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SEMILINEAR HILLE–YOSIDA THEORY: THE APPROXIMATION THEOREM AND GROUPS OF OPERATORS

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INTRODUCTION

OF GREAT interest in nonlinear functional analysis are *necessary and sufficient conditions* for the generation of one parameter semigroups and groups of nonlinear operators. There is no full analogue of the Hille–Yosida generation theorem for nonlinear contraction semigroups on *arbitrary* Banach spaces. (There are such analogues known for contraction semigroups on Hilbert space and on special Banach spaces, but this is not our concern here.)

Recently Ôharu and Takahashi [11] made a start on a Hille–Yosida theory of *necessary and sufficient conditions* for nonlinear (not necessarily contractive) semigroups by studying a class of semilinear equations of the form $du/dt = Au + Bu$ where the linear operator A generates a (C_0) semigroup on the (arbitrary) Banach space X and B is locally Lipschitz continuous on X. A precise version of this (nonlinear) semigroup result is stated in Section 1. In the present paper we extend this result from semigroups to groups. The extension is not entirely trivial. We also obtain the characterization of when the semigroup S generated by $A + B$ is continuously Fréchet differentiable in terms of properties of B. Furthermore we establish an analogue of the Neveu–Trotter–Kato approximation theorem in the context of the work of Ôharu and Takahashi [11]. This is presented in Section 2.

Our Hille–Yosida type theorems for groups are presented in Sections 3 and 5. Section 4 contains the approximation theorem for groups. Two simple examples are given in Section 5.

We are also interested in the second order equation version of these results, that is, in necessary and sufficient conditions for global wellposedness with certain types of bounds for

$$d^2u/dt^2 = Au + Bu + C\,du/dt,$$

where the linear operator A generates a strongly continuous cosine function and B and C are locally Lipschitzian operators on the underlying Hilbert (or Banach) space. This will be the subject of a future paper [6], which will be based on the present work.

1. THE BACKGROUND

Let $(X, |\cdot|)$ be a Banach space. We begin by defining the classes Ξ_0, Ξ_1, Ξ_2.
$(C, p) \in \Xi_0$ means
(i) C is a convex set in X;
(ii) p is a proper convex lower semicontinuous function from X to $[0, \infty]$;
(iii) $C \subset \{x \in X : p(x) < \infty\}$;
(iv) the convex set

$$C_r := \{x \in C : p(x) \leq r\}$$

is closed for all $r > 0$.

$(B, C, p) \in \Xi_1$ means $(C, p) \in \Xi_0$ and

(v) $B : C \to X$ is locally Lipschitzian in the sense that for all $r > 0$ there is a positive constant $w(r)$ such that $|Bx - By| \leq w(r)|x - y|$ whenever $x, y \in C_r$.

$(A, B, C, p) \in \Xi_2$ means $(B, C, p) \in \Xi_1$ and

(vi) A generates a linear (C_0) contraction semigroup

$$T = \{T(t) : t \in \mathbb{R}^+ = [0, \infty)\} \qquad \text{on } X.$$

The adjective "contraction" need not be present in (vi) above, but it can be inserted without loss of generality by replacing $A + B$ by $(A - \omega I) + (B + \omega I)$ for suitable $\omega \geq 0$ and by replacing $|\cdot|$ by an equivalent norm.

Sometimes we shall write Ξ_j as $\Xi_j(X)$ if we want to emphasize the dependence on X.

Associated with $(A, B, C, p) \in \Xi_2$ is the Cauchy problem

$$u'(t) = (A + B)u(t) \qquad [t \in \mathbb{R}^+],\ u(0) = X \in C. \tag{CP}$$

We want (CP) to be globally well-posed. Thus we expect (CP) to be governed by a nonlinear semigroup $S = \{S(t) : t \in \mathbb{R}^+\}$ on C which provides unique mild solutions to (CP) on $\mathbb{R}^+$. This means that for each $x \in C$, $u(t) = S(t)x$ is strongly continuous in $t \in \mathbb{R}^+$ and satisfies the integral equation

$$u(t) = T(t)x + \int_0^t T(t - s)Bu(s)\mathrm{d}s, \qquad t \geq 0. \tag{IE}$$

Let $(C, p) \in \Xi_0$. We write $S \in \mathfrak{S}(C, p)$ and we call S a (C, p) *locally Lipschitzian semigroup* provided that $S = \{S(t) : t \in \mathbb{R}^+\}$ is a family of mappings from C to C satisfying
(vii) $S(t + s)x = S(t)S(s)x$, $S(0)x = x$ for all $t, s \in \mathbb{R}^+$ and all $x \in C$;
(viii) $S(\cdot)x$ is strongly continuous on $\mathbb{R}^+$ for each $x \in C$;
(ix) for all $r, \tau > 0$, there is a positive constant $\omega(r, \tau)$ such that

$$|S(t)x - S(t)y| \leq e^{\omega(r,\tau)t}|x - y|$$

holds whenever $0 \leq t \leq \tau$ and $x, y \in C_r$.

The main result of Ôharu-Takahashi [11] is a semilinear version of the Hille-Yosida theorem which can be stated as follows.

THEOREM 1. Let $a \geq 0, b \geq 0$ and let $(A, B, C, p) \in \Xi_2$. The following three statements are equivalent.

[I] There is an $S \in \mathfrak{S}(C, p)$ satisfying

$$S(t)x = T(t)x + \int_0^t T(t-s)BS(s)x\,ds \tag{I.1}$$

for all $t \in \mathbb{R}^+$ and all $x \in C$,

$$p(S(t)x) \leq e^{at}(p(x) + bt) \qquad \text{holds for all } x \in C \text{ and all } t \geq 0. \tag{I.2}$$

[II] For every $r > 0$ there is a $\lambda_0(r) > 0$ such that for all $x \in C_r$ and all $\lambda \in (0, \lambda_0(r))$ there exists an $x_\lambda \in \mathfrak{D}(A) \cap C$ such that

$$x_\lambda - \lambda(A + B)x_\lambda = x, \tag{II.1}$$

$$p(x_\lambda) \leq (1 - a\lambda)^{-1}(p(x) + b\lambda). \tag{II.2}$$

[III] For all $x \in C$ and all $\varepsilon > 0$ there is a pair $(h, x_h) \in (0, \varepsilon) \times C$ such that

$$|T(h)x + hBx - x_h| \leq h\varepsilon, \tag{III.1}$$

$$p(x_h) \leq e^{ah}(p(x) + bh). \tag{III.2}$$

Property [I] says that the semigroup provides global mild solutions to (CP) which have certain growth properties (determined by p, a, b). The equivalence [I] iff [II] is a semilinear analogue of the Hille–Yosida generation theorem, and it is the first such result (giving *necessary* and sufficient conditions) for nonlinear semigroups in a *general* Banach space setting.

2. THE APPROXIMATION THEOREM

The approximation theorem says that the semigroup S of theorem 1 depends continuously on A, B, C, p. We deduce this result from the generation theorem (theorem 1) by applying Kisyński's sequence space trick [9]. A similar proof was given in the nonlinear *contraction* semigroup case by Goldstein [3].

The idea is simple, so we sketch it before stating the approximation theorem. For each $n \in \mathbb{N}_0 = \{0, 1, 2, \ldots\}$ let $(A_n, B_n, C_n, p_n) \in \Xi_2(X)$ and suppose $D_n \to D_0$ as $n \to \infty$ (in a certain sense) for $D \in \{A, B, C, p\}$. Let

$$\mathfrak{X} = c(X) = \{\mathfrak{x} = (x_n)_0^\infty : x_n \in X, |x_n - x_0| \to 0 \text{ as } n \to \infty\}$$

be the space of convergent sequences in X. $\mathfrak{X}$ is a Banach space under the norm $\|\mathfrak{x}\| = \sup_n |x_n|$.

From the sequence $\{(A_n, B_n, C_n, p_n)\}$ in $\Xi_2(X)$ we want to construct an element $(\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{p})$ of $\Xi_2(\mathfrak{X})$. We do this as follows. Define $\mathfrak{A}$ by: $\mathfrak{A}\mathfrak{x} = \mathfrak{y}$ means $\mathfrak{x} = (x_n)_0^\infty \in \mathfrak{X}$, $\mathfrak{y} = (y_n)_0^\infty \in \mathfrak{X}$, and for all $n \in \mathbb{N}_0$, $A_n x_n = y_n$. Note that necessarily $A_n x_n = y_n \to y_0 = A_0 x_0$ as $n \to \infty$. Then $\mathfrak{A}$ is a densely defined (linear) m-dissipative operator on $\mathfrak{X}$ iff $T_n(t)x \to T_0(t)x$ as $n \to \infty$ for each $x \in X$ and each $t \in \mathbb{R}^+$ iff $(I - \lambda A_n)^{-1}x \to (I - \lambda A_0)^{-1}x$ for each $x \in X$ and each $\lambda > 0$ iff for all $x_0 \in \mathfrak{D}(A_0)$ there is an $x_n \in \mathfrak{D}(A_n)$ such that $x_n \to x_0$ and $A_n x_n \to A_0 x_0$. This follows from the Neveu–Trotter–Kato approximation theorem for semigroups. (See e.g. Goldstein [5], Davies [1]; the assertion involving $A_n x_n \to A_0 x_0$ is due to Kurtz [10].) In this case $\mathfrak{A}$ generates a (C_0) contraction semigroup $\mathfrak{T} = \{\mathfrak{T}(t) : t \in \mathbb{R}^+\}$ on $\mathfrak{X}$ given by

$$\mathfrak{I}(t) = \lim_{m \to \infty} (I - \frac{t}{m}\mathfrak{A})^{-m}\mathfrak{x} = (T_n(t)x_n)_0^\infty.$$

Note that $\mathfrak{I}(t)\mathfrak{x} \in \mathfrak{X}$ for $\mathfrak{x} = (x_n)_0^\infty \in \mathfrak{X}$, so that $T_n(t)x_n \to T_0(t)x_0$ whenever $x_n \to x_0$.

Define $\mathfrak{B} : \mathfrak{D}(\mathfrak{B}) \subset \mathfrak{X} \to \mathfrak{X}$ by: $\mathfrak{B}\mathfrak{x} = \mathfrak{y}$ means $\mathfrak{x}, \mathfrak{y} \in \mathfrak{X}, x_n \in C_n = \mathfrak{D}(B_n)$ for each $n \in \mathbb{N}_0$, and $y_n = B_n x_n \to y_0 = B_0 x_0$ as $n \to \infty$. Define $\tilde{\mathfrak{C}} = \{\mathfrak{x} \in \mathfrak{X} : x_n \in C_n$ for each $n \in \mathbb{N}_0\}$. Define $\tilde{p}: \tilde{\mathfrak{C}} \to [0, \infty]$ by $\tilde{p}(\mathfrak{x}) = \sup_n p_n(x_n)$. Then $\tilde{p}$ is a proper convex lower semicontinuous function. Let $\mathfrak{C} = \{\mathfrak{x} \in \tilde{\mathfrak{C}} : \tilde{p}(\mathfrak{x}) < \infty\}$, and let $\mathfrak{p}$ be the restriction of $\tilde{p}$ to $\mathfrak{C}$.

We want to make assumptions to guarantee that $(\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{p}) \in \Xi_2(\mathfrak{x})$. Since we take $a \geq 0$ and $b \geq 0$ to be independent of n we apply theorem 1 in the space $\mathfrak{X}$. We get a semigroup $\mathcal{S} \in \mathfrak{S}(\mathfrak{C}, \mathfrak{p})$ satisfying (I.1), (I.2) iff [II] holds. But $\mathcal{S}(t)\mathfrak{x} = (S_n(t)x_n)$ belongs to $\mathfrak{X}$, whence $S_n(t)x_n \to S_0(t)x_0$ as $n \to \infty$. Equation (II.1) says $\mathfrak{x}_\lambda - \lambda(\mathfrak{A} + \mathfrak{B})\mathfrak{x}_\lambda = \mathfrak{x}$, which is equivalent to $x_{\lambda n} - \lambda(A_n + B_n)x_{\lambda n} = x_n$ holds for $n \in \mathbb{N}_0$ and $0 < \lambda < \lambda_0(r)$ (independent of n), and $x_n \to x_0, x_{\lambda n} \to x_{\lambda 0}$ as $n \to \infty$.

Thus let $\mathfrak{x} \in \mathfrak{X}, 0 < \lambda < \lambda_0(r)$. Let $y_n = (I - \lambda(A_n + B_n))^{-1}x_n$ for $n \in \mathbb{N}_0$, so that $(I - \lambda(A_n + B_n))y_n = x_n$. Suppose that there exists a sequence (z_n) such that $z_n \in C_n$ for $n \in \mathbb{N}_0, z_n \to y_0$ and $B_n z_n \to B_0 y_0$. Set $w_n = (I - \lambda A_n)^{-1}(x_n + B_n z_n)$. Then, by our assumption on $\mathfrak{A}$, $w_n \to y_0$ and

$$|y_n - w_n| \leqq \lambda|B_n y_n - B_n z_n| \leqq \lambda L|y_n - z_n| \qquad \text{for } n \in \mathbb{N}_0$$

provided that B_n satisfies a uniform Lipschitz condition on a set containing $\{z_n\} \cup \{y_n\}$. Therefore $(1 - \lambda L)|y_n - w_n| \leqq \lambda L|w_n - z_n|$ and

$$|y_n - y_0| \leqq |y_n - w_n| + |w_n - y_0| \to 0$$

as $n \to \infty$ since $z_n \to y_0$ and $w_n \to y_0$ provided that $\lambda L < 1$. This then implies that [II] of theorem 1 holds for $\mathfrak{A}$, $\mathfrak{B}$ on $\mathfrak{X}$. We can then apply theorem 1 to conclude by [I] that $S_n(t)x_n \to S_0(t)x_0$.

Theorem 2 below is our desired result; the hypotheses guarantee the existence of the (local) Lispschitz constant L above.

THEOREM 2. For each $n \in \mathbb{N}_0$ let $(A_n, B_n, C_n, p_n) \in \Xi_2(X)$. Let $a, b \geq 0$.

(a) Suppose $(I - \alpha A_n)^{-1}x \to (I - \alpha A_0)^{-1}x$ for each $x \in X$ and each $\alpha > 0$.

(b) For $R > 0$ there is an $R' > 0$ such that for each $x_0 \in C_0$ with $p_0(x_0) \leq R$ there is an $x_n \in C_n$ such that $x_n \to x_0$ and $\sup_n p_n(x_n) \leq R'$.

(c) For $x_n \in C_n$ with $x_n \to x_0 \in C_0$ and $\sup_n p_n(x_n) < \infty$, it follows that $B_n x_n \to B_0 x_0$.

(d) For each n and each $r > 0$ the smallest Lipschitz constant ω_{nr} of B_n on the set $C_{nr} \equiv C_n \cap \{x \in X : p_n(x) \leqq r\}$ satisfies $\sup_n \omega_{nr} < \infty$.

(e) Suppose that [II] of theorem 1 holds for (A_n, B_n, C_n, p_n) with $\lambda_0(r)$ depending on $r > 0$ but not on n.

Then if $x_n \in C_n, x_n \to x_0 \in C_0$, and $\sup_n p_n(x_n) < \infty$, it follows that $S_n(t)x_n \to S_0(t)x_0$ as $n \to \infty$ for each $t \geq 0$.

Note that assumptions (b) and (c) ensure that $\mathfrak{C}$, which is the domain of $\mathcal{S}(t)$, consists of all $\mathfrak{x} \in \mathfrak{X}$ satisfying the conditions of (c).

We now finish the proof of theorem 2. Let $r > 0$ and let $\lambda_0 = \lambda_0(r) > 0$ be a number given by condition [II]. Here we may assume that $\lambda_0 a < 1$. Let $\mathfrak{x} \in \mathfrak{C}, \mathfrak{x} = (x_n), \mathfrak{p}(\mathfrak{x}) \leqq r$ and let

$R \geqq (1 - a\lambda_0)^{-1}(r + b\lambda_0)$. Then $\sup_n p_n(x_n) \leqq r$ and $x_n \to x_0 \in C_0$. Let (y_n) be a sequence defined by $(I - \lambda(A_n + B_n))y_n = x_n$, $n \in \mathbb{N}_0$. Then $\sup_n(p_n(y_n)) \leqq R$. Therefore, to obtain the desired assertion, it sufficies to show that $y_n \to y_0$ as $n \to \infty$.

Since $y_0 \in C_0$ and $p_0(y_0) \leqq R$, condition (b) implies that there is an $R' \geqq R$ and a sequence (z_n) satisfying $z_n \in C_n$, $z_n \to y_0$ and $\sup_n p_n(z_n) \leqq R'$. Put $\tilde{\omega} = \sup_n \omega_{nR'}$; $\tilde{\omega} < \infty$ by (d). Let $\lambda \in (0, \lambda_0)$ be such that $\lambda\tilde{\omega} < 1$ and define a sequence (w_n) by $w_n = (I - \lambda A_n)^{-1}(x_n + \lambda B_n z_n)$, $n \in \mathbb{N}_0$. By (c), $x_n + \lambda B_n z_n \to x_0 + \lambda B_0 y_0$, and so it follows from (a) that $w_n \to (I - \lambda A_0)^{-1}(x_0 + \lambda B_0 y_0) = y_0$ as $n \to \infty$. Hence, applying (d), we have

$$|y_n - w_n| \leqq \lambda|B_n y_n - B_n z_n| \leqq \lambda\tilde{\omega}|y_n - z_n|$$
$$\leqq \lambda\tilde{\omega}(|y_n - w_n| + |w_n - z_n|)$$

and so

$$(1 - \lambda\tilde{\omega})|y_n - w_n| \leqq \lambda\tilde{\omega}(|w_n - y_0| + |z_n - y_0|) \to 0 \qquad \text{as } n \to \infty.$$

This shows that $y_n \to y_0$.

Theorem 2 now follows. ■

Let (b), (c) be strengthened to: for each $x_0 \in C_0$ there is a sequence $\{x_n\}$ in X such that $x_n \in C_n$ for $n \in \mathbb{N}_0$, $x_n \to x_0$ and $\sup_n p_n(x_n) < \infty$; then $p_n(x_n) \to p_0(x_0)$ and $B_n x_n \to B_0 x_0$. Under this assumption we can conclude, in addition, that $p_n(S_n(t)x_n) \to p_0(S_0(t)x_0$.

Consider (d) can be replaced by the following condition which is stronger than (d) but is useful for the applications.

($\tilde{\text{d}}$) $\{B_n\}$ is uniformly approximable by globally Lipschitzian operators in the following sense. For each n and each $r > 0$ there is a globally Lipschitzian operator L_n which agrees with B_n on the set C_{nr}. The smallest Lipschitz constant ω_{nr} of L_n (or of B_n on C_{nr}) satisfies $\sup_n \omega_{nr} < \infty$ for each $r > 0$.

In fact, condition ($\tilde{\text{d}}$) motivated (d). The usual notion of approximability by globally Lipschitzian operators is given in terms of the norm of X. An operator $B : \mathfrak{D}(B) \subset X \to X$ is said to be *approximable by globally Lipschitzian operators* if for each $R > 0$ there is a globally Lipschitzian operator on X which agrees with B on $\mathfrak{D}(B) \cap \{x \in X : |x| \leqq R\}$ (cf. Goldstein [4]). Any locally Lipschitzian operator on X is approximable by globally Lipschitzian operators when X is a Hilbert space (see Valentine [16]), but this is not necessarily true in the general Banach space case (cf. Hayden–Wells [7]). Still the locally Lipschitzian operators that arise in the applications are frequently approximable by globally Lipschitzian operators. (See Goldstein [4] for an example.) In condition ($\tilde{\text{d}}$) we introduced the notion of approximability by globally Lipschitzian operators in a different setting (both notions agree when p_n is the norm). The point is that both $(I - \lambda(A_n + B_n))^{-1}x_n$ and $(I - \lambda(A_n + L_n))^{-1}x_n$ are well-defined, but the mere *m*-dissipativity of $A_n + L_n - \omega(r)I$ (for some $\omega(r)$, a Lipschitz constant of L_n) does not guarantee that both elements are contained in the same level set of p_n and coincide if p_n is a general l.s.c. convex functional.

The following variant of theorem 2 shows that theorem 2 is actually more general than theorem 1.

THEOREM 2′. Let $(A_n, B_n, C_n, p_n) \in \Xi_2(X)$ for each n and let $a, b, \geqq 0$. Under conditions (a)–(e) as above, the following two statements are equivalent:

[I] There is a $(\mathcal{C}, \mathfrak{p})$ locally Lipschitzian semigroup $\{\mathcal{S}(t)\}$ such that

$$\mathcal{S}(t)x = (S_n(t)x_n)_{n=0}^{\infty}, \; S_n(t)x_n \to S_0(t)x_0 \qquad \text{for } t \geqq 0, \; x \in \mathcal{C} \tag{I.1}$$

and the convergence is uniform on bounded subintervals of $[0, \infty)$,

$$S_n(t)x_n = T_n(t)x_n + \int_0^t T_n(t-s)B_n S_n(s)x_n \, ds \tag{I.2}$$

for $t \geqq 0$, $x_n \in C_n$ and $n \in \mathbb{N}_0$,

$$p_n(S_n(t)x_n) \leqq e^{at}(p_n(x_n) + bt) \qquad \text{for } t \geqq 0, \; x_n \in C_n \qquad \text{and} \qquad n \in \mathbb{N}_0. \tag{I.3}$$

[II] For every $r > 0$ there is a $\lambda_0(r) > 0$ such that for $x = (x_n) \in \mathcal{C}_r = \{x \in \mathcal{C} : \mathfrak{p}(x) \leqq r\}$ and $\lambda \in (0, \lambda_0(r))$ there exists an $x_\lambda = (x_{n\lambda}) \in \mathcal{D}(\mathcal{A}) \cap \mathcal{C}$ with the following properties:

$$x_\lambda - \lambda(\mathcal{A} + \mathcal{B})x_\lambda = x, \; x_{n\lambda} \to x_{0\lambda} \qquad \text{for } \lambda \in (0, \lambda_0(r)) \qquad \text{and} \qquad x \in C_r, \tag{II.1}$$

$$x_{n\lambda} - \lambda(A_n + B_n)x_{n\lambda} = x_n \qquad \text{for } \lambda \in (0, \lambda_0(r)), \; x_n \in C_n \qquad \text{and} \qquad n \in \mathbb{N}_0, \tag{II.2}$$

$$p_n(x_{n\lambda}) \leqq (1 - a\lambda)^{-1}(p_n(x_n) + b\lambda) \qquad \text{for } \lambda \in (0, \lambda_0(r)), \; x_n \in C_n \qquad \text{and} \qquad n \in \mathbb{N}_0. \tag{II.3}$$

The sequence space trick used to prove theorems 2 and 2′ show that such approximation theorems can be regarded as generation theorems. In fact, theorem 1 follows from theorem 2′ by setting $A_n \equiv A$, $B_n \equiv B$, $C_n \equiv C$ and $p_n \equiv p$. But we emphasize that theorem 2′ cannot be deduced from theorem 1 directly.

3. GROUPS OF OPERATORS

In the context of theorem 1, suppose that A generates a (C_0) group rather than a (C_0) semigroup. We want to show that $A + B$ determines a group rather than a semigroup. Applying theorem 1 to $(\pm A, \pm B, C, p)$ gives a semigroup $\{S_\pm(t) : t \geq 0\}$. Setting $S(t)$ equal to $S_+(t)$ or $S_-(-t)$, according as $t \geq 0$ or $t \leq 0$, formally gives a group. To prove that it is a group it suffices to establish that $S_-(t) = S_+(t)^{-1}$, which follows from $(d/dt)S_+(t)S_-(t)x = 0$ for all x, t.

Let A be a nonlinear operator in a Banach space X and suppose that A and $-A$ generate semigroups $S_+ = \{S_\pm(t) : t \geqq 0\}$ on some subset C of X. If

$$(d/dt)S_+(t)S_-(t)x = 0 \qquad \text{for } t \geqq 0 \qquad \text{and} \qquad x \in C, \tag{*}$$

then $S_-(t) = S_+(t)^{-1}$ for $\geqq 0$ and a nonlinear group $S = \{S(t) : t \in \mathbb{R}\}$ on C can be obtained by defining $S(t) = S_+(t)$ for $t \geqq 0$ and $S(t) = S_-(-t)$ for $t < 0$. In general, (*) does not hold even if both A and $-A$ are m-dissipative in X and S_+, S_- are generated in the sense of Crandall and Liggett by $+A$, $-A$, respectively. However (*) does hold for the continuous semigroups S_+, S_- constructed respectively via theorem 1 for the semilinear operators $\pm(A + B)$ such that $(\pm A, \pm B, C, p) \in \Xi_2$.

LEMMA 1. Let $(\pm A, \pm B, C, p) \in \Xi_2$. Then (*) holds for all $x \in C$ and all $t \geq 0$.

Proof. Suppose that $(\pm A, \pm B, C, p) \in \Xi_2$ and $\pm(A + B)$ generate nonlinear semigroups

$S_\pm \in \mathfrak{S}(C, p)$, respectively. Let $t \geqq 0$ and $x \in C$. Put $y = S_-(t)x$. Then $S_+(t+h)S_-(t+h)x = S_+(t)S_+(h)S_-(h)S_-(t)x$ for $h > 0$ and one can find a (Lipschitz) constant $L = L(t, x)$ such that

$$|S_+(t+h)S_-(t+h)x - S_+(t)S_-(t)x| \leqq L|S_+(h)S_-(h)y - y|$$

for $h \in (0, 1)$. Now S_+ and S_- satisfy the integral equations (I.1) with the linear semigroups $\{T_\pm(t) : t \geqq 0\}$ generated by $\pm A$ and locally Lipschitzian operators $\pm B$, respectively. Hence

$$S_+(h)S_-(h)y = T_+(h)S_-(h)y + \int_0^h T_+(h-\xi)BS_+(\xi)S_-(h)y\,\mathrm{d}\xi,$$

$$S_-(h)y = T_-(h)y - \int_0^h T_-(h-\xi)BS_-(\xi)y\,\mathrm{d}\xi,$$

and so we obtain

$$S_+(h)S_-(h)y = T_+(h)T_-(h)y - T_+(h)\int_0^h T_-(h-\xi)BS_-(\xi)y\,\mathrm{d}\xi + \int_0^h T_+(h-\xi)BS_+(\xi)S_-(h)y\,\mathrm{d}\xi$$

or, since $T_+(h)T_-(h)y = y$,

$$S_+(h)S_-(h)y - y = \int_0^h [T_+(h-\xi)BS_+(\xi)S_-(h)y - T_+(h)T_-(h-\xi)BS_-(\xi)y]\,\mathrm{d}\xi.$$

Therefore $|S_+(h)S_-(h)y - y| = o(h)$ as $h \downarrow 0$ and

$$\lim_{h\downarrow 0} h^{-1}|S_+(t+h)S_-(t+h)x - S_+(t)S_-(t)x| = 0.$$

This implies that $(\mathrm{d}^+/\mathrm{d}t)S_+(t)S_-(t)x = 0$ for $t \geqq 0$, where $(\mathrm{d}^+/\mathrm{d}t)$ stands for differentiation from the right. ■

4. DIFFERENTIABILITY

The semigroup constructed in theorem 1 is not in general differentiable unless further restrictions are placed on B. This is what we do next.

Let $(B, C, p) \in \Xi_1$. We say that B is *F-differentiable on C relative to p* iff for each $r > 0$ and each $x \in C_r$ there is a bounded linear operator $B'(x)$ on X (i.e. $B'(x) \in \mathcal{L}(X)$) and a function w from $C_r \times \{v \in X : |v| < \varepsilon\}$ to X (for some $\varepsilon > 0$) such that $w(x, 0) = 0$ and

$$B(x+v) - B(x) = B'(x)v + |v|w(x, v)$$

for $|v| < \varepsilon$ with $x + v \in C_r$ and $|w(x, v)| \to 0$ as $|v| \to 0$. Next, B is *continuously F-differentiable on C relative to p* if B is F-differentiable on C relative to p and if $B'(\cdot) : C \to \mathcal{L}(X)$ is continuous on each of the level sets C_n, $r > 0$.

Let $(A, B, C, p) \in \Xi_2 = \Xi_2(X)$ and suppose that the conditions of theorem 1 hold. We say that $(A, B, C, p) \in \Xi_3$ if, in addition, B is Fréchet differentiable on C relative to p and $B'(S(\cdot)x)$ is strongly measurable and locally bounded in the space $\mathcal{L}(X)$ for each $x \in C$. If in addition B is continuously F-differentiable relative to p, we say that $(A, B, C, p) \in \Xi_4$.

We shall show that $S(\cdot)x$ is strongly differentiable provided $(A, B, C, p) \in \Xi_3$. From the point of view of the applications, the continuous Fréchet differentiability of B is not a serious restriction, and one can often demonstrate easily that $(A, B, C, p) \in \Xi_4$. According to theorem 3 below, this will ensure that $S(\cdot)x$ is strongly continuously differentiable on $\mathbb{R}^+$ if $x \in \mathfrak{D}(A) \cap C$.

LEMMA 2. Let $(A, B, C, p) \in \Xi_3$ [resp. $(A, B, C, p) \in \Xi_4$]. Then for all $x \in C$ and all $t \in \mathbb{R}^+, y \to S(t)y$ is Fréchet differentiable [resp. continuously Fréchet differentiable] at x.

Proof. (See also Dorroh and Graff [2].) Let $r > 0, \tau > 0, R \geq e^{a\tau}(r + b\tau)$, and let $x \in C_r$. Then there exists $\omega_R \in \mathbb{R}$ such that $|B(u + z) - Bu| \leq \omega_R |z|$ and

$$B(u + z) - Bu = B'(u)z + |z| w(u, z) \tag{1}$$

for $u \in C_R$ and $z \in X$ with $u + z \in C_R$ and $|w(u, z)| \to 0$ as $z \to 0$. Further, $S(t)x \in C_R$ for $t \in [0, \tau]$ and $|B'(S(t)x)| \leq \gamma(x, \tau)$ for $t \in [0, \tau]$ and some constant $\gamma(x, \tau)$.

Let

$$\Delta(t, x, v) = S(t)(x + v) - S(t)x. \tag{2}$$

Since $S(t)x + \Delta(t, x, v) = S(t)(x + v) \in C_R$ and $|S(t)(x + v) - S(t)x| \leq \exp(t\omega(r, \tau))|v|$ for $t \in [0, \tau]$ and $v \in X$ with $x + v \in C_r$ and for some constant $\omega(r, \tau)$, it follows with the help of (1) that

$$|w(S(t)x, \Delta(t, x, v)) \leq \omega_R + \gamma(x, \tau)$$

for $t \in [0, \tau]$ and $v \in X$ with $x + v \in C_r$. Thus $w(S(\cdot)x, \Delta(\cdot, x, v))$ is bounded on $[0, \tau]$. It is also strongly measurable since $(A, B, C, p) \in \Xi_3$.

Next, by (2) and (I.1) of theorem 1,

$$\Delta(t, x, v) = T(t)v + \int_0^t T(t - s)B'(S(s)x)\Delta(s, x, v)ds + \zeta \tag{3}$$

where

$$\zeta = \int_0^t T(t - s)|\Delta(s, x, v)| w(S(s)x, \Delta(s, x, v))ds. \tag{4}$$

Strictly speaking the set of t for which (3), (4) hold depends on x and v, but we shall later apply this result in the context of theorem 1, which means that $t \in \mathbb{R}^+$ can be arbitrary, so we assume now global existence for (I.1) (or for (3)).

We now pass to the first variation of (I.1). There is a function $u: \mathbb{R}^+ \to X$ satisfying

$$u(t) = T(t)v + \int_0^t T(t - s)B'(S(s)x)u(s)ds \tag{5}$$

for each $t \geq 0$. The function u is a mild solution of

$$du(t)/dt = (A + P(t))u(t) \ [t \geq 0], \qquad u(0) = x; \tag{6}$$

here $P(t) = B'(S(t)x)$ is a locally (in t) bounded measurable perturbation of A. The existence of u follows from (an obvious extension of) either of two 1953 results, due to Kato [8] and Phillips [14].

Using evolution operator notation we can write $u(t)$ as $U(t, 0; x)x$, where $\{U(t, s; x)\}$ is the evolution operator governing (6). Let

$$z(t) = \Delta(t, x, v) - U(t, 0; x)v, \qquad t \in \mathbb{R}^+.$$

Then by (3) and (5),

$$z(t) = \int_0^t T(t-s)B'(S(s)x)z(s)\mathrm{d}s + \zeta.$$

Applying Gronwall's inequality gives the conclusion

$$|z(t)| \leq ce^{tk} \qquad \text{for } 0 \leq t \leq \tau, \tag{7}$$

where c is an upper bound for $\{|\zeta| = |\zeta(t)| : 0 \leq t \leq \tau\}$ and k is an upper bound for $\{|B'(S(s)x)| : 0 \leq s \leq \tau\}$. Since

$$|\Delta(t, x, v)| = |S(t)(x+v) - S(t)x| \to 0 \qquad \text{and} \qquad |w(S(t)x, \Delta(t, x, v))| \to 0$$

whenever $x + v \in C_r$ and $v \to 0$, from (4) it follows that

$$\zeta = o(\sup\{|\Delta(s, x, v)| : 0 \leq s \leq t\}), \qquad S(t)(x+v) - S(t)x - U(t, 0; x)v = o(|v|) \tag{8}$$

as $x + v \in C_r$ and $v \to 0$. This enables us to conclude that each $S(t)$ is F-differentiable relative to p.

If in particular $(A, B, C, p) \in \Xi_4$, then the F-derivative $U(t, 0; x)$ of $S(t)$ at x is continuous on each C_r with respect to x. Hence, in this case, each $S(t)$ is continuously F-differentiable on C relative to p. ∎

THEOREM 3. Suppose that condition [I] (or [II]) holds and that $(A, B, C, p) \in \Xi_3$. Then for each $x \in \mathfrak{D}(A) \cap C$, the function $S(\cdot)x$ is strongly continuously differentiable on $\mathbb{R}^+$ and satisfies

$$(\mathrm{d}/\mathrm{d}t)S(t)x = S'(t, x)(A + B)x = (A + B)S(t)x$$

for $t \geq 0$, where $S'(t, x) = U(t, 0; x)$ is the derivative of $S(t)x$ with respect to x.

We must show that the assumption of strong continuous differentiability of B implies that the semigroup trajectory $S(\cdot)x$ is continuously differentiable from $\mathbb{R}^+$ to X for each $x \in \mathfrak{D}(A) \cap C$. For this we use lemma 2.

We have

$$S(t)(x+v) - S(t)x = S'(t, x)v + |v|w(t, x, v)$$

where $w(t, x, v) \to 0$ as $v \to 0$. For $x \in C \cap \mathfrak{D}(A)$ and $t \in \mathbb{R}^+$, $h > 0$, let $v = S(h)x - x$. Then (see (8)),

$$\begin{aligned} h^{-1}[S(t+h)x - S(t)x] &= h^{-1}[S(t)(x+v) - S(t)x] \\ &= S'(t, x)(h^{-1}v) + |h^{-1}v|w(t, x, v) \\ &\to S'(t, x)(A + B)x + 0 \end{aligned}$$

as $h \to 0$ by (I.1), since $x \in \mathfrak{D}(A)$.

Thus for $x \in \mathfrak{D}(A) \cap C$, the right derivative of $S(t)x$ with respect to t exists and equals $(A + B)S(t)x$, which is strongly continuous; whence $S(\cdot)x$ is strongly continuously differentiable on $\mathbb{R}^+$ for $x \in \mathfrak{D}(A) \cap C$ and

$$(\mathrm{d}/\mathrm{d}t)S(t)x = (A + B)S(t)x = S'(t, x)(A + B)x.$$

This completes the proof of theorem 3. ■

COROLLARY 1. Let $(A, B, C, p) \in \Xi_3$. Then $A + B$ is the (strong) infinitesimal generator of a semigroup $S \in \mathfrak{S}(X, p)$ satisfying [I] iff [II] holds.

We next relate the continuous Fréchet differentiability of B and the F-differentiability of the semigroup S.

Let $(A, B, C, p) \in \Xi_2$ and let [I] hold. Then using (5) we obtain

$$S(t)(x + v) - S(t)x - U(t, 0; x)v = \int_0^t T(t - \xi)[BS(\xi)(x + v) - BS(\xi)x - P(\xi, x)U(\xi, 0; x)v]\mathrm{d}\xi,$$

where

$$P(t, x) = B'(S(t)x) \qquad \text{for } t \geqq 0.$$

We then set

$$w(t, x; v) = \frac{1}{t}\int_0^t T(t - \xi)[BS(\xi)(x + v) - BS(\xi)x - P(\xi, x)U(\xi, 0; x)v]\mathrm{d}\xi,$$

$$z(t, x; v) = \frac{1}{t}\int_0^t T(t - \xi)[P(\xi, x)U(\xi, 0; x)v - P(0, x)v]\mathrm{d}\xi.$$

If B is continuously Fréchet differentiable, then we get

$$|t^{-1}[(S(t) - T(t))(x + v) - (S(t) - T(t))x] - P(0, x)v| \leqq |w(t, x; v)| + |z(t, x; v)|, \tag{9}$$

$$\sup_{0 \leqq t \leqq \tau} |w(t, x; v)| = o(|v|) \text{ as } v \to 0 \qquad \text{and} \qquad \lim_{t \downarrow 0} |z(t, x; v)| = 0. \tag{10}$$

THEOREM 4. Suppose that condition [I] holds and that $(A, B, C, p) \in \Xi_3$. Then B is continuously F-differentiable on C relative to p iff for each $r > 0$ and each $x \in C_r$, condition (9) holds for $v \in X$ with $x + v \in C_r$ and some $w(t, x; v)$ and $z(t, x; v)$ satisfying (10). Moreover, in this case, each $S(t)$ is continuously F-differentiable on C relative to p.

We continue the proof. Since condition [I] implies

$$\lim_{t \downarrow 0} t^{-1}(S(t) - T(t))x = Bx \qquad \text{for } x \in C, \tag{11}$$

the semilinear operator $A + B$ is the full infinitesimal generator of S (cf. [12, definition 3.1]), and for each $x \in C$ and each v with $x + v \in C$, (9) holds for some $w(t, x; v)$ and $z(t, x; v)$ satisfying (10).

Conversely, let $(A, B, C, p) \in \Xi_3$, so that by corollary 1, $A + B$ is the full infinitesimal generator of a semigroup S satisfying [I]. Assume then that (9) and (10) hold for some continuous operator $P(\cdot)$ from C into $L(X)$. Then $A + B$ satisfies condition [II] by theorem 1. Let $x \in C$ and $x + v \in C$. Letting $t \downarrow 0$ in (9) and using (11) we get

$$|B(x+v) - Bx - P(x)v| = o(|v|) \qquad \text{as } v \to 0,$$

which shows that B is continuously F-differentiable on C. ■

Theorems 3 and 4 address the differentiability of $S(t)x$, both as a function of t and as a function of x. These theorems help clarify the relationship between the classes Ξ_3 and Ξ_4.

5. GENERATION OF GROUPS

Let $(C,p) \in \Xi_0$. We write $S \in \mathcal{G}(C,p)$ and we call S a (C,p) *locally Lipschitzian group* provided that $S = \{S(t): t \in \mathbb{R}\}$ is a family of mappings from C to C satisfying

(vi′) $S(t+s)x = S(t)S(s)x$, $S(0)x = x$ for all $t, s \in \mathbb{R}$ and all $x \in C$;

(vii′) $S(\cdot)x$ is strongly continuous on $\mathbb{R}$ for each $x \in C$;

(viii′) for all $r, \tau > 0$ there is a positive constant $\omega(r,\tau)$ such that

$$|S(t)x - S(t)y| \le e^{\omega(r,\tau)|t|}|x-y|$$

holds whenever $|t| \le \tau$ and $x, y \in C_r$.

The following results are the group analogues of theorem 1.

THEOREM 5. Let $a \ge 0$, $b \ge 0$ and let $(\pm A, \pm B, C, p) \in \Xi_2$. The following two statements are equivalent.

[I′] There is an $S \in \mathcal{G}(C,p)$ satisfying

$$S(t)x = T(x)x + \int_0^t T(t-s)BS(s)x\,ds \tag{I.1′}$$

for all $t \in \mathbb{R}$ and all $x \in C$,

$$p(S(t)x) \le e^{a|t|}(p(x) + b|t|) \tag{I.2′}$$

holds for all $x \in C$ and $t \in \mathbb{R}$.

[II′] For every $r > 0$ there is a $\lambda_0(r) > 0$ such that for all $x \in C_r$ and all real λ with $|\lambda| < \lambda_0(r)$ there exists an $x_\lambda \in \mathfrak{D}(A)$ such that

$$x_\lambda - \lambda(A+B)x_\lambda = x, \tag{II.1′}$$

$$p(x_\lambda) \le (1 - a|\lambda|)^{-1}(p(x) + b|\lambda|). \tag{II.2′}$$

We omit the statement corresponding to [III], which is equivalent to [I′] and [II′].

THEOREM 5′. Let $a \ge 0$, $b \ge 0$ and let $(\pm A, \pm B, C, p) \in \Xi_2$. Then [I′] holds together with (9), (10) (for $t \in \mathbb{R}$, i.e. with t replaced by $\pm t$) if and only if (II′) holds and B is continuously Fréchet differentiable on C, i.e. $(A, B, C, p) \in \Xi_4$. In this case S consists of Fréchet differentiable operators and (5) holds (with t replaced by $\pm t$).

Proofs. That [I′] implies [II′] follows easily from theorem 1. For the converse implication, taking $\lambda > 0$ [resp. $\lambda < 0$] in [II′] and using Theorem 1 gives a semigroup $S_+ \in \mathfrak{S}(C,p)$ [resp. $S_- \in \mathfrak{S}(C,p)$] satisfying [I]. It suffices to establish the group property for S defined by

$$S(t) = \begin{cases} S_+(t) & \text{if} \quad t \ge 0 \\ S_-(-t) & \text{if} \quad t \le 0. \end{cases}$$

It suffices to show that $S_-(t) = S_+(t)^{-1}$ for $t > 0$. If this holds and $0 < s < -t$,

$$S(t+s) = S_-(-s-t) = S_-(-t)S(s)^{-1}$$
$$= S(t)S_+(s) = S(t)S(s),$$

and similarly for all other cases, so that $S(t+s) = S(t)S(s)$ holds for all t, s in $\mathbb{R}$.

But $S_-(t) = S_+(t)^{-1}$ follows from lemma 1. This completes the proof. ■

Theorems 5 and 5′ are semilinear Hille–Yosida theorems for groups. These will be applied to second order equations in a future paper [6].

6. APPROXIMATION OF GROUPS

The group S of theorem 5 (or 5′) depends continuously on (A, B, C, p). A precise version of this assertion is the group analogue of theorem 2. This result (stated below as theorems 6 and 6′) can be proved by the sequence space trick used in Section 2 and thus deduced as a consequence of theorems 5, 5′. The proofs are so close to that of theorem 2 that we can safely omit the details.

THEOREM 6. For each $n \in \mathbb{N}_0$ let $(\pm A_n, \pm B_n, C_n, p_n) \in \Xi_2(X)$. Let $a, b \geq 0$.

(a′) Suppose $(I - \alpha A_n)^{-1}x \to (I - \alpha A_0)^{-1}x$ for each $x \in X$ and each $\alpha \in (-\varepsilon_0, \varepsilon_0)$ for some fixed $\varepsilon_0 > 0$.

(b′) Suppose for each x_0 in C_0 there is an $x_n \in C_n$ such that $x_n \to x_0$ and $\sup_n p_n(x_n) < \infty$.

(c′) For $x_n \in C_n$ with $x_n \to x_0 \in C_0$ and $\sup_n p_n(x_n) < \infty$, it follows that $p_n(x_n) \to p_0(x_0)$, $B_n x_n \to B_0 x_0$, and $B'_n(x_n)y \to B'_0(x_0)y$ for all $y \in X$.

(d′) $\{B_n\}$ is uniformly approximable by globally C^1 functions in the sense that for each $r > 0$ there is a strongly continuously differentiable function L_{nr} on X whose C^1 norm is bounded by a function of r, independently of n, and which agrees with B_n on $C_{nr} = C_n \cap \{x \in X : |x| \leq r\}$.

(e′) Suppose that [II′] of theorem 5 holds for (A_n, B_n, C_n, p_n) with $\lambda_0(r)$ depending on $r > 0$ but not on n.

Then if $x_n \in C_n$ and $x_n \to x_0$, we have $S_n(t)x_n \to S_0(t)x_0$ as $n \to \infty$ for each $t \in \mathbb{R}$.

THEOREM 6′. Let $(\pm A_n, \pm B_n, C_n, p_n) \in \Xi_2(X)$ for each $n \in \mathbb{R}_0$ and let $a \geq 0, b \geq 0$. Assume conditions (a′) – (e′) of theorem 6. Then the following two statements are equivalent.

[I] There is a $(\mathcal{C}, \mathfrak{p})$ locally Lipschitzian group $\{\mathcal{S}(t)\}$ such that

$$\mathcal{S}(t)x = (S_n(t)x_n)_{n=0}^{\infty}, \; S_n(t)x_n \to S_0(t)x_0 \qquad \text{for } t \geq 0, x \in \mathcal{C}, \tag{I.1}$$

and the convergence is uniform on bounded subintervals of $[0, \infty)$,

$$S_n(t)x_n = T_n(t)x_n + \int_0^t T_n(t-s)B_n S_n(s)x_n \mathrm{d}s \tag{I.2}$$

for $t \geq 0$, $x_n \in C_n$, and $n \in \mathbb{N}$,

$$p_n(S_n(t)x_n) \leq e^{at}(p_n(x_n) + bt) \qquad \text{for } t \geq 0, x_n \in C_n, \quad \text{and} \quad n \in \mathbb{N}_0. \tag{I.3}$$

[II] For every $r > 0$ there is a $\lambda_0(r) > 0$ such that for $x = (x_n) \in C_r = \{x \in \mathcal{C} : \mathfrak{p}(x) \leq r\}$ and

$\lambda \in (-\lambda_0(r))$ there exists an $\mathfrak{x}_\lambda = (x_{\lambda n}) \in \mathfrak{D}(\mathfrak{A}) \cap \mathfrak{C}$ with the following properties:

$$\mathfrak{x}_\lambda - \lambda(A + B)\mathfrak{x}_\lambda = \mathfrak{x},\ x_{\lambda n} \to x_{\lambda 0} \quad \text{for } \lambda \in (-\lambda_0(r), \lambda_0(r)) \quad \text{and} \quad \mathfrak{x} \in \mathfrak{C}_r, \tag{II.1}$$

$$x_{\lambda n} - \lambda(A_n + B_n)x_{\lambda n} = x_n \quad \text{for } \lambda \in (-\lambda_0(r), \lambda_0(r)), x_n \in C_n \quad \text{and} \quad n \in \mathbb{N}_0, \tag{II.2}$$

$$p_n(x_{n\lambda}) \leq (1 - a\lambda)^{-1}(p_n(x_n) + b\lambda) \quad \text{for } \lambda \in (-\lambda_0(r), \lambda_0(r)), x_n \in C_n \quad \text{and} \quad n \in \mathbb{N}_0. \tag{II.3}$$

Combining theorem 2 and lemma 1 leads to the following variant of theorem 6.

THEOREM 6_{bis}. For each $n \in \mathbb{N}_0$ let $(\pm A_n, \pm B_n, C_n, p_n) \in \Xi_2(X)$. Let $a, b \geqq 0$. Suppose that (a′) of theorem 6, (b), (c), (d) of theorem 2 and (e′) of theorem 6 are satisfied. Then if $x_n \in C_n$, $x_n \to x_0 \in C_0$ and $\sup_n p_n(x_n) < \infty$, it follows that $S_n(t)x_n \to S_0(t)x_0$ as $n \to \infty$ for each $t \geqq 0$.

Variants of theorems 6 exist with Ξ_2 is replaced by Ξ_3 or Ξ_4. We omit the details.

7. EXAMPLES

For $1 \leq q < \infty$ let $X_q = L^q(\mathbb{R})$, and let $X_\infty = C_0(\mathbb{R})$. Fix $q \in [1, \infty]$. Consider the parabolic equation

$$\partial u/\partial t = \partial^2 u/\partial x^2 + F(u)$$

for $x \in \mathbb{R}, t \geq 0$, where $F: \mathbb{R} \to \mathbb{R}$ is a locally Lipschitzian function satisfying $F(0) = 0$. Suppose further that the ordinary differential equation $y' = F(y)$ has the property that if y is a solution and $y(0) \geq 0$, then $y(t) \geq 0$ for all $t \geq 0$ for which y is defined. With further assumptions on F, we shall apply theorem 1 with $A = \mathrm{d}^2/\mathrm{d}x^2$, $X = X_q$, $C = \{u \in X_q \cap C_0(\mathbb{R}) : u \geq 0\}$, $(Bu)(x) = F(u(x))$ for $u \in X_q$ and $x \in \mathbb{R}$, and $p(u) = \|u\|_\infty$.

First suppose $F(x) - F(x) \leq \omega|x - y|$ holds for some real ω and all $x, y \in \mathbb{R}$. Then $F - \omega I$ is nonincreasing and continuous on $\mathbb{R}$; thus it follows from standard semigroup theory that [II] holds with $p(u) = \|u\|_q$, $a = \omega$ and $b = 0$. This works for all q and we could equally well replace $p(u) = \|u\|_q$ by $\|u\|_\infty$.

We next construct an example of an F to which theorem 1 applies, while the Crandall-Liggett theorem does not apply directly to $A = \mathrm{d}^2/\mathrm{d}x^2 + F$.

Let $F \in C^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ (or let F be locally Lipschitzian and bounded) but suppose that F' is not bounded above and $F(0) = 0$. (For instance take $F(y) = \sin(\alpha y^2)$ with α a positive constant.) Take $X = X_q$ (for q fixed in $[1, \infty]$) and $p(u) = \|u\|_\infty$, so that $C \cap \mathfrak{D}(A) = \{y \in X_q : y \in C^2(\mathbb{R}), y'' \in X_q$ and $y \geq 0$ on $\mathbb{R}\}$. For each $h \in C$ and $\lambda > 0$ sufficiently small we shall construct a unique y in $\mathfrak{D}(A) \cap C$ such that $y - \lambda(A + B)y = h$, i.e. $y - \lambda(y'' + F(y)) = h$.

Let $F_n : \mathbb{R} \to \mathbb{R}$ be globally Lipschitzian, bounded by $\|F\|_\infty$, and agree with F on $[-n, n]$. Then there is a unique $y_n \in C \cap \mathfrak{D}(A)$ such that $y_n - \lambda(y_n'' + F_n(y_n)) = h$. It follows from that maximum principle that $\|y_n\|_\infty \leq \|h\|_\infty + \lambda\|F\|_\infty$. Also, consequently, $\|y_n''\|_\infty$ is bounded, independently of n. Thus so is $\|y_n'\|_\infty$, by the Kallman-Rota inequally (see [5]). Using these facts, $y_n = (I - \lambda A)^{-1}(\lambda F_n(y_n) + h)$, and compactness criteria, we may pass to a convergent subsequence (still denoted by $\{y_n\}$) satisfying

$$y_n \to y \text{ uniformly on } \mathbb{R},$$

$$y_n' \to y' \text{ uniformly on } \mathbb{R},$$

$$y_n'' \to y'' \text{ uniformly on compacta,}$$

$$\text{and} \qquad y - \lambda(y'' + F(y)) = h \qquad \text{on } \mathbb{R}.$$

Letting $x \to \pm\infty$ and noting $h(\pm\infty = y(\pm\infty) = 0 = F(0)$, we conclude that $y(\pm\infty) = 0$. Thus y belongs to $\mathfrak{D}(A) \cap C$ and satisfies $y - \lambda(A + B)y = h$. To establish the uniqueness of y, let y_1, y_2 be two solutions and let $w = y_1 - y_2$. Then in the same way above we have

$$\|w\|_\infty \leqq \lambda \|Fy_1 - Fy_2\| \leqq \lambda\omega_r \|w\|_\infty$$

provided that $\|y_1\|_\infty, \|y_2\|_\infty \leqq r$ and ω_r denotes the Lipschitz constant of F on the interval $[-r, r]$. This shows that if $\lambda_0(r) = 1/\omega_r$ and $\lambda \in (0, \lambda_0(r))$ the solution y is unique.

Now choose $x_0 \in \mathbb{R}$ such that $y(x_0) = \pm\|y\|_\infty$. Then from $y(x_0) - \lambda(y''(x_0) + F(y(x_0)) = h(x_0)$ and $\pm y''(x_0) \leq 0$ it follows that

$$\|y\|_\infty \leq \|h\|_\infty + \lambda\|F\|_\infty.$$

Thus [II] holds with $X = X_q$, $C = \{u \in X_q \cap C^2(\mathbb{R}) : u \geq 0 \text{ on } \mathbb{R}\}$, $p = \|\cdot\|_\infty$, $a = 0$ and $b = \|F\|_\infty$. Consequently [I] holds in this case. Note that the solution of

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + F(u), \qquad u(0, x) = f(x)$$

satisfies $u \in C^1([0, \infty), C)$ provided $f \in C$. Also, u depends continuously on F, according to theorem 2.

For a group version of this phenomenon, the above reasoning applies to the example

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x} + F(u)$$

with everything as before except this time $A = \mathrm{d}/\mathrm{d}x$ generates a (C_0) group of isometries on X_q. The example $F(y) = \sin(xy^2)$ again works.

By taking $X = L^2(\mathbb{R})$ we can also handle some problems of the form

$$u_t = u_{xxx} + F(u)$$

(using $|F(x)| \leq K|x|$ for some k and all x), but unfortunately the Kortweg–de Vries equation does not seem to fit into the framework of this paper.

Nevertheless, it is possible to formulate semilinear evolution equations which approximate the K–dV equation. Such equations fit into the framework of this paper and weak solutions of the K–dV equation are obtained as limits of "smooth" solutions of the approximate equations. See for instance Takahashi [15].

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