M1 MAS
Statistics

TD Chapter 3

Exercise 1 Could you give the likelihood and/or loglikelihood in the following cases:

Q:

(1) Poisson distribution, with one single observation $x \Rightarrow X \sim \mathcal{P}(\theta)$

(2) Exponential distribution, with an n-sample: $X=(X_1,\dots,X_n)$, $X_i \sim \mathcal{E}xp(\theta)$

(3) Cauchy distribution, with a n-sample: $X=(X_1,\dots,X_n)$, $X_i \sim Cauchy(\underset{\text{center}}{\theta})$

(4) Geometric distribution, with a n-sample: $X=(X_1,\dots,X_n)$, $X_i \sim Geo(\theta)$.

A:

(1) $\cdot\ L(\theta;x) = f_X(x;\theta) = \mathbb{P}(X=x;\theta) = e^{-\theta}\dfrac{\theta^x}{x!}$

$\cdot\ \ln L(\theta;x) = x\ln(\theta) - \ln(x!) - \theta$

(2) $\cdot\ L(\theta,x) = L(\theta;(x_1,\dots,x_n)) = f_{(X_1,\dots,X_n)}((x_1,\dots,x_n);\theta) \overset{X_i \perp\!\!\!\perp}{=} \prod_{i=1}^n f_{X_i}(x_i;\theta)$

$\overset{i.i.d}{=} \prod_{i=1}^n \theta e^{-\theta x_i}\mathbb{1}_{\{x_i>0\}}$; given that $(x_1,\dots,x_n)$ is a sequence of positive values.

$= \theta^n e^{-\theta\sum_{i=1}^n x_i}$

$\cdot\ \ln L(\theta;x) = n\ln(\theta) - \theta\sum_{i=1}^n x_i$

(3) $\cdot\ L(\theta;x) = \prod_{i=1}^n f_{X_i}(x_i;\theta) = \prod_{i=1}^n \dfrac{1}{\pi(1+(x_i-\theta)^2)}\mathbb{1}_{\mathbb{R}}(x_i)$

$= \pi^{-n}\prod_{i=1}^n \dfrac{1}{1+(x_i-\theta)^2}$

$\cdot\ \ln L(\theta;x) = -n\ln(\pi) + \sum_{i=1}^n \ln\left(\dfrac{1}{1+(x_i-\theta)^2}\right) = -n\ln(\pi) - \sum_{i=1}^n \ln(1+(x_i-\theta)^2)$

(4) • $L(\theta;x) = \prod_{i=1}^{n} \mathbb{P}(X_i = x_i;\theta) = \prod_{i=1}^{n} (1-\theta)^{x_i-1}\,\theta = \theta^n \prod_{i=1}^{n} \frac{(1-\theta)^{x_i}}{(1-\theta)}$

$= \left(\frac{\theta}{1-\theta}\right)^n (1-\theta)^{\sum_{i=1}^{n} x_i}$

• $\ln L(\theta;x) = n\ln(\theta) - n\ln(1-\theta) + \left(\sum_{i=1}^{n} x_i\right)\ln(1-\theta)$

$= n\ln(\theta) + \left(\sum_{i=1}^{n} x_i - n\right)\ln(1-\theta).$

Exercise 2 : on the computer...

Exercise 3 Compute the maximum likelihood estimator from a n-sample

Q: in the following cases:

(1) $f(x;\theta) = \theta x^{\theta-1}\mathbb{1}_{\{0<x<1\}}$, with $\theta>0$.

(2) $f(x;\theta) = \theta^2 x e^{-\theta x}\mathbb{1}_{\{x>0\}}$, with $\theta>0$.

(3) $f(x;\theta) = (\theta+1)x^{-\theta-2}\mathbb{1}_{\{x>1\}}$, with $\theta>0$.

A: (1) $\ell(\theta;x) = \sum_{i=1}^{n}\ln\left(f_{X_i}(x_i;\theta)\right) = \sum_{i=1}^{n}\ln\left(\theta x_i^{\theta-1}\right) = \sum_{i=1}^{n}\left[\ln(\theta) + (\theta-1)\ln(x_i)\right]$

log-likelihood

$= n\ln(\theta) + (\theta-1)\sum_{i=1}^{n}\ln(x_i)$

$\hat\theta$ is such that $\left.\frac{\partial \ell(\theta;x)}{\partial\theta}\right|_{\theta=\hat\theta} = 0 \iff \frac{n}{\hat\theta} + \sum_{i=1}^{n}\ln(x_i) = 0 \Rightarrow \hat\theta = -\frac{n}{\sum_{i=1}^{n}\ln(x_i)}$

(2) $\ell(\theta;x) = \sum_{i=1}^{n}\ln\left(\theta^2 x_i e^{-\theta x_i}\right) = \sum_{i=1}^{n}\left[2\ln(\theta) + \ln(x_i) - \theta x_i\right]$

$= 2n\ln(\theta) + \sum_{i=1}^{n}\ln(x_i) - \theta\sum_{i=1}^{n} x_i$

Then $\frac{2n}{\hat\theta} - \sum_{i=1}^{n} x_i = 0 \iff \hat\theta = \frac{2n}{\sum_{i=1}^{n} x_i}$

(3) $\ell(\theta; x) = \sum_{i=1}^{n} \ln\left((\theta+1)\, x_i^{-\theta-2}\right) = \sum_{i=1}^{n} \left[\ln(\theta+1) - (\theta+2)\ln(x_i)\right]$

$= n \ln(\theta+1) - (\theta+2) \sum_{i=1}^{n} \ln(x_i)$

Then $\frac{n}{\hat\theta+1} - \sum_{i=1}^{n} \ln(X_i) = 0 \iff \frac{n}{\hat\theta+1} = \sum_{i=1}^{n} \ln(X_i) \iff \hat\theta+1 = \frac{n}{\sum_{i=1}^{n} \ln(X_i)}$

Hence $\hat\theta = \frac{n}{\sum_{i=1}^{n} \ln(X_i)} - 1$

Exercice 4 Compute the Fisher information when dealing with a

Q: n-sample $X = (X_1, \dots, X_n)$, where (1) $X_i \sim \mathcal{B}(\pi)$

(2) $X_i \sim \mathcal{N}(\mu, \sigma^2)$

A: (1) We have a n-sample, i.e. $(X_i)_{1\le i\le n}$ are iid realizations, and because they are independent r.v. we know that $I_n(\pi) = n I_1(\pi)$.

Moreover, by definition, $I_n(\pi) = \mathrm{Var}\left(\frac{\partial}{\partial\pi} \ln L(\pi; (X_1, \dots, X_n))\right)$

Sometimes, it is simpler to use:

$$I_n(\pi) = -\mathbb{E}\left[\frac{\partial^2}{\partial\pi^2} \ln L(\pi; (X_1, \dots, X_n))\right]$$

Here, we have:

$$\ell(\pi; X_1) = \ln(L(\pi; X_1)) = \ln\left(\pi^{X_1}(1-\pi)^{1-X_1}\right) = X_1 \ln(\pi) + (1-X_1)\ln(1-\pi)$$

therefore $I_n(\pi) = n I_1(\pi) = n \,\mathrm{Var}\left(\frac{\partial}{\partial\pi} \ell(\pi; X_1)\right) = n\, \mathrm{Var}\left(\frac{X_1}{\pi} - \frac{1-X_1}{1-\pi}\right)$

$$= n\left[\frac{1}{\pi^2}\mathrm{Var}(X_1) + \frac{1}{(1-\pi)^2}\mathrm{Var}(X_1)\right] = n\left(\frac{\cancel{\pi}(1-\pi)}{\pi^{\cancel{2}}} + \frac{\pi(\cancel{1-\pi})}{(1-\pi)^{\cancel{2}}}\right)$$

$$= n\left(\frac{1-\pi}{\pi} + \frac{\pi}{1-\pi}\right) = n\left(\frac{(1-\pi)^2 + \pi^2}{\pi(1-\pi)}\right) = n\left(\frac{1-2\pi+\pi^2+\pi^2}{\pi(1-\pi)}\right) = \frac{n}{\pi(1-\pi)}$$

(2) : $X_i \sim \mathcal{N}(\mu, \sigma^2)$.

Here θ is a two-dimensional vector of parameters. The definition thus has to be adapted, considering partial derivatives. Taking

$$I_1(\theta) = I_1((\mu,\sigma^2)) = -E\left[\begin{pmatrix} \frac{\partial^2}{\partial\mu^2}\ln L(\theta;X_1) & \frac{\partial^2}{\partial\mu\partial\sigma^2}\ln L(\theta;X_1) \\ \frac{\partial^2}{\partial\sigma^2\partial\mu}\ln L(\theta;X_1) & \frac{\partial^2}{(\partial\sigma^2)^2}\ln L(\theta;X_1)\end{pmatrix}\right]$$

- $$\ell(\theta;X_1) = \ln L((\mu,\sigma^2);X_1)$$
$$= \ln f(X_1;(\mu,\sigma^2))$$
$$= \ln\left(\frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(X_1-\mu)^2}{2\sigma^2}}\right) = -\frac{1}{2}\ln(2\pi) - \frac{1}{2}\ln(\sigma^2) - \frac{(X_1-\mu)^2}{2\sigma^2}$$

- $$\frac{\partial}{\partial\mu}\ell(\theta;X_1) = \cancel{-}\cancel{(-1)}\cancel{2}(X_1-\mu)\frac{1}{\cancel{2}\sigma^2} = \frac{X_1-\mu}{\sigma^2}$$

- $$\frac{\partial^2}{\partial\mu\partial\sigma^2}\ell(\theta;X_1) = (X_1-\mu)\times -(\sigma^2)^{-2} = -\frac{X_1-\mu}{(\sigma^2)^2} = \frac{\mu-X_1}{\sigma^4} \Rightarrow -E\left[\frac{\mu-X_1}{\sigma^4}\right] = 0$$

- $$\frac{\partial^2}{\partial\mu^2}\ell(\theta;X_1) = -\frac{1}{\sigma^2} \Rightarrow -E\left[-\frac{1}{\sigma^2}\right] = \frac{1}{\sigma^2}$$

- $$\frac{\partial}{\partial\sigma^2}\ell(\theta;X_1) = -\frac{1}{2\sigma^2} - \frac{(X_1-\mu)^2}{2}\times(-1)(\sigma^2)^{-2} = -\frac{1}{2\sigma^2} + \frac{(X_1-\mu)^2}{2\sigma^4}$$

- $$\frac{\partial^2}{(\partial\sigma^2)^2}\ell(\theta;X_1) = -\frac{1}{2}(-1)(\sigma^2)^{-2} + \frac{(X_1-\mu)^2}{\cancel{2}}(\cancel{-2})(\sigma^2)^{-3}$$
$$= \frac{1}{2\sigma^4} - \frac{(X_1-\mu)^2}{\sigma^6} \Rightarrow -E\left[\frac{1}{2\sigma^4} - \frac{(X_1-\mu)^2}{\sigma^6}\right] = \frac{-1}{2\sigma^4} + \frac{1}{\sigma^6}\underbrace{E[(X_1-\mu)^2]}_{Var(X_1)}$$

$$= -\frac{1}{2\sigma^4} + \frac{1}{\sigma^6}\sigma^2 = -\frac{1}{2\sigma^4} + \frac{1}{\sigma^4} = \frac{-1+2}{2\sigma^4} = \frac{1}{2\sigma^4}$$

We then get the result: $I_n(\mu,\sigma^2) = \begin{pmatrix} \frac{n}{\sigma^2} & 0 \\ 0 & \frac{n}{2\sigma^4}\end{pmatrix}$

Exercice 5 Let $x = (x_1, \dots, x_n)$ a n-sample of iid r.v. such that ③

Q: $X_i \sim \mathcal{B}(\pi)$. Show that $S = X_1 + X_2 + \dots + X_n$ is an exhaustive statistic. Is it also minimal?

A: • $X_i \sim \mathcal{B}(\pi)$, $X_i \in \{0;1\}$, $\mathbb{P}(X_i = x_i) = \pi^{x_i}(1-\pi)^{1-x_i}$

$$L(\pi; x) = \prod_{i=1}^{n} \mathbb{P}(X_i = x_i) = \prod_{i=1}^{n} \pi^{x_i}(1-\pi)^{1-x_i} = \pi^{\sum_{i=1}^{n} x_i}(1-\pi)^{n-\sum_{i=1}^{n} x_i}$$

We can conclude from the factorization theorem by Fisher-Neyman; since there exist two measurable functions g and h such that: $\forall x \in \mathcal{X}$, $\forall \pi \in \Pi$,

$$L(\pi; x) = \underbrace{\left(\frac{\pi}{1-\pi}\right)^{\sum x_i}(1-\pi)^n}_{g(\sum x_i; \pi)} \times \underbrace{1}_{h(x)} \Rightarrow t(x) = \sum_{i=1}^{n} X_i \text{ is an exhaustive statistic for } \pi.$$

this statistic is also minimal, Indeed, we need at least this information to be able to compute the likelihood.

• Same question with $X_i \sim \mathcal{N}(\theta, 1)$:

$$L(\theta; x) = \prod_{i=1}^{n} f_{X_i}(x_i; \theta) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}} e^{-\frac{(x_i-\theta)^2}{2}} = \left(\frac{1}{\sqrt{2\pi}}\right)^n e^{-\sum_{i=1}^{n}\frac{(x_i-\theta)^2}{2}}$$

$$= (2\pi)^{-\frac{n}{2}} e^{-\sum_{i=1}^{n} \frac{x_i^2 - 2\theta x_i + \theta^2}{2}} = (2\pi)^{-\frac{n}{2}} \underbrace{e^{-\frac{1}{2}\sum_{i=1}^{n} x_i^2}}_{h(x)} \underbrace{e^{\theta \sum_{i=1}^{n} x_i} e^{-\frac{n}{2}\theta^2}}_{g(\sum x_i, \theta)}$$

Therefore, $\sum_{i=1}^{n} X_i = T(X)$ is a sufficient (exhaustive) statistic for the estimation of θ. It is also minimal.

Exercice 6 Let $X = (X_1, \dots, X_n)$ a n-sized sample, where $X_i \sim \mathcal{U}([0;\theta])$.

Denote by $\begin{cases} \hat{\theta}_n = \max(X_1, \dots, X_n) \\ \tilde{\theta}_n = 2\bar{X}_n = \frac{2}{n}\sum_{i=1}^{n} X_i \end{cases}$

1) Show that $\tilde{\theta}_n$ is an unbiased and consistent estimator of θ, and that $\sqrt{n}(\tilde{\theta}_n - \theta) \underset{n\to+\infty}{\overset{\mathcal{L}}{\sim}} \mathcal{N}(0, \theta^2/3)$.

A: • $\mathbb{E}[\tilde{\theta}_n] = \mathbb{E}[2\bar{X}_n] = 2\,\mathbb{E}[\bar{X}_n] = 2\,\mathbb{E}\left[\frac{1}{n}\sum_{i=1}^{n} X_i\right] = \frac{2}{n}\sum_{i=1}^{n}\mathbb{E}[X_i]$

$= \frac{2\cancel{n}}{\cancel{n}}\mathbb{E}[X_i] = 2\,\mathbb{E}[X_i] = 2\left(\frac{b-a}{2}\right)$ with $\begin{pmatrix} a=0 \\ b=\theta \end{pmatrix}$

$= 2\,\frac{\theta}{2} = \theta \Rightarrow$ unbiased.

• $\tilde{\theta}_n$ is unbiased: we now check the variance to study the quadratic risk:

$\mathrm{Var}(\tilde{\theta}_n) = \mathrm{Var}(2\bar{X}_n) = 4\,\mathrm{Var}\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right) = \frac{4}{n^2}\mathrm{Var}(\Sigma X_i) = \frac{4}{n}\mathrm{Var}(X_i)$

$= \frac{4}{n}\frac{(b-a)^2}{12} = \frac{4}{12n}\theta^2 \underset{n\to+\infty}{\longrightarrow} 0$

then $R(\tilde{\theta}_n) = \mathbb{E}[(\tilde{\theta}_n - \theta)^2] = \mathrm{Var}(\tilde{\theta}_n) \underset{n\to+\infty}{\longrightarrow} 0 \Rightarrow \tilde{\theta}_n \underset{n\to+\infty}{\overset{MQ}{\longrightarrow}} \theta$

And therefore $\tilde{\theta}_n \underset{n\to+\infty}{\overset{\mathbb{P}}{\longrightarrow}} \theta$. $\tilde{\theta}_n$ is a consistent estimator of θ.

• X_i are iid, with finite mean m_i and variance σ_i^2: from the CLT,

$\sum_{i=1}^{n} X_i \underset{n\to+\infty}{\overset{\mathcal{L}}{\sim}} \mathcal{N}(n m_i, n\sigma_i^2) \Rightarrow \sum_{i=1}^{n} X_i \sim \mathcal{N}\left(n\frac{\theta}{2}, n\frac{\theta^2}{12}\right)$

$\Rightarrow \underbrace{\frac{2}{n}\Sigma X_i}_{\tilde{\theta}_n} \sim \mathcal{N}\left(\theta, \frac{\theta^2}{3n}\right) \Rightarrow \sqrt{n}(\tilde{\theta}_n - \theta) \sim \mathcal{N}\left(0, \frac{\theta^2}{3}\right).$

(2) Build an asymptotic confidence interval on θ, based on $\tilde{\theta}_n$, with security coefficient $1-\alpha$.

(4)

We thus look for bounds L and U such that:

$$\mathbb{P}(L \leq \theta \leq U) = 1-\alpha$$

$$\mathbb{P}(\hat{\theta}_n - U \leq \tilde{\theta}_n - \theta \leq \hat{\theta}_n - L) = 1-\alpha$$

$$\mathbb{P}(\sqrt{n}(\tilde{\theta}_n - U) \leq \sqrt{n}(\tilde{\theta}_n - \theta) \leq \sqrt{n}(\tilde{\theta}_n - L)) = 1-\alpha$$

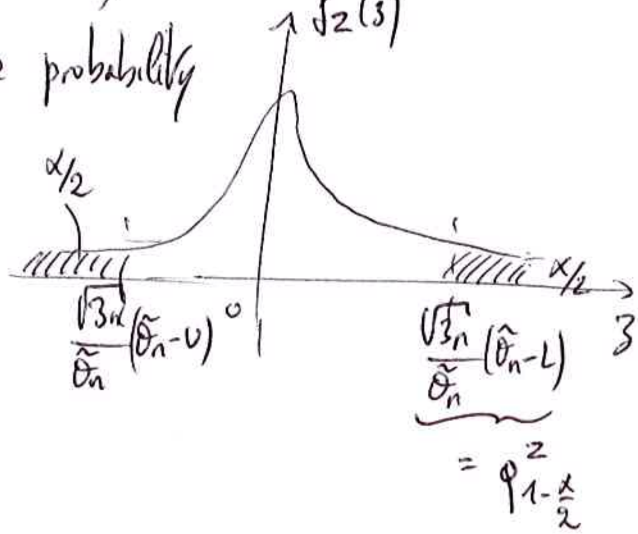
$$\mathbb{P}\left(\frac{\sqrt{n}(\tilde{\theta}_n - U)}{\sqrt{\frac{\theta^2}{3}}} \leq \underbrace{\frac{\sqrt{n}(\tilde{\theta}_n - \theta)}{\sqrt{\frac{\theta^2}{3}}}}_{Z \underset{n\to+\infty}{\sim} \mathcal{N}(0,1)} \leq \frac{\sqrt{n}(\tilde{\theta}_n - L)}{\sqrt{\frac{\theta^2}{3}}} \right) = 1-\alpha$$

$$\mathbb{P}\left(\frac{\sqrt{3n}}{\theta}(\tilde{\theta}_n - U) \leq Z \leq \frac{\sqrt{3n}}{\theta}(\hat{\theta}_n - L) \right) = 1-\alpha$$

We know that $\tilde{\theta}_n \xrightarrow[n\to+\infty]{\mathbb{P}} \theta$, thus we use the Slutsky theorem:

$$F_Z\left(\frac{\sqrt{3n}}{\tilde{\theta}_n}(\hat{\theta}_n - L) \right) - F_Z\left(\frac{\sqrt{3n}}{\tilde{\theta}_n}(\hat{\theta}_n - U) \right) = 1-\alpha$$

Z has a symetric density $\Rightarrow$ we can share probability α between two different zones with same probability $\frac{\alpha}{2} \Rightarrow L = -U$



We thus have

$$\underbrace{q^Z_{0,975}}_{=1,96} = q^Z_{1-\frac{\alpha}{2}} = \frac{\sqrt{3n}}{\tilde{\theta}_n}(\hat{\theta}_n - L)$$

Therefore $\tilde{\theta}_n \, q^{Z}_{1-\frac{\alpha}{2}} = \sqrt{3n}\,(\tilde{\theta}_n - L) \Leftrightarrow \frac{\hat{\theta}_n}{\sqrt{3n}} q^{Z}_{1-\frac{\alpha}{2}} = \tilde{\theta}_n - L$

Finally,
$$\begin{cases} L = \tilde{\theta}_n - \frac{\hat{\theta}_n}{\sqrt{3n}} q^{Z}_{1-\frac{\alpha}{2}} \\ U = \tilde{\theta}_n + \frac{\hat{\theta}_n}{\sqrt{3n}} q^{Z}_{1-\frac{\alpha}{2}} \end{cases} \Rightarrow IC_{1-\alpha}(\theta) = \left[\tilde{\theta}_n \pm \frac{\hat{\theta}_n}{\sqrt{3n}} q^{Z}_{1-\frac{\alpha}{2}} \right]$$

(3) Show that $\frac{\hat{\theta}_n}{\theta}$ is a pivotal function to estimate θ: (remind that a pivotal function has a distribution which does not depend on θ).

A: We have $\hat{\theta}_n = \max(X_1, \dots, X_n)$, when $X_i \overset{iid}{\sim} \mathcal{U}([0,\theta])$.

This means that $\mathbb{P}\left(\frac{\hat{\theta}_n}{\theta} \le x\right) = \mathbb{P}\left(\frac{\max(X_1,\dots,X_n)}{\theta} \le x\right) = \mathbb{P}\left(\frac{X_1}{\theta} \le x, \dots, \frac{X_n}{\theta} \le x\right)$

$= \mathbb{P}\left(\frac{X_1}{\theta} \le x, \dots, \frac{X_n}{\theta} \le x\right)$, where $\frac{X_i}{\theta} \sim \mathcal{U}([0,1])$

$\left(\begin{array}{l} X \sim \mathcal{U}([0,\theta]) \\ Y = \frac{X}{\theta} \sim ? \\ \mathbb{P}(Y \le y) = \mathbb{P}\left(\frac{X}{\theta} \le y\right) \\ = \mathbb{P}(X \le \theta y) = \int_0^{\theta y} \frac{1}{\theta} \\ = y \text{ if } y \le 1 \\ \Rightarrow Y \sim \mathcal{U}([0,1]) \end{array} \right.$

Hence $\mathbb{P}\left(\frac{\hat{\theta}_n}{\theta} \le x\right) \overset{\perp\!\!\!\perp}{=} \mathbb{P}\left(\frac{X_1}{\theta} \le x\right) \mathbb{P}\left(\frac{X_2}{\theta} \le x\right) \dots \mathbb{P}\left(\frac{X_n}{\theta} \le x\right)$

$= \mathbb{P}(U_1 \le x) \dots \mathbb{P}(U_n \le x) = x^n$ if $0 \le x \le 1$.

Finally, $F_{\frac{\hat{\theta}_n}{\theta}}(x) = \mathbb{P}\left(\frac{\hat{\theta}_n}{\theta} \le x\right) = \begin{cases} 0 & \text{if } x < 0 \\ x^n & \text{if } 0 \le x \le 1 \\ 1 & \text{if } x > 1 \end{cases} \Rightarrow$ this distribution does not depend on θ.

(4) Build a (non-asymptotic) confidence interval of θ, with security coefficient $1-\alpha$, based on $\hat{\theta}_n$.

We look for bounds U and L such that:

$$\mathbb{P}(L \le \theta \le U) = 1 - \alpha.$$

We know that $\mathbb{P}(\theta > 0) = 1$.

We thus get $\mathbb{P}(0 < \theta \leq U) = 1-\alpha \Leftrightarrow \mathbb{P}(\theta \leq U) - \underbrace{\mathbb{P}(\theta < 0)}_{=0} = 1-\alpha$

And then $\mathbb{P}\left(\frac{1}{\theta} \geq \frac{1}{U}\right) = 1-\alpha \Leftrightarrow 1 - \mathbb{P}\left(\frac{\hat{\theta}_n}{\theta} \leq \frac{\hat{\theta}_n}{U}\right) = 1-\alpha$

Therefore $\left(\frac{\hat{\theta}_n}{U}\right)^n = \alpha \Leftrightarrow \frac{\hat{\theta}_n}{U} = \alpha^{1/n} \Leftrightarrow U = \frac{\hat{\theta}_n}{\sqrt[n]{\alpha}}$

We thus have $IC_{1-\alpha}(\theta) = \left[0, \frac{\hat{\theta}_n}{\alpha^{1/n}}\right]$.

(5) Compare the two obtained intervals:

$$IC_1 = \left[\tilde{\theta}_n - \frac{\tilde{\theta}_n}{\sqrt{3n}} q^{z}_{1-\frac{\alpha}{2}}, \tilde{\theta}_n + \frac{\tilde{\theta}_n}{\sqrt{3n}} q^{z}_{1-\frac{\alpha}{2}}\right]$$

$$IC_2 = \left[0, \frac{\hat{\theta}_n}{\alpha^{1/n}}\right].$$

Exercice 7 Let $\theta \in]0, 1[$, and X be a discrete random variable with mass function: $\mathbb{P}_\theta(X=k) = (k+1)(1-\theta)^2\theta^k$, $k \in \mathbb{N}$.

We have $\mathbb{E}[X] = \frac{2\theta}{1-\theta}$ and $Var(X) = \frac{2\theta}{(1-\theta)^2}$.

The goal is to estimate θ from the sample of iid r.v. $X = (X_1, .., X_n)$.

1. Give $\hat{\theta}_n$ the moment estimator of θ.
2. Is $\tilde{\theta}_n$ the maximum likelihood estimator well defined?
3. Study the consistency of $\hat{\theta}_n$, and give its limiting distribution.

A: ① - By the SLLN, we have $\overline{X_n} \xrightarrow[n\to+\infty]{p.s.} \mathbb{E}[X]$.

Here we have $\mathbb{E}[X] = \dfrac{2\theta}{1-\theta} \iff 2\theta = (1-\theta)\,\mathbb{E}[X]$

$\iff \theta\left(2+\mathbb{E}[X]\right) = \mathbb{E}[X] \iff \theta = \dfrac{\mathbb{E}[X]}{2+\mathbb{E}[X]}$.

We thus get $\hat{\theta}_n = \dfrac{\overline{X_n}}{2+\overline{X_n}}$.

② - $\ell(\theta;x) = \ln L(\theta;x) = \sum_{i=1}^{n} \ln \mathbb{P}_\theta(X_i) = \sum_{i=1}^{n} \ln\left((X_i+1)(1-\theta)^2\theta^{X_i}\right)$

$= 2n\ln(1-\theta) + n\overline{X_n}\ln(\theta) + \sum_{i=1}^{n}\ln(X_i+1)$

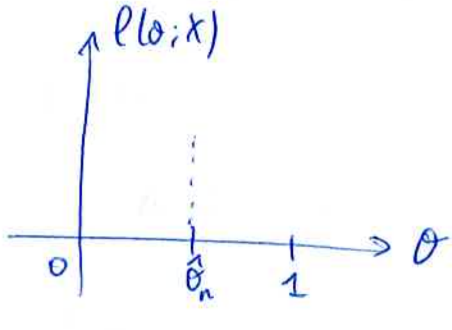
$\Rightarrow \dfrac{\partial \ell(\theta;x)}{\partial\theta} = -\dfrac{2n}{1-\theta} + n\dfrac{\overline{X_n}}{\theta}$.

- this function is positive? $-\dfrac{2n}{1-\theta} + n\dfrac{\overline{X_n}}{\theta} \geq 0 \iff n\dfrac{\overline{X_n}}{\theta} \geq \dfrac{2n}{1-\theta}$

$\iff \dfrac{1-\theta}{\theta} \geq \dfrac{2n}{n\overline{X_n}} \iff \dfrac{1}{\theta} - 1 \geq \dfrac{2n}{n\overline{X_n}} \iff \dfrac{1}{\theta} \geq 1 + \dfrac{2n}{n\overline{X_n}}$

$\iff \theta \leq \left(1+\dfrac{2n}{n\overline{X_n}}\right)^{-1} = \left(\dfrac{\cancel{n}\overline{X_n}+2\cancel{n}}{\cancel{n}\overline{X_n}}\right)^{-1} = \dfrac{\overline{X_n}}{2+\overline{X_n}} = \hat{\theta}_n$ - (MME) Method Moment Estimator

Hence $\ell'(\theta;x) \geq 0 \iff \theta \leq \hat{\theta}_n = \text{MME}$



- If $\hat{\theta}_n > 0$ (i.e. $\overline{X_n} > 0$), then the maximum is located in $\hat{\theta}_n$ on $]0,1[$.
  Then $\tilde{\theta}_n = \hat{\theta}_n$.
- If $\hat{\theta}_n = 0$ (i.e. $\overline{X_n} = 0$), then the likelihood is strictly decreasing on $]0,1[$ and the MLE ~~is~~ ~~not~~ does not exist.

3. We know that $\overline{X}_n \xrightarrow[n\to+\infty]{p.s.} \mathbb{E}[X_i] = \mathbb{E}[X]$.

By the continuity theorem, we thus have $\hat{\theta}_n \xrightarrow[n\to+\infty]{p.s.} \theta$.

Then $\hat{\theta}_n$ is consistent. Moreover, provided that $\begin{pmatrix} Var(X_i) < +\infty \\ \mathbb{E}[X_i] < +\infty \\ X_i \text{ iid.} \end{pmatrix}$, we have

from CLT: $\sum_{i=1}^{n} X_i \underset{n\to+\infty}{\overset{\mathcal{L}}{\sim}} \mathcal{N}\left(\sum_{i=1}^{n} \mathbb{E}[X_i], \sum_{i=1}^{n} Var(X_i)\right)$.

$$\overline{X}_n \sim \mathcal{N}\left(\mathbb{E}[X_i], \frac{Var(X_i)}{n}\right) \Leftrightarrow \overline{X}_n \underset{n\to+\infty}{\sim} \mathcal{N}\left(\underbrace{\frac{2\theta}{1-\theta}}_{=\theta'}, \frac{2\theta}{n(1-\theta)^2}\right)$$

$$\sqrt{n}\left(\overline{X}_n - \overset{\theta'}{\frac{2\theta}{1-\theta}}\right) \sim \mathcal{N}\left(0, \frac{2\theta}{(1-\theta)^2}\right)$$, then we use the Delta method

with function $g(x) = \frac{x}{2+x} = \frac{2+x-2}{2+x} = 1 - \frac{2}{2+x}$, at point $\theta' = \frac{2\theta}{1-\theta}$.

This function has a derivative in $x = \frac{2\theta}{(1-\theta)}$, with derivative given by:

$$g'(x) = \frac{2}{(2+x)^2}, \text{ thus } g'(2\theta/1-\theta) = \frac{2}{(2+(2\theta/1-\theta))^2} = \frac{(1-\theta)^2}{2}$$

Finally, we get $\sqrt{n}\left(\hat{\theta}_n - \theta\right) \underset{n\to+\infty}{\overset{\mathcal{L}}{\sim}} \mathcal{N}\left(0, \frac{2\theta}{(1-\theta)^2} \times \left(\frac{(1-\theta)^2}{2}\right)^2\right) = \mathcal{N}\left(0, \frac{\theta(1-\theta)^2}{2}\right)$.

Exercice 8 Let $X = (X_1, \dots, X_{25})$ an iid. sample of random variables with $X_i \sim \mathcal{N}(\mu, \sigma^2 = 81)$. We know that $\overline{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$ gave a realization $\overline{x}_n = 81{,}2$.

Find the confidence interval with confidence probability 95% for the mean μ.

This is the case of Gaussian random variable with known variance.

Hence, $\overline{X}_n \sim \mathcal{N}\left(\mu, \frac{\sigma^2}{n}\right) \Leftrightarrow \sqrt{n}\,\frac{\overline{X}_n-\mu}{\sigma} \sim \mathcal{N}(0,1)$

(Remember this is an exact confidence interval, not an asymptotic one!).

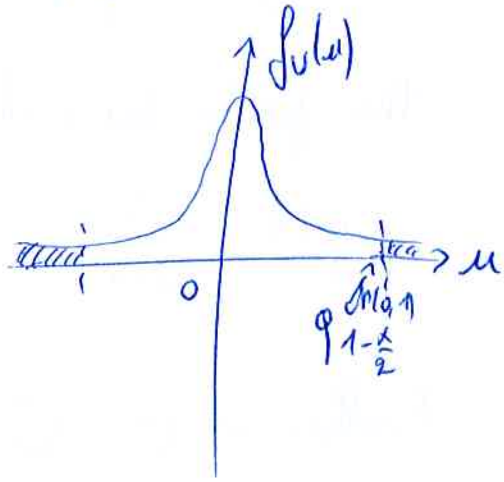
Therefore $\sqrt{n}\,\frac{\overline{X}_n-\mu}{\sigma}$ is a pivot (a function of the observations X_i and the parameter of interest μ, whose distribution is independent from μ).

Thus, we look for bounds Z_1 and Z_2 such that

$$\mathbb{P}(Z_1 \leq \mu \leq Z_2) = 1-\alpha.$$

But we know that $U = \sqrt{n}\,\frac{\overline{X}_n-\mu}{\sigma} \sim \mathcal{N}(0,1)$, thus

$$\mathbb{P}\left(|U| \leq q_{1-\frac{\alpha}{2}}^{\mathcal{N}(0,1)}\right) = 1-\alpha$$



$$\Leftrightarrow \mathbb{P}\left(\sqrt{n}\,\frac{|\overline{X}_n-\mu|}{\sigma} \leq q_{1-\frac{\alpha}{2}}^{\mathcal{N}(0,1)}\right) = 1-\alpha$$

$$\Leftrightarrow \mathbb{P}\left(|\overline{X}_n-\mu| \leq \frac{\sigma}{\sqrt{n}}\, q_{1-\frac{\alpha}{2}}^{\mathcal{N}(0,1)}\right) = 1-\alpha$$

$$\Leftrightarrow \mathbb{P}\left(\underbrace{\overline{X}_n - \frac{\sigma}{\sqrt{n}}\, q_{1-\frac{\alpha}{2}}^{\mathcal{N}(0,1)}}_{Z_1} \leq \mu \leq \underbrace{\overline{X}_n + \frac{\sigma}{\sqrt{n}}\, q_{1-\frac{\alpha}{2}}^{\mathcal{N}(0,1)}}_{Z_2}\right) = 1-\alpha$$

$$\Rightarrow IC_{1-\alpha}(\mu) = \left[\overline{X}_n - \frac{\sigma}{\sqrt{n}}\, q_{1-\frac{\alpha}{2}}^{\mathcal{N}(0,1)} ; \overline{X}_n + \frac{\sigma}{\sqrt{n}}\, q_{1-\frac{\alpha}{2}}^{\mathcal{N}(0,1)}\right].$$

<u>A.N</u>: $\left(q_{1-\frac{\alpha}{2}}^{\mathcal{N}(0,1)} = q_{97,5\%}^{\mathcal{N}(0,1)} = 1,96 \right.$; $n = 25$; $\sigma = 9$; $\overline{x}_n = 81,2$ $\Rightarrow IC_{1-\alpha}(\mu) \simeq \left[81,2 - \frac{9}{5}\times 2, 81,2 + \frac{9}{5}\times 2\right]$

Exercice 9 Let $X = (X_1, .., X_n)$ a n-sample of iid random variables, where $X_i \sim \mathcal{N}(\mu, \sigma^2 = 16)$. Find the minimal number of observations such that $[\bar{X}_n - 1; \bar{X}_n + 1]$ would be the 90% confidence interval of μ.

A: We are exactly int the same framework as in Exercice 7. Thus,

$$\left[\bar{X}_n - \frac{\sigma}{\sqrt{n}} q_{1-\frac{\alpha}{2}}^{\mathcal{N}(0,1)} ; \bar{X}_n + \frac{\sigma}{\sqrt{n}} q_{1-\frac{\alpha}{2}}^{\mathcal{N}(0,1)}\right] \equiv \left[\bar{X}_n - 1 ; \bar{X}_n + 1\right].$$

Therefore, we can identify the bounds; which yields:

$$\frac{\sigma}{\sqrt{n}} q_{1-\frac{\alpha}{2}}^{\mathcal{N}(0,1)} = 1 \iff \sqrt{n} = \sigma\, q_{1-\frac{\alpha}{2}}^{\mathcal{N}(0,1)} \iff n = \left(\sigma\, q_{1-\frac{\alpha}{2}}^{\mathcal{N}(0,1)}\right)^2$$

A.N: $\left.\begin{array}{l} \alpha = 10\% \\ \sigma = 4 \\ q_{95\%}^{\mathcal{N}(0,1)} = 1,64 \end{array}\right\} \Rightarrow n = (4 \times 1,64)^2 = 6,56^2 \simeq 43,..$

We thus require $n \geqslant 44$.