Exer. 1.1 :

1. $r(z, x) = \mathbb{E}\left(\ell(Y, z) \mid X = x \right)$

$$R(f) = \mathbb{E}\left(r(f(X), X) \right)$$

2. $r(a + bx, x) = \mathbb{E}\left[\ell(\alpha + \beta x + \varepsilon, a + bx) \cancel{\mid X = x} \right]$

This only rv, which is $\perp X$

Quadratic loss ; L^2-loss

If $\ell(y, z) = (y - z)^2$

$$= \mathbb{E}\left(\left(\underbrace{(\alpha - a) + (\beta - b)x}_{\circledast} + \varepsilon\right)^2\right)$$

$$= \mathbb{E}\left(\circledast^2 + 2\circledast\varepsilon + \varepsilon^2\right)$$

$$= \circledast^2 + 2\circledast\underbrace{\mathbb{E}(\varepsilon)}_{=0} + \underbrace{\mathbb{E}(\varepsilon^2)}_{\sigma^2}$$

$$= \left[(\alpha - a) + (\beta - b)x\right]^2 + \sigma^2$$

- If $f_{a,b}(x) = a + bx$, then $R(f_{a,b}) = \mathbb{E}\left(\left[\overbrace{\left[(\alpha - a) + (\beta - b)X\right]^2}^{(\alpha - a)^2 + 2(\alpha - a)(\beta - b)X + (\beta - b)^2 X^2} + \sigma^2\right]\right)$

$$R(f_{a,b}) = \sigma^2 + (\alpha - a)^2 + 2(\alpha - a)(\beta - b)\mathbb{E}(X) + (\beta - b)^2\mathbb{E}(X^2)$$

Exer. 1.2

$$\mathbb{E}(\hat{R}_n(f)) = R(f)$$

Empirical risk is an unbiased estimator of $R(f)$.

Exer 1.3

1. Pick a measurable f. We have

$$R(f) \overset{\text{Ex 1.1 Q1}}{=} \mathbb{E}(\underbrace{r(f(X), X)}_{\geq \min_z r(z,X) = r(f^*(X), X)}) \geq \mathbb{E}(r(f^*(X), X)) = R^*$$

2. ▶ $R(f) = \mathbb{P}(Y \neq f(X))$

▶ $r(z; x) = \mathbb{P}(Y \neq z \mid X = x) \quad z \in \{0, 1\}$

$$= \begin{cases} \mathbb{P}(Y=0 \mid X=x) & \text{if } z = 1 \\ \mathbb{P}(Y=1 \mid X=x) & \text{if } z = 0 \end{cases}$$

Note : • $\mathbb{P}(Y=0 \mid X=x) + \mathbb{P}(Y=1 \mid X=x) = 1$

• $r(f^*(x), x) = \mathbb{P}(Y=0 \mid X=x) \wedge \mathbb{P}(Y=1 \mid X=x)$

means min

So $f^*(x) = \begin{cases} 1 & \text{if } \mathbb{P}(Y=0 \mid X=x) < \frac{1}{2} \iff \mathbb{P}(Y=1 \mid X=x) > \frac{1}{2} \\ 0 & \text{if } \mathbb{P}(Y=1 \mid X=x) < \frac{1}{2} \iff \mathbb{P}(Y=0 \mid X=x) > \frac{1}{2} \\ \text{0 or 1 at random} & \text{if } \mathbb{P}(Y=0 \mid X=x) = \frac{1}{2}. \end{cases}$

$$R(f^*) = \mathbb{P}(Y \neq f^*(X)) = \mathbb{E}\left(\mathbb{P}(Y=0 \mid X) \wedge \mathbb{P}(Y=1 \mid X)\right)$$

▶ Excess Risk $= R(f) - R^*$

$$3.\quad R(f) = \mathbb{E}\left[(Y - f(X))^2 \right]$$

$$r(z, x) = \mathbb{E}\left[\underbrace{(Y - z)^2}_{Y^2 - 2zY + z^2} \,\middle|\, X = x \right]$$

$$= \underbrace{\mathbb{E}(Y^2 \mid X = x) - 2z\, \mathbb{E}(Y \mid X = x) + z^2}_{\text{is a quadratic fun. of } z \text{ that looks like}}$$

It has a unique min at $z = \mathbb{E}(Y \mid X = x) = f^*(x)$