Volatility interpolation and Yosida Approximation

Vladimir Lucic
Equity Derivatives Quant Research
London, UK

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Abstract

The purpose of this note is to point out a link between the Yosida approximation for linear operators and the method for volatility interpolation developed in Andreasen and Huge (2011).

0 Preliminaries

This section gathers some well-known facts form the theory of Markov processes. A full account can be found in Ethier and Kurtz (1986).

Let E be a separable metric space and let $\bar{C}(E)$ denote the space of all bounded continuous functions on E. Let A be a linear operator defined on a subspace $D(A) \subset \bar{C}(E)$. Assume that a process X defined on a filtered probability space $(\Omega, \mathcal{F}, P, \{\mathcal{F}_t\})$ is a solution of the martingale problem for A, i.e.

$$f(X(t)) - \int_0^t Af(X(u))\, du$$

is an $\{\mathcal{F}_t\}$-martingale for every $f \in D(A)$.

The *Yosida approximation* of A is defined for each $\lambda > 0$ by

$$A_\lambda := \lambda A(\lambda - A)^{-1}, \quad D(A_\lambda) := R(\lambda - A).$$

This is a family of *bounded* linear operators approximating (typically unbounded) operator A, and as such plays a key role in the general theory of Markov processes. For example, Yosida approximation is closely related to approximation of X (in the sense of weak convergence) by a Markov jump process $\hat{X}(t) := Y(V(t))$, where $Y(n)$ is the Markov chain with a transition function $\mu_\lambda(x, \Gamma)$ and $V(t)$ is an independent Poisson process with parameter λ. The trasition function defines the contraction $P_\lambda f(\cdot) := \int f(y)\mu_\lambda(\cdot, dy)$, $f \in \bar{C}(E)$, extending $\lambda(\lambda - A)^{-1}$ from $D(A_\lambda)$ to $\bar{C}(E)$:

$$P_\lambda f = \lambda(\lambda - A)^{-1} f, \quad f \in D(A_\lambda).$$

As a consequence, $Y(n)$ solves the following discrete-time martingale problem: for every $f \in R(\lambda - A)$

$$f(Y(n)) - \sum_{i=0}^{n-1} \frac{1}{\lambda} A_\lambda f(Y(i)) \tag{1}$$

is an $\{\mathcal{F}_n^Y\}$-martingale (here $\mathcal{F}_n^Y$ is the σ-algebra generated by $Y(0), Y(1), \ldots, Y(n)$).

1 A discrete-time forward equation

Fix $g \in D(A)$, and let $f := (I - A)g$. By taking $\lambda = 1$ in (1) we get for a fixed n

$$E[f(Y(n+1)) - f(Y(n)) - A_1 f(Y(n))] = 0. \tag{2}$$

Since $A_1 f \equiv Ag$, so from (2) we get

$$E[(I - A)g(Y(n+1))] = E[g(Y(n))].$$

Assuming that Y_n has a density[1] p_n we now have

$$\int p_{n+1}(y)g(y)\,dy - \int p_{n+1}(y)Ag(y)\,dy = \int p_n(y)g(y).$$

Assuming A^* is the adjoint of A we have $\int p_{n+1}(y)Ag(y)\,dy = \int g(y)A^* p_{n+1}(y)\,dy$, which gives

$$\int [p_{n+1}(y) - A^* p_{n+1}(y)]g(y)\,dy = \int p_n(y)g(y)\,dy.$$

[1]See Appendix for technical conditions.

This is a weak form of[2]

$$p_{n+1} - A^* p_{n+1} = p_n. \tag{3}$$

For the special case of the one-dimensional diffusion $A = \frac{1}{2} x^2 \sigma^2(x) \frac{\partial^2}{\partial x^2}$, (3) gives a key equation used in Andreasen and Huge (2011) for devising a volatility interpolation scheme and "large jumps" local volatility model. It is also developed, under different conditions, in Carr and Cousot (2012).

From the foregoing we therefore see that the equation (3) arises naturally as discrete forward equation for density of the approximating Markov chain for the process with generator A. As such, (3) always has a solution which is probability density as long as the underlying continuous-time Markov process has a density and admits a Markov jump process approximation (conditions for the latter are typically quite mild – see Section 4.3 of Ethier and Kurtz (1986)).

References

Andreasen, J. and Huge, B. (2011), 'Voatility interpolation', *RISK* .

Carr, P. and Cousot, L. (2012), 'Explicit construction of martingales calibrated to given implied volatility smiles', *SIAM J. Finan. Math.* **3**. Electronic copy available at: http://ssrn.com/abstract=1699002.

Ethier, S. N. and Kurtz, T. G. (1986), *Markov Processes: Characterization and Convergence*, Wiley.

Gihman, I. I. and Skorohod, A. V. (1972), *Stochastic Differential Equations*, Springer.

A Conditions for existence of density

Assume that the transition function $P(t; x, y)$ of the underlying solution of the martingale problem for A admits density $p(t; x, y)$, Then the transition

[2] In all sensible applications (e.g. if A generates a semigroup) $D(A)$ will be dense in $\bar{C}(E)$.

function $\mu_\lambda(\cdot, \Gamma)$ has a density as well, and $\mu_\lambda(\cdot, dy) = \lambda \int_0^\infty e^{-\lambda t} p(t; \cdot, y)\, dt\, dy$. In the case of the familiar second-order parabolic operator

$$L = b(x)\frac{\partial}{\partial x} + \frac{1}{2}a(x)\frac{\partial^2}{\partial x^2}$$

a set of sufficient conditions for existence of the density p are given by Theorem I.13.2 of Gihman and Skorohod (1972).