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Boundary Control Systems with Yosida Type Approximate Boundary Observation

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Abstract: A model of boundary control system with boundary observation is described and analyzed, which involves no unbounded operator except for the dynamics generator. This model consists of an extended abstract differential equation whose state components are the boundary input, the state (up to an affine transformation) and a Yosida-type approximation of the output of the nominal system. It is shown that, under suitable conditions, the model is well-posed and, in particular, that the dynamics operator is the generator of an analytic C_0-semigroup and the model is observable. Moreover the model is shown to be approximately reachable provided that so is the nominal system with respect to an additional distributed input. It is expected that this approach will lead hopefully to a good trade-off between the cost of modelling and the efficiency of methods of resolution of control problems for such systems.

Keywords: Boundary control, boundary observation, Yosida approximation, C_0-semigroup, convection-diffusion-reaction system.

1. INTRODUCTION

Boundary control systems with boundary observation typically feature unbounded observation and control operators along with the homogeneous dynamics generator. The unboundedness property leads to technical difficulties which make the modelling and analysis of such systems very hard, see e.g. Weiss (1994), Tucsnak-Weiss (2009) and references therein, and the design of control laws as well, especially when boundary control and boundary observation are both present in the model, see e.g. Weiss-Weiss (1997). If possible, this characteristic should better be avoided in order to achieve an acceptable trade-off between the cost of modelling and the efficiency of analytic and/or numerical methods of resolution of control problems.

Our aim is to establish an extended differential system involving no unbounded operator except for the dynamics generator. We consider, under suitable conditions, a change of variables for the state and input of the nominal system, as well as a Yosida type approximation of the output which is based on the resolvent operator of the dynamics generator, Weiss (1994). This choice yields a well-posed extended system with the same dynamical properties than those of the nominal one.
More specifically, we consider an abstract boundary control system with boundary observation of the form

$$\dot{x}(t) = \mathcal{A}x(t) + B_d u_d(t), \quad x(0) = x_0,$$

$$\mathcal{B}x(t) = u_b(t),$$

$$y(t) = \mathcal{C}x(t),$$

with a bounded linear distributed control operator B_d and with unbounded boundary control and observation linear operators $\mathcal{B}$ and $\mathcal{C}$, respectively. Under suitable conditions, an appropriate change of variables yields the following extended differential system

$$\dot{x}^e(t) = A^e x^e(t) + B^e u^e(t), \quad x^e(0) = x_0^e,$$
$$y^e(t) = C^e x^e(t)$$

where the (extended) state is given, under suitable initial conditions, by

$$x^e(t) = [u_b(t) \;\; v(t) \;\; y_\alpha(t)]^T,$$

with $v(t)$ depending on the nominal state $x(t)$ and

$$u^e(t) = [\dot{u}_b(t) \;\; u_d(t)]^T$$

as new input and where $y_\alpha(t)$ could be interpreted as an approximation of $y(t)$ based on the resolvent operator, and the control and observation operators, B^e and C^e respectively, are bounded. This extended system description can be seen as an extension of the one established in Curtain-Zwart (1995) for boundary control alone. The novelty here is that the boundary input and approximate output trajectories are given jointly by the dynamics equation, thanks to the fact that both are components of the extended state. Observe that the variation rate of the boundary input is now an input of the extended system.

Although the new output is only an approximation of $y(t)$, the proposed extension preserves many interesting properties of the nominal system, including (approximate) reachability and observability and the spectral structure of the dynamics generator, while keeping the state-space framework, with bounded linear operators and the dynamics generating a C_0-semigroup.

More precisely, it is shown that, under suitable conditions, the model is well-posed and, in particular, that the dynamics operator is the generator of an analytic C_0-semigroup and the model is observable. Moreover the model is shown to be approximately reachable provided that so is the nominal system with respect to the distributed input. The theoretical results are illustrated by an example of convection-diffusion-reaction system.

It is expected that this approach will lead hopefully to a good trade-off between the cost of modelling and the efficiency of methods of resolution of control problems for such systems, like the LQ-optimal control problem. This question is currently under investigation for a class of convection-diffusion-reaction systems, Dehaye JR - Winkin (2013).

2. MODEL DESCRIPTION

We consider a dynamical system with boundary and distributed control and boundary observation. The associated abstract boundary control model with boundary observation is described by

$$\dot{x}(t) = \mathcal{A}x(t) + B_d u_d(t), \quad x(0) = x_0 \tag{1}$$

$$\mathcal{B}x(t) = u_b(t) \tag{2}$$

$$y(t) = \mathcal{C}x(t) \tag{3}$$

where $\mathcal{A} : D(\mathcal{A}) \subset X \to X$ is an unbounded linear operator and $B_d \in \mathcal{L}(U_d, X)$ is a bounded linear operator, where U_d and X are Hilbert spaces, and $\mathcal{B}$ and $\mathcal{C}$ are unbounded linear operators on X taking values in Hilbert spaces U_b and Y respectively and whose domains contain the one of the operator $\mathcal{A}$, i.e. $D(\mathcal{A}) \subset D(\mathcal{B})$ and $D(\mathcal{A}) \subset D(\mathcal{C})$.

Definition 1. An abstract boundary control model (1)–(3) is said to be *a boundary control system with boundary observation (BCBO)* if the following conditions hold:

[C1] the operator $A : D(A) \to X$ defined by

$$Ax = \mathcal{A}x ,$$

for all x in its domain

$$D(A) = D(\mathcal{A}) \cap \text{ Ker } \mathcal{B} ,$$

is the infinitesimal generator of an analytic C_0-semigroup of bounded linear operators on X,

[C2] there exists a bounded linear operator $B_b \in \mathcal{L}(U_b, X)$ such that for all $u \in U_b$, $B_b u \in D(\mathcal{A})$, the operator $\mathcal{A}B_b \in \mathcal{L}(U_b, X)$ and

$$\mathcal{B}B_b u = u \quad \text{for all } u \in U_b , \tag{4}$$

[C3] there exist constants $a, b \geq 0$ such that, for all $x \in D(A)$,

$$\|\mathcal{C}x\| \leq a\|Ax\| + b\|x\|. \tag{5}$$

Now, for any parameter α in the resolvent set $\rho(A)$, let us define the operator C_α by

$$C_\alpha : X \to Y$$
$$x \mapsto C_\alpha x := \alpha\mathcal{C}(\alpha I - A)^{-1}x. \tag{6}$$

The operator C_α (more specifically, its limit as the parameter α tends to infinity, whenever it exists) is useful in the analysis of the well-posedness of infinite-dimensional systems with an unbounded observation operator, see e.g. Weiss (1994), Tucsnak-Weiss (2009) and references therein. It is related to a concept known in the literature as the *Yosida approximation*, which plays an important role in the proof of the Hille-Yosida theorem, see e.g. Curtain-Zwart (1995), Jacob-Zwart (2010). In the sequel, the operator C_α will be interpreted as a *Yosida type approximate boundary observation operator*. For technical reasons (see below), it is useful to consider a weighted operator defined for every $x \in X$ by

$$\widetilde{C}_\alpha \, x \ := \ \rho_\alpha \ C_\alpha \, x , \tag{7}$$

where $\rho_\alpha > 0$ is a suitably chosen parameter.

Under these conditions, the following abstract differential equations are well posed for $u_d \in C^1([0,\tau], U_d)$ and $u_b \in C^2([0,\tau], U_b)$, where τ is any fixed final time (see below):

$$\dot{v_1}(t) = Av_1(t) - B_b\dot{u}_b(t) + \mathcal{A}B_b u_b(t) + B_d u_d(t) \tag{8}$$

$$v_1(0) = v_{1_0} \tag{9}$$

and

$$\dot{v_2}(t) = \widetilde{C}_\alpha A v_1(t) + \widetilde{C}_\alpha \mathcal{A}B_b u_b(t) + \widetilde{C}_\alpha B_d u_d(t) \tag{10}$$

$$v_2(0) = v_{2_0}. \tag{11}$$

Then one can define the extended system

$$\dot{x}^e(t) = A^e x^e(t) + B^e u^e(t) \ , \ x^e(0) = [x^e_{0_1} \ \ x^e_{0_2} \ \ x^e_{0_3}]^T \tag{12}$$

$$y^e(t) = C^e x^e(t) \tag{13}$$

on the extended state space

$$\widetilde{X}^e := U_b \oplus X \oplus Y ,$$

where the (extended) state is defined, under suitable initial conditions, by

$$x^e(t) = [u_b(t) \ \ v_1(t) \ \ v_2(t)]^T \tag{14}$$

$$= [u_b(t) \ \ x(t) - B_b u_b(t) \ \ y_\alpha(t)]^T, \tag{15}$$

the (extended) input is defined by

$$u^e(t) = [\dot{u}_b(t) \ \ u_d(t)]^T \ \in \ U^e := U_b \oplus U_d \tag{16}$$

and where

$$A^e = \begin{pmatrix} 0 & 0 & 0 \\ \mathcal{A}B_b & A & 0 \\ \widetilde{C}_\alpha \mathcal{A}B_b & \widetilde{C}_\alpha A & 0 \end{pmatrix}, \; B^e = \begin{pmatrix} I & 0 \\ -B_b & B_d \\ 0 & \widetilde{C}_\alpha B_d \end{pmatrix}, \tag{17}$$

$$C^e = \begin{pmatrix} \rho_1 I & 0 & 0 \\ \rho_2 B_b & \rho_2 I & 0 \\ 0 & 0 & \rho_3 \, {\rho_\alpha}^{-1} I \end{pmatrix} . \tag{18}$$

The domain of the operator A^e is given by

$$D(A^e) := U_b \oplus D(A) \oplus Y$$

and ρ_i, $i = 1, 2, 3$, are weighing factors. The observation operator C^e is considered with a view to solving an LQ-optimal feedback control problem with a quadratic cost involving the (extended) output $y^e(t)$, Dehaye JR - Winkin (2013).

3. WELL-POSEDNESS AND DYNAMICAL ANALYSIS

The well-posedness analysis of the extended model of a BCBO system (12)-(18) is based on the following three preliminary results.

In what follows, for any $\delta > 0$, Σ_δ denotes the sector $\{z \in \mathbb{C} : |\arg z| < \delta\}\backslash\{0\}$ of the complex plane. For a given linear operator $\Gamma : D(\Gamma) \subset X \to X$ on a Banach space X, a linear operator $\Delta : D(\Delta) \subset X \to X$ is said to be Γ-bounded if $D(\Gamma) \subset D(\Delta)$ and if there exist nonnegative constants γ and δ such that, for all $x \in D(\Gamma)$,

$$\|\Delta x\| \leq \gamma \, \|\Gamma x\| + \delta \, \|x\| \, . \tag{19}$$

The Γ-bound of the operator Δ is given by
$\gamma_0 := \inf\{\gamma \geq 0 : \text{ there exists } \delta \geq 0 \text{ such that (19) holds}\}$.

Remark 1. In view of condition (**C3**), the operator $\mathcal{C}$ is A-bounded with A-bound less than or equal to a.

Lemma 2. (Engel-Nagel (2006), Theorem 2.10, p.130) Let the linear operator $\Gamma : D(\Gamma) \subset X \to X$ be the infinitesimal generator of an analytic C_0-semigroup $(T(z))_{z\in\Sigma_\delta\cup\{0\}}$ on a Banach space X. Then a) there exists a constant $c > 0$ such that $\Gamma + \Delta : D(\Gamma) \to X$ is the infinitesimal generator of an analytic C_0-semigroup for every Γ-bounded operator Δ having Γ-bound $\gamma_0 < c$; b) for every Γ-bounded operator Δ, the operator $\Gamma + \rho\,\Delta$ is the infinitesimal generator of an analytic C_0-semigroup provided that the parameter $\rho \in \mathbb{R}$ be such that $|\rho|$ is sufficiently small.

Remark 3. In Lemma 2b, the operator $\Gamma + \rho\ \Delta$ is the infinitesimal generator of an analytic C_0-semigroup for every $\rho \in \mathbb{R}$ and for every Γ-bounded operator Δ whose Γ-bound is zero. This is clearly the case when the operator Δ is Γ-compact, see (Engel-Nagel (2006), Corollary 2.14, p. 133).

Lemma 4. Let the operators $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ define a BCBO system (1)-(3) such that conditions (**C1**)-(**C3**) hold. Consider the operator $\widetilde{C}_\alpha$ given by (7), where ρ_α is chosen sufficiently small. Then, for any distributed input $u_d \in C^1([0,\tau], U_d)$ and for any boundary input $u_b \in C^2([0,\tau], U_b)$, where $\tau > 0$ is any fixed final time, the abstract differential equations (8)-(9) and (10)-(11) are well-posed, i.e. for all initial conditions v_{1_0} and v_{2_0} in $D(A)$, the Cauchy problems (8)-(9) and (10)-(11) have unique classical solutions $v_1 \in C^1([0,\tau], X)$ and $v_2 \in C^1([0,\tau], Y)$, respectively, with $v_1(t) \in D(A)$ and $v_2(t) \in Y$ for all $t \in [0, \tau]$.

Proof. By (Curtain-Zwart (1995), Theorem 3.1.3, p.103) and by Lemma 2b applied to the operators

$$\Gamma = \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} \text{ and } \Delta = \begin{pmatrix} 0 & 0 \\ C_\alpha A & 0 \end{pmatrix} ,$$

the result follows directly from condition (**C1**) and from the fact that, thanks to conditions (**C2**) and (**C3**), the operator $C_\alpha A$ is A-bounded and the operators B_b, $\mathcal{A}B_b$, B_d, $\widetilde{C}_\alpha \mathcal{A} B_b$ and $\widetilde{C}_\alpha B_d$ are bounded.

Observe that the third equation of the abstract differential equation (12) should correspond to the dynamics of the output trajectories of its two first equations through the output operator $\widetilde{C}_\alpha$. In order to take this feature into account in the description of the extended system, let us consider the bounded linear operator $C : U_b \oplus X \to Y$ defined for all $(x_1^e, x_2^e) \in U_b \oplus X$ by

$$C \begin{pmatrix} x_1^e \\ x_2^e \end{pmatrix} = \begin{pmatrix} \widetilde{C}_\alpha B_b & \widetilde{C}_\alpha \end{pmatrix} \begin{pmatrix} x_1^e \\ x_2^e \end{pmatrix} = \widetilde{C}_\alpha (x_2^e + B_b x_1^e). \tag{20}$$

Thanks to the fact that the graph $G(C)$ of the operator C is a closed subspace of $\widetilde{X}^e$, from now on, we will use

$$X^e := G(C) \subset \widetilde{X}^e = U_b \oplus X \oplus Y \; ,$$

as new extended (Hilbert) state space for the extended system.

Lemma 5. The restriction of the operator A^e, given by (17), (7) where $\rho_\alpha \in \mathbb{R}$ is such that $|\rho_\alpha|$ is sufficiently small, to the subspace $X^e = G(C)$, whose domain is given by $D(A^e) \cap G(C)$, is the infinitesimal generator of a C_0-semigroup of bounded linear operators on X^e.

Proof. First observe that the operator A^e, with domain $D(A^e) = U_b \oplus D(A) \oplus Y$, is the infinitesimal generator of a C_0-semigroup of (bounded linear operators) on $\widetilde{X}^e$. Indeed, let us define the operator

$$A_0^e = \begin{pmatrix} 0 & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

on its domain $D(A_0^e) = D(A^e)$, and the perturbation operators

$$P_1^e = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & C_\alpha A & 0 \end{pmatrix}, \quad \widetilde{P}_2^e = \begin{pmatrix} 0 & 0 & 0 \\ \mathcal{A}B_b & 0 & 0 \\ \widetilde{C}_\alpha \mathcal{A}B_b & 0 & 0 \end{pmatrix}.$$

Since A is the infinitesimal generator of an analytic C_0-semigroup $(T(t))_{t\geq 0}$ on X, the operator A_0^e is the infinitesimal generator of the analytic C_0-semigroup $(T_0^e(t))_{t\geq 0}$ on $\widetilde{X}^e$, given for all $t \geq 0$ by

$$T_0^e(t) = \begin{pmatrix} I & 0 & 0 \\ 0 & T(t) & 0 \\ 0 & 0 & I \end{pmatrix} .$$

It follows from the fact that the operator P_1^e is A_0^e-bounded that, by Lemma 2b, the operator $A_0^e + \rho_\alpha P_1^e$ (with domain $D(A^e)$) is still the infinitesimal generator of an analytic C_0-semigroup for $|\rho_\alpha|$ sufficiently small. Since $\widetilde{P}_2^e \in \mathcal{L}(\widetilde{X}^e)$, the operator $A^e = A_0^e + \rho_\alpha P_1^e + \widetilde{P}_2^e$ is the infinitesimal generator of a C_0-semigroup $(T^e(t))_{t\geq 0}$ on $\widetilde{X}^e$. The conclusion follows by (Engel-Nagel (2006), Corollary p. 48), since X^e is a $T^e(t)$-invariant closed subspace of $\widetilde{X}^e$.

In what follows, we will use the same notation for the C_0-semigroup $(T^e(t))_{t\geq 0}$ and its restriction to the subspace X^e.

Theorem 6. [Well-posedness of the extended system]
For any distributed input $u_d \in C^1([0,\tau], U_d)$ and for any boundary input $u_b \in C^2([0,\tau], U_b)$, where $\tau > 0$ is any fixed final time, the dynamics (12) of the extended system (12)-(18), where $\rho_\alpha \in \mathbb{R}$ is such that $|\rho_\alpha|$ is sufficiently small, are well-posed, i.e. the abstract differential equation

$$\dot{x}^e(t) = A^e x^e(t) + B^e u^e(t) ,$$

with initial condition

$$x^e(0) = x_0^e = \begin{pmatrix} x_{0_1}^e \\ x_{0_2}^e \\ x_{0_3}^e \end{pmatrix} = \begin{pmatrix} u_b(0) \\ v_{1_0} \\ v_{2_0} \end{pmatrix} \in D(A^e) \cap G(C)$$

and input given by (16), has the unique classical solution

$$x^e(t) = \begin{pmatrix} u_b(t) \\ v_1(t) \\ v_2(t) \end{pmatrix} ,$$

where $v_1(t)$ and $v_2(t)$ are the unique classical solutions of the abstract differential equations (8)-(9) and (10)-(11), respectively.
Moreover, if $x_0 = v_{1_0} + B_b u_b(0)$, whence $v_{2_0} = \widetilde{C}_\alpha x_0$, then the state trajectory $x(t)$ of the BCBO system (1)-(3) is related to the one of the extended system (12)-(18), for all $t \geq 0$, by

$$\begin{cases} x(t) \ = v_1(t) + B_b u_b(t) \\ v_2(t) \ = C_\alpha x(t) \ . \end{cases}$$

Proof. It suffices to observe that the third component $x_3^e(t)$ of the extended state is the solution of the abstract differential equation (10)-(11) with initial condition $v_{2_0} = x_{0_3}^e$, whence $v_2(t) \equiv x_3^e(t)$. The result follows from Lemma 4 and 5 by an analysis going along the lines of (Curtain-Zwart (1995), Section 3.3).

We conclude this section with the analysis of system theoretic properties of the extended system, namely reachability and observability. This analysis is based on the following auxiliary result.

Lemma 7. If the pair (A, B_d) is (approximately) reachable, then the pair $(\tilde{A}^e, \tilde{B}^e)$ given by

$$\tilde{A}^e = \begin{pmatrix} 0 & 0 \\ \mathcal{A}B_b & A \end{pmatrix}, \quad \tilde{B}^e = \begin{pmatrix} I & 0 \\ -B_b & B_d \end{pmatrix}$$

is reachable.

Proof. The operator $\tilde{A}^e$ is the infinitesimal generator of a C_0-semigroup $(\tilde{T}^e(t))_{t\geq 0}$ of the following form:

$$\tilde{T}^e(t) = \begin{pmatrix} I & 0 \\ \tilde{S}(\cdot) & T(\cdot) \end{pmatrix} ,$$

where $(T(t))_{t\geq 0}$ is the C_0-semigroup generated by A. Observe that the dual observability map is given for all $\tilde{x}^e = [x_1^e \; x_2^e]^T \in U_b \oplus X$ by

$$\begin{aligned} \tilde{\mathcal{B}}_t^e \tilde{x}^e &= (\tilde{B}^e)^* (\tilde{T}^e(\cdot))^* \tilde{x}^e \\ &= \begin{pmatrix} x_1^e + (\tilde{S}(\cdot))^* x_2^e + -B_b^* (T(\cdot))^* x_2^e \\ B_d^* (T(\cdot))^* x_2^e \end{pmatrix} . \end{aligned}$$

Hence $\tilde{\mathcal{B}}_t^e \tilde{x}^e = 0$ if and only if $\tilde{x}^e = 0$. The conclusion follows by a standard duality argument, see e.g. Curtain-Zwart (1995).

Proposition 8. a) The extended system (12)-(18), i.e. the pair (C^e, A^e), is observable.
b) If the pair (A, B_d) is (approximately) reachable, then the extended system (12)-(18), i.e. the pair (A^e, B^e), is reachable.

Proof. a) First observe that the C_0-semigroup $(T^e(t))_{t\geq 0}$ generated by A^e has the following form:

$$T^e(t) = \begin{pmatrix} I & 0 & 0 \\ S_1(\cdot) & T(\cdot) & 0 \\ S_2(\cdot) & S_3(\cdot) & I \end{pmatrix} ,$$

where $(T(t))_{t\geq 0}$ is the C_0-semigroup generated by A. It follows that the observability map is given for all $x^e \in X^e$ by

$$\mathcal{C}_t^e x^e = C^e T^e(\cdot) x^e = \begin{pmatrix} \rho_1 x_1^e \\ \rho_2 (B_b x_1^e + S_1(\cdot) x_1^e + T(\cdot) x_2^e) \\ \rho_3 \rho_\alpha^{-1} (S_2(\cdot) x_1^e + S_3(\cdot) x_2^e + x_3^e) \end{pmatrix} .$$

Hence $\mathcal{C}_t^e x^e = 0$ if and only if $x^e = 0$.
b) For any fixed time $t > 0$ and any state $z = (z_1, z_2, z_3) = (z_1, z_2, C(z_1, z_2)) \in X^e$, it should be shown that, for an arbitrarily fixed $\epsilon > 0$, there exists an input function $u^e(\cdot)$ (defined on the time interval $[0, t]$) such that

$$\|x^e(t) - z\| < \epsilon , \tag{21}$$

where $x^e(\cdot)$ is the state trajectory of the extended system corresponding to the input $u^e(\cdot)$ with zero initial condition. Since $C \in \mathcal{L}(U_b \oplus X, Y)$, one can define

$$\widetilde{\epsilon} := \frac{\epsilon}{\sqrt{1 + \|C\|^2}} > 0 .$$

By Lemma 7, there exists an input function $u^e(\cdot)$ such that $\|(x_1^e(t), x_2^e(t)) - (z_1, z_2)\| < \widetilde{\epsilon}$. It follows that

$$\|x_3^e(t) - z_3\| = \|C(x_1^e(t), x_2^e(t)) - C(z_1, z_2)\| \leq \|C\| \, \widetilde{\epsilon} ,$$

whence (21) holds.

4. APPLICATION

The results of the previous section are illustrated by a system which is notably useful for modeling chemical and biochemical reactors, see e.g. Winkin et al. (2000), Delattre et al. (2003).

Let us consider a convection-diffusion-reaction (CDR) system with boundary control and observation of the form

$$\begin{cases} \frac{\partial x}{\partial t}(z,t) &= D\frac{\partial^2 x}{\partial z^2}(z,t) - v\frac{\partial x}{\partial z}(z,t) - kx(z,t) \\ & \quad + \chi_{[1-\epsilon_u,1]}(z)u_d(t) \\ -D\frac{\partial x}{\partial z}(0,t) &= v(u_b(t) - x(0,t)) \\ \frac{\partial x}{\partial z}(1,t) &= 0 \\ x(z,0) &= x_0(z) \\ y(t) &= x(1,t) \end{cases} \tag{22}$$

where $t \geq 0$ and $z \in [0,1]$ denote the time and the spatial variable, respectively, D, v and k are positive constants and $\epsilon_u \in]0,1]$ is a parameter.

This dynamical system can be interpreted as an abstract boundary control model with boundary observation described by (1)-(3) where the operator $\mathcal{A} : D(\mathcal{A}) \subset X = L^2([0,1]) \to L^2([0,1])$ is given by

$$\mathcal{A}x = D\frac{d^2x}{dz^2} - v\frac{dx}{dz} - kx$$

on its domain $D(\mathcal{A})$, which is defined as the set of all $x \in L^2([0,1])$ such that x and $\frac{dx}{dz}$ are absolutely continuous (a.c.), $\frac{d^2x}{dz^2} \in L^2([0,1])$ and $\frac{dx}{dz}(1) = 0$, the operator $B_d : U_d = \mathbb{R} \to L^2([0,1])$ is given for all $u_d \in U_d$ and $z \in [0,1]$, by

$$(B_d u_d)(z) = \chi_{[1-\epsilon_u,1]}(z)u_d , \tag{23}$$

where $\chi_{[1-\epsilon_u,1]}$ denotes the characteristic function of the interval $[1-\epsilon_u, 1]$, the operator $\mathcal{B} : D(\mathcal{B}) \supset D(\mathcal{A}) \to U_b = \mathbb{R}$ is given by

$$\mathcal{B}x = -\frac{D}{v}\frac{dx}{dz}(0) + x(0)$$

and the operator $\mathcal{C} : D(\mathcal{C}) \supset D(\mathcal{A}) \to Y = \mathbb{R}$ is given by

$$\mathcal{C}x = x(1).$$

It can be shown that this model is a BCBO system. Indeed, it is well-known that condition (**C1**) holds, where the operator A is given by $Ax = \mathcal{A}x$ for all $x \in D(A) = D(\mathcal{A}) \cap \text{Ker } \mathcal{B}$, where

$$D(A) = \Big\{ x \in L^2([0,1]) : x, \frac{dx}{dz} \text{ are a. c., } \frac{d^2x}{dz^2} \in L^2([0,1]), \\ D\frac{dx}{dz}(0) - vx(0) = 0 = \frac{dx}{dz}(1) \Big\}.$$

Moreover condition (**C2**) holds with the operator B_b defined as the multiplication operator by the unit step function, i.e. $B_b u = 1(\cdot)u$, where $1(z) \equiv 1$ on $[0,1]$.

Finally, by arguments similar to those used in Deutscher (2013), it can be shown that the operator $\mathcal{C}$ is A-bounded. More precisely, it can be shown that there exists $f \in L^2([0,1])$ such that, for all $x \in D(A)$,

$$\mathcal{C}x = x(1) = \langle f, (I - A)x \rangle . \tag{24}$$

Such a function f is given by

$$f = \sum_{n=1}^{\infty} \frac{1}{1-\lambda_n} \psi_n(1)\, \phi_n ,$$

where $(\phi_n)_{n\in\mathbb{N}}$ is a Riesz basis of eigenvectors of the operator A, $(\psi_n)_{n\in\mathbb{N}}$ is a corresponding dual Riesz basis such that the vectors ϕ_n and ψ_n are bi-orthonormal, and the real numbers λ_n are the eigenvalues of A. Then inequality (5) can be easily derived from (24). Hence condition (**C3**) holds.

It follows that the analysis and all the results of the previous section apply to this model.

In order to make this analysis complete it remains to be shown that the pair (A, B_d) is reachable for an appropriate choice of the parameter ϵ_u.

Lemma 9. Consider a CDR system with boundary control and observation described by (22). Assume that the distributed control operator B_d is given by (23), where the window width is

$$\epsilon_u := \frac{1}{j} , \tag{25}$$

with $j \in \mathbb{N}_0$. Then the pair (A, B_d) is reachable.

Proof. It is known (see Winkin et al. (2000)) that the operator A is self-adjoint with respect to the equivalent inner product $\langle \cdot, \cdot \rangle_\rho$ defined for all $f, g \in L^2([0,1])$ by

$$\langle f, g \rangle_\rho = \int_0^1 e^{-\frac{v}{D}z} f(z)\overline{g(z)}dz,$$

and its eigenvectors $(\phi_n)_{n\in\mathbb{N}_0}$ form an orthonormal basis of $L^2([0,1])$ equiped with this inner product. By Curtain-Zwart (1995), the CDR system (22) is reachable if and only if for all $n \in \mathbb{N}_0$, $\langle b, \phi_n \rangle_\rho \neq 0$, that is

$$\left[e^{-\frac{v}{2D}z} \left[\frac{s_n^2 - v^2}{vs_n} \sin\left(\frac{s_n}{2D}z\right) - 2\cos\left(\frac{s_n}{2D}z\right) \right] \right]_{1-\epsilon_u}^{1} \neq 0 ,$$

where the parameters s_n are the solutions of

$$\tan\left(\frac{s}{2D}\right) = \frac{2vs}{s^2 - v^2}, \quad s > 0.$$

A straightforward computation reveals that

$$\langle b, \phi_n \rangle_\rho = 0$$
$$\Leftrightarrow \frac{s_n^2 - v^2}{vs_n} \sin\left(\frac{s_n}{2D}(1-\epsilon_u)\right) - 2\cos\left(\frac{s_n}{2D}(1-\epsilon_u)\right) = 0$$
$$\Leftrightarrow \qquad \tan\left(\frac{s_n}{2D}(1-\epsilon_u)\right) = \tan\left(\frac{s_n}{2D}\right).$$

The last assertion holds if and only if there exists $k \in \mathbb{Z}$ such that

$$\frac{s_n}{2D}(1-\epsilon_u) + k\pi = \frac{s_n}{2D},$$

or equivalently

$$\epsilon_u s_n = 2Dk\pi, \ ,$$

i.e., in view of (25), $s_n = 2Dkj\pi$, where $kj \in \mathbb{N}$. It follows that

$$0 = \tan(kj\pi) = \tan\left(\frac{s_n}{2D}\right) = \frac{2vs_n}{s_n^2 - v^2} \ .$$

Clearly this is a contradiction with the fact that $s_n > 0$ and $v > 0$.

5. CONCLUDING REMARKS

In view of the results stated in Sections 3 and 4, the LQ-optimal control problem is well-posed and solvable for BCBO systems described by an extended model of the form (12)-(18) and, in particular, for CDR systems described by (22). This problem is currently studied, see Dehaye JR - Winkin (2013), following the methodology of Callier-Winkin (1992) and Winkin et al. (2005).

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