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Moreau–Yosida Regularization of Maximal Monotone Operators of Type (D)

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Abstract We propose a Moreau–Yosida regularization for maximal monotone operators of type (D), in non-reflexive Banach spaces. It generalizes the classical Moreau–Yosida regularization as well as Brezis–Crandall–Pazy's extension of this regularization to strictly convex (reflexive) Banach spaces with strictly convex duals. Our main results are obtained by making use of recent results by the authors on convex representations of maximal monotone operators in non-reflexive Banach spaces.

Keywords Maximal monotone operator · Operators of type (D) · Moreau–Yosida regularization · Non-reflexive Banach spaces

Mathematics Subject Classifications (2000) 47H05 · 49J52 · 47N10

1 Introduction

Let X be a real Banach space, X^* its topological dual and X^{**} its topological bidual. The norms on X, X^* and X^{**} will be denoted by $\|\cdot\|$. The notation $\langle \cdot, \cdot \rangle$ stands for the duality product in both $X \times X^*$ and $X^* \times X^{**}$,

$$\langle x, x^* \rangle = x^*(x), \quad \langle x^*, x^{**} \rangle = x^{**}(x^*), \quad x \in X,\ x^* \in X^*, x^{**} \in X^{**}.$$

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A point-to-set operator $T : X \rightrightarrows X^*$ is a relation on X to X^*:

$$T \subset X \times X^*$$

and $x^* \in T(x)$ means $(x, x^*) \in T$. The domain of T is defined as

$$\mathrm{Dom}(T) = \{x \in X \mid T(x) \neq \emptyset\}.$$

An operator $T : X \rightrightarrows X^*$ is *monotone* if

$$\langle x - y, x^* - y^* \rangle \geq 0, \qquad \forall (x, x^*), (y, y^*) \in T.$$

The operator T is *maximal monotone* if it is monotone and maximal in the family of monotone operators of X into X^* (with respect to order of inclusion). In a real Hilbert space H, the *Moreau–Yosida regularization* and the *resolvent operator* of a maximal monotone operator $T : H \rightrightarrows H$, with parameter $\lambda > 0$ are, respectively, the operators

$$T_\lambda = \frac{I - R_\lambda}{\lambda}, \quad R_\lambda = (\lambda T + I)^{-1}. \tag{1}$$

The operator T_λ is defined in the whole H, it is point-to-point and Lipschitz continuous. For properties and applications of the Moreau–Yosida regularization of maximal monotone operators in *Hilbert* spaces we refer [10].

The *subdifferential* of $f : X \to \mathbb{R} \cup \{\infty\}$ is the point-to-set operator $\partial f : X \rightrightarrows X^*$,

$$\partial f(x) = \{x^* \in X^* \mid f(y) \geq f(x) + \langle y - x, x^* \rangle, \ \forall \ y \in X\}.$$

Moreau–Yosida regularization has been extended for maximal monotone operators in *strictly convex* reflexive Banach spaces with strictly convex duals by Brezis, Crandall and Pazy in [1]. In this setting, the identity operator used in Eq. 1 is replaced by the duality mapping

$$J = \partial \frac{1}{2} \| \cdot \|^2,$$

which is point-to-point. Brezis–Crandall–Pazy's regularization has some of the properties of the classical Moreau–Yosida regularization [1, Lemma 1.3] and in Hilbert spaces coincides with the classical Moreau–Yosida regularization.

Our aim is to extend the Brezis–Crandall–Pazy's version of the Moreau–Yosida regularization for maximal monotone operators of type (D), in a general Banach space, without re-norming it. In a *reflexive* Banach space any maximal monotone operator is of type (D). Therefore, in particular, we will generalize Brezis–Crandall–Pazy's version of Moreau–Yosida regularization for maximal monotone operators in reflexive spaces *without* re-norming the space to a strictly convex and smooth norm.

2 Basic Results and Notation

We use the notation $\overline{\mathbb{R}}$ for the extended real numbers:

$$\overline{\mathbb{R}} = \{-\infty\} \cup \mathbb{R} \cup \{\infty\}.$$

A convex function $f : X \to \overline{\mathbb{R}}$ is said to be proper if $f > -\infty$ and there exists a point $\hat{x} \in X$ for which $f(\hat{x}) < \infty$.

The *Fenchel–Legendre conjugate* of $f : X \to \overline{\mathbb{R}}$ is $f^* : X^* \to \overline{\mathbb{R}}$ defined by

$$f^*(x^*) = \sup_{x\in X}\langle x, x^*\rangle - f(x). \tag{2}$$

Notice that f^* is always convex and lower semicontinuous. If f is proper, convex and lower semicontinuous, then f^* is proper and its *biconjugate function*, $f^{**} := (f^*)^*$, is well defined and $f^{**}(x) = f(x)$, for all $x \in X$. We will identify $x \in X$ with its canonical inclusion into X^{**}. Moreover, from the definition of f^*, follows directly the *Fenchel–Young inequality*: for all $x \in X$, $x^* \in X^*$,

$$f(x) + f^*(x^*) \geq \langle x, x^*\rangle \text{ and } f(x) + f^*(x^*) = \langle x, x^*\rangle \text{ iff } x^* \in \partial f(x). \tag{3}$$

In the particular case $f(x) = \|x\|^2/(2\lambda)$, $\lambda > 0$, we obtain a characterization of the duality mapping $J : X \rightrightarrows X^*$

$$\begin{aligned} &\frac{1}{2\lambda}\|x\|^2 + \frac{\lambda}{2}\|x^*\|^2 \geq \langle x, x^*\rangle, \\ &\frac{1}{2\lambda}\|x\|^2 + \frac{\lambda}{2}\|x^*\|^2 = \langle x, x^*\rangle \text{ iff } x^* \in \lambda^{-1}J(x). \end{aligned} \tag{4}$$

An important tool that will be used in the next sections is the classical Fenchel duality formula, which we present now.

Theorem 2.1 *Let $f, g : X \to \overline{\mathbb{R}}$ be proper convex and lower semicontinuous functions. If there exists $\hat{x} \in X$ such that f(or g) is continuous at $\hat{x} \in X$ and $f(\hat{x}) < \infty$, $g(\hat{x}) < \infty$, then,*

$$\inf_{x\in X} f(x) + g(x) = \max_{x^*\in X^*} -f^*(-x^*) - g^*(x^*).$$

Fitzpatrick proved [3] that associated to each maximal monotone operator $T : X \rightrightarrows X^*$ there exists a family $\mathcal{F}_T$ of convex, proper lower semicontinuous functions which majorizes the duality product and coincides with it on T:

$$\mathcal{F}_T = \left\{ h : X \times X^* \to \overline{\mathbb{R}} \;\middle|\; \begin{array}{l} h \text{ is convex and lower semicontinuous} \\ h(x, x^*) \geq \langle x, x^*\rangle, \quad \forall (x, x^*) \in X \times X^* \\ (x, x^*) \in T \Rightarrow h(x, x^*) = \langle x, x^*\rangle \end{array} \right\}. \tag{5}$$

Fitzpatrick also gave an explicit formula for the smallest element of $\mathcal{F}_T$ and proved that for any $h \in \mathcal{F}_T$

$$(x, x^*) \in T \iff h(x, x^*) = \langle x, x^*\rangle.$$

In view of the above equation, from now on we will call any $h \in \mathcal{F}_T$ a *convex representation* of T.

The next theorem is due to Marques Alves and Svaiter [5]. It has been proved for *reflexive* Banach spaces by Burachik and Svaiter in [2].

Theorem 2.2 ([5, Corollary 4.4, Theorem 4.2]) *Let X be a real Banach space. If $h : X \times X^* \to \overline{\mathbb{R}}$ is convex and*

$$\begin{aligned} h(x, x^*) &\geq \langle x, x^*\rangle, \quad \forall (x, x^*) \in X \times X^* \\ h^*(x^*, x^{**}) &\geq \langle x^*, x^{**}\rangle, \quad \forall (x^*, x^{**}) \in X^* \times X^{**} \end{aligned}$$

then the operator $T : X \rightrightarrows X^*$ defined by

$$T = \{(x, x^*) \in X \times X^* \mid h^*(x^*, x) = \langle x, x^* \rangle\}$$

is maximal monotone and $g(x, x^*) := h^*(x^*, x)$ is a convex representation of T. Moreover, if h is lower semicontinuous, then

$$T = \{(x, x^*) \in X \times X^* \mid h(x, x^*) = \langle x, x^* \rangle\}$$

and h is also a convex representations of T.

Next we enunciate a trivial consequence of the above theorem. To prove it, just recall that the duality product is continuous and the conjugation is invariant under the lower semicontinuous closure operation.

Corollary 2.3 *For h and T as in Theorem 2.2, $\mathrm{cl}\, h$, the lower semicontinuous closure of h, is also a convex representation of T.*

3 Gossez Type (D) Operators

Gossez defined the class of type (D) operators in order to extend to nonreflexive spaces some properties of maximal monotone operators on reflexive Banach spaces.

Gossez's monotone closure [4] of $T : X \rightrightarrows X^*$ is the operator $\widetilde{T} : X^{**} \rightrightarrows X^*$,

$$\widetilde{T} = \{(x^{**}, x^*) \in X^{**} \times X^* \mid \langle x^{**} - y, x^* - y^* \rangle \geq 0, \quad \forall (y, y^*) \in T\}. \tag{6}$$

The operator T is of type (D) if any element in $\widetilde{T}$ is the limit in the weak-$*$ $\times$strong topology of $X^{**} \times X^*$ of a *bounded* net $\{(x_i, x_i^*)\}$ on T. Note that in a reflexive Banach space any maximal monotone operator is of type (D). The next two theorems are due to Gossez [4].

Theorem 3.1 *If T is maximal monotone of type (D) then $\widetilde{T}$ is the unique maximal monotone extension of T to $X^{**} \times X^*$.*

Theorem 3.2 *If $f : X \to \overline{\mathbb{R}}$ is a proper, lower semicontinuous convex function then ∂f is of type (D) and*

$$\widetilde{\partial f} = (\partial f^*)^{-1}.$$

Fitzpatrick's convex representations of maximal monotone operators can be used to characterize maximal monotone operators of type (D).

Theorem 3.3 ([7, 8, Theorem 4.4], [6, Theorem 1.4]) *Let $T : X \rightrightarrows X^*$ be maximal monotone. Then the properties below are equivalent:*

1. *T is of type (D),*
2. *for any $h \in \mathfrak{F}_T$,*

$$h^*(x^*, x^{**}) \geq \langle x^{**}, x^* \rangle, \qquad \forall (x^*, x^{**}) \in X^* \times X^{**},$$

3. *there exists $h \in \mathcal{F}_T$ such that,*

$$h^*(x^*, x^{**}) \geq \langle x^{**}, x^* \rangle, \qquad \forall\, (x^*, x^{**}) \in X^* \times X^{**},$$

Moreover, if any of these conditions holds and $h \in \mathcal{F}_T$, then

$$h^* \in \mathcal{F}_{\widetilde{T}^{-1}}$$

4 Moreau–Yosida Regularization in General Banach Spaces

If X is a strictly convex, smooth and reflexive Banach space, then the duality map

$$J = \partial \frac{1}{2} \| \cdot \|^2$$

is a bijection between X and X^*. In this setting, for $T : X \rightrightarrows X^*$ maximal monotone, Brezis–Crandall–Pazy's generalization of the resolvent and Moreau–Yosida regularization of T, with parameter $\lambda > 0$, are respectively

$$R_\lambda : X \to X, \qquad T_\lambda : X \to X^*$$

defined as follows. For each $x \in X$, $\lambda > 0$, $(z, x^*) \in X \times X^*$ is the *unique* solution of

$$\lambda\, x^* + J(z - x) = 0, \qquad x^* \in T(z), \tag{7}$$

and so that

$$R_\lambda(x) := z, \qquad T_\lambda(x) := x^*. \tag{8}$$

The first attempt to generalize Brezis–Crandall–Pazy's version of Moreau–Yosida regularization 7, Eq. 8 would be to substitute the first of these equations by

$$\lambda\, x^* + J(z - x) \ni 0, \qquad x^* \in T(z).$$

In a nonreflexive Banach space J is not onto. Therefore, if $T \equiv x^*$ and $x^* \notin J(X)$, the above inclusion would not have a solution for any x and we would obtain an empty T_λ. To circumvent this problem, we will work with Gossez's monotone closures of T and J, that is, $\widetilde{T}$ and $\widetilde{J}$. Note that since J is a subdifferential, in view of Theorem 3.2 it is of type (D) and

$$\widetilde{J} = (J_{X^*})^{-1}, \tag{9}$$

where J_{X^*} denotes the duality map of X^*.

Definition 4.1 Let X be a real Banach space and $T : X \rightrightarrows X^*$ a maximal monotone operator of type (D). The *Moreau–Yosida regularization* and the *resolvent* of T, with regularization-parameter $\lambda > 0$, are respectively $T_\lambda : X \rightrightarrows X^*$ and $R_\lambda : X \rightrightarrows X^{**}$

$$T_\lambda = \left\{ (x, x^*) \in X \times X^* \;\middle|\; \begin{array}{l} \exists\, z^{**} \in X^{**} \text{ s.t.} \\ \lambda\, x^* + \widetilde{J}(z^{**} - x) \ni 0, \quad x^* \in \widetilde{T}(z^{**}) \end{array} \right\}, \tag{10}$$

$$R_\lambda = \left\{ (x, z^{**}) \in X \times X^{**} \;\middle|\; \begin{array}{l} \exists\, x^* \in X^* \text{ s.t.} \\ \lambda\, x^* + \widetilde{J}(z^{**} - x) \ni 0, \quad x^* \in \widetilde{T}(z^{**}) \end{array} \right\}. \tag{11}$$

In a strictly convex, smooth and reflexive Banach space, the above definition trivially yields Brezis–Crandall–Pazy's version of Moreau–Yosida regularization. The main tools for proving maximal monotonicity of T_λ will be the Fitzpatrick's convex representations of T, Theorems 2.2 and 3.3.

Define for $h : X \times X^* \to \overline{\mathbb{R}}$ and $\lambda > 0$

$$h_\lambda(x, x^*) = \inf_{z \in X} \left\{ h(z, x^*) + \frac{1}{2\lambda}\|z - x\|^2 \right\} + \frac{\lambda}{2}\|x^*\|^2. \tag{12}$$

Lemma 4.2 *Let $h : X \times X^* \to \overline{\mathbb{R}}$ be a convex, proper and lower semicontinuous function. Then*

$$h_\lambda^*(x^*, x^{**}) = \min_{z^{**} \in X^{**}} \left\{ h^*(x^*, z^{**}) + \frac{1}{2\lambda}\|z^{**} - x^{**}\|^2 \right\} + \frac{\lambda}{2}\|x^*\|^2 \tag{13}$$

and h_λ^*, h_λ are convex and proper functions, being h_λ^* lower semicontinuous. Moreover, if X is reflexive, then h_λ is also lower semicontinuous and the inf *in Eq. 12 is a* min.

Proof Direct calculation yields

$$\begin{aligned} h_\lambda^*(x^*, x^{**}) &= \sup_{(y, y^*, z)} \langle y, x^* \rangle + \langle y^*, x^{**} \rangle - h(z, y^*) - \frac{1}{2\lambda}\|z - y\|^2 - \frac{\lambda}{2}\|y^*\|^2 \\ &= \sup_{(z, y^*)} \left\{ \sup_y \langle y, x^* \rangle - \frac{1}{2\lambda}\|y - z\|^2 \right\} + \langle y^*, x^{**} \rangle - h(z, y^*) - \frac{\lambda}{2}\|y^*\|^2 \\ &= \frac{\lambda}{2}\|x^*\|^2 + \sup_{(z, y^*)} \langle z, x^* \rangle + \langle y^*, x^{**} \rangle - h(z, y^*) - \frac{\lambda}{2}\|y^*\|^2 \end{aligned}$$

Define $g(z, y^*) = \frac{\lambda}{2}\|y^*\|^2$. Then, the sup on the last term of the above equation is

$$(h + g)^*(x^*, x^{**}).$$

As g is convex and continuous and h is convex, proper and lower semicontinuous, the above conjugate is the (exact) inf-convolution of the conjugates of h and g. Hence

$$\begin{aligned} h_\lambda^*(x^*, x^{**}) &= \frac{\lambda}{2}\|x^*\|^2 + \inf_{(u^*, u^{**})} h^*(x^* - u^*, x^{**} - u^{**}) + g^*(u^*, u^{**}) \\ &= \frac{\lambda}{2}\|x^*\|^2 + \inf_{(u^*, u^{**})} h^*(x^* - u^*, x^{**} - u^{**}) + \frac{1}{2\lambda}\|u^{**}\|^2 + \delta_0(u^*) \\ &= \frac{\lambda}{2}\|x^*\|^2 + \inf_{u^{**}} h^*(x^*, x^{**} - u^{**}) + \frac{1}{2\lambda}\|u^{**}\|^2 \end{aligned}$$

Using the substitution $z^{**} = x^{**} - u^{**}$ we get Eq. 13, which proves the first part of the lemma.

It is easy to check that h_λ is convex. Since h is finite at some point, h_λ is proper. The function h_λ^* is convex and lower semicontinuous, because it is the conjugate of h_λ. Using that h_λ is proper we conclude that $h_\lambda^* > -\infty$. Moreover, from Eq. 13 it follows that $h_\lambda^* < \infty$ at some point. So, h_λ^* is proper. Now suppose that X is reflexive. For proving that h_λ is lower semicontinuous and that the inf in its definition is a min, apply the first part of the lemma to the function $\tilde{h}(x, x^*) = h^*(x^*, x)$ to obtain $(\tilde{h}_\lambda)^* = h_\lambda$. □

Theorem 4.3 *Let $T : X \rightrightarrows X^*$ be a maximal monotone operator of type (D). For any $\lambda > 0$, the operator T_λ is maximal monotone of type (D). Moreover, if h is a convex representation of T and h_λ is defined as in Eq. 12,*

$$h_\lambda(x, x^*) = \inf_{z \in X} \left\{ h(z, x^*) + \frac{1}{2\lambda}\|z - x\|^2 \right\} + \frac{\lambda}{2}\|x^*\|^2,$$

then

$$T_\lambda = \{(x, x^*) \in X \times X^* \mid h^*_\lambda(x^*, x) = \langle x, x^* \rangle\}$$

and the functions $\operatorname{cl} h_\lambda$ and $g(x, x^*) := h^*_\lambda(x^*, x)$ are convex representations of T_λ. If X is reflexive, then h_λ is also a convex representation of T_λ.

Proof Let h be a convex representation of T. Since h majorizes the duality product, for any $x, z \in X$, $x^* \in X^*$ we have

$$h(z, x^*) + \frac{1}{2\lambda}\|z - x\|^2 + \frac{\lambda}{2}\|x^*\|^2 \geq \langle z, x^* \rangle + \langle x - z, x^* \rangle = \langle x, x^* \rangle. \tag{14}$$

Therefore h_λ majorizes the duality product in $X \times X^*$. Using Theorem 3.3, item 2, we have that h^* majorizes the duality product in $X^* \times X^{**}$. Therefore, for any $x^* \in X^*$, $x^{**}, z^{**} \in X^{**}$

$$\begin{aligned} h^*(x^*, z^{**}) + \frac{1}{2\lambda}\|z^{**} - x^{**}\|^2 + \frac{\lambda}{2}\|x^*\|^2 &\geq \langle z^{**}, x^* \rangle + \langle z^{**} - x^{**}, x^* \rangle \\ &= \langle x^{**}, x^* \rangle. \end{aligned} \tag{15}$$

Combining the above equation with Lemma 4.2, Eq. 13, we conclude that h^*_λ also majorizes the duality product.

According to Lemma 4.2, h_λ and h^*_λ are convex and h^*_λ is lower semicontinuous. Since h_λ and h^*_λ majorizes the duality product in their respective domains, defining

$$S = \{(x, x^*) \in X \times X^* \mid h^*_\lambda(x^*, x) = \langle x, x^* \rangle\}$$

and using Theorem 2.2 and Corollary 2.3 we conclude that S is maximal monotone of type (D) and $g(x, x^*) = h^*_\lambda(x^*, x)$, $\operatorname{cl} h_\lambda$ are convex representation of S.

To prove that $T_\lambda = S$, use Eq. 13 to conclude that $(x, x^*) \in S$ if, and only if, there exists $z^{**} \in X^{**}$ such that

$$h^*(x^*, z^{**}) + \frac{1}{2\lambda}\|z^{**} - x\|^2 + \frac{\lambda}{2}\|x^*\|^2 = \langle x, x^* \rangle. \tag{16}$$

Using Theorem 2.2 we have

$$(z^{**}, x^*) \in \widetilde{T} \iff h^*(x^*, z^{**}) = \langle z^{**}, x^* \rangle.$$

Combining the two above equations with Eq. 15, the fact that h^* majorizes the duality product, Eqs.9 and 4 we conclude that Eq. 16 is equivalent to

$$x^* \in \widetilde{T}(z^{**}), \qquad -\lambda x^* \in \widetilde{J}(z^{**} - x).$$

Therefore, $T_\lambda = S$ is maximal monotone and g, $\operatorname{cl} h_\lambda$ are convex representation of S. As $(\operatorname{cl} h_\lambda)^* = h_\lambda^*$ and h_λ^* majorizes the duality product, using again Theorem 2.2 we conclude that T_λ is of type (D). Finally, if X is reflexive, use Lemma 4.2 to conclude that h_λ is lower semicontinuous and so that $h_\lambda = \operatorname{cl} h_\lambda$. □

We have proved that if T is maximal monotone of type (D), the T_λ is also (maximal monotone) of type (D). Therefore is natural to search for expressions of $\widetilde{T_\lambda}$.

Corollary 4.4 *If $T : X \rightrightarrows X^*$ is maximal monotone of type (D), then*

$$\widetilde{T_\lambda} = \left(\widetilde{T}^{\,-1} + \lambda^{-1}\widetilde{J}^{\,-1}\right)^{-1}, \quad T_\lambda = \left(\widetilde{T}^{\,-1} + \lambda^{-1}\widetilde{J}^{\,-1}\right)^{-1} \bigcap X \times X^*.$$

If X is reflexive, then

$$T_\lambda = \left(T^{\,-1} + \lambda^{-1} J^{\,-1}\right)^{-1}$$

and if X is a Hilbert space, then $T_\lambda = (T^{-1} + \lambda^{-1} I)^{-1}$.

For $h : X \times X^* \to \overline{\mathbb{R}}$ and $(z, z^*) \in X \times X^*$ define [5, 9]

$$h_{(z,z^*)}(x, x^*) = h(x + z, x^* + z^*) - [\langle x, z^*\rangle + \langle z, x^*\rangle + \langle z, z^*\rangle]. \tag{17}$$

Lemma 4.5 *Let $T : X \rightrightarrows X^*$ be a maximal monotone operator of type (D). Then for any $\lambda > 0$, $\operatorname{Dom}(T_\lambda) = X$.*

Proof Take $x_0 \in X$. Choose $h \in \mathfrak{F}_T$ and let

$$\beta := \inf_{x^* \in X^*} h_\lambda(x_0, x^*) - \langle x_0, x^*\rangle,$$

where h_λ is defined as in Eq. 12. Define also $f : X \times X^* \to \overline{\mathbb{R}}$,

$$f(y, y^*) = h(y + x_0, y^*) - \langle x_0, y^*\rangle.$$

Note that f is convex, lower semicontinuous and

$$f^*(y^*, y^{**}) = h^*(y^*, y^{**} + x_0) - \langle x_0, y^*\rangle.$$

Since h and h^* majorizes the duality product, f and f^* majorizes the duality product in their respective domains and $\beta \geq 0$.

Direct use of Eq. 12 yields

$$\begin{aligned}\beta &= \inf_{z \in X, x^* \in X^*} f(z - x_0, x^*) + \frac{1}{2\lambda}\|z - x_0\|^2 + \frac{\lambda}{2}\|x^*\|^2 \\ &= \inf_{x \in X, x^* \in X^*} f(x, x^*) + \frac{1}{2\lambda}\|x\|^2 + \frac{\lambda}{2}\|x^*\|^2\end{aligned}$$

Using Theorem 2.1 we conclude that there exist $w^* \in X^*$, $w^{**} \in X^{**}$ such that

$$\beta = -\left[f^*(w^*, w^{**}) + \frac{\lambda}{2}\|w^*\|^2 + \frac{1}{2\lambda}\|w^{**}\|^2\right] \leq 0$$

where the last inequality follows from the fact that f^* majorizes the duality product. As $\beta \geq 0$ we conclude that the above inequality holds as an equality. Therefore, using also Eqs. 9 and 4 we have

$$h^*(w^*, w^{**} + x_0) - \langle x_0, w^* \rangle = \langle w^*, w^{**} \rangle, \quad -\lambda w^* \in \widetilde{J}(w^{**})$$

Defining $z^{**} = w^{**} + x_0$ we conclude that $h^*(w^*, z^{**}) = \langle z^{**}, w^* \rangle$. Since h^* is a convex representation of $\widetilde{T}^{-1}$, we conclude that $w^* \in \widetilde{T}(w^*)$ and

$$0 \in \lambda w^* + \widetilde{J}(z^{**} - x_0)$$

which means $w^* \in T_\lambda(x_0)$. □

Theorem 4.6 *Let $T : X \rightrightarrows X^*$ be a maximal monotone operator of type (D). Then, for all $\lambda > 0$, the following statements holds:*

1. *T_λ is maximal monotone operator of type (D),*
2. $\mathrm{Dom}(T_\lambda) = X$,
3. *T_λ maps bounded sets on bounded sets.*

Proof Item 1 is proved in Theorem 4.3. Combining Theorem 4.3 and Lemma 4.5 we obtain item 2. For ending the proof, it remains to prove item 3. Fix some $(y, y^*) \in T$. Take $(x, x^*) \in T_\lambda$. By the definition of T_λ there exists $z^{**} \in X^{**}$ such that

$$x^* \in \widetilde{T}(z^{**}) \ \text{ and } \ -\lambda x^* \in \widetilde{J}(z^{**} - x). \tag{18}$$

Let $(y, y^*) \in T$. Using both inclusions in Eq. 18 and the fact that $T \subset \widetilde{T}$ we obtain

$$\begin{aligned}
0 &= \frac{\lambda}{2}\|x^*\|^2 + \frac{1}{2\lambda}\|z^{**} - x\|^2 + \langle x^*, z^{**} - x\rangle \\
&= \frac{\lambda}{2}\|x^*\|^2 + \frac{1}{2\lambda}\|z^{**} - x\|^2 + \langle x^* - y^*, z^{**} - y\rangle + \langle x^* - y^*, y - x\rangle + \langle y^*, z^{**} - x\rangle \\
&\geq \frac{\lambda}{2}\|x^*\|^2 + \frac{1}{2\lambda}\|z^{**} - x\|^2 + \langle x^* - y^*, y - x\rangle + \langle y^*, z^{**} - x\rangle \\
&\geq \frac{\lambda}{2}\|x^*\|^2 - \frac{\lambda}{2}\|y^*\|^2 + \langle x^* - y^*, y - x\rangle \\
&\geq \frac{\lambda}{2}\|x^*\|^2 - \|x^*\|\|y - x\| - \frac{\lambda}{2}\|y^*\|^2 - \|y^*\|\|y - x\|.
\end{aligned}$$

Therefore,

$$0 \geq \|x^*\|^2 - \frac{2}{\lambda}\|y - x\|\|x^*\| - \left(\|y^*\|^2 + \frac{2}{\lambda}\|y^*\|\|y - x\|\right)$$

and so that

$$\|x^*\| \leq \frac{2}{\lambda}\|y - x\| + \|y^*\| \leq \frac{2}{\lambda}\|x\| + \left(\frac{2}{\lambda}\|y\| + \|y^*\|\right).$$

□

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