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Weak convergence of probability measures of Yosida approximate mild solutions of neutral SPDEs

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ABSTRACT

In this paper, a neutral stochastic partial differential equation is studied in real separable Hilbert spaces. The aim here is to introduce Yosida approximations for this class of equations and to show the convergence of the mild solutions of the Yosida approximating systems in the pth-mean ($p \geq 2$) to the mild solutions of such equations. Moreover, in the special case when the neutral term has no delay, the weak convergence of probability measures induced by the mild solutions of the Yosida approximating system to the probability measures induced by mild solutions of the original equation is proved for the case $p = 2$.

1. Introduction

Existence, stability and approximation problems of stochastic partial differential equations (SPDEs) have been investigated by several authors, see Ahmed (1991), Da Prato and Zabczyk (1992a), Ichikawa (1982) and McKibben (2011), among others. SPDEs are well known to model stochastic processes observed in the study arising from many areas of science, engineering and finance.

In this paper, a neutral stochastic partial functional differential equation is considered in a real Hilbert space X of the form:

$$d[x(t) + f(t, x_t)] = [Ax(t) + a(t, x_t)]dt + b(t, x_t)dw(t), \quad t > 0; \tag{1.1}$$

$$x(t) = \varphi(t), \quad t \in [-r, 0]\ (0 \leq r < \infty); \tag{1.2}$$

where $x_t(s) = x(t+s)$, $-r \leq s \leq 0$, represents a finite history of x at t and $x(t)$ its value at time t, $a : R^+ \times X \to X (R^+ = [0, \infty))$, $b : R^+ \times X \to L(Y, X)$ and $f : R^+ \times X \to D(A^{-\alpha})$, $0 < \alpha \leq 1$, are Borel measurable, and $A : D(A) \subset X$ is the infinitesimal generator of a strongly continuous semigroup $\{S(t), t \geq 0\}$ defined on X. Here $w(t)$ is a Y-valued Q-Wiener process, and the past stochastic process $\{\varphi(t), t \in [-r, 0]\}$ has almost sure (a.s.) continuous sample paths with $E\|\varphi\|_C^p < \infty$, $p \geq 2$. When $f = 0$ this class of systems has been well studied, see Ichikawa (1982), Da Prato and Zabczyk (1992a) and McKibben (2011). For a motivation to this class of equations in the deterministic case, see Hernández and Henríquez (1998) and also Govindan (2005).

In the deterministic case, the classical Yosida approximate mild solutions have been well studied for evolution equations, see Pazy (1983). In the stochastic case, in other words, for stochastic evolution equations, Ichikawa (1982), Ahmed

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(1991), Ahmed and Ding (1995), Da Prato and Zabczyk (1992a) and McKibben (2011), to mention a few, considered Yosida approximations to solutions of such equations. Kannan and Bharucha-Reid (1985) and Govindan (1994) studied Yosida approximate solutions to stochastic integrodifferential equations and also the convergence of their corresponding probability measures. In this article, our goal is to introduce Yosida approximations for neutral stochastic partial functional differential equations (1.1), and to prove the convergence of mild solutions of this approximating system to the mild solutions of Eq. (1.1) in the pth-mean ($p \geq 2$). In the special case when the neutral term has no delay, more precisely, when $f(t, x_t) = f(t, x(t))$, it is interesting to consider the probability measure induced by the Yosida approximating system and to prove its weak convergence to the probability measure induced by the mild solution of equation (1.1) for the case $p = 2$.

The format of the rest of the paper is as follows: In Section 2, we give the preliminaries from Govindan (2005) and Taniguchi (1995) and a reference therein. In Section 3, we present the Yosida approximation results. Lastly, in Section 4, the weak convergence of the induced probability measures is shown.

2. Preliminaries

Let X, Y be real separable Hilbert spaces and $L(Y, X)$ be the space of bounded linear operators mapping Y into X. We shall use the same notation $|\cdot|$ to denote norms in X, Y and $L(Y, X)$. Let $\left(\Omega, B, P, \{B_t\}_{t\geq 0}\right)$ be a complete probability space with an increasing right continuous family $\{B_t\}_{t\geq 0}$ of complete sub-σ-algebras of B. Let $\beta_n(t)$ $(n = 1, 2, 3, \ldots)$ be a sequence of real-valued standard Brownian motions mutually independent defined on this probability space. Set $w(t) = \sum_{n=1}^{\infty} \sqrt{\lambda_n}\beta_n(t)e_n$, $t \geq 0$, where $\lambda_n \geq 0$ $(n = 1, 2, 3, \ldots)$ are nonnegative real numbers and $\{e_n\}$ $(n = 1, 2, 3, \ldots)$ is a complete orthonormal basis in Y. Let $Q \in L(Y, Y)$ be an operator defined by $Qe_n = \lambda_n e_n$. The above Y-valued stochastic process $w(t)$ is called a Q-Wiener process. Let $h(t)$ be an $L(Y, X)$-valued function and let λ be a sequence $\left\{\sqrt{\lambda_1}, \sqrt{\lambda_2}, \ldots\right\}$. Then we define $|h(t)|_\lambda = \left\{\sum_{n=1}^{\infty} |\sqrt{\lambda_n}h(t)e_n|^2\right\}^{1/2}$. If $|h(t)|_\lambda^2 < \infty$, then $h(t)$ is called λ-Hilbert–Schmidt operator.

Next, we defined the X-valued stochastic integral with respect to the Y-valued Q-Wiener process, see Taniguchi (1995).

Definition 2.1. Let $\Phi : [0, \infty) \to \sigma(\lambda)(Y, X)$ be a B_t adapted process. Then for any Φ satisfying $\int_0^t E|\Phi(s)|_\lambda^2 ds < \infty$, we define the X-valued stochastic integral $\int_0^t \Phi(s)dw(s) \in X$ with respect to $w(t)$ by

$$\left(\int_0^t \Phi(s)dw(s), h\right) = \int_0^t \langle \Phi^*(s)h, dw(s)\rangle, \quad h \in X, \tag{2.3}$$

where Φ^* is the adjoint operator of Φ.

A semigroup $\{S(t), t \geq 0\}$ is said to be exponentially stable if there exist positive constants M and a such that $\|S(t)\| \leq M\exp(-at)$, $t \geq 0$, where $\|.\|$ denotes the operator norm in X. If $M = 1$, the semigroup is said to be a contraction.

Let $C := C([-r, 0]; X)$ denote the space of continuous functions $\varphi : [-r, 0] \to X$ endowed with the norm $\|\varphi\|_C = \sup_{-r\leq s\leq 0} |\varphi(s)|$.

If $\{S(t), t \geq 0\}$ is an analytic semigroup with infinitesimal generator A such that $0 \in \rho(A)$ (the resolvent set of A), then it is possible to define the fractional power A^α, for $0 < \alpha \leq 1$, as a closed linear operator on its domain $D(A^\alpha)$. Further, the subspace $D(A^\alpha)$ is dense in X and the expression $\|x\|_\alpha = |A^\alpha x|$, $x \in D(A^\alpha)$, defines a norm on $D(A^\alpha)$.

Definition 2.2. A stochastic process $\{x(t), t \in [-r, T]\}(0 < T < \infty)$ is called a mild solution of equation (1.1) if

(i) $x(t)$ is B_t-adapted with $\int_0^T |x(t)|^2 dt < \infty$, a.s., and
(ii) $x(t)$ satisfies the integral equation

$$\begin{aligned} x(t) &= S(t)[\varphi(0) + f(0, \varphi)] - f(t, x_t) - \int_0^t AS(t-s)f(s, x_s)ds + \int_0^t S(t-s)a(s, x_s)ds \\ &\quad + \int_0^t S(t-s)b(s, x_s)dw(s), \quad \text{a.s., } t \in [0, T]. \end{aligned}$$

For convenience of the reader, we will state below some results that will be needed in the sequel.

Lemma 2.1 (Da Prato and Zabczyk, 1992b)**.** *Let $W_A^\Phi(t) = \int_0^t S(t-s)\Phi(s)dw(s)$, $t \in [0, T]$. For any arbitrary $p > 2$ there exists a constant $c(p, T) > 0$ such that for any $T \geq 0$ and a proper modification of the stochastic convolution W_A^Φ one has:*

$$E\sup_{t\leq T} |W_A^\Phi(t)|^p \leq c(p, T)\sup_{t\leq T} \|S(t)\|^p E\int_0^T |\Phi(s)|_\lambda^p ds.$$

Moreover if $E\int_0^T |\Phi(s)|_\lambda^p ds < \infty$ then there exists a continuous version of the process $\{W_A^\Phi, t \geq 0\}$.

Lemma 2.2 (*Da Prato and Zabczyk, 1992a, Theorem 6.10*). *Suppose A generates a contraction semigroup. Then the process $W_A^{\Phi}(.)$ has a continuous modification and there exists a constant $\kappa > 0$ such that*

$$E \sup_{s\in[0,t]} |W_A^{\Phi}(s)|^2 \leq \kappa E \int_0^t |\Phi(s)|_{\lambda}^2 ds, \quad t \in [0, T].$$

We impose the following hypothesis to consider the main results.

Hypothesis (H). Let the following assumptions hold a.s.:

(H1) A is the infinitesimal generator of an analytic semigroup of bounded linear operators $\{S(t), t \geq 0\}$ in X and that the semigroup is exponentially stable.

(H2) For $p \geq 2$, the functions $a(t, u)$ and $b(t, u)$ satisfy the Lipschitz and linear growth conditions:

$$|a(t, u) - a(t, v)|^p \leq C_1|u - v|^p, \quad C_1 > 0,$$
$$|b(t, u) - b(t, v)|_{\lambda}^p \leq C_2|u - v|^p, \quad C_2 > 0,$$

for all $u, v \in X$, and

$$|a(t, u)|^p + |b(t, u)|_{\lambda}^p \leq C_3(1 + |u|^p), \quad C_3 > 0,$$

for all $u \in X$.

(H3) $f(t, u)$ is a function continuous in t and satisfies:

$$|A^{\alpha}f(t, u) - A^{\alpha}f(t, v)| \leq C_4|u - v|, \quad C_4 > 0,$$

for all $u, v \in X$, and

$$|A^{\alpha}f(t, u)| \leq C_5(1 + |u|), \quad C_5 > 0,$$

for all $u \in X$.

3. Yosida approximations

In this section, we introduce the Yosida approximations to Eq. (1.1). We first study the existence and uniqueness of mild solutions of equation (1.1) in Theorem 3.1. Then we show that the mild solution of the Yosida approximating system converges in the pth-mean ($p \geq 2$) to the mild solution of equation (1.1), see Theorem 3.2 for the precise details.

The following result establishes the existence and uniqueness of mild solutions of equation (1.1).

Theorem 3.1. *Let the Hypothesis (H) hold. Suppose that for the case $p = 2$, the semigroup $\{S(t), t \geq 0\}$ is a contraction. Then, there exists a unique mild solution x in $C([0, \infty), L^p(\Omega, X))$ of Eq. (1.1) provided $L\|A^{-\alpha}\| < 1$, where $L = \max\{C_4, C_5\}$ and $1/p < \alpha < 1$.*

Proof of Theorem 3.1. It can be proved as in Govindan (2005, 2009).

Next, we introduce Yosida approximations to Eq. (1.1) by another similar equation:

$$d[x^{\lambda}(t) + R(\lambda)f(t, x_t^{\lambda})] = [Ax^{\lambda}(t) + R(\lambda)a(t, x_t^{\lambda})]dt + R(\lambda)b(t, x_t^{\lambda})dw(t), \quad t > 0; \tag{3.4}$$
$$x(t) = R(\lambda)\varphi(t), \quad t \in [-r, 0], \ \lambda \in \rho(A), \tag{3.5}$$

where $\rho(A)$ is the resolvent set of A with $R(\lambda) = \lambda R(\lambda; A)$, where $R(\lambda; A) = (I - \lambda A)^{-1}$, the resolvent of A. We name (3.4)–(3.5) as the Yosida approximations of (1.1).

The next theorem guarantees the existence and uniqueness of a mild solution to Eq. (3.4). Moreover, the mild solution of (3.4) converges to the mild solution of (1.1) in the pth-mean ($p \geq 2$), uniformly in $t \in [0, T]$ for each $T > 0$. This result is motivated by the classic work of Ichikawa (1982) and the proof of its second part follows using some arguments from Ichikawa (1982) and Liu (1998).

Theorem 3.2. *Under the Hypothesis (H), the Yosida approximating Eq. (3.4) has a unique mild solution $x^{\lambda}(t) \in C([0, \infty), L^p(\Omega, X))$, $p \geq 2$ and $\lambda \in \rho(A)$ provided $2L\|A^{-\alpha}\| < 1$, and $1/p < \alpha < 1$. Moreover, for each $0 < T < \infty$,*

$$\lim_{\lambda\to\infty} \sup_{0\leq s\leq t} E|x^{\lambda}(s) - x(s)|^p = 0, \quad t \in [0, T],$$

where $x(t)$ is a mild solution of equation (1.1).

Proof. The first part is a direct consequence of Theorem 3.1. To prove the second part, we first claim, for each $T > 0$ there exists a positive constant $C(T) > 0$ such that the mild solution of equation (1.1) satisfies

$$\sup_{0\leq t\leq T} E|x(t)|^p < C(T).$$

For this, considering the mild solution of (1.1) and using assumption (H3), we obtain

$$\begin{aligned}|x(s)| \leq{}& Me^{-as}|\varphi(0)| + Me^{-as}\|A^{-\alpha}\|C_5(1+\|\varphi\|_C) + \|A^{-\alpha}\|C_5(1+|x_s|) + \left|\int_0^s AS(s-\tau)f(\tau,x_\tau)d\tau\right| \\ &+ \left|\int_0^s S(s-\tau)a(\tau,x_\tau)d\tau\right| + \left|\int_0^s S(s-\tau)b(\tau,x_\tau)dw(\tau)\right|, \quad \text{a.s. } s\in[0,T].\end{aligned}$$

Hence, by using Lemma 2.1 (or Lemma 2.2 for $p = 2$) and Lemma 3.2 (Govindan, 2009), we get

$$\begin{aligned}[1-C_5\|A^{-\alpha}\|]^p \sup_{0\leq s\leq t} E|x(s)|^p \leq 4^p\Big\{&E[M|\varphi(0)| + M\|A^{-\alpha}\|C_5(1+\|\varphi\|_C) + C_5\|A^{-\alpha}\|]^p \\ &+ k(p,a,\alpha)E\int_0^t |A^\alpha f(s,x_s)|^p ds + T^{p-1}M^pE\int_0^t |a(s,x_s)|^p ds \\ &+ M^p c(p,T)E\int_0^t |b(s,x_s)|_\lambda^p ds\Big\},\end{aligned}$$

where $k(p,a,\alpha) > 0$ is a constant. Assumption (H2) yields

$$\begin{aligned}[1-C_5\|A^{-\alpha}\|]^p \sup_{0\leq s\leq t} E|x(s)|^p \leq 4^p\Big\{&E[M|\varphi(0)| + M\|A^{-\alpha}\|C_5(1+\|\varphi\|_C) + C_5\|A^{-\alpha}\|]^p \\ &+ TC_5k(p,a,\alpha) + T^pM^pC_3 + TM^pc(p,T)C_3 \\ &+ [C_5k(p,a,\alpha) + T^{p-1}M^pC_3 + M^pc(p,T)C_3]\int_0^t \sup_{0\leq t\leq s} E|x(\tau)|^p ds\Big\}.\end{aligned}$$

Invoking the well known Bellman–Gronwall's lemma, we get

$$\begin{aligned}\sup_{0\leq s\leq t} E|x(s)|^p &\leq \frac{C_1(T)}{[1-C_5\|A^{-\alpha}\|]^p}\exp\left\{\frac{tC_2(T)}{[1-C_5\|A^{-\alpha}\|]^p}\right\} \\ &< C(T),\end{aligned}$$

where $C_1(T)$, $C_2(T)$ and $C(T)$ are some positive constants and the claim follows.

Next, considering the difference of mild solutions $x^\lambda(s) - x(s)$, we have

$$\begin{aligned}[1-2C_4\|A^{-\alpha}\|]^p \sup_{0\leq s\leq t} E|x^\lambda(s)-x(s)|^p \leq 9^p\Big\{&E\left|\int_0^t AS(t-\tau)R(\lambda)[f(\tau,x_\tau^\lambda)-f(\tau,x_\tau)]d\tau\right|^p \\ &+ E\left|\int_0^t S(t-\tau)R(\lambda)[a(\tau,x_\tau^\lambda)-a(\tau,x_\tau)]d\tau\right|^p + \sup_{0\leq s\leq t} E\left|\int_0^s S(s-\tau)R(\lambda)[b(\tau,x_\tau^\lambda)-b(\tau,x_\tau)]dw(\tau)\right|^p \\ &+ E|S(t)[R(\lambda)-I]\varphi(0)|^p + E|S(t)[R(\lambda)-I]f(0,\varphi)|^p - \sup_{0\leq s\leq t} E|[R(\lambda)-I]f(s,x_s)|^p \\ &+ E\left|\int_0^t AS(t-\tau)[R(\lambda)-I]f(\tau,x_\tau)d\tau\right|^p + E\left|\int_0^t S(t-\tau)[R(\lambda)-I]a(\tau,x_\tau)d\tau\right|^p \\ &+ \sup_{0\leq s\leq t} E\left|\int_0^s S(s-\tau)[R(\lambda)-I]b(\tau,x_\tau)dw(\tau)\right|^p\Big\}. \qquad (3.6)\end{aligned}$$

Let us now estimate each term on the RHS of (3.6). First, by using Lemma 3.2 (Govindan, 2009), assumption (H3) and noting that $\|R(\lambda)\| \leq 2$ for large λ, see Ichikawa (1982), we have

$$\begin{aligned}E\left|\int_0^t AS(s-\tau)R(\lambda)[f(\tau,x_\tau^\lambda)-f(\tau,x_\tau)]d\tau\right|^p &\leq k(p,a,\alpha)E\int_0^t |R(\lambda)A^\alpha[f(s,x_s^\lambda)-f(s,x_s)]|^p ds \\ &\leq 2^pk(p,a,\alpha)C_4\int_0^t \sup_{0\leq\tau\leq s} E|x^\lambda(\tau)-x(\tau)|^p ds.\end{aligned}$$

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Next, by assumption (H2), it follows that

$$E\left|\int_0^t S(t-s)R(\lambda)[a(s,x_s^\lambda)-a(s,x_s)]ds\right|^p \le 2^pM^pC_1\int_0^t \sup_{0\le\tau\le s} E|x^\lambda(\tau)-x(\tau)|^p ds.$$

Using Lemma 2.1 (or Lemma 2.2 for $p=2$) and by assumption (H2), we estimate the stochastic integral as follows:

$$\sup_{0\le s\le t} E\left|\int_0^s S(s-\tau)R(\lambda)[b(\tau,x_\tau^\lambda)-b(\tau,x_\tau)]dw(\tau)\right|^p \le 2^pc(p,T)M^pC_2\int_0^t \sup_{0\le\tau\le s} E|x^\lambda(\tau)-x(\tau)|^p ds.$$

Note that $R(\lambda)x\to x$ as $\lambda\to\infty$ for all $x\in X$, see Ichikawa (1982). Using this and the hypothesis, we consider estimates of the rest of the terms in (3.6) together below:

$$E|S(t)[R(\lambda)-I]\varphi(0)|^p \le M^pE|[R(\lambda)-I]\varphi(0)|^p \to 0 \quad \text{as } \lambda\to\infty, \text{ and}$$
$$E|S(t)[R(\lambda)-I]f(0,\varphi)|^p \le M^pE|[R(\lambda)-I]f(0,\varphi)|^p \to 0 \quad \text{as } \lambda\to\infty.$$

Next, by assumption (H3), we obtain

$$\begin{aligned}\sup_{0\le s\le t} E|[R(\lambda)-I]f(s,x_s)|^p &\le \|R(\lambda)-I\|^pC_5(1+\sup_{0\le s\le T}E|x(s)|^p)\\ &\le \|R(\lambda)-I\|^pC_5(1+C(T))\to 0 \quad \text{as } \lambda\to\infty.\end{aligned}$$

By assumption (H3) and Lemma 3.2 (Govindan, 2009), we get

$$E\left|\int_0^t AS(t-s)[R(\lambda)-I]f(s,x_s)ds\right|^p \le k(p,a,\alpha)M^pE\int_0^t |[R(\lambda)-I]A^\alpha f(s,x_s)|^p ds \to 0 \quad \text{as } \lambda\to\infty,$$

by using the dominated convergence theorem, and so also

$$E\left|\int_0^t S(t-s)[R(\lambda)-I]a(s,x_s)ds\right|^p \le T^{p-1}M^pE\int_0^t |[R(\lambda)-I]a(s,x_s)|^p ds \to 0 \quad \text{as } \lambda\to\infty,$$

by the same theorem. Lastly, using Lemma 2.1 (or Lemma 2.2 for $p=2$), we have

$$\sup_{0\le s\le t} E\left|\int_0^s S(s-\tau)[R(\lambda)-I]b(\tau,x_\tau)dw(\tau)\right|^p \le c(p,T)M^pE\int_0^t |[R(\lambda)-I]b(s,x_s)|^p ds \to 0 \quad \text{as } \lambda\to\infty,$$

again by the dominated convergence theorem. Thus, we can write

$$\sup_{0\le s\le t} E|x^\lambda(s)-x(s)|^p \le K(T)\int_0^t \sup_{0\le\tau\le s} E|x^\lambda(\tau)-x(\tau)|^p ds+\epsilon(\lambda),$$

where $K(T)=9^p\left\{2^pk(p,a,\alpha)C_4+2^pM^pC_1+2^pc(p,T)M^pC_2\right\}$ and $\lim_{\lambda\to\infty}\epsilon(\lambda)=0$. The claim now follows from Gronwall's inequality.

4. Weak convergence of induced probability measures

In the rest of the paper, we consider only a special case, namely, a nondelay $f(t,x_t)=f(t,x(t))$.

The approximate result given in Theorem 3.2 can be used to derive another approximation result. Note that x and x^λ are elements of $C([0,\infty)), L^2(\Omega,X))$. Let P and P^λ be the probability measures on $C([0,\infty)), L^2(\Omega,X)$ induced by x and x^λ, respectively. In the next result, we shall show that P^λ converges weakly to P as $\lambda\to\infty$. Towards this, first note that Theorem 3.2 implies that every finite dimensional (joint) distribution of P^λ converges weakly to the corresponding one of P. Our claim will be proved once we establish the tightness of the family P^λ, $\lambda\in\rho(A)$. This requires some moment estimates.

Theorem 4.1. *Under Hypothesis (H), P^λ converges weakly to P as $\lambda\to\infty$.*

Proof. The proof is divided into three steps.
Step (1). We claim that for each $T<\infty$ and $p\ge 1$, we have

$$\sup_{\lambda\in\rho(A)}\sup_{0\le t\le T} E|x^\lambda(t)|^{2p}<\infty.$$

This can be accomplished using the Gronwall's lemma by considering the mild solution of the Yosida approximating system (3.4) and proceeding along similar lines as in the proof of Theorem 3.2. So we omit it.
Step (2). Let us arbitrarily fix a $0<T<\infty$. We claim that, for every $\lambda\in\rho(A)$ and $0\le s<t\le T$, there exists a constant $K>0$ such that

$$E|x^\lambda(t)-x^\lambda(s)|^4 \le K(t-s)^2. \tag{4.7}$$

To show this, considering the difference of mild solutions $x^\lambda(t)-x^\lambda(s)$, exploiting assumption (H3) and proceeding as before, we get

$$\begin{aligned}\left[1-2C_4\|A^{-\alpha}\|\right]^4 E|x^\lambda(t)-x^\lambda(s)|^4 \le 5^4\Big\{&E|[S(t)-S(s)]R(\lambda)\varphi(0)|^4+E\Big|\int_0^t AS(t-s)R(\lambda)f(s,x^\lambda(s))ds\\ &-\int_0^s AS(s-\tau)R(\lambda)f(\tau,x^\lambda(\tau))d\tau\Big|^4+E\Big|\int_0^t S(t-s)R(\lambda)a(s,x^\lambda_s)ds\\ &-\int_0^s S(s-\tau)R(\lambda)a(\tau,x^\lambda_\tau)d\tau\Big|^4+E\Big|\int_0^t S(t-s)R(\lambda)b(s,x^\lambda_s)dw(s)\\ &-\int_0^s S(s-\tau)R(\lambda)b(\tau,x^\lambda_\tau)dw(\tau)\Big|^4+E|[S(t)-S(s)]R(\lambda)f(0,\varphi)|^4\Big\}\\ =&\sum_{i=1}^5 E|I_i|^4, \quad \text{respectively, say.}\end{aligned} \tag{4.8}$$

We shall now estimate each term on the RHS of (4.8):

$$\begin{aligned}|I_1| = |[S(t)-S(s)]R(\lambda)\varphi(0)| &\le \int_s^t |S(u)AR(\lambda)\varphi(0)|du\\ &\le \int_s^t Me^{-au}|AR(\lambda)\varphi(0)|du \le M|AR(\lambda)\varphi(0)|(t-s).\end{aligned}$$

Therefore, $E|I_1|^4 \le K_1(t-s)^2$, where $K_1 \ge M^4T^2\|AR(\lambda)\|^4E\|\varphi\|_C^4$. Next,

$$\begin{aligned}I_2 =& \int_0^s A^{1-\alpha}[S(t-u)-S(s-u)]R(\lambda)A^\alpha f(u,x^\lambda(u))du\\ &+\int_s^t A^{1-\alpha}S(t-u)R(\lambda)A^\alpha f(u,x^\lambda(u))du = J_1+J_2, \quad \text{respectively, say.}\end{aligned}$$

But, by Theorem 6.13, p. 74 from Pazy (1983) and assumption (H3), we have

$$\begin{aligned}E|J_2|^4 &\le (t-s)^3E\int_s^t |A^{1-\alpha}S(t-u)R(\lambda)A^\alpha f(u,x^\lambda(u))|^4du\\ &\le (t-s)^3E\int_s^t M^4_{1-\alpha}t^{-4(1-\alpha)}e^{-4a(t-u)}\|R(\lambda)\|^4|A^\alpha f(u,x^\lambda(u))|^4du\\ &\le 16M^4_{1-\alpha}T^{2(1+\alpha)}e^{4aT}C_5(T+\sup_{0\le t\le T}E|x^\lambda(t)|^4)(t-s)^2, \quad \text{and}\end{aligned}$$

$$\begin{aligned}E|J_1|^4 &\le E\int_0^s\Big|A^{1-\alpha}\int_{s-u}^{t-u}S(v)AR(\lambda)A^\alpha f(u,x^\lambda(u))dv\Big|^4du\\ &\le \int_0^s\Big|\int_0^{t-s}M_{1-\alpha}(w+s-u)^{-(1-\alpha)}e^{-a(w+s-u)}AR(\lambda)A^\alpha f(u,x^\lambda(u))dw\Big|^4du\\ &\le M^4_{1-\alpha}T^{3+4\alpha}\|AR(\lambda)\|^4C_5(T+\sup_{0\le t\le T}E|x^\lambda(t)|^4)(t-s)^2.\end{aligned}$$

Thus, $E|I_2|^4 \le K_2(t-s)^2$, where $K_2 > 0$ is a suitable constant. Next, consider

$$\begin{aligned}I_3 =& \int_0^s[S(t-u)-S(s-u)]R(\lambda)a(u,x^\lambda_u)du+\int_s^t S(t-u)R(\lambda)a(u,x^\lambda_u)du\\ =& J_3+J_4, \quad \text{respectively, say.}\end{aligned}$$

By assumption (H2), we have

$$\begin{aligned}E|J_4|^4 &\le (t-s)^3E\int_s^t M^4e^{-4a(t-u)}\|R(\lambda)\|^4C_3(1+|x^\lambda_u|^2)du\\ &\le TM^4e^{4aT}\|R(\lambda)\|^4C_3(T+\sup_{0\le t\le T}E|x^\lambda(t)|^4)(t-s)^2, \quad \text{and}\end{aligned}$$

$$
\begin{aligned}
E|J_3|^4 &\le E\int_0^s \left| \int_0^{t-s} M e^{-a(v+s-u)} AR(\lambda) a(u, x_u^\lambda) dv \right|^4 du \\
&\le M^4 T^3 \|AR(\lambda)\|^4 C_3 (T + \sup_{0\le t\le T} E|x^\lambda(t)|^4)(t-s)^2.
\end{aligned}
$$

Thus, $E|I_3|^4 \le K_3(t-s)^2$, where $K_3 > 0$ is a suitable constant. Next, for the stochastic integral term, we have

$$
\begin{aligned}
I_4 &= \int_0^s [S(t-u) - S(s-u)] R(\lambda) b(u, x_u^\lambda) dw(u) + \int_s^t S(t-u) R(\lambda) b(u, x_u^\lambda) dw(u) \\
&= J_5 + J_6, \quad \text{respectively, say.}
\end{aligned}
$$

By applying Lemma 7.7 from Da Prato and Zabczyk (1992a, p. 194), Holder's inequality and Lemma 3.2 (Taniguchi, 1995), we get

$$
\begin{aligned}
E|J_6|^4 &\le c\left(\int_s^t \left[E|S(t-u)R(\lambda)b(u, x_u^\lambda)|^4 \right]^{1/2} du \right)^2 \\
&\le cM^4 \left\{ \left(\int_s^t e^{-a(t-u)} du \right)^{1/2} \left(\int_s^t e^{-a(t-u)} E|R(\lambda)b(u, x_u^\lambda)|^4 \right)^{1/2} \right\}^2 \\
&\le 16cM^4 e^{2aT} T^2 C_3 (T + \sup_{0\le t\le T} E|x^\lambda(t)|^4)(t-s)^2,
\end{aligned}
$$

where $c > 0$ is a constant and by Lemma 7.2 from Da Prato and Zabczyk (1992a, p. 182), we get

$$
\begin{aligned}
E|J_5|^4 &= E\left| \int_0^s \int_{s-u}^{t-u} S(v) AR(\lambda) b(u, x_u^\lambda) dv dw(u) \right|^4 \\
&\le C_p E\left(\int_0^t \left| \int_{s-u}^{t-u} S(v) AR(\lambda) b(u, x_u^\lambda) dv \right|^2 du \right)^2 \\
&\le C_p \left(E\int_0^t \left| \int_0^{t-s} M e^{-a(v+s-u)} AR(\lambda) b(u, x_u^\lambda) dv \right|^2 du \right)^2 \\
&\le e^{4aT} C_p C_3 M^4 T^2 \|AR(\lambda)\|^4 (T + \sup_{0\le t\le T} E|x^\lambda(t)|^4)(t-s)^2,
\end{aligned}
$$

where $C_p > 0$ is a constant. Thus, $E|I_4|^4 \le K_4(t-s)^2$, where $K_4 > 0$ is a suitable constant. Lastly, proceeding as above for I_1, it can be easily shown that $E|I_5|^4 \le K_5(t-s)^2$, where $K_5 > 0$ is a suitable constant. Combining all the above estimates, we get (4.7).

Step (3). The proof can be completed as in Kannan and Bharucha-Reid (1985).

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