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LQ-optimal boundary control of infinite-dimensional systems with Yosida-type approximate boundary observation[☆]

Jérémy R. Dehaye[1], Joseph J. Winkin

Namur Center for Complex Systems (naXys) and Department of Mathematics, University of Namur, Namur, B-5000, Belgium

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ABSTRACT

A class of boundary control systems with boundary observation is considered, for which the unbounded operators often lead to technical difficulties. An extended model for this class of systems is described and analyzed, which involves no unbounded operator except for the dynamics generator. A method for the resolution of the LQ-optimal control problem for this model is described and the solution provides a stabilizing feedback for the nominal system with unbounded operators, in the sense that, in closed-loop, the state trajectories converge to zero exponentially fast. The model consists of an extended abstract differential equation whose state components are the boundary input, the state (up to an affine transformation) and a Yosida-type approximation of the output of the nominal system. It is shown that, under suitable conditions, the model is well-posed and, in particular, that the dynamics operator is the generator of a C_0-semigroup. Moreover, the model is shown to be observable and to carry controllability, stabilizability and detectability properties from the nominal system. A general method of resolution based on the problem of spectral factorization of a multi-dimensional operator-valued spectral density is described in order to solve a LQ-optimal control problem for this model. It is expected that this approach will lead hopefully to a good trade-off between the cost of modeling and the efficiency of methods of resolution of control problems for such systems.

1. Introduction

Boundary control systems with boundary observation typically feature unbounded observation and control operators along with the homogeneous dynamics generator. In this paper, we consider an abstract boundary control system with boundary observation of the form

$$\dot{x}(t) = \mathcal{A}x(t) + B_d u_d(t), \quad x(0) = x_0 \tag{1}$$

$$\mathcal{B}x(t) = u_b(t) \tag{2}$$

$$y(t) = \mathcal{C}x(t) \tag{3}$$

with a bounded linear distributed control operator B_d and with unbounded boundary control and observation linear operators $\mathcal{B}$ and $\mathcal{C}$, respectively. Our main goal is to solve a LQ-optimal control problem related to such systems by using the method of spectral factorization by symmetric extraction.

Because of technical difficulties caused by the unboundedness, it is often difficult to solve specific control problems and design control laws for such systems. If possible, this characteristic should then better be avoided in order to achieve an acceptable trade-off between the cost of modeling and the efficiency of analytic and/or numerical methods of resolution of control problems. In order to solve the LQ-optimal control problem, the first step consists of building an extended differential system involving no unbounded operator except for the dynamics generator. We consider, under suitable conditions, a change of variables for the state and input of the nominal system, as well as a Yosida type approximation of the output which is based on the resolvent operator of the dynamics generator (Weiss, 1994). This choice yields a well-posed extended system with the same dynamical properties than those of the nominal one. More precisely, under suitable conditions, an appropriate change of variables yields the following extended differential system:

$$\begin{cases}\dot{x}^e(t) = A^e x^e(t) + B^e u^e(t), \quad x^e(0) = x^e_0, \\ y^e(t) = C^e x^e(t)\end{cases} \tag{4}$$

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E-mail addresses: jeremy.dehaye@unamur.be (J.R. Dehaye), joseph.winkin@unamur.be (J.J. Winkin).

[1] Tel.: +32 (0)81 72 53 12; fax: +32 (0)81 72 49 14.

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where the (extended) state is given, under suitable initial conditions, by $x^e(t) = (u_b(t) \;\; v(t) \;\; y_\alpha(t))^T$, with $v(t)$ depending on the nominal state $x(t)$ and $u^e(t) = (\dot{u}_b(t) \;\; u_d(t))^T$ as new input and where $y_\alpha(t)$ can be interpreted as an approximation of $y(t)$ based on the resolvent operator, and the control and observation operators, B^e and C^e respectively, are bounded. Although the new output is an approximation of $y(t)$, the proposed extension preserves some useful properties of the nominal system, including (approximate) reachability (when the nominal system has this property with respect to the distributed input u_d) and the spectral structure of the dynamics generator. It is shown that, under suitable conditions, the model is well-posed and, in particular, that the dynamics operator is the generator of a C_0-semigroup and the model is observable. In addition, the model is shown to be approximately reachable provided that so is the nominal system with respect to the distributed input. In the main part of this paper, a LQ-optimal (pure) boundary control problem is posed for the extended model with the associated cost functional given by

$$J(u^e, x_0^e, \infty) = \frac{1}{2}\int_0^{+\infty} \left[\|C^e x^e(t)\|^2 + u^e(t)^* Q u^e(t) \right] dt \tag{5}$$

where Q is a positive-definite weighting matrix and it is shown how it can be interpreted in the framework of the nominal system. A method of resolution of this problem is then developed. This method is based on the resolution of the problem of spectral factorization $\hat{F}(j\omega) = \hat{R}_*(j\omega)\hat{R}(j\omega)$ of an appropriate spectral density matrix $\hat{F}$. It is shown that this problem has a unique solution and can be solved by the resolution of the problem of spectral factorization of a specific coercive spectral density and the resolution of a Diophantine equation. Moreover, it is shown that the solution can be interpreted as a dynamical feedback for the nominal system. The key result of this paper is that this dynamical feedback designed for the extended system exponentially stabilizes the nominal system in the sense that, in closed-loop, the state trajectories of the nominal system converge to zero exponentially fast. The theoretical results are illustrated by an example of convection–diffusion–reaction system, for which spectral factorization is achieved by a semi-heuristic algorithm of symmetric extraction of pole-zero elementary matrix factors.

The idea behind this methodology was introduced for finite-dimensional systems with input derivative constraints, where the time derivative of the input enters the quadratic cost functional for the resolution of a LQ-optimal control problem, see e.g. Moore and Anderson (1967). In the infinite-dimensional framework, this approach is motivated by the fact that the unboundedness property leads to technical difficulties which make the modeling and analysis of such systems very hard, see e.g. Weiss (1994), Tucsnak and Weiss (2009), Staffans (2005) and references therein, and the design of control laws as well, especially when boundary control and boundary observation are both present in the model, see e.g. Weiss and Weiss (1997), or when the control operator is very unbounded, see e.g. Opmeer (2014) and references therein. A change of variables was already considered for boundary control alone (without observation) in Curtain and Zwart (1995, Section 3.3, pp. 121–128) and Fattorini (1968) in order to deal with this problem. In Tucsnak and Weiss (2009, Section 4.7, p. 147) and Weiss (1994), the Lebesgue and Yosida extensions were presented in the framework of well-posed linear systems. The LQ-optimal control problem has been studied extensively for systems with bounded control and observation operators, see e.g. Callier and Winkin (1992), Curtain and Zwart (1995, Chapter 6, pp. 269–334) and (Aksikas, Winkin, & Dochain, 2007) (and references therein). A part of the core components of this paper has been studied in Dehaye and Winkin (2013a) and Dehaye and Winkin (2013b), where, more specifically, Theorems 1–3 as well as some of the numerical results have been reported in a slightly different form. The results presented in Section 2.2 are an extension of the results obtained in Dehaye and Winkin (2013a), where the well-posedness analysis of the model required the C_0-semigroup generated by the dynamics operator to be analytic and the approximate output operator to be weighted by a well-chosen parameter, which could turn out to be hard to determine in practical situations. The assumptions have been weakened and the proofs adapted in consequence in order to obtain a more general model. Moreover, new results concerning the exponential stabilizability and detectability of the extended system have been added. New numerical tests have been performed in order to illustrate the good stabilization properties of such a dynamical controller when applied to a standard unstable parabolic system. Theorem 2 and its proof have been revised compared to their former versions which can be found in Dehaye and Winkin (2013b). A detailed proof of Theorem 3 has been added as well.

The novelty here is that the boundary input and approximate output trajectories are given jointly by the dynamics equation, thanks to the fact that both are components of the extended state. Observe that the variation rate of the boundary input is now an input of the extended system, without requiring too restrictive additional assumptions. Though this implies that control laws designed for the extended system are dynamical, it is seen with an example that it does not cause additional problems in practical situations. Moreover, numerical simulations show that, in this framework, the variation rate of the boundary input can be adjusted thanks to a weight parameter in the quadratic cost. A contribution of this paper is the analysis of some dynamical properties of the extended system with these joint dynamics. The main addition is the resolution of the LQ-optimal control problem by spectral factorization for this system with assumptions less restrictive than those in Aksikas et al. (2007) and Callier and Winkin (1992), and the proof that its solution can be interpreted as a stabilizing dynamical feedback for the nominal system. The semi-heuristic algorithm of symmetric extraction for the convection–diffusion–reaction system and the comparison between the extended and nominal systems in the last section are new as well.

It is expected that this approach will lead hopefully to a good trade-off between the cost of modeling and the efficiency of methods of resolution of control problems for such systems, like the LQ-optimal control problem.

The paper is organized as follows. In Section 2, we introduce a class of abstract differential linear systems with unbounded control and observation operators and we show that they can be described as in (4). In Section 3, an infinite horizon LQ-optimal control problem is defined for the system (4) and we show that it is related to a LQ-optimal control problem for the nominal system and its solution can be interpreted as a dynamical feedback. In Section 4, the theoretical results presented in the previous sections are illustrated by the numerical resolution of a particular LQ-optimal control problem for a convection–diffusion–reaction system with boundary control and boundary observation. In Section 5, we introduce additional assumptions in order to show that the outputs and transfer functions of both systems are related. More precisely, it is shown that, when some real parameter goes to infinity, the output and transfer function of the nominal system are the limits of the output and transfer function of the extended system, respectively. Finally, in Section 6, we make some concluding remarks and give some perspectives for future work.

2. Modeling and analysis

2.1. Construction of the model

We consider a dynamical system with boundary and distributed control and boundary observation. The associated abstract boundary control model with boundary observation is described by (1)–(3) where $\mathcal{A} : D(\mathcal{A}) \subset X \to X$ is an unbounded linear operator and $B_d \in \mathcal{L}(U_d, X)$ is a bounded linear operator, where U_d and X are Hilbert spaces, and $\mathcal{B}$ and $\mathcal{C}$ are unbounded linear operators on X taking values in Hilbert spaces U_b and Y respectively and whose domains contain the one of the operator $\mathcal{A}$, i.e. $D(\mathcal{A}) \subset D(\mathcal{B})$ and $D(\mathcal{A}) \subset D(\mathcal{C})$. Observe that, even though the distributed control operator B_d is important and may bring interesting properties, it is not required to perform the following analysis. Hence, throughout this section, it can be considered as 0 when one wants to deal with pure boundary control without affecting the main results which remain valid in this particular case. Section 3 deals with this specific situation and the consequences of using the more general model presented here are stated in separate remarks.

Definition 1. An abstract boundary control model (1)–(3) is said to be *a boundary control system with boundary observation (BCBO)* if the following conditions hold:

[C1] the operator $A : D(A) \to X$ defined by $Ax = \mathcal{A}x$ for all x in its domain $D(A) = D(\mathcal{A}) \cap \operatorname{Ker} \mathcal{B}$ is the infinitesimal generator of a C_0-semigroup $(\mathbb{T}_t)_{t\geq 0} = (T(t))_{t\geq 0}$ of bounded linear operators on X,

[C2] the operator $\mathcal{B}$ is onto, such that there exists a bounded linear operator $B_b \in \mathcal{L}(U_b, X)$ such that for all $u \in U_b$, $B_b u \in D(\mathcal{A})$, the operator $\mathcal{A}B_b \in \mathcal{L}(U_b, X)$ and for all $u \in U_b$, $\mathcal{B}B_b u = u$,

[C3] there exist constants $a, b \geq 0$ such that, for all $x \in D(A)$,

$$\|\mathcal{C}x\| \leq a\|Ax\| + b\|x\|. \tag{6}$$

Remark 1. Condition **[C3]** is equivalent to the fact that $\mathcal{C} \in \mathcal{L}(X_1, Y)$, where $X_1 = D(A)$ equipped with the norm $\|x\|_1 = \|(\beta I - A)x\|$ for some β in the resolvent set $\rho(A)$ of the operator A.

Now, for any parameter α in $\rho(A)$, let us define the operator C_α by

$$C_\alpha : X \to Y : x \mapsto C_\alpha x := \alpha\mathcal{C}(\alpha I - A)^{-1}x. \tag{7}$$

Observe that, by condition **[C3]**, $C_\alpha \in \mathcal{L}(X, Y)$, i.e. C_α is a bounded observation operator from X to Y. This operator (more specifically, its limit as the parameter α tends to infinity, whenever it exists) is useful in the analysis of the well-posedness of infinite-dimensional systems with an unbounded observation operator, see e.g. Weiss (1994), Tucsnak and Weiss (2009) and references therein. It is related to a concept known in the literature as the *Yosida approximation*, which plays an important role in the proof of the Hille–Yosida theorem, see e.g. Curtain and Zwart (1995, Theorem 2.1.12, p. 26) and Jacob and Zwart (2012, Theorem 6.1.3, p. 66). In the sequel, the operator C_α will be interpreted as a *Yosida type approximate boundary observation operator*. Under these conditions, consider the following abstract differential equations:

$$\dot{v}_1(t) = Av_1(t) - B_b\dot{u}_b(t) + \mathcal{A}B_b u_b(t) + B_d u_d(t) \tag{8}$$

$$v_1(0) = v_{1_0}, \tag{9}$$

$$\dot{v}_2(t) = C_\alpha A v_1(t) + C_\alpha \mathcal{A}B_b u_b(t) + C_\alpha B_d u_d(t) \tag{10}$$

$$v_2(0) = v_{2_0}. \tag{11}$$

The question of the well-posedness of these equations will be studied and answered positively in Section 2.2. Then one can define the extended system

$$\dot{x}^e(t) = A^e x^e(t) + B^e u^e(t), \qquad x^e(0) = \left(x^e_{0_1},\ x^e_{0_2},\ x^e_{0_3}\right)^T \tag{12}$$

$$y^e(t) = C^e x^e(t) \tag{13}$$

on the extended state space $\widetilde{X}^e := U_b \oplus X \oplus Y$, where the (extended) state is defined, under suitable initial conditions, by

$$x^e(t) = (u_b(t) \ \ v_1(t) \ \ v_2(t))^T \tag{14}$$

$$= (u_b(t) \ \ x(t) - B_b u_b(t) \ \ y_\alpha(t))^T, \tag{15}$$

the (extended) input is defined by

$$u^e(t) = (\dot{u}_b(t) \ \ u_d(t))^T \in U^e := U_b \oplus U_d \tag{16}$$

and where

$$A^e = \begin{pmatrix} 0 & 0 & 0 \\ \mathcal{A}B_b & A & 0 \\ C_\alpha \mathcal{A}B_b & C_\alpha A & 0 \end{pmatrix}, \tag{17}$$

$$B^e = \begin{pmatrix} I & 0 \\ -B_b & B_d \\ 0 & C_\alpha B_d \end{pmatrix}, \qquad C^e = \begin{pmatrix} \rho_1 I & 0 & 0 \\ \rho_2 B_b & \rho_2 I & 0 \\ 0 & 0 & \rho_3 I \end{pmatrix}. \tag{18}$$

The domain of the operator A^e is given by $D(A^e) := U_b \oplus D(A) \oplus Y$ and $\rho_i > 0$, $i = 1, 2, 3$, are weighting factors. The observation operator C^e is considered with a view to solving an LQ-optimal feedback control problem with a quadratic cost involving the (extended) output $y^e(t)$, see Section 3.

Remark 2. At first glance, one may be worried that it is not recommended to differentiate $y(t)$ (or its approximation) because measurement noise may be amplified by the differentiation of the output in the model. However, it is shown later that, when designing a state feedback control law, the dynamics of the output are only used in the computation of the feedback operator. Without loss of generality, the control law can then be rewritten as acting only on the boundary input and the state of the nominal system (see (28)). The approximate output y_α is not used directly in the feedback, which is useful to avoid such problems in practical applications.

It should be noted that Eqs. (12)–(15) are equivalent to (1)–(3) with $\mathcal{C}$ replaced by C_α, as shown in Theorem 1(b).

2.2. Well-posedness and dynamical analysis

The well-posedness analysis of the extended model of a BCBO system (12)–(18) is based on the following preliminary results.

For a given linear operator $\Gamma : D(\Gamma) \subset X \to X$ on a Banach space X, a linear operator $\Delta : D(\Delta) \subset X \to X$ is said to be Γ-bounded if $D(\Gamma) \subset D(\Delta)$ and if there exist nonnegative constants γ and δ such that, for all $x \in D(\Gamma)$,

$$\|\Delta x\| \leq \gamma\|\Gamma x\| + \delta\|x\|. \tag{19}$$

The Γ-bound of the operator Δ is given by

$$\gamma_0 := \inf\{\gamma \geq 0 \ : \ \text{there exists } \delta \geq 0 \text{ such that (19) holds}\}.$$

Remark 3. In view of condition (**C3**), the operator $\mathcal{C}$ is A-bounded with A-bound less than or equal to a.

Observe that the third equation of the abstract differential equation (12) should correspond to the dynamics of the output trajectories of its two first equations through the output operator C_α. In order to take this feature into account in the description of the extended system, let us consider the bounded linear operator $C \in \mathcal{L}(U_b \oplus X, Y)$ defined for all $(x^e_1, x^e_2) \in U_b \oplus X$ by $C\left(x^e_1,\ x^e_2\right)^T = (C_\alpha B_b \ \ C_\alpha)\left(x^e_1,\ x^e_2\right)^T = C_\alpha(x^e_2 + B_b x^e_1)$. Thanks to the fact that the graph $G(C)$ of the operator C is a closed subspace of $\widetilde{X}^e$, from now on, we will use $X^e := G(C) \subset \widetilde{X}^e = U_b \oplus X \oplus Y$ as new extended (Hilbert) state space for the extended system. This restriction is particularly useful when analyzing some subsequent properties such as stability, reachability and stabilizability, since, as it will be studied later, the state-output relationship $y_\alpha = C_\alpha(v + B_b u_b) = C_\alpha x$ always holds in X^e.

The proof of the following result, and more particularly the well-posedness of (8)–(11), is based on Engel and Nagel (2006, Corollary 1.5, p. 119) and follows the lines of Dehaye and Winkin (2013a, Lemma 4, Lemma 5 and Theorem 6) with a less restrictive operator C_α.

Theorem 1 (*Well-Posedness of the Extended System*)**.** (a) *The restriction of the operator A^e, given by (17), (7), to the subspace $X^e = G(C)$, whose domain is given by $D(A^e) \cap G(C)$, is the infinitesimal generator of a C_0-semigroup $(T^e(t))_{t\geq 0}$ of bounded linear operators on X^e.*

(b) *For any distributed input $u_d \in C^1([0,\tau], U_d)$ and for any boundary input $u_b \in C^2([0,\tau], U_b)$, where $\tau > 0$ is any fixed final time, the dynamics (12) of the extended system (12)–(18) are well-posed, i.e. the abstract differential equation $\dot{x}^e(t) = A^e x^e(t) + B^e u^e(t)$ with initial condition $x^e(0) = x_0^e = \left(x_{0_1}^e,\ x_{0_2}^e,\ x_{0_3}^e\right)^T = \left(u_b(0),\ v_{1_0},\ v_{2_0}\right)^T \in D(A^e) \cap G(C)$ and input given by (16), has the unique classical solution $x^e(t) = (u_b(t) \quad v_1(t) \quad v_2(t))^T$, where $v_1(t)$ and $v_2(t)$ are the unique classical solutions of the abstract differential equations (8)–(9) and (10)–(11), respectively.*

Moreover, if $x_0 = v_{1_0} + B_b u_b(0)$, hence $v_{2_0} = C_\alpha x_0$, then the state trajectory $x(t)$ of the BCBO system (1)–(3) is related to the one of the extended system (12)–(18), for all $t \geq 0$, by

$$x(t) = v_1(t) + B_b u_b(t) \quad \text{and} \quad v_2(t) = C_\alpha x(t). \tag{20}$$

Though the proof of a similar result can be found in Dehaye and Winkin (2013a), the assumptions, and notably the analyticity of the C_0-semigroup generated by A, have been weakened and the main tool used to establish the well-posedness is different. Since this part of the proof is not a straightforward adaptation, it is detailed in the sequel.

Proof. (a) First observe that the operator A^e, with domain $D(A^e) = U_b \oplus D(A) \oplus Y$, is the infinitesimal generator of a C_0-semigroup (of bounded linear operators) on $\widetilde{X}^e$. Indeed, let us define the operator $A_0^e = \mathrm{diag}(0, A, 0)$ on its domain $D(A_0^e) = D(A^e)$, and the perturbation operators

$$P_1^e = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & C_\alpha A & 0 \end{pmatrix}, \qquad P_2^e = \begin{pmatrix} 0 & 0 & 0 \\ \mathcal{A}B_b & 0 & 0 \\ C_\alpha \mathcal{A} B_b & 0 & 0 \end{pmatrix}.$$

Since A is the infinitesimal generator of a C_0-semigroup $(T(t))_{t\geq 0}$ on X, the operator A_0^e is the infinitesimal generator of the C_0-semigroup $(T_0^e(t))_{t\geq 0}$ on $\widetilde{X}^e$, given for all $t \geq 0$ by $T_0^e(t) = \mathrm{diag}(I, T(t), I)$. Observe that $\mathrm{Ran}\, P_1^e \subset D(A_0^e)$. Moreover, let $\beta \in \rho(A_0^e)$. For any $x^e \in D(A_0^e) = D(A^e)$, $\|(\beta I - A_0^e)P_1^e x^e\| \leq |\beta|\|P_1^e x^e\| + \|A_0^e P_1^e x^e\| = |\beta|\|P_1^e x^e\|$, which implies that $(\beta I - A_0^e)P_1^e \in \mathcal{L}(X_1^e, \widetilde{X}^e)$ since P_1^e is A_0^e-bounded, and hence $P_1^e \in \mathcal{L}(X_1^e)$, where $X_1^e = D(A_0^e) = D(A^e)$. By Engel and Nagel (2006, Corollary 1.5, p. 119), the operator $A_0^e + P_1^e$ (with domain $D(A^e)$) is the infinitesimal generator of a C_0-semigroup. Since $P_2^e \in \mathcal{L}(\widetilde{X}^e)$, the operator $A^e = A_0^e + P_1^e + P_2^e$ is the infinitesimal generator of a C_0-semigroup $(\widetilde{T}^e(t))_{t\geq 0}$ on $\widetilde{X}^e$. The conclusion follows by Engel and Nagel (2006, Corollary p. 48), since X^e is a $\widetilde{T}^e(t)$-invariant closed subspace of $\widetilde{X}^e$.

(b) It suffices to observe that the third component $x_3^e(t)$ of the extended state is the solution of the abstract differential equation (10)–(11) with initial condition $v_{2_0} = x_{0_3}^e$, hence $v_2(t) \equiv x_3^e(t)$. The result is based on Engel and Nagel (2006, Corollary 1.5, p. 119) and follows by an analysis going along the lines of Curtain and Zwart (1995, Section 3.3, pp. 121–128), Dehaye and Winkin (2013a). ■

Remark 4. (a) By using a scaling of the operators C_α and C^e, the well-posedness of the extended system holds with A^e generating an analytic C_0-semigroup provided that the C_0-semigroup generated by A is analytic too, see Dehaye and Winkin (2013a).

(b) The well-posedness of some specific classes of boundary control systems, in particular port-hamiltonian systems, has been studied in detail in several existing works, see e.g. Villegas (2007), Le Gorrec, Maschke, Villegas, and Zwart (2006) and Jacob and Zwart (2012).

In the next section, some system theoretic properties of the extended system will be analyzed, namely (approximate) reachability and observability, and (exponential) stabilizability and detectability. See e.g. Curtain and Zwart (1995, Definitions 4.1.3 a., p. 143, 4.1.12 a., p. 154, 4.1.17, p. 157) for the definitions of (approximate) reachability and (approximate) observability, and Curtain and Zwart (1995, Definition 5.2.1, p. 227) for the definitions of (exponential) stabilizability and (exponential) detectability.

In the sequel, the extended system (12)–(18) will be considered on the state space X^e as in Theorem 1.

3. LQ-optimal control

In this section, we assume that $B_d = 0$ in order to consider LQ-optimal pure boundary control. However, when significant differences arise in the following analysis between the cases $B_d = 0$ and $B_d \neq 0$, it will be mentioned in specific remarks. We consider the following LQ-optimal control problem for the extended system (12)–(18), which is assumed from now on to be reachable and stabilizable (see Dehaye & Winkin, 2013b, for the full analysis with $B_d \neq 0$):

$$\min_{u^e} J(u^e, x_0^e, \infty) \tag{21}$$

where the quadratic cost functional is given by (5) with the weighting operator $Q = \eta I$, with $\eta > 0$. Observe that

$$\begin{aligned} J(u^e, x_0^e, \infty) &= \frac{1}{2}\int_0^{+\infty} \left[\rho_1\|u_b(t)\|^2 + \rho_2\|v_1(t) + B_b u_b(t)\|^2 \right. \\ &\quad \left. + \rho_3\|v_2(t)\|^2 + \eta\|\dot{u}_b(t)\|^2\right] dt \\ &= \frac{1}{2}\int_0^{+\infty} \left[\rho_1\|u_b(t)\|^2 + \rho_2\|x(t)\|^2 \right. \\ &\quad \left. + \rho_3\|y_\alpha(t)\|^2 + \eta\|\dot{u}_b(t)\|^2\right] dt \end{aligned} \tag{22}$$

for a suitable initial condition, where ρ_2 and η can be chosen small. As a consequence, with suitably chosen parameters, the cost functional for the extended system can be interpreted as a cost functional for the nominal system, with a non standard term involving the norm of the variation rate of the boundary input $u_b(t)$ for which the corresponding parameter η can be adjusted. (Observe that, when $B_d \neq 0$, an additional term of the form $\eta_2\|u_d(t)\|^2$, with $\eta_2 > 0$, appears in the cost functional.) The following result shows how this problem can be solved by using the methodology of Callier and Winkin (1992) extended to possibly infinite-dimensional input and output Hilbert spaces. This result was reported in Dehaye and Winkin (2013b) for the particular case of an analytic extended C_0-semigroup.

In what follows, for any operator valued function $\hat{F}(s)$, the expression $\hat{F}_*$ is the parahermitian adjoint of $\hat{F}$, i.e. $\hat{F}_*(s) = \hat{F}(-\bar{s})^*$. Moreover, for any Hilbert space $\mathcal{H}$, the usual Hardy space of $\mathcal{L}(\mathcal{H})$-valued functions that are holomorphic and bounded on the open right half-plane is denoted by $H^\infty(\mathcal{L}(\mathcal{H}))$, see Curtain and Zwart (1995, Definition A.6.14, p. 643).

In the sequel, we show how this problem can be solved when Q is the identity operator.

First, observe that the transfer function of the extended system is given by

$$\hat{G}^e(s) = \begin{pmatrix} \rho_1 \frac{1}{s} I \\ \rho_2 E(s) \\ \rho_3 C_\alpha E(s) \end{pmatrix}$$

where $E(s) = \frac{1}{s}\left[(sI - A)^{-1}\mathcal{A} - A(sI - A)^{-1}\right] B_b$. This transfer function is unstable due to $\hat{G}^e_{11}(s)$.

The first step consists of prestabilizing the system in order to find a right coprime fraction of $\hat{G}^e(s)$. For this purpose, let us consider any stabilizing feedback

$$K^e = \begin{pmatrix} k_1 & k_2 & k_3 \end{pmatrix} \tag{23}$$

where $k_1 \in \mathcal{L}(U_b)$, $k_2 \in \mathcal{L}(X, U_b)$ and $k_3 \in \mathcal{L}(Y, U_b)$.

Remark 5. When $B_d \neq 0$ and the pair (A, B_d) is stabilizable, the feedback can be chosen as $K^e = \begin{pmatrix} k_1 & 0 & 0 \\ 0 & k_2 & 0 \end{pmatrix}$ where $k_1 \in \mathcal{L}(U_b)$ is the bounded infinitesimal generator of a stable C_0-semigroup on U_b (such an operator always exists, take e.g. $k_1 = -I$) and $k_2 \in \mathcal{L}(X, U_d)$ is a stabilizing feedback for the pair (A, B_d).

A right coprime fraction of $\hat{G}^e(s)$ is then given by

$$\hat{N}^e(s) = C^e(sI - A^e - B^eK^e)^{-1}B^e,$$
$$\hat{D}^e(s) = I + K^e(sI - A^e - B^eK^e)^{-1}B^e.$$

Then, one can compute the spectral density $\hat{F}^e(s) = \hat{N}^e_*(s)\hat{N}^e(s) + \hat{D}^e_*(s)\hat{D}^e(s)$ and solve the spectral factorization problem, i.e. find $\hat{R}^e \in H^\infty(\mathcal{L}(U_b))$ such that $(\hat{R}^e)^{-1} \in H^\infty(\mathcal{L}(U_b))$ and for all $\omega \in \mathbb{R}$, $\hat{F}^e(j\omega) = \hat{R}^e_*(j\omega)\hat{R}^e(j\omega)$.

Remark 6. When $B_d \neq 0$, solving the problem consists of finding $\hat{R}^e \in H^\infty(\mathcal{L}(U_b \oplus U_d))$ such that $(\hat{R}^e)^{-1} \in H^\infty(\mathcal{L}(U_b \oplus U_d))$ and for all $\omega \in \mathbb{R}$, $\hat{F}^e(j\omega) = \hat{R}^e_*(j\omega)\hat{R}^e(j\omega)$ with

$$\hat{F}^e(s) = \begin{pmatrix} \hat{F}_1(s) & \hat{F}_2(s) \\ \hat{F}_{2*}(s) & \hat{F}_4(s) \end{pmatrix} \quad \text{and} \quad \hat{R}^e(s) = \begin{pmatrix} \hat{R}_1(s) & \hat{R}_2(s) \\ \hat{R}_3(s) & \hat{R}_4(s) \end{pmatrix}.$$

It is known that this problem admits a solution if the operator spectral density $\hat{F}^e(s)$ is (uniformly) coercive on the imaginary axis, i.e. there exists $\eta > 0$ such that for all $\omega \in \mathbb{R}$, $\hat{F}^e(j\omega) \geq \eta I$. When the input space is finite-dimensional, this solution is known to be in $\text{Mat}(\hat{\mathcal{A}}_-)$ together with its inverse, see e.g. Callier and Winkin (1992, 1999). For the existence of a spectral factor $\hat{R}^e \in H^\infty(\mathcal{L}(U_b \oplus U_d))$, see Weiss and Weiss (1997), which is based on Rosenblum and Rovnyak (1985, Theorem 3.7). Finally, one has to find the unique constant solution of the Diophantine equation $\mathcal{U}\hat{D}^e(s) + \mathcal{V}\hat{\mathcal{N}}^e(s) = \hat{R}^e(s)$, where $\hat{\mathcal{N}}^e(s) := (sI - A^e - B^eK^e)^{-1}B^e$, and compute the optimal feedback given by $K^e_0 = -\mathcal{U}^{-1}\mathcal{V} = -\mathcal{U}^*\mathcal{V}$. The computation of the spectral factor and the constant solution of the diophantine equation often requires adapted numerical schemes, such as the algorithm presented in Section 4. It should be noted that the optimal feedback provided by this methodology is unique and independent of the choice of the prestabilizing feedback (23). In fact, even though the poles of the spectral density (and hence the spectral factor) depend on this choice, they are canceled in the Diophantine equation and hence may only play a role in the numerical conditioning of the problem. This methodology is summarized in Theorem 2. This result is based on the following lemma, which states that the pair (C^e, A^e) is observable and detectable by construction, without any assumption on the pair $(\mathcal{C}, A)$ or (C_α, A).

Lemma 1. (a) *The extended system (12)–(18), i.e. the pair (C^e, A^e), is observable.*

(b) *The extended system (12)–(18), i.e. the pair (C^e, A^e), is detectable.*

Proof. (a) See the proof of Dehaye and Winkin (2013a, Proposition 8).

(b) Consider the operator $L^e := -\kappa I \in \mathcal{L}(Y^e, X^e)$, where $\kappa \geq 0$. It is easy to see that the operator

$$A^e + L^eC^e = \begin{pmatrix} -\kappa\rho_1 I & 0 & 0 \\ (\mathcal{A} - \kappa\rho_2 I)B_b & A - \kappa\rho_2 I & 0 \\ C_\alpha\mathcal{A}B_b & C_\alpha A & -\kappa\rho_3 I \end{pmatrix}$$

is the generator of a stable C_0-semigroup on X^e if κ is chosen large enough. ■

Theorem 2. *Consider the extended system (12)–(18) with $B_d = 0$. Assume that the pair (A^e, B^e) is reachable and stabilizable.*

Let us consider the LQ-optimal control problem (21) with the cost functional (5) where $Q = I$.

Consider (a) a stabilizing feedback K^e and the spectral density given by

$$\hat{F}^e(s) = \hat{N}^e_*(s)\hat{N}^e(s) + \hat{D}^e_*(s)\hat{D}^e(s), \tag{24}$$

where the pair $(\hat{N}^e, \hat{D}^e)$ defined by

$$\hat{N}^e(s) = C^e(sI - A^e - B^eK^e)^{-1}B^e \tag{25}$$

$$\hat{D}^e(s) = I + K^e(sI - A^e - B^eK^e)^{-1}B^e \tag{26}$$

is a right coprime fraction of the operator-valued transfer function $\hat{G}^e(s) = C^e(sI - A^e)^{-1}B^e$, and (b) the spectral factorization problem

$$\hat{F}^e(j\omega) = \hat{R}^e_*(j\omega)\hat{R}^e(j\omega), \tag{27}$$

and the (standard) invertible stable spectral factor $\hat{R}^e \in H^\infty(\mathcal{L}(U_b \oplus U_d))$ such that $(\hat{R}^e)^{-1} \in H^\infty(\mathcal{L}(U_b \oplus U_d))$ and $\hat{R}^e(\infty) = I$.

Then the solution of the LQ-optimal control problem (21) is the stabilizing feedback control law $u^e = K^e_0 x^e$ with the state feedback operator $K^e_0 = \begin{pmatrix} k_{0_1} & k_{0_2} & k_{0_3} \end{pmatrix}$ where $k_{0_1} \in \mathcal{L}(U_b)$, $k_{0_2} \in \mathcal{L}(X, U_b)$ and $k_{0_3} \in \mathcal{L}(Y, U_b)$ are bounded linear operators. Without loss of generality, the solution has the following structure:

$$K^e_0 = \begin{pmatrix} k_{0_1} & k_{0_2} & 0 \end{pmatrix}. \tag{28}$$

The feedback operator K^e_0 is given by

$$K^e_0 = -\mathcal{U}^{-1}\mathcal{V} = -\mathcal{U}^*\mathcal{V} \tag{29}$$

where $(\mathcal{U}, \mathcal{V})$ is the unique constant solution of the diophantine equation

$$\mathcal{U}\hat{D}^e(s) + \mathcal{V}\hat{\mathcal{N}}^e(s) = \hat{R}^e(s), \tag{30}$$

with $\hat{\mathcal{N}}^e(s) := (sI - A^e - B^eK^e)^{-1}B^e$.

This result is an extension of the results established in Aksikas et al. (2007) and Callier and Winkin (1992), where the input space is assumed to be finite-dimensional and the C_0-semigroup is assumed to be stable, respectively. In this paper, the theorem is stated without these assumptions. Its proof is a direct adaptation of the proof of Dehaye and Winkin (2013b, Theorem 1). This theorem is stated for the normalized case, i.e. where the weighting operator Q is the identity. However, one can define the positive-definite weighting operator $\tilde{Q} := \eta I > 0$ and the modified bounded control operator $\tilde{B}^e = B^e\tilde{Q}^{-\frac{1}{2}}$ such that the input is given by $\tilde{u}^e = \tilde{Q}^{\frac{1}{2}}u^e$. Hence $\tilde{u}^{e*}\tilde{u}^e = u^{e*}Qu^e = \eta\|u^e\|^2$.

Remark 7. If a distributed control input is present (i.e. $B_d \neq 0$), different conditions on the nominal system can imply reachability and/or stabilizability of the extended system. For example, reachability and stabilizability of the pair (A, B_d) imply reachability and stabilizability of the extended system (12)–(18), i.e. the pair (A^e, B^e), respectively. The first assertion is proved in Dehaye and Winkin (2013a, Proposition 8). For the second one, observe that, since (A, B_d) is stabilizable, there exists a stabilizing feedback operator $K_d \in \mathcal{L}(X, U_d)$ such that $A + B_d K_d$ is the generator of a stable C_0-semigroup on X. Hence the operator $K^e \in \mathcal{L}(X^e, U^e)$ defined by $K^e = \begin{pmatrix} -I & 0 & 0 \\ 0 & K_d & 0 \end{pmatrix}$ is a stabilizing feedback operator for the pair (A^e, B^e). Indeed, the operator

$$A^e + B^e K^e = \begin{pmatrix} -I & 0 & 0 \\ (\mathcal{A} + I)B_b & A + B_d K_d & 0 \\ C_\alpha \mathcal{A} B_b & C_\alpha (A + B_d K_d) & 0 \end{pmatrix}$$

is the generator of a stable C_0-semigroup on X^e. For this purpose, observe that, in closed loop, the first two equations generate a stable C_0-semigroup on $U_b \oplus X$. Hence, there exist $M_1, M_2 > 0$ and $\alpha_1, \alpha_2 < 0$ such that for all $(x^e_{1_0}, x^e_{2_0}) \in U_b \oplus D(A)$ and for all $t \geq 0$, $\|x^e_1(t)\| \leq M_1 e^{\alpha_1 t} \|x^e_{1_0}\|$ and $\|x^e_2(t)\| \leq M_2 e^{\alpha_2 t} \|x^e_{2_0}\|$, where x^e_1 and x^e_2 are the state trajectories corresponding to the C_0-semigroup generated by

$$\begin{pmatrix} -I & 0 \\ (\mathcal{A} + I)B_b & A + B_d K_d \end{pmatrix}.$$

Now, observe that, on X^e, for all $t \geq 0$,

$$\begin{aligned} \|x^e_3(t)\| &= \|C_\alpha (x^e_2(t) + B_b x^e_1(t))\| \\ &\leq \|C_\alpha\| (M_2 e^{\alpha_2 t} \|x^e_{2_0}\| + \|B_b\| M_1 e^{\alpha_1 t} \|x^e_{1_0}\|). \end{aligned}$$

Finally, taking the max norm $\|\cdot\|_\infty$ on the product space $U_b \oplus X \oplus Y$ yields the inequality $\|(x^e_1(t), x^e_2(t), x^e_3(t))\|_\infty \leq [(1 + \|C_\alpha\| \|B_b\|) M_1 + (1 + \|C_\alpha\|) M_2] e^{\alpha t} \|(x^e_{1_0}, x^e_{2_0}, x^e_{3_0})\|_\infty$, where $\alpha := \max\{\alpha_1, \alpha_2\}$.

Remark 8. More interestingly, in Curtain and Zwart (1995, Exercise 4.20., p. 201–204), it is stated that, if $\mathcal{A} B_b = 0$, the pair (A^e, B^e) is reachable if and only if the pair (A, B_b) is. Even though this model does not include the approximate output in the state, it can be generalized to fit the BCBO system. Stabilizability can be achieved as well in this case when the pair (A, B_b) is stabilizable, without the requirement of distributed control in the model, as seen in Curtain and Zwart (1995, Exercise 5.25., pp. 262–264). More generally, if $0 \in \rho(A)$ and (A, B_b) is stabilizable, stabilizability of the extended system holds if and only if $\mathrm{Ker}\,(sI \;\; (\mathcal{A} B_b)^*) \cap \mathrm{Ker}\,(0 \;\; sI - A^*) \cap \mathrm{Ker}\,(I \;\; -B_b^*) = \{0\}$ for all $s \in \overline{\mathbb{C}_0^+}$.

Remark 9. (a) In general, for infinite-dimensional systems, reachability does not imply stabilizability (see e.g. Curtain & Zwart, 1995, Example 5.2.2, p. 228). Similarly, by duality, observability does not guarantee that detectability holds.

(b) The reachability assumption guarantees that the solution of (30) is unique, which is not necessarily the case in general, and that (29) holds. However, without this assumption, any constant solution of (30) provides the reachable restriction of the optimal feedback operator. In these conditions, (29) only holds on the reachable subspace, i.e. for all x^e in the reachable subspace $\mathcal{R}(A^e, B^e)$, $K^e_0 x^e = -\mathcal{U}^{-1} \mathcal{V} x^e = -\mathcal{U}^* \mathcal{V} x^e$, see Callier and Winkin (1992), Aksikas et al. (2007).

Remark 10. It is well known that, under suitable assumptions, the LQ-optimal control problem can alternatively be solved by the resolution of an operator algebraic Riccati equation, see e.g. Alizadeh Moghadam, Aksikas, Dubljevic, and Forbes (2013), Curtain and Zwart (1995, Section 6.2, pp. 292–303) and Pritchard and Salamon (1987). Efficient numerical methods, including standard and modified Newton–Kleinman algorithms, were developed and analyzed in order to solve this problem for approximate finite-dimensional systems in practical applications, in particular for distributed parameter systems where the dynamics are described by partial differential equation, see e.g. Grad and Morris (1996), Morris and Navasca (2010) and references therein.

So far, it has been shown that the LQ-optimal control problem is solvable for the extended system. However, it is still unclear at this point how its solution is useful for the stabilization of the nominal one. The following crucial theorem shows that the (static) optimal feedback for the extended system can be seen as a stabilizing dynamical feedback compensator for the nominal system, with an exponential rate of convergence, provided that the C_0-semigroup generated by A is analytic or, more generally, immediately differentiable (see e.g. Engel & Nagel, 2000, Definition 4.13, p. 109). This fact is illustrated by Fig. 1, where the black part corresponds to the distributed input and can be plugged out when the system only features boundary control.

Theorem 3. *Assume that the C_0-semigroup $(\mathbb{T}_t)_{t\geq 0} = (T(t))_{t\geq 0}$ generated by A is analytic. Then, under the assumptions of Theorem 2, the optimal control law for the extended system is given by the dynamic compensator for the nominal system described by*

$$\dot{u}_b(t) = K^e_0 \left(u_b(t),\; x(t) - B_b u_b(t),\; y_\alpha(t)\right)^T \tag{31}$$

$$= (k_{0_1} - k_{0_2} B_b) u_b(t) + k_{0_2} x(t) \tag{32}$$

with $u_b(0)$ satisfying $x_0 - B_b u_b(0) \in D(A)$, whose state is the boundary input $u_b(t)$ and with input $x(t)$.

In addition, this dynamic compensator is (exponentially) stabilizing, i.e. in closed loop, there exist $M > 0$ and $\alpha < 0$ such that the state trajectory $x(t)$ given by (20) is such that for all $t \geq 0$, $\|x(t)\| \leq M e^{\alpha t} r(x_0)$, where $r(x_0) > 0$ depends on $x_0 = x(0) \in D(\mathcal{A})$.

Moreover, this compensator is optimal with respect to the cost (22) for the nominal system among all dynamic compensators of the form (32) where k_{0_1} and k_{0_2} are bounded linear operators.

Proof. First, we show that the assumptions of Theorem 1(b) hold for the closed-loop system, more specifically the fact that $u_b \in C^2([0, +\infty), U_b)$. For this purpose, let us consider the auxiliary homogeneous differential system

$$\dot{\tilde{x}}(t) = \tilde{A} \tilde{x}(t) \tag{33}$$

with the initial condition
$\tilde{x}(0) = \tilde{x}_0 = (u_b(0),\; x_0 - B_b u_b(0))^T \in D(\tilde{A})$, where
$\tilde{A} : D(\tilde{A}) \subset U_b \oplus X \to U_b \oplus X$ is defined by

$$\tilde{A} = \begin{pmatrix} k_{0_1} & k_{0_2} \\ \mathcal{A} B_b - B_b k_{0_1} & A - B_b k_{0_2} \end{pmatrix} \tag{34}$$

on its domain $D(\tilde{A}) = U_b \oplus D(A)$. Since A is the generator of an analytic C_0-semigroup on X, the operator $\mathrm{diag}(0, A)$ is the generator of an analytic C_0-semigroup on $U_b \oplus X$. Now, since the perturbation operator $\tilde{A} - \mathrm{diag}(0, A) \in \mathcal{L}(U_b \oplus X)$, the operator $\tilde{A}$ is the generator of an analytic C_0-semigroup $(\tilde{T}(t))_{t\geq 0}$ on $U_b \oplus X$, see e.g. Engel and Nagel (2000, Theorem 2.10, p. 176). Hence, the classical solution of (33)–(34) satisfies $\tilde{x}(\cdot) = \tilde{T}(\cdot)\tilde{x}_0 \in C^\infty([0, +\infty), U_b \oplus X)$. Observe that the second equation of (33)–(34) is equivalent to (8) under (31) with $v_1 := \tilde{x}_2$ and $u_b := \tilde{x}_1$ and, since $\tilde{x}_{0_2} = x_0 - B_b u_b(0)$ and $\tilde{x} \in C^\infty([0, +\infty), U_b \oplus X)$ (hence $\tilde{x}_1 \in C^2([0, +\infty), U_b \oplus X)$), the classical solution is given for all $t \geq 0$ by $\tilde{x}_2(t) = x(t) - B_b u_b(t)$

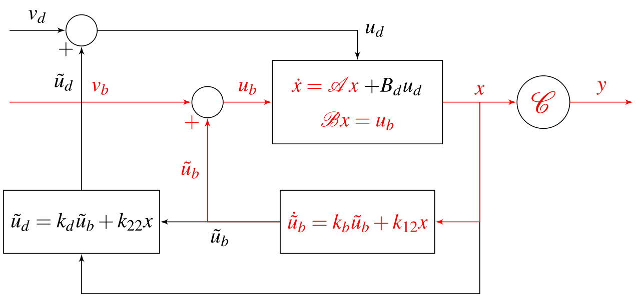

Fig. 1. Dynamical interpretation of the LQ-optimal feedback.

by Theorem 1(b). The first equation of (33)–(34) is then equivalent to the dynamics of (32), which shows that (20) holds for the closed-loop system.

Now, it is well known that the solution of the LQ-optimal control problem is a stabilizing feedback operator. Hence, there exist $M_u, M_v > 0$ and $\alpha_u, \alpha_v < 0$ such that, in closed loop, for all $t \geq 0$, $\|u_b(t)\| \leq M_u e^{\alpha_u t}\|u_b(0)\|$ and $\|v_1(t)\| \leq M_v e^{\alpha_v t}\|v_1(0)\|$. Since $x(t) = v_1(t) + B_b u_b(t)$ by (20), for all $t \geq 0$, $\|x(t)\| \leq Me^{\alpha t}r(x_0)$, where $M := 2\max\{\|B_b\|M_u, M_v\} > 0$, $\alpha := \max\{\alpha_u, \alpha_v\} < 0$ and $r(x_0) := \max\{\|u_b(0)\|, \|v_1(0)\|\}$.

Finally, since it has been shown that (20) holds for the closed-loop system, any dynamic compensator of the form (32) can be interpreted as a static feedback law of the form (28) for the extended system. Hence (32), where k_{0_1} and k_{0_2} form the optimal feedback operator, minimizes the cost (22). ∎

Remark 11. The assumption that the C_0-semigroup generated by A is analytic, which is the case for the application studied in Section 4, can be replaced by the following:

(i) the C_0-semigroup $(\mathbb{T}_t)_{t\geq 0} = (T(t))_{t\geq 0}$ generated by A is immediately differentiable,

(ii) $\limsup_{t\to 0^+} \frac{t\log\|AT(t)\|}{\log\left(\frac{1}{t}\right)} = 0$,

see e.g. Doytchinov, Hrusa, and Watson (1997, Theorem 1) and Engel and Nagel (2000, Definition 4.13, p. 109). The first assumption is satisfied in particular when the C_0-semigroup is analytic. However, the converse is not true. Some multiplication semigroups, for example, are immediately differentiable but not analytic, see e.g. Engel and Nagel (2000, Counterexample 4.33, p. 123).

As a consequence of this result, in closed loop, the nominal system becomes

$$\begin{cases} \dot{x}(t) = \mathcal{A}x(t), \\ \mathcal{B}x(t) = u_b(t), \\ \dot{u}_b(t) = (k_{0_1} - k_{0_2}B_b)u_b(t) + k_{0_2}x(t), \end{cases} \tag{35}$$

with the initial conditions $x(0) = x_0$ and $u_b(0) = \mathcal{B}x_0$, where the interconnection between both dynamics only appears in the boundary conditions as expected. It should be observed that an additional constraint of this approach is the fact that the initial condition for the dynamic compensator has to be compatible with the boundary conditions at the time $t = 0$.

Remark 12. It is interesting to notice that, when distributed control is present in the model, the optimal feedback law Eq. (31) reads

$$(\dot{u}_b(t),\ u_d(t))^T = K_0^e\,(u_b(t),\ x(t) - B_b u_b(t),\ y_\alpha(t))^T$$

whereas the dynamic compensator (32) has an additional output equation and becomes

$$\begin{cases} \dot{u}_b(t) = (k_{11} - k_{12}B_b)u_b(t) + k_{12}x(t) \\ u_d(t) = (k_{21} - k_{22}B_b)u_b(t) + k_{22}x(t). \end{cases}$$

The boundary control u_b then plays the role of the state of the dynamic compensator (32), whereas the distributed control u_d is its output and the closed-loop nominal system becomes

$$\begin{cases} \dot{x}(t) = (\mathcal{A} + k_{22})x(t) + B_d(k_{21} - k_{22}B_b)u_b(t), \\ \mathcal{B}x(t) = u_b(t), \\ \dot{u}_b(t) = (k_{11} - k_{12}B_b)u_b(t) + k_{12}x(t), \end{cases}$$

where the control input u_b now acts both in the boundary conditions and in the dynamics via a bounded operator and u_d is uniquely determined by u_b and x.

In general, one can expect that this dynamic feedback law is not optimal with respect to the standard cost for the nominal system. However, it has been shown that, under the assumptions of Theorem 3, the dynamic compensator minimizes the modified cost (22). The question of optimality with respect to a modified performance index has been studied in detail for finite-dimensional systems, see e.g. Ikeda and Šiljak (1990) for nonlinear systems and Moore and Anderson (1967) for linear systems with input derivative constraints.

4. Application

4.1. Description and analysis

The results of the previous section are illustrated by a system which is inspired by the literature and may correspond to a linearization around an unstable equilibrium. The systems on which this application is based are notably useful for modeling chemical and biochemical reactors, see e.g. Dramé, Dochain, and Winkin (2008), Winkin, Dochain, and Ligarius (2000), Delattre, Dochain, and Winkin (2003). Let us consider a convection–diffusion-reaction (CDR) system with boundary control and observation of the form

$$\begin{cases} \dfrac{\partial x}{\partial t}(z,t) = D\dfrac{\partial^2 x}{\partial z^2}(z,t) - v\dfrac{\partial x}{\partial z}(z,t) - kx(z,t) \\ \quad + \chi_{[1-\varepsilon_u,1]}(z)u_d(t) \\ -D\dfrac{\partial x}{\partial z}(0,t) = v(u_b(t) - x(0,t)) \\ \dfrac{\partial x}{\partial z}(1,t) = 0 \\ x(z,0) = x_0(z) \\ y(t) = x(1,t) \end{cases} \tag{36}$$

where $t \geq 0$ and $z \in [0,1]$ denote the time and the spatial variable, respectively, D, v and k are constants and $\varepsilon_u \in [0,1]$ is a parameter. This dynamical system can be interpreted as an abstract boundary control model with boundary observation described by (1)–(3) where the operator $\mathcal{A} : D(\mathcal{A}) \subset X = L^2(0,1) \to L^2(0,1)$ is given by $\mathcal{A}x = D\frac{d^2x}{dz^2} - v\frac{dx}{dz} - kx$ on its domain $D(\mathcal{A})$, which is defined as the set of all $x \in L^2(0,1)$ such that x and $\frac{dx}{dz}$ are absolutely continuous (a.c.), $\frac{d^2x}{dz^2} \in L^2(0,1)$ and $\frac{dx}{dz}(1) = 0$, the operator $B_d : U_d = \mathbb{R} \to L^2(0,1)$ is given for all $u_d \in U_d$ and $z \in [0,1]$, by

$$(B_d u_d)(z) = \chi_{[1-\varepsilon_u,1]}(z)u_d, \tag{37}$$

where $\chi_{[1-\varepsilon_u,1]}$ denotes the characteristic function of the interval $[1-\varepsilon_u, 1]$, the operator $\mathcal{B} : D(\mathcal{B}) \supset D(\mathcal{A}) \to U_b = \mathbb{R}$ is given by $\mathcal{B}x = -\frac{D}{v}\frac{dx}{dz}(0) + x(0)$ and the operator $\mathcal{C} : D(\mathcal{C}) \supset D(\mathcal{A}) \to Y = \mathbb{R}$ is given by $\mathcal{C}x = x(1)$. It should be noted that the parameter ε_u is allowed to be zero, such that $B_d = 0$, resulting in a pure boundary control model. An LQ-optimal control problem is solved in Section 4.2 for this particular case. It can be shown that this model is a BCBO system. Indeed, it is well-known that condition (**C1**) holds, where the operator A is given by $Ax = \mathcal{A}x$ for all $x \in D(A) = D(\mathcal{A}) \cap \text{Ker}\,\mathcal{B}$, where

$$D(A) = \left\{ x \in L^2(0,1) \,:\, x, \frac{dx}{dz} \text{ are a. c., } \frac{d^2x}{dz^2} \in L^2(0,1), \right.$$
$$\left. D\frac{dx}{dz}(0) - vx(0) = 0 = \frac{dx}{dz}(1) \right\}.$$

Moreover condition (**C2**) holds with the operator B_b defined as the multiplication operator by the unit step function, i.e. $B_b u = 1(\cdot)u$, where $1(z) \equiv 1$ on $[0,1]$. Finally, by arguments similar to those used in Deutscher (2013), it can be shown that the operator $\mathcal{C}$ is A-bounded. More precisely, it can be shown that there exists $f \in L^2(0,1)$ such that, for all $x \in D(A)$,

$$\mathcal{C}x = x(1) = \langle f, (I - A)x \rangle. \tag{38}$$

Such a function f is given by $f = \sum_{n=1}^{+\infty} \frac{1}{1-\lambda_n} \psi_n(1)\phi_n$, where $(\phi_n)_{n\in\mathbb{N}}$ is a Riesz basis of eigenvectors of the operator A, $(\psi_n)_{n\in\mathbb{N}}$ is a corresponding dual Riesz basis such that the vectors ϕ_n and ψ_n are bi-orthonormal, and the real numbers λ_n are the eigenvalues of A. Then inequality (6) can be easily derived from (38). Hence condition (**C3**) holds.

It follows that the analysis and all the results of the previous section apply to this model. In order to make this analysis complete, we state the following lemma which shows that the pair (A, B_d) is reachable for an appropriate choice of the parameter ε_u.

Remark 13. (a) Consider a CDR system with boundary control and observation, as well as distributed control, described by (36). If the distributed control operator B_d is given by (37), where the window width is $\varepsilon_u := 1/j$ with $j \in \mathbb{N}_0$, then the pair (A, B_d), and hence (A^e, B^e), is reachable. The proof of this result can be found in Dehaye and Winkin (2013a).

(b) Stabilizability of the pair (A^e, B^e) can be easily established even without distributed input (when $B_d = 0$) by using e.g. the criterion mentioned in Remark 8, since $0 \in \rho(A)$ (except for a very specific value of k) and (A, B_b) is stabilizable.

In view of the results stated in Sections 2 and 3, the LQ-optimal control problem is well-posed and solvable for CDR systems described by (36). This problem has been studied in e.g. Mohammadi, Aksikas, Dubljevic, and Forbes (2012).

4.2. Numerical results

This problem is solved numerically by computing successively the eigenvalues of the operator A, the zeros of the spectral density and the spectral factors and by solving the diophantine equation with a residue computation. In this case, the diophantine Eq. (30) is equivalent to

$$(K^e - K_0^e)(sI - A^e - B^eK^e)^{-1}B^e = \hat{R}^e(s) - I \tag{39}$$

where the stabilizing feedback K^e is given by (23). Since for all $x \in X$ and for all $s \in \rho(A)$,

$$(sI - A)^{-1}x = \sum_{n=1}^{+\infty} \frac{1}{s - \lambda_n} \langle x, \psi_n \rangle \phi_n,$$

the scalar feedback component k_{0_1} and the decomposition of the functional component k_{0_2} in the Riesz basis $(\phi_n)_{n\in\mathbb{N}_0}$ of eigenvectors of the operator A with corresponding dual Riesz basis $(\psi_n)_{n\in\mathbb{N}_0}$, can be found by computing the residues of the spectral factor at the pole K_1 and at each of the selected dominant eigenvalues λ_n, $n = 1, \ldots, N-1$, where $N \geq 2$. It is known that the eigenvalues $(\lambda_n)_{n\geq 1}$ of A are given for all $n \geq 1$ by

$$\lambda_n = -\frac{s_n^2 + v^2}{4D} - k \tag{40}$$

where the s_n, $n \geq 1$, are the solutions of the resolvent equation

$$\tan\left(\frac{s}{2D}\right) = \frac{2vs}{s^2 - v^2}, \quad s > 0, \tag{41}$$

and can be computed numerically with standard algorithms. The truncated modal decomposition of the solution is then integrated with the ODE solver ode45 in MATLAB. The numerical process is described by the following computational algorithm (inspired by Vandewalle and Dewilde (1975) and Winkin, Callier, Jacob, and Partington (2005)) of an approximate optimal feedback by symmetric extraction of elementary matrix spectral factors. Even though this algorithm has been designed for the general case of MIMO CDR systems, it can be easily adapted and simplified for the case of a single boundary input, which is the case that is treated here.

Algorithm:

1. Fix the number $N \geq 2$ of elementary spectral factors that will be computed. This number can be determined e.g. by performing an error analysis on the determinant of the spectral density, see Winkin et al. (2005), or when the H^∞ norm of the difference between two consecutive estimated spectral factors becomes sufficiently small.

2. Compute the $N-1$ first solutions of the resolvent Eq. (41) and the $N-1$ associated eigenvalues of A given by (40).

3. Compute the N dominant zeros of the determinant of the spectral density (24).

4. Compute the elementary spectral factors $\hat{W}_n^e$, $n = 1, \ldots, N$, associated to the N dominant pole-zero pairs (p_n, z_n) and given by $\hat{W}_n^e(s) = I - \frac{1}{s-p_n} uv^*$, where u is in the range of the Hankel matrix $H(\hat{F}^e, p_n)$, v^* is a linear combination of the lines of $H((\hat{F}^e)^{-1}, z_n)$ and $v^*u = z_n - p_n$, such that $(\det \hat{W}_n^e)(z_n) = 0$. The Hankel matrix of the spectral density $\hat{F}^e$ at the pole p_i is defined by

$$H(\hat{F}^e, p_i) = \begin{pmatrix} F_{i1} & F_{i2} & \cdots & F_{il_i-1} & F_{i1_i} \\ F_{i2} & F_{i3} & \cdots & F_{il_i} & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ F_{i1_i} & 0 & \cdots & 0 & 0 \end{pmatrix}$$

where the F_{ik} are the coefficients of the Laurent series of $\hat{F}^e$ given by $\hat{F}^e(s) = \sum_{k=1}^{l_i} F_{ik}(s - p_i)^{-k} + F_{i0} + \sum_{k=1}^{+\infty} \tilde{F}_{ik}(s - p_i)^k$ such that its elements correspond to the residues of $\hat{F}^e$ at p_i. See Vandewalle and Dewilde (1975) for the definition in the case of rational spectral densities.

5. Compute the approximate spectral factor

$$\hat{R}_N^e(s) = \hat{W}_N^e(s)\hat{W}_{N-1}^e(s) \cdots \hat{W}_1^e(s) \tag{42}$$

such that the sequence $(\det \hat{R}_N^e)_{N\in\mathbb{N}_0}$ converges to $\det \hat{R}^e$ in $\hat{\mathcal{A}}_-$.

6. Solve the Diophantine equation

$$\mathcal{U}_N \hat{D}^e(s) + \mathcal{V}_N \hat{\mathcal{N}}^e(s) = \hat{R}_N^e(s)$$

by using modal decomposition with (39).

7. Compute the approximate optimal feedback

$$K_{0_N}^e = \begin{pmatrix} k_{11_N} & k_{12_N} & 0 \\ k_{21_N} & k_{22_N} & 0 \end{pmatrix}$$

of the form (28) given by $K^e_{0_N} = -\mathcal{U}_N^{-1}\mathcal{V}_N = -\mathcal{U}_N^*\mathcal{V}_N$, where, for $i = 1, 2$,

$$k_{i2_N} = \sum_{n=1}^{N} \langle \tilde{k}_{i2_N}, \phi_n \rangle \psi_n. \quad (43)$$

Remark 14. This algorithm can be readily extended to a more general class of differential linear systems, where the determinant of the spectral density satisfies the assumptions described in Winkin et al. (2005), i.e. more specifically, to the case of a Riesz-spectral operator A with discrete spectrum $\sigma(A) = \sigma_p(A) = \{\lambda_n : n \in \mathbb{N}\} \subset \mathbb{C}$ consisting of simple eigenvalues such that

$$\inf\{|\lambda_n - \lambda_m| : n, m \in \mathbb{N},\ n \neq m\} > 0, \quad \text{and}$$
$$\sup\left\{\sum_{l=1,\, l\neq n}^{+\infty} \frac{1}{|\lambda_l - \lambda_n|^2} : n \in \mathbb{N}\right\} < +\infty.$$

Remark 15. This spectral factorization algorithm is semi-heuristic and is inspired by a method described in Vandewalle and Dewilde (1975) for parahermitian rational spectral densities. Moreover, there is currently no proof of convergence of this algorithm for multiple input systems. However, it has been shown that, under suitable conditions, the method of spectral factorization by symmetric extraction is convergent for the determinant of the multidimensional spectral density (Winkin et al., 2005).

Remark 16. Even though this approach shares a connection with classical modal control (which is sometimes referred to as the direct approach), it should be noted that it is less prone to error propagation since the modal approximation is done in closed loop, after the prestabilization and computation of the spectral density (late lumping, or indirect approach), see e.g. Balas (1986, 1988) and Christofides and Daoutidis (1997). Moreover, the transfer function of the closed-loop system with the truncated feedback functionals (43) computed by the algorithm is given by $G^e_{cl} = N^e(R^e_N)^{-1}$, where R^e_N is the approximate spectral factor given by (42).

The convergence of the algorithm of spectral factorization by symmetric extraction seems to be fast for the convection–diffusion–reaction system. In general, three elementary spectral factors are sufficient in order to obtain an accurate solution. It should be noted that the numerical conditioning heavily relies on the relative sizes of the diffusion and convection parameters. If the convection parameter is larger than the diffusion one, the model is numerically ill-conditioned and the numerical algorithm can fail quickly, especially when finding zeros of the determinant of the spectral density. However, a relatively large diffusion parameter makes the numerical conditioning much better. A nearly pure diffusive system (close to the heat equation) allows the computation of more spectral factors and/or with a higher parameter α.

The following figures illustrate the computation of the eigenvalues of the operator A and the asymptotic behavior of the state and input trajectories of the extended system with the application of the optimal feedback with the following parameters: $D = 8$, $v = 4$, $k = -9$, $\rho_1 = 2$, $\rho_2 = 1e-6$, $\rho_3 = 1.5$, $\eta = 1$, $\varepsilon = 0$ or $\varepsilon = 0.4$, $\alpha = 3100$, $N = 6$. The prestabilizing feedback (whose existence is guaranteed, see Remark 13) is given by (23) with, in this case,

$$k_1 = 0.1 \quad \text{and} \quad k_2 = 20\langle\cdot, \psi_1\rangle. \quad (44)$$

The stability of the corresponding closed-loop system has been checked numerically. Again, it should be noted that the prestabilizing feedback (44) is only used as a computational tool and its influence is completely removed as soon as the Diophantine equation is solved, yielding the optimal feedback law. As such, this prestabilizing feedback is not implemented in the closed-loop system at any point in the resolution of the problem. The initial condition is given by $u_{b0} = u_b(0) = 0$ and for all $z \in [0, 1]$, $v_0(z) = \sin[(7\pi/2)z] + 7D\pi/(2v)\cos(6\pi z)$. Note that $u_b(0) = 0$ must hold for the classical solution, since $u_b(0) = -(D/v)x_0(0) + x_0(0)$ and $x_0 \in D(A)$. However, the mild solution is well defined for all $x_0 \in X$ and for the corresponding u_{b0}.

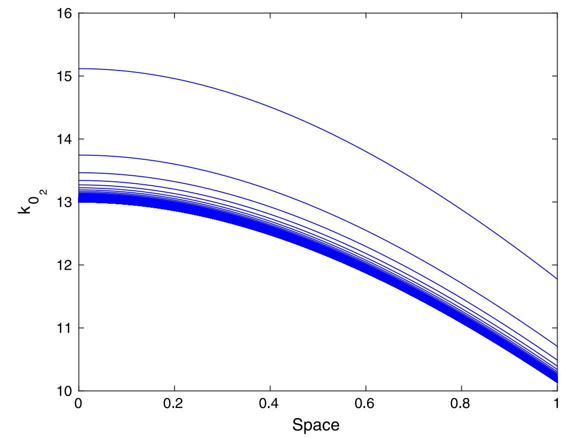

Fig. 2. Profile of the feedback functional k_{0_2} with varying α.

The poles and zeros of $\hat{F}^e$ are given below.

Poles	Zeros
−1.60295	1.90984
−1.13281e+01	1.13086e+01
−7.82578e+01	−7.82579e+01
−3.15273e+02	−3.15273e+02
−7.10087e+02	−7.10087e+02
−1.26279e+03	−1.26279e+03

Remark 17. The poles correspond to the dominant closed-loop eigenvalues with the prestabilizing feedback (23), (44). It should be noted that the distance between the poles and zeros is converging towards 0 at a fast rate.

The optimal feedback approximate coefficients are $k_{0_1} = 1.61244e-01$, $\langle k_{0_2}, \phi_1\rangle = 1.35439e+01$, $\langle k_{0_2}, \phi_2\rangle = -5.13287e-03$, $\langle k_{0_2}, \phi_3\rangle = 3.44387e-04$, $\langle k_{0_2}, \phi_4\rangle = -5.90957e-05$, $\langle k_{0_2}, \phi_5\rangle = 1.55667e-05$.

Fig. 2 illustrates the numerical convergence of the optimal feedback when α goes to infinity. As the parameter α increases, the distance between two successive feedback functionals becomes smaller as expected.

Figs. 3–5 illustrate the behavior of the closed-loop system, with a comparison between the closed-loop state trajectories with and without distributed control. One can clearly see in Figs. 3 and 4 the influence of the boundary and distributed actuation, respectively.

In Fig. 5, it is seen that, as the ratio ρ_1/η increases, the norm of the boundary input is reduced compared to its variation rate due to the higher weight in the cost functional, which leads to a faster stabilization at the cost of higher variations.

Remark 18. Some additional numerical simulations suggest that the optimal cost could be a monotonically convergent function of the parameter α.

Finally, it should be noted that, even though $B_d = 0$ in these simulations, considering a nonzero but suitably scaled distributed control window can be used as a way to obtain a good approximation of the solution while providing an easy mean to guarantee

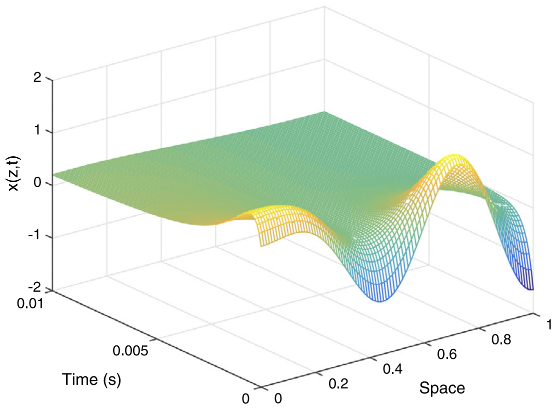

Fig. 3. Closed-loop state trajectory $x(t)$ with pure boundary control.

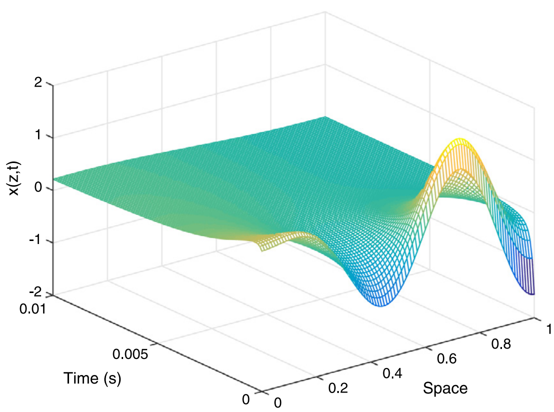

Fig. 4. Closed-loop state trajectory $x(t)$ with mixed boundary-distributed control ($\varepsilon = 0.4$).

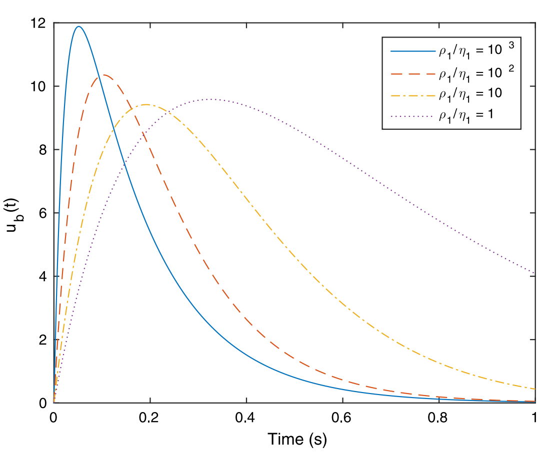

Fig. 5. Closed-loop boundary input $u_b(t)$ with varying ratio ρ_1/η.

reachability, and hence uniqueness of the solution, see Remark 13. Figs. 6 and 7 show the comparison between the boundary input and output trajectories obtained with $B_d = 0$ (bold blue line) and $B_d = \delta\chi_{[1-\varepsilon,1]}$, with δ and ε converging towards zero. As expected, pure boundary control is slightly slower to stabilize the output than when it is helped by distributed control.

5. Comparison with the nominal system

The analysis and results presented in this section deal with the general case where B_d and u_d are present in the model but remain valid in the specific case where $B_d = 0$ and can be adapted in a very straightforward way.

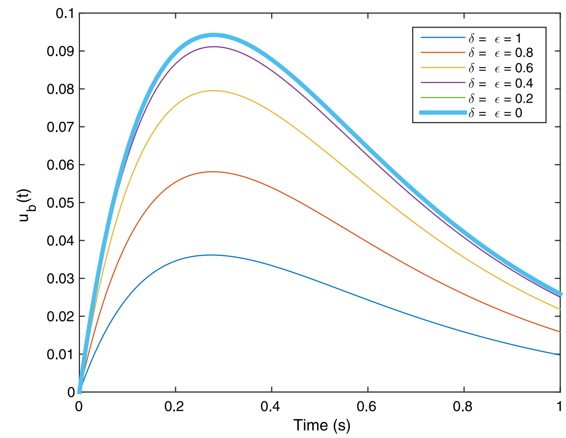

Fig. 6. Closed-loop boundary input $u_b(t)$ with $B_d = 0$ and $B_d = \varepsilon\chi_{[1-\varepsilon,1]}$ for different values of ε.

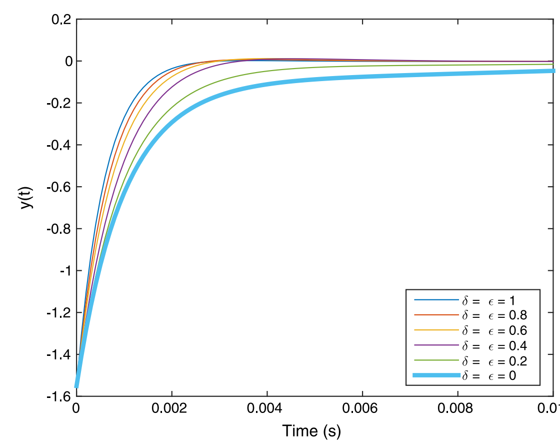

Fig. 7. Closed-loop output $y(t)$ with $B_d = 0$ and $B_d = \varepsilon\chi_{[1-\varepsilon,1]}$ for different values of ε.

We assume that the nominal system (1)–(3) is a regular linear system (Weiss, 1994). Under this assumption, the state trajectories are almost everywhere in the domain of the Yosida extension, i.e. for all $x_0 \in D(A)$ and for almost all $t \geq 0$, $C_\alpha x(t) \to \mathcal{C}x(t)$ when $\alpha \to +\infty$ along the real axis.

First, as in Weiss (1994), let us define the Yosida extension $C_\Lambda : D(C_\Lambda) \subset X \to Y$ given by

$$C_\Lambda x_0 = \lim_{\alpha\to+\infty} C_\alpha x_0 \tag{45}$$

where $\alpha \in \mathbb{R}$, for all x_0 in its domain $D(C_\Lambda) = \{x_0 \in X :$ the limit in (45) exists$\}$. It should be noted that the operator C_Λ is an extension of the operator C_L defined by

$$C_L x_0 = \lim_{\tau\to 0} \mathcal{C}\frac{1}{\tau}\int_0^\tau \mathbb{T}_\sigma x_0 d\sigma \tag{46}$$

on its domain $D(C_L) = \{x_0 \in X :$ the limit in (46) exists$\}$, which is often referred to as the *Lebesgue extension* of $\mathcal{C}$, in the sense that $D(C_L) \subset D(C_\Lambda) \subset X$ and for all $x \in D(C_L)$, $C_\Lambda x = C_L x$, see Weiss (1994), where it is also emphasized that in most cases C_L and C_Λ are interchangeable and one can choose the most suitable one. In this case, the Yosida extension C_Λ was chosen because of the importance and the recurrent role of the frequency domain and the resolvent operator of A in this theory. In fact, the computation of the resolvent operator of A (and hence of A^e) is a crucial step in order to obtain the right coprime fraction (25)–(26) and the spectral density (24). Observe that both C_L and C_Λ are extensions of the admissible observation operator $\mathcal{C}$, which can be written as $\mathcal{C}x = \lim_{t\to 0} \frac{1}{t}\int_0^t (\Psi_\tau x)(\sigma)d\sigma$ for all $x \in X_1$. Now, we assume that

(1) the operator $(\mathcal{A}-A)B_b$ is an admissible control operator for the semigroup $\mathbb{T}=(\mathbb{T}_t)_{t\geq 0}=(T(t))_{t\geq 0}$ (see condition **[C1]** in Definition 1), i.e. for some $\tau>0$, Ran $\tilde{\Phi}_\tau\subset X$, where for all $u_b\in U_b$, $\tilde{\Phi}_\tau u_b=\int_0^\tau T(\tau-\sigma)(\mathcal{A}-A)B_b u_b(\sigma)d\sigma$,

(2) the operator $\mathcal{C}$ is an admissible observation operator for $\mathbb{T}$, i.e. for some $\tau>0$, there exists a constant $K_\tau\geq 0$ such that for all $x\in D(A)$, $\int_0^\tau\|\mathcal{C}T(t)x\|_Y^2dt\leq K_\tau^2\|x\|_X^2$.

Under these conditions, the BCBO system (1)–(3) is a *well-posed linear system* whose state, input-state, state-output and input–output mappings defined in Weiss (1994) are given by $\mathbb{T}_\tau=T(\tau)$ for all $\tau\geq 0$,

$$\Phi_\tau u=\int_0^\tau T(\tau-\sigma)(\mathcal{A}-A)B_b u_b(\sigma)d\sigma+\int_0^\tau T(\tau-\sigma)B_d u_d(\sigma)d\sigma$$

for all $\tau\geq 0$ and $u\in\Omega=L^2([0,+\infty),U)$, where $U:=U_b\oplus U_d$,

$$(\Psi_\tau x)(t)=\begin{cases}\mathcal{C}T(t)x & \text{if } t\in[0,\tau)\\ 0 & \text{if } t\geq\tau\end{cases}$$

for all $\tau>0$ and $x\in D(A)$, with $\Psi_0=0$, and

$$(\mathbb{F}_\tau u)(t)=\begin{cases}\mathcal{C}\int_0^t[T(t-\sigma)\mathcal{A}-AT(t-\sigma)]B_b u_b(\sigma)d\sigma\\ \quad+\mathcal{C}\int_0^t T(t-\sigma)B_d u_d(\sigma)d\sigma \quad \text{if } t\in[0,\tau)\\ 0\quad \text{if } t\geq\tau\end{cases}$$

for all $\tau\geq 0$ and $u\in\Omega$, with $\mathbb{F}_0=0$, respectively.

In what follows, for any Hilbert space W, for all $u,v\in L^2([0,+\infty),W)$ and for all $\tau\geq 0$, the τ-concatenation of u and v is defined by $(u\diamond_\tau v)(t)=\begin{cases}u(t) & \text{if } t\in[0,\tau)\\ v(t-\tau) & \text{if } t\geq\tau.\end{cases}$

The mappings introduced above satisfy the assumptions of Weiss (1994). Indeed,

(i) $\mathbb{T}=(\mathbb{T}_t)_{t\geq 0}=(T(t))_{t\geq 0}$ is a C_0-semigroup of bounded linear operators on X.

(ii) The family $\Phi=(\Phi_t)_{t\geq 0}$ is a family of bounded linear operators from Ω to X. Indeed, $(\mathcal{A}-A)B_b\in\mathcal{L}(U_b,X_{-1})$ since, for all $u_b\in U_b$,

$$\begin{aligned}\|(\mathcal{A}-A)B_b u_b\|_{-1}&=\|(\beta I-A)^{-1}(\mathcal{A}-A)B_b u_b\|\\&\leq\left(\|(\beta I-A)^{-1}\mathcal{A}B_b\|+|\beta|\|(\beta I-A)^{-1}B_b\|+\|B_b\|\right)\|u_b\|\end{aligned}$$

where $\beta\in\rho(A)$. Moreover, $(\mathcal{A}-A)B_b$ is admissible by assumption, hence we deduce from Tucsnak and Weiss (2009, Proposition 4.2.2., p. 126) that for all $t\geq 0$, $\Phi_t\in\mathcal{L}(\Omega,X)$. In addition, for all $u_1,u_2\in\Omega$ and for all $\tau,t\geq 0$, $\Phi_{\tau+t}(u_1\diamond_\tau u_2)=T(t)\Phi_\tau u_1+\Phi_t u_2=\mathbb{T}_t\Phi_\tau u_1+\Phi_t u_2$.

(iii) The family $\Psi=(\Psi_t)_{t\geq 0}$ is a family of bounded linear operators from X to $\Gamma=L^2([0,+\infty),Y)$. Indeed, since $\mathcal{C}\in\mathcal{L}(X_1,Y)$ is admissible, it follows by Tucsnak and Weiss (2009, Proposition 4.3.2., p. 132) that for all $t\geq 0$, $\Psi_t\in\mathcal{L}(X,\Gamma)$. Moreover, since $(T(t))_{t\geq 0}$ satisfies the semigroup property, it is easy to check that for all $x\in X$ and for all $\tau,t\geq 0$, $\Psi_{\tau+t}x=\Psi_\tau x\diamond_\tau\Psi_t T(\tau)x=\Psi_\tau x\diamond_\tau\Psi_t\mathbb{T}_\tau x$. Finally, $\Psi_0=0$.

(iv) The family $\mathbb{F}=(\mathbb{F}_t)_{t\geq 0}$ is a family of bounded linear operators from Ω to Γ, which follows from (ii) and (iii), and for all $u_1,u_2\in\Omega$ and for all $\tau,t\geq 0$, $\mathbb{F}_{\tau+t}(u_1\diamond_\tau u_2)=\mathbb{F}_\tau u_1\diamond_\tau(\Psi_t\Phi_\tau u_1+\mathbb{F}_t u_2)$. Finally, $\mathbb{F}_0=0$.

Proposition 4. *Let us assume that*

(i) *the transfer function of the nominal system* (1)–(3) *given by* $\hat{H}(s)=\left(C_\Lambda(sI-A)^{-1}(\mathcal{A}B_b-sB_b)+C_\Lambda B_b\quad C_\Lambda(sI-A)^{-1}B_d\right)$ *has a strong limit along the real axis, i.e., for all* $u=(u_b,u_d)\in U_b\oplus U_d$, $\lim_{\mathbb{R}\ni s\to\infty}\hat{H}(s)u\in Y$, *which implies that the system is regular, and*

(ii) *for all* $(u_b,u_d)\in U_b\oplus U_d$, $B_b u_b\in D(C_\Lambda)$ *and*

$$\lim_{\alpha\to+\infty}C_\alpha B_b u_b=\mathcal{C}B_b u_b,$$
$$\lim_{\alpha\to+\infty}C_\alpha(sI-A)^{-1}B_b u_b=\mathcal{C}(sI-A)^{-1}B_b u_b,$$
$$\lim_{\alpha\to+\infty}C_\alpha(sI-A)^{-1}B_d u_d=\mathcal{C}(sI-A)^{-1}B_d u_d.$$

Then, the following properties hold:

(a) *for all state trajectories* $x(\cdot)$ *of the BCBO system and for a.e.* $t\geq 0$, $x(t)$ *is in the domain of the Yosida extension, i.e.* $y_\alpha(t)=C_\alpha x(t)$ *converges in* Y *as* α *goes to* $+\infty$, *and*

$$\lim_{\alpha\to+\infty}y_\alpha(t)=\lim_{\alpha\to+\infty}C_\alpha x(t)=\mathcal{C}x(t),$$

(b) *for all* $s\in\rho(A)$, $\lim_{\alpha\to+\infty}\hat{\hat{G}}^e_\alpha(s)=\hat{G}(s)$, *where the input–output transfer functions* $\hat{G}(s)$ *and* $\hat{\hat{G}}^e_\alpha$ *of the nominal and extended systems respectively are defined by* $\hat{y}(s)=\hat{G}(s)\left(\hat{u}_b(s),\ \hat{u}_d(s)\right)^T$ *and* $\hat{y}_\alpha(s)=\hat{\hat{G}}^e_\alpha(s)\left(\hat{u}_b(s),\ \hat{u}_d(s)\right)^T$.

Proof. Since (1)–(3) is a well-posed linear system, (a) follows immediately from Weiss (1994, Theorem 5.8.) and Weiss (1994, Remark 6.2.).

(b) follows from (ii) and Weiss (1994, Theorem 5.8.) since the regularity implies that for all $s\in\rho(A)$ and for all $(u_b,u_d)\in U_b\oplus U_d$, $(sI-A)^{-1}B_b u_b$, $(sI-A)^{-1}\mathcal{A}B_b u_b$ and $(sI-A)^{-1}B_d u_d\in D(C_\Lambda)$. ■

Corollary 1. *The convection–diffusion–reaction system* (36) *is such that*

(a) *for all* $s\in\rho(A)$, $\lim_{\alpha\to+\infty}\hat{\hat{G}}^e_\alpha(s)=\hat{G}(s)$, *and*

(b) *for a.e.* $t\geq 0$, $y_\alpha(t)=C_\alpha x(t)$ *converges in* $\mathbb{C}$ *as* α *goes to* $+\infty$, *and* $\lim_{\alpha\to+\infty}y_\alpha(t)=\lim_{\alpha\to+\infty}C_\alpha x(t)=x(1,t)$.

The proof of this result can be found in the Appendix.

Remark 19. Transfer functions and input–output maps have been studied for boundary control systems in factor form with an admissible observation operator in Grabowski and Callier (2001).

6. Conclusion

We are currently investigating the convergence of the method of spectral factorization by symmetric extraction for a class of MIMO distributed parameter systems including parabolic systems such as convection–diffusion–reaction systems, extending to the multi-input case the one studied in Winkin et al. (2005). This would provide a guarantee that the solution generated by the semi-heuristic algorithm will converge to the spectral factor. We hope that this analysis will provide an estimate of the rate of convergence.

Moreover, in view of the promising numerical results obtained in the application (see e.g. Fig. 2), it is expected that, possibly under additional conditions, the closed-loop stable transfer function $(\hat{u}_b,\hat{u}_d)\mapsto\hat{y}_\alpha$ (which depends on α) will converge to a certain closed-loop transfer function corresponding to the nominal system with respect to an appropriate norm as α tends to $+\infty$. This conjecture could also be studied in detail and proved, potentially resulting in a proof of convergence of the optimal feedback for the extended system towards the solution of an appropriate LQ-optimal control problem for the nominal system as α tends to $+\infty$ and η_1 tends to 0, see (7) and (22). This question is beyond the scope of this article and is an interesting topic for future research. The continuity of the spectral factorization with respect to a parameter could also play an important role here, see Jacob, Winkin, and Zwart (1999).

Finally, there are ongoing developments in order to compare the numerical results obtained in Section 4 with a priori discretization methods, such as finite differences and the resolution of the finite-dimensional LQ-optimal control problem by the resolution of the Riccati equation.

Acknowledgments

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Appendix. Proof of Corollary 1

The transfer function of the convection–diffusion–reaction system (36) is given by $\hat{G}(s) = \begin{pmatrix}\hat{G}_1(s) & \hat{G}_2(s)\end{pmatrix}$ where $\hat{G}_1(s) = \mathcal{C}(sI-A)^{-1}(\mathcal{A}B_b - sB_b) + \mathcal{C}B_b = \frac{ve^{\frac{v}{2D}}\sqrt{\rho}}{g(s,1)}$ and $\hat{G}_2(s) = \mathcal{C}(sI-A)^{-1}B_d = -\frac{e^{\frac{v}{2D}\varepsilon_u}}{s+k}\frac{g(s,1-\varepsilon_u)}{g(s,1)} + \frac{1}{s+k}$ with $\rho := \rho(s) := v^2 + 4D(k+s)$ and $g(s,z) = [v^2 + 2D(k+s)]\sinh\left(z\frac{\sqrt{\rho(s)}}{2D}\right) + v\sqrt{\rho(s)}\cosh\left(z\frac{\sqrt{\rho(s)}}{2D}\right)$. This transfer function is strictly proper and thus admits a strong limit at $+\infty$ along the real axis, i.e. for all $(u_b, u_d) \in \mathbb{C}\oplus\mathbb{C}$, $\hat{G}(s)\,(u_b, u_d)^T = \hat{G}_1(s)u_b + \hat{G}_2(s)u_d \to 0$ as $|s| \to +\infty$. The input–output transfer function $\hat{\hat{G}}^e_\alpha$ of the extended system defined by $\hat{y}_\alpha(s) = \hat{\hat{G}}^e_\alpha(s)\left(\hat{u}_b(s), \hat{u}_d(s)\right)^T$ for all $s \in \mathbb{C}$ is given by $\hat{\hat{G}}^e_\alpha(s) = \begin{pmatrix}\hat{\hat{G}}^e_{\alpha 1}(s) & \hat{\hat{G}}^e_{\alpha 2}(s)\end{pmatrix}$ where $\hat{\hat{G}}^e_{\alpha_i}(s) = \frac{\alpha}{s-\alpha}[\hat{G}_i(\alpha) - \hat{G}_i(s)]$, $i = 1, 2$, by the resolvent identity, see Jacob and Zwart (2012, Proposition 5.2.4, p. 59). Since

$$\lim_{\alpha\to+\infty}\mathcal{C}(\alpha I - A)^{-1}B_b = \lim_{\alpha\to+\infty}\mathcal{C}(\alpha I - A)^{-1}B_d = 0 \quad\text{and}$$

$$\lim_{\alpha\to+\infty} C_\alpha B_b = \lim_{\alpha\to+\infty}\frac{\alpha}{k+\alpha}\left[-\frac{ve^{\frac{v}{2D}}\sqrt{\rho(\alpha)}}{g(\alpha,1)} + 1\right] = 1,$$

it is easy to see that for all $s \in \mathbb{C}$, $\lim_{\alpha\to+\infty}\hat{\hat{G}}^e_\alpha(s) = \hat{G}(s)$.

Now we show that the operators $(\mathcal{A} - A)B_b$ and $\mathcal{C}$ are admissible for $(T(t))_{t\geq 0}$. Since A is a Riesz-spectral operator, it is diagonalizable, i.e. isomorphic to a diagonal operator on l^2, see Tucsnak and Weiss (2009, Section 2.6, pp. 49–56). This isomorphism is $Q \in \mathcal{L}(L^2(0,1), l^2)$, which is defined by $Qx = (\langle x, \psi_n\rangle)_{n\in\mathbb{N}_0}$ for all $x \in L^2(0,1)$. Obviously, for all $n \geq 1$, $Q\phi_n = e_n$, where $\{e_n : n \in \mathbb{N}_0\}$ is the canonical basis of l^2. By this similarity transformation, the operator A can be seen as the diagonal operator $\tilde{A} : D(\tilde{A}) \subset l^2 \to l^2$ defined by $\tilde{A}w = (\lambda_n w_n)_{n\in\mathbb{N}_0}$ for all $w \in D(\tilde{A})$, where $D(\tilde{A}) = \left\{w \in l^2 : \sum_{n\in\mathbb{N}_0}(1+\lambda_n^2)|w_n|^2 < +\infty\right\}$. The operator $\mathcal{C} \in \mathcal{L}(X_1, \mathbb{C})$ can be seen as the operator $\tilde{\mathcal{C}} \in \mathcal{L}(l_1^2, \mathbb{C})$ defined by $\tilde{\mathcal{C}}w = \langle w, \bar{c}\rangle = \sum_{n\in\mathbb{N}_0} c_n w_n = \sum_{n\in\mathbb{N}_0}\phi_n(1)w_n$ for all $w \in l_1^2$, where the sequence $(c_n)_{n\in\mathbb{N}_0} = (\phi_n(1))_{n\in\mathbb{N}_0} \in l^2_{-1}$. We show that this sequence satisfies the Carleson measure criterion for the sequence $(\lambda_n)_{n\in\mathbb{N}_0}$, i.e. there exists $M > 0$ such that for all $h > 0$ and $\omega \in \mathbb{R}$,

$$\sum_{-\bar{\lambda}_n\in R(h,\omega)}|c_n|^2 \leq Mh, \tag{A.1}$$

where $R(h,\omega) = \{s \in \mathbb{C} : 0 < \operatorname{Re} s \leq h,\ |\operatorname{Im} s - \omega| \leq h\}$. Actually, let $h > 0$ and $\omega \in \mathbb{R}$. Since all the eigenvalues of A are real, we see that, if $|\omega| > h$, then, for all $n \in \mathbb{N}_0$, $-\bar{\lambda}_n = -\lambda_n \notin R(h,\omega)$ and $\sum_{-\bar{\lambda}_n\in R(h,\omega)}|c_n|^2 = 0$. If $|\omega| \leq h$, then $-\bar{\lambda}_n = -\lambda_n \in R(h,\omega)$ if and only if $-\lambda_n \leq h$, which is true if and only if $s_n^2 \leq 4Dh - 4Dk - v^2$. Since the sequence $(s_n)_{n\in\mathbb{N}_0}$ is such that for all $n \in \mathbb{N}_0$, $s_{n+1} \geq s_n > 0$, we can distinguish three cases.

(1) If $h < \frac{s_1^2+v^2}{4D} + k$, then the condition $-\bar{\lambda}_n \in R(h,\omega)$ is never fulfilled, hence

$$\sum_{-\bar{\lambda}_n\in R(h,\omega)}|c_n|^2 = 0. \tag{A.2}$$

(2) If $\frac{s_1^2+v^2}{4D} + k \leq h < \frac{s_2^2+v^2}{4D} + k$, then one observes that $-\bar{\lambda}_1 = -\lambda_1 \in R(h,\omega)$ and for all $n \geq 2$, $-\bar{\lambda}_n = -\lambda_n \notin R(h,\omega)$. Hence,

$$\sum_{-\bar{\lambda}_n\in R(h,\omega)}|c_n|^2 = |c_1|^2 = |\phi_1(1)|^2. \tag{A.3}$$

Observe that, for all $n \in \mathbb{N}_0$,

$$|\phi_n(1)|^2 = \left|K_n e^{\frac{v}{2D}}\left(\cos\frac{s_n}{2D} + \frac{v}{s_n}\sin\frac{s_n}{2D}\right)\right|^2$$
$$\leq e^{\frac{v}{2D}}\sup_{n\in\mathbb{N}_0}|K_n|^2\left(1 + \frac{2v}{s_1} + \frac{v^2}{s_1^2}\right) \tag{A.4}$$

since $(K_n)_{n\in\mathbb{N}_0}$ is a bounded sequence. Moreover,

$$h \geq \frac{s_1^2+v^2}{4D} + k > \frac{v^2}{4D} + k. \tag{A.5}$$

From (A.3) to (A.5), we deduce that

$$\sum_{-\bar{\lambda}_n\in R(h,\omega)}|c_n|^2 \leq m \leq \frac{m}{K}h \tag{A.6}$$

where $m = e^{\frac{v}{2D}}\sup_{n\in\mathbb{N}_0}|K_n|^2\left(1 + \frac{2v}{s_1} + \frac{v^2}{s_1^2}\right)$ and $K = \frac{v^2}{4D} + k$.

(3) If $\frac{s_2^2+v^2}{4D} + k \leq h$, observe that, for all $n \geq 2$, $\frac{s_n}{2D\pi} \geq 1$ since $2D(n-1)\pi \leq s_n \leq 2Dn\pi$, which implies that for all $n \geq 2$, $\frac{s_n}{2D\pi} \leq \frac{s_n^2}{4D^2\pi^2}$. Now, it can be shown that

$$\sum_{-\bar{\lambda}_n\in R(h,\omega)}|c_n|^2 \leq \left(\frac{m}{D\pi^2} + \frac{m}{K}\right)h \tag{A.7}$$

where $\left\lfloor\frac{h}{D\pi^2} + 1\right\rfloor$ is the integer part of $\frac{h}{D\pi^2} + 1$.

Finally, combining (A.2), (A.6) and (A.7), one concludes that there exists $M = \frac{m}{D\pi^2} + \frac{m}{K} > 0$, which is independent of h and ω, such that (A.1) holds. Since $(T(t))_{t\geq 0}$ is an exponentially stable C_0-semigroup, it follows from Tucsnak and Weiss (2009, Theorem 5.3.2., p. 159) and Tucsnak and Weiss (2009, Remark 5.3.4., p. 162) that $\mathcal{C}$ is an admissible observation operator for $(T(t))_{t\geq 0}$.

In order to show that $(\mathcal{A} - A)B_b$ is an admissible control operator for $(T(t))_{t\geq 0}$, we show that its adjoint $[(\mathcal{A} - A)B_b]^*$ is an admissible observation operator for $(T(t)^*)_{t\geq 0}$. Let us consider again the operator $\tilde{A}$ and observe that it is a self-adjoint operator. Now we compute $[(\mathcal{A} - A)B_b]^*$ by using the identity $\langle\mathcal{A}x, \psi\rangle = \langle x, A^*\psi\rangle + \langle\mathcal{B}x, [(\mathcal{A} - A)B_b]^*\psi\rangle$ for all $x \in D(\mathcal{A})$, $\psi \in D(A^*)$, see Tucsnak and Weiss (2009, Remark 10.1.6., p. 330). In this case, the operator $A^* : D(A^*) \subset L^2(0,1) \to L^2(0,1)$ is given by $A^*x = D\frac{d^2x}{dz^2} + v\frac{dx}{dz} - kx$ for all $x \in D(A^*)$, where

$$D(A^*) = \left\{x \in L^2(0,1) : x, \frac{dx}{dz} \text{ are a. c.}, \frac{d^2x}{dz^2} \in L^2(0,1),\right.$$
$$\left.\frac{dx}{dz}(0) = 0 = D\frac{dx}{dz}(1) + vx(1)\right\}.$$

Using integration by parts and the conditions of $D(\mathcal{A})$ and $D(A^*)$, we obtain

$$\langle \mathcal{A}x, \psi\rangle - \langle x, A^*\psi\rangle = \left[-\frac{D}{v}\frac{dx}{dz}(0) + x(0)\right] v\overline{\psi(0)} \tag{A.8}$$

for all $x \in D(\mathcal{A})$, $\psi \in D(A^*)$. But (A.8) should be equal to

$$\langle \mathcal{B}x, [(\mathcal{A} - A)B_b]^*\psi\rangle = \left[-\frac{D}{v}\frac{dx}{dz}(0) + x(0)\right] \overline{[(\mathcal{A} - A)B_b]^*\psi}.$$

Hence, for all $\psi \in D(A^*)$, $[(\mathcal{A} - A)B_b]^*\psi = v\psi(0)$. The corresponding observation operator on l^2 is $\tilde{B} \in \mathcal{L}(l^2_1, \mathbb{C})$ defined by $\tilde{B}w = \langle w, \overline{b}\rangle = \sum_{n\in\mathbb{N}_0} b_n w_n = \sum_{n\in\mathbb{N}_0} v\phi_n(0)w_n$ for all $w \in l^2_1$, where the sequence $(b_n)_{n\in\mathbb{N}_0} = (v\phi_n(0))_{n\in\mathbb{N}_0} \in l^2_{-1}$. By an analysis similar to the one performed previously for the operator $\mathcal{C}$, one can show that the sequence $(b_n)_{n\in\mathbb{N}_0}$ satisfies the Carleson measure criterion for the sequence $(\lambda_n)_{n\in\mathbb{N}_0}$ with $\tilde{M} = \frac{\tilde{m}}{D\pi^2} + \frac{\tilde{m}}{K} > 0$, where $\tilde{m} = v\sup_{n\in\mathbb{N}_0} |K_n|^2$. Hence $(\mathcal{A} - A)B_b$ is an admissible control operator for $(T(t))_{t\geq 0}$. The conclusion then follows from Proposition 4. ■

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Jérémy R. Dehaye was born on November 28, 1986, in Ath, Belgium. He received a Master degree in Applied Mathematics in 2009 and a Ph.D. in Applied Mathematics in 2015 from the University of Namur (Belgium). From 2009 to 2015, he was a teaching assistant at the Department of Mathematics, University of Namur, where he is currently working as a collaborateur didactique. His main research interests are infinite-dimensional (distributed parameter) systems theory, boundary control, linear–quadratic optimal control, spectral factorization and applications to parabolic and hyperbolic systems.

Joseph J. Winkin is professor and head of the Department of Mathematics at the University of Namur, Belgium. He is also member of the Namur Center for Complex Systems (naXys), University of Namur, Belgium, and of the Center for Systems Engineering and Applied Mechanics (CESAME), UCL, Louvain-la-Neuve, Belgium. His main research interest is in the area of system and control theory, especially distributed parameter (infinite-dimensional) system theory and applications, linear–quadratic optimal control, spectral factorization techniques and dynamical analysis and control of (bio)chemical reactor models. He has also research interest in nonlinear system analysis and control. J. J. Winkin has contributed to numerous technical papers in these fields. He is member of the IEEE and IFAC Technical Committees on Distributed Parameter Systems. He is also member of IEEE, IBRA (Institut Royal Belge de Régulation et d'Automatisme), BMS (Belgian Mathematical Society) and EMS (European Mathematical Society), and serves as a reviewer for several journals in the system and control area. He is an associate editor of the Journal Européen des Systèmes Automatisés and a past associate editor of the IEEE Transactions on Automatic Control.