# On Random Walks with Geometric Lifetimes

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**Abstract.** We consider the following stochastic process. Initially, there is a single particle placed at the origin of $\mathbb{Z}$. This particle moves as a discrete-time simple random walk on $\mathbb{Z}$, which takes one step to the right with probability $p$. Furthermore, the particle has a probability of death $(1-\alpha)$ before each step. Whenever this particle returns to the origin, it gives birth to a new particle, which performs the same dynamics. For which values of the pair $(p, \alpha)$ does this model survive with positive probability? Can we obtain the survival probability as a function of $(p, \alpha)$? By answering these questions, we review some fundamental concepts and tools in probability, such as the return times of a simple random walk on the integers, probability generating functions, branching processes, and coupling.

**1. INTRODUCTION.** Stochastic processes are very important in scientific modeling, and an analysis of their properties depends on techniques from various fields of mathematics. The theory starts from the classical processes: discrete-time Markov chains, Poisson processes, continuous-time Markov chains, and martingales. These include fundamental models, such as the Bienaymé–Galton–Watson branching process and random walks. As introductory references, we cite the books of Dobrow [5], Grimmett and Stirzaker [10], Pinsky and Karlin [20], and Schinazi [21].

In addition to the classical models, the field includes various special stochastic processes, such as the percolation model (see Bollobás and Riordan [3], Grimmett [8]), and the contact process (Grimmett [9], Liggett [17, 18]). These processes can be regarded as *toy models*: mathematical models proposed in order to describe in a simplified way a natural phenomenon, for example from biology, physics, or sociology. Thus percolation theory was founded by Broadbent and Hammersley [4] with the aim of formulating a simple mathematical model for the spread of a fluid through a porous medium. The contact process, introduced by Harris [11], is a *toy model* for epidemics, where each vertex of a graph can be infected or healthy. Healthy vertices are infected at a rate $\lambda$ times the number of infected neighbors, whereas infected vertices become healthy at a constant rate, equal to 1. In the same way, several research fields in probability (such as the random graphs theory and interacting particle systems) have been developed that are mathematically rich and useful for illuminating problems in many areas.

In this paper, we study the following discrete-time stochastic process on the integers $\mathbb{Z}$. Assume that at time zero there is a unique particle at the origin of $\mathbb{Z}$. This particle performs a simple random walk on $\mathbb{Z}$, changing its position at each step by $+1$ with probability $p$ or by $-1$ with probability $(1-p)$. Whenever the particle returns to the origin, it gives birth to a new particle that performs the same process. Furthermore,

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before each step, each living particle chooses either to survive with probability $\alpha$, or to disappear with probability $(1-\alpha)$, independently of everything else. Therefore its lifetime has a geometric distribution with parameter $(1-\alpha)$. We call this model the *returns model* on $\mathbb{Z}$ with parameter $(p, \alpha)$. Naturally, we say that a particular realization of the model *survives* if there is at least one particle at every instant of time. Otherwise, it *dies out*. Our main purpose is to determine under which conditions the returns model has a positive probability of survival. We also establish an explicit formula for the extinction probability as a function of $(p, \alpha)$.

By studying this process, we review some basic definitions and techniques, such as probability generating functions, properties of the simple random walk on $\mathbb{Z}$, branching processes, and the coupling method. We recall what we need in Section 2. The study of the returns model, presented in Section 3, also allows us to discuss critical phenomena. By this we mean that the returns model, like percolation models and some interacting particle systems, shows an abrupt change in behavior as the parameter varies. We hope that this paper will help readers consolidate these probabilistic ideas and inspire them to create and solve other problems.

## 2. SOME PROBABILISTIC TOOLS.

**Probability generating functions.** Generating functions are a valuable tool in mathematics, especially in combinatorics, number theory, and probability. The main idea is to map a class of objects (such as the probability distributions) into a new class of objects whose computations are simpler. In probability theory, these transformations include the probability generating functions, moment generating functions, and characteristic functions. They are primarily helpful to study the sum of independent random variables and convergence in distribution. Here we focus on probability generating functions, which are useful for studying random variables assuming integer values. For more details, we refer the reader to Feller [**7**, Chapter XI] and Grimmett and Stirzaker [**10**, Chapter 5]. An extensive treatment of generating functions can be found in Wilf [**25**].

**Definition 1.** Let $X$ be a random variable taking values in the set $\mathbb{N}_0 := \{0, 1, 2, \dots\}$. The *probability generating function* (or *pgf*) of $X$ is the function $G_X(s)$ defined by

$$G_X(s) = E(s^X) = \sum_{j=0}^{\infty} s^j P(X = j),$$

for all values of $s$ for which the right-hand side converges absolutely.

As $G_X(1) = 1$, the function $G_X$ is defined at least for $|s| \leq 1$. From the elementary theory of power series, one can prove the following result.

**Theorem 2.** *The function $G_X$ is infinitely often continuously differentiable on the interval $(-1, 1)$. Moreover, for $k \geq 1$, the kth derivative $G_X^{(k)}$ fulfills*

$$G_X^{(k)}(1) := \lim_{s \nearrow 1} G_X^{(k)}(s) = E\big[X(X-1)\dots(X-k+1)\big],$$

*where both sides can be equal to $+\infty$.*

For an example, let $X$ be a geometrically distributed random variable with parameter $p \in (0, 1]$, that is,

$$P(X = j) = p\,(1-p)^j \quad \text{for any } j \in \mathbb{N}_0.$$

Then, $X + 1$ is the waiting time for the first success in a sequence of independent Bernoulli trials, each yielding a success with probability $p$. The pgf of $X$ is given by

$$G_X(s) = p \sum_{j=0}^{\infty} [s(1-p)]^j = \frac{p}{1 - s(1-p)} \quad \text{if } |s| < (1-p)^{-1}. \tag{1}$$

Consequently, from Theorem 2, we obtain that the expected value of $X$ is

$$E(X) = G'_X(1) = \lim_{s \nearrow 1} \frac{p\,(1-p)}{[1 - s(1-p)]^2} = \frac{1-p}{p}. \tag{2}$$

Similarly, the variance of $X$ is

$$\mathrm{Var}(X) = E(X^2) - [E(X)]^2 = G''_X(1) + G'_X(1) - [G'_X(1)]^2 = \frac{1-p}{p^2}.$$

**Random walks on $\mathbb{Z}$.** A random walk is a stochastic process designed to describe the trajectory of a particle or object, whose movement occurs through a sequence of random steps in a space, as the set of integers $\mathbb{Z} = \{\dots, -1, 0, 1, \dots\}$. Random walks are widely used to model several phenomena in biology, computer science, economics, and physics. The term random walk was proposed in 1905 by the British statistician Karl Pearson, in a letter to the journal *Nature* (Pearson [**19**]). Recently, the Abel Prize 2020 was awarded by the Norwegian Academy of Science and Letters to mathematicians Hillel Furstenberg and Gregory Margulis, for their pioneering research in random walk techniques which unveiled the power of random walks to solve many important problems in group theory, number theory, and combinatorics.

**Definition 3.** Let $X_1, X_2, \dots$ be a sequence of independent and identically distributed random variables, with distribution given by

$$P(X_i = 1) = 1 - P(X_i = -1) = p, \quad p \in [0, 1], \quad i \geq 1.$$

Define $S_0 = 0$ and $S_n = \sum_{i=1}^{n} X_i$ for $n \geq 1$. The sequence of random variables $\{S_n\}_{n \geq 0}$ is called a *simple random walk* on $\mathbb{Z}$, starting from 0. If $p = 1/2$, we say that the simple random walk on $\mathbb{Z}$ is *symmetric*.

Thus, $\{S_n\}_{n \geq 0}$ describes the successive positions of a particle that starts at the origin of the integer line, and at each unit of time moves either to the right or to the left, with respective probabilities $p$ and $(1 - p)$. The random walk is called simple because the size of the steps is restricted to the set $\{-1, +1\}$, that is, the particle can only jump to one of the two sites of $\mathbb{Z}$ that are nearest neighbors of its current position. General models can be obtained by assuming that the steps take values in $\mathbb{Z}^d$ or $\mathbb{R}^d$ for some $d \geq 1$. We cite Feller [**7**, Chapters III and XIV] and Grimmett and Stirzaker [**10**, Sections 3.9, 3.10, and 5.3] as introductory references for the subject; more complete treatments can be found in Lawler and Limic [**13**] and Spitzer [**22**].

A natural question concerns whether the probability that the random walk never revisits its starting point is positive or not. Let $\{S_n\}_{n \geq 0}$ be a simple random walk on $\mathbb{Z}$, starting at the origin. We define

$$T_0 = \inf\{n \geq 1 : S_n = 0\},$$

where $\inf \varnothing = \infty$. Thus $T_0$ is the random amount of time (number of steps) until the particle returns to the origin. Note that the event $\{T_0 = \infty\}$ means "never revisiting the origin," so $T_0$ may be a *defective random variable*. That is, it may happen that $P(T_0 < \infty) < 1$.

Now let $p_0(n) = P(S_n = 0)$ denote the probability that the particle is at the origin after $n$ steps, and let

$$f_0(n) = P(T_0 = n) = P(S_1 \neq 0, \ldots, S_{n-1} \neq 0, S_n = 0)$$

be the probability that it returns to the origin for the first time after $n$ steps. Of course, $p_0(n) = f_0(n) = 0$ if $n$ is odd. We define the generating functions of the sequences $\{p_0(n)\}$ and $\{f_0(n)\}$ by setting

$$P_0(s) = \sum_{n=0}^{\infty} p_0(n)s^n \quad \text{and} \quad F_0(s) = \sum_{n=1}^{\infty} f_0(n)s^n,$$

respectively. Notice that $F_0$ is the pgf of $T_0$. Our study of the returns model makes essential use of the following result, whose proof can be found in Grimmett and Stirzaker [**10**, p. 163].

**Theorem 4.** *We have that*

(a) $P_0(s) = 1 + P_0(s)F_0(s)$,

(b) $P_0(s) = (1 - 4p(1-p)s^2)^{-1/2}$,

(c) $F_0(s) = 1 - \left[1 - 4p(1-p)s^2\right]^{1/2}$.

As a fundamental consequence of Theorem 4, we conclude that the probability that the particle ever returns to its starting point is given by

$$\sum_{n=1}^{\infty} f_0(n) = F_0(1) = 1 - \left[(2p-1)^2\right]^{1/2} = 1 - |2p-1|.$$

Consequently, if $p = 1/2$, then $F_0(1) = 1$, so an eventual return to the origin occurs almost surely. Once the particle reenters the origin, the process behaves as if it just started anew (this property is known as the *strong Markov property*). Thus with probability 1 the particle returns infinitely many times to its starting point. For this reason, the simple symmetric random walk on $\mathbb{Z}$ is said to be *recurrent*. On the other hand, if $p \neq 1/2$, then there is a positive probability, namely, $1 - F_0(1) = |2p-1|$, that the random walk will never again revisit the origin, starting from there. The process is *transient*, since the number of returns to the origin is finite with probability 1; this is a geometric random variable with parameter $|2p-1|$.

We also observe that if $p = 1/2$ then the pgf of $T_0$ is given by

$$F_0(s) = 1 - (1 - s^2)^{1/2}, \ |s| \leq 1.$$

From Theorem 2, it follows that the expected time for the first return is

$$E(T_0) = F_0'(1) = \lim_{s \nearrow 1} \frac{s}{(1-s^2)^{1/2}} = +\infty.$$

Thus generating functions provide a powerful tool for analyzing the question of recurrence or transience of the simple random walk on $\mathbb{Z}$. A different approach to this topic relies on the theory of Markov chains; see Dobrow [**5**, Example 3.13]. We also refer the reader to van der Hofstad and Keane [**12**] for a direct proof of the random walk hitting time theorem, and to Doyle and Snell [**6**] for a study of the problem of the recurrence of random walks on finite and infinite graphs by applying the classical theory of electricity.

**Branching processes.** An important area of probability deals with stochastic models for the growth of a population. Here we review a fundamental process of this class which was independently formulated by the French statistician Irénée-Jules Bienaymé in the 1840s, the English scientist Sir Francis Galton, and the English mathematician Reverend Henry William Watson in the 1870s. Their original motivation was to investigate the problem of the extinction of family names. Nowadays branching processes are applied in diverse scientific disciplines, such as ecology, genetics, medicine, and nuclear physics. For more details on the subject, see Athreya and Ney [2], Feller [7, Chapter XII], and Schinazi [21, Chapter 2].

The Bienaymé–Galton–Watson branching process is a discrete-time stochastic process $\{Z_n\}_{n\geq 0}$ which describes how a population of individuals of the same type (such as bacteria or elementary particles) evolves. For each $n \geq 0$, the random variable $Z_n$ stands for the size of the $n$th generation. We assume that the process starts with a single individual, so $Z_0 = 1$. Individuals reproduce according to the following rules:

(i) Each individual gives birth to exactly $j$ new individuals with probability $p_j$, $j \in \mathbb{N}_0$, where $p_j \geq 0$ and $\sum_{j=0}^{\infty} p_j = 1$.

(ii) The random numbers of children produced by different individuals are independent.

Thus the individuals of the $(n+1)$st generation are the direct descendants of those individuals present in the $n$th generation (if any). To exclude trivial cases, we suppose that $0 < p_0 < 1$ and $p_0 + p_1 < 1$.

The main result on branching processes concerns the probability that the population becomes extinct. For $n \geq 0$, let $A_n = \{Z_n = 0\}$ be the event that the population is extinct at or prior to the $n$th generation. Then, $A_0 = \varnothing \subset A_1 \subset A_2 \subset \cdots$, and

$$A = \bigcup_{n=0}^{\infty} A_n = \{Z_n = 0 \text{ for some } n \geq 1\}$$

is the event that the population ultimately dies out. Defining $\delta_n = P(A_n)$, $n \geq 0$, and $\delta = P(A)$, it follows from the continuity of probability that

$$\delta = P(A) = \lim_{n\to\infty} P(A_n) = \lim_{n\to\infty} \delta_n. \tag{3}$$

Now let

$$\mu = \sum_{j=1}^{\infty} j\, p_j \quad \text{and} \quad G(s) = \sum_{j=0}^{\infty} p_j\, s^j,\ |s| \leq 1,$$

denote the mean and the pgf of the offspring distribution, respectively. By conditioning on $Z_1$, we have that for every $n \geq 0$,

$$\begin{aligned}\delta_{n+1} &= P(Z_{n+1} = 0) = \sum_{j=0}^{\infty} P(Z_{n+1} = 0 \,|\, Z_1 = j)\, P(Z_1 = j) \\ &\overset{(*)}{=} \sum_{j=0}^{\infty} (\delta_n)^j\, p_j = G(\delta_n).\end{aligned} \tag{4}$$

To understand $(*)$, notice that, given that $Z_1 = j$, the event $\{Z_{n+1} = 0\}$ occurs if and only if all the $j$ subpopulations originated by the individuals present at time 1 die out in $n$ generations. Since $G$ is a continuous function on $[0, 1]$, by making $n \to \infty$ in (4) and using (3), we conclude that

$$\delta = G(\delta).$$

Then, using that $G$ is a convex function satisfying $G(0) = p_0$, $G(1) = 1$, and $G'(1) = \mu$, one can establish the following result.

**Theorem 5.** *Let $\{Z_n\}_{n\geq 0}$ be a Bienaymé–Galton–Watson branching process with mean number of descendants per individual $\mu$.*

(a) *If $\mu \leq 1$, then the process dies out with probability* 1.

(b) *If $\mu > 1$, then the extinction probability $\delta$ is the unique solution in* $(0, 1)$ *of the equation $G(s) = s$.*

As an example, suppose that the distribution of the number of children of a single individual is geometric with parameter $p \in (0, 1)$, that is,

$$p_j = p\,(1 - p)^j, \; j \in \mathbb{N}_0.$$

From (2), we have that $\mu = (1 - p)/p$. If $p \geq 1/2$, then $\mu \leq 1$, so the process dies out almost surely. On the other hand, recalling (1), we see that for $p < 1/2$ the extinction probability is the unique solution in $(0, 1)$ of the equation

$$\frac{p}{1 - s(1 - p)} = s.$$

This reduces to a quadratic equation in $s$, whose roots are $p/(1 - p)$ and 1. Therefore the probability that a branching process with Geometric($p$) offspring distribution becomes extinct is given by

$$\begin{cases} 1 & \text{if } p \geq \frac{1}{2}, \\ \frac{p}{1-p} & \text{if } p < \frac{1}{2}. \end{cases} \tag{5}$$

**Coupling.** Coupling is an essential probabilistic tool that consists in constructing two random variables (or stochastic processes) simultaneously by means of the same random device. This technique was first formulated in 1938 by the French mathematician Wolgang Doeblin, in order to prove the convergence to equilibrium for regular finite state Markov chains. A collection of illustrative examples can be found in Thorisson [23]; see Lindvall [16] and Thorisson [24] for a thorough treatment of the subject.

**Definition 6.** Let $X_1$ and $X_2$ be two random variables. A *coupling* of $X_1$ and $X_2$ is a random vector $(\hat{X}_1, \hat{X}_2)$ such that for every $i = 1, 2$,

$$\hat{X}_i \stackrel{\mathcal{D}}{=} X_i,$$

where $\stackrel{\mathcal{D}}{=}$ denotes identity in distribution.

We observe that the random variables $X_1$ and $X_2$ need not be defined on a common probability space, whereas $\hat{X}_1$ and $\hat{X}_2$ must be constructed on the same probability space. Hence coupling refers to the fact that $\hat{X}_1$ and $\hat{X}_2$ have a joint distribution; only their marginal distributions are preassigned. The joint distribution can be appropriately chosen according to one's intention, for example to demonstrate a property of the random variables or a relationship between them.

By way of illustration, consider $0 < \alpha_1 < \alpha_2 < 1$, and for $i = 1, 2$, let $X_i$ be a geometrically distributed random variable with parameter $(1 - \alpha_i)$. That is,

$$P(X_i = j) = (1 - \alpha_i)\,(\alpha_i)^j,\ j \in \mathbb{N}_0.$$

Thus $X_i$ can be interpreted as the number of failures that occur before the first success is obtained, in a series of independent Bernoulli trials with the probability of success equal to $(1 - \alpha_i)$. To define a coupling of $X_1$ and $X_2$, let $U$ be a random variable with uniform distribution on the interval $(0, 1)$, and define

$$\hat{X}_i = \left[\frac{\log U}{\log \alpha_i}\right],\ i = 1, 2, \tag{6}$$

where $[x]$ is the integer part of $x$ (i.e., the largest integer less than or equal to $x$). It is not difficult to see that $\hat{X}_i \stackrel{\mathcal{D}}{=} X_i$ for $i = 1, 2$, and that $\hat{X}_1 \leq \hat{X}_2$. Thus we construct $\hat{X}_1$ and $\hat{X}_2$ simultaneously, with the required marginal distributions, in such a way that $\hat{X}_1$ is *pointwise dominated* by $\hat{X}_2$ (the number of failures counted in $\hat{X}_1$ is always less than or equal to that counted in $\hat{X}_2$).

**3. THE RETURNS MODEL.** In this section, we study for which values of the parameter $(p, \alpha)$ the returns model dies out almost surely or survives with positive probability. We denote the probability that the returns model with parameter $(p, \alpha)$ becomes extinct by $\psi(p, \alpha)$, and let $\theta(p, \alpha) = 1 - \psi(p, \alpha)$ be the survival probability. First, we observe that $\theta(p, 1) = 1$ for every $p \in [0, 1]$, since the initial particle lives forever if $\alpha = 1$.

In order to obtain $\psi(p, \alpha)$ for $\alpha < 1$, let us define a Bienaymé–Galton–Watson branching process $\{Z_n\}_{n\geq 0}$ embedded in the returns model. We define $Z_0 = 1$, as we start with a single particle. Each time this initial particle returns to the origin, it gives birth to an individual; the generation 1 of the branching process is formed by all these individuals. Thus $Z_1$ is defined to be the number of returns to the origin that the initial particle makes throughout its lifespan. If $Z_1 = 0$, then both the returns model and the branching process have died out. If $Z_1 = j \geq 1$, each of the $j$ particles (which are the direct descendants of the original parent) has its own trajectory and lifetime, acting independently of the others. In this case, we define $Z_2$ as the sum of the numbers of times that these $j$ particles return to the origin. The definitions of $Z_3, Z_4, \ldots$ go in the same way. Therefore, for $\alpha < 1$, we couple the returns model and the branching process $\{Z_n\}_{n\geq 0}$ so that the former survives if and only if the latter does.

Now, for $\alpha < 1$, let $\mu(p, \alpha)$ denote the mean number of returns to the origin of the initial particle. That is, $\mu(p, \alpha) = E(\eta)$, where the random variable $\eta$ stands for the number of offspring of a single individual in the branching process $\{Z_n\}_{n\geq 0}$. To ease the understanding, we split the analysis into two parts: I. $p = 1/2$, and II. $(p, \alpha) \in [0, 1]^2$.

**I. Symmetric random walks.** First, we assume that $p = 1/2$. To simplify the notation, here we write $\psi(\alpha) = \psi(1/2, \alpha)$, $\theta(\alpha) = \theta(1/2, \alpha)$, and $\mu(\alpha) = \mu(1/2, \alpha)$.

Of course, $\theta(0) = 0$ and $\theta(1) = 1$. In addition, it is intuitively clear that $\theta$ is a nondecreasing function of $\alpha$. This monotonicity follows essentially from the coupling between the two geometric random variables $\hat{X}_1$ and $\hat{X}_2$ given by (6). Indeed, for

$0 < \alpha_1 < \alpha_2 < 1$, let us formally define a coupling of two realizations 1 and 2 of the returns model with respective parameters $(1/2, \alpha_1)$ and $(1/2, \alpha_2)$, so that model 2 survives whenever model 1 does. Toward this end, let

$$\begin{aligned} &\{\{S_n(k)\}_{n\geq 0};\ k \geq 1\}, && \{U(k);\ k \geq 1\}, \\ &\{\{S'_n(k)\}_{n\geq 0};\ k \geq 1\}, \quad \text{and} && \{\tau'_{\alpha_2}(k);\ k \geq 1\} \end{aligned}$$

be independent sets of random objects defined as follows. For each $k \geq 1$, $\{S_n(k)\}_{n\geq 0}$ and $\{S'_n(k)\}_{n\geq 0}$ are simple symmetric random walks on $\mathbb{Z}$, starting from the origin, $U(k)$ is a random variable with uniform distribution on the interval $(0, 1)$, and $\tau'_{\alpha_2}(k)$ is a geometrically distributed random variable with parameter $(1 - \alpha_2)$. For each $i = 1, 2$ and $k \geq 1$, we define

$$\tau_{\alpha_i}(k) = \left\lfloor \frac{\log U(k)}{\log \alpha_i} \right\rfloor.$$

Now we jointly construct versions of model 1 and model 2 in such a way that:

(i) whenever a new particle is created in model 1 (including the initial one), the same happens in model 2, and

(ii) these two twin particles perform the same random walk, and the lifespan of the particle born in model 1 is less than or equal to that associated to its twin particle in model 2.

More precisely, the $k$th particle created in model 1 follows the simple random walk $\{S_n(k)\}_{n\geq 0}$, and disappears after $\tau_{\alpha_1}(k)$ steps. Its twin particle in model 2 follows the same random walk, disappearing after $\tau_{\alpha_2}(k)$ steps. Notice that there may exist particles in model 2 with no corresponding twin in model 1. The $k$th such particle is assigned the random objects $\{S'_n(k)\}_{n\geq 0}$ and $\tau'_{\alpha_2}(k)$, which stand respectively for its trajectory and lifetime. Hence we couple the two processes in such a way that model 2 has always more particles than model 1. Consequently, the survival of model 1 implies the survival of model 2, and $\theta(\alpha_1) \leq \theta(\alpha_2)$.

Thus we conclude that there exists a critical value $\alpha_c$ of $\alpha$ such that

$$\theta(\alpha) \begin{cases} = 0 & \text{if } \alpha < \alpha_c, \\ > 0 & \text{if } \alpha > \alpha_c. \end{cases}$$

Formally, the *critical parameter* $\alpha_c$ is defined by

$$\alpha_c = \inf\{\alpha \in [0, 1] : \theta(\alpha) > 0\}.$$

From Theorem 5, we have that $\theta(\alpha) > 0$ if and only if $\mu(\alpha) > 1$.

In order to compute $\mu(\alpha) = E(\eta)$ for $\alpha < 1$, let $\{S_n\}_{n\geq 0}$ be a simple symmetric random walk on $\mathbb{Z}$, starting from the origin, and $\tau$ be a random variable whose law is given by

$$P(\tau = j) = (1 - \alpha)\,\alpha^j,\ j \in \mathbb{N}_0,$$

with $\{S_n\}_{n\geq 0}$ and $\tau$ independent. Then, $\{S_n\}_{n\geq 0}$ describes the trajectory of the initial particle (if it never died), and $\tau$ stands for the number of steps it takes until it disappears. Now we write $\eta$ as a sum of indicator random variables. For each $i = 1, 2 \ldots,$

we define

$$Y_i = \begin{cases} 1 & \text{if the initial particle is at the origin at time } i, \\ 0 & \text{otherwise.} \end{cases}$$

Notice that $\{Y_i = 1\} = \{S_i = 0, \tau \geq i\}$ for every $i$, and $\eta = \sum_{i=1}^{\infty} Y_i$. Consequently, from Theorem 4 (b), the expected number of returns to the origin that the initial particle makes is given by

$$\begin{aligned} \mu(\alpha) &= E\left(\sum_{i=1}^{\infty} Y_i\right) = \sum_{i=1}^{\infty} E(Y_i) = \sum_{i=1}^{\infty} P(Y_i = 1) \\ &= \sum_{i=1}^{\infty} P(S_i = 0) P(\tau \geq i) = \sum_{i=1}^{\infty} \alpha^i P(S_i = 0) \\ &= P_0(\alpha) - 1 = (1 - \alpha^2)^{-1/2} - 1. \end{aligned} \tag{7}$$

We observe that $\mu$ is an increasing function, with $\mu(0) = 0$, and $\lim_{\alpha \nearrow 1} \mu(\alpha) = \infty$. The graph of $\mu$ is shown in Figure 1.

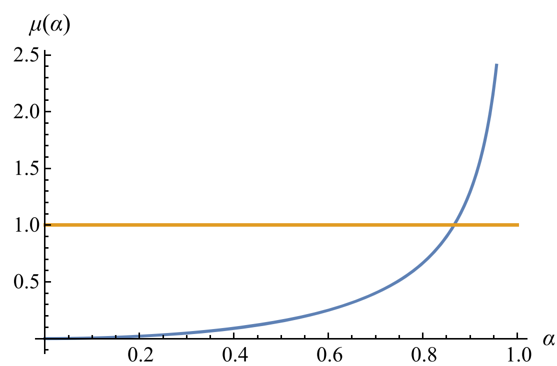


**Figure 1.** Graph of $\mu(\alpha)$.

By solving $\mu(\alpha) = 1$, we obtain that the critical parameter is

$$\alpha_c = \frac{\sqrt{3}}{2} \approx 0.866.$$

Therefore, the returns model with $p = 1/2$ dies out almost surely for $\alpha \leq \sqrt{3}/2$, while it survives with positive probability for $\alpha > \sqrt{3}/2$. As $\alpha_c \in (0, 1)$, we say that this model undergoes a nontrivial *phase transition*.

The study of phase transition phenomenon forms a central and fascinating topic in probability theory. In words, a system exhibits *phase transition* if there is an abrupt change in a qualitative property of the system, after a parameter of interest (like a probability or temperature) passes through a critical value. For most models, obtaining the exact value of the critical parameter is a very difficult task. Besides showing that a given model exhibits phase transition, the main objectives are to study its properties in the two distinct phases and near criticality, and to determine bounds for the critical parameter. The percolation model and the contact process are classical examples. More details on these processes can be found in Grimmett [8, 9], Liggett [17, 18], and Schinazi [21].

To get the probability of extinction $\psi(\alpha)$, we argue that for $\alpha < 1$, $\eta$ has a geometric distribution. In fact, if $\alpha < 1$, then the initial particle stops jumping at some finite instant of time, so it has a finite number of children (although the random walk is recurrent). Starting from the origin, let $\rho(\alpha)$ denote the probability that the particle has no offspring (that is, it disappears before returning for the first time to the origin). By the Markov property of $\{S_n\}_{n\geq 0}$ and the memoryless property of the geometric distribution of $\tau$, we have that the distribution of $\eta$ is given by

$$P(\eta = j) = \rho(\alpha)\,(1-\rho(\alpha))^j,\ j \in \mathbb{N}_0.$$

From (2) and (7), we conclude that $\rho(\alpha) = (1-\alpha^2)^{1/2}$. Hence, using (5), we obtain that, in the case $p = 1/2$, the probability of extinction is given by

$$\psi(\alpha) = \begin{cases} 1 & \text{if } 0 \leq \alpha \leq \dfrac{\sqrt{3}}{2}, \\ \dfrac{(1-\alpha^2)^{1/2}}{1-(1-\alpha^2)^{1/2}} & \text{if } \dfrac{\sqrt{3}}{2} < \alpha \leq 1. \end{cases}$$

The graph of the survival probability $\theta(\alpha) = 1 - \psi(\alpha)$ is depicted in Figure 2.

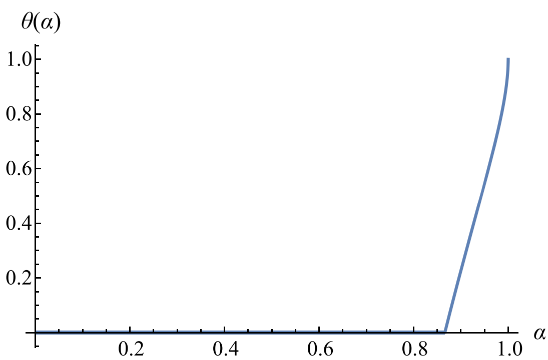


**Figure 2.** Graph of $\theta(\alpha)$.

**II. General case.** Now we deal with the general case, with a bidimensional parameter $(p, \alpha) \in [0, 1]^2$. As mentioned earlier, if $\alpha = 1$ then the returns model survives almost surely. For $\alpha < 1$, by Theorem 4 (b) and the same line of reasoning as that presented for the case where $p = 1/2$, it follows that $\eta$ has geometric distribution with mean

$$\mu(p, \alpha) = (1 - 4p(1-p)\alpha^2)^{-1/2} - 1.$$

Figure 3 shows the graph of $\mu(p, \alpha)$. We conclude that the region of parameters for which the returns model has a positive probability of survival is the union of the set $[0, 1] \times \{1\}$ plus the set of points $(p, \alpha) \in [0, 1] \times [0, 1)$ such that the graph of $\mu(p, \alpha)$ lies above the pink plane.

For $\alpha < 1$, by simplifying the inequality $\mu(p, \alpha) > 1$, we obtain that the returns model has a positive probability of survival if and only if the following condition

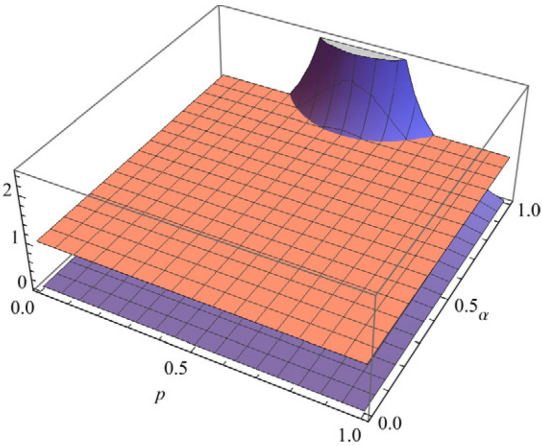


**Figure 3.** Graph of $\mu(p, \alpha)$.

holds:

$$\frac{1}{4} < p < \frac{3}{4} \quad \text{and} \quad \alpha > \phi(p) := \frac{\sqrt{3}}{4}\sqrt{\frac{1}{p(1-p)}}.$$

The situation is shown in the *phase diagram* illustrated in Figure 4: for values of $(p, \alpha)$ in the blue region, there is positive probability of survival, whereas for those values in the pink region the returns model dies out almost surely. The set $\{(p, \alpha) : \alpha = \phi(p)\}$ defines the critical curve in the phase space $[0, 1]^2$ that separates these regions. Notice also that the survival region contains the set

$$\left\{\frac{1}{2}\right\} \times \left(\frac{\sqrt{3}}{2}, 1\right] \bigcup\, [0, 1] \times \{1\}\,.$$

Using (5), we get that the probability that the returns model becomes extinct is given by

$$\psi(p, \alpha) = \begin{cases} \dfrac{(1-4p(1-p)\alpha^2)^{1/2}}{1-(1-4p(1-p)\alpha^2)^{1/2}} & \text{if } \dfrac{1}{4} < p < \dfrac{3}{4} \text{ and } \phi(p) < \alpha < 1, \\ 0 & \text{if } 0 \le p \le 1 \text{ and } \alpha = 1, \\ 1 & \text{otherwise.} \end{cases}$$

The graph of $\theta(p, \alpha) = 1 - \psi(p, \alpha)$ is depicted in Figure 5.

**4. WEB SUPPLEMENT.** Symbolic mathematical software is an invaluable aid in both teaching and research. It can be very helpful to reinforce the usage of key definitions and results, complementing the traditional learning process. In the Mathematica Notebook we provide, we start off with the geometric distribution, by calculating its mean, variance, pgf, and analyzing the Bienaymé–Galton–Watson branching process with this offspring distribution. Then, we perform the computations related to the returns model in the two cases discussed in Section 3. The commands used to generate the figures presented in the paper are also included.

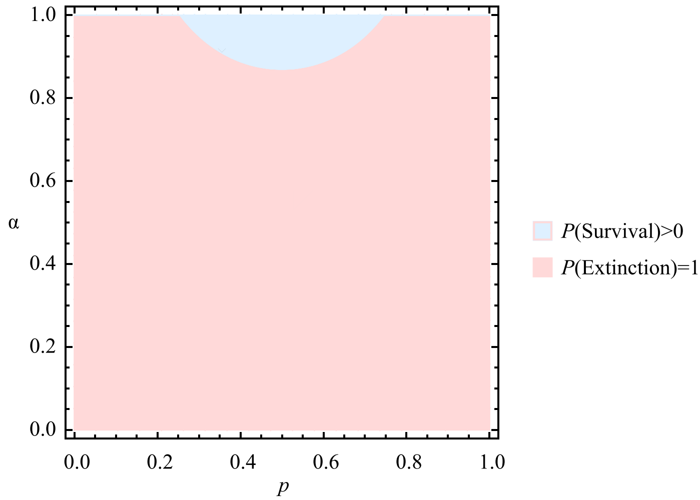


**Figure 4.** Phase diagram for the returns model. The region of almost sure extinction is represented in pink, and the region of survival with positive probability in blue.

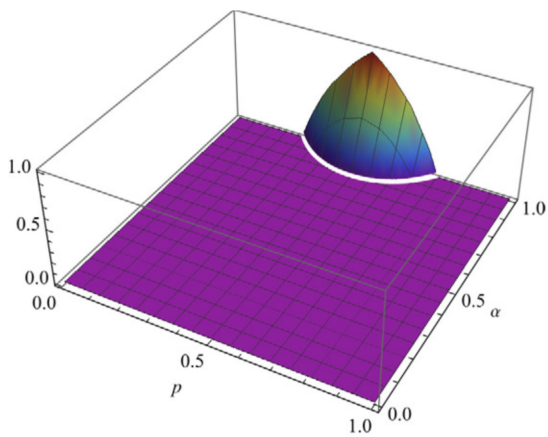


**Figure 5.** Graph of $\theta(p, \alpha)$.

**5. CONCLUDING REMARKS.** In this paper, we define and study the returns model, a system of random walks on $\mathbb{Z}$ with geometrically distributed lifetimes. Each time a particle returns to the origin, a new particle is generated. The formulation of the returns model was inspired by the interacting particle system known as the *frog model*. This is a toy model for an infection spreading (or rumor transmission) through a population, which has been studied since the end of the 1990s. The main difference between the models is that, in the frog model, a new particle is created at a vertex only at the first moment when this vertex is reached by a living particle. The frog model with geometric lifetimes was first considered by Alves et al. [1], who analyze the question of phase transition on infinite graphs, especially on the hypercubic lattices $\mathbb{Z}^d$, $d \geq 1$, and homogeneous trees. For more details on the subject and other topics of research, see Lebensztayn and Utria [14, 15], and references therein.

We underline that, thanks to the way that particles reproduce in the returns model, we can determine completely the parameters that lead to a positive probability of survival, and obtain a closed formula for the survival probability. Moreover, along our study we showcase several essential themes, such as branching processes, coupling, phase transition, and properties of the simple random walk on $\mathbb{Z}$. We expect that this article may inspire readers to pursue their own adventures in probability.

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## 100 Years Ago This Month in *The American Mathematical Monthly*
## Edited by Vadim Ponomarenko

*The Reorganization of Mathematics in Secondary Education*. A report by the National Committee on Mathematical Requirements. Published by the Mathematical Association of America, 1923. x + 652 pages.

In these days, when state legislation and survey reports are opposing required mathematics in secondary schools, it is highly essential that arguments be presented in justification of the retention of these courses in the high school curriculum. Pupils and parents are constantly asking the question "What's the stuff good for?" It is incumbent upon the teachers of mathematics in colleges and secondary schools satisfactorily to answer this question. This report furnishes ample material for meeting these demands. The second chapter in particular is devoted to a discussion of the valid aims and purposes of instruction in mathematics.

In chapters three and four, the committee outlines the work in mathematics to be covered in grades seven to twelve inclusive. The introduction in grades seven to nine of such topics as line, bar, and circle graphs; statistical graphs; simple frequency distributions; intuitive geometry; and numerical trigonometry may at first cause somewhat of a shock to some of our more conservative educators.

In the discussion of "College entrance requirements" the attitude of the committee is unusually fair and unbiased. The position is taken that the high school courses in mathematics should prepare for college courses in general, not mathematics courses only. With this in view, a tabulation is given showing the findings of a committee that endeavored to determine what topics are essential for other college courses.

In accordance with the present popularity of the testing movement, the subject of "Standardized tests in mathematics for secondary schools" is given by far the most space of any subject. This report by C. B. Upton occupies one hundred and fifty pages, or nearly one-fourth of the entire publication. It gives a rather minute description of the various tests in arithmetic, algebra, and geometry with an explanation of the methods of using them. In addition to the standardized educational tests in these three subjects, the Rogers test of mathematical ability, and the Thurstone vocational guidance tests are discussed.

—Excerpted from "Recent Publications",
D. C. Gillespie (1924). 31(2): 91–100.

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