LEGENDRE Antoine M1DS

Exercice 1:

$z_1 = 3+2i \quad \mathrm{Re}(z_1) = 3 \quad \mathrm{Im}(z_1) = 2$

Polar form: $\sqrt{3^2+2^2}\, e^{i \arctan\left(\frac{2}{3}\right)} = \sqrt{13}\, e^{i \arctan\left(\frac{2}{3}\right)}$

$z_2 = \frac{2-i}{2-3i} = \frac{(2-i)(2+3i)}{(2-3i)(2+3i)} = \frac{4+6i-2i+3}{4+6i-6i+9} = \frac{7+4i}{13} \quad \mathrm{Re}(z_2) = \frac{7}{13} \quad \mathrm{Im}(z_2) = \frac{4}{13}$

Polar form: $\sqrt{\left(\frac{7}{13}\right)^2+\left(\frac{4}{13}\right)^2}\, e^{i \arctan\left(\frac{4}{7}\right)} = \frac{\sqrt{65}}{13}\, e^{i \arctan\left(\frac{4}{7}\right)}$

$z_3 = \frac{1}{(3-i)^2} = \frac{1}{9-6i-1} = \frac{1}{8-6i} = \frac{8+6i}{(8-6i)(8+6i)} = \frac{8+6i}{64+36} = \frac{8+6i}{100} \quad \mathrm{Re}(z_3) = \frac{8}{100} \quad \mathrm{Im}(z_3) = \frac{6}{100}$

Polar form: $\sqrt{\left(\frac{8}{100}\right)^2+\left(\frac{6}{100}\right)^2}\, e^{i \arctan\left(\frac{6}{8}\right)} = \frac{\sqrt{100}}{100}\, e^{i \arctan\left(\frac{3}{4}\right)} = \frac{1}{10}\, e^{i \arctan\left(\frac{3}{4}\right)}$

$z_4 = e^{i2\pi} = 1 \quad \mathrm{Re}(z_4) = 1 \quad \mathrm{Im}(z_4) = 0$ Polar form: $e^{i2\pi}$

$z_5 = e^{i\pi} = -1 \quad \mathrm{Re}(z_5) = -1 \quad \mathrm{Im}(z_5) = 0$ Polar form: $e^{i\pi}$

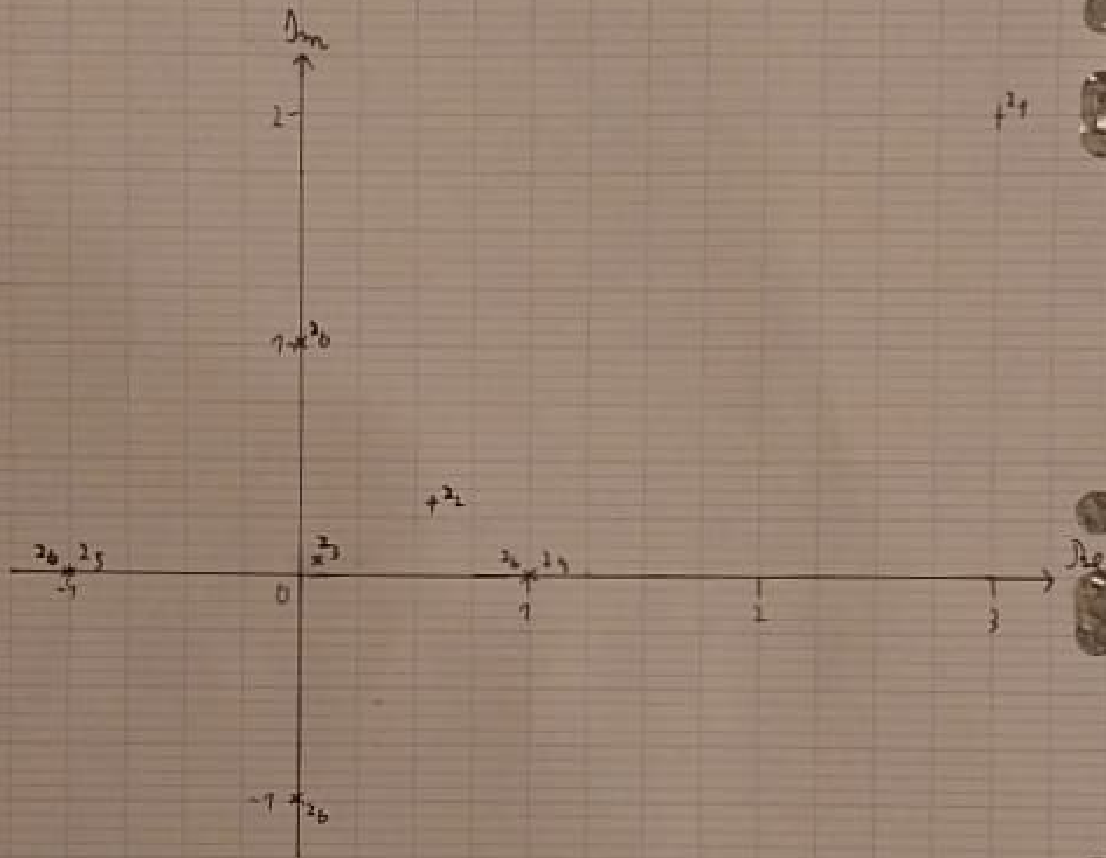
$z_6 = i^n \ (n \in \mathbb{N})$

If $n \equiv 1[4]$: $i^n = i \quad \mathrm{Re}(z_6) = 0 \quad \mathrm{Im}(z_6) = 1$ Polar form: $e^{i\frac{\pi}{2}}$

If $n \equiv 2[4]$: $i^n = i^2 = -1 \quad \mathrm{Re}(z_6) = -1 \quad \mathrm{Im}(z_6) = 0$ Polar form: $e^{i\pi}$

If $n \equiv 3[4]$: $i^n = i^3 = -i \quad \mathrm{Re}(z_6) = 0 \quad \mathrm{Im}(z_6) = -1$ Polar form: $e^{-i\frac{\pi}{2}}$

If $n \equiv 0[4]$: $i^n = i^4 = 1 \quad \mathrm{Re}(z_6) = 1 \quad \mathrm{Im}(z_6) = 0$ Polar form: $e^{i2\pi}$



Exercise 2:

a: Each signal has 20 elements.

b: S_1 : $\ell_0 = 19$; $\ell_1 = 3+4+5+4+1+2+4+5+0+2+3+4+5+6+7+3+1+2+1+3 = 62$;

$\ell_2 = \sqrt{3^2+4^2+5^2+4^2+1^2+2^2+4^2+5^2+0^2+2^2+3^2+1^2+5^2+6^2+7^2+3^2+1^2+2^2+1^2+3^2} = \sqrt{260}$;

$\ell_\infty = 7$

S_2 : $\ell_0 = 9$; $\ell_1 = 23$; $\ell_2 = \sqrt{93}$; $\ell_\infty = 7$ $\qquad$ S_3 : $\ell_0 = 1$; $\ell_1 = 1$; $\ell_2 = 1$; $\ell_\infty = 1$

S_1 has the highest number of non-zero elements and the highest values for all norms. S_3 has only one non-zero element and all its norms are equal to 1.

c: S_3 is the sparser signal with only one non-zero element. The difference between the three signals is that S_1 is slightly sparse, S_2 is moderately sparse and S_3 is very sparse.

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suite. d: $S_1 - S_2 = \{2,-4,5,4,-8,2,6,5,0,-2,2,-3,4,6,8,3,-1,-2,5,3\}$

$\|S_1-S_2\|_1 = 75$; $\|S_1-S_2\|_2 = \sqrt{375}$; $\|S_1-S_2\|_\infty = 8$

$S_1 - S_3 = \{3,-4,5,4,-1,2,4,5,-1,-1,3,-1,5,6,7,3,-1,2,1,3\}$

$\|S_1-S_3\|_1 = 63$; $\|S_1-S_3\|_2 = \sqrt{267}$; $\|S_1-S_3\|_\infty = 7$

$S_2 - S_3 = \{1,0,0,0,7,0,-2,0,-1,0,1,2,1,0,-1,0,0,4,-4,0\}$

$\|S_2-S_3\|_1 = 24$; $\|S_2-S_3\|_2 = \sqrt{94}$; $\|S_2-S_3\|_\infty = 7$

Exercise 3:

To verify if $\|x\|_\alpha$ satisfies the properties of a norm for $\alpha = 0, 1, 2$ and ∞, we need to check the following three properties:
- non-negativity: $\|x\|_\alpha \geq 0$ for all x and $\|x\|_\alpha = 0$ if and only if $x = 0$.
- scalar multiplication: $\|\lambda x\|_\alpha = |\lambda| \, \|x\|_\alpha$, for any scalar λ.
- triangle inequality: $\|x+y\|_\alpha \leq \|x\|_\alpha + \|y\|_\alpha$ for all x and y.

l_0: Let $x = (1,0)$ and λ a scalar, then $\|\lambda x\|_0 = \|(\lambda,0)\|_0 = 1$ but $|\lambda| \, \|x\|_0 = |\lambda| \, \|(1,0)\|_0 = |\lambda|$. So we don't have $\|\lambda x\|_0 = |\lambda| \, \|x\|_0$ for any scalar λ, so l_0 doesn't satisfies the scalar multiplication and l_0 isn't a norm.

l_1: Non-negativity: Each term $|x_k|$ is non-negative. Therefore, $\|x\|_1 = \sum |x_k| \geq 0$. If $\|x\|_1 = 0$, it means that all terms $|x_k| = 0$, which implies $x_k = 0$ for all k. Thus $x = 0$.

Scalar multiplication: Let λ a scalar. $\|\lambda x\|_1 = \sum_k |\lambda x_k| = \sum_k |\lambda||x_k| = |\lambda| \sum_k |x_k| = |\lambda| \, \|x\|_1$

Triangle inequality: $\|x+y\|_1 = \sum_k |x_k + y_k| \leqslant \sum_k |x_k| + |y_k| = \sum_k |x_k| + \sum_k |y_k| = \|x\|_1 + \|y\|_1$

l_1 satisfies all the properties so l_1 is a norm.

l_2: Non-negativity: Since each term $|x_k|^2$ is non-negative, $\|x\|_2 \geqslant 0$. If $\|x\|_2 = 0$, each $|x_k|$ must be zero, which implies $x = 0$.

Scalar multiplication: Let λ a scalar:
$\|\lambda x\|_2 = \sqrt{\sum_k |\lambda x_k|^2} = \sqrt{\sum_k |\lambda|^2 |x_k|^2} = \sqrt{|\lambda|^2 \sum_k |x_k|^2} = |\lambda| \sqrt{\sum_k |x_k|^2} = |\lambda| \, \|x\|_2$

Triangle inequality:
$\|x+y\|_2^2 = \sum_k |x_k + y_k|^2 = \sum_k |x_k|^2 + |y_k|^2 + 2\,\mathrm{Re}(x_k \bar{y}_k) = \sum_k |x_k|^2 + \sum_k |y_k|^2 + 2 \sum_k \mathrm{Re}(x_k \bar{y}_k)$
By the Cauchy-Schwarz inequality, we have:
$2 \sum_k \mathrm{Re}(x_k \bar{y}_k) \leqslant 2 \sqrt{\sum_k |x_k|^2} \sqrt{\sum_k |y_k|^2} = 2 \|x\|_2 \|y\|_2$
So now we have:
$\|x+y\|_2^2 \leqslant \sum_k |x_k|^2 + \sum_k |y_k|^2 + 2\|x\|_2 \|y\|_2 = \|x\|_2^2 + \|y\|_2^2 + 2\|x\|_2\|y\|_2 = (\|x\|_2 + \|y\|_2)^2$
Finally, taking the square root of both sides gives us the triangle inequality: $\|x+y\|_2 \leqslant \|x\|_2 + \|y\|_2$.

l_2 satisfies all the properties so l_2 is a norm.

l_∞: Non-negativity: The supremum of absolute values is non-negative: $\|x\|_\infty \geqslant 0$. If $\|x\|_\infty = 0$, then $|x_k| = 0$ for all k, implying $x = 0$.

Scalar multiplication: Let λ a scalar: $\|\lambda x\|_\infty = \sup_k |\lambda x_k| = |\lambda| \sup_k |x_k| = |\lambda| \, \|x\|_\infty$

Triangle inequality: Following the properties of suprema, we have:
$\|x+y\|_\infty = \sup_k |x_k + y_k| \leqslant \sup_k (|x_k| + |y_k|) \leqslant \sup_k |x_k| + \sup_k |y_k| = \|x\|_\infty + \|y\|_\infty$

l_∞ satisfies all the properties so l_∞ is a norm.

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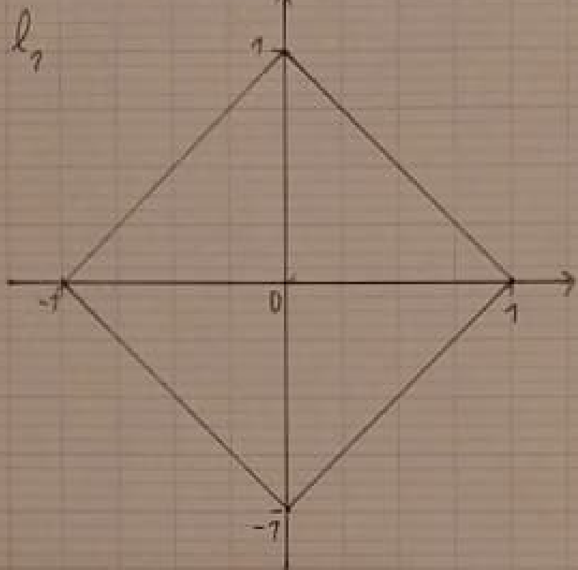
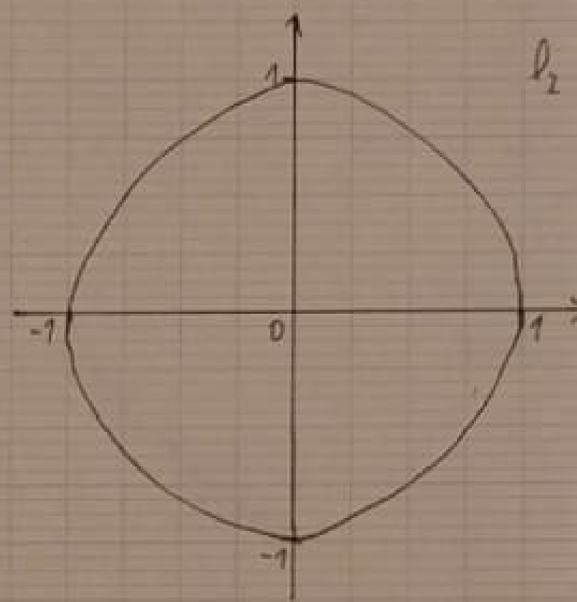
suite: Exercice 4:

l_1



l_2



l_∞