Integrated project : FFT : theory and implementation

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1 Exercise 1

a)

$$x_n = e^{i\omega_0 n} = e^{i2\pi\alpha n} = e^{i2\pi\frac{p}{q}n}$$

The sequence x_n is periodic with period q if:

$$x_{n+q} = x_n \quad \text{for all } n$$

$$x_{n+q} = e^{i\omega_0(n+q)} = e^{i2\pi\frac{p}{q}(n+q)} = e^{i2\pi\frac{p}{q}n} \cdot e^{i2\pi\frac{p}{q}q} = x_n \cdot e^{i2\pi p} = x_n$$

because

$$e^{i2\pi p} = \left(e^{i2\pi}\right)^p = 1^p = 1$$

Since $x_{n+q} = x_n$ for all n, the sequence x_n is periodic with period q. This holds because $e^{i2\pi p} = 1$, which results from the fact that p and q are integers and $e^{i2\pi k} = 1$ for any integer k.

b) We assume that x_n is periodic with some period T, meaning:

$$x_{n+T} = x_n \quad \text{for all } n$$

Thus:

$$e^{i\omega_0(n+T)} = e^{i\omega_0 n} \Leftrightarrow e^{i\omega_0 n} \cdot e^{i\omega_0 T} = e^{i\omega_0 n} \Leftrightarrow e^{i\omega_0 T} = 1$$

For $e^{i\omega_0 T} = 1$, the argument of the exponential must be an integer multiple of 2π:

$$\omega_0 T = 2\pi k \quad \text{for some integer } k$$

$\omega_0 = 2\pi\alpha$, so:

$$\omega_0 T = 2\pi k \Leftrightarrow 2\pi\alpha T = 2\pi k \Leftrightarrow \alpha T = k \Leftrightarrow \alpha = \frac{k}{T}$$

This implies that α is rational (since $\frac{k}{T}$ is a ratio of integers). However, by assumption, α is irrational. Thus, no such integer T can exist, meaning x_n cannot be periodic.

Therefore, if α is irrational, the complex exponential sequence $x_n = e^{i\omega_0 n}$ is not periodic.

c) Show that if x_n and y_n are two periodic sequences with periods M and N, respectively, then $z_n = x_n + y_n$ is periodic with period $\text{lcm}(M, N)$.
x_n is periodic with period M, meaning:

$$x_{n+M} = x_n \quad \text{for all } n$$

y_n is periodic with period N, meaning:

$$y_{n+N} = y_n \quad \text{for all } n$$

Assume that T is the period of z. Show that $T = \text{lcm}(M, N)$.
$z_{n+T} = x_{n+T} + y_{n+T}$ so:

$$z_{n+T} = z_n \Leftrightarrow x_{n+T} + y_{n+T} = x_n + y_n \Leftrightarrow x_{n+T} = x_n \text{ and } y_{n+T} = y_n$$

$x_{n+T} = x_n$ if $T = kM$ for some integer k. $y_{n+T} = y_n$ if $T = lN$ for some integer l.
Thus, T must be a common multiple of both M and N. Therefore, since T is the period of z, T is the lowest common multiple of M and N, that is $\text{lcm}(M, N)$.

2 Exercise 2

a) Fourier Transform:

$$X = Fx = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & W_N & W_N^2 & \cdots & W_N^{N-1} \\ 1 & W_N^2 & W_N^4 & \cdots & W_N^{2(N-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & W_N^{N-1} & W_N^{2(N-1)} & \cdots & W_N^{(N-1)^2} \end{pmatrix} \begin{pmatrix} x_0 \\ \vdots \\ \vdots \\ \vdots \\ x_{N-1} \end{pmatrix}$$

Inverse Fourier Transform:

$$x = F^{-1}X = \frac{1}{N} \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & W_N^{-1} & W_N^{-2} & \cdots & W_N^{-(N-1)} \\ 1 & W_N^{-2} & W_N^{-4} & \cdots & W_N^{-2(N-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & W_N^{-(N-1)} & W_N^{-2(N-1)} & \cdots & W_N^{-(N-1)^2} \end{pmatrix} \begin{pmatrix} X_0 \\ \vdots \\ \vdots \\ \vdots \\ X_{N-1} \end{pmatrix}$$

b)

$$F_{k,n} = W_N^{kn} = e^{-i\frac{2\pi}{N}kn}, \quad \text{for } k,n = 0,1,\ldots,N-1$$

$$F^*_{k,n} = W_N^{-kn} = e^{i\frac{2\pi}{N}kn} \quad \text{for } k,n = 0,1,\ldots,N-1$$

$$(F \cdot \frac{1}{N}F^*)_{k,m} = \frac{1}{N}\sum_{n=0}^{N-1} F_{k,n} \cdot F^*_{n,m} = \frac{1}{N}\sum_{n=0}^{N-1} e^{-i\frac{2\pi}{N}kn} \cdot e^{i\frac{2\pi}{N}nm} = \frac{1}{N}\sum_{n=0}^{N-1} e^{i\frac{2\pi}{N}(m-k)n}$$

When $k = m$:

$$\frac{1}{N}\sum_{n=0}^{N-1} e^{i\frac{2\pi}{N}\cdot 0} = \frac{1}{N}\sum_{n=0}^{N-1} 1 = 1$$

When $k \neq m$:

$$\sum_{n=0}^{N-1} e^{i\frac{2\pi}{N}(m-k)n} = 0 \quad \text{(geometric series sum)}$$

In conclusion:

$$F \cdot \frac{1}{N}F^* = I$$

Hence, $F^{-1} = \frac{1}{N}F^*$.

c) To prove that both the Discrete Fourier Transform (DFT) and its Inverse (IDFT) are linear operators, we need to check two properties for each operator:
- 1. Additivity: $F(x+y) = F(x) + F(y)$
- 2. Homogeneity: $F(\alpha x) = \alpha F(x)$, where $\alpha \in \mathbb{C}$

Let $x, y \in \mathbb{C}^N$ and $\alpha \in \mathbb{C}$.

DFT:
Additivity:

$$(F(x+y))_k = \sum_{n=0}^{N-1}(x_n+y_n)e^{-i\frac{2\pi}{N}kn} = \sum_{n=0}^{N-1} x_n e^{-i\frac{2\pi}{N}kn} + \sum_{n=0}^{N-1} y_n e^{-i\frac{2\pi}{N}kn} = (Fx)_k + (Fy)_k$$

Homogeneity:

$$(F(\alpha x))_k = \sum_{n=0}^{N-1}(\alpha x_n)e^{-i\frac{2\pi}{N}kn} = \alpha\sum_{n=0}^{N-1} x_n e^{-i\frac{2\pi}{N}kn} = \alpha(Fx)_k$$

Therefore, the DFT is linear.

IDFT:
Additivity:

$$(F^{-1}(X+Y))_n = \frac{1}{N}\sum_{k=0}^{N-1}(X_k+Y_k)e^{i\frac{2\pi}{N}kn} = \frac{1}{N}\left(\sum_{k=0}^{N-1}X_ke^{i\frac{2\pi}{N}kn}+\sum_{k=0}^{N-1}Y_ke^{i\frac{2\pi}{N}kn}\right) = (F^{-1}X)_n+(F^{-1}Y)_n$$

Homogeneity:

$$(F^{-1}(\alpha X))_n = \frac{1}{N}\sum_{k=0}^{N-1}(\alpha X_k)e^{i\frac{2\pi}{N}kn} = \alpha\left(\frac{1}{N}\sum_{k=0}^{N-1}X_ke^{i\frac{2\pi}{N}kn}\right) = \alpha(F^{-1}X)_n$$

Therefore, the IDFT is linear.

d) DFT:

$$X_{k+N} = \sum_{n=0}^{N-1}x_ne^{-i\frac{2\pi}{N}(k+N)n} = \sum_{n=0}^{N-1}x_ne^{-i\frac{2\pi}{N}kn}\cdot e^{-i\frac{2\pi}{N}Nn}$$

Since $e^{-i\frac{2\pi}{N}Nn} = e^{-i2\pi n} = 1$ for any integer n:

$$X_{k+N} = \sum_{n=0}^{N-1}x_ne^{-i\frac{2\pi}{N}kn} = X_k$$

Thus, X_k is N-periodic:

$$X_{k+lN} = X_k \quad \text{for all } l \in \mathbb{Z}$$

IDFT:

$$x_{n+N} = \frac{1}{N}\sum_{k=0}^{N-1}X_ke^{i\frac{2\pi}{N}nk}\cdot e^{i\frac{2\pi}{N}Nk}$$

Since $e^{i\frac{2\pi}{N}Nk} = e^{i2\pi k} = 1$ for any integer k:

$$x_{n+N} = \frac{1}{N}\sum_{k=0}^{N-1}X_ke^{i\frac{2\pi}{N}nk} = x_n$$

Thus, x_n is also N-periodic:

$$x_{n+lN} = x_n \quad \text{for all } l \in \mathbb{Z}$$

e) DFT:
For each output element X_k, we need to compute the sum over N terms, and each term involves a complex multiplication:
- Number of complex multiplications per X_k: N
- Number of output values X_k: N
Thus, the total number of complex multiplications for the DFT is:

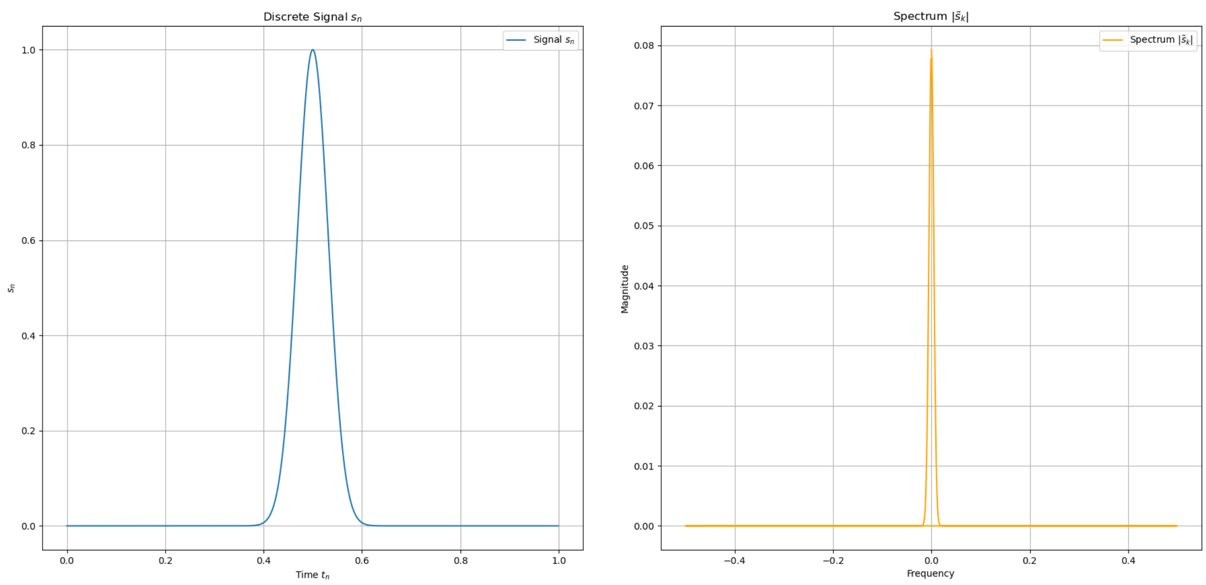
$$N \times N = N^2$$

IDFT:
The IDFT has a similar structure to the DFT, requiring the same number of operations:

$$N \times N = N^2$$

In conclusion:
- Computational complexity of DFT: $\mathcal{O}(N^2)$
- Computational complexity of IDFT: $\mathcal{O}(N^2)$

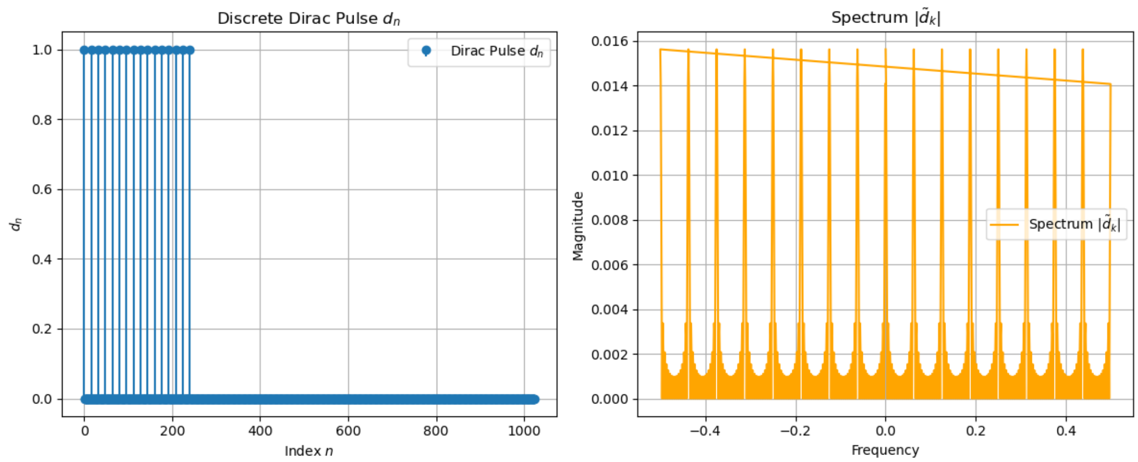
3 Exercise 3



The discrete signal s_n, showing a Gaussian-shaped curve centered around $t_0 = 0.5$.

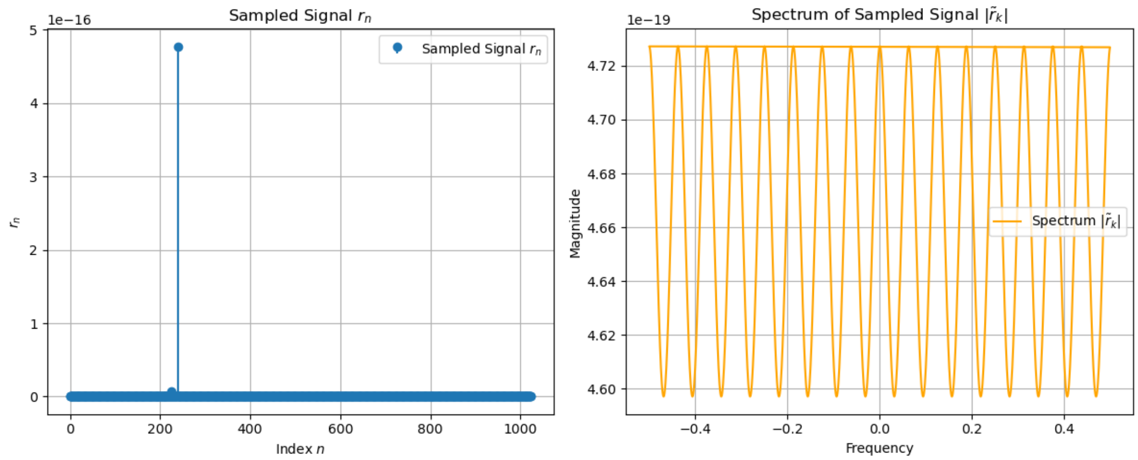
The spectrum $|\tilde{s}_k|$, representing the magnitudes of the Fourier coefficients. The peak in the spectrum corresponds to the dominant frequency components of the signal.

4 Exercise 4

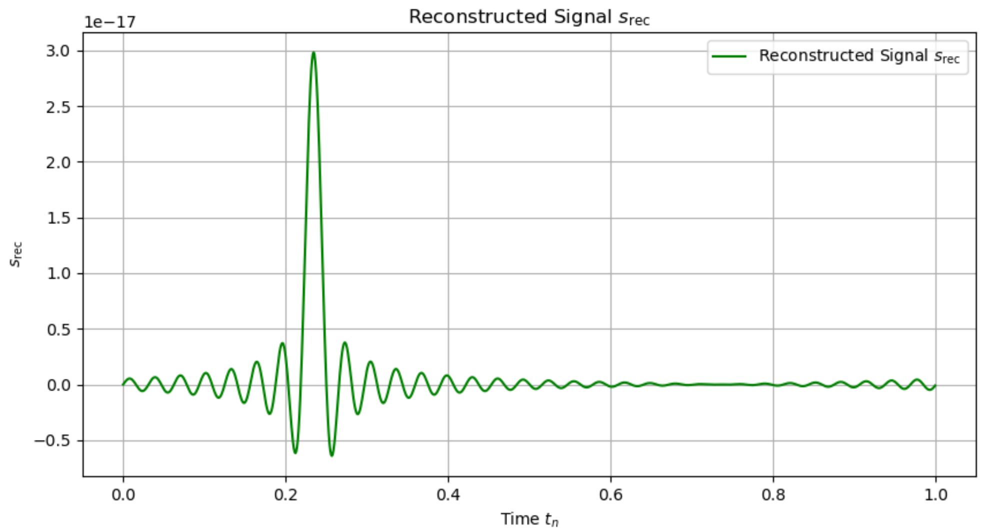


The Dirac pulse d_n with impulses at regular intervals determined by $f_0 = 64$. The spectrum $|\tilde{d}_k|$, showing periodic peaks corresponding to the harmonic structure of the Dirac pulse. The peaks in the spectrum reflect the fundamental frequency f_0 and its harmonics.

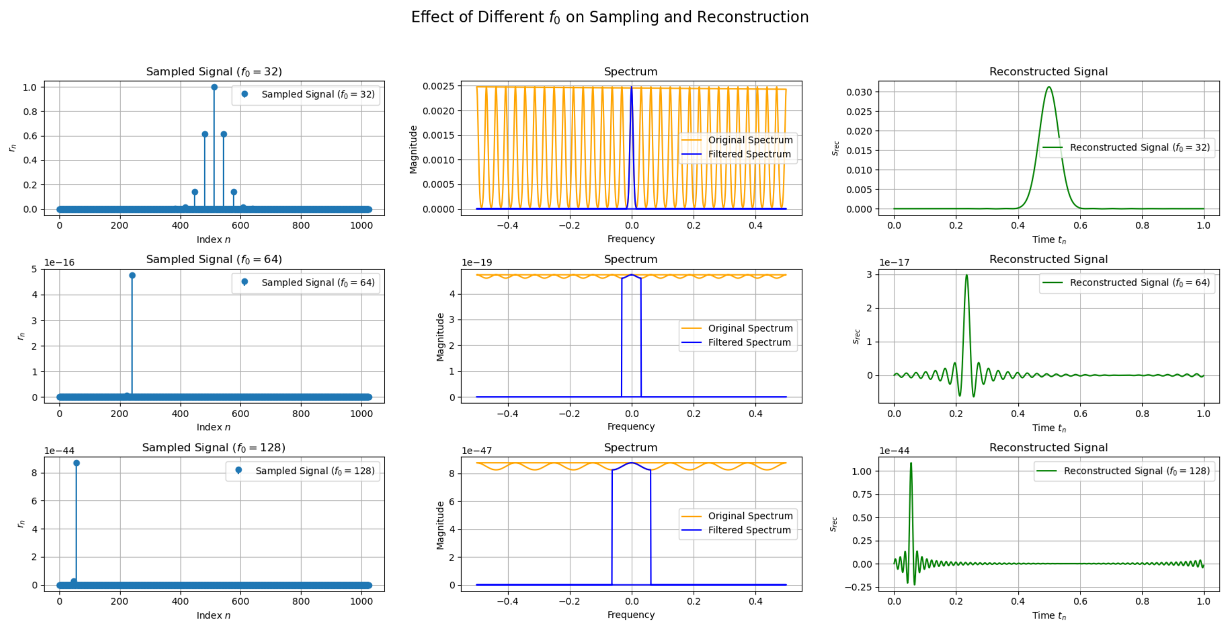
5 Exercise 5



6 Exercise 6



7 Exercise 7



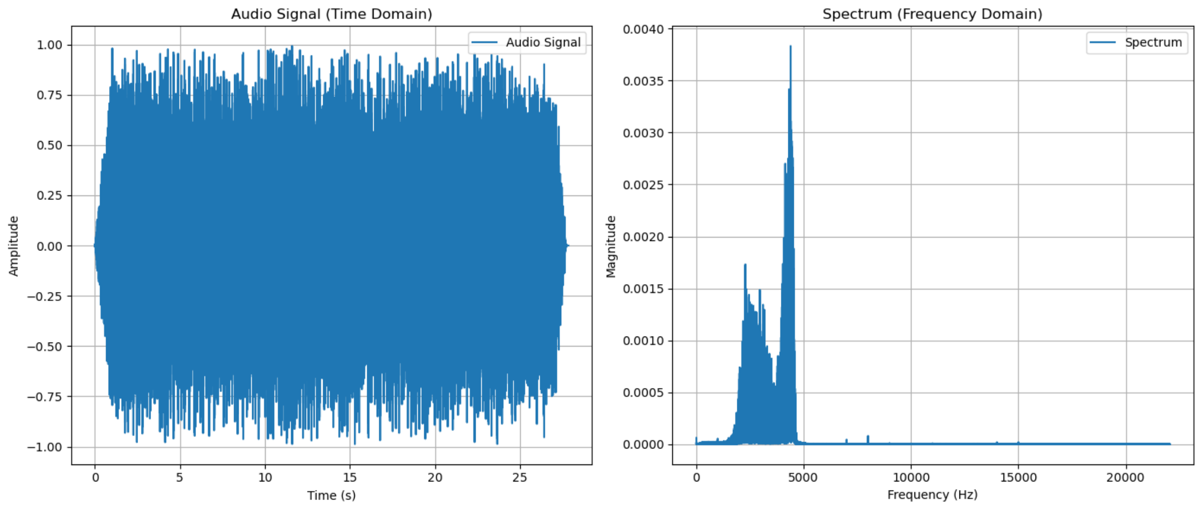
Observations:

Higher f_0 : ($f_0 > 64$):
- Sampling is more frequent.
- Spectrum $|\tilde{r}_k|$ becomes wider.
- Reconstruction better approximates the original signal.

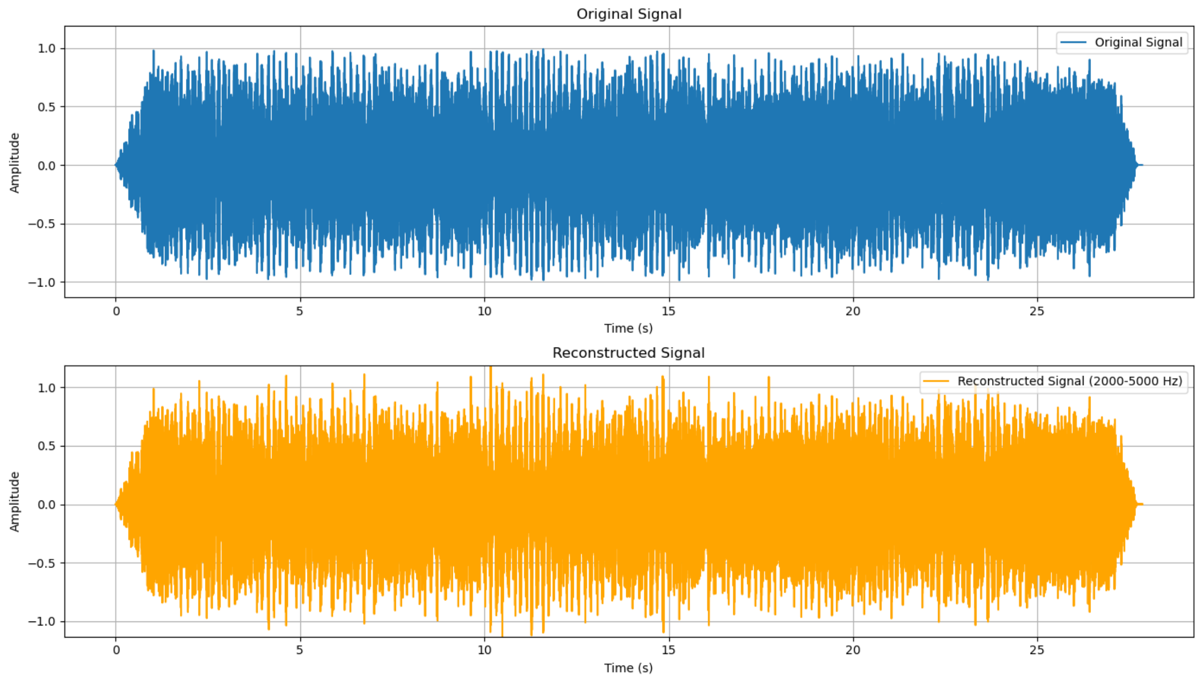
Lower f_0 : ($f_0 < 64$):
- Sampling is less frequent.
- Spectrum $|\tilde{r}_k|$ shrinks.
- Reconstruction may lose some details (aliasing).

8 Bonus

We will apply the DFT to the cicada's song. First, we retrieve an audio file of a cicada's song (Cicadas_in_Greece.ogg, available on Wikipedia), then we apply the DFT and we display the original audio signal and its spectrum on graphs.



We notice that the frequencies in the cicada's song are mainly located between 2000 and 5000 Hertz. Therefore, we select the frequencies between these two frequencies and reconstruct the signal using the IDFT.



We notice that the reconstruction is quite close to the original signal which is normal because almost all the sounds of cicadas have been retained and only some parasitic noises whose frequencies were distant from those of cicadas have been eliminated.
By creating a new audio file from the reconstructed signal, we observe that the two audio files are very similar, but the background noise is no longer present in the reconstructed audio signal.
Of course, the selected frequency values can be changed to conduct other observations. For example, in some cases, it may be interesting to isolate the background noise.

The Python code, the original audio file, and the reconstructed audio file are uploaded on Ametice.