Aix–Marseille Université
Faculté des Sciences
Master M1 MAS
Prof. Kai Schneider

Faculté
des Sciences
Aix★Marseille Université

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Homework: Integrated project, FFT, theory and implementation

Exercise 1: Properties of complex exponential sequences

We consider a complex exponential sequence of the form

$$x_n = e^{i\omega_0 n}, \quad n \in \mathbb{Z}$$

where $i = \sqrt{-1}$.

a) Show that if $\alpha = \omega_0/(2\pi)$ is a rational number, $\alpha = p/q$, with p and q coprime integers, then x is periodic with period q.

b) Show that if $\alpha = \omega_0/(2\pi)$ is irrational, then x is not periodic.

c) Show that if x and y are two periodic sequences with periods M and N, respectively, then $x + y$ is periodic with period $lcm(M, N)$.

N.B.: p and q are coprime integers if the only positive integer that divides both of them is 1. Consequently, any prime number that divides one does not divide the other. This is equivalent to their greatest common divisor being 1.
A number L is called a common multiple of M and N if both M and N divide L. The smallest such L is called the least common multiple of M and N and is denoted by $lcm(M, N)$, e.g., $lcm(8, 12) = 24$.

Exercise 2: Discrete Fourier transform (DFT)

We consider the linear operator $F : \mathbb{C}^N \to \mathbb{C}^N$ defined by

$$X_k = (Fx)_k = \sum_{n=0}^{N-1} x_n W_N^{kn}$$

for $k = 0, 1, ..., N-1$ and $x \in \mathbb{C}^N$, where $W_N^{kn} = e^{-i(2\pi/N)kn}$ and $i = \sqrt{-1}$. The vector $X_k = (Fx)_k \in C^N$ is called the discrete Fourier transform (DFT) of x.

The inverse discrete Fourier transform (IDFT) is defined by

$$x_n = (F^{-1}X)_n = \frac{1}{N}\sum_{k=0}^{N-1} X_k W_N^{-kn}$$

for $n = 0, 1, ..., N-1$ and $X \in \mathbb{C}^N$

a) Write the Fourier transform and its inverse as a matrix-vector product ($X = Fx$ and $x = F^{-1}X$).

b) Show that $F^{-1} = \frac{1}{N}F^\star$

c) Verify that the DFT and its inverse are indeed linear operators.

d) Show that X_k computed by DFT is N-periodic, i.e. $X_k = X_{k+\ell N}$ for $\ell \in Z$ and that x_n computed via the IDFT is likewise N-periodic.

e) What is the computational complexity (number of multiplications) of the DFT (and its inverse)?

Exercise 3:

We consider the signal $s(t) = \exp(-(t - t_0)^2/\sigma^2)$ and the discrete signal $s_n = s(t_n)$ with $t_n = n/N$ for $n = 0, ..., N - 1$ and with $t_0 = 0.5$, $\sigma^2 = 1/500$ and $N = 2^{10} = 1024$.

- Plot the discrete signal s_n.

- Compute its discrete Fourier transform

$$\widehat{s}_k = \frac{1}{N} \sum_{n=0}^{N-1} s_n \exp(-i2\pi kn/N) \quad , \quad k = 0, ..., N - 1$$

(using FFT)

- Plot the spectrum (modulus of the Fourier coefficients).
  $|\widehat{s}_k|$, $k = 0, ..., N - 1$.

Exercise 4:

Given a discrete Dirac pulse

$$d_n = \begin{cases} 1 & \text{for } n = mN/f_0 , \quad m \in \mathbb{N} \\ 0 & \text{else} \end{cases}$$

for $n = 0, ..., N - 1$ with $N = 2^{10} = 1024$, and frequency $f_0 = 2^6 = 64$.

- Plot the Dirac pulse.

- Compute its discrete Fourier transform.

- Plot the spectrum.

Exercise 5:

Multiply the signal s with the Dirac pulse and plot the resulting sampled signal $r_n = s_n d_n$

for $n = 0, ..., N - 1$.
- Plot the spectrum of r.

- What do you observe?

Exercise 6:

Reconstruct the signal s^{rec} from the sampled signal r by applying an ideal low pass filter in Fourier space with cut off frequency $k_c = f_0/2 = 32$, i.e.

$$\widehat{s}_k^{rec} = \begin{cases} \widehat{r}_k & \text{for } |k| \leq k_c \\ 0 & \text{else} \end{cases}$$

for $k = 0, ..., N - 1$.

Apply the inverse discrete Fourier transform (e.g. using fft(s^{rec}, inverse=TRUE) / length(s^{rec})) to get

$$s_n^{rec} = \sum_{k=0}^{N-1} \widehat{s}_k^{rec} \exp(i2\pi kn/N) \quad , \quad n = 0, ..., N - 1$$

and plot s^{rec}.

Exercise 7:

Repeat exercises 4-6 with different values $f_0 > 64$ and $f_0 < 64$. What do you observe?