Linear Programming

STATISTICS 186

Part 3

Last week?

- Some more of Linear Programming
- Touched on the extreme point method
- Realised that there are cases that more than one optimal solution exists

Today?

- Shadow Prices

Shadow Prices

- First need to know what a scare resource is
- Scarce Resource definition
  - A resource which limits a production process
- Shadow Price definition for a scarce resource:
  - The marginal contribution that the resource makes to profit OR in other words
    - Amount that profit increases if one additional unit of the resource was made available
- Will discuss two methods of calculating the shadow price of a scarce resource

Revisiting Example 1

- In example 1 scarce resource was capacity of machine and finishing time (used in finding optimal solution)
- Remember constraint equations
  - $6\mathrm{A} + 12\mathrm{B} \leq 1200$ (Machine)
  - $10A + 6B \leq 1200$ (Finishing)
- Now asked to calculate shadow price of Machine time

Method 1

APPLY DEFINITION DIRECTLY

Steps

1. State the critical constraints
2. Obtain the original optimal profit
3. Obtain the optimal profit by adding one additional unit of resource
4. Subtract the two quantities to determine shadow price
5. Provide interpretation

Step 1 – State the Critical Constraints

- $6A + 12B \leq 1200$ (Machine)
- $10A + 6B \leq 1200$ (Finishing)

Step 2 – Obtain the Original Optimal Profit

- When calculating shadow prices the number of units should **NOT** be rounded
- Therefore must recalculate profit
  - Optimal solution for example 1 (unrounded)
    - $A = 85.714$
    - $B = 57.143$
  - Hence profit
    - $P = 2(85.714) + 2.5(57.1429) = 314.2859$

Step 3 – Obtain the new optimal profit by adding one additional unit of resource

- $6\mathrm{A} + 12\mathrm{B} \leq 1200 + 1$ (Machine)
- $10A + 6B \leq 1200$ (Finishing)

- Hence new constraint equations becomes

  - $6\mathrm{A} + 12\mathrm{B} \leq 1201$ (Machine)
  - $10A + 6B \leq 1200$ (Finishing)

Step 3 – Obtain the new optimal profit by adding one additional unit of resource

- Following all steps to find an optimal solution we obtain solutions
  - $A = 85.6429$
  - $B = 57.2619$
- New optimal contribution of
  - $P = 2(85.6429) + 2.5(57.2619) = 314.4406$

Step 4 – Subtract the two quantities to determine shadow price

- $314.4406 - 314.2859 = 0.1547$
- Shadow price for machine time = $R0.1547$ per minute
- For an exercise work out that the shadow price for the finishing time is $= R0.1071$ per minute

Step 5 - Interpretation

- Shadow Price definition for a scarce resource:
  - Amount that profit increases if one additional unit of the resource was made available
- Hence definition
- Machine time
  - For every additional minute of machine time, optimal profit will increase by $R0.1547$
- Finishing time
  - For every additional minute of finishing time, optimal profit will increase by $R0.1071$

Method 2

PREFERABLE

Example 1

- Saw that the optimal solution was found where the following two constraint lines intersected

$$6A + 12B \leq 1200 \ (Machine)$$
$$10A + 6B \leq 1200 \ (\text{Finishing})$$

- Above two equations are called critical constraints

Method 2: Step 1 – Set Up Table

- Set up table
  - **First rows** will be constraint equations
  - **Last row** will be objective function
  - **First column** will be first product
  - **Second column** will be second product
  - **Third column** will be a third product

Step 1

$$6A + 12B \leq 1200\ (Machine)$$
$$10A + 6B \leq 1200\ (\text{Finishing})$$
$$P = 2A + 2.5B$$

	Product A	Product B	Limit
Machine time	6	12	1200
Finishing time	10	6	1200
Contribution	2	2.5	

Step 2 – Obtain shadow price equations

- Use columns of Product A and Product B to create the following equations

	Product A	Product B	Limit
Machine time	6	12	1200
Finishing time	10	6	1200
Contribution	2	2.5	

- Equations:

$$6M + 10F = 2$$

$$12M + 6F = 2.5$$

Step 3 – Solve Shadow Equation

- $M = R0.1548$ per minute
- $F = R0.1071$ per minute

- Remember definition of shadow prices
  - Amount that profit increases if one additional unit of the resource was made available
  - Here the additional resource was time
- If we compare units of shadow equations we see

Step 3 – Solve Shadow Equation

- If we compare units of shadow equations, we see

$$\underbrace{6M}_{\text{Min/unit}} + \underbrace{10F}_{\text{Min/unit}} = \underbrace{2}_{\text{R/unit}}$$

- Since left hand side is equal to right hand side it agrees per definition of shadow prices
- Note that since raw material was not a critical constraint – the shadow price is zero

Application of Shadow Prices

- Say as a business you receive a special offer?
  - Good for business since you obtain a bulk order which you can profit from
  - Need to be smart about situation
  - Shadow prices can be used to calculate the minimum selling price that needs to be quoted
- Will see how through an example

Example 1 - Revisted

- Saw amount that achieved maximum profit was
  - 85 units of A
  - 57 units of B
- Now the company receives a special order of 100 units of another product C

Example 1 - Revisited

- Following resources are used to produce 100 units
  - Raw material – 90kg
  - Machine time – 200 minutes
  - Finishing time – 140 minutes
- Cost of resources are
  - Raw material – R10/kg
  - Machine time – R5/minute
  - Finishing time - R10/minute
- Question now
  - Lowest selling price for special offer?

Step 1 – Work out cost price for new product

- Important to distinguish shadow prices and cost of machine and finishing time
  - Shadow prices deals with change in profit per unit
    - Already takes costs into account
  - Costs given calculated as the sum of labour, electricity involved in operating the machinery
    - Does not involve profits

Step 1 – Work out cost price for new product

- Cost price can be calculated as
  - $90 \times 10 = 900$ – Raw Material
  - $200 \times 5 = 1000$ – Machine time
  - $140 \times 10 = 1400$ – Finishing time
  - Total cost = 900 + 1000 + 1400 = R3300

- However this is not minimum amount
  - Since now there will be less resources for product A and B
  - Need to add the loss in contributions
    - For this we use shadow prices

Step 2 – Calculate loss resulting from fewer contributions from other products

- Raw Material
- Only used to produce product B
- Produce 57 units in optimal solution
- Requires 3 kg per unit

- Constraint was $3B \leq 270$ hence material available is 270kg
- Raw material used to produce B = $3 \times 57 = 171$kg
- Leaves us 99kg for product C
- Since only need 90kg for product C – does not cause loss of contribution

Step 2 – Calculate loss resulting from fewer contributions from other products

- Machine time and Finishing time
- Already know that these are scarce resources
- Shadow prices
  - $M = R0.1548$ per minute
  - $F = R0.1071$ per minute
- Now every minute lost to produce product C will result in a loss not profit

Step 2 – Calculate loss resulting from fewer contributions from other products

- Need 200 minutes machine and 140 minutes finishing hence loss in contribution
  - $200 \times R0.1548 = R30.96$
  - $R140 \times R0.1071 = R14.99$

- Hence lowest selling price for producing 100 units of product C

Manufacturing Cost	R3300
Loss in contribution:	
- Machine time	R30.96
- Finishing time	R14.99
Total Cost	R3345.95

But wait another method…

FOR STEP 2

Alternative method

- Use linear programming by considering new set of constraints
  - Obtained by subtracting resources required for product C
  - New constrain equations becomes

  - $6A + 12B \leq 1200 - 200 \Rightarrow 6A + 12B \leq 1000$
  - $10A + 6B \leq 1200 - 140 \Rightarrow 10A + 6B \leq 1060$
  - $3B \leq 270 - 90 \Rightarrow 3B \leq 180$

- Optimal profit obtained is then subtracted from original optimal profit
- **Must always use decimals when calculating contribution** (Only difference)

Further Remarks on Alternate Method

- If new opportunity or special offer affects a material that is not scarce
  - Example if product C required 150kg of raw material but on 99kg left over after catering for product A and B
  - Then not enough left to produce C
- Calculating loss in contribution must be using alternative method

Summary

- Tackled the idea of shadow prices
- Used it in an application of a special offer of a new product
  - Used methods to determine minimum selling price of special offer