Research Article

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Composite relaxed resolvent operator and Yosida approximation operator for solving a system of Yosida inclusions

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Abstract: In this paper, we first study a composite relaxed resolvent operator and prove some of its properties. After that, we introduce a Yosida approximation operator based on the composite relaxed resolvent operator and demonstrate some properties of the Yosida approximation operator. Finally, we obtain the solution of a system of Yosida inclusions by applying these concepts. We construct a conjoin example in support of many concepts derived in this paper. Our concepts and results are new in the literature and can be used for further research.

Keywords: Inclusion, relaxed, system, solution, Yosida

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1 Introduction

The concept of variational inequality was introduced by Hartmann and Stampacchia [7] in the early sixties, and was later expanded by Stampacchia [12] in his other works. The theory of variational inequalities is being developed very fast, having as model the variational theory of boundary value problems for partial differential equations. While the variational theory of boundary value problems has its starting point in the method of orthogonal projection, the theory of variational inequalities has its starting point in the projection on a convex set. It is worth mentioning that the increasing use of game theory in many economic and engineering applications leads to investigating optimization problems described by the solution of variational inequality problems and their generalization. An important generalization of variational inequality is known as variational inclusion, which is mainly due to Hassouni and Moudafi [8].

Several types of generalized monotonicities are introduced in the literature and applied to solve many inclusion problems. More precisely, Verma [13–15] introduced relatively A-maximal monotonicity, Fang and Huang [6] introduced H-monotonicity and then they applied their notions to solve a general class of inclusion problems, i.e.,

$$0 \in T(x),$$

where $T : H \to 2^H$ is a set-valued mapping on a real Hilbert space H. The set-valued variational inclusion problems are interesting and have many applications in pure and applied sciences. Proximal mapping and

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resolvent operator techniques are used for computing solutions of many problems related to variational inclusions; see, e.g., [11].

It is well known in the literature that Yosida approximations are instrumental for solving general variational inclusion problems in the context of non-expansive resolvent operators, non-linear inhomogeneous evolution inclusions etc. For more details, we refer to [2–4, 9, 10] and references therein.

The aim of this work is twofold. First we introduce a composite relaxed resolvent operator and prove that it is single-valued as well as Lipschitz continuous. After that we introduce a Yosida approximation based on the composite relaxed resolvent operator and prove that it is strongly monotone and Lipschitz continuous. We apply these new concepts to solve a system of Yosida inclusions. An example is constructed in support of many concepts discussed in this paper.

2 Preliminaries

Throughout the paper, we take X to be a real Hilbert Space endowed with a norm $\|\cdot\|$ and an inner product $\langle\cdot,\cdot\rangle$.

The following known definitions and results are essential to achieve the goal of this paper.

Definition 2.1. Let $A, H : X \to X$ be two single-valued mappings.
(i) A is said to be strongly monotone if there exists a constant $\delta_A > 0$ such that

$$\langle A(x) - A(y), x - y\rangle \geq \delta_A \|x - y\|^2 \quad \text{for all } x, y \in X.$$

(ii) H is said to be relaxed Lipschitz continuous if there exists a constant $\alpha_1 > 0$ such that

$$\langle H(x) - H(y), x - y\rangle \leq -\alpha_1 \|x - y\|^2 \quad \text{for all } x, y \in X.$$

(iii) H is said to be relaxed Lipschitz continuous with respect to A if there exists a constant $\alpha_2 > 0$ such that

$$\langle H(A(x)) - H(A(y)), x - y\rangle \leq -\alpha_2 \|x - y\|^2 \quad \text{for all } x, y \in X.$$

Definition 2.2 ([5]). A functional $f : X \times X \to \mathbb{R}$ is said to be 0-diagonally quasi concave (in short 0-DQCV) in x if, for any finite set $\{x_1, \ldots, x_n\} \subset X$ and for any $y = \sum_{i=1}^n \lambda_i x_i$ with $\lambda_i \geq 0$ and $\sum_{i=1}^n \lambda_i = 1$, one has

$$\min_{1\leq i\leq n} f(x_i, y) \leq 0.$$

Definition 2.3. Let $\phi : X \to \mathbb{R} \cup \{+\infty\}$ be a proper functional. A vector $f^* \in X$ is said to be a subgradient of ϕ at $x \in \operatorname{dom}\phi$ if

$$\langle f^*, y - x\rangle \leq \phi(y) - \phi(x) \quad \text{for all } y \in X.$$

Each ϕ can be associated with the following map $\partial\phi$, called the subdifferential of ϕ at x, defined by

$$\partial\phi(x) = \begin{cases} f^* \in X : \langle f^*, y - x\rangle \leq \phi(y) - \phi(x) & \text{for all } y \in X,\ x \in \operatorname{dom}\phi, \\ \emptyset, & x \notin \operatorname{dom}\phi. \end{cases}$$

Lemma 2.4 ([16]). *Let D be a nonempty convex subset of a topological vector space and let $f : D \times D \to [-\infty, \infty]$ be such that the following conditions hold:*
(i) *For each $x \in D$, the mapping $y \to f(x, y)$ is lower semicontinuous on each compact subset of D.*
(ii) *For each finite set $\{x_1, \ldots, x_n\} \subset D$ and for each $y = \sum_{i=1}^n \lambda_i x_i$ with $\lambda_i \geq 0$ and $\sum_{i=1}^n \lambda_i = 1$,*

$$\min_{1\leq i\leq n} f(x_i, y) \leq 0.$$

(iii) *There exist a nonempty compact convex subset D_0 of D and a nonempty compact subset K of D such that for each $y \in D \setminus K$ there is an $x \in Co(D_0 \cup \{y\})$ satisfying $f(x, y) > 0$, where Co denotes the convex hull.*
Then there exists $\hat{y} \in D$ such that $f(x, \hat{y}) \leq 0$ for all $x \in D$.

Definition 2.5. Let $A, H, I : X \to X$ be the mappings such that I is an identity mapping. Let $\phi : X \to \mathbb{R} \cup \{+\infty\}$ be a proper functional. If for any $z \in X$ and $\rho > 0$ there exists a unique $x \in X$ satisfying

$$\langle [(I-H)\circ A]x - z, y - x\rangle + \rho\phi(y) - \rho\phi(x) \geq 0 \quad \text{for all } y \in X,$$

then the mapping $z \to x$, denoted by $R^{\partial\phi}_{\rho,I}(z)$, is said to be the composite relaxed resolvent operator of ϕ. We have $z - [(I-H)\circ A]x \in \rho\partial\phi(x)$. It follows that

$$R^{\partial\phi}_{\rho,I}(z) = \big[[(I-H)\circ A] + \rho\partial\phi\big]^{-1}(z).$$

Now we prove some suitable conditions which ensure the existence and Lipschitz continuity of the composite relaxed resolvent operator $R^{\partial\phi}_{\rho,I}$.

Theorem 2.6. *Let X be a real Hilbert space. Suppose that $A, H, I : X \to X$ are the continuous mappings such that A is strongly monotone with constant δ_A and Lipschitz continuous with constant γ_A, H is continuous and relaxed Lipschitz continuous with respect to A with constant α, and I is an identity mapping. Let $\phi : X \to \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous, subdifferentiable, proper functional which may not be convex. For any $z, x \in X$ let the mapping $h(y, x) = \langle z - [(I-H)\circ A]x, y - x\rangle$ be 0-DQCV in y. Then for any $\rho > 0$ and for any $z \in X$,*

$$\langle [(I-H)\circ A]x - z, y - x\rangle + \rho\phi(y) - \rho\phi(x) \geq 0 \quad \textit{for all } y \in X, \tag{2.1}$$

i.e., $x = R^{\partial\phi}_{\rho,I}(z)$, and hence the composite relaxed resolvent operator of ϕ is well-defined.

Proof. For any $\rho > 0$ and $z \in X$, define a functional $f : X \times X \to \mathbb{R} \cup \{+\infty\}$ by

$$f(y, x) = \langle z - [(I-H)\circ A]x, y - x\rangle + \rho\phi(x) - \rho\phi(y) \quad \text{for all } x, y \in X.$$

Using the continuity of the mappings A, H, I and the lower semicontinuity of ϕ, we have for each $y \in X$ that $x \to f(x, y)$ is lower semicontinuous on X.

We claim that $f(y, x)$ satisfies Lemma 2.4 (ii). If it is false, then there exists a finite set $\{y_1, \dots, y_n\} \subset X$ and $x_0 = \sum_{i=1}^n t_i y_i$ with $t_i \geq 0$ and $\sum_{i=1}^n t_i = 1$ such that we have

$$\langle z - [(I-H)\circ A]x_0, y_i - x_0\rangle + \rho\phi(x_0) - \rho\phi(y_i) > 0 \quad \text{for all } i = 1, 2, \dots, n.$$

Since ϕ is subdifferentiable at x_0, there exists a point $f^*_{x_0} \in X$ such that

$$\phi(y_i) - \phi(x_0) \geq \langle f^*_{x_0}, y_i - x_0\rangle \quad \text{for all } i = 1, 2, \dots, n.$$

It follows that

$$\langle z - [(I-H)\circ A]x_0, y_i - x_0\rangle > \rho\phi(y_i) - \rho\phi(x_0) \geq \rho\langle f^*_{x_0}, y_i - x_0\rangle \quad \text{for all } i = 1, 2, \dots, n.$$

Thus, we have

$$\langle z - [(I-H)\circ A]x_0 - \rho f^*_{x_0}, y_i - x_0\rangle > 0 \quad \text{for all } i = 1, 2, \dots, n. \tag{2.2}$$

On the other hand, by the assumption that $h(y, x) = \langle z - [(I-H)\circ A]x, y - x\rangle$ is 0-DQCV in y, we have

$$\min_{1\leq i\leq n} \langle z - [(I-H)\circ A]x_0 - \rho f^*_{x_0}, y_i - x_0\rangle \leq 0,$$

which contradicts inequality (2.2). Hence $f(y, x)$ satisfies Lemma 2.4 (ii).

We take a point $\bar{y} \in \operatorname{dom}\phi$. As ϕ is subdifferentiable at $\bar{y}$, there exists a point $f^*_{\bar{y}} \in X$ such that

$$\phi(x) - \phi(\bar{y}) \geq \langle f^*_{\bar{y}}, x - \bar{y}\rangle \quad \text{for all } x \in X.$$

Since A is strongly monotone with constant δ_A and H is relaxed Lipschitz continuous with respect to A with constant α, we have

$$\begin{aligned}
f(\overline{y}, x) &= \langle z - [(I - H) \circ A]x, \overline{y} - x\rangle + \rho\phi(x) - \rho\phi(\overline{y}) \\
&= \langle z - [(I - H) \circ A]\overline{y} + [(I - H) \circ A]\overline{y} - [(I - H) \circ A]x, \overline{y} - x\rangle + \rho\phi(x) - \rho\phi(\overline{y}) \\
&= \langle z - [A(\overline{y}) - H(A(\overline{y}))] + [A(\overline{y}) - H(A(\overline{y}))] - [A(x) - H(A(x))], \overline{y} - x\rangle + \rho\phi(x) - \rho\phi(\overline{y}) \\
&\geq \langle A(\overline{y}) - A(x), \overline{y} - x\rangle - \langle H(A(\overline{y})) - H(A(x)), \overline{y} - x\rangle + \langle z, \overline{y} - x\rangle + \langle H(A(\overline{y})) - A(\overline{y}), \overline{y} - x\rangle + \rho\langle f^*_{\overline{y}}, x - \overline{y}\rangle \\
&\geq \delta_A\|\overline{y} - x\|^2 + \alpha\|\overline{y} - x\|^2 - \|z\|\|\overline{y} - x\| - (\|H(A(\overline{y}))\| + \|A(\overline{y})\|)\|\overline{y} - x\| - \rho\|f^*_{\overline{y}}\|\|\overline{y} - x\| \\
&= \|\overline{y} - x\|\big[(\delta_A + \alpha)\|\overline{y} - x\| - \{\|z\| + \|H(A(\overline{y}))\| + \|A(\overline{y})\| + \rho\|f^*_{\overline{y}}\|\}\big].
\end{aligned}$$

Let

$$r = \frac{1}{(\delta_A + \alpha)}\{\|z\| + \|H(A(\overline{y}))\| + \|A(\overline{y})\| + \rho\|f^*_{\overline{y}}\|\} \quad \text{and} \quad K = \{x \in X : \|\overline{y} - x\| \leq r\}.$$

Then $D_0 = \{\overline{y}\}$ and K are both weakly compact convex subsets of X, and for each $x \in X \setminus K$ there exists a $\overline{y} \in Co(D_0 \cup \{\overline{y}\})$ such that $f(\overline{y}, x) > 0$. Hence, all the conditions of Lemma 2.4 are satisfied. Then there exists an $\overline{x} \in X$ such that $f(\overline{y}, x) \leq 0$ for all $y \in X$, i.e.,

$$\langle[(I - H) \circ A]\overline{x} - z, y - \overline{x}\rangle + \rho\phi(y) - \rho\phi(\overline{x}) \geq 0 \quad \text{for all } y \in X.$$

Now, we show that $\overline{x}$ is a unique solution of problem (2.1). Suppose that $x_1, x_2 \in X$ are two arbitrary solutions of problem (2.1). Then we have

$$\langle[(I - H) \circ A]x_1 - z, y - x_1\rangle + \rho\phi(y) - \rho\phi(x_1) \geq 0 \quad \text{for all } y \in X, \tag{2.3}$$
$$\langle[(I - H) \circ A]x_2 - z, y - x_2\rangle + \rho\phi(y) - \rho\phi(x_2) \geq 0 \quad \text{for all } y \in X. \tag{2.4}$$

Taking $y = x_2$ in (2.3) and $y = x_1$ in (2.4) and adding the resulting inequalities, we obtain

$$\langle[(I - H) \circ A]x_1 - z, x_2 - x_1\rangle + \langle[(I - H) \circ A]x_2 - z, x_1 - x_2\rangle \geq 0$$

or

$$\begin{aligned}
0 &\leq \langle -[(I - H) \circ A]x_1 + z + [(I - H) \circ A]x_2 - z, x_1 - x_2\rangle \\
&\geq \langle A(x_1) - A(x_2), x_1 - x_2\rangle - \langle H(A(x_1)) - H(A(x_2)), x_1 - x_2\rangle.
\end{aligned}$$

As A is strongly monotone with constant δ_A and H is relaxed Lipschitz continuous with respect to A with constant α, it follows that

$$\delta_A\|x_1 - x_2\|^2 + \alpha\|x_1 - x_2\|^2 \leq \langle A(x_1) - A(x_2), x_1 - x_2\rangle - \langle H(A(x_1)) - H(A(x_2)), x_1 - x_2\rangle \leq 0,$$

and hence we must have $x_1 = x_2$. This completes the proof. □

Theorem 2.7. *If all the conditions of Theorem 2.6 are satisfied, then the composite relaxed resolvent operator $R^{\partial\phi}_{\rho,I}$ of ϕ is $\frac{1}{[\delta_A+\alpha]}$-Lipschitz continuous.*

Proof. By Theorem 2.6, the composite relaxed resolvent operator $R^{\partial\phi}_{\rho,I}$ of ϕ is well-defined.

For any $z_1, z_2 \in X$, let $x_1 = R^{\partial\phi}_{\rho,I}(z_1)$ and $x_2 = R^{\partial\phi}_{\rho,I}(z_2)$ such that

$$\langle[(I - H) \circ A]x_1 - z_1, y - x_1\rangle \geq \rho\phi(x_1) - \rho\phi(y) \quad \text{for all } y \in X, \tag{2.5}$$
$$\langle[(I - H) \circ A]x_2 - z_2, y - x_2\rangle \geq \rho\phi(x_2) - \rho\phi(y) \quad \text{for all } y \in X. \tag{2.6}$$

Taking $y = x_2$ in (2.5) and $y = x_1$ in (2.6) and adding the resulting inequalities, we have

$$\langle[(I - H) \circ A]x_1 - z_1, x_2 - x_1\rangle + \langle[(I - H) \circ A]x_2 - z_2, x_1 - x_2\rangle \geq 0$$

or

$$\begin{aligned}
&\langle A(x_1) - A(x_2), x_2 - x_1\rangle + \langle H(A(x_2)) - H(A(x_1)), x_2 - x_1\rangle - \langle z_1 - z_2, x_2 - x_1\rangle \geq 0, \\
&\langle A(x_1) - A(x_2), x_2 - x_1\rangle + \langle H(A(x_2)) - H(A(x_1)), x_2 - x_1\rangle \geq \langle z_1 - z_2, x_2 - x_1\rangle, \\
&\langle A(x_2) - A(x_1), x_2 - x_1\rangle - \langle H(A(x_2)) - H(A(x_1)), x_2 - x_1\rangle \leq \langle z_1 - z_2, x_2 - x_1\rangle.
\end{aligned}$$

As A is strongly monotone with constant δ_A and H is relaxed Lipschitz continuous with respect to A with constant α, we get

$$(\delta_A + \alpha)\|x_2 - x_1\|^2 \leq \|z_2 - z_1\|\|x_2 - x_1\|,$$

i.e.,

$$\|x_2 - x_1\| \leq \theta_1\|z_2 - z_1\|,$$

where $\theta_1 = \frac{1}{[\delta_A+\alpha]}$.

Therefore, the composite relaxed resolvent operator $R^{\partial\phi}_{\rho,I}$ of ϕ is $\frac{1}{[\delta_A+\alpha]}$-Lipschitz continuous. This completes the proof. □

Now, we define the Yosida approximation operator based on the composite relaxed resolvent operator and demonstrate some of its properties.

Definition 2.8. The Yosida approximation operator of ϕ is defined by

$$J^{\partial\phi}_{\rho,I}(z) = \frac{1}{\rho}[A - R^{\partial\phi}_{\rho,I}](z) \quad \text{for all } z \in X, \tag{2.7}$$

where $R^{\partial\phi}_{\rho,I}$ is the composite relaxed resolvent operator of ϕ defined in Definition 2.5.

Theorem 2.9. *The Yosida approximation operator $J^{\partial\phi}_{\rho,I}$ of ϕ is*
(i) *θ_2-strongly monotone, where $\theta_2 = \frac{1}{\rho}[\delta_A - \frac{1}{\delta_A+\alpha}]$ and $\delta_A > \frac{1}{\delta_A+\alpha}$,*
(ii) *θ_3-Lipschitz continuous, where $\theta_3 = \frac{1}{\rho}[\gamma_A + \frac{1}{\delta_A+\alpha}]$.*

Proof. (i) Since A is δ_A-strongly monotone, using Theorem 2.7, we have

$$\begin{aligned}
\langle J^{\partial\phi}_{\rho,I}(x) - J^{\partial\phi}_{\rho,I}(y), x - y\rangle &= \frac{1}{\rho}\langle A(x) - R^{\partial\phi}_{\rho,I}(x) - [A(y) - R^{\partial\phi}_{\rho,I}(y)], x - y\rangle \\
&= \frac{1}{\rho}[\langle A(x) - A(y), x - y\rangle - \langle R^{\partial\phi}_{\rho,I}(x) - R^{\partial\phi}_{\rho,I}(y), x - y\rangle] \\
&\geq \frac{1}{\rho}(\delta_A\|x - y\|^2 - \|R^{\partial\phi}_{\rho,I}(x) - R^{\partial\phi}_{\rho,I}(y)\|\|x - y\|) \\
&\geq \frac{1}{\rho}\Big(\delta_A\|x - y\|^2 - \frac{1}{[\delta_A + \alpha]}\|x - y\|\|x - y\|\Big) \\
&= \frac{1}{\rho}\Big[\delta_A - \frac{1}{\delta_A + \alpha}\Big]\|x - y\|^2,
\end{aligned}$$

i.e., the Yosida approximation operator $J^{\partial\phi}_{\rho,I}$ of ϕ is θ_2-strongly monotone.

(ii) Since A is γ_A-Lipschitz continuous, using Theorem 2.7, we have

$$\begin{aligned}
\|J^{\partial\phi}_{\rho,I}(x) - J^{\partial\phi}_{\rho,I}(y)\| &= \rho^{-1}\|A(x) - R^{\partial\phi}_{\rho,I}(x) - [A(y) - R^{\partial\phi}_{\rho,I}(y)]\| \\
&\leq \rho^{-1}[\|A(x) - A(y)\| + \|R^{\partial\phi}_{\rho,I}(x) - R^{\partial\phi}_{\rho,I}(y)\|] \\
&\leq \rho^{-1}\Big[\gamma_A\|x - y\| + \frac{1}{[\delta_A + \alpha]}\|x - y\|\Big] \\
&= \frac{1}{\rho}\Big[\gamma_A + \frac{1}{\delta_A + \alpha}\Big]\|x - y\|,
\end{aligned}$$

i.e., the Yosida approximation operator $J^{\partial\phi}_{\rho,I}$ of ϕ is θ_3-Lipschitz continuous. □

The Yosida approximation operator of ϕ defined by (2.7) can also be expressed in some other form and in this regard, we state the following theorem.

Theorem 2.10. *The Yosida approximation operator $J^{\partial\phi}_{\rho,I}$ of ϕ defined by* (2.7) *is equivalent to*

$$[\rho A + \rho\{A \circ (I - H) \circ A + \rho(A \circ \partial\phi) - I\}^{-1} \circ A]^{-1}(z) \quad \textit{for all } z \in X.$$

Proof. Let

$$u \in \frac{1}{\rho}[A - [(I - H) \circ A + \rho\partial\phi]^{-1}](z).$$

This is equivalent to the following assertions:

$$\begin{aligned}
&A(z) - \rho u \in [(I - H) \circ A + \rho\partial\phi]^{-1}(z)\\
&\quad\Longleftrightarrow\ [(I - H) \circ A + \rho\partial\phi](A(z) - \rho u) = z\\
&\quad\Longleftrightarrow\ [A \circ (I - H) \circ A + \rho(A \circ \partial\phi)](A(z) - \rho u) - \rho u = (A(z) - \rho u)\\
&\quad\Longleftrightarrow\ -\rho u = (A(z) - \rho u) - [A \circ (I - H) \circ A + \rho(A \circ \partial\phi)](A(z) - \rho u)\\
&\quad\Longleftrightarrow\ \rho u = \big[[A \circ (I - H) \circ A + \rho(A \circ \partial\phi)] - I\big](A(z) - \rho u)\\
&\quad\Longleftrightarrow\ A(z) - \rho u \in [A \circ (I - H) \circ A + \rho(A \circ \partial\phi)]^{-1}(\rho u)\\
&\quad\Longleftrightarrow\ A(z) \in \big[\rho\{A \circ (I - H) \circ A + \rho(A \circ \partial\phi) - I\}^{-1} + \rho I\big](u)\\
&\quad\Longleftrightarrow\ u \in \big[\rho A + \rho\{A \circ (I - H) \circ A + \rho(A \circ \partial\phi) - I\}^{-1} \circ A\big]^{-1}(z) \quad \text{for all } z \in X.
\end{aligned}$$

This completes the proof. □

Remark 2.11. It is to be noted that one can easily prove that

$$\|J^{\partial\phi}_{\rho,I}\| \le |\partial\phi|$$

by using the approach of [1, 9].

In support of Definition 2.1 (i) and (iii), Definition 2.8, Theorem 2.7, Theorem 2.9 (i) and (ii) and Remark 2.11, we construct the following consolidated example.

Example 2.12. Let $X = \mathbb{R}$ and let $A, H : X \to X$ be the mappings defined by

$$A(x) = x + 1, \quad H(x) = -\frac{x}{2} \qquad \text{for all } x \in X.$$

Then it can be easily shown that A is strongly monotone and Lipschitz continuous, and H is continuous and relaxed Lipschitz continuous with respect to A.

We define the mapping $\phi : X \to \mathbb{R} \cup \{+\infty\}$ by $\phi(x) = x^2$ and its subdifferential operator by $\partial\phi = 2x$ for all $x \in X$. Then the composite relaxed resolvent operator $R^{\partial\phi}_{\rho,I}$ for $\rho = 1$ discussed in Definition 2.5 has the form

$$R^{\partial\phi}_{\rho,I}(x) = [(I - H) \circ A + \rho\partial\phi]^{-1}(x) = \frac{2x - 3}{7} \quad \text{for all } x \in X,$$

and the Yosida approximation operator $J^{\partial\phi}_{\rho,I}$ of ϕ defined by (2.7) for $\rho = 1$ is expressed as

$$J^{\partial\phi}_{\rho,I}(x) = \frac{1}{\rho}[A - R^{\partial\phi}_{\rho,I}](x) = \frac{5x + 10}{7} \quad \text{for all } x \in X.$$

Now, it can be easily verified that Theorem 2.7 and Theorem 2.9 (i) and (ii) are satisfied, i.e., the composite relaxed resolvent operator $R^{\partial\phi}_{\rho,I}$ of ϕ is Lipschitz continuous and the Yosida approximation operator $J^{\partial\phi}_{\rho,I}$ of ϕ is strongly monotone and Lipschitz continuous.

3 Formulation of the problem and existence results

Let X_1 and X_2 be two real Hilbert spaces and let $\phi_1, \psi_1 : X_1 \times X_2 \to \mathbb{R} \cup \{+\infty\}$, $\phi_2 : X_1 \times X_1 \to \mathbb{R} \cup \{+\infty\}$ and $\psi_2 : X_2 \times X_2 \to \mathbb{R} \cup \{+\infty\}$ be the lower semicontinuous, subdifferentiable, proper functionals which may not be convex. Suppose that $g_1, H_1, A_1 : X_1 \to X_1$ and $g_2, H_2, A_2 : X_2 \to X_2$ are the single-valued mappings. We consider the following problem: For some $(f_1, f_2) \in X_1 \times X_2$, find $(x, y) \in X_1 \times X_2$ such that

$$\begin{cases} f_1 \in J_{\rho,I}^{\partial\phi_1(g_1(\cdot),y)}(x) - \dfrac{1}{\rho}(H_1 \circ A_1)x + \partial\phi_2(g_1(x), x), \\ f_2 \in J_{\rho,I}^{\partial\psi_1(x,g_2(\cdot))}(y) - \dfrac{1}{\rho}(H_2 \circ A_2)y + \partial\psi_2(y, g_2(y)). \end{cases} \tag{3.1}$$

We call system (3.1) a system of Yosida inclusions.

The following result shows that the system of Yosida inclusions (3.1) is equivalent to a set of fixed point problems, which can be easily proved by using Definition 2.5.

Theorem 3.1. *The set of elements $(x, y) \in X_1 \times X_2$ satisfies (3.1) if and only if $(x, y) \in X_1 \times X_2$ satisfies the relations*

$$\begin{aligned} x &= R_{\rho,I}^{\partial\phi_2(g_1(\cdot),\cdot)}[\rho f_1 + R_{\rho,I}^{\partial\phi_1(g_1(\cdot),y)}(x)], \\ y &= R_{\rho,I}^{\partial\psi_2(\cdot,g_2(\cdot))}[\rho f_2 + R_{\rho,I}^{\partial\psi_1(x,g_2(\cdot))}(y)]. \end{aligned}$$

Proof. Suppose that the first relation holds. Then

$$\begin{aligned} x = R_{\rho,I}^{\partial\phi_2(g_1(\cdot),\cdot)}&[\rho f_1 + R_{\rho,I}^{\partial\phi_1(g_1(\cdot),y)}(x)] \\ &\iff x = [(I - H) \circ A + \rho\partial\phi_2(g_1(\cdot), \cdot)]^{-1}[\rho f_1 + R_{\rho,I}^{\partial\phi_1(g_1(\cdot),y)}(x)] \\ &\iff A(x) - (H \circ A)(x) + \rho\partial\phi_2(g_1(x), x) = \rho f_1 + R_{\rho,I}^{\partial\phi_1(g_1(\cdot),y)}(x) \\ &\iff \frac{1}{\rho}[A - R_{\rho,I}^{\partial\phi_1(g_1(\cdot),y)}](x) - \frac{1}{\rho}(H \circ A)(x) + \partial\phi_2(g_1(x), x) = f_1, \end{aligned}$$

which implies that

$$f_1 \in J_{\rho,I}^{\partial\phi_1(g_1(\cdot),y)}(x) - \frac{1}{\rho}(H_1 \circ A_1)x + \partial\phi_2(g_1(x), x).$$

Similarly, we can prove that y is the fixed point in the second relation of (3.1). This completes the proof. □

Theorem 3.2. *Let X_1 and X_2 be two real Hilbert spaces. For $i = 1, 2$, let $g_i, H_i, A_i, I_i : X_i \to X_i$ be the mappings and let $\phi_1, \psi_1 : X_1 \times X_2 \to \mathbb{R} \cup \{+\infty\}$, $\phi_2 : X_1 \times X_1 \to \mathbb{R} \cup \{+\infty\}$ and $\psi_2 : X_2 \times X_2 \to \mathbb{R} \cup \{+\infty\}$ be the lower semicontinuous, subdifferentiable, proper functionals which may not be convex such that following conditions are satisfied:*

(i) *The mapping A_1 is strongly monotone with constant δ_{A_1} and γ_{A_1}-Lipschitz continuous, and the mapping A_2 is strongly monotone with constant δ_{A_2} and γ_{A_2}-Lipschitz continuous.*
(ii) *The mappings H_1 is continuous and relaxed Lipschitz continuous with constant α_1, and H_2 is continuous and relaxed Lipschitz continuous with constant α_2.*
(iii) *$(H_1 \circ A_1)$ is ξ_1-Lipschitz continuous and $(H_2 \circ A_2)$ is ξ_2-Lipschitz continuous.*
(iv) *g_1 and g_2 are continuous mappings.*
(v) *For $i = 1, 2$, let the mapping $h_i(y, x) = \langle z - [(I_i - H_i) \circ A_i]x, y - x\rangle$ be 0-DQCV in y.*

In addition, if for each $\rho > 0$ the conditions

$$\begin{cases} \|R_{\rho,I}^{\partial\phi_1(g_1(\cdot),y_1)}(x_1) - R_{\rho,I}^{\partial\phi_1(g_1(\cdot),y_2)}(x_1)\| \le \mu_1\|y_1 - y_2\|, \\ \|R_{\rho,I}^{\partial\psi_1(x_1,g_2(\cdot))}(y_1) - R_{\rho,I}^{\partial\psi_1(x_2,g_2(\cdot))}(y_1)\| \le \mu_2\|x_1 - x_2\| \end{cases} \tag{3.2}$$

and

$$\begin{cases} L_1 = [\theta_1'(\gamma_{A_1} + \rho\theta_3') + \theta_1''\mu_2] < 1, \\ L_2 = [\theta_1''(\gamma_{A_2} + \rho\theta_3'') + \theta_1'\mu_1] < 1 \end{cases} \tag{3.3}$$

are satisfied, then the system of Yosida inclusions (3.1) admits a solution $(x, y) \in X_1 \times X_2$ and, moreover, the solution set is compact.

Proof. By Theorem 2.7 and Theorem 2.9 we know that the composite relaxed resolvent operators $R_{\rho,I}^{\partial\phi_2(g_1(\cdot),\cdot)}$ and $R_{\rho,I}^{\partial\psi_2(\cdot,g_2(\cdot))}$ are θ_1'-Lipschitz continuous and θ_1''-Lipschitz continuous, respectively, the Yosida approximation operator $J_{\rho,I}^{\partial\phi_1(\cdot,g_1(\cdot))}$ is θ_2'-strongly monotone and θ_3'-Lipschitz continuous, and the Yosida approximation operator $J_{\rho,I}^{\partial\psi_1(\cdot,g_2(\cdot))}$ is θ_2''-strongly monotone and θ_3''-Lipschitz continuous, where

$$\theta_1' = \frac{1}{[\delta_{A_1}+\alpha_1]}, \quad \theta_1'' = \frac{1}{[\delta_{A_2}+\alpha_2]}, \quad \theta_2' = \frac{1}{\rho}\Big[\delta_{A_1} - \frac{1}{\delta_{A_1}+\alpha_1}\Big], \quad \theta_2'' = \frac{1}{\rho}\Big[\delta_{A_2} - \frac{1}{\delta_{A_2}+\alpha_2}\Big],$$
$$\theta_3' = \frac{1}{\rho}\Big[\gamma_{A_1} + \frac{1}{\delta_{A_1}+\alpha_1}\Big], \quad \theta_3'' = \frac{1}{\rho}\Big[\gamma_{A_2} + \frac{1}{\delta_{A_2}+\alpha_2}\Big], \quad \delta_{A_1} > \frac{1}{\delta_{A_1}+\alpha_1}, \quad \delta_{A_2} > \frac{1}{\delta_{A_2}+\alpha_2}.$$

Now, we define a mapping $N : X_1 \times X_2 \to X_1 \times X_2$ by

$$N(x,y) = (T(x,y), S(x,y)) \quad \text{for all } (x,y) \in X_1 \times X_2, \tag{3.4}$$

where $T : X_1 \times X_2 \to X_1$ and $S : X_1 \times X_2 \to X_2$ are defined by

$$T(x,y) = R_{\rho,I}^{\partial\phi_2(g_1(\cdot),\cdot)}[\rho f_1 + R_{\rho,I}^{\partial\phi_1(g_1(\cdot),y)}(x)], \tag{3.5}$$

$$S(x,y) = R_{\rho,I}^{\partial\psi_2(\cdot,g_2(\cdot))}[\rho f_2 + R_{\rho,I}^{\partial\psi_1(x,g_2(\cdot))}(y)], \tag{3.6}$$

where $\rho > 0$ is a constant.

For any $(x_1,y_1), (x_2,y_2) \in X_1 \times X_2$, using the Lipschitz continuity of A_1, (3.5), (3.6), Theorem 2.7, condition (ii) of Theorem 2.9 and condition (3.2), we have

$$\begin{aligned}
\|T(x_1,y_1) - T(x_2,y_2)\| &= \|R_{\rho,I}^{\partial\phi_2(g_1(\cdot),\cdot)}[\rho f_1 + R_{\rho,I}^{\partial\phi_1(g_1(\cdot),y_1)}(x_1)] - R_{\rho,I}^{\partial\phi_2(g_1(\cdot),\cdot)}[\rho f_1 + R_{\rho,I}^{\partial\phi_1(g_1(\cdot),y_2)}(x_2)]\| \\
&\le \theta_1'\|R_{\rho,I}^{\partial\phi_1(g_1(\cdot),y_1)}(x_1) - R_{\rho,I}^{\partial\phi_1(g_1(\cdot),y_2)}(x_2)\| \\
&\le \theta_1'\|R_{\rho,I}^{\partial\phi_1(g_1(\cdot),y_1)}(x_1) - R_{\rho,I}^{\partial\phi_1(g_1(\cdot),y_2)}(x_1) + R_{\rho,I}^{\partial\phi_1(g_1(\cdot),y_2)}(x_1) - R_{\rho,I}^{\partial\phi_1(g_1(\cdot),y_2)}(x_2)\| \\
&\le \theta_1'\|R_{\rho,I}^{\partial\phi_1(g_1(\cdot),y_1)}(x_1) - R_{\rho,I}^{\partial\phi_1(g_1(\cdot),y_2)}(x_1)\| + \theta_1'\|R_{\rho,I}^{\partial\phi_1(g_1(\cdot),y_2)}(x_1) - R_{\rho,I}^{\partial\phi_1(g_1(\cdot),y_2)}(x_2)\| \\
&\le \theta_1'\mu_1\|y_1 - y_2\| + \theta_1'\rho\Big\|\frac{1}{\rho}[A_1(x_2) - R_{\rho,I}^{\partial\phi_1(g_1(\cdot),y_2)}(x_2)] - \frac{1}{\rho}[A_1(x_1) - R_{\rho,I}^{\partial\phi_1(g_1(\cdot),y_2)}(x_1)] \\
&\qquad + \frac{1}{\rho}[A_1(x_1) - A_1(x_2)]\Big\| \\
&\le \theta_1'\mu_1\|y_1 - y_2\| + \theta_1'\rho\|J_{\rho,I}^{\partial\phi_1(g_1(\cdot),y_2)}(x_2) - J_{\rho,I}^{\partial\phi_1(g_1(\cdot),y_2)}(x_1)\| + \theta_1'\|A(x_1) - A(x_2)\| \\
&\le \theta_1'\mu_1\|y_1 - y_2\| + \theta_1'\rho\theta_3'\|x_1 - x_2\| + \theta_1'\gamma_{A_1}\|x_1 - x_2\| \\
&= \theta_1'\mu_1\|y_1 - y_2\| + [\theta_1'(\gamma_{A_1} + \rho\theta_3')]\|x_1 - x_2\|
\end{aligned} \tag{3.7}$$

and

$$\begin{aligned}
\|S(x_1,y_1) - S(x_2,y_2)\| &= \|R_{\rho,I}^{\partial\psi_2(\cdot,g_2(\cdot))}[\rho f_2 + R_{\rho,I}^{\partial\psi_1(x_1,g_2(\cdot))}(y_1)] - R_{\rho,I}^{\partial\psi_2(\cdot,g_2(\cdot))}[\rho f_2 + R_{\rho,I}^{\partial\psi_1(x_2,g_2(\cdot))}(y_2)]\| \\
&\le \theta_1''\|R_{\rho,I}^{\partial\psi_1(x_1,g_2(\cdot))}(y_1) - R_{\rho,I}^{\partial\psi_1(x_2,g_2(\cdot))}(y_2)\| \\
&\le \theta_1''\|R_{\rho,I}^{\partial\psi_1(x_1,g_2(\cdot))}(y_1) - R_{\rho,I}^{\partial\psi_1(x_2,g_2(\cdot))}(y_1) + R_{\rho,I}^{\partial\psi_1(x_2,g_2(\cdot))}(y_1) - R_{\rho,I}^{\partial\psi_1(x_2,g_2(\cdot))}(y_2)\| \\
&\le \theta_1''\|R_{\rho,I}^{\partial\psi_1(x_1,g_2(\cdot))}(y_1) - R_{\rho,I}^{\partial\psi_1(x_2,g_2(\cdot))}(y_1)\| + \theta_1''\|R_{\rho,I}^{\partial\psi_1(x_2,g_2(\cdot))}(y_1) - R_{\rho,I}^{\partial\psi_1(x_2,g_2(\cdot))}(y_2)\| \\
&\le \theta_1''\mu_2\|x_1 - x_2\| + \theta_1''\rho\Big\|\frac{1}{\rho}[A_2(y_2) - R_{\rho,I}^{\partial\psi_1(x_2,g_2(\cdot))}(y_2)] \\
&\qquad - \frac{1}{\rho}[A_2(y_1) - R_{\rho,I}^{\partial\psi_1(x_2,g_2(\cdot))}(y_1)] + \frac{1}{\rho}[A_2(y_1) - A_2(y_2)]\Big\| \\
&\le \theta_1''\mu_2\|x_1 - x_2\| + \theta_1''\rho\|J_{\rho,I}^{\partial\psi_1(x_2,g_2(\cdot))}(y_2) - J_{\rho,I}^{\partial\psi_1(x_2,g_2(\cdot))}(y_1)\| + \theta_1''\|A_2(y_1) - A_2(y_2)\| \\
&\le \theta_1''\mu_2\|x_1 - x_2\| + \theta_1''\rho\theta_3''\|y_1 - y_2\| + \theta_1''\gamma_{A_2}\|y_1 - y_2\| \\
&= \theta_1''\mu_2\|x_1 - x_2\| + [\theta_1''(\gamma_{A_2} + \rho\theta_3'')]\|y_1 - y_2\|.
\end{aligned} \tag{3.8}$$

From (3.7) and (3.8) we have

$$\begin{aligned}
&\|T(x_1,y_1) - T(x_2,y_2)\| + \|S(x_1,y_1) - S(x_2,y_2)\| \\
&\quad \le [\theta_1'(\gamma_{A_1} + \rho\theta_3') + \theta_1''\mu_2]\|x_1 - x_2\| + [\theta_1''(\gamma_{A_2} + \rho\theta_3'') + \theta_1'\mu_1]\|y_1 - y_2\| \\
&\quad = L_1\|x_1 - x_2\| + L_2\|y_1 - y_2\| \\
&\quad \le \max\{L_1, L_2\}(\|x_1 - x_2\| + \|y_1 - y_2\|),
\end{aligned} \tag{3.9}$$

where

$$L_1 = [\theta_1'(\gamma_{A_1} + \rho\theta_3') + \theta_1''\mu_2], \quad L_2 = [\theta_1''(\gamma_{A_2} + \rho\theta_3'') + \theta_1'\mu_1]$$

and

$$\theta_1' = \frac{1}{[\delta_{A_1} + \alpha_1]}, \quad \theta_1'' = \frac{1}{[\delta_{A_2} + \alpha_2]}, \quad \theta_3' = \frac{1}{\rho}\Big[\gamma_{A_1} + \frac{1}{\delta_{A_1} + \alpha_1}\Big], \quad \theta_3'' = \frac{1}{\rho}\Big[\gamma_{A_2} + \frac{1}{\delta_{A_2} + \alpha_2}\Big].$$

Now, we define a norm $\|(x, y)\|_*$ on $X_1 \times X_2$ by

$$\|(x, y)\|_* = \|x\| + \|y\| \quad \text{for all } (x, y) \in X_1 \times X_2. \tag{3.10}$$

It is easy to see that $(X_1 \times X_2, \|\cdot\|_*)$ is a Banach space. Hence from (3.4), (3.9) and (3.10) we have

$$\|N(x_1, y_1) - N(x_2, y_2)\|_* \le \max\{L_1, L_2\}\|(x_1, y_1) - (x_2, y_2)\|_*. \tag{3.11}$$

By (3.3) we know that $\max\{L_1, L_2\} < 1$. It follows from (3.11) that N is a contraction operator. Hence there exists a unique $(x, y) \in X_1 \times X_2$ such that

$$N(x, y) = (x, y),$$

which implies that

$$\begin{aligned} x &= R_{\rho,I}^{\partial\phi_2(g_1(\cdot),\cdot)}[\rho f_1 + R_{\rho,I}^{\partial\phi_1(g_1(\cdot),y)}(x)], \\ y &= R_{\rho,I}^{\partial\psi_2(\cdot,g_2(\cdot))}[\rho f_2 + R_{\rho,I}^{\partial\psi_1(x,g_2(\cdot))}(y)]. \end{aligned}$$

It follows from Theorem 3.1 that (x, y) is a unique solution of the system of Yosida inclusions (3.1).

Let us denote the solution set of the system of Yosida inclusions (3.1) by SOLSYI. From the above arguments it is clear that SOLSYI is closed.

Now, we will show that SOLSYI is bounded. For any $(x_1, y_1) \in X_1 \times X_2$ and $\rho > 0$, let

$$\begin{cases} z_\rho \in J_{\rho,I}^{\partial\phi_1(g_1(\cdot),y)}(x_1) - \dfrac{1}{\rho}(H_1 \circ A_1)x_1 + \partial\phi_2(g_1(x_1), x_1), \\ w_\rho \in J_{\rho,I}^{\partial\psi_1(x,g_2(\cdot))}(y_1) - \dfrac{1}{\rho}(H_2 \circ A_2)y_1 + \partial\psi_2(y_1, g_2(y_1)). \end{cases} \tag{3.12}$$

Combining the first line of (3.1) and the first line of (3.12), we have

$$\begin{aligned} z_\rho - J_{\rho,I}^{\partial\phi_1(g_1(\cdot),y)}(x_1) - f_1 + J_{\rho,I}^{\partial\phi_1(g_1(\cdot),y)}(x) - \frac{1}{\rho}(H_1 \circ A_1)x + \frac{1}{\rho}(H_1 \circ A_1)x_1 \\ \in \partial\phi_2(g_1(x_1), x_1) - \partial\phi_2(g_1(x), x). \end{aligned}$$

As the subdifferential operator $\partial\phi_2$ is a multi-valued maximal monotone operator, $H_1 \circ A_1$ is strongly monotone with constant ξ_1. Using the Lipschitz continuity of the Yosida operator $J_{\rho,I}^{\partial\phi_1}$ with constant θ_2', we have

$$\begin{aligned} 0 &\le \Big\langle z_\rho - f_1 - J_{\rho,I}^{\partial\phi_1(g_1(\cdot),y)}(x_1) + J_{\rho,I}^{\partial\phi_1(g_1(\cdot),y)}(x) - \frac{1}{\rho}[(H_1 \circ A_1)x - (H_1 \circ A_1)x_1], x - x_1\Big\rangle \\ &= \langle z_\rho - f_1, x - x_1\rangle - \langle J_{\rho,I}^{\partial\phi_1(g_1(\cdot),y)}(x) - J_{\rho,I}^{\partial\phi_1(g_1(\cdot),y)}(x_1), x - x_1\rangle \\ &\qquad - \frac{1}{\rho}\langle (H_1 \circ A_1)x - (H_1 \circ A_1)x_1, x - x_1\rangle \\ &\le \|z_\rho - f_1\|\|x - x_1\| - \langle J_{\rho,I}^{\partial\phi_1(g_1(\cdot),y)}(x) - J_{\rho,I}^{\partial\phi_1(g_1(\cdot),y)}(x_1), x - x_1\rangle - \frac{1}{\rho}\xi_1\|x - x_1\|^2 \\ &\le \|z_\rho - f_1\|\|x - x_1\| - \theta_2'\|x - x_1\|^2 - \frac{1}{\rho}\xi_1\|x - x_1\|^2 \\ &\le \|z_\rho - f_1\|\|x - x_1\| - \Big(\theta_2' + \frac{\xi_1}{\rho}\Big)\|x - x_1\|^2. \end{aligned} \tag{3.13}$$

Using condition (3.2) and Remark 2.11, we have

$$\begin{aligned}\|J_{\rho,I}^{\partial\phi_1(g_1(\cdot),y)}(x_1)\| &= \|J_{\rho,I}^{\partial\phi_1(g_1(\cdot),y)}(x_1) - J_{\rho,I}^{\partial\phi_1(g_1(\cdot),y_1)}(x_1) + J_{\rho,I}^{\partial\phi_1(g_1(\cdot),y_1)}(x_1)\| \\ &\leq \Big\|\frac{1}{\rho}(A_1(x_1) - R_{\rho,I}^{\partial\phi_1(g_1(\cdot),y)}(x_1) - A_1(x_1) + R_{\rho,I}^{\partial\phi_1(g_1(\cdot),y_1)}(x_1))\Big\| + |\partial\phi_1(g_1(x_1), y_1)| \\ &\leq \frac{1}{\rho}\mu_1\|y_1 - y\| + |\partial\phi_1(g_1(x_1), y_1)|. \end{aligned} \tag{3.14}$$

For any $\rho > 0$, take $\mu_\rho \in \partial\phi_2(g_1(x_1), x_1))$ and $\nu_\rho \in \partial\psi_2(y_2, g_2(y_2))$ such that

$$z_\rho = J_{\rho,I}^{\partial\phi_1(g_1(\cdot),y)}(x_1) - \frac{1}{\rho}(H_1 \circ A_1)x_1 + \mu_\rho$$

and

$$w_\rho = J_{\rho,I}^{\partial\psi_1(x,g_2(\cdot))}(y_1) - \frac{1}{\rho}(H_1 \circ A_1)y_1 + \nu_\rho.$$

Since $\{\mu_\rho\} \subset \partial\phi_2(g_1(x_1), x_1)$ and $\{\nu_\rho\} \subset \partial\psi_2(y_2, g_2(y_1))$, and $\partial\phi_2$ and $\partial\psi_2$ are maximal monotone multi-valued operators, it is clear that $\{\mu_\rho\}$ and $\{\nu_\rho\}$ are bounded.

Combining (3.13) and (3.14), we have

$$\Big(\theta_2' + \frac{\xi_1}{\rho}\Big)\|x - x_1\| \leq \|z_\rho - f_1\|,$$

which implies that

$$\begin{aligned}\|x - x_1\| &\leq \frac{\rho}{[\rho\theta_2' + \xi_1]}\|z_\rho - f_1\| \\ &\leq \frac{\rho}{[\rho\theta_2' + \xi_1]}(\|z_\rho\| + \|f_1\|) \\ &\leq \frac{\rho}{[\rho\theta_2' + \xi_1]}\Big(\|J_{\rho,I}^{\partial\phi_2(g_1(\cdot),y)}(x_1)\| + \frac{1}{\rho}\|(H_1 \circ A_1)x_1\| + \|\mu_\rho\| + \|f_1\|\Big) \\ &\leq \frac{\rho}{\rho\theta_2' + \xi_1}\Big(\frac{1}{\rho}\mu_1\|y_1 - y\| + |\partial\phi_1(g_1(x_1, y_1))| + \frac{1}{\rho}\|(H_1 \circ A_1)x_1\| + \|\mu_\rho\| + \|f_1\|\Big).\end{aligned}$$

By using similar arguments, we obtain

$$\|y_1 - y\| \leq \frac{\rho}{\rho\theta_2'' + \xi_2}\Big(\frac{1}{\rho}\mu_2\|x - x_1\| + |\partial\psi_1(y_1, x_1)| + \frac{1}{\rho}\|(H_2 \circ A_2)y_1\| + \|\nu_\rho\| + \|f_2\|\Big).$$

It follows that SOLSYI, i.e., the solution set $\{x, y\}$ of the system of Yosida inclusions is bounded, and hence it is compact. This completes the proof. □

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