amU Aix Marseille Université **Année universitaire 2024/2025**

Site : ☐ Luminy ☒ St-Charles ☐ St-Jérôme ☐ Cht-Gombert ☐ Aix-Montperrin

Sujet de : ☐ 1$^{\text{er}}$ semestre ☒ 2$^{\text{ème}}$ semestre ☐ Session 2 Durée de l'épreuve : 2h

Examen de : M1 Nom du diplôme : Master MAS

Code du module : SMSBU09C Libellé du module : Markov chains, martingales.

Calculatrices autorisées : OUI Documents autorisés : OUI, une feuille A4 recto verso

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Exercise 1. 4-states MC (5 points)

We consider the state space $E = \{1, 2, 3, 4\}$ of a homogeneous Markov chain.

1) Complete the following matrix so that it becomes a transition matrix :

$$P = \begin{pmatrix} \cdot & \frac{1}{2} & 0 & 0 \\ \cdot & \frac{2}{3} & 0 & 0 \\ 0 & \cdot & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{4} & \cdot & 0 & \frac{1}{2} \end{pmatrix}$$

2) Represent the associated graph.
3) Is the chain irreducible? Are the states recurrent, transient? periodic, aperiodic?
4) Determine the stationary distribution(s).
5) Suppose that at time 0, we are in state 3. For a large number of time units n, what are the probabilities of being in each of the four states?

Exercise 2. Moran model (7pts)

Consider a population of fixed size N, consisting of two types of individuals, denoted a and A. We note X_n the number of individuals of type a at time n. Between times n and $n+1$, an individual in the population splits, while another one dies. These two individuals are chosen randomly in the population and can be identical.

1) Give the transitions matrix of $(X_n)_{n\geqslant 0}$ and draw the transition graph.
2) Classify the chain in terms of recurrent and transient states/classes. Justify this classification.
3) Is the model irreducible? aperiodic?
4) Give the invariant measures. Interpret in terms of the long time behavior of the model.

Exercise 3. Loser loses it all (8 pts)

We consider a player whose probability of giving a correct answer is p independently of the question asked, and the probability of giving an incorrect answer is $1-p$, with $0 < p < 1$. We assume that the answers are independent of each other and that if the player gives a correct answer, he earns 1 point, whereas if he gives an incorrect answer, he loses everything. The initial score of the player is $X_0 = 0$, and for all $n \geqslant 1$, we denote by X_n his score at time n.

1) Provide the graph as well as the transition matrix associated with the Markov chain $(X_n)_{n\geqslant 0}$. Specify the state space of the chain.
2) Is the chain irreducible?
3) Let $T_0 = \inf\{n > 0 \colon X_0 = 0\}$ be the hitting time of 0. For all $n \in \mathbb{N}^*$, compute $\mathbb{P}_0(T_0 = n)$.
4) Is the chain recurrent?

-> Please turn

5) Does a unique stationary probability measure exist? If so, can you recognize a known distribution?
6) Analyze the limiting distribution of the chain at time n. What can you conclude about the probability of the player's score when he plays for a long time?