1 = q_2 \cdot r_2 + r_3 \quad (6.3)$$

$\vdots$

$$r_{k-1} = q_k \cdot r_k + r_{k+1} \quad (6.4)$$

$$r_k = q_{k+1} \cdot r_{k+1} + r_{k+2} \quad (6.5)$$

$\vdots$

$$r_{n-1} = q_n \cdot r_n + r_{n+1} \quad (6.6)$$

$$r_n = q_{n+1} \cdot r_{n+1} + 0 \quad (6.7)$$

Such that  $\gcd(a, b) := r_{n+1}$ .

## 6.2 $\gcd(a, b)$ as a Linear Combination of a and b

**Proposition 6.2.1.** Given  $a, b \in \mathbb{Z}$ , then for some  $k_1, k_2 \in \mathbb{Z}$ , and some  $d \in \mathbb{Z}$ ,

$$d = \gcd(a, b) = k_1 a + k_2 b \quad (6.8)$$

**Remark.** To solve the congruence  $4 * x = 1 \pmod{17}$  for  $x$ , find  $x$  in the form of  $x = 4^{-1} \pmod{17}$ .

For instance, to find  $\gcd(34, 13)$  as a linear combination  $k_1 a + k_2 b$ , then first use the Euclidean algorithm to find  $\gcd(34, 13)$ :

$$\begin{array}{l|l} 34 = 2 \cdot 13 + 8 & a = 2 \cdot b + r_1 \\ 13 = 8 + 5 & b = r_1 + r_2 \\ 8 = 5 + 3 & r_1 = r_2 + r_3 \\ 5 = 3 + 2 & r_2 = r_3 + r_4 \\ 3 = 2 + \boxed{1} & r_3 = r_4 + \boxed{r_5} \\ 2 = 2 \cdot 1 + 0 & r_4 = 2 \cdot r_5 + 0 \end{array} \quad (6.9)$$

Note that

$$\begin{array}{ll}
 a = 2 \cdot b + r_1 & r_1 = a - 2b \\
 b = r_1 + r_2 & r_2 = b - r_1 \\
 r_1 = r_2 + r_3 & r_3 = r_1 - r_2 \\
 r_2 = r_3 + r_4 & r_4 = r_2 - r_3 \\
 r_3 = r_4 + \boxed{r_5} & \boxed{r_5} = r_3 - r_4 \\
 r_4 = 2 \cdot r_5 + 0 & 
 \end{array} \quad (6.10)$$

It is now possible to *collect*  $k_1$  and  $k_2$  in a bottom-up manner:

$$\boxed{r_5} = r_3 - r_4 \quad (6.11)$$

$$= r_3 - (r_2 - r_3) \quad (6.12)$$

$$= -r_2 + 2r_3 \quad (6.13)$$

$$= -r_2 + 2(r_1 - r_2) \quad (6.14)$$

$$= 2r_1 - 3r_2 \quad (6.15)$$

$$= 2r_1 - 3(b - r_1) \quad (6.16)$$

$$= -3b + 5r_1 \quad (6.17)$$

$$= -3b + 5(a - 2b) \quad (6.18)$$

$$= 5a - 13b \quad (6.19)$$

Hence  $\gcd(34, 13) = \gcd(a, b) = 5a - 13b$  for some  $a, b \in \mathbb{Z}$ . One may verify this by checking that

$$5 \cdot 34 - 13 \cdot 13 = 170 - 169 = 1 \quad (6.20)$$

### 6.3 Problems for Integers Modulo $m$

- $\boxed{a * x = b \pmod{m} \Leftrightarrow x = a^{-1} * b \pmod{m}}$   
For  $\mathbb{R}^+$ , given some  $a, b, m \in \mathbb{Z}$

$$a * x = b \pmod{m} \quad (6.21)$$

$$\Leftrightarrow a^{-1} * a * x = a^{-1} * b \pmod{m} \quad (6.22)$$

$$\Leftrightarrow x = a^{-1} * b \pmod{m} \quad (6.23)$$

- $\boxed{a^n \pmod{m} \Leftrightarrow (a \cdot a^2 \cdot a^4 \cdot a^8 \cdot \dots) \pmod{m}}$

That is, to decompose the exponent into smaller equivalences.

- $\boxed{x^a = b \pmod{m} \Leftrightarrow x = b^{a^{-1}} \pmod{m}}$

For  $\mathbb{R}^+$ , given some  $a, b, m \in \mathbb{Z}$

$$x^a = b \pmod{m} \quad (6.24)$$

$$x = \sqrt[a]{b} \pmod{m} \quad (6.25)$$

$$x = b^{\frac{1}{a}} \pmod{m} \quad (6.26)$$

$$x = b^{a^{-1}} \pmod{m} \quad (6.27)$$

- For the discrete logarithm:  $a^x = b \pmod{m} \Leftrightarrow x = \log_a b \pmod{m}$

## 6.4 Multiplicative Group of Integers Modulo $m$

**Definition 6.4.1** (Relatively Prime, Coprime). Two integers  $a, b \in \mathbb{Z}$  are *relatively prime* (or *coprime*) if

$$\gcd(a, b) = 1 \quad (6.28)$$

**Definition 6.4.2** (Multiplicative Group of mod  $m$ ). Given  $m \in \mathbb{Z}$ , then

$$G_m^\times := \{a \in \mathbb{Z} \mid (1 \leq a < m) \wedge (\gcd(a, m) = 1)\} \quad (6.29)$$

Forms a group  $(G_m^\times, * \pmod{m})$  under *multiplicative modulo  $m$* .

### 1. Closure

$$\forall a, b, m \in G_m^\times : (\gcd(a, m) = 1) \wedge (\gcd(b, m) = 1) \rightarrow (\gcd(a * b, m) = 1) \quad (6.30)$$

### 2. Associativity

Given by multiplication on integers modulo  $m$ .

### 3. Neutral Element

$$\forall m \in G_m^\times : \gcd(1, m) = 1 \quad (6.31)$$

### 4. Invertibility

$$\forall a \in G_m^\times : \exists y \in G_m^\times : a * y = 1 \pmod{m} \quad (6.32)$$

For which the inverse element  $y$  is denoted  $a^{-1}$ , giving

$$\forall a \in G_m^\times : a * a^{-1} = 1 \pmod{m} \quad (6.33)$$

**Theorem 6.4.1** (Euler Totient Function). Given the *multiplicative modulo group*  $G_m^\times$ , then

$$\phi(m) := |G_m^\times| \quad (6.34)$$

**Theorem 6.4.2.** If  $p$  is prime then

$$\phi(p) \equiv p - 1 \quad (6.35)$$

**Theorem 6.4.3.** If  $p$  is prime and  $k \geq 1$  then

$$\phi(p^k) \equiv p^{k-1}(p - 1) \quad (6.36)$$

**Theorem 6.4.4.** If  $a, b \in \mathbb{Z}$  and  $a, b$  are *relatively prime* (i.e.  $\gcd(a, b) = 1$ ) then

$$\phi(ab) \equiv \phi(a)\phi(b) \quad (6.37)$$

**Theorem 6.4.5.** If  $a, m \in \mathbb{Z}$  are *relatively prime* (i.e.  $\gcd(a, m) = 1$ ) then

$$a^{\phi(m)} \equiv 1 \pmod{m} \quad (6.38)$$

**Theorem 6.4.6** (Fermat's Little Theorem). Given  $p$  is a prime number, then for any  $a \in \mathbb{Z}$

$$a^p \equiv a \pmod{p} \quad (6.39)$$

Additionally, if  $a, p \in \mathbb{Z}$  are *relatively prime*,  $\gcd(a, p) = 1$ ,

$$a^{p-1} \equiv 1 \pmod{p} \quad (6.40)$$

**Remark.** Given  $a \in G_m^\times$ , to find  $x$  such that

$$a * x = b \pmod{m} \quad (6.41)$$

Find  $a^{-1} \pmod{m}$ .

For example, for

$$13 * x = 6 \pmod{34} \quad (6.42)$$

Since

$$x = 13^{-1} * 6 \pmod{34} \quad (6.43)$$

Find  $13^{-1} \pmod{34}$  via the *Euclidean algorithm* which gives

$$13^{-1} = 21 \pmod{34} \quad (6.44)$$

Then

$$x = 21 * 6 \pmod{34} \quad (6.45)$$

$$= 126 - 3 * 34 \pmod{34} \quad (6.46)$$

$$= 24 \pmod{34} \quad (6.47)$$

**Remark.** To compute expressions of the form

$$a^n \pmod{m} \quad (6.48)$$

One should decompose  $a^n$  to  $a^n = a \cdot a^2 \cdot a^4 \cdot \dots$ , and use Fermat's Little Theorem and Euler Totient Function Identities whenever possible.

**Remark.** For equations of the form

$$x^a = b \pmod{m} \quad (6.49)$$

Then

$$x = b^{a^{-1}} \pmod{m} \quad (6.50)$$



If  $\gcd(a, \phi(m)) = 1$  then

$$a * y = 1 \pmod{\phi(m)} \quad (6.51)$$

$$x = b^y \pmod{m} \quad (6.52)$$

if  $\gcd(b, m) = 1$ , that is if  $b, m$  are *relatively prime*

$$x^a = (b^y)^a \pmod{m} \quad (6.53)$$

$$= b^{a*y} \pmod{m} \quad (6.54)$$

$$= b^{1+k\phi(m)} \pmod{m} \quad (6.55)$$

$$= b * (b^{\phi(m)})^k \pmod{m} \quad (6.56)$$

$$= b * 1^k \pmod{m} \quad (6.57)$$

$$= b \pmod{m} \quad (6.58)$$

## 6.5 Rivest–Shamir–Adleman (RSA) Cryptography

**Definition 6.5.1** (RSA, Public Keys and Private Keys). Given actors Alice and Bob, the process of RSA is

1. Alice provides *secrete* primes  $p$  and  $q$ .

$$n = p * q \quad (6.59)$$

2. Alice provides two integers  $d$  and  $e$  such that

$$d * e = 1 \pmod{\phi(p * q)} \quad (6.60)$$

3. Alice distributes the pair  $(n, e)$  to everyone.
4. Encryption and Decryption is then

$$\text{encrypt}_{n,e}(m) := m^e \pmod{n} \quad (6.61)$$

$$\text{decrypt}_{n,d}(m) := c^d \pmod{n} \quad (6.62)$$

5. Bob *encrypts* message  $m$  as the encrypted message  $c$  where

$$c := \text{encrypt}_{n,e}(m) \quad (6.63)$$

And sends  $c$  to Alice.

6. Alice *decrypts*  $c$  as

$$m' = \text{decrypt}_{n,d}(c) \quad (6.64)$$

Check that  $\gcd(m, n) = 1$ , that is if  $m, n$  are *relatively prime*, then

$$m' \pmod{n} = c^d \pmod{n} \quad (6.65)$$

$$= (m^e)^d \pmod{n} \quad (6.66)$$

$$= m^{d*e} \pmod{n} \quad (6.67)$$

$$= m^{1+k\phi(p*q)} \pmod{n} \quad (6.68)$$

$$= m \pmod{n} \quad (6.69)$$

Then *only* Alice can decrypt the encrypted message  $c$  in polynomial time.

**Remark.** An example of the RSA process:

1. Alice provides secret primes  $p = 3, q = 41$

$$n = 3 * 41 = 123 \quad (6.70)$$

2. Alice provides two integers  $d = 27, e = 3$

$$d * e \pmod{\phi(3 * 41)} = 27 * 3 \pmod{\phi(3 * 41)} \quad (6.71)$$

$$= 81 \pmod{[\phi(3) * \phi(41)]} \quad (6.72)$$

$$= 81 \pmod{[2 * 40]} \quad (6.73)$$

$$= 81 \pmod{80} \quad (6.74)$$

$$= 1 \pmod{80} \quad (6.75)$$

3. Alice distributes  $(n, e) = (123, 3)$  to everyone.

4. The encryption and decryption functions are

$$\text{encrypt}_{n,e}(m) = m^3 \pmod{n} \quad (6.76)$$

$$\text{decrypt}_{n,d}(c) = c^{27} \pmod{n} \quad (6.77)$$

5. Given a message  $m = 5$  then Bob sends

$$c = 5^3 \pmod{123} \quad (6.78)$$

$$= 125 \pmod{123} \quad (6.79)$$

$$= 2 \pmod{123} \quad (6.80)$$

6. Alice receives the encrypted message  $c = 2$  and decrypts with the fact that  $\gcd(123, 5) = 1$

$$m' \pmod{123} = 2^{27} \pmod{123} \quad (6.81)$$

$$= 5 \pmod{123} \quad (6.82)$$

# 7 Linear Algebra

## 7.1 Matrix Basics

**Definition 7.1.1** (Matrix). A  $(n \times m)$ -dimension matrix  $A$  has  $n$  rows and  $m$  columns, and each of its entries  $a_{j,k}$ , for  $1 \leq j \leq n$  and  $1 \leq k \leq m$  are denoted as

$$A = \begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,m} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n,1} & a_{n,2} & \cdots & a_{n,m} \end{bmatrix} \quad (7.1)$$

**Definition 7.1.2** (Set of Matrices of Dimension  $n \times m$ ). Let  $\mathcal{M}(n, m)$  denote the set of all matrices with dimension  $n \times m$ , that is, having  $n$  rows and  $m$  columns.

**Definition 7.1.3** (Square Matrix). A *square matrix* is a matrix with dimension  $n \times n$ .

**Definition 7.1.4** (Matrix Addition). Let  $A, B \in \mathcal{M}(n, m)$  be two matrices of the same dimension  $n \times m$ . Then the sum matrix  $C = A + B$  is defined to have entries

$$c_{j,k} = a_{j,k} + b_{j,k} \quad (7.2)$$

That is,

$$\begin{aligned} & \begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,m} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n,1} & a_{n,2} & \cdots & a_{n,m} \end{bmatrix} + \begin{bmatrix} b_{1,1} & b_{1,2} & \cdots & b_{1,m} \\ b_{2,1} & b_{2,2} & \cdots & b_{2,m} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n,1} & b_{n,2} & \cdots & b_{n,m} \end{bmatrix} \\ & := \begin{bmatrix} a_{1,1} + b_{1,1} & a_{1,2} + b_{1,2} & \cdots & a_{1,m} + b_{1,m} \\ a_{2,1} + b_{2,1} & a_{2,2} + b_{2,2} & \cdots & a_{2,m} + b_{2,m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n,1} + b_{n,1} & a_{n,2} + b_{n,2} & \cdots & a_{n,m} + b_{n,m} \end{bmatrix} \end{aligned} \quad (7.3)$$

**Definition 7.1.5** (Matrix Multiplication). Let  $A$  be an  $(l \times m)$  matrix and  $B$  be an  $(m \times n)$  matrix. Then their product  $C = A \cdot B$  is the  $(l \times n)$  matrix where each entry  $c_{j,k}$  is

$$c_{j,k} := \sum_{s=1}^m a_{j,s} b_{s,k} \quad (7.4)$$

Note that matrix multiplication is *not commutative*, that is, for most cases  $A \cdot B \neq B \cdot A$

**Definition 7.1.6** (Identity Matrix). Let  $I_n$  denote the *identity* matrix with dimension  $n \times n$

$$I_n := \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} \quad (7.5)$$

Notice that all diagonal entries  $i_{j,k}$  with indices  $j = k$  is 1, while all other entries are 0.

Alternatively, the *identity* matrix can be defined with entries  $\delta_{j,k}$  where  $\delta$  is the *Kronecker symbol* such that

$$\delta_{j,k} := \begin{cases} 1 & j = k \\ 0 & j \neq k \end{cases} \quad (7.6)$$

**Definition 7.1.7** (Matrix Multiplication by Scalar  $\lambda$ ). Let  $\lambda \in \mathbb{R}$  be a constant, then the multiplication of an  $(n \times m)$ -dimension matrix  $A$  by  $\lambda$  is defined as

$$\lambda A := \begin{bmatrix} \lambda a_{1,1} & \lambda a_{1,2} & \cdots & \lambda a_{1,m} \\ \lambda a_{2,1} & \lambda a_{2,2} & \cdots & \lambda a_{2,m} \\ \vdots & \vdots & \ddots & \vdots \\ \lambda a_{n,1} & \lambda a_{n,2} & \cdots & \lambda a_{n,m} \end{bmatrix} \quad (7.7)$$

If the dimension of  $A$  is  $n \times n$ , i.e.  $A$  is a *square matrix*, then  $\lambda A$  is equivalently

$$\lambda A := \begin{bmatrix} \lambda & 0 & \cdots & 0 \\ 0 & \lambda & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda \end{bmatrix} \begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n,1} & a_{n,2} & \cdots & a_{n,n} \end{bmatrix} \quad (7.8)$$

**Lemma 7.1.1.** If  $A$  is a matrix with dimension  $n \times n$ ,  $A$  is a *square matrix*, then

$$AI \equiv IA \equiv A \quad (7.9)$$

Where  $I$  is the *identity* matrix with dimension  $n \times n$ .

*Proof.* Let  $B = AI$ , then

$$b_{j,k} = \sum_{s=1}^n a_{j,s} \delta_{s,k} \quad (7.10)$$

Only  $\delta_{k,k}$  is non-zero, thus  $b_{j,k} = a_{j,k}$ . The same is true for  $IA$ . ■

### 7.1.1 Matrix Addition and Multiplication Properties

**Proposition 7.1.1** (Associative Matrix Multiplication). Given matrices  $A \in \mathcal{M}(n, m)$ ,  $B \in \mathcal{M}(m, p)$  and  $C \in \mathcal{M}(p, q)$  then

$$(AB)C \equiv A(BC) \quad (7.11)$$

*Proof.* The entry  $t_{j,l}$  of  $T = (AB)C$  is

$$t_{j,l} = \sum_{k=1}^p \left( \sum_{s=1}^m a_{j,s} b_{s,k} \right) c_{k,l} \equiv \sum_{k=1}^p a_{j,s} \left( \sum_{s=1}^m b_{s,k} c_{k,l} \right) = u_{j,l} \quad (7.12)$$

Where  $u_{j,l}$  are entries of the matrix  $U = A(BC)$  ■

**Proposition 7.1.2** (Distributive Matrix Multiplication). Given matrices  $A \in \mathcal{M}(n, m)$ ,  $B \in \mathcal{M}(m, p)$  and  $C \in \mathcal{M}(p, q)$  then

$$A(B + C) = AB + AC \quad (7.13)$$

$$(A + B)C = AC + BC \quad (7.14)$$

*Proof.* Let  $S = A(B + C)$  and  $E = AB + AC$ , then each entry  $s_{j,l}$  from  $S$  is

$$s_{j,l} = \sum_{s=1}^m a_{j,s} (b_{s,l} + c_{s,l}) \equiv \sum_{s=1}^m a_{j,s} b_{s,l} + \sum_{s=1}^m a_{j,s} c_{s,l} = e_{j,l} \quad (7.15)$$

Where  $e_{j,l}$  are entries from  $E$ .

Let  $T = (A + B)C$  and  $F = AC + BC$ , then each entry  $t_{j,l}$  from  $T$  is

$$t_{j,l} = \sum_{s=1}^m (a_{j,s} + b_{j,s}) c_{s,l} \equiv \sum_{s=1}^m a_{j,s} c_{s,l} + \sum_{s=1}^m b_{j,s} c_{s,l} = f_{j,l} \quad (7.16)$$

Where  $f_{j,l}$  are entries from  $F$ . ■

## 7.1.2 Determinant of a Square Matrix

**Definition 7.1.8** (Determinant of a  $2 \times 2$  Matrix). Given a  $2 \times 2$  square matrix  $A \in \mathcal{M}(2, 2)$

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad (7.17)$$

Then the determinant of  $A$ , denoted  $\det(A)$  or  $|A|$  is calculated with

$$\det(A) = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc \quad (7.18)$$

**Definition 7.1.9** (Determinant of a  $3 \times 3$  Matrix). Given a  $3 \times 3$  square matrix  $A \in \mathcal{M}(3, 3)$

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \quad (7.19)$$

Then the determinant of  $A$ , denoted  $\det(A)$  or  $|A|$  is calculated with

$$\det(A) = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a \begin{vmatrix} \square & \square & \square \\ \square & e & f \\ \square & h & i \end{vmatrix} - b \begin{vmatrix} \square & \square & \square \\ d & \square & f \\ g & \square & i \end{vmatrix} + c \begin{vmatrix} \square & \square & \square \\ d & e & \square \\ g & h & \square \end{vmatrix} \quad (7.20)$$

$$= a \begin{vmatrix} e & f \\ h & i \end{vmatrix} - b \begin{vmatrix} d & f \\ g & i \end{vmatrix} + c \begin{vmatrix} d & e \\ g & h \end{vmatrix} \quad (7.21)$$

$$= aei - afh + bfg - bdi + cdh - ceg \quad (7.22)$$

**Definition 7.1.10** (Upper Triangular Matrix). An  $n \times n$  matrix  $A \in \mathcal{M}(n, n)$  is called a *upper triangular* (or *right triangular*) matrix if it has the form

$$A = \begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ & a_{2,2} & \cdots & a_{2,n} \\ & & \ddots & \vdots \\ 0 & & & a_{n,n} \end{bmatrix} \quad (7.23)$$

Where all the lower triangular part are 0s.

**Lemma 7.1.2** (Determinant of an Upper Triangular Matrix). Given an  $n \times n$  *upper triangular* matrix  $A$ , then its *determinant*  $\det(A)$  can be calculated as

$$\det(A) = \begin{vmatrix} \gamma_1 & * & * & \cdots & * \\ \vdots & \gamma_2 & * & \ddots & \vdots \\ \vdots & \cdots & \gamma_3 & * & * \\ \vdots & \ddots & \vdots & \ddots & * \\ 0 & \cdots & \cdots & \cdots & \gamma_n \end{vmatrix} = \gamma_1 \gamma_2 \cdots \gamma_n \quad (7.24)$$

Where  $*$  represents arbitrary entries.

**Corollary 7.1.2.1.** A specialization of this lemma is the case for  $3 \times 3$  *upper triangular* matrix  $A$ :

$$\det(A) = \begin{vmatrix} \gamma_1 & * & * \\ 0 & a & b \\ 0 & c & d \end{vmatrix} = \begin{vmatrix} \gamma_1 & * & * \\ 0 & a & b \\ 0 & 0 & d - b \cdot \frac{c}{a} \end{vmatrix} = \gamma_1(ad - bc) \quad (7.25)$$

## 7.2 Solving Linear System of Equations

**Definition 7.2.1.** Matrices are useful for solving a *linear system of equations* of the form

$$\begin{cases} a_{1,1}x_1 + a_{1,2}x_2 + \cdots + a_{1,n}x_n = b_1 \\ a_{2,1}x_1 + a_{2,2}x_2 + \cdots + a_{2,n}x_n = b_2 \\ \vdots \\ a_{n,1}x_1 + a_{n,2}x_2 + \cdots + a_{n,n}x_n = b_n \end{cases} \quad (7.26)$$

Then, the matrix of the *coefficients* is denoted as  $A$  with dimension  $n \times n$  where

$$A = \begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n,1} & a_{n,2} & \cdots & a_{n,n} \end{bmatrix} \quad (7.27)$$

The *unknowns* are denoted as  $X$  with dimension  $n \times 1$  where

$$X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \quad (7.28)$$

The *constants* are denoted as  $B$  with dimension  $n \times 1$  where

$$B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} \quad (7.29)$$

Together, they yield the matrix equation

$$A \cdot X = B \quad (7.30)$$

To solve for  $X$ , one needs to find the *inverse* matrix  $A^{-1}$  of  $A$  such that

$$A \cdot X = B \quad (7.31)$$

$$A^{-1} \cdot A \cdot X = A^{-1} \cdot B \quad (7.32)$$

$$I \cdot X = A^{-1} \cdot B \quad (7.33)$$

$$X = A^{-1} \cdot B \quad (7.34)$$

Where  $I$  is the *identity* matrix.

## 7.3 Gaussian Elimination

**Definition 7.3.1** (Augmented Matrix). Given a system of linear equations

$$\begin{cases} a_{1,1}x_1 + a_{1,2}x_2 + \cdots + a_{1,n}x_n = b_1 \\ a_{2,1}x_1 + a_{2,2}x_2 + \cdots + a_{2,n}x_n = b_2 \\ \vdots \\ a_{n,1}x_1 + a_{n,2}x_2 + \cdots + a_{n,n}x_n = b_n \end{cases} \quad (7.35)$$

Then its *augmented* matrix  $A|B$  is

$$\left[ \begin{array}{cccc|c} a_{1,1} & a_{1,2} & \cdots & a_{1,n} & b_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} & b_{2,n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n,1} & a_{n,2} & \cdots & a_{n,n} & b_{n,n} \end{array} \right] \quad (7.36)$$

**Definition 7.3.2** (Row Operations).

1. **Multiply and Add Row**

Multiply row by scalar  $\gamma$  then add the result to another row.

$$\det(A') = \det(A) \quad (7.37)$$

2. **Swap Rows**

$$\det(A') = -\det(A) \quad (7.38)$$

3. **Multiply Row**

Multiply a row by scalar  $\gamma$ .

$$\det(A') = \gamma \det(A) \quad (7.39)$$

**Definition 7.3.3** (Gaussian Elimination). Using the *row operations* applied to  $A|B$  then one transforms  $AX = B$  into an equivalent system

$$A'X = B' \quad (7.40)$$

If it is the case that

$$A' = I \quad (7.41)$$

Then there exists a *solution*  $X = B'$  to the system

$$B' = A'X = IX = X \quad (7.42)$$

**Definition 7.3.4** (Inverse Matrix). The *inverse* matrix  $A^{-1}$  of  $A$  is the matrix for which under multiplication yields the *identity* matrix  $I$

$$AA^{-1} \equiv A^{-1}A \equiv I \quad (7.43)$$

With *Gaussian Elimination* applied to  $A|I$  then one transforms

$$AA^{-1} = I \Rightarrow A'A^{-1} = B' \quad (7.44)$$

If

$$A' = I \quad (7.45)$$

Then there exists a solution to  $A^{-1} = B'$

$$B' = A'A^{-1} = IA^{-1} = A^{-1} \quad (7.46)$$



## 7.4 Linear Maps

**Definition 7.4.1** ( $\mathbb{R}^n$ ).

$$\mathbb{R}^n := \overbrace{\mathbb{R} \times \mathbb{R} \times \cdots \times \mathbb{R}}^n \quad (7.47)$$

**Definition 7.4.2** ( $\mathbb{R}^{m,n}$ ). Is the domain of a matrix with  $m$  rows and  $n$  columns.

**Lemma 7.4.1** (Linear Mapping and Matrices). Any matrix defines a linear mapping.

Given a matrix  $A \in \mathbb{R}^{m,n}$ , then  $A$  defines a *linear mapping*  $f: \mathbb{R}^n \rightarrow \mathbb{R}^m$  if entries of  $\mathbb{R}^n$  are treated as *column vectors* then for  $V \in \mathbb{R}^{n,1}$

$$f(V) = AV \quad (7.48)$$

**Remark.** For example, for the  $\mathbb{R}^{2,3}$  matrix  $A$  where

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \in \mathbb{R}^{2,3} \quad (7.49)$$

$A$  defines a *linear mapping*  $f$  such that

$$f: \mathbb{R}^3 \rightarrow \mathbb{R}^2 \quad (7.50)$$

Since column vectors are used, then an  $m \times n$  matrix defines a mapping from  $\mathbb{R}^n \rightarrow \mathbb{R}^m$  with  $m, n$  reversed.

Then the mapping  $f$  is defined as

$$f \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \end{pmatrix} \quad f \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 5 \end{pmatrix} \quad f \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 6 \end{pmatrix} \quad (7.51)$$

Then the  $i$ th column of  $A$  represents the image of the  $i$ th element of  $\mathbb{R}^{n,1}$

**Remark.** Let there be an system of linear equations

$$\begin{cases} x'_1 = a_{1,1}x_1 + a_{1,2}x_2 + \cdots + a_{1,n}x_n \\ x'_2 = a_{2,1}x_1 + a_{2,2}x_2 + \cdots + a_{2,n}x_n \\ \vdots \\ x'_n = a_{n,1}x_1 + a_{n,2}x_2 + \cdots + a_{n,n}x_n \end{cases} \quad (7.52)$$

With

$$A = \begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n,1} & a_{n,2} & \cdots & a_{n,n} \end{bmatrix} \quad X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \quad X' = \begin{bmatrix} x'_1 \\ x'_2 \\ \vdots \\ x'_n \end{bmatrix} \quad (7.53)$$

Then there is a linear map

$$X' = AX \quad (7.54)$$

## 7.5 Eigenvalues and Eigenvectors

**Definition 7.5.1** (Eigenvalue and Eigenvector).

1. A real number  $\lambda \in \mathbb{R}$  is an *eigenvalue* of  $A$
2. A non-zero vector  $\vec{v}$  is an *eigenvector*

If

$$A\vec{v} = \lambda\vec{v}, \vec{v} \neq \vec{0} \quad (7.55)$$

Since

$$A\vec{v} - \lambda\vec{v} = (A - \lambda I) \cdot \vec{v} = \vec{0} \implies |A - \lambda I| = 0 \quad (7.56)$$

Hence, to solve for  $\lambda$ , use the equality

$$|A - \lambda I| = 0 \quad (7.57)$$

**Remark.** An example.

For the system of linear equations

$$\begin{cases} x' = 2x + 2y \\ y' = 2x + 5y \end{cases} \quad (7.58)$$

$$A = \begin{bmatrix} 2 & 2 \\ 2 & 5 \end{bmatrix} \quad (7.59)$$

$$|A - \lambda I| = \begin{vmatrix} 2 - \lambda & 2 \\ 2 & 5 - \lambda \end{vmatrix} = \lambda^2 - 7\lambda + 6 = 0 \quad (7.60)$$

Then there exist two *eigenvalues*

$$\lambda^2 - 7\lambda + 6 \implies \lambda_1 = 1, \lambda_2 = 6 \quad (7.61)$$

Then

$$A - \lambda_1 I = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \quad (7.62)$$

And

$$A - \lambda_2 I = \begin{bmatrix} -4 & 2 \\ 2 & -1 \end{bmatrix} \quad (7.63)$$

To find the *eigenvector* associated with each *eigenvalue*:

1. Case  $\lambda_1 = 1$

From the system, to find the *eigenvector*  $\vec{v}_{\lambda_1}$

$$(A - \lambda_1 I) \begin{bmatrix} \vec{v}_1 \\ \vec{v}_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} \vec{v}_1 \\ \vec{v}_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad (7.64)$$

Via Gaussian elimination,

$$\Leftrightarrow \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \vec{v}_1 \\ \vec{v}_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \implies \begin{cases} 1v_1 + 2v_2 = 0 \\ 0 + 0 = 0 \end{cases} \quad (7.65)$$

Then there exists an *infinite* number of solutions where

$$v_1 = -2v_2 \quad (7.66)$$

Taking one of them is sufficient, e.g.

$$\vec{v}_{\lambda_1} = \begin{bmatrix} -2 \\ 1 \end{bmatrix} \quad (7.67)$$

Check that for the *eigenvalue-eigenvector* pair that

$$A\vec{v}_{\lambda_1} = \lambda_1\vec{v}_{\lambda_1} \quad (7.68)$$

$$\begin{bmatrix} 2 & 2 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} -2 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \end{bmatrix} = \lambda_1 \begin{bmatrix} -2 \\ 1 \end{bmatrix} \quad (7.69)$$

2. Case  $\lambda_2 = 6$

Repeat the same procedure, and the *eigenvector* takes the value

$$\vec{v}_{\lambda_2} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad (7.70)$$

**Remark.** With  $A$  being symmetric, then *eigenvectors*  $\vec{v}_{\lambda_1}$  and  $\vec{v}_{\lambda_2}$  are *orthogonal*

$$\begin{bmatrix} \vec{v}_{\lambda_1} & \vec{v}_{\lambda_2} \end{bmatrix} \begin{bmatrix} \vec{v}_{\lambda_1} \\ \vec{v}_{\lambda_2} \end{bmatrix} \equiv \vec{0} \quad (7.71)$$

**Remark.** For the system of linear equations

$$\begin{cases} x' = \frac{\sqrt{2}}{2}x + \frac{\sqrt{2}}{2}y \\ y' = -\frac{\sqrt{2}}{2}x + \frac{\sqrt{2}}{2}y \end{cases} \quad (7.72)$$

$$A = \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix} \quad (7.73)$$

$$|A - \lambda I| = \begin{vmatrix} \frac{\sqrt{2}}{2} - \lambda & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} - \lambda \end{vmatrix} = \left( \frac{\sqrt{2}}{2} - \lambda \right)^2 + \frac{1}{2} = 0 \quad (7.74)$$

And thus there is no *real eigenvalues*; this  $A$  is in fact a *rotation*.



# 8 Counting

## 8.1 Counting Basics

### 8.1.1 Multiplication Principle

**Definition 8.1.1** (Multiplication Principle). The *multiplication principle* is used to count number of *tuples*  $(t_1, t_2, t_3, \dots)$  where  $t_i$  are selected from *independent* sources.

For any sets  $A_1, A_2, \dots, A_n$ , their Cartesian product

$$|A_1 \times A_2 \times \dots \times A_n| \equiv |A_1| \cdot |A_2| \cdot \dots \cdot |A_n| \quad (8.1)$$

**Remark.** For the set  $E_2 = \{0, 1\}$ ,

$$|E_2^3| = |E_2 \times E_2 \times E_2| = 2^3 = 8 \quad (8.2)$$

**Remark.** The number of boolean  $n$ -tuples is  $2^n$

$$|E_2^n| = \underbrace{|E_2 \times E_2 \times \dots \times E_2|}_n = 2^n \quad (8.3)$$

*Proof.* For the Cartesian product  $A \times B$  between any sets  $A$  and  $B$ ,

$$|A \times B| \equiv |A| \cdot |B| \quad (8.4)$$

|          | $a_1$        | $a_2$        | $\dots$  | $a_n$        |
|----------|--------------|--------------|----------|--------------|
| $b_1$    | $(a_1, b_1)$ | $(a_2, b_1)$ | $\dots$  | $(a_n, b_1)$ |
| $b_2$    | $(a_1, b_2)$ | $(a_2, b_2)$ | $\dots$  | $(a_n, b_2)$ |
| $\vdots$ | $\vdots$     | $\vdots$     | $\ddots$ | $\vdots$     |
| $b_k$    | $(a_1, b_k)$ | $(a_2, b_k)$ | $\dots$  | $(a_n, b_k)$ |

■

### 8.1.2 Addition Principle

**Definition 8.1.2** (Addition Principle (Inclusion-Exclusion Principle)). For any sets  $A$  and  $B$ ,

$$|A \cup B| \equiv |A| + |B| - |A \cap B| \quad (8.5)$$

**Remark.** This is used in probability where for any events  $A$  and  $B$

$$P(A \vee B) \equiv P(A) + P(B) - P(A \wedge B) \quad (8.6)$$



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