COMP0147 Discrete Mathematics for Computer Scientists Notes

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Notes adapted from lecture notes by Max Kanovich and Robin Hirsch [1].

Contents

1 Foundations **5**
1.1 Set Theory 5
1.1.1 Set Notations 5
1.1.2 Properties 5
1.1.3 Set Equality 5
1.1.4 Set Operations 6
1.1.5 Boolean Algebra 6
1.1.6 Set Algebra 7
1.2 Functions 9

1 Foundations

Contents

1.1 Set Theory **5**
- 1.1.1 Set Notations 5
- 1.1.2 Properties 5
- 1.1.3 Set Equality 5
- 1.1.4 Set Operations 6
- 1.1.5 Boolean Algebra 6
- 1.1.6 Set Algebra 7

1.2 Functions **9**

1.1 Set Theory

1.1.1 Set Notations

- Set definition: $A = \{a, b, c\}$
- Set membership (element-of): $a \in A$
- Set builder notation: $\{x \mid x \in \mathbb{R} \land x^2 = x\}$
- Empty set: $\emptyset$

1.1.2 Properties

- No structure
- No order
- No copies

For example, a, b, c are references to actual objects in

$$\{a, b, c\} \Leftrightarrow \{c, a, b\} \Leftrightarrow \{a, b, c, b\}$$

1.1.3 Set Equality

Definition 1.1.1 (Set Equality)**.** Set $A = B$ iff:

1. $A \subseteq B \implies \forall\, x(x \in A \to x \in B)$

2. $B \subseteq A \implies \forall\, y(y \in B \to y \in A)$

Remark. $A = B \Leftrightarrow A \subseteq B \wedge B \subseteq A$

1.1.4 Set Operations

- *Union*: $A \cup B \equiv \{x \mid x \in A \vee x \in B\}$
- *Intersection*: $A \cap B \equiv \{x \mid x \in A \wedge x \in B\}$
- *Relative Complement*: $A \setminus B \equiv \{x \mid x \in A \wedge x \notin B\}$
- *Absolute Complement*: $A^c \equiv U \setminus A \equiv \{x \mid x \in U \wedge x \notin A\}$
- *Symmetric Difference*: $A \Delta B \equiv (A \setminus B) \cup (B \setminus A) \equiv (A \cup B) \setminus (A \cap B)$
- *Cartesian Product*: $A \times B \equiv \{(x, y) \mid x \in A \wedge y \in B\}$

1.1.5 Boolean Algebra

Definition 1.1.2 (De Morgan's Laws)**.**

$$\neg(p \vee q) \equiv \neg p \wedge \neg q \tag{1.1}$$
$$\neg(p \wedge q) \equiv \neg p \vee \neg q \tag{1.2}$$

Definition 1.1.3 (Idempotent Laws)**.**

$$p \vee p \equiv p \tag{1.3}$$
$$p \wedge p \equiv p \tag{1.4}$$

Definition 1.1.4 (Commutative Laws)**.**

$$p \vee q \equiv q \vee p \tag{1.5}$$
$$p \wedge q \equiv q \wedge p \tag{1.6}$$

Definition 1.1.5 (Associative Laws)**.**

$$p \vee (q \vee r) \equiv (p \vee q) \vee r \tag{1.7}$$
$$p \wedge (q \wedge r) \equiv (p \wedge q) \wedge r \tag{1.8}$$

Definition 1.1.6 (Distributive Laws)**.**

$$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r) \tag{1.9}$$
$$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r) \tag{1.10}$$

Definition 1.1.7 (Identity Laws)**.**

$$p \vee \mathrm{F} \equiv p \tag{1.11}$$
$$p \vee \mathrm{T} \equiv \mathrm{T} \tag{1.12}$$
$$p \wedge \mathrm{T} \equiv p \tag{1.13}$$
$$p \wedge \mathrm{F} \equiv \mathrm{F} \tag{1.14}$$

Definition 1.1.8 (Absorption Laws).

$$p \vee (p \wedge q) \equiv p \tag{1.15}$$
$$p \wedge (p \vee q) \equiv p \tag{1.16}$$

Definition 1.1.9 (Implication and Negation Laws).

- *Identity*: $p \to q \equiv \neg p \vee q$
- *Counter-example*: $\neg(p \to q) \equiv p \wedge \neg q$
- *Equivalences*: $p \to q \to r \equiv (p \wedge q) \to r \equiv q\ to(p \to r)$
- *Absorption*:
  $p \to \mathrm{T} \equiv T$
  $p \to \mathrm{F} \equiv \neg p$
  $\mathrm{T} \to p \equiv p$
  $\mathrm{F} \to p \equiv T$
- *Contrapositive*: $p \to q \equiv \neg q \to \neg p$
- *Law of Excluded Middle*:
  $p \vee \neg p \equiv \mathrm{T}$
  $p \wedge \neg p \equiv \mathrm{F}$
- *Double Negation*: $\neg\neg p \equiv p$
- *Reduction to Absurdity*: $\neg p \to \mathrm{F} \equiv p$

1.1.6 Set Algebra

Definition 1.1.10 (De Morgan's Laws).

$$(A \cup B)^c \equiv A^c \cap B^c \tag{1.17}$$
$$(A \cap B)^c \equiv A^c \cup B^c \tag{1.18}$$

Definition 1.1.11 (Idempotent Laws).

$$A \cup A \equiv A \tag{1.19}$$
$$A \cap A \equiv A \tag{1.20}$$

Definition 1.1.12 (Commutative Laws).

$$A \cup B \equiv B \cup A \tag{1.21}$$
$$A \cap B \equiv B \cap A \tag{1.22}$$

Definition 1.1.13 (Associativity Laws).

$$A \cup (B \cup C) \equiv (A \cup B) \cup C \tag{1.23}$$
$$A \cap (B \cap C) \equiv (A \cap B) \cap C \tag{1.24}$$

Definition 1.1.14 (Distributive Laws).

$$A \cap (B \cup C) \equiv (A \cap B) \cup (B \cap C) \tag{1.25}$$
$$A \cup (B \cap C) \equiv (A \cup B) \cap (B \cup C) \tag{1.26}$$

Definition 1.1.15 (Identity Laws).

$$A \cup \emptyset \equiv A \tag{1.27}$$
$$A \cap \emptyset \equiv \emptyset \tag{1.28}$$
$$A \cap U \equiv A \tag{1.29}$$
$$A \cup U \equiv U \tag{1.30}$$

Definition 1.1.16 (Absorption Laws).

$$A \cup (A \cap B) \equiv A \tag{1.31}$$
$$A \cap (A \cup B) \equiv A \tag{1.32}$$

Definition 1.1.17 (Difference Identity Laws).

$$C \setminus (A \cup B) \equiv (C \setminus A) \cap (C \setminus B) \tag{1.33}$$
$$C \setminus (A \cap B) \equiv (C \setminus A) \cup (C \setminus B) \tag{1.34}$$

Definition 1.1.18 (Complement-Difference Identity Law).

$$C \setminus D \equiv C \cap D^c \tag{1.35}$$

Definition 1.1.19 (Double Complement Law).

$$(D^c)^c \equiv D \tag{1.36}$$

Definition 1.1.20 (Contraposition).

$$C \subseteq D \Leftrightarrow D^c \subseteq C^c \tag{1.37}$$
$$C = D \Leftrightarrow C^c = D^c \tag{1.38}$$

Definition 1.1.21 (Arbitrary Union). Given sets $A_1, A_2, \dots, A_n$ where $I = \{1, 2, \dots, n\}$

$$A_1 \cup A_2 \cup \dots \cup A_n \equiv \bigcup_{i \in I} A_i \tag{1.39}$$

Then

$$x \in \bigcup_{i \in I} A_i \Leftrightarrow \exists\, i \in I : x \in A_i \tag{1.40}$$

Definition 1.1.22 (Arbitrary Intersection). Given sets $A_1, A_2, \dots, A_n$ where $I = \{1, 2, \dots, n\}$

$$A_1 \cap A_2 \cap \dots \cap A_n \equiv \bigcap_{i \in I} A_i \tag{1.41}$$

Then

$$x \in \bigcap_{i \in I} A_i \Leftrightarrow \forall\, i \in I : x \in A_i \tag{1.42}$$

1.2 Functions

Definition 1.2.1 (Function). A function f is a mapping from X to Y

$$f\colon X \mapsto Y \tag{1.43}$$

- $\mathrm{domain}(f) = X$
- $\mathrm{image}(f) = f(X)$

Definition 1.2.2 (Total Function). A function is *total* if

$$\mathrm{domain}(f) = X \tag{1.44}$$

Definition 1.2.3 (Partial Function). A function is *partial* if

$$\mathrm{domain}(f) \subseteq X \tag{1.45}$$

Definition 1.2.4 (Surjection). A function $f\colon X \mapsto Y$ is *surjective* iff

$$f(X) = Y \Leftrightarrow \forall\, y \in Y\colon\ \exists\, x \in X\colon\ f(x) = y \tag{1.46}$$

Namely each $y \in Y$ has a corresponding $x \in X$.

Definition 1.2.5 (Injection (Encodings, One-to-one)). A function $f\colon X \mapsto Y$ is *injective* iff

$$\forall\, x_1, x_2 \in X\colon x_1 \neq x_2 \to f(x_1) \neq f(x_2) \tag{1.47}$$
$$\Leftrightarrow \forall\, x_1, x_2 \in X\colon f(x_1) = f(x_2) \to x_1 = x_2 \tag{1.48}$$

Namely each distinct element $x \in X$ maps to a different element in Y.

Definition 1.2.6 (Bijection). A function $f\colon X \mapsto Y$ is *bijective* iff f is both *injective* and *surjective*.

$$\mathrm{Bijective}(f) \equiv \mathrm{Injective}(f) \wedge \mathrm{Surjective}(f) \tag{1.49}$$

The *inverse bijection* $f^{-1}\colon Y \mapsto X$ does exist.

Bibliography

[1] Max Kanovich and Robin Hirsch.
"Lecture Notes on Discrete Mathematics for Computer Scientists".
URL: http://www.cs.ucl.ac.uk/1819/a4u/t2/comp0147_discrete_mathematics_for_computer_scientists/.