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The Yosida approximation iterative technique for split monotone Yosida variational inclusions

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Abstract
In this article, we introduce a new type of split monotone Yosida inclusion problem in the setting of infinite-dimensional Hilbert spaces. To calculate the approximate solutions of split monotone Yosida inclusion problem, first we develop a new iterative algorithm and then study the weak as well strong convergence analysis of iterative sequences generated by the proposed iterative algorithm by using demicontractive property, nonexpansive property, and strongly positive bounded linear property of mappings. A numerical example is formulated to explain our main result through MATLAB programming.

Keywords Split monotone Yosida variational inclusions · Yosida operator · Convergence · Demicontractive mapping · Nonexpansive mappings

Mathematics Subject Classification (2010) 47H05 · 47H09 · 47J25

1 Introduction

The general framework of variational inclusion is to find an element $x \in H$ such that $0 \in G(x)$, where $G : H \longrightarrow 2^H$ is a multi-valued mapping and H is a Hilbert space,

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which covers minimization or maximization problems, variational inequalities, equilibrium problems, and many more variational and optimization systems in applied sciences. It is well known that general maximal monotonicity has played a crucial role for developing and using suitable proximal point algorithms to study convex programming, variational inequalities, and related problems.

It is worth to mention that every monotone mapping on Hilbert spaces can be regularized into single-valued, nonexpansive, Lipschitz continuous monotone mappings by means of Yosida approximation notion. Such operators have been studied broadly due to its significant role in convex analysis, partial differential equations, variational inclusions, etc. Another possible scheme of solving multi-valued differential equations, elliptic boundary value problems is based upon the Yosida approximation approach. For example, in 1995, Petterson [17] proved the existence of multi-valued stochastic differential equations with a maximal monotone operator through the Yosida approximation method.

The convex feasibility problem (CFP) considered in Chang et al. [10] is a central problem in applied mathematics, which can be formulated in various ways to find a common point of closed and convex sets, to find a common fixed point of nonexpansive mappings, to find a common minimum of convex functionals, to solve a system of variational inequalities. The CFP is to find a member of the intersection of finitely many closed convex sets in Euclidean spaces. When the intersection is empty, one can minimize a proximity mapping to obtain an approximate solution to the problem. The precise mathematical formulation is as follows: Let H be a Hilbert space and $C_{\mathrm{n}}, n = 1, 2, \cdots, N$ be the closed convex subsets of H. The CFP is to find some point

$$x \in \bigcap_{n=1}^{N} C_n,$$

where the intersection is nonempty.

In 1994, Censor and Elfving [8] introduced the following split feasibility problem (SFP) in finite dimensional Hilbert spaces which is generalization of the (CFP). Let H_1 and H_2 be the real Hilbert spaces. Let C and Q be two nonempty closed convex subsets of real Hilbert spaces H_1 and H_2, respectively, $A : H_1 \longrightarrow H_2$ be a bounded linear mapping. The (SFP) is to find a point x^* such that

$$x^* \in C \ \ and \ \ Ax^* \in Q. \tag{1.1}$$

The formalism of (SFP) arise for modeling phase retrieval-like real world problems, and it later was investigated extensively as an extremely robust tool for the exploration of many far-flung inverse problems such as image recovery problems, computerized tomography, data compression, and intensity-modulated radiation therapy treatment planning (see, e.g., [3, 7–9, 16]). For signal restoration, image reconstruction with positive constrained problems, following ill-posed linear problem, can be formulated which is a special case of the SFP. This problem is to find a point x^* such that

$$x^* \in C \ \ and \ \ Ax^* = b,$$

where $C \subseteq H_1$ is a nonempty closed convex subset and b is a given element of H_2. This problem is known as convexly constrained linear inverse problem [11] where x represents unknown signal to be recovered and b is the collection of measurements.

One of the most powerful extensions of CFP and SFP is the multiple-set split feasibility problem [8]. Its precise mathematical formulation is to find a point x^* with the property

$$x^* \in \bigcap_{i=1}^{m} C_{\mathrm{i}} \;\; and \;\; Ax^* \in \bigcap_{i=1}^{n} Q_{\mathrm{i}},$$

where $\{C_{\mathrm{i}}\}_{i=1}^m$ and $\{Q_{\mathrm{i}}\}_{i=1}^n$ are the families of nonempty closed convex subsets of H_1 and H_2, respectively. This problem can serve as a model for inverse problems where constraints are imposed on the solutions in the domain of the linear operator and in the operator's range.

In 2011, Moudafi [14] introduced the following split monotone variational inclusion problem (SMVIP), i.e., the problem of finding $x^* \in H_1$ such that

$$0 \in f_1(x^*) + B_1(x^*), \tag{1.2}$$

and such that

$$y^* = Ax^* \in H_2 \;\; solves \;\; 0 \in f_2(y^*) + B_2(y^*), \tag{1.3}$$

where $B_1 : H_1 \longrightarrow 2^{H_1}$ and $B_2 : H_2 \longrightarrow 2^{H_2}$ are multi-valued maximal monotone mappings, and $f_1 : H_1 \longrightarrow H_1$ and $f_2 : H_2 \longrightarrow H_2$ are single-valued mappings. If $f_1 = f_2 \equiv 0$, then SMVIP reduces to the following split variational inclusion problem (SVIP) of finding $x^* \in H_1$ such that

$$0 \in B_1(x^*), \tag{1.4}$$

and such that

$$y^* = Ax^* \in H_2 \;\; solves \;\; 0 \in B_2(y^*). \tag{1.5}$$

We notice that the solution x^* of (1.4) is the pre-image of the solution y^* of (1.5) through a bounded linear operator A in two different Hilbert spaces.

In order to obtain the weak and strong convergence results of SVIP, Byrne et al. [4] introduced the following iterative algorithm: For initial point x_0 and $\lambda > 0$, compute iterative sequence $\{x_n\}$ generated by the following scheme

$$x_{\mathrm{n}+1} = R_\lambda^{B_1}\left(x_{\mathrm{n}} + \gamma A^*\left(R_\lambda^{B_2} - I\right)Ax_{\mathrm{n}}\right), \tag{1.6}$$

where $R_\lambda^{B_1} = (I + \lambda B_1)^{-1}$ and $R_\lambda^{B_2} = (I + \lambda B_2)^{-1}$ are the resolvent operators of the mappings B_1 and B_2, respectively, $\gamma \in \left(0, \frac{1}{\|AA^*\|}\right)$, A^* is the adjoint operator of A. It is well known that resolvent operator is single-valued and firmly nonexpansive.

In 2015, Sitthithakerngkiet et al. [20] presented the following hybrid steepest descent method

$$\begin{cases} u_{\mathrm{n}} = R_\lambda^{B_1}\left(x_{\mathrm{n}} + \gamma A^*\left(R_\lambda^{B_2} - I\right)Ax_{\mathrm{n}}\right), \\ x_{\mathrm{n}+1} = \alpha_{\mathrm{n}}\xi f(x_{\mathrm{n}}) + (I - \alpha_{\mathrm{n}}D)S_{\mathrm{n}}u_{\mathrm{n}}, \end{cases} \tag{1.7}$$

where S_{n} is a sequence of nonexpansive mappings, D is a strongly positive bounded linear operator, f is a contractive mapping, α_{n} is a positive real number satisfying certain conditions, and $\xi > 0$. The sequence $\{x_{\mathrm{n}}\}$ generated by the iterative scheme

(1.7) converges strongly to a point z which is a unique solution of the variational inequalities

$$\langle (D - \xi f)z, z - x \rangle \leq 0.$$

Very recently, Ahmad et al. [1] introduced the following Yosida inclusion problem to find $x \in X$ such that

$$0 \in J_{M,\lambda}^{H(\cdot,\cdot)}(x) + M(x), \lambda > 0, \tag{1.8}$$

where X is the smooth Banach space, $J_{M,\lambda}^{H(\cdot,\cdot)}$ is the generalized Yosida approximation operator defined by

$$J_{M,\lambda}^{H(\cdot,\cdot)}(u) = \frac{1}{\lambda}\left[I - R_{M,\lambda}^{H(\cdot,\cdot)}\right](u), \forall u \in X, \tag{1.9}$$

where I is the identity mapping on X and $R_{M,\lambda}^{H(\cdot,\cdot)}$ is the resolvent operator associated with the mappings $H(\cdot,\cdot)$ and M [1]. It is also shown that generalized Yosida approximation operator is Lipschitz continuous and strongly monotone. They proved the following fixed point formulation of Yosida inclusion problem (1.8):

$$x = R_{M,\lambda}^{H(\cdot,\cdot)}\left[H(A,B)x - \lambda J_{M,\lambda}^{H(\cdot,\cdot)}(x)\right], \forall x \in X, \quad \lambda > 0.$$

In order to study the strong convergence properties of the solution of Yosida inclusion problem (1.8), Ahmad et al. [1] proposed the following iterative algorithm: For any $x_0 \in X$, compute the sequence $\{x_n\} \subset X$ by the following scheme:

$$x_{\mathrm{n}+1} = R_{M_{\mathrm{n}},\lambda}^{H(\cdot,\cdot)}\left[H(A,B)x_{\mathrm{n}} - \lambda J_{M_{\mathrm{n}},\lambda}^{H(\cdot,\cdot)}(x_{\mathrm{n}})\right].$$

(for more details regarding various algorithms and approaches for the split variational inclusion problems and related problems, one may refer to [2, 6, 18, 19, 22] and related references.)

Motivated and influenced by the work of Ahmad et al. [1], Moudafi [14], Byrne et al. [4], and Sitthithakerngkiet et al. [20], in this paper, we introduce a new type of split monotone Yosida variational inclusion problem. We introduce a new iterative scheme to approximate the solutions of split monotone Yosida variational inclusion problem. Under suitable conditions on parameter and mappings, we establish that the sequences generated by the suggested iterative scheme converge weakly as well as strongly to the solution of split monotone Yosida variational inclusion problem. A numerical example is constructed and through MATLAB programming, which describes the convergence of the suggested iterative algorithm. The results proved in this article are new addition and extension of previously known results.

2 Formulation of problem and preliminaries

For $i = 1, 2$, let H_i be a Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$, and let $A : H_1 \longrightarrow H_2$ be a bounded linear operator. Let $B_i : H_i \longrightarrow 2^{H_i}$ be the multi-valued maximal monotone mapping and $f_i : H_i \longrightarrow H_i$ be the single-valued mapping. We introduce the following problem: Find $x^* \in H_1$ such that

$$0 \in f_1(x^*) + B_1(x^*) - J_{\lambda_1}^{B_1}(x^*), \tag{2.1}$$

and such that

$$y^* = Ax^* \in H_2 \ \ solves \ \ 0 \in f_2(y^*) + B_2(y^*) - J_{\lambda_2}^{B_2}(y^*), \tag{2.2}$$

where $J_{\lambda_i}^{B_i} = \frac{1}{\lambda_i}\left(I_i - R_{\lambda_i}^{B_i}\right)$ is the Yosida approximation operator of the mapping B_i, $R_{\lambda_i}^{B_i} = (I_i + \lambda_i B_i)^{-1}$ is the resolvent operator of the mapping B_i for $\lambda_i > 0$, and I_i is identity mapping on H_i. The set of all solutions of the problems (2.1) and (2.2) is denoted by Ω, i.e.,

$$\Omega = \{x \in H_1 \ solves \ (2.1) : Ax \ solves \ (2.2)\}. \tag{2.3}$$

We call this problem as split monotone Yosida variational inclusion problem (SMYVIP).

Remark 2.1 For the parameter $\lambda_i > 0$, the resolvent operator is single-valued and nonexpansive mapping whereas Yosida approximation operator defined above is Lipschitz continuous with constant $\frac{1}{\lambda_i}$ and strongly monotone.

In the sequel, let us recall some basic definitions and facts which will be needed for the proof of our main result. A point $x \in H$ is said to be a fixed point of T provided $Tx = x$, where $T : H \longrightarrow H$ is a mapping and H is a Hilbert space. We denote the set of fixed points of the mapping T by $Fix(T)$. We denote the strong convergence and weak convergence of $\{x_n\}$ to $x \in H$ by $x_n \rightarrow x$ and $x_n \rightharpoonup x$, respectively. It is well known that

$$\|\alpha x + (1-\alpha)y\|^2 = \alpha\|x\|^2 + (1-\alpha)\|y\|^2 - \alpha(1-\alpha)\|x-y\|^2, \tag{2.4}$$

for all $x, y \in H$, and $\alpha \in [0, 1]$.

Definition 2.1 A mapping $T : H \longrightarrow H$ is called contraction if there exists a constant $\xi \in (0, 1)$ such that

$$\|T(x) - T(y)\| \leq \xi\|x - y\|, \ \forall x, y \in H.$$

If $\xi = 1$, then T is said to be nonexpansive mapping.

Definition 2.2 A mapping $T : H \longrightarrow H$ is said to be firmly quasi-nonexpansive if $Fix(T) \neq \emptyset$ and

$$\|Tx - x^*\|^2 \leq \|x - x^*\|^2 - \|x - Tx\|^2, \ \ \forall x \in H, x^* \in Fix(T).$$

Definition 2.3 A mapping $T : H \longrightarrow H$ is said to be k-demicontractive if $Fix(T) \neq \emptyset$ and there exists a constant $k \in (0, 1)$ such that

$$\|Tx - x^*\|^2 \leq \|x - x^*\|^2 + k\|x - Tx\|^2, \ \forall x \in C, x^* \in Fix(T). \tag{2.5}$$

Remark 2.2 Clearly, the class of firmly quasi-nonexpansive mappings is a subclass of demicontractive mappings. Note also that the mapping T satisfying (2.5) with $k = 1$ is called hemicontractive. It is easy to observe from (2.5) that

$$\begin{aligned}\|Tx - x^*\| &\leq \|x - x^*\| + \sqrt{k}\|x - Tx\| \\ &\leq \left(1 + \sqrt{k}\right)\|x - x^*\| + \sqrt{k}\|Tx - x^*\| \\ &\leq \left(\frac{1+\sqrt{k}}{1-\sqrt{k}}\right)\|x - x^*\| \\ &= L\|x - x^*\|,\end{aligned}$$

where $L = \dfrac{1+\sqrt{k}}{1-\sqrt{k}}$, so that

$$\|Tx - x^*\| \leq L\|x - x^*\|.$$

Definition 2.4 A mapping $T : H \longrightarrow H$ is demiclosed at a point $z \in H$ if the weak convergence of any sequence $\{x_k\}$ to some point $\bar{x}$ and the strong convergence of $\{T(x_k)\}$ to z implies that $T(\bar{x}) = z$.

Theorem 2.1 (The Demiclosedness principle) *Let C be a closed and convex subset of H and $S : C \longrightarrow H$ a nonexpansive mapping. Then, $I - S$ (I is the identity mapping of H) is demiclosed at $z \in H$.*

If $I - T$ is demiclosed at 0, we get $x_k \rightharpoonup \bar{x}$ and $(I - T)x_k \to z$ implies that $\bar{x} \in Fix(T)$.

Definition 2.5 Let $B : H \longrightarrow 2^H$ be a multi-valued mapping with graph $\mathcal{G}(B) = \{(x, y) : y \in B(x)\}$. Then, B is called monotone if for all $(x, u), (y, v) \in \mathcal{G}(B)$ such that

$$\langle x - y, u - v\rangle \geq 0.$$

The monotone mapping B is said to be maximal if its graph $\mathcal{G}(B)$ is not properly contained in the graph of any other monotone mapping.

Definition 2.6 A mapping $T : H \longrightarrow H$ is said to be

(i) strongly nonexpansive if T is nonexpansive and

$$\lim_{n\to\infty} \|(x_{\mathrm{n}} - y_{\mathrm{n}}) - (Tx_{\mathrm{n}} - Ty_{\mathrm{n}})\| = 0,$$

whenever $\{x_{\mathrm{n}}\}$ and $\{y_{\mathrm{n}}\}$ are bounded sequences in H and

$$\lim_{n\to\infty} \|(x_{\mathrm{n}} - y_{\mathrm{n}}) - (Tx_{\mathrm{n}} - Ty_{\mathrm{n}})\| = 0,$$

$$\lim_{n\to\infty} (\|(x_{\mathrm{n}} - y_{\mathrm{n}})\| - \|(Tx_{\mathrm{n}} - Ty_{\mathrm{n}})\|) = 0,$$

(ii) averaged nonexpansive if it can be written as $T = (1 - \alpha)I + \alpha S$, where $\alpha \in (0, 1)$, I is the identity mapping on H, and $S : H \longrightarrow H$ is a nonexpansive mapping,

(iii) α-inversely strongly monotone if there exists a constant $\alpha > 0$ such that

$$\langle Tx - Ty, x - y\rangle \geq \alpha \|Tx - Ty\|^2, \ \ \forall x, y \in H.$$

Lemma 2.1 ([21]) *Let $\{\gamma_{\mathrm{n}}\}$ be a sequence in $(0, 1)$ and $\{\delta_{\mathrm{n}}\}$ be a sequence in $\mathbb{R}$ satisfying*

(i) $\sum_{n=1}^{\infty} \gamma_{\mathrm{n}} = \infty$,
(ii) $\limsup_{n\to\infty} \gamma_{\mathrm{n}} \leq 0$ *or* $\sum_{n=1}^{\infty} |\gamma_{\mathrm{n}}\delta_{\mathrm{n}}| < \infty$.

If $\{a_{\mathrm{n}}\}$ is a sequence of non-negative real numbers such that

$$a_{\mathrm{n}+1} \leq (1 - \gamma_{\mathrm{n}})a_{\mathrm{n}} + \gamma_{\mathrm{n}}\delta_{\mathrm{n}},$$

for each $n \geq 0$, then $\lim_{n\to\infty} a_{\mathrm{n}} = 0$.

Lemma 2.2 ([12]) *Let $\{s_{\mathrm{n}}\}$ be a sequence of real numbers that does not decrease at infinity, in the sense that there exists a subsequence $\{s_{\mathrm{n_i}}\}$ of $\{s_{\mathrm{n}}\}$ such that $s_{\mathrm{n_i}} \leq s_{\mathrm{n_{i+1}}}$ for all $i \geq 0$. For every $n \in \mathbb{N}$, define an integer sequence $\{\tau(n)\}$ as*

$$\tau(n) = \max\{k \leq n : s_k < s_{k+1}\}$$

Then, $\tau(n) \to \infty$ and $\max\{s_{\tau(n)}, s_{\mathrm{n}}\} \leq s_{\tau(n)+1}$.

Lemma 2.3 ([15]) *Let C be a nonempty, closed, and convex subset of a real Hilbert space H and $T : C \to C$ be a nonexpansive operator with $Fix(T) \neq \phi$, if the sequence $\{x_{\mathrm{n}}\} \subseteq C$ converges weakly to x and the sequence $\{(I - T)x_{\mathrm{n}}\}$ converges strongly to y, then $(I - T)x = y$. In particular, if $y = 0$, then $x \in Fix(T)$.*

Definition 2.7 A linear bounded operator A on a real Hilbert space H is said to be strongly positive if there exists a constant $\bar{\gamma} > 0$ with the property

$$\langle Ax, x\rangle \geq \bar{\gamma}\|x\|^2, \ \forall x \in H.$$

Lemma 2.4 ([13]) *Assume that A is a strongly positive self-adjoint bounded linear operator on a Hilbert space H with coefficient $\bar{\gamma} > 0$ and $0 < \rho \leq \|A\|^{-1}$. Then, $\|I - \rho A\| \leq 1 - \rho\bar{\gamma}$.*

Lemma 2.5 (Opial's Lemma [15]) *Let H be a real Hilbert space and $\{\mu_{\mathrm{n}}\}$ be a sequence in H such that there exists a nonempty set $C \subset H$ satisfying the following conditions:*

(i) *for every $\mu \in C$, $\lim_{n\to\infty} \|\mu_{\mathrm{n}} - \mu\|$ exists,*
(ii) *any weak cluster point of the sequence $\{\mu_{\mathrm{n}}\}$ belongs to C.*

Then, there exists $w^* \in C$ such that $\{\mu_{\mathrm{n}}\}$ converges weakly to w^*.

3 Main results

In this section, we first give fixed point formulation of the problem and based on it, we establish an iterative algorithm to prove the convergence analysis of the scheme.

Lemma 3.1 *The split monotone Yosida variational inclusion problem* $(SMYVIP)$ *have the solutions* x^* *and* y^* *if and only if*

$$x^* = R_{\lambda_1}^{B_1}\left[I + \lambda_1\left(J_{\lambda_1}^{B_1} - f_1\right)\right](x^*),$$

and

$$y^* = R_{\lambda_2}^{B_2}\left[I + \lambda_2\left(J_{\lambda_2}^{B_2} - f_2\right)\right](y^*).$$

Proof For $\lambda_i > 0$, we can see that $0 \in f_i(x) + B_i(x) - J_{\lambda_i}^{B_i}(x)$ if and only if $\left[I + \lambda_i\left(J_{\lambda_i}^{B_i} - f_i\right)\right](x) \in x + \lambda_i B_i(x)$. This directly follows from the definitions of resolvent operator $R_{\lambda_i}^{B_i}$, Yosida operator $J_{\lambda_i}^{B_i}$ and the split monotone Yosida variational inclusion problem $(SMYVIP)$. □

Based on above fixed point formulation, we have the following convergence theorem.

Theorem 3.1 *Let* H_1 *and* H_2 *be two real Hilbert spaces,* $A : H_1 \longrightarrow H_2$ *be a bounded linear operator, and* A^* *be the adjoint of* A*. Let* $B_1 : H_1 \longrightarrow 2^{H_1}$ *and* $B_2 : H_2 \longrightarrow 2^{H_2}$ *be two multi-valued maximal monotone mappings with nonempty values,* $f_1 : H_1 \longrightarrow H_1$ *and* $f_2 : H_2 \longrightarrow H_2$ *be two inverse strongly monotone mappings. Let* $\{g_n\}$ *be a family of k-demicontractive mappings and uniformly convergent for any* $x \in K$*, where* K *is any bounded subset of* H_1*,* $f : H_1 \longrightarrow H_1$ *be a* ξ*-contraction mapping,* D *be a strongly positive bounded linear operator on* H_1 *with coefficient* $\bar{r} > 0$ *and* $\beta \in \left(0, \frac{\bar{r}}{k}\right)$*. Assume that* $\Omega \neq \emptyset$ *and* $\tau > 0$*, and let the sequence* $\{x_n\}$ *be generated by the following iterative algorithm:*

$$\begin{cases} x_0 & \in H_1, \\ u_n & = T[x_n + \gamma A^*(S - I)Ax_n], \\ v_n & = \delta_n u_n + \tau g_n(u_n), \\ x_{n+1} & = (1 - \alpha_n D)v_n + \alpha_n \beta f(v_n), \end{cases} \tag{3.1}$$

where the operators $T = R_{\lambda_1}^{B_1}\left[I + \lambda_1\left(J_{\lambda_1}^{B_1} - f_1\right)\right]$ *and* $S = R_{\lambda_2}^{B_2}\left[I + \lambda_2\left(J_{\lambda_2}^{B_2} - f_2\right)\right]$*,* $\gamma \in \left(0, \frac{1}{\|A\|^2}\right)$*. Assume that the sequences* $\{\alpha_n\} \in [0, 1)$ *and* $\{\delta_n\} \in [0, 1)$ *satisfy the following conditions:*

(i) $\lim_{n\to\infty} \alpha_n = 0,\ \sum_{n=0}^{\infty} \alpha_n = \infty,\ \sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty,$

(ii) $\lim_{n\to\infty} \delta_n = 0,\ \sum_{n=0}^{\infty} \delta_n = \infty.$

Then, the sequence $\{x_n\}$ *converges strongly to a point* $x^* \in \Omega$ *which is a unique solution of the variational inequality problem:*

$$\langle (D - \beta f)x^*, x - x^* \rangle \geq 0, \quad \forall x \in \Omega. \tag{3.2}$$

If H *satisfies the Opial's property (see, Lemma 2.5), the the sequence* $\{x_n\}$ *converges weakly to an element* $x^* \in \Omega$.

Proof It is well known that $R_{\lambda_i}^{B_i}$ is firmly nonexpansive, and hence it is averaged. Since the composition of averaged mappings is again average, therefore $R_{\lambda_i}^{B_i}\left[I + \lambda_i\left(J_{\lambda_i}^{B_i} - f_i\right)\right]$ is averaged. As every averaged mapping is strongly nonexpansive [5], it follows that $R_{\lambda_i}^{B_i}\left[I + \lambda_i\left(J_{\lambda_i}^{B_i} - f_i\right)\right]$ is also strongly nonexpansive, and therefore T and S are nonexpansive.

Now, we have the following steps to prove the conclusion of the theorem.

Step I: We show that $\{x_n\}$ is bounded. Let us take $p \in \Omega$. As $p = Tp$ and $Ap = S(Ap)$, then by using the algorithm (3.1), we have

$$\begin{aligned}
\|u_n - p\|^2 &= \|T[x_n + \gamma A^*(S - I)Ax_n] - p\|^2 \\
&= \|T[x_n + \gamma A^*(S - I)Ax_n] - Tp\|^2 \\
&\leq \|x_n + \gamma A^*(S - I)Ax_n - p\|^2 \\
&\leq \Big\{\|x_n - p\|^2 + \gamma^2\|A^*(S - I)Ax_n\|^2 \\
&\quad +2\gamma\langle x_n - p, A^*(S - I)Ax_n\rangle\Big\} \\
&\leq \|x_n - p\|^2 + \gamma^2\|A\|^2\|(S - I)Ax_n\|^2 \\
&\quad +2\gamma\langle x_n - p, A^*(S - I)Ax_n\rangle.
\end{aligned} \tag{3.3}$$

Now,

$$\begin{aligned}
&2\gamma\langle x_n - p, A^*(S - I)Ax_n\rangle \\
&= 2\gamma\langle A(x_n - p), (S - I)Ax_n\rangle \\
&= 2\gamma\langle Ax_n - Ap, (S - I)Ax_n\rangle \\
&= 2\gamma\langle Ax_n - Ap + (S - I)Ax_n - (S - I)Ax_n, (S - I)Ax_n\rangle \\
&= 2\gamma\langle S(Ax_n) - Ap, (S - I)Ax_n\rangle - 2\gamma\|(S - I)Ax_n\|^2 \\
&= \gamma\Big\{\|S(Ax_n) - Ap\|^2 + \|(S - I)Ax_n\|^2 - \|Ax_n - Ap\|^2\Big\} \\
&\quad -2\gamma\|(S - I)Ax_n\|^2 \\
&\leq \gamma\|(S - I)Ax_n\|^2 - 2\gamma\|(S - I)Ax_n\|^2 \\
&= -\gamma\|(S - I)Ax_n\|^2.
\end{aligned} \tag{3.4}$$

Using (3.4), (3.3) becomes

$$\|u_n - p\|^2 = \|x_n - p\|^2 - \gamma(1 - \gamma\|A\|^2)\|(S - I)Ax_n\|^2, \tag{3.5}$$

which implies that

$$\|u_n - p\|^2 \leq \|x_n - p\|^2. \tag{3.6}$$

Using (3.1) and (3.6), we evaluate

$$
\begin{aligned}
\|v_n - p\|^2 &= \|\delta_n u_n + \tau g_n(u_n) - p\|^2 \\
&= \|\delta_n u_n - p + \tau g_n(u_n)\|^2 \\
&\le \delta_n \|u_n - p\|^2 + \tau \|g_n(u_n) - p\|^2 - \tau\delta_n \|u_n - g_n(u_n)\|^2 \\
&= \delta_n \|u_n - p\|^2 + \tau\{\|u_n - p\|^2 + k\|u_n - g_n(u_n)\|^2\} \\
&\quad -\tau\delta_n \|u_n - g_n(u_n)\|^2 \\
&= \delta_n \|u_n - p\|^2 + \tau \|u_n - p\|^2 + \tau k\|u_n - g_n(u_n)\|^2 \\
&\quad -\tau\delta_n \|u_n - g_n(u_n)\|^2 \\
&= (\delta_n + \tau)\|u_n - p\|^2 + \tau(k - \delta_n)\|u_n - g_n(u_n)\|^2 \qquad (3.7) \\
&\le (\delta_n + \tau)\|x_n - p\|^2 + \tau(k - \delta_n)\|u_n - g_n(u_n)\|^2 \\
&\le \|x_n - p\|^2 - \tau(\delta_n - k)\|u_n - g_n(u_n)\|^2. \qquad (3.8)
\end{aligned}
$$

Hence,

$$\|v_n - p\| \le \|x_n - p\|. \tag{3.9}$$

Again using (3.1) and (3.9), we obtain

$$
\begin{aligned}
\|x_{n+1} - p\| &= \|(1 - \alpha_n D)v_n + \alpha_n \beta f(v_n) - p\| \\
&= \|(1 - \alpha_n D)(v_n - p) + \alpha_n(\beta f(v_n) - Dp)\| \\
&\le \alpha_n \Big\{\beta\|f(v_n) - f(p)\| + \|\beta f(p) - Dp\|\Big\} \\
&\quad +(1 - \alpha_n \bar{r})\|v_n - p\| \\
&\le \alpha_n \beta\alpha \|v_n - p\| + \alpha_n \|\beta f(p) - Dp\| + (1 - \alpha_n \bar{r})\|v_n - p\| \\
&= [\alpha_n\beta\alpha + (1 - \alpha_n\bar{r})]\|v_n - p\| + \alpha_n\|\beta f(p) - Dp\| \\
&= [\alpha_n\beta\alpha + (1 - \alpha_n\bar{r})]\|x_n - p\| + \alpha_n\|\beta f(p) - Dp\| \\
&\le [1 - \alpha_n(\bar{r} - \alpha\beta)]\|x_n - p\| + \alpha_n\|\beta f(p) - Dp\| \\
&\le [1 - \alpha_n(\bar{r} - \alpha\beta)]\|x_n - p\| + (\bar{r} - \alpha\beta)\frac{\|\beta f(p) - Dp\|}{\bar{r} - \alpha\beta} \\
&= \max\{\|x_n - p\|, \frac{\|\beta f(p) - Dp\|}{\bar{r} - \alpha\beta}\},
\end{aligned}
$$

Therefore $\{x_n\}$ is a bounded sequence. Furthermore, $\{v_n\}$, $\{f(v_n)\}$, and $\{u_n\}$ are also bounded sequences.

Step II: Making use of step-I, we estimate

$$\begin{aligned}
\|x_{n+1}-p\|^2 &= \|(I-\alpha_n D)v_n+\alpha_n\beta f(v_n)-p\|^2\\
&= \|(I-\alpha_n D)(v_n-p)+\alpha_n(\beta f(v_n)-Dp)\|^2\\
&\le \|(I-\alpha_n D)\|^2\|v_n-p\|^2+\alpha_n^2\|\beta f(v_n)-Dp\|^2\\
&\quad +2\alpha_n(1-\alpha_n\bar{r})\|\beta f(v_n)-Dp\|\|v_n-p\|\\
&\le (1-\alpha_n\bar{r})^2\Big\{(\tau+\delta_n)\|u_n-p\|^2\\
&\quad +\tau(k-\delta_n)\|u_n-g_n(u_n)\|^2\Big\}\\
&\quad +\alpha_n{}^2\|\beta f(v_n)-Dp\|^2+2\alpha_n(1-\alpha_n\bar{r})\|\beta f(v_n)\\
&\quad -Dp\|\|v_n-p\|
\end{aligned}$$

$$\begin{aligned}
&\le (1-\alpha_n\bar{r})^2\Big\{(\tau+\delta_n)\big[\|x_n-p\|^2-\gamma(1-\gamma\|A\|^2)\|(S-I)Ax_n\|^2\big]\\
&\quad +\tau(k-\delta_n)\|u_n-g_n(u_n)\|^2\Big\}+\alpha_n{}^2\|\beta f(v_n)-Dp\|^2\\
&\quad +2\alpha_n(1-\alpha_n\bar{r})\|\beta f(v_n)-Dp\|\|v_n-p\|.
\end{aligned}$$

This implies that

$$\begin{aligned}
&(1-\alpha_n\bar{r})^2\gamma(1-\gamma\|A\|^2)(\tau+\delta_n)\|(S-I)Ax_n\|^2\\
&+(1-\alpha_n\bar{r})^2\tau(\delta_n-k)\|u_n-g_n(u_n)\|^2\\
\le\ &(\|x_n-p\|^2-\|x_{n+1}-p\|^2)+\alpha_n{}^2\|\beta f(v_n)-Dp\|^2\\
&+2\alpha_n(1-\alpha_n\bar{r})\|\beta f(v_n)-Dp\|\|v_n-p\|.
\end{aligned} \tag{3.10}$$

Now, we consider two cases:

Case I: Suppose that $\{\|x_n-p\|\}$ is a monotone sequence. Then, there exists natural number n_0 such that $n>n_0$, we have $\{\|x_n-p\|\}$ is either nondecreasing or nonincreasing. As $\{\|x_n-p\|\}$ is bounded sequence, it is convergent. Also, as $\lim_{n\to\infty}\alpha_n=0$, $\{f(v_n)\}$, $\{v_n\}$ and $\{x_n\}$ are bounded, from (3.10), we get

$$\lim_{n\to\infty}(1-\alpha_n\bar{r})^2\gamma(1-\gamma\|A\|^2)(\tau+\delta_n)\|(S-I)Ax_n\|^2=0 \tag{3.11}$$

and

$$\lim_{n\to\infty}(1-\alpha_n\bar{r})^2\tau(\delta_n-k)\|u_n-g_n(u_n)\|^2=0. \tag{3.12}$$

By the conditions that $\gamma\in\left(0,\frac{1}{\|A\|^2}\right)$, $(1-\alpha_n\bar{r})>0$, $\tau(\delta_n-k)>0$, we can see from (3.11) and (3.12) that

$$\lim_{n\to\infty}\|(S-I)Ax_n\|=0 \tag{3.13}$$

and

$$\lim_{n\to\infty} \|u_{\mathrm{n}} - g_n(u_{\mathrm{n}})\| = 0. \tag{3.14}$$

Also, we can see from (3.1) that

$$\begin{aligned}
\|u_{\mathrm{n}} - p\|^2 &= \|T(x_{\mathrm{n}} + \gamma A^*(S-I)Ax_{\mathrm{n}}) - p\|^2 \\
&= \|T(x_{\mathrm{n}} + \gamma A^*(S-I)Ax_{\mathrm{n}}) - T(p)\|^2 \\
&\le \langle u_{\mathrm{n}} - p, x_{\mathrm{n}} + \gamma A^*(S-I)Ax_{\mathrm{n}} - p\rangle \\
&= \frac{1}{2}\{\|u_{\mathrm{n}} - p\|^2 + \|x_{\mathrm{n}} + \gamma A^*(S-I)Ax_{\mathrm{n}} - p\|^2 \\
&\quad -\|u_{\mathrm{n}} - p - (x_{\mathrm{n}} + \gamma A^*(S-I)Ax_{\mathrm{n}} - p)\|^2\} \\
&= \frac{1}{2}\{\|u_{\mathrm{n}} - p\|^2 + \|x_{\mathrm{n}} + \gamma A^*(S-I)Ax_{\mathrm{n}} - p\|^2 \\
&\quad -\|u_{\mathrm{n}} - x_{\mathrm{n}} - \gamma A^*(S-I)Ax_{\mathrm{n}}\|^2\}
\end{aligned}$$

$$\begin{aligned}
&\le \frac{1}{2}\{\|u_{\mathrm{n}} - p\|^2 + \|x_{\mathrm{n}} - p\|^2 - \|u_{\mathrm{n}} - x_{\mathrm{n}}\|^2 - \gamma^2\|A^*(S-I)Ax_{\mathrm{n}}\|^2 \\
&\quad +2\gamma\langle u_{\mathrm{n}} - x_{\mathrm{n}}, A^*(S-I)Ax_{\mathrm{n}}\rangle\} \\
&= \frac{1}{2}\{\|u_{\mathrm{n}} - p\|^2 + \|x_{\mathrm{n}} - p\|^2 - \|u_{\mathrm{n}} - x_{\mathrm{n}}\|^2 - \gamma^2\|A^*\|^2\|(S-I)Ax_{\mathrm{n}}\|^2 \\
&\quad +2\gamma\|A(u_{\mathrm{n}}) - A(x_{\mathrm{n}})\|\|(S-I)Ax_{\mathrm{n}}\|^2\},
\end{aligned}$$

which implies that

$$\begin{aligned}
\|u_{\mathrm{n}} - p\|^2 \le\ &\|x_{\mathrm{n}} - p\|^2 - \|u_{\mathrm{n}} - x_{\mathrm{n}}\|^2 - \gamma^2\|A^*\|^2\|(S-I)Ax_{\mathrm{n}}\|^2 \\
&+2\gamma\|A(u_{\mathrm{n}}) - A(x_{\mathrm{n}})\|\|(S-I)Ax_{\mathrm{n}}\|.
\end{aligned} \tag{3.15}$$

On the other hand, we have

$$\begin{aligned}
\|x_{\mathrm{n}+1} - p\|^2 &= \|(1-\alpha_{\mathrm{n}}D)v_{\mathrm{n}} + \alpha_{\mathrm{n}}\beta f(v_{\mathrm{n}}) - p\|^2 \\
&= \|(1-\alpha_{\mathrm{n}}D)(v_{\mathrm{n}} - p) + \alpha_{\mathrm{n}}(\beta f(v_{\mathrm{n}}) - Dp)\|^2 \\
&\le \|(I-\alpha_{\mathrm{n}}D)\|^2\|v_{\mathrm{n}} - p\|^2 + {\alpha_{\mathrm{n}}}^2\|\beta f(v_{\mathrm{n}}) - Dp\|^2 \\
&\quad +2\alpha_{\mathrm{n}}\|I-\alpha_{\mathrm{n}}D\|\|v_{\mathrm{n}} - p\|\|\beta f(v_{\mathrm{n}}) - Dp\| \\
&\le (1-\alpha_{\mathrm{n}}\bar{r})^2\|v_{\mathrm{n}} - p\|^2 + {\alpha_{\mathrm{n}}}^2\|\beta f(v_{\mathrm{n}}) - Dp\|^2 \\
&\quad +2\alpha_{\mathrm{n}}(1-\alpha_{\mathrm{n}}\bar{r})\|v_{\mathrm{n}} - p\|\|\beta f(v_{\mathrm{n}}) - Dp\| \\
&\le (1-\alpha_{\mathrm{n}}\bar{r})^2\{\|u_{\mathrm{n}} - p\|^2 + \tau(k-\delta_{\mathrm{n}})\|u_{\mathrm{n}} - g_n(u_{\mathrm{n}})\|^2\} \\
&\quad +{\alpha_{\mathrm{n}}}^2\|\beta f(v_{\mathrm{n}}) - Dp\|^2 + 2\alpha_{\mathrm{n}}(1-\alpha_{\mathrm{n}}\bar{r})\|v_{\mathrm{n}} - p\|\|\beta f(v_{\mathrm{n}}) - Dp\| \\
&\le (1-\alpha_{\mathrm{n}}\bar{r})^2\{\|x_{\mathrm{n}} - p\|^2 - \|u_{\mathrm{n}} - x_{\mathrm{n}}\|^2 - \gamma^2\|A\|^2\|(S-I)Ax_{\mathrm{n}}\|^2 \\
&\quad +2\gamma\|A(u_{\mathrm{n}}) - A(x_{\mathrm{n}})\|\|(S-I)Ax_{\mathrm{n}}\|\} \\
&\quad +\tau(1-\alpha_{\mathrm{n}}\bar{r})^2(k-\delta_{\mathrm{n}})\|u_{\mathrm{n}} - g_n(u_{\mathrm{n}})\|^2 \\
&\quad +{\alpha_{\mathrm{n}}}^2\|\beta f(v_{\mathrm{n}}) - Dp\|^2 + 2\alpha_{\mathrm{n}}(1-\alpha_{\mathrm{n}}\bar{r})\|v_{\mathrm{n}} - p\|\|\beta f(v_{\mathrm{n}}) - Dp\|,
\end{aligned}$$

and hence

$$
\begin{aligned}
(1-\alpha_{\mathrm{n}}\bar{r})^2\|u_{\mathrm{n}}-x_{\mathrm{n}}\|^2 \leq\ & \|x_n-p\|^2-\|x_{\mathrm{n}+1}-p\|^2 \\
& -(1-\alpha_{\mathrm{n}}\bar{r})^2\gamma^2\|A\|^2\|(S-I)Ax_{\mathrm{n}}\|^2 \\
& +2\gamma(1-\alpha_{\mathrm{n}}\bar{r})^2\|A(u_{\mathrm{n}})-A(x_{\mathrm{n}})\|\|(S-I)Ax_{\mathrm{n}}\| \\
& +\tau(1-\alpha_{\mathrm{n}}\bar{r})^2(k-\delta_{\mathrm{n}})\|u_{\mathrm{n}}-g_{\mathrm{n}}(u_{\mathrm{n}})\|^2 \\
& +\alpha_{\mathrm{n}}{}^2\|\beta f(v_{\mathrm{n}})-Dp\|^2+2\alpha_{\mathrm{n}}(1-\alpha_{\mathrm{n}}\bar{r})\|v_{\mathrm{n}} \\
& -p\|\|\beta f(v_{\mathrm{n}})-Dp\|.
\end{aligned}
\tag{3.16}
$$

Since $\{x_{\mathrm{n}}-p\}$ is convergent, α_n, δ_n to 0; $1-\alpha\bar{r}>0$ $\{f(v_{\mathrm{n}})\}$, $\{v_{\mathrm{n}}\}$, and $\{x_{\mathrm{n}}\}$ are bounded and using the facts (3.13) and (3.14), we conclude that

$$
\lim_{n\to\infty}\|u_{\mathrm{n}}-x_{\mathrm{n}}\|=0. \tag{3.17}
$$

Next, we want to show that $\lim\limits_{n\to\infty}\|v_n-x_n\|=0$.

First, by using algorithm (3.1) and contractivity of f, we have

$$
\begin{aligned}
\|x_{n+1}-x_n\| =\ & \|(I-\alpha_n D)v_n+\alpha_n\beta f(v_n)-(I-\alpha_{n-1}D)v_{n-1} \\
& +\alpha_{n-1}\beta f(v_{n-1})\| \\
=\ & \|\alpha_n\beta f(v_n)+\alpha_n\beta f(v_{n-1})-\alpha_n\beta f(v_{n-1})-\alpha_{n-1}\beta f(v_{n-1}) \\
& +(I-\alpha_n D)v_n+(I-\alpha_n D)v_{n-1}-(I-\alpha_n D)v_{n-1} \\
& -(I-\alpha_n D)v_{n-1}\| \\
\leq\ & \alpha_n\beta\|f(v_n)-f(v_{n-1})\|+|\alpha_n-\alpha_{n-1}|\beta\|f(v_{n-1})\| \\
& +(1-\alpha_n\bar{r})\|v_n-v_{n-1}\|+|\alpha_n-\alpha_{n-1}|\|Dv_{n-1}\| \\
\leq\ & (\alpha_n\beta\xi+1-\alpha_n\bar{r})\|v_n-v_{n-1}\|+|\alpha_n-\alpha_{n-1}|(\beta\|f(v_{n-1})\| \\
& +\|Dv_{n-1}\|).
\end{aligned}
\tag{3.18}
$$

Next, we will estimate $\|v_n-v_{n-1}\|$. Using (3.1), we have

$$
\begin{aligned}
\|v_n-v_{n-1}\| =\ & \|\delta_n u_n+\tau g_n(u_n)-\delta_{n-1}u_{n-1}-\tau g_{n-1}(u_{n-1})\| \\
\leq\ & \|\delta_n u_n-\delta_{n-1}u_{n-1}\|+\tau\|g_n(u_n)-g_{n-1}(u_{n-1})\| \\
=\ & \|\delta_n u_n-\delta_n u_{n-1}+\delta_n u_{n-1}-\delta_{n-1}u_{n-1}\| \\
& +\tau\|g_n(u_n)-g_n(u_{n-1})+g_n u_{n-1}-g_{n-1}(u_{n-1})\| \\
=\ & \delta_n\|u_n-u_{n-1}\|+|\delta_n-\delta_{n-1}|\|u_{n-1}\|+\tau\|g_n(u_n)-g_n(u_{n-1})\| \\
& +\tau\|g_n u_{n-1}-g_{n-1}(u_{n-1})\| \\
\leq\ & (\delta_n+\tau L_n)\|u_n-u_{n-1}\|+|\delta_n-\delta_{n-1}|\|u_{n-1}\| \\
& +\tau\|g_n u_{n-1}-g_{n-1}(u_{n-1})\|.
\end{aligned}
\tag{3.19}
$$

Furthermore, we can see from the nonexpansivity of T and S that

$$\begin{aligned}
\|u_n - u_{n-1}\| &= \|T(x_n + \gamma A^*(S-I)Ax_n) - T(x_{n-1} + \gamma A^*(S-I)Ax_{n-1})\| \\
&\leq \|x_n + \gamma A^*(S-I)Ax_n - x_{n-1} - \gamma A^*(S-I)Ax_{n-1}\| \\
&\leq \|x_n - x_{n-1}\| + \gamma\|A^*(S-I)Ax_n - A^*(S-I)Ax_{n-1}\| \\
&\leq \|x_n - x_{n-1}\| + \gamma\|A\|\|(S-I)Ax_n - (S-I)Ax_{n-1}\| \\
&= \|x_n - x_{n-1}\| + \gamma\|A\|\|S(Ax_n) - Ax_n - S(Ax_{n-1}) - Ax_{n-1}\| \\
&\leq \|x_n - x_{n-1}\| + \gamma\|A\|\{\|S(Ax_n) - S(Ax_{n-1})\| + \|Ax_n - Ax_{n-1}\|\} \\
&\leq \|x_n - x_{n-1}\| + \gamma\|A\|\{\|Ax_n - Ax_{n-1}\| + \|Ax_n - Ax_{n-1}\|\} \\
&\leq \|x_n - x_{n-1}\| + 2\gamma\rho\|A\|\|x_n - x_{n-1}\| \\
&= (1 + 2\gamma\rho\|A\|)\|x_n - x_{n-1}\|. \qquad (3.20)
\end{aligned}$$

Thus, we have

$$\begin{aligned}
\|x_{n+1} - x_n\| &= (\alpha_n\beta\xi + 1 - \alpha_n\bar{r})\Big[(\delta_n + \tau L_n)(1 + 2\gamma\rho\|A\|)\|x_n - x_{n-1}\| \\
&\quad + |\delta_n - \delta_{n-1}|\|u_{n-1}\| + \tau\|g_n u_{n-1} - g_{n-1}(u_{n-1})\|\Big] \\
&\quad + |\alpha_n - \alpha_{n-1}|\Big(\beta\|f(v_{n-1})\| + \|Dv_{n-1}\|\Big) \\
&= (1 - \alpha_n(\bar{r} - \beta\xi))(\delta_n + \tau L_n)(1 + 2\gamma\rho\|A\|)\|x_n - x_{n-1}\| \\
&\quad + (1 - \alpha_n(\bar{r} - \beta\xi))|\delta_n - \delta_{n-1}|\|u_{n-1}\| \\
&\quad + (1 - \alpha_n(\bar{r} - \beta\xi))\tau\|g_n u_{n-1} - g_{n-1}(u_{n-1})\| \\
&\quad + |\alpha_n - \alpha_{n-1}|\Big(\beta\|f(v_{n-1})\| + \|Dv_{n-1}\|\Big).
\end{aligned}$$

Using uniform convergence of $\{g_n(x)\}$ and Lemma 2.1, we have

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \qquad (3.21)$$

From (3.1), we have

$$\begin{aligned}
\|x_{n+1} - v_n\| &= \|(I - \alpha_n D)v_n + \alpha_n\beta f(v_n) - v_n\| \\
&= \|\alpha_n\beta f(v_n) - \alpha_n Dv_n\| \\
&\leq \alpha_n\Big\{\|\beta f(v_n)\| + \|Dv_n\|\Big\},
\end{aligned}$$

which implies that

$$\lim_{n\to\infty} \|x_{n+1} - v_n\| = 0. \qquad (3.22)$$

Further from (3.21) and (3.22), we get

$$\begin{aligned}
\lim_{n\to\infty} \|x_n - v_n\| &= \lim_{n\to\infty} \|x_n - x_{n+1} + x_{n+1} - v_n\| \\
&\leq \lim_{n\to\infty} \|x_{n+1} - x_n\| + \lim_{n\to\infty} \|x_{n+1} - v_n\| \\
&= 0. \qquad (3.23)
\end{aligned}$$

Hence using (3.17) and (3.23), we get

$$\begin{aligned}\lim_{n\to\infty}\|u_n - v_n\| &= \lim_{n\to\infty}\|u_n - x_n + x_n - v_n\|\\ &\le \lim_{n\to\infty}\|u_n - x_n\| + \lim_{n\to\infty}\|x_n - v_n\|\\ &= 0.\end{aligned}$$

Step III: We will prove that $\{x_n\}$ converges weakly to $x^* \in \Omega$ and

$$\lim_{n\to\infty}\sup\langle (D-\beta f)x^*, x_n - x^*\rangle \ge 0, \tag{3.24}$$

where x^* is a unique solution of the variational inequality.

To prove the above inequality, we can take a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that

$$0 \le \lim_{n\to\infty}\sup\langle (D-\beta f)x^*, x_n - x^*\rangle = \lim_{n\to\infty}\langle (D-\beta f)x^*, x_{n_i} - x^*\rangle.$$

Since $\{x_{n_i}\}$ is a bounded sequence, there exists a subsequence $\{x_{n_{i_j}}\}$ of $\{x_{n_i}\}$ which converges weakly to z. Without loss of generality, we suppose that $\{x_{n_i}\} \rightharpoonup z$. Since $x^* = P_\Omega(I - D + \beta f)x^*$, it follows that

$$\begin{aligned}\lim_{n\to\infty}\sup\langle (D-\beta f)x^*, x_n - x^*\rangle &= \lim_{i\to\infty}\langle (D-\beta f)x^*, x_{n_i} - x^*\rangle\\ &= \langle (D-\beta f)x^*, z - x^*\rangle \ge 0,\end{aligned}$$

which shows that (3.24) is proved.

Finally, we show that $x_n \to x^* \in \Omega$.

Further, since $S = R_{\lambda_2}^{B_2}\left[I + \lambda_2\left(J_{\lambda_2}^{B_2} - f_2\right)\right]$ is nonexpansive, by Lemma 3.1 and from (3.13), we have

$$S(Ax^*) = Ax^*,$$

this means that

$$\begin{aligned}Ax^* &\in Fix(S)\\ &= Fix\left(R_{\lambda_2}^{B_2}\left[I + \lambda_2\left(J_{\lambda_2}^{B_2} - f_2\right)\right]\right).\end{aligned}$$

Now, we consider $w_n = x_n + \gamma A^*(S - I)Ax_n$. Then, from inequality (3.13), we can see that

$$\|w_n - x_n\| = 0. \tag{3.25}$$

Combining (3.17) with (3.25), we deduce that

$$\begin{aligned}\lim_{n\to\infty}\|u_n - w_n\| &= \lim_{n\to\infty}\|u_n - x_n + x_n - w_n\|\\ &\le \lim_{n\to\infty}\|u_n - x_n\| + \lim_{n\to\infty}\|w_n - x_n\|.\\ &= 0.\end{aligned}$$

As we can see that $u_n = T(w_n)$, by using similar arguments as above, we establish that $x^* \in Fix(T) = Fix\left(R_{\lambda_1}^{B_1}[I + \lambda_1(J_{\lambda_1}^{B_1} - f_1)]\right)$. This implies that $x^* \in \Omega$.

Assume that there exists another subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\{x_{n_k}\}$ converges weakly to $y^* \in H_1$. Using the same arguments as above, we also conclude that $y^* \in \Omega$. Since each Hilbert space possesses Opial's condition, we have

$$
\begin{aligned}
\lim_{n_i \to \infty} \inf \|x_{n_i} - x^*\| &< \lim_{n_i \to \infty} \inf \|x_{n_i} - y^*\| \\
&= \lim_{n \to \infty} \inf \|x_n - y^*\| \\
&= \lim_{n_j \to \infty} \inf \|x_{n_j} - y^*\| \\
&< \lim_{n_j \to \infty} \inf \|x_{n_j} - x^*\| \\
&= \lim_{n \to \infty} \inf \|x_n - x^*\| \\
&= \lim_{n_i \to \infty} \inf \|x_{n_i} - x^*\|,
\end{aligned}
$$

which is a contradiction. Therefore, $\{x_n\}$ converges weakly to $x^* \in \Omega$.

Now,

$$
\begin{aligned}
\|x_{n+1} - x^*\|^2 &= \|(I - \alpha_n D)v_n + \alpha_n \beta f(v_n) - x^*\|^2 \\
&= \|(I - \alpha_n D)(v_n - x^*) + \alpha_n(\beta f(v_n) - Dx^*)\|^2 \\
&\leq \|(I - \alpha_n D)(v_n - x^*)\|^2 + 2\alpha_n \langle \beta f(v_n) - Dx^*, x_{n+1} - x^* \rangle \\
&\leq (1 - \alpha_n \bar{r})^2 \|v_n - x^*\|^2 + 2\alpha_n \langle \beta f(v_n) - \beta f(x^*), x_{n+1} - x^* \rangle \\
&\quad + 2\alpha_n \langle \beta f(x^*) - D(x^*), x_{n+1} - x^* \rangle \\
&\leq (1 - \alpha_n \bar{r})^2 \|x_n - x^*\|^2 + 2\alpha_n \beta \|f(v_n) - f(x^*)\| \|x_{n+1} - x^*\| \\
&\quad + 2\alpha_n \langle \beta f(x^*) - D(x^*), x_{n+1} - x^* \rangle \\
&\leq (1 - \alpha_n \bar{r})^2 \|x_n - x^*\|^2 + 2\alpha_n \beta \xi \{\|v_n - x^*\| \|x_{n+1} - x^*\|\} \\
&\quad + 2\alpha_n \langle \beta f(x^*) - D(x^*), x_{n+1} - x^* \rangle \\
&\leq (1 - \alpha_n \bar{r})^2 \|x_n - x^*\|^2 + 2\alpha_n \beta \xi \Big\{\|x_n - x^*\| \|x_{n+1} - x^*\|\Big\} \\
&\quad + 2\alpha_n \langle \beta f(x^*) - D(x^*), x_{n+1} - x^* \rangle \\
&\leq \Big((1 - \alpha_n \bar{r})^2 + \alpha_n \beta \xi\Big) \|x_n - x^*\|^2 + \alpha_n \beta \xi \|x_{n+1} - x^*\|^2 \\
&\quad + 2\alpha_n \langle \beta f(x^*) - D(x^*), x_{n+1} - x^* \rangle,
\end{aligned}
$$

which implies that

$$
\begin{aligned}
(1 - \alpha_n \beta \xi)\|x_{n+1} - x^*\|^2 &\leq \big(1 - 2\alpha_n \bar{r} + \alpha_n^2 \bar{r}^2 + \alpha_n \beta \xi\big) \|x_n - x^*\|^2 \\
&\quad + 2\alpha_n \langle \beta f(x^*) - D(x^*), x_{n+1} - x^* \rangle,
\end{aligned}
$$

which shows that

$$
\begin{aligned}
\|x_{n+1} - x^*\|^2 &\le \frac{(1 - 2\alpha_n \bar{r} + \alpha_n \beta \xi)}{1 - \alpha_n \beta \xi}\|x_n - x^*\|^2 + \frac{(\alpha_n^2 \bar{r}^2)}{1 - \alpha_n \beta \xi}\|x_n - x^*\|^2 \\
&\quad + \frac{2\alpha_n}{1 - \alpha_n \beta \xi}\langle \beta f(x^*) - D(x^*), x_n - x^* \rangle \\
&\le \left(1 - \frac{2(\bar{r} - \beta\xi)\alpha_n}{1 - \alpha_n \beta \xi}\right)\|x_n - x^*\|^2 + \frac{(\alpha_n^2 \bar{r}^2)}{1 - \alpha_n \beta \xi}\|x_n - x^*\|^2 \\
&\quad + \frac{2\alpha_n}{1 - \alpha_n \beta \xi}\langle \beta f(x^*) - D(x^*), x_n - x^* \rangle \\
&= (1 - C_n)\|x_n - x^*\|^2 + d_n,
\end{aligned}
$$

where $C_n = \dfrac{2(\bar{r} - \beta\xi)\alpha_n}{1 - \alpha_n \beta \xi}$ and $d_n = \dfrac{\alpha_n}{1 - \alpha_n \beta \xi}\Big\{\alpha_n \bar{r}\|x_n - x^*\|^2 + 2\langle \beta f(x^*) - D(x^*), x_n - x^* \rangle\Big\}$.

Now, from the given conditions and (3.24) we can see that $C_n \to 0$ and $\sum_{n=1}^{\infty} C_n = \infty$ and $\lim_{n\to\infty} \sup d_n \le 0$. Therefore by Lemma 2.4, we establish that the sequence $\{x_n\}$ converges strongly to $x^* \in \Omega$.

Case II: Suppose that $\{\|x_n - x^*\|\}$ is not a monotone sequence. Then, by definition, we can determine a sequence of positive integers $\{\varphi(n)\}$, for $n \ge n_0$ (where n_0 is large enough) by

$$
\varphi(n) = \max\{k \le x : \|x_k - x^*\| \le \|x_{k+1} - x^*\|\}.
$$

Then, evidently $\{\varphi(n)\}$ is a nondecreasing sequence such that $\varphi(n) \to \infty$ as $n \to \infty$, and for any $n \ge n_0$, we have

$$
\|x_{\varphi(n)} - x^*\| \le \|x_{\varphi(n)+1} - x^*\|.
$$

Hence, $\{\|x_{\varphi(n)} - x^*\|\}$ is a decreasing sequence. According to case I, we have $\lim_{n\to\infty} \|x_{\varphi(n)} - x^*\| = 0$ and $\lim_{n\to\infty} \|x_{\varphi(n)+1} - x^*\| = 0$. By using Lemma 2.2, we conclude that

$$
0 \le \|x_n - x^*\| \le \max\Big\{\|x_n - x^*\|, \|x_{\varphi(n)} - x^*\|\Big\} \le \|x_{\varphi(n)+1} - x^*\| \to 0 \ as \ n \to \infty.
$$

This implies that $\{x_n\}$ converges strongly to $x^* \in \Omega$. This completes the proof. $\square$

Remark 3.1 Since D is a strongly positive linear-bounded operator on H_1 with coefficient $\bar{r} > 0$, then from [13] for $\beta \in \left(0, \frac{\bar{r}}{k}\right)$, we can observe that $D - \beta f$ is strongly monotone with coefficient $(\bar{r} - \beta k)$ which indeed shows the uniqueness of the solution of variational inequality (3.2).

Next, we construct a numerical example to illustrate the algorithm (3.1) and convergence analysis of the sequences of our main result.

Example 3.1 Let $B_1 : H_1 \to H_1$ defined by $B_1(x) = 2(x-1)$ and $B_2 : H_2 \to H_2$ defined by $B_2(x) = x+1$, $f_1 : H_1 \to H_1$ defined by $f_1(x) = 4x$, $f_2 : H_2 \to H_2$ defined by $f_2(x) = \frac{x}{2}$, $\lambda_1 = \lambda_2 = \frac{1}{4}$. Let the mapping $A : H_1 \to H_2$ be defined by $A(x) = -2x$. Let the iterative sequences $\{u_n\}$, $\{v_n\}$, and $\{x_n\}$ be computed by the following iterative scheme:

$$\begin{cases} u_n &= T[x_n + \gamma A^*(S-I)Ax_n], \\ v_n &= \delta_n u_n + \tau g_n(u_n), \\ x_{n+1} &= (1-\alpha_n D)v_n + \alpha_n \beta f(v_n), \end{cases}$$

First, we compute the resolvent operator and the Yosida approximation operator as

$$\begin{aligned} R_{\lambda_1}^{B_1} &= [I + \lambda_1 B_1]^{-1} = \frac{2x}{3} + \frac{1}{3}, \\ R_{\lambda_2}^{B_2} &= [I + \lambda_2 B_2]^{-1} = \frac{4x}{5} - \frac{1}{5}, \\ J_{\lambda_1}^{B_1} &= \frac{1}{\lambda_1}[I - R_{\lambda_1}^{B_1}](x) = \frac{4x}{3} + \frac{4}{3}, \\ J_{\lambda_2}^{B_2} &= \frac{1}{\lambda_2}[I - R_{\lambda_2}^{B_2}](x) = \frac{4x}{5} + \frac{4}{5}. \end{aligned}$$

Then, the operator

$$\begin{aligned} T(x) &= R_{\lambda_1}^{B_1}[I + \lambda_1(J_{\lambda_1}^{B_1} - f_1)] \\ &= \frac{2x}{9} + \frac{1}{9}, \end{aligned}$$

and

$$\begin{aligned} S(x) &= R_{\lambda_2}^{B_2}[I + \lambda_2(J_{\lambda_2}^{B_2} - f_2)] \\ &= \frac{43x}{50} - \frac{1}{25}. \end{aligned}$$

Since A is a bounded linear operator on $\mathbb{R}$ with adjoint operator A^* and $\|A\| = \|A^*\| = 2$, and hence, $\gamma \in \left(0, \frac{1}{4}\right)$. Therefore, we chose $\gamma = 0.1$. Further, let the mapping $g_n : H_1 \to H_1$ defined by $g_n(x) = \frac{1}{5}x + \frac{1}{11(n+1)}$, the mapping $D : H_1 \to H_1$ defined by $D(x) = \frac{1}{2}x$ and the mapping $f : H_1 \to H_1$ defined by $f(x) = 2x$. Let

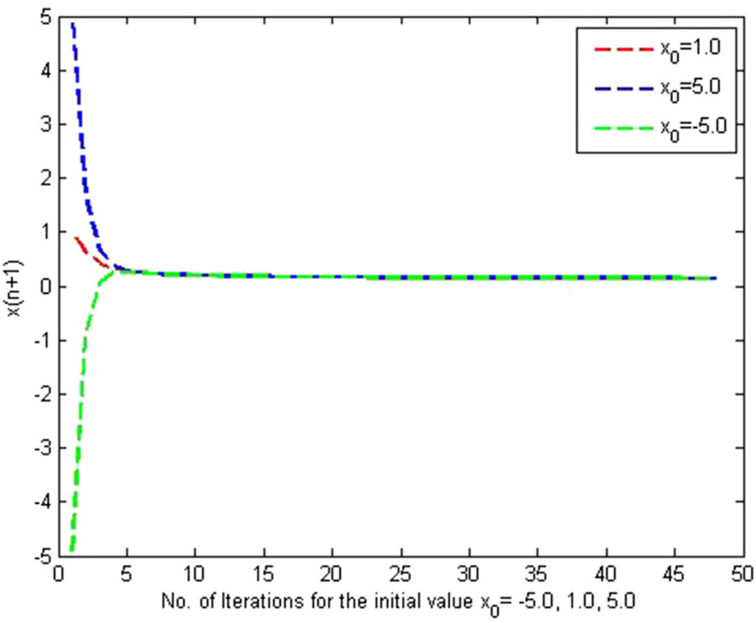

Fig. 1 The convergence of $\{x_n\}$ with initial values $x_0 = 1$, $x_0 = 5$ and $x_0 = -5$

Table 1 Computational results for different initial values

No. of iterations	$x_0 = 1.0$ x_n	No. of iterations	$x_0 = 5.0$ x_n	No. of iterations	$x_0 = -5.0$ x_n
1	1.0000	1	5.0000	1	−5.0000
5	0.2711	5	0.2800	5	0.2577
10	0.1947	10	0.1947	10	0.1947
15	0.1722	15	0.1722	15	0.1722
20	0.1613	20	0.1613	20	0.1613
25	0.1549	25	0.1549	25	0.1549
30	0.1506	30	0.1506	30	0.1506
35	0.1476	35	0.1476	35	0.1476
40	0.1454	40	0.1454	40	0.1454
45	0.1436	45	0.1436	45	0.1436
46	0.1433	46	0.1433	46	0.1433
47	0.1430	47	0.1430	47	0.1430
48	0.1428	48	0.1428	48	0.1428
49	0.1428	49	0.1428	49	0.1428
50	0.1428	50	0.1428	50	0.1428

$\tau = \frac{47}{10}$, $\delta_n = \frac{47}{110(n+1)}$, $\alpha_n = \frac{1}{4(n+1)}$ and $\beta = \frac{1}{2}$, then the sequences $\{u_n\}$, $\{v_n\}$, $\{x_n\}$ computed by the iterative schemes:

$$\begin{aligned} u_n &= T[x_n + \gamma A^*(S-I)Ax_n] = \frac{1}{9}\left(\frac{236x_n}{125} + \frac{127}{125}\right) \\ v_n &= \delta_n u_n + \tau g_n(u_n) = \left(\frac{47}{10}\right)\left(\frac{1}{11(n+1)} + \frac{1}{5}\right)u_n + \frac{1}{22(n+1)} \\ x_{n+1} &= (1-\alpha_n D)v_n + \alpha_n \beta f(v_n) = \left(1 + \frac{1}{8(n+1)}\right)v_n \end{aligned}$$

Then, the sequence $\{x_n\}$ converges to a point $x^* = \dfrac{1}{7}$ (Fig. 1).

All codes are written in MATLAB 2012; we have the following different initial values $x_0 = 1.0; 5.0; -5.0$ which shows that the sequence $\{x_n\}$ converge to $x^* = \dfrac{1}{7}$ (Table 1).

4 Conclusion

The purpose of this article is to introduce a new type of split monotone Yosida inclusion problem in Hilbert spaces and to construct an iterative algorithm to estimate the approximate solutions of our problem. Our problem is a new problem and much more general than split monotone variational inclusion problem (SMVIP), split variational inclusion problem (SVIP), and Yosida inclusion problem (1.8). Since generalized Yosida approximations associated with maximal monotone mappings performed extrusive part in Doulous-Rachford splitting method for finding zero of the sum of two monotone mappings, it motivated us to use the Yosida approximate iterative technique for convergence analysis. The converges theorem, Theorem 3.1, generalizes and extends some well-known results of Moudafi [14], Byrne et al. [4], and Sitthithakerngkiet et al. [20] as we considered the class of demicontractive mappings, which is much substantial than the class of nonexpansive mappings, firmly quasi-nonexpansive mappings.

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