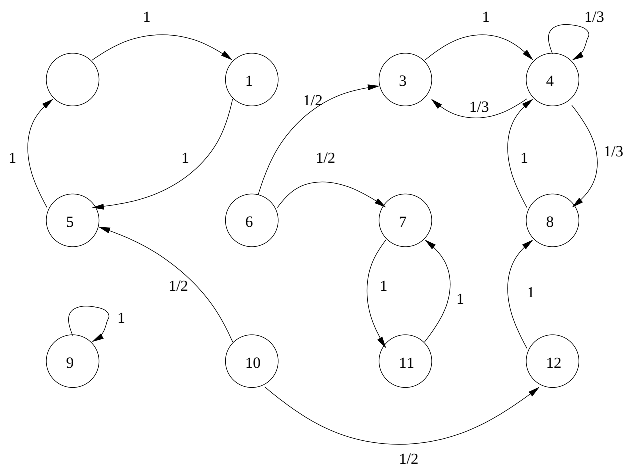
Sheet 2 - Markov Chains : main properties

Exercise 1 : Let's consider a Markov chain with the following transition graph



1. What are the recurrent states and the transient states ?
2. Give the irreducible recurrent classes.
3. What are the periodic classes ? For which initial states does the chain law converge ?
4. What is the limit law of the chain if its initial state is the 12 state ?

Exercise 2 : Transmitter - length of successes

We consider a transmitter which emits binary signals in an independent way, the value 1 going out with probability $p \in]0,1[$. At any time n, we note X_n the length of the sequence of 1 before the instant n. Thus, if the sequence of transmitted signals is

$$00110101110\ldots,$$

the corresponding sequence $(X_n)_{n>0}$ is

$$00120101230.$$

1. Show that $(X_n)_{n>0}$ is a Markov chain on $\mathbb{N}$. Specify its transition matrix.
2. Show that any invariant measure μ is of the form $\mu_i = \alpha p^i$, for a $\alpha > 0$. Deduce that there is a unique invariant probability π.

Exercise 3 : Dam lake

A dam lake has a capacity of 3 volume units. Let X_n be the quantity of water retained at the beginning of the n-th day, $n \in \mathbb{N}$. Each day, a certain quantity of water arrives, given by a random variable Y whose distribution is :

k	0	1	2	3
$\mathbb{P}(Y = k)$	$0,2$	$0,5$	$0,2$	$0,1$

Then in the evening the dam empties by one unit of volume (when it is not already empty).

1. Explain why the process $(X_n)_{n\geq 0}$ is a Markov chain. Give its transition matrix.

2. Calculate its stationary distribution.

Exercise 4 : Urn and Balls

We consider 4 numbered balls from 1 to 4, distributed in two urns A and B. At each instant, we randomly draw a number k between 1 and 4, remove ball k from the urn it is in, and randomly place it into one of the two urns. We denote X_n as the number of balls in urn A at time n.

1. Provide the transition matrix and graph of (X_n).

2. Is the chain irreducible ? Aperiodic ?

3. Provide the stationary distribution(s). Do you recognize a known distribution ?

4. We start with urn A empty. After a sufficiently long time, we observe the number of balls in urn A. What is (approximately) the probability that this number is even ?

5. We start with urn A full. We observe a realization $(X_0, X_1, \ldots, X_n, \ldots)$ of the chain. What proportion of the time are there strictly fewer balls in A than in B ?

6. Generalize the previous study with M balls numbered from 1 to M.

Exercise 5 : Scarab Beetle

A scarab beetle moves along the edges of a regular tetrahedron with vertices A, B, C, D. Regardless of the vertex it is at any given moment, it randomly and equally likely chooses the vertex to which it will move next. It takes one unit of time to reach the chosen vertex. Additionally, we assume that the scarab beetle moves continuously, meaning it never stops at a vertex. We denote X_n as the position of the scarab beetle at time n.

1. Determine the transition matrix of the Markov chain, as well as the stationary distributions.

2. Do we have convergence in law of (X_n) ?

3. The scarab beetle pays 1€ each time it passes through vertex A, 2€ at vertex B, 3€ at vertex C, and 4€ at vertex D. Let C_N be the cost of its trajectory up to time N. What can you say about the convergence of $\frac{C_N}{N}$?

Exercise 6 : Epidemiological Model

We consider the evolution of a virus in a population of size N. On day n, there are X_n infected individuals and $S_n = (N - X_n)$ healthy individuals. The following day, the X_n individuals previously infected are healthy, but each of the S_n individuals previously healthy has a probability p of encountering one of the I_n infected from the previous day and thus contracting the infection, with all these encounters being independent of each other. It is clear that if nobody is infected, the same will be true the next day. The parameter p is called the infectious contact rate.

1. We assume that the population consists of only 3 individuals and that the infectious contact rate is $p = 1/3$.

a) Give the transition matrix and graph of the Markov chain (X_n).

b) Is the chain irreducible ?

c) Determine the stationary distribution(s). Interpret.

2. We still assume a population of 3 individuals, but the infectious contact rate is the parameter $p \in]0,1[$. Denoting $q = 1-p$, provide the transition graph and stationary distributions.

3. We now assume a population of N individuals, with an infectious contact rate $p \in]0,1[$. Justify the fact that X_n has transition probabilities :

$$p_{i,j} = C_{N-i}^{j}(1-q^i)^j q^{i(N-i-j)} \mathbb{1}_{\{i+j \leq N\}}.$$

Exercise 7 : Simple random walk on $\mathbb{Z}$

Let (M_n) be the simple random walk on $\mathbb{Z}$, i.e. the Markov chain such that $M_0 = 0$, and with transition probabilities defined by

$$\forall x \in \mathbb{Z}, \begin{cases} p_{x,x+1} = p, \\ p_{x,x-1} = 1-p, \end{cases}$$

where $p \in]0,1[$ is fixed.

1. Show that the chain is irreducible.

2. We suppose for the moment $p \neq \frac{1}{2}$. Use the law of large numbers to show that (M_n) is transient.

3. In what follows, we now assume that $p = 1/2$. Compute for $n \geq 0$, the probability $P^n(0,0) = \mathbb{P}(M_n = 0)$. Show then that (M_n) is recurent. (Recall Stirling's formula $n! = \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$.)

Exercise 8 : Ehrenfest urn

We consider 2 urns, in which initially we randomly distribute N balls. We note X_0 the initial number of balls in the urn 1 (there are thus initially $N - X_0$ balls in the urn 2). At each moment, we choose at random (uniformly) one of the N balls, and we change the urn. Let us note X_n the number of balls in the urn 1, at the time n.

1. Explain why the sequence (X_n) is a Markov chain and give its transition matrix.

2. Is this chain irreducible ? recurrent ? periodic ? If yes, what is its period ?

3. Show that the only invariant probability measure is the binomial measure $B(N, \frac{1}{2})$.

4. Let $P_n = \frac{X_n}{N}$ be the proportion of balls that are in the urn 1. Show that if we put $\alpha = 1 - \frac{2}{N}$, and $\beta = 1 - \frac{4}{N}$, we have

$$\mathbb{E}(P_n) = \frac{1}{2} + \big(\mathbb{E}(P_0) - \frac{1}{2}\big)\alpha^n,$$

$$\mathbb{V}(P_n) = \frac{1}{4N} + \big(\mathbb{V}(P_0) - \frac{1}{4N}\big)\beta^n + \big(\mathbb{E}(P_0) - \frac{1}{2}\big)(\beta^n - \alpha^{2n}).$$

5. Let $T_k = \inf\{n \geq 1 : X_n = k\}$, the return time in k. Show that

* $\mathbb{E}_0[T_0] = 2^N$
* $\mathbb{E}_{N/2}[T_{N/2}] \underset{N\to\infty}{\sim} \sqrt{\frac{\pi N}{2}}$.

Compare these two results.