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Remark on the Yosida approximation iterative technique for split monotone Yosida variational inclusions

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Abstract
In an attractive article, Rahman et al. introduced the split monotone Yosida variational inclusions (SMYVI) and estimate the approximate solution of the split monotone Yosida variational inclusions using nonexpansive property of operators. The main result of this paper has flaw and not correct in the present form. We modify the SMYVI and give the strong convergence theorem under some new assumptions. We also give a weak convergence theorem to solve modified split Yosida variational inclusion problem using properties of averaged operators with three new supporting lemmas.

Keywords Split monotone Yosida variational inclusions · Inverse strongly monotone operator · Averaged operator · Nonexpansive operator

Mathematics Subject Classification 47H05 · 47H09 · 47J25

1 Introduction

The evolution equation $x^{'}(t) + A(x) = 0, x(0) = x_0$ is the mathematical model of several physical problems of practical utilizations. It is not easy to solve if the function A is not continuous. To overcome this obstacle, Yosida introduced an idea to find a sequence of Lipschitz functions that approximate A in some sense. On the other hand, it is noted that a monotone operator in Hilbert spaces can be regularized into a single-valued Lipschitzian

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monotone operators by means of Yosida approximation. The Yosida approximation operators are advantageous to estimate the solution of variational inclusion problem using resolvent operators. In recent time, many authors applied Yosida approximation operators to solve variational inclusion and system of variational inclusion problems, see (Cao 2003; Lan 2013; Ahmad et al. 2017; Akram et al. 2018) and references therein.

Let H be a Hilbert space and B be a multi-valued monotone operator, and then, the Yosida approximation of B is defined by $J_\lambda^B = \frac{1}{\lambda}[I - R_\lambda^B]$, where $R_\lambda^B = [I + \lambda B]^{-1}$ be the resolvent of B, $\lambda > 0$ and the Yosida inclusion problem is to find $x \in H$, such that:

$$0 \in J_\lambda^B(x) + B(x).$$

A similar Yosida inclusion problem and a system of Yosida inclusion problems have been investigated in Ahmad et al. (2017) and Akram et al. (2018) with generalized monotone mappings. The above Yosida inclusion problem has been also studied in the setting of Hadamard manifold (Dilshad 2019).

Suppose that H_1, H_2 be Hilbert spaces and $B_1 : H_1 \to 2^{H_1}$, $B_2 : H_2 \to 2^{H_2}$ be maximal monotone mappings. Let $f_1 : H_1 \to H_1$ and $f_2 : H_2 \to H_2$ be inverse strongly monotone operators and $A : H_1 \to H_2$ be a bounded linear operator. The split monotone Yosida variational inclusions (SMYVI) introduced by Rahman et al. (2018) are described as: find $x^* \in H_1$, such that:

$$0 \in B_1(x^*) + f_1(x^*) - J_{\lambda_1}^{B_1}(x^*) \tag{1}$$

$$\text{such that } Ax^* = y^* \in H_2 \text{ solves } 0 \in B_2(y^*) + f_2(y^*) - J_{\lambda_2}^{B_2}(y^*), \tag{2}$$

where $J_{\lambda_1}^{B_1}$ is Yosida approximation operator of B_1 and $J_{\lambda_2}^{B_2}$ is Yosida approximation operator of B_2 which are defined as $J_{\lambda_1}^{B_1} = \frac{1}{\lambda_1}(I - R_{\lambda_1}^{B_1})$, $J_{\lambda_1 2}^{B_2} = \frac{1}{\lambda_2}(I - R_{\lambda_2}^{B_2})$, with $\lambda_1 > 0, \lambda_2 > 0$ and $R_{\lambda_1}^{B_1} = (I + \lambda_1 B_1)^{-1}$, $R_{\lambda_2}^{B_2} = (I + \lambda_2 B_2)^{-1}$ are the resolvent of B_1 and B_2, respectively. They proved (Rahman et al. 2018, Lemma 3.1) that x^* and y^* solve SMYVI (1)–(2) if and only if:

$$x^* = R_{\lambda_1}^{B_1}\big[I + \lambda_1(J_{\lambda_1}^{B_1} - f_1)\big](x^*),$$
$$\text{and } y^* = R_{\lambda_2}^{B_2}\big[I + \lambda_2(J_{\lambda_2}^{B_2} - f_2)\big](y^*).$$

The solution set of SMYVI (1)–(2) is denoted by $\Omega = \{x \in H_1 \text{ solves}(1) : Ax \text{ solves } (2)\}$. Based on above fixed point equations, Rahman et al. (2018) studied the following convergence theorem to solve SMYVI (1)–(2).

Theorem 1 (Rahman et al. 2018, Theorem 3.1) *Let H_1 and H_2 be two real Hilbert spaces, $A : H_1 \to H_2$ be a bounded linear operator, and A^* be the adjoint of A. Let $B_1 : H_1 \to 2^{H_1}$ and $B_2 : H_2 \to 2^{H_2}$ be two nonempty multi-valued maximal monotone mappings, $f_1 : H_1 \to H_1$ and $f_2 : H_2 \to H_2$ be two inverse strongly monotone mappings. Let $\{g_n\}$ be a family of k-contractive mapping and uniformly convergent for any $x \in K$, where K is any bounded subset of H_1, $f : H_1 \to H_1$ be a ξ-contraction mapping, and D be a strongly positive bounded linear operator on H_1 with coefficient $\bar{r} > 0$ and $\beta \in (0, \frac{\bar{r}}{k})$. Assume that $\Omega \neq \emptyset$ and $\tau > 0$ and let the sequence $\{x_n\}$ be generated by the following iterative algorithm:*

$$\begin{aligned} x_0 &\in H_1 \\ u_n &= T[x_n + \gamma A^*(S - I)Ax_n], \\ v_n &= \delta_n u_n + \tau g_n(u_n), \\ x_{n+1} &= (1 - \alpha_n D)v_n + \alpha_n \beta f(v_n), \end{aligned}$$

where the operators $T = R^{B_1}_{\lambda_1}\big[I + \lambda_1(J^{B_1}_{\lambda_1} - f_1)\big]$ *and* $S = R^{B_2}_{\lambda_2}\big[I + \lambda_2(J^{B_2}_{\lambda_2} - f_2)\big]$, $\gamma \in (0, \frac{1}{\|A\|^2})$. *Suppose that the sequences* $\alpha_n \in [0, 1)$ *and* $\delta_n \in [0, 1)$ *satisfy the following conditions:*

$$\text{(i)} \lim_{n\to\infty} \alpha_n = 0, \sum_{n=0}^{\infty} \alpha_n = \infty, \sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty,$$

$$\text{(ii)} \lim_{n\to\infty} \delta_n = 0, \sum_{n=0}^{\infty} \delta_n = \infty.$$

Then, the sequence $\{x_n\}$ *converges strongly to a point* $x^* \in \Omega$ *which is unique solution of the variational inequality problem:*

$$\langle (D - \beta f)x^*, x - x^* \rangle \geq 0, \ \forall\, x \in \Omega.$$

If H *satisfies the Opial's property, then the sequence* $\{x_n\}$ *converges weakly to an element* $x^* \in \Omega$.

To prove Theorem 1 (Theorem 3.1 of Rahman et al. 2018), authors stated that $T = R^{B_1}_{\lambda_1}\big[I + \lambda_1(J^{B_1}_{\lambda_1} - f_1)\big]$ and $S = R^{B_2}_{\lambda_2}\big[I + \lambda_2(J^{B_2}_{\lambda_2} - f_2)\big]$ are averaged being the composition of two averaged operators and, hence, nonexpansive. However, they did not prove that how the operators $[I + \lambda_1(J^{B_1}_{\lambda_1} - f_1)\big]$ and $[I + \lambda_2(J^{B_2}_{\lambda_2} - f_2)\big]$ are averaged and, hence, nonexpansive.

Our claim is that the operators $T = R^{B_1}_{\lambda_1}\big[I + \lambda_1(J^{B_1}_{\lambda_1} - f_1)\big]$ and $S = R^{B_2}_{\lambda_2}\big[I + \lambda_2(J^{B_1}_{\lambda_1} - f_1)\big]$ are neither averaged nor nonexpansive in general and, hence, the proof of Theorem 1 is not correct in present form.

In this paper, our motive is to present a counter example to ensure that the operators $T = R^{B_1}_{\lambda_1}\big[I + \lambda_1(J^{B_1}_{\lambda_1} - f_1)\big]$ and $S = R^{B_2}_{\lambda_2}\big[I + \lambda_2(J^{B_1}_{\lambda_1} - f_1)\big]$ are neither averaged nor nonexpansive in general as mentioned in Rahman et al. (2018). Furthermore, we modify the SMYVI (1)–(2) and present the convergence theorem under new suitable conditions. A weak convergence theorem is also presented to solve modified split Yosida variational inclusion problem with three supporting lemmas.

2 Preliminaries

Let us recall some definitions and results which are used throughout this paper.

Definition 1 Let H be a Hilbert space and $T : H \to H$ be an operator, then:

(i) T is said to be contraction, if there exists a constant $\xi \in (0, 1)$, such that:

$$\|Tx - Ty\| \leq \xi \|x - y\|, \ \forall\, x, y \in H.$$

(ii) T is said to be k-demicontractive, if $\text{Fix}(T) \neq \emptyset$ and there exists a constant $k \in (0, 1)$, such that:

$$\|Tx - x^*\| \leq \|x - x^*\|^2 + k\|x - Tx\|^2, \ \forall\, x \in H \text{ and } x^* \in Fix(T).$$

(iii) T is said to be nonexpansive, if

$$\|Tx - Ty\| \leq \|x - y\|, \ \forall\, x, y \in H.$$

(iv) T is said to be firmly nonexpansive, if

$$\langle Tx - Ty, x - y \rangle \geq \|Tx - Ty\|^2, \ \forall\, x, y \in H.$$

(iv) T is said to be t-inverse strongly monotone (t-ism), if

$$\langle Tx - Ty, x - y\rangle \geq t\|Tx - Ty\|^2,\ t > 0,\ \forall\, x, y \in H.$$

Definition 2 (Baillon et al. 1978) An operator $T : H \to H$ is said to be an averaged operator if it can be written as the average of identity I and a nonexpansive mapping S, that is:

$$T = (1 - \alpha)I + \alpha S,$$

where $\alpha \in (0, 1)$.

Thus, firmly nonexpansive mappings (in particular, projections on nonempty closed convex subsets and resolvent operators of maximal monotone operators) are averaged (Moudafi 2011).

We assembled some important conclusions and properties, which are required to prove our main results.

Lemma 1 (Moudafi 2011; Xu 2011; Reich 1985)

(i) *The composition of finitely many averaged mappings is averaged. In particular, if T_i is α_i-averaged, where $\alpha_i \in (0, 1)$ for $i = 1, 2$, then the composition T_1T_2 is α-averaged, where $\alpha = \alpha_1 + \alpha_2 + \alpha_1\alpha_2$.*

(ii) *If the mappings $\{T_i\}_{i=1}^N$ averaged and have a nonempty common fixed point, then:*

$$\bigcap_{i=1}^{N} \mathrm{Fix}T_i = \mathrm{Fix}(T_1T_2 \ldots T_N).$$

In particular, for $n = 2$:

$$\bigcap_{i=1}^{2} \mathrm{Fix}T_i = \mathrm{Fix}(T_1) \bigcap Fix(T_2) = \mathrm{Fix}(T_1T_2).$$

(iii) *If T is t-ism and $\gamma \in (0, t]$, then $I - \gamma T$ is firmly nonexpansive.*

(iv) *T is nonexpansive if and only if $I - T$ is $\frac{1}{2}$-ism. Indeed, for $\alpha \in (0, 1)$, T is averaged if and only if $I - T$ is $\frac{1}{2\alpha}$-ism.*

(v) *If T is t-ism, then for $\gamma > 0$, γT is $\frac{t}{\gamma}$-ism.*

(vi) *T is averaged if and only if $I - T$ is t-ism for some $t > \frac{1}{2}$.*

(vii) *T is firmly nonexpansive if and only if $I - T$ is firmly nonexpansive.*

Definition 3 (Rahman et al. 2018) A bounded linear operator D on Hilbert space H is said to be strongly positive if there exists a constant $\bar{\gamma} > 0$, such that:

$$\langle Dx, x\rangle \geq \bar{\gamma}\|x\|^2,\ \forall\, x \in H.$$

Lemma 2 (Opial's Lemma Opial 1967) *Let H be real Hilbert space and $\{\mu_n\}$ be a sequence in H, such that there exists a nonempty subset $C \subset H$ satisfying the following conditions:*

(i) *for every $\mu \in C$, $\lim_{n\to\infty} \|\mu_n - \mu\|$ exists,*

(ii) *any weak cluster point of the sequence $\{\mu_n\}$ belongs to C.*

Then, there exists $w^* \in C$, such that $\{\mu_n\}$ converges weakly to w^*.

Lemma 3 *If x, y and z are positive real numbers, then the following inequality holds:*

$$x^2 + \frac{y^2}{z} \geq \frac{(x + y)^2}{1 + z}. \tag{3}$$

Theorem 2 (Mann 1953, Krasnosel'skii–Mann Theorem) *Let $M : H \to H$ be an averaged and assume $Fix(M) \neq \emptyset$. Then, for any starting point $x_0 \in H$, the sequence $\{M^k x_0\}$ converges weakly to a fixed point of M.*

3 Main results

In this section, first of all, we show with the help of counter example that the operators $P = \big[I + \lambda_1 (J_{\lambda_1}^{B_1} - f_1)\big]$ and $Q = \big[I + \lambda_2 (J_{\lambda_2}^{B_2} - f_2)\big]$ are neither averaged nor nonexpansive in general.

Example 1 Let $B_1 : R \to R$ be a maximal monotone mapping defined as $B_1(x) = 2x + 3$ and let $\lambda_1 = 1$, then:

$$R_{\lambda_1}^{B_1} = [I + \lambda_1 B_1]^{-1} = [I + B_1]^{-1} = \tfrac{x-3}{3}$$

and

$$J_{\lambda_1}^{B_1} = \frac{1}{\lambda_1}[I - R_{\lambda_1}^{B_1}] = \frac{2x+3}{3}.$$

Also, let $f_1 : R \to R$ be defined by $f_1(x) = \frac{x}{3}$. Then:

$$\begin{aligned}\langle f_1(x) - f_1(y),\ x - y\rangle &= \langle \tfrac{x}{3} - \tfrac{y}{3},\ x - y\rangle = \tfrac{1}{3}\|x - y\|^2\\ &\geq \tfrac{1}{3}\tfrac{1}{3}\|x - y\|^2 = \|f(x) - f(y)\|^2;\end{aligned}$$

that is, f_1 is 1-inverse strongly monotone. Then:

$$\lambda_1\Big[J_{\lambda_1}^{B_1} - f_1\Big](x) = \frac{x+3}{3},$$

and

$$P = [I + \lambda_1 (J_{\lambda_1}^{B_1} - f_1)](x) = \frac{4x+3}{3}.$$

We can see that the mappings B_1, f_1 and λ_1 all satisfy the conditions assumed in Theorem 3.1 of Rahman et al. (2018). From Lemma 1(vi), we know that P is averaged if and only if $I - P$ is t-ism for some $t > \frac{1}{2}$. Therefore:

$$\begin{aligned}\langle (I-P)x - (I-P)y,\ x - y\rangle &= \left\langle \frac{-x-3}{3} - \frac{-y-3}{3},\ x - y\right\rangle\\ &= \frac{-1}{3}\langle x - y, x - y\rangle = \frac{-1}{3}\|x - y\|^2 \end{aligned}\tag{4}$$

and

$$\|(I-P)x - (I-P)y\|^2 = \left\|\frac{-x-3}{3} - \frac{-y-3}{3}\right\|^2 = \frac{1}{9}\|x - y\|^2. \tag{5}$$

From (4) and (5), we conclude that:

$$\langle (I-P)x - (I-P)y,\ x - y\rangle \geq t\|(I-P)x - (I-P)y\|^2 \tag{6}$$

will never be satisfied for any $t > \frac{1}{2}$. This implies that P is not an averaged operator.

Furthermore:

$$\begin{aligned}\|Px - Py\| &\leq \left\| \frac{4x+3}{3} - \frac{4y+3}{3} \right\| \\ &= \frac{4}{3}\|x-y\| \\ &\geq \|x-y\|.\end{aligned}$$

This implies that P is also not a nonexpansive operator.

Therefore, the assumptions in Theorem 3.1 of Rahman et al. (2018) that $T = R_{\lambda_1}^{B_1} P = R_{\lambda_1}^{B_1}[I + \lambda_1(J_{\lambda_1}^{B_1} - f_1)]$ and $S = R_{\lambda_2}^{B_2} Q = R_{\lambda_2}^{B_2}[I + \lambda_2(J_{\lambda_2}^{B_2} - f_2)]$ are averaged, and hence, nonexpansive are wrong and, hence, the proof of Theorem 3.1 of Rahman et al. (2018) is not correct in present form.

Above example motivate us to modify the Rahman's split monotone Yosida variational inclusions in the following manner: find $x^* \in H_1$, such that:

$$0 \in B_1(x^*) + f_1(x^*) + J_{\lambda_1}^{B_1}(x^*) \tag{7}$$

$$\text{such that } Ax^* = y^* \in H_2 \text{ solves } 0 \in B_2(y^*) + f_2(y^*) + J_{\lambda_2}^{B_2}(y^*). \tag{8}$$

We called it Modified split Yosida variational inclusion problem (MSYVIP). Let $\Gamma = \{x \in H_1 \text{ solves (7)} : Ax \in H_2 \text{ solves (8)}\}$ be the solution set of MSYVIP (7)–(8).

The fixed point equivalence of MSYVIP (7)–(8) can be stated in the following lemma which can be proved easily using the definition of resolvents of B_1 and B_2.

Lemma 4 *$(x^*, y^*) \in H_1 \times H_2$ is a solution of MSYVIP (7)–(8) if and only if*

$$x^* = R_{\lambda_1}^{B_1}\big[I - \lambda_1(J_{\lambda_1}^{B_1} + f_1)\big](x^*),$$

and

$$y^* = R_{\lambda_2}^{B_2}\big[I - \lambda_2(J_{\lambda_2}^{B_2} + f_2)\big](y^*).$$

Lemma 5 *Let H_1 be a Hilbert space, $B_1 : H_1 \to 2^{H_1}$ be a maximal monotone operator, and $f_1 : H_1 \to H_1$ be a α_1-inverse strongly monotone operator. Let $J_{\lambda_1}^{B_1}$ be Yosida approximation operator of B_1 which is defined as $J_{\lambda_1}^{B_1} = \frac{1}{\lambda_1}(I - R_{\lambda_1}^{B_1})$, where $R_{\lambda_1}^{B_1} = (I + \lambda_1 B_1)^{-1}$ is the resolvent of B_1. If $0 < \lambda_1 < \alpha_1$, then $[I - \lambda_1(J_{\lambda_1}^{B_1} + f_1)]$ is nonexpansive operator.*

Proof It is given that f_1 is a α_1-inverse strongly monotone, then there exists $\alpha_1 > 0$, such that:

$$\langle f_1(x) - f_1(y),\ x - y\rangle \geq \alpha_1 \|f_1(x) - f_2(y)\|^2.$$

Since $R_{\lambda_1}^{B_1}$ of maximal monotone mapping, B_1 is 1-ism (firmly nonexpansive); therefore, by Lemma 1 (vii), $I - R_{\lambda_1}^{B_1}$ is also 1-ism (firmly nonexpansive). Therefore, by Lemma 1 (v), $\frac{1}{\lambda_1}(I - R_{\lambda_1}^{B_1})$ is λ_1-ism. That is, $J_{\lambda_1}^{B_1}$ is λ_1-ism. Now:

$$\begin{aligned}&\langle (J_{\lambda_1}^{B_1} + f_1)(x) - (J_{\lambda_1}^{B_1} + f_1)(y),\ x - y\rangle \\ &\quad = \langle J_{\lambda_1}^{B_1}(x) - J_{\lambda_1}^{B_1}(y),\ x - y\rangle + \langle f_1(x) - f_1(y),\ x - y\rangle \\ &\quad \geq \lambda_1 \|J_{\lambda_1}^{B_1}(x) - J_{\lambda_1}^{B_1}(y)\|^2 + \alpha_1 \|f_1(x) - f_1(y)\|^2 \\ &\quad \geq \min\{\lambda_1, \alpha_1\}\{\|J_{\lambda_1}^{B_1}(x) - J_{\lambda_1}^{B_1}(y)\|^2 + \|f_1(x) - f_1(y)\|^2\} \\ &\quad \geq \frac{\min\{\lambda_1, \alpha_1\}}{2}\{\|(J_{\lambda_1}^{B_1} + f_1)(x) - (J_{\lambda_1}^{B_1} + f_1)(y)\|^2\}.\end{aligned}$$

If $0 < \lambda_1 < \alpha_1$, then we see that $(J_{\lambda_1}^{B_1} + f_1)$ is $\frac{\lambda_1}{2}$-ism. This implies that by Lemma 1 (v), $\lambda_1(J_{\lambda_1}^{B_1} + f_1)$ is $\frac{1}{2}$-ism. Therefore, by Lemma 1 (iv), $[I - \lambda_1(J_{\lambda_1}^{B_1} + f_1)]$ is nonexpansive. □

Remark 1 Similarly, one can show with the assumption $0 < \lambda_2 < \alpha_2$ that the operator $[I - \lambda_2(J_{\lambda_2}^{B_2} + f_2)]$ is also nonexpansive.

Now, we are able to state the main convergence theorem to solve MSYVIP (7)–(8), with some new assumptions.

Theorem 3 *Let H_1 and H_2 be two real Hilbert spaces; $A : H_1 \to H_2$ be a bounded linear operator and A^* be the adjoint of A. Let $B_1 : H_1 \to 2^{H_1}$ and $B_2 : H_2 \to 2^{H_2}$ be multi-valued maximal monotone mappings; $f_1 : H_1 \to H_1$ be α_1-inverse strongly monotone and $f_2 : H_2 \to H_2$ be α_2-inverse strongly monotone mappings. Let $\{g_n\}$ be a family of k-contractive mapping and uniformly convergent for any $x \in K$, where K is any bounded subset of H_1, $f : H_1 \to H_1$ be a ξ-contraction mapping, D be a strongly positive bounded linear operator on H_1 with coefficient $\bar{r} > 0$ and $\beta \in (0, \frac{\bar{r}}{k})$. Assume that $\Omega \neq \emptyset$ and $\tau > 0$ and let the sequence $\{x_n\}$ be generated by the following iterative algorithm:*

$$\begin{aligned} x_0 &\in H_1, \\ u_n &= T[x_n + \gamma A^*(S - I)Ax_n], \\ v_n &= \delta_n u_n + \tau g_n(u_n), \\ x_{n+1} &= (1 - \alpha_n D)v_n + \alpha_n \beta f(v_n), \end{aligned}$$

where $T = R_{\lambda_1}^{B_1}\big[I - \lambda_1(J_{\lambda_1}^{B_1} + f_1)\big]$ and $S = R_{\lambda_2}^{B_2}\big[I - \lambda_2(J_{\lambda_2}^{B_2} + f_2)\big]$, $\gamma \in (0, \frac{1}{\|A\|^2})$, $\alpha_n \in [0, 1)$ and $\delta_n \in [0, 1)$. Suppose that the following conditions are satisfied:

(i) $0 < \lambda_1 < \alpha_1$ *and* $0 < \lambda_2 < \alpha_2$,
(ii) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=0}^{\infty} \alpha_n = \infty$, $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$,
(iii) $\lim_{n\to\infty} \delta_n = 0$, $\sum_{n=0}^{\infty} \delta_n = \infty$.

Then, the sequence $\{x_n\}$ converges strongly to $x^* \in \Omega$ which is unique solution of the variational inequality problem:

$$\langle (D - \beta f)x^*, x - x^* \rangle \geq 0, \ \forall\, x \in \Gamma.$$

If each Hilbert spaces possesses the Opial's property, then the sequence $\{x_n\}$ converges weakly to $x^* \in \Omega$.

Proof By Lemma 5, $\big[I - \lambda_1(J_{\lambda_1}^{B_1} + f_1)\big]$ and $\big[I - \lambda_2(J_{\lambda_2}^{B_2} + f_2)\big]$ are nonexpansive. Since the composition of nonexpansive mappings is nonexpansive, the operators $T = R_{\lambda_1}^{B_1}\big[I - \lambda_1(J_{\lambda_1}^{B_1} + f_1)\big]$ and $S = R_{\lambda_2}^{B_2}\big[I - \lambda_2(J_{\lambda_2}^{B_2} + f_2)\big]$ are nonexpansive. Then using T and S are nonexpansive, the rest of the proof can be done similarly as in Theorem 3.1 of Rahman et al. (2018). □

Now, we will state and prove two supporting lemmas which are used in weak convergence theorem of MSYVIP.

Lemma 6 *Let $B_1 : H_1 \to 2^{H_1}$, $B_2 : H_2 \to 2^{H_2}$ be maximal monotone mappings and $J_{\lambda_1}^{B_1}$, $J_{\lambda_2}^{B_2}$ be Yosida approximation operators of B_1 and B_2, respectively. Let $A : H_1 \to H_2$ be a bounded linear operator. Then, the operator $I - \theta L$ is firmly nonexpansive, where $L = [(I - T) + A^*(I - S)A]$, $\theta = \frac{1}{2(1+\|A\|^2)}$, and $T = R_{\lambda_1}^{B_1}[I - \lambda_1(J_{\lambda_1}^{B_1} + f_1)]$, $S = R_{\lambda_2}^{B_2}[I - \lambda_2(J_{\lambda_2}^{B_2} + f_2)]$, are same as described in Theorem 3.*

Proof Let $x, y \in H_1$, Then, using Lemma 1 (iv) and Lemma 3, we have:

$$\begin{aligned}
\langle x-y, Lx-Ly\rangle &= \langle x-y, [(I-T)+A^*(I-S)A]x-[(I-T)+A^*(I-S)A]y\rangle \\
&= \langle x-y, (I-T)x-(I-T)y\rangle + \langle Ax-Ay, (I-S)Ax-(I-S)Ay\rangle \\
&\geq \frac{1}{2}\|(I-T)x-(I-T)y\|^2 + \frac{1}{2}\|(I-S)Ax-(I-S)Ay\|^2 \\
&\geq \frac{1}{2}\|(I-T)x-(I-T)y\|^2 + \frac{1}{2}\frac{\|A^*(I-S)Ax-A^*(I-S)Ay\|^2}{\|A^*\|^2} \\
&\geq \frac{1}{2}\left\{\frac{\|[(I-T)+A^*(I-S)A]x-[(I-T)+A^*(I-S)A]y\|^2}{1+\|A\|^2}\right\} \\
&\geq \frac{1}{2(1+\|A\|^2)}\left\{\|Lx-Ly\|^2\right\} \\
&= \theta\|Lx-Ly\|^2;
\end{aligned}$$

that is, L is θ-ism, where $\theta = \frac{1}{2(1+\|A\|^2)}$. Then, by Lemma 1 (iii), $I-\theta L$ is firmly nonexpansive and, hence, averaged. □

Lemma 7 *Let* $\Gamma = \{x \in H_1 \text{ solves } (7) : Ax \in H_2 \text{ solves } (8)\} \neq \emptyset$, *then* $\mathrm{Fix}(I-\theta L) = \Gamma$.

Proof It can be easily seen that $\Gamma \subseteq \mathrm{Fix}(I-\theta L)$. To show that $\mathrm{Fix}(I-\theta L) \subset \Gamma$, let $u \in \mathrm{Fix}(I-\theta L)$ and $p \in \Gamma$. Thus, we have $Lu = 0$ and $Lp = 0$. Now:

$$\begin{aligned}
0 &= \langle u-p, Lu-Lp\rangle \\
&\geq \frac{1}{2}\|(I-T)u-(I-T)p\|^2 + \frac{1}{2}\|(I-S)Au-(I-S)Ap\|^2 \\
&= \frac{1}{2}\|(I-T)u\|^2 + \frac{1}{2}\|(I-S)Au\|^2 \geq 0;
\end{aligned}$$

this implies that $u \in \mathrm{Fix}(T)$ and $Au \in \mathrm{Fix}(S)$. Thus, $u \in \Gamma$. This completes the proof. □

Theorem 4 *Let H_1 and H_2 be two real Hilbert spaces; $A : H_1 \to H_2$ be a bounded linear operator and A^* be the adjoint of A. Let $B_1 : H_1 \to 2^{H_1}$ and $B_2 : H_2 \to 2^{H_2}$ be multi-valued maximal monotone mappings and $f_1 : H_1 \to H_1$ be α_1-inverse strongly monotone and $f_2 : H_2 \to H_2$ be α_2-inverse strongly monotone mappings. If $0 < \lambda_1 < \alpha_1$ and $0 < \lambda_2 < \alpha_2$, then, for any arbitrary point x_0, the sequence $\{(I-\theta L)x_0\}$ converges to a fixed point of $\{I-\theta L\}$, which is a solution of MSYVIP (7)–(8), where $L = [(I-T)+A^*(I-S)A]$, $T = R^{B_1}_{\lambda_1}[I-\lambda_1(J^{B_1}_{\lambda_1}+f_1)]$ and $S = R^{B_2}_{\lambda_2}[I-\lambda_2(J^{B_2}_{\lambda_2}+f_2)]$.*

Proof From Lemma 6, the operator $\{(I-\theta L)\}$ is firmly nonexpansive and, hence, averaged; therefore, Theorem 2 implies that the sequence generated by $\{(I-\theta L)x_0\}$ for an arbitrary $x_0 \in H_1$ converges weakly to fixed point of $\{(I-\theta L)\}$ which, by Lemma 7, is a solution of MSYVIP (7)–(8). □

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