Homework 3

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Exercise 1: Introduction to DS

i) Data Science: Data Science is an interdisciplinary field that combines statistics, computer science, and domain expertise to extract insights and knowledge from structured and unstructured data. It involves various techniques, including data mining, predictive analytics, and machine learning, to analyze data and inform decision-making.

ii) Data Scientist: A Data Scientist is a professional who uses advanced analytical techniques, programming skills, and domain knowledge to interpret complex data and derive actionable insights. They are proficient in statistical analysis, machine learning, and data visualization, often working to solve business problems or enhance products and services.

iii) Data Analysis: Data Analysis is the process of systematically applying statistical and logical techniques to evaluate and interpret data sets. It involves cleaning, transforming, and modeling data to discover useful information, inform conclusions, and support decision-making.

iv) Machine Learning: Machine Learning is a subset of artificial intelligence that focuses on the development of algorithms and statistical models that enable computers to learn from and make predictions or decisions based on data. It allows systems to improve their performance over time without explicit programming.

v) Big Data: Big Data refers to extremely large and complex data sets that traditional data processing software cannot efficiently manage or analyze. These data sets are characterized by their volume, velocity, and variety, often requiring advanced technologies and tools for storage, processing, and analysis to uncover valuable insights.

Exercise 2: Projection and bases

A)a) Let's calculate the inner product $\langle x, \Phi \rangle$:

$$\langle x, \Phi \rangle = \frac{1}{\sqrt{2}}(2 \times 1 + 1 \times 1) = \frac{3}{\sqrt{2}}$$

The projection of x is therefore:

$$\hat{x} = \frac{3}{\sqrt{2}} \times \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{3}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{3}{2} \\ \frac{3}{2} \end{bmatrix}$$

Let's calculate the inner product $\langle y, \Phi \rangle$:

$$\langle y, \Phi \rangle = \frac{1}{\sqrt{2}}((-2 \times 1) + 1 \times 1) = \frac{-1}{\sqrt{2}}$$

The projection of y is therefore:

$$\hat{y} = \frac{-1}{\sqrt{2}} \times \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = -\frac{1}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ -\frac{1}{2} \end{bmatrix}$$

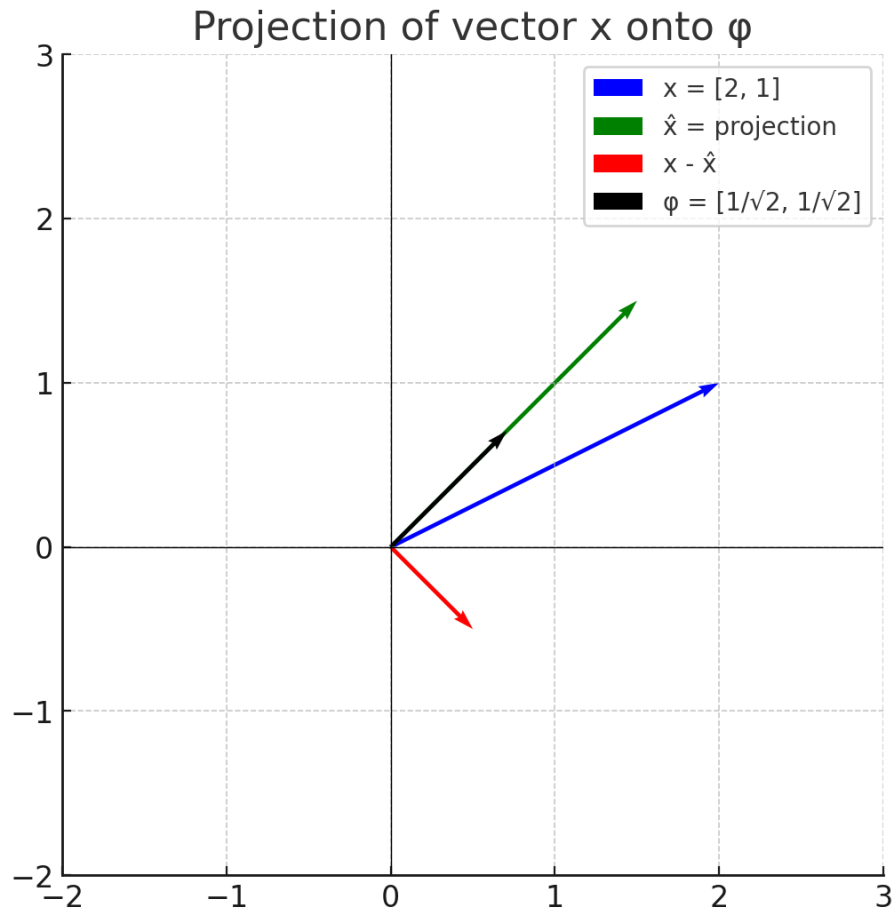A)b) Let's show that $x - \hat{x}$ is orthogonal to Φ.

$$x - \hat{x} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} - \begin{bmatrix} \frac{3}{2} \\ \frac{3}{2} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \end{bmatrix}$$

Let's calculate the inner product $\langle x - \hat{x}, \Phi \rangle$:

$$\langle x - \hat{x}, \Phi \rangle = \frac{1}{\sqrt{2}} \left(\frac{1}{2} \times 1 + (-\frac{1}{2}) \times 1 \right) = \frac{1}{\sqrt{2}} \times 0 = 0$$

Thus, $x - \hat{x}$ is indeed orthogonal to Φ.

A)c)

Projection of vector x onto φ

Legend:
- x = [2, 1]
- x̂ = projection
- x - x̂
- φ = [1/√2, 1/√2]

B) We need to show that the linear combination:

$$\alpha_1 e_1 + \alpha_2 e_2 + \alpha_3 e_3 = 0$$

implies that $\alpha_1 = \alpha_2 = \alpha_3 = 0$. By writing this equation component by component, we have:

$$\alpha_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \alpha_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \alpha_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$
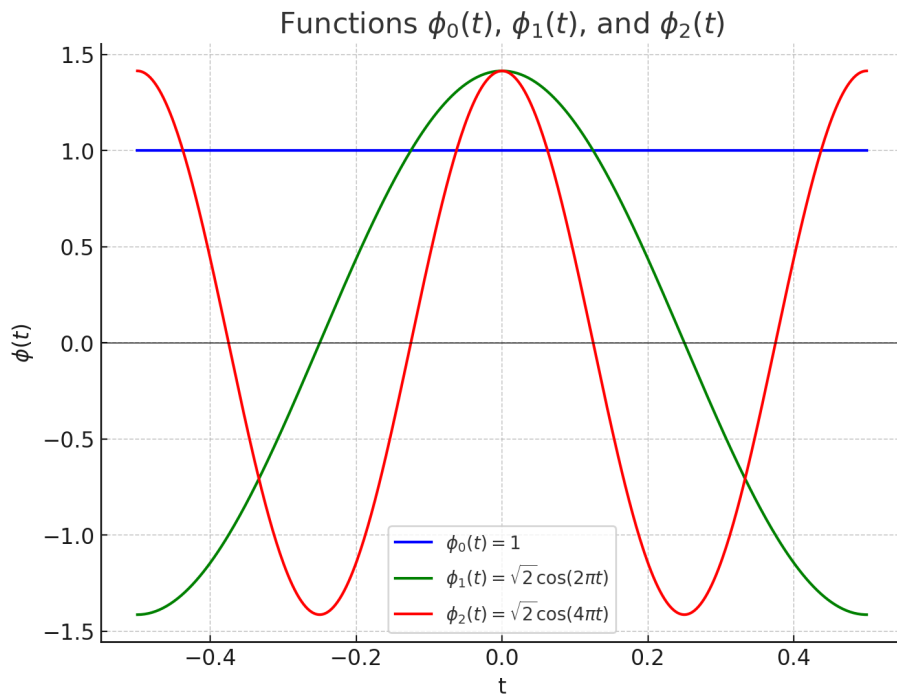
This gives three equations:

$$\alpha_1 = 0, \quad \alpha_2 = 0, \quad \alpha_3 = 0$$

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Thus, the vectors e_1, e_2, e_3 are linearly independent.

Any vector $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{R}^3$ can be written as a linear combination of the vectors e_1, e_2, e_3:

$$x = x_1 e_1 + x_2 e_2 + x_3 e_3$$

C)a)

C)b) Let's show that the set of vectors Φ is orthogonal, meaning that:

$$\langle \Phi_k, \Phi_\ell \rangle = \delta_{k,\ell}$$

For $k \neq \ell$, let's calculate the inner product $\langle \Phi_k, \Phi_\ell \rangle$:

$$\langle \Phi_k, \Phi_\ell \rangle = \int_{-1/2}^{1/2} \sqrt{2} \cos(2\pi k t) \cdot \sqrt{2} \cos(2\pi \ell t)\, dt$$

Using the trigonometric identity:

$$\cos(A) \cos(B) = \frac{1}{2}[\cos(A - B) + \cos(A + B)]$$

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we have:

$$\langle \Phi_k, \Phi_\ell \rangle = 2 \int_{-1/2}^{1/2} \left(\frac{1}{2} [\cos(2\pi(k-\ell)t) + \cos(2\pi(k+\ell)t)] \right) dt$$

The cosine function is even, so:

$$\langle \Phi_k, \Phi_\ell \rangle = 0 \quad \text{for } k \neq \ell$$

And for $k = \ell$:

$$\int_{-1/2}^{1/2} 2\cos^2(2\pi k t)\, dt = \int_{-1/2}^{1/2} (1 + \cos(4\pi k t))\, dt = \int_{-1/2}^{1/2} 1\, dt + \int_{-1/2}^{1/2} \cos(4\pi k t)\, dt = 1 + 0 = 1$$

since the cosine function is even.
Thus, the set Φ is orthogonal.

C)c) Let's show that each function $\Phi_k(t)$ is orthogonal to the set of odd
functions $S_{\text{odd}} = \{s \mid s(t) = -s(-t) \text{ for } t \in [-1/2, 1/2]\}$.
Let $s(t) \in S_{\text{odd}}$. We need to show that:

$$\langle \Phi_k, s \rangle = \int_{-1/2}^{1/2} \Phi_k(t) s(t)\, dt = 0$$

Since $\Phi_k(t) = \sqrt{2}\cos(2\pi k t)$ is an even function and $s(t)$ is odd, their product
$\Phi_k(t)s(t)$ is an odd function. The integral of an odd function over a symmetric
interval is zero:

$$\langle \Phi_k, s \rangle = \int_{-1/2}^{1/2} \Phi_k(t) s(t)\, dt = 0$$

Thus, each $\Phi_k(t)$ is orthogonal to all odd functions $s(t) \in S_{\text{odd}}$.

Exercise 3: Projection onto a subspace

a) We are given:

$$A = \frac{1}{2} \begin{bmatrix} 1 & 1 & -1 \\ -1 & 1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}$$

To show that A is a left inverse of B, we compute the matrix product AB and
check if it equals the identity matrix I_2.
First, we compute AB:

$$AB = \frac{1}{2} \begin{bmatrix} 1 & 1 & -1 \\ -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2$$

This shows that A is a left inverse of B.

b) We will compute the matrix product $P = BA$:

$$P = BA = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} \cdot \frac{1}{2} \begin{bmatrix} 1 & 1 & -1 \\ -1 & 1 & 1 \end{bmatrix} == \frac{1}{2} \begin{bmatrix} 1 & 1 & -1 \\ 0 & 2 & 0 \\ -1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ 0 & 1 & 0 \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

Next, we need to verify that P is a projection operator by checking if $P^2 = P$:

$$P^2 = P \cdot P = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ 0 & 1 & 0 \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix} \cdot \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ 0 & 1 & 0 \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ 0 & 1 & 0 \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix} = P$$

Therefore, since $P^2 = P$, we conclude that P is a projection operator.

c) We have already computed P as follows:

$$P = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ 0 & 1 & 0 \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

To determine if P is an orthogonal projection operator, we need to check if $P^T = P$.
Calculating the transpose of P:

$$P^T = \begin{bmatrix} \frac{1}{2} & 0 & -\frac{1}{2} \\ \frac{1}{2} & 1 & \frac{1}{2} \\ -\frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}$$

Since $P \neq P^T$, we conclude that P is not an orthogonal projection operator. Therefore, since $P^2 = P$ holds true, but $P^T \neq P$, we can conclude that the projection operator P is an oblique projection operator.

d) To compute $P\mathbf{x}$, we perform the matrix multiplication:

$$P\mathbf{x} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ 0 & 1 & 0 \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 6 \\ 6 \\ 8 \end{bmatrix} = \begin{bmatrix} 2 \\ 6 \\ 4 \end{bmatrix}.$$

e) To find the kernel of P, we solve $Pv = 0$, where $v = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$.

$$\frac{1}{2} \begin{bmatrix} 1 & 1 & -1 \\ 0 & 2 & 0 \\ -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = 0$$

This gives the system of equations:

$$\frac{1}{2}(v_1 + v_2 - v_3) = 0 \quad \Rightarrow \quad v_1 + v_2 - v_3 = 0$$

$$\frac{1}{2}(2v_2) = 0 \quad \Rightarrow \quad v_2 = 0$$

$$\frac{1}{2}(-v_1 + v_2 + v_3) = 0 \quad \Rightarrow \quad -v_1 + v_3 = 0 \quad \Rightarrow \quad v_1 = v_3$$

Thus, the kernel is:

$$\text{Ker}(P) = \left\{ \begin{bmatrix} v_1 \\ 0 \\ v_1 \end{bmatrix} \mid v_1 \in \mathbb{R} \right\} = \text{span}\left(\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right)$$