Aix-Marseille University — Markov chains, martingales
M1 MAS DS/CMB — 2024-2025

---

**Sheet 1 : First steps with Markov Chains**

---

**Exercise 1 :** Let $(X_n)_{n\geq 0}$ a homogeneous Markov chain on $E = \{e_0, e_1, \cdots, e_n, \cdots\}$ with transition matrix $P = (p_{i,j})_{i,j\in\mathbb{N}}$ and initial law $\pi = (\pi_i)_{i\in\mathbb{N}}$.
**1.** Show that the following three properties are equivalents :

$$\begin{aligned}&\mathbb{P}(X_{n+1} = e_{i_{n+1}} \mid X_n = e_{i_n}, X_{n-1} = e_{i_{n-1}}, \cdots, X_0 = e_{i_0})\\ &\quad = \mathbb{P}(X_{n+1} = e_{i_{n+1}} \mid X_n = e_{i_n}) = p_{i_n, i_{n+1}}\end{aligned} \qquad (*)$$

$$\begin{aligned}&\mathbb{P}(X_{n+m} = e_{i_{n+m}}, \cdots, X_{n+1} = e_{i_{n+1}} \mid X_n = e_{i_n}, X_{n-1} = e_{i_{n-1}}, \cdots, X_0 = e_{i_0})\\ &\quad = \mathbb{P}(X_{n+m} = e_{i_{n+m}}, \cdots, X_{n+1} = e_{i_{n+1}} \mid X_n = e_{i_n}) = p_{i_n, i_{n+1}} \times \cdots \times p_{i_{n+m-1}, i_{n+m}};\end{aligned} \qquad (**)$$

$$\mathbb{P}(X_n = e_{i_n}, \cdots, X_1 = e_{i_1}, X_0 = e_{i_0}) = \pi_{i_0} p_{i_0, i_1} \cdots p_{i_{n-1}, i_n}. \qquad (***)$$

**2.** Check that $(**) = \mathbb{P}(X_m = e_{i_{n+m}}, \cdots, X_1 = e_{i_{n+1}} \mid X_0 = e_{i_n})$.

**Exercise 2 :** Let $\alpha$, $\beta \in [0, 1]$ and $(X_n)_{n\geq 0}$ a two states Markov chain with transition matrix given by

$$P = \begin{pmatrix} 1-\alpha & \alpha \\ \beta & 1-\beta \end{pmatrix}.$$

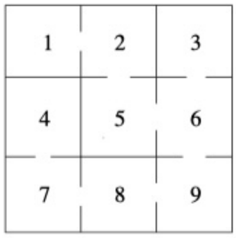
**1.** Give a graphical representation of the chain transitions.
**2.** Show that if $\alpha + \beta > 0$, then for all $n \geq 1$,

$$P^n = \frac{1}{\alpha+\beta}\begin{pmatrix} \beta & \alpha \\ \beta & \alpha \end{pmatrix} + \frac{(1-(\alpha+\beta))^n}{\alpha+\beta}\begin{pmatrix} \alpha & -\alpha \\ -\beta & \beta \end{pmatrix}$$

**3.** Compute $\lim_{n\to\infty} P^n$.

**Exercise 3 :** We toss a fair coin : the outcomes are independent random variables $Y_0, Y_1, \ldots$ taking values 0 or 1. For all $n \geq 1$, we denote $X_n = Y_n + Y_{n-1}$.
**1.** Compute $\mathbb{P}(X_3 = 0 \mid X_1 = 0\, X_2 = 1)$ and $\mathbb{P}(X_3 = 0 \mid X_2 = 1)$.
**2.** Is $(X_n)_{n\geq 0}$ a Markov chain ?

**Exercise 4 :** A rat moves in a maze with nine compartments (see figure below). At each step, it moves to a different compartment. When it is in a compartment with $k$ doors, it chooses one of these $k$ doors with equal probability. Let $X_n$ be the number of the compartment where the rat is at step $n$. Provide the transition matrix and the associated transition graph.



**Exercise 5 :** We play roulette : there are 18 red numbers, 18 black numbers, and one green, the number 0. We bet on red for 1€, each time. We start with 50€, and stop if we have 100€, or if we go bankrupt. Let $X_n$ be our fortune after $n$ rounds. Provide the transition matrix and the associated transition graph.

**Exercise 6 :** A zoo received six gorillas, three males, and three females randomly distributed into two cages of three monkeys each. The presbyopic director, unable to discern the sexes, decides to promote their reproduction by swapping two inmates each week, one from each cage, chosen at random. Let $X_n$ be the number of female monkeys present in the first cage in week $n$. Provide the transition matrix and the associated transition graph.

**Exercise 7 :** There are two boxes and $2d$ balls, $d$ of which are black and $d$ red. At the beginning, $d$ balls are placed randomly in the 1 box and the remaining $d$ balls in the 2 box. At each instant, we draw a ball at random from each box and reverse them. Let $X_n$ be the number of black balls in the 1 box after $n$ draws. Provide the transition matrix and the associated transition graph.

**Exercise 8 :** Let $(X_n)_{n\geq 0}$ a homogeneous Markov chain on $E = \{e_0, e_1, \cdots, e_n, \cdots\}$ with transition matrix $P = (p_{i,j})_{i,j\in\mathbb{N}}$

**1.** Show that for all $n \geq 0$ and all $n$-uplet $(A_0, \cdots, A_{n-1})$ of subsets of $E$,

$$\mathbb{P}(X_{n+1} = e_j \mid X_n = e_i,\ X_{n-1} \in A_{n-1}, \cdots,\ X_0 \in A_0) = p_{i,j}.$$

**2.** Find a counter-example such that le following equality

$$\mathbb{P}(X_{n+1} = e_j \mid X_n \in A_n,\ X_{n-1} \in A_{n-1}, \cdots,\ X_0 \in A_0) = \mathbb{P}(X_{n+1} = e_j \mid X_n \in A_n)$$

is not satisfied.

**Exercise 9 :** Let $(\epsilon_n)_{n\geq 0}$ be a sequence of random variables which are independent and identically distributed. From $(\epsilon_n)_{n\geq 0}$, we construct $(X_n)_{n\geq 0}$ as follows :

$$X_n = X_{n-1} + bX_{n-2} + \epsilon_n, \qquad n \geq 2,$$

$b$ being any real, $X_0$ and $X_1$ two random variables to be specified. Let's assume $b = 0$.

**1.** If $X_0 = 0$, show that $(X_n)_{n\geq 0}$ defines a Markov chain.

**2.** More generally, let $Y_0$ be a discrete random variable independent of the sequence $(\epsilon_n)_{n\geq 0}$ and $G$ a measurable function with integer values. If

$$Y_n = G(Y_{n-1}, \epsilon_n), \qquad n \geq 1,$$

show that $(Y_n)_{n\geq 0}$ is a Markov chain.
Let's assume $b \neq 0$.
**3.** if $X_0 = X_1 = 0$, show that $(X_n)_{n\geq 0}$ is not a Markov chain.
**4.** Construct from the sequence $(X_n)_{n\geq 0}$ a Markov chain.

**Exercise 10 :** Let $P$ be the transition matrix

$$P = \begin{pmatrix} 0 & 1/2 & \times \\ 1/3 & 1/3 & \times \\ \times & 1/4 & 1/4 \end{pmatrix}.$$

**1.** Complete the values of $P$.
Let $(U_n)_{n\geq 0}$ be a sequence of random variables i.i.d. of uniform distribution on $[0, 1]$.
**2.** Find a function $F$ such that $X_n = F(X_{n-1}, U_n)$ defines a Markov chain with transition matrix $P$.

**Exercise 11 :** The eigenvalues of a stochastic matrix $P$ have interesting properties. They also give us an information on the evolution in time of the Markov chain defined by Markov chain defined by $P$. Let the stochastic matrix be

$$P = \begin{pmatrix} 1/2 & 1/2 & 0 \\ 1/3 & 1/3 & 1/3 \\ 0 & 1/2 & 1/2 \end{pmatrix}$$

**1.** Draw the corresponding transition graph.
**2.** Compute the eigenvalues of $P$, and verify that there is only one eigenvalue equal to 1, while the others are of modulus strictly less than 1. Determine the eigenvectors on the left associated with the three eigenvalues. Which of these three vectors define probability distributions ?
**3.** Also determine the eigenvectors to the right of $P$, and Verify that $P = SQS^{-1}$ where : $S$ has as columns the eigenvectors on the right of $P$, $S^{-1}$ has as rows the left eigenvectors of $P$, $Q$ is a diagonal matrix including the eigenvalues of $P$. Don't forget to normalize these eigenvectors so that the eigenproduct of the eigenvectors (right and left) corresponding to each eigenvalue is equal to 1.
**4.** Use the above relationship to calculate $P^2$ and $P^3$.

**Exercise 12 :** We are interested, in a population of flies, in an allele, which can be $a$ or $A$. Each fly carries two alleles (one in each chromosome of the pair concerned) and thus its genotype can be $aa$, $aA$, or $AA$. We start with two individuals. They reproduce, and among their offspring one chooses two individuals of opposite sex, who reproduce, and so on. One specifies that the genotype of a child is determined by choosing a letter at random in each of the two parents (as for the humans). The pairs of genotypes at the $n$-th generation define a Markov chain which can take six states : $AA, AA,\quad AA, Aa\quad AA, aa,\quad Aa, Aa,\quad Aa, aa\quad aa, aa$. Determine the transition probabilities.