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Queueing theory - Part 1 / Toustaanteorie - Deel 1

LSCM344 - July 2022
Lecturer - HW Freiboth

Slides adapted from those originally compiled for the module by Dr Neil Jacobs (2019)

Inleiding / Introduction

Die wêreld is vol van voorbeelde waar mense en "dinge" tyd in toue spandeer. Niemand se tyd is verniet nie, nie die diensverskaffer of die diensontvanger s'n nie - daarom is dit belangrik om tou-gedrag te bestudeer.

Verloop van die toustaanteorie lesingreeks:

1. Inleidende konsepte en terminologie.
2. Die eksponensiële verdeling.
3. Eienskappe en berekenings met 'n paar algemene toue.

The world is full of examples where people and "things" spend time in queues. Nobody's time is free, neither the service provider's nor the service receiver's - therefore it is important to study queue-behaviour.

Course of the queueing theory lecture series:

1. Introductory concepts and terminology.
2. The exponential distribution
3. Characteristics and calculations with some common queues.

Cars queueing / Voertuig tou

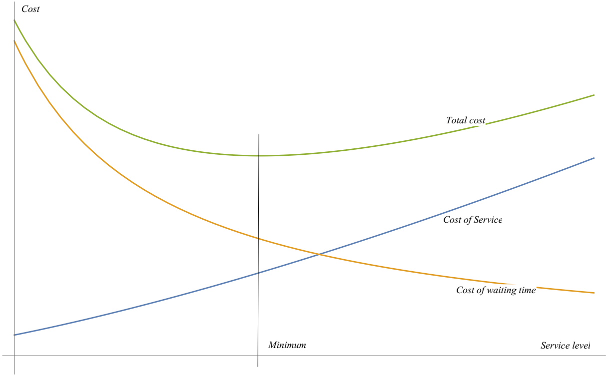
Service point / Dienspunt

Warehouse / Pakhuis

Tyd kostes / Time costs

Nie kliënte tyd of -diens is verniet nie...

Neither client time nor service is free...



Voorbeeld van tyd koste / Example of time costs

Blitz Verspreiders huur voertuie om bestellings wat by hom geplaas word af te lewer. Voertuie ry in sy perseel in, word gelaai en vertrek dan op die afleweringsrit. Voertuie word een-vir-een op 'n eerste-kom-eerste-bedien basis gelaai. Voertuie wat wag om gelaai te word staan in 'n tou. Daar is opgemerk dat hoe meer senior die span is wat die laaiwerk doen, hoe vinniger en gladder verloop die proses.

Blitz Distributors rents vehicles to deliver orders that were placed on him. Vehicles drive into his premises, are loaded and departs on a delivery run. Vehicles are loaded one-by-one on a first-come-first-serve basis. Vehicles waiting to be loaded stand in a queue. It has been observed that the more senior a team is that does the loading the faster and smoother the process goes.

Voorbeeld van tyd koste / Example of time costs

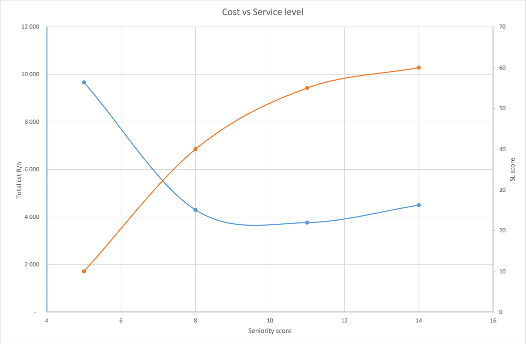
Koste ramings vir Blitz van verskillende laai-spanne word in die onderstaande tabel getoon.

Cost estimates for different loading teams are shown in the table below.

Blitz Distributors - Hourly cost vs Service level

Team seniority score	5	8	11	14
Arrival rate (v/h)	50	50	50	50
Avg Waiting time (minutes)	55	20	16	15
Tot waiting time per hour (h)	46	17	13	13
Waiting cost per veh.hour (R)	200	200	200	200
Tot waiting cost(R)	9 167	3 333	2 667	2 500
Total salary per hour (R)	500	967	1100	2000
Total cost per hour (R)	9 667	4 300	3 767	4 500
Service level score	10	40	55	60

Grafiese effekte van tyd koste / Graphical effects of time costs



Aard en eienskappe van toustaan-stelsels verskil / Nature and characteristics of queueing systems differ

Dink aan:

- Aankomste
  - Populasie
  - Patroon
  - Tussenaankomstye
  - Gedrag
- Waglyn
  - Lengte
  - Bedieningsreëls
- Dienspunte
  - Bedieningskanale
  - Fases
  - Dienstye

Voorbeelde ?

Think about:

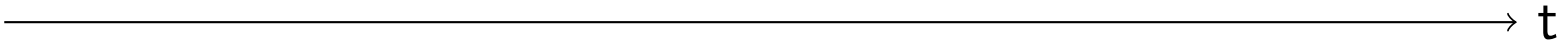
- Arrivals
  - Population
  - Pattern
  - Interarrivaltimes
  - Behaviour
- Queue
  - Length
  - Service rules
- Service points
  - Service channels
  - Phases
  - Service times

Examples ?

Tussenaankomstye / Interarrival times

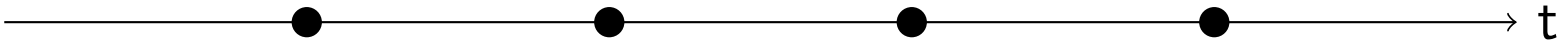
Wat is 'n "tussenaankomstyd"?

What is an "interarrival time"?



Tussenaankomstye / Interarrival times

Wat is 'n "tussenaankomstyd"? What is an "interarrival time"?



Tussenaankomstye / Interarrival times

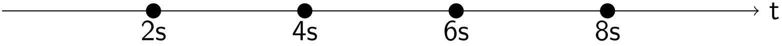
Wat is 'n "tussenaankomstyd"?

What is an "interarrival time"?

Aankomstye / Arrival times: $\bar{t} = \{2, 4, 6, 8\}$

Tussenaankomstye / Interarrival times

Wat is 'n "tussenaankomstyd"? What is an "interarrival time"?

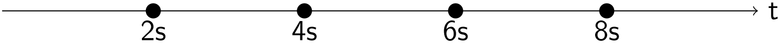


Aankomstye / Arrival times: $\bar{t} = \{2, 4, 6, 8\}$

Tussenaankomstyd / Interarrival times: $T_i = t_{i+1} - t_i$

Tussenaankomstye / Interarrival times

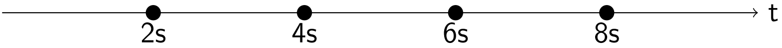
Wat is 'n "tussenaankomstyd"? What is an "interarrival time"?



Aankomstye / Arrival times: $\bar{t} = \{2, 4, 6, 8\}$
Tussenaankomstyd / Interarrival times: $T_3 = t_4 - t_3 = 8 - 6 = 2$

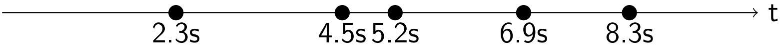
Tussenaankomstye / Interarrival times

Wat is 'n "tussenaankomstyd"? What is an "interarrival time"?



Aankomstye / Arrival times: $\bar{t} = \{2, 4, 6, 8\}$
Tussenaankomstyd / Interarrival times:: $\mathbf{T} = \{2, 2, 2\}$
Deterministies as die patroon dieselfde bly. / Deterministic if the pattern always remains the same.

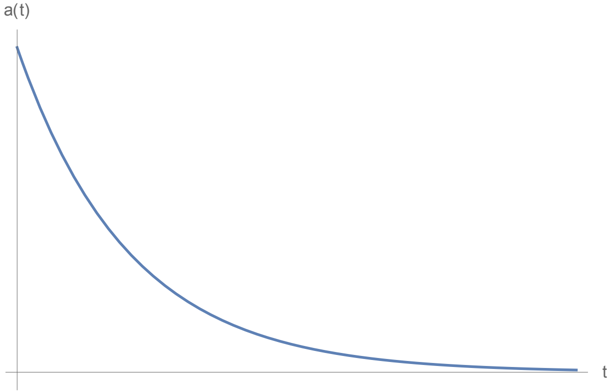
Stogastiese tussenaankomstye / Stochastic interarrival times



Stogastiese tussenaankomstye / Stochastic interarrival times

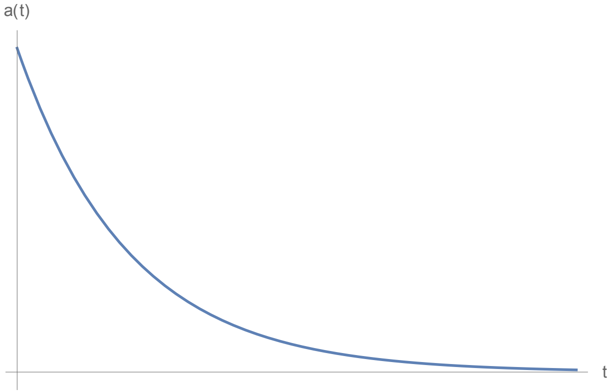
Tussenaankomstyd-waarskynlikheidsverdeling

In die modellering van tussenaankomstye word veronderstel dat die tussenaankomstye onafhanklik is van mekaar en beskryf kan word as 'n kansveranderlike **A** wat almal 'n identiese waarskynlikheidsverdeling volg met waarskynlikheidsdigtheidsfunksie $a(t)$.



Interarrival time-probability distribution

In the modelling of interarrival times it is supposed that the interarrival times are independent of each other and can be described by a random variable **A** which all follows the identical probability density function $a(t)$

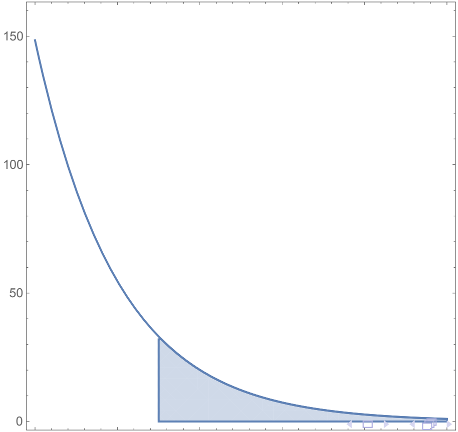


Tussenaankomstyd waarskynlikheidsverdeling / Interarrival time probability distribution

Met $a(t)$ bekend is

With $a(t)$ known it one can say

$$P(A > x) = \int_x^\infty a(t)dt$$



Die eksponensiële verdeling / The exponential distribution

Uiters belangrike waarskynlikheidsverdeling in toustaanteorie en ander modelleringsomgewings.

As die aantal aankomste per tydseenheid Poisson verdeel is, is die tussenaankomstye eksponensieel verdeel en vice versa.

Extremely important probability distribution in queuing theory and other modelling environment.

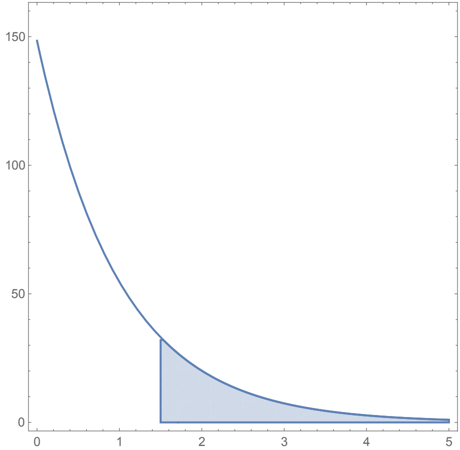
If the number of arrivals per time unit is Poisson distributed, the interarrival times are exponential distributed and vice versa

Die eksponensiële verdeling / The exponential distribution

Vir die eksponensiële verdeling is $a(t) = \lambda e^{-\lambda t}$ en die integraal

For the exponential $a(t) = \lambda e^{-\lambda t}$ and the integral

$$P(A > x) = \int_x^\infty a(t)dt = \int_x^\infty \lambda e^{-\lambda x} dt = e^{-\lambda x}$$



Oor die eksponensiële verdeling / On the exponential distribution

As / If

$$P_{Poisson}(N_{\lambda t} = n) = \frac{\mathrm{e}^{-\lambda t}(\lambda t)^n}{n!}, \tag{1}$$

dan is / then

$$P_{Expon}(A > t) = \int_t^\infty \lambda \mathrm{e}^{-\lambda x} dx = \mathrm{e}^{-\lambda t}. \tag{2}$$

en vice versa. / and vice versa.

$$\tag{3}$$

Oor die eksponensiële verdeling / On the exponential distribution

Dus gestel $\lambda = 0.5$ aankomste per minuut en die vraag is "Wat is die waarskynlikheid dat die volgende aankoms eers oor 3 minute sal plaasvind" dan sê die eksponensiële verdeling:

Thus say $\lambda = 0.5$ arrivals per minute and the question is "What is the probability that the next arrival will take place after 3 minutes" then the exponential distribution tells us:

$$P_{Expon}(A > 3) = \mathrm{e}^{-0.5 \cdot 3} = \mathrm{e}^{-1.5}$$

.

Oor die eksponensiële verdeling / On the exponential distribution

Om te sê dat die volgende aankoms eers na 3 minute plaasvind is dieselfde as om te sê daar het geen aankoms voor die verloop van 3 minute plaasgevind nie en gee die Poisson verdeling die antwoord op dieselfde vraag so:

By saying that the next arrival will only take place after 3 minutes is the same as to say that no arrival will take place before 3 minutes elapsing and the Poisson distribution answers the same question thus:

$$P_{Poisson}(N_{0.5\cdot 3} = 0) = \frac{\mathrm{e}^{-0.5\cdot 3}(0.5\cdot 3)^0}{0!} = \mathrm{e}^{-1.5} \tag{4}$$

en dus / and thus

$$P_{Poisson}(N_{0.5\cdot 3} = 0) = P_{Expon}(A > 3) \tag{5}$$

Wat is λ? / What is λ?

λ is die gemiddelde aankomstempo en word **altyd** uitgedruk in die vorm **aantal aankomste van iets per tydseenheid**, soos bv.
3 kliënte / minuut of
12 motors / uur of
3 transaksies / maand of
0.5 skepe / jaar, ens.

$\frac{1}{\lambda}$ is die gemiddelde tussenaankomstyd soos bv. 20 sekondes tussen kliënte se aankomste, 5 minute tussen motors, $\frac{1}{3}$ maand tussen transaksies, 2 jaar per skip, ens.

λ is the mean arrival rate and is **always** expressed in the form **number of arrivals of something per time unit**, for example
3 customers / minute or
12 cars / hour or
3 transactions / month or
0.5 ships / year

$\frac{1}{\lambda}$ is the mean interarrival time for example 20 seconds between customer arrivals, 5 minutes between cars, $\frac{1}{3}$ month between transactions, 2 year per ship, etc.

Redelike bemaring van die aankomstempo - $\hat{\lambda}$ / Reasonable estimate of the arrival rate - $\hat{\lambda}$

Gegewe 'n reeks van n tussen aankomstye $\{t_1, t_2, \ldots, t_n\}$, dan is

Given a series of n interarrival times $\{t_1, t_2, \ldots, t_n\}$, then

$$\hat{\lambda} = \frac{n}{\sum_{i=1}^{n} t_i} \tag{3}$$

Wat is μ? / What is μ?

μ is die gemiddelde dienstempo soos bereken vir die tyd wat die dienspunt in bedryf is en besig is. Dus as 'n dienspunt 10 kliënte gemiddeld in 'n uur help het maar gemiddeld vir 5 minute ledig is (daar is maw gemiddeld vir 5 minute van die uur geen kliënte by hom om te bedien nie) dan word die dienstempo bereken oor 55 minute.

μ is the average service rate as calculated for the time that a service point is in operation and busy. Thus if a service point helps 10 customers per hour on average but is on average 5 minutes of the hour idle (in other words for 5 minutes of the hour there are no clients there to serve) then the service rate is calculated over 55 minutes.

$$\mu = \frac{10}{55} \text{ kl/min} \tag{6}$$

$$= 0.\dot{1}\dot{8} \text{ kl/min} \tag{7}$$

$$= \frac{10}{55} \text{ kl/min } \cdot 60 \text{ min/h} \tag{8}$$

$$= 10.\dot{9}\dot{0} \text{ kl/h} \tag{9}$$

Hoe word $\hat{\mu}$ normaalweg bereken? / How is $\hat{\mu}$ normally calculated?

Deur waarneming van 'n groot aantal kliënte deur die dienspunt en bepaling van die gemiddelde dienstyd van die waargenome versameling bv. vir 'n versameling van n waarnemings, waar X_i die i^{de} waarneming is, dan kan $\frac{1}{\mu}$ benader word deur

By observing a large number of customers through the service point and finding the average service time of the observed set eg. for a set of n observations, where X_i is the i^{th} observation, then $\frac{1}{\mu}$ can be approximated by

$$\bar{X} = \frac{\sum_{i=1}^{n} X_i}{n},$$

waaruit dit volg dat

from which it follows that

$$\hat{\mu} = \frac{1}{\bar{X}}.$$

Oor λ en μ? / About λ and μ?

λ en μ is parameters van die onderskeie eksponensiële verdelings wat die tussenaankomstye en dienstye volg (volgens 'n aanname).

Van uiterste belang is dat λ en μ van presies dieselfde eenhede moet wees, bv die een kan nie wees skepe/jaar and die ander skepe per maand nie of die een transaksies/maand en die ander transaksies/week nie, om nie eers te praat van maande/skip of weke/transaksie nie.

λ en μ is **altyd** in terme van aantal kliënte per tydseenheid ($\#c/t$)

λ and μ are the parameters of the respective exponential distributions which the interarrival times and service times follow (according an assumption).

Of utmost importance λ and μ must be of exactly the same units, eg the one can not be ships/year and the other in ships/month or the one in transactions/month and the other in transaction/week, not even to mention months/ship or weeks/transaction.

λ and μ is **always** in terms of number of customers per time unit ($\#c/t$)

Tut vrae vir 25 Julie / Tut 1 questions for 25 July

Vraag 1
As die aantal aankomste per tydsdeenheid Poisson verdeel is en aankomste geskied teen 4 aankomste per uur, wat is die waarskynlikheid dat die volgende aankoms na 20 minute sal plaasvind?

Vraag 2
As die tussenaankomstyd eksponensieel verdeel is en aankomste geskied teen 4 aankomste per uur, wat is die waarskynlikheid dat die volgende aankoms na 20 minute sal plaasvind?

Question 1
If the number of arrivals per time unit is Poisson distributed and arrivals occur at 4 arrivals per hour, what is the probability that the next arrival will take place after 20 minutes?

Question 2
If the interarrival time is exponential distributed and arrivals occur at 4 arrivals per hour, what is the probability that the next arrival will take place after 20 minutes?