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Positivity

Approximating a Feller Semigroup by Using the Yosida Approximation of the Symbol of its Generator

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Abstract. We show that if the pseudodifferential operator $-q(x,D)$ generates a Feller semigroup $(T_t)_{t\geq 0}$ then the Feller semigroups $\big(T_t^{(\nu)}\big)_{t\geq 0}$ generated by the pseudodifferential operators with symbol $-\frac{\nu q(x,\xi)}{\nu+q(x,\xi)}$ will converge strongly to $(T_t)_{t\geq 0}$ as $\nu \to \infty$.

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1. Introduction

Let $(T_t)_{t\geq 0}$ be a Feller semigroup on $C_\infty(\mathbb{R}^n)$, the space of all continuous functions vanishing at infinity, i.e. $(T_t)_{t\geq 0}$ is a strongly continuous contraction semigroup which is positivity preserving. Denote by $\big(A, D(A)\big)$, $D(A) \subset C_\infty(\mathbb{R}^n)$, the generator of $(T_t)_{t\geq 0}$. In case that $C_0^\infty(\mathbb{R}^n) \subset D(A)$ a result due to Ph. Courrège [1] states that A has on $C_0^\infty(\mathbb{R}^n)$ the representation as a pseudodifferential operator

$$Au(x) = -q(x,D)u(x) = -(2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{ix\cdot\xi} q(x,\xi)\widehat{u}(\xi)\,\mathrm{d}\xi \tag{1.1}$$

where $\widehat{u}$ is the Fourier transform of u and the symbol $q : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C}$ is measurable, locally bounded and for $x \in \mathbb{R}^n$ fixed, the function $q(x,\cdot) : \mathbb{R}^n \to \mathbb{C}$ is continuous and negative definite. Recall that a function $\psi : \mathbb{R}^n \to \mathbb{C}$ is negative definite if $\psi(0) \geq 0$ and

$$\xi \mapsto e^{-t\psi(\xi)} \quad \text{is positive definite for } t > 0. \tag{1.2}$$

Equivalently ψ is a continuous negative definite function if $\psi : \mathbb{R}^n \to \mathbb{C}$ is continuous and has the Lévy-Khinchin representation

$$\psi(\xi) = a + id \cdot \xi + Q(\xi) + \int_{\mathbb{R}^n \setminus \{0\}} \left(1 - e^{-i\langle y, \xi \rangle} - \frac{i\langle y, \xi\rangle}{1 + |y|^2}\right) \mu(\mathrm{d}y) \tag{1.3}$$

where $a \geq 0$, $d \in \mathbb{R}^n$, $Q(\xi) = \sum_{k,l=1}^n a_{kl}\xi_k\xi_l \geq 0$, $a_{kl} = a_{lk} \in \mathbb{R}$, and μ is a Borel measure on $\mathbb{R}^n \setminus \{0\}$ such that $\int_{\mathbb{R}^n \setminus \{0\}} \left(1 \wedge |y|^2\right) \mu(\mathrm{d}y) < \infty$. We call a pseudodifferential operator of type (1.1) a *pseudodifferential operator with negative definite symbol* if for all $x \in \mathbb{R}^n$ it holds that $q(x, \cdot) : \mathbb{R}^n \to \mathbb{C}$ is a continuous negative definite function. Clearly, it is of interest to solve the converse problem: Given a continuous negative definite symbol q and define $q(x, D)$ on $C_0^\infty(\mathbb{R}^n)$. When does $-q(x, D)$ extend to a generator of a Feller semigroup? Meanwhile many results in this direction are known. We refer to [10] and [11], and the references given therein. The most general results so far are due to W. Hoh obtained either by using a martingale problem approach [2] and [3] combined with [4], or by using an adapted symbolic calculus, see [5] and [7].

It is easy to see that under mild conditions $q(x, D)$ is a bounded operator on $C_\infty(\mathbb{R}^n)$ if $q(x, \xi)$ is bounded, i.e. $|q(x, \xi)| \leq M$ for all $x, \xi \in \mathbb{R}^n$. In this case $-q(x, D)$ will always generate a strongly continuous contraction semigroup on $C_\infty(\mathbb{R}^n)$ and the fact that $\xi \mapsto q(x, \xi)$ is a continuous negative definite function implies that this semigroup is indeed a Feller semigroup. We refer also to A. Potrykus [14] as well as to our joint paper [12] where for a bounded symbol the corresponding semigroup is constructed by adapting Roth's method to our situation, see J.-P. Roth [15]. This was done by using some ideas of E. Popescu [13]. So far we could not make Roth's method work for general pseudodifferential operators with a negative definite symbol. However, since the Yosida approximation of the symbol $-q(x, \xi)$, i.e. the symbol

$$-q^{(\nu)}(x, \xi) = -\frac{\nu q(x, \xi)}{\nu + q(x, \xi)}, \tag{1.4}$$

is bounded and negative definite, one may ask whether the corresponding Feller semigroups, i.e. the semigroups $\left(T_t^{(\nu)}\right)_{t \geq 0}$ generated by $-q^{(\nu)}(x, D)$, will converge to the semigroup $(T_t)_{t \geq 0}$ (in the strong sense for example) generated by $-q(x, D)$. This problem consists in fact of two subproblems:

1. Will the semigroups $\left(T_t^{(\nu)}\right)_{t \geq 0}$ converge (strongly) as ν tends to infinity to a semigroup $(T_t)_{t \geq 0}$?
2. In the affirmative case in i) will $-q(x, D)$ be the generator of the limiting semigroup?

So far we can not solve both problems, but as main result we will show, compare Theorem 3.4 and Corollary 3.5, that under suitable, but still quite general conditions on $q(x, \xi)$ it holds:

If $(T_t)_{t\geq 0}$ is generated by $-q(x,D)$ then for all $u \in C_\infty(\mathbb{R}^n)$ it holds

$$\lim_{\nu\to\infty} \left\| T_t^{(\nu)} u - T_t u \right\|_\infty = 0, \tag{1.5}$$

i.e. in the strong sense $(T_t)_{t\geq 0}$ is approximated in $C_\infty(\mathbb{R}^n)$ by the semigroups $\big(T_t^{(\nu)}\big)_{t\geq 0}$ where $T_t^{(\nu)} u = e^{-\left(\frac{\nu q}{\nu+q}\right)(x,D)} u$.

Note that this result emphasized once again the natural approach a symbolic calculus is for solving the construction problem:
By the Hille-Yosida theorem we know that the semigroups generated by the Yosida-approximations $\nu A R_\nu$ of a generator (or pre-generator) A converge strongly to the semigroup generated by A (or its extension). By our result we may substitute for the approximation result the operator $-\nu q(x,D)\big(\nu + q(x,D)\big)^{-1}$ by the much simpler operator $-\left(\frac{\nu q}{\nu+q}\right)(x,D)$.

2. Some Preparations

Let $\psi : \mathbb{R}^n \to \mathbb{R}$ be a fixed continuous negative definite function and for $s \in \mathbb{R}$ we set

$$\lambda^{s,\psi}(\xi) := \big(1+\psi(\xi)\big)^{\frac{s}{2}}. \tag{2.1}$$

We introduce the norm

$$\|u\|_{H^{s,\psi}} := \left\| \lambda^{s,\psi}(\cdot)\widehat{u} \right\|_{L^2}, \tag{2.2}$$

i.e.

$$\|u\|_{H^{s,\psi}}^2 = \int_{\mathbb{R}^n} \big(1+\psi(\xi)\big)^s \left|\widehat{u}(\xi)\right|^2 \, \mathrm{d}\xi, \tag{2.3}$$

and define the function spaces

$$H^{s,\psi}(\mathbb{R}^n) := \left\{ u \in \mathcal{S}'(\mathbb{R}^n);\ \|u\|_{H^{s,\psi}} < \infty \right\}, \tag{2.4}$$

where $\mathcal{S}'(\mathbb{R}^n)$ denotes the space of temperate distributions. The spaces $H^{s,\psi}(\mathbb{R}^n)$ are Hilbert spaces and many results for this scale of anisotropic Bessel potential spaces can be found in [9], Section 3.10. In particular we have

Theorem 2.1. *If for some $c > 0$ and $r > 0$ the continuous negative definite function $\psi : \mathbb{R}^n \to \mathbb{R}$ satisfies $\psi(\xi) \geq c|\xi|^r$ for $|\xi|$ large, then for $s > \frac{n}{r}$ the space $H^{s,\psi}(\mathbb{R}^n)$ is continuously embedded into $C_\infty(\mathbb{R}^n)$ and it holds*

$$\|u\|_\infty \leq c\|u\|_{s,\psi}. \tag{2.5}$$

All our considerations will take place in function spaces over $\mathbb{R}^n$ and therefore in the following we simply write L^2, $H^{s,\psi}$, C_∞, $\mathcal{S}'$, etc. for $L^2(\mathbb{R}^n)$, $H^{s,\psi}(\mathbb{R}^n)$, $C_\infty(\mathbb{R}^n)$, $\mathcal{S}'(\mathbb{R}^n)$, etc.

Since we want to use Hoh's symbolic calculus we need some further preparations, compare W. Hoh [5] and [6], or [10].

Definition 2.2. A. We say that a continuous negative definite function belongs to the class Λ if for all $\alpha \in \mathbb{N}_0^n$ it holds

$$\left|\partial_\xi^\alpha \left(1+\psi(\xi)\right)\right| \leq C_{|\alpha|}\left(1+\psi(\xi)\right)^{\frac{2-\rho(|\alpha|)}{2}} \tag{2.6}$$

where $\rho(k) = k \wedge 2$.
B. Let $m \in \mathbb{R}$ and $\psi \in \Lambda$. A C^∞-function $q : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C}$ is called a symbol in the class $S_\rho^{m,\psi}$ if for all $\alpha,\ \beta \in \mathbb{N}_0^n$

$$\left|\partial_\xi^\alpha \partial_x^\beta q(x,\xi)\right| \leq c_{\alpha\beta}\left(1+\psi(\xi)\right)^{\frac{m-\rho(|\alpha|)}{2}} \tag{2.7}$$

holds with some constants $c_{\alpha\beta}$.
C. If for $\psi \in \Lambda$ a C^∞-function $q : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C}$ satisfies for all $\alpha,\ \beta \in \mathbb{N}_0^n$ the estimates

$$\left|\partial_\xi^\alpha \partial_x^\beta q(x,\xi)\right| \leq \widetilde{c_{\alpha\beta}}\left(1+\psi(\xi)\right)^{\frac{m}{2}} \tag{2.8}$$

where $m \in \mathbb{R}$ and $\widetilde{c_{\alpha\beta}}$ are constants, then we call q an element in the class $S_0^{m,\psi}$.

Remark 2.3. In order to establish a full symbolic calculus one needs these smoothness assumptions. However as pointed out in [6] and [7] by W. Hoh there is a natural way to decompose a pseudodifferential operator $q(x,D)$ with a non-smooth negative definite symbol into a sum of two pseudodifferential operators where the first one has a symbol in $S_\rho^{2,\psi}$ (or $S_0^{2,\psi}$) and the second one is a bounded perturbation.

Next we introduce amplitudes:

Definition 2.4. For $m \geq 0$ we call $a : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C}$ an amplitude in the class A^m if for all $\alpha,\ \beta \in \mathbb{N}_0^n$ the estimates

$$\left|\partial_y^\alpha \partial_\eta^\beta a(y,\eta)\right| \leq c'_{\alpha\beta}\left(1+|y|^2\right)^{\frac{m}{2}}\left(1+|\eta|^2\right)^{\frac{m}{2}} \tag{2.9}$$

holds with some constants $c'_{\alpha\beta} \geq 0$.

The following result is quite standard.

Theorem 2.5. *Let $a \in A^m$ and $\chi \in \mathcal{S}(\mathbb{R}^n \times \mathbb{R}^n)$ such that $\chi(0,0) = 1$. Then the limit*

$$\lim_{\varepsilon\to 0}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{-i\langle y,\eta\rangle} a(y,\eta)\chi(\varepsilon y,\varepsilon\eta)\,\mathrm{d}y\,\mathrm{d}\eta$$

exists, is independent of χ, and is denoted by $\mathrm{Os}\text{–}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{-i\langle y,\eta\rangle}a(y,\eta)\,\mathrm{d}y\,\mathrm{d}\eta$. *One has the estimate*

$$\left|\mathrm{Os}\text{–}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{-i\langle y,\eta\rangle} a(y,\eta)\,\mathrm{d}y\,\mathrm{d}\eta\right| \leq C_m |\!|\!| a |\!|\!|_{m+2n+1}, \tag{2.10}$$

where

$$|\!|\!| a |\!|\!|_k = \max_{|\alpha|\leq k}\sup_{y,\eta\in\mathbb{R}^n}\left|\left(1+|y|^2\right)^{-\frac{m}{2}}\left(1+|\eta|^2\right)^{-\frac{m}{2}}\partial_x^\alpha\partial_\xi^\beta a(y,\eta)\right|.$$

Note that if $q \in S_0^{m,\psi}$ then q belongs to A^m. The next result assures the L^2-continuity of certain pseudodifferential operators, we refer to I. L. Hwang [8].

Theorem 2.6. *Let $q \in C^{2n}(\mathbb{R}^n)$ be such that*

$$|\partial_x^\alpha \partial_\xi^\beta q(x,\xi)| \le C_{\alpha\beta}$$

for all $\alpha, \beta \in \{0,1\}^n$. Then the pseudodifferential operator $q(x,D)$ satisfies

$$\|q(x,D)u\|_{L^2} \le 2^{-\frac{3}{4}n}\, \pi^{\frac{3}{4}n} \sum_{\alpha,\beta,\gamma \in \{0,1\}^n} \binom{\gamma}{\alpha} C_{\alpha\beta} \|u\|_{L^2}. \tag{2.11}$$

For $q_1 \in S_0^{m_1,\psi}$ and $q_2 \in S_0^{m_2,\psi}$ we may consider the composition of the corresponding pseudodifferential operators $q_1(x,D)$ and $q_2(x,D)$. For the symbol $q_1 \# q_2$ of the operator $q_1(x,D) \circ q_2(x,D)$ we find

$$(q_1\#q_2)(x,\xi) = (2\pi)^{-n}\, \mathrm{Os}\!-\!\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{-i\langle y,\eta\rangle} q_1(x,\xi-\eta) q_2(x-y,\xi)\, \mathrm{d}y\, \mathrm{d}\eta. \tag{2.12}$$

In case of $q_1 \in S_\rho^{m_1,\psi}$ and $q_2 \in S_\rho^{m_2,\psi}$ we have in addition

$$(q_1\#q_2)(x,\xi) = q_1(x,\xi)q_2(x,\xi) + \sum_{j=1}^n \partial_{\xi_j} q_1(x,\xi) D_{x_j} q_2(x,\xi) + q_{r_2}(x,\xi) \tag{2.13}$$

where $q_{r_2} \in S_0^{m_1+m_2-2,\psi}$ and $D_{x_j} = -i\partial_{x_j}$.

3. Approximating Feller Semigroups

In this section we will always assume

$$\psi \text{ is of class } \Lambda, \tag{3.1}$$

and

$$\psi(\xi) \ge c|\xi|^r \text{ for some } c > 0,\ r > 0 \text{ and large } |\xi|. \tag{3.2}$$

Further for $q \in S_0^{m,\psi}$, $m \in \mathbb{R}$, ψ satisfying (3.1) and (3.2), we will require

$$q(x,\xi) \ge K\lambda^{m,\psi}(\xi) \tag{3.3}$$

for all $x, \xi \in \mathbb{R}^n$ with some $K > 0$. In addition when we want to assure to work with a Feller semigroup generated by $-q(x,D)$ we need to assume that $\xi \mapsto q(x,\xi)$ is a continuous negative definite function.

For $\nu > 0$ we introduce the symbols

$$q^{(\nu)} := \frac{\nu q}{\nu + q}, \tag{3.4}$$

which yields

$$q - q^{(\nu)} = \frac{q^2}{\nu + q}. \tag{3.5}$$

Clearly we have $q^{(\nu)} \in S_0^{0,\psi}$ and $q - q^{(\nu)} \in S_0^{m,\psi}$.

Theorem 3.1. *Let ψ be a continuous negative definite function satisfying* (3.1) *and* (3.2). *Moreover assume $q \in S_0^{m,\psi}$, $m > 0$, satisfying* (3.3) *is given.*
A. For every $s \in \mathbb{R}$ there exists a constant C_s' such that $\left(q - q^{(\nu)}\right)(x, D)u \in H^{s,\psi}$ for all $u \in H^{m+s,\psi}$ and

$$\left\| \left(q - q^{(\nu)}\right)(x, D)u \right\|_{H^{s,\psi}} \leq C_s' \|u\|_{H^{m+s,\psi}} \tag{3.6}$$

holds, i.e. the operator $\left(q - q^{(\nu)}\right)(x, D)$ maps the space $H^{m+s,\psi}$ continuously into the space $H^{s,\psi}$.
B. For every $s \in \mathbb{R}$ there exists a constant C_s independent of ν such that $\left(q - q^{(\nu)}\right)(x, D)u \in H^{s,\psi}$ for all $u \in H^{2m+s,\psi}$ and

$$\left\| \left(q - q^{(\nu)}\right)(x, D)u \right\|_{H^{s,\psi}} \leq \frac{1}{\nu} C_s \|u\|_{H^{2m+s,\psi}}.$$

Proof. A. This part of the theorem follows from the general calculus for operators with symbols in $S_0^{m,\psi}$, compare [10], Theorem 2.5.4.
B. We follow a standard technique used in the theory of pseudodifferential operators and reduce the problem to estimate an operator of order zero. In fact we have to deal with a family $r^{(\nu)}(x, D)$, $\nu \in \mathbb{N}$, of operators, and for their symbols we eventually prove the estimate

$$\left|\partial_x^\alpha \partial_\xi^\beta r^{(\nu)}(x, \xi)\right| \leq \frac{1}{\nu} C_{\alpha\beta}^{'}.$$

Since not all readers of this journal or all mathematicians being interested in Feller semigroups or Feller processes are used to techniques from the theory of pseudodifferential operators we prefer to give the proof very detailed. Define

$$r^{(\nu)} := \lambda^{s,\psi} \# \left(q - q^{(\nu)}\right) \# \lambda^{-2m-s,\psi}.$$

As $r^{(\nu)} \in S_0^{0,\psi}$ we may use Theorem 2.6 to find

$$\begin{aligned}
\left\| \left(q - q^{(\nu)}\right)(x, D)u \right\|_{H^{s,\psi}} &= \left\| \lambda^{s,\psi}(D) \left(q - q^{(\nu)}\right)(x, D)u \right\|_{L^2} \\
&= \left\| \lambda^{s,\psi}(D) \left(q - q^{(\nu)}\right)(x, D) \lambda^{-2m-s,\psi}(D) \lambda^{2m+s,\psi}(D)u \right\|_{L^2} \\
&\leq \left\| r^{(\nu)}(x, D) \right\|_{L^2 \to L^2} \|u\|_{H^{2m+s,\psi}}.
\end{aligned}$$

Now it remains to show that

$$\left\| r^{(\nu)}(x, D)u \right\|_{L^2} \leq \frac{1}{\nu} C_s \|u\|_{L^2}.$$

In principle this is just an application of Theorem 2.6, but we are particularly interested in the $\frac{1}{\nu}$-decay of the constant

$$2^{-\frac{3}{4}n} \pi^{\frac{3}{4}n} \sum_{\alpha,\beta,\gamma \in \{0,1\}^n} \binom{\gamma}{\alpha} C_{\alpha\beta}$$

compare (2.11). In the above expression we want to explicitly calculate each constant $C_{\alpha\beta}$. Going back to Theorem 2.6, we find that this constant is derived by estimating $\partial_x^\alpha \partial_\xi^\beta r^{(\nu)}(x,\xi)$. We start by writing

$$r^{(\nu)} = \lambda^{s,\psi} \# \left(q - q^{(\nu)}\right) \# \lambda^{-2m-s,\psi} = \lambda^{s,\psi} \# \left(\left(q - q^{(\nu)}\right) \# \lambda^{-2m-s,\psi}\right)$$

and it follows that

$$\left(\left(q - q^{(\nu)}\right) \# \lambda^{-2m-s,\psi}\right)(x,\xi) = \left(q - q^{(\nu)}\right)(x,\xi)\lambda^{-2m-s,\psi}(\xi).$$

Since for $q_1, q_2 \in S_0^{\infty,\psi} = \bigcup\limits_m S_0^{m,\psi}$

$$\begin{aligned} &\partial_x^\alpha \partial_\xi^\beta \left(q_1 \# q_2\right)(x,\xi) \\ &= \sum_{\beta_1 \le \beta} \sum_{\alpha_1 \le \alpha} \binom{\beta}{\beta_1} \binom{\alpha}{\alpha_1} \left(\left(\partial_x^{\alpha_1} \partial_\xi^{\beta_1} q_1\right) \# \left(\partial_x^{\alpha-\alpha_1} \partial_\xi^{\beta-\beta_1} q_2\right)\right)(x,\xi), \end{aligned}$$

we get

$$\begin{aligned} &\left|\partial_x^\alpha \partial_\xi^\beta r^{(\nu)}(x,\xi)\right| \\ &\le \sum_{\beta_1 \le \beta} \sum_{\alpha_1 \le \alpha} \binom{\beta}{\beta_1} \binom{\alpha}{\alpha_1} \left| \mathrm{Os-}\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{-i\langle y,\eta\rangle} \left(\partial_x^{\alpha_1} \partial_\xi^{\beta_1} \lambda^{s,\psi}(\xi-\eta)\right) \right. \\ &\left. \cdot \left(\partial_x^{\alpha-\alpha_1} \partial_\xi^{\beta-\beta_1} \left(q - q^{(\nu)}\right)(x-y,\xi)\lambda^{-2m-s,\psi}(\xi)\right) \mathrm{d}y\,\mathrm{d}\eta \right|. \end{aligned}$$

Set

$$c_{x,\xi}(y,\eta) := \left(\partial_x^{\alpha_1} \partial_\xi^{\beta_1} \lambda^{s,\psi}(\xi-\eta)\right)\left(\partial_x^{\alpha-\alpha_1} \partial_\xi^{\beta-\beta_1} \left(q - q^{(\nu)}\right)(x-y,\xi)\lambda^{-2m-s,\psi}(\xi)\right);$$

we want to check that $c_{x,\xi}(y,\eta)$ is an amplitude and then we use estimate (2.10) with the oscillatory integral above. We need the estimate

$$\begin{aligned} &\left|\partial_y^\gamma \partial_\eta^\delta c_{x,\xi}(y,\eta)\right| \\ &\le \sum_{\delta_1 \le \delta} \sum_{\gamma_1 \le \gamma} \binom{\delta}{\delta_1} \binom{\gamma}{\gamma_1} \left|\partial_y^{\gamma_1} \partial_\eta^{\delta_1} \partial_x^{\alpha_1} \partial_\xi^{\beta_1} \lambda^{s,\psi}(\xi-\eta)\right| \\ &\cdot \left|\partial_y^{\gamma-\gamma_1} \partial_\eta^{\delta-\delta_1} \partial_x^{\alpha-\alpha_1} \partial_\xi^{\beta-\beta_1} \left(q - q^{(\nu)}\right)(x-y,\xi)\lambda^{-2m-s,\psi}(\xi)\right|. \end{aligned}$$

It is easy to see that

$$\left|\partial_y^{\gamma_1} \partial_\eta^{\delta_1} \partial_x^{\alpha_1} \partial_\xi^{\beta_1} \lambda^{s,\psi}(\xi-\eta)\right| \le C_1 \lambda^{s-|\beta_1|-|\delta_1|,\psi}(\xi-\eta) \le C_1 \lambda^{s,\psi}(\xi-\eta). \tag{3.7}$$

For the second factor we get

$$\begin{aligned}
&\left|\partial_y^{\gamma-\gamma_1}\partial_\eta^{\delta-\delta_1}\partial_x^{\alpha-\alpha_1}\partial_\xi^{\beta-\beta_1}\left(q-q^{(\nu)}\right)(x-y,\xi)\lambda^{-2m-s,\psi}(\xi)\right|\\
&\leq \sum_{\beta_2\leq\beta-\beta_1}\binom{\beta-\beta_1}{\beta_2}\left|\partial_\xi^{\beta_2}\lambda^{-2m-s,\psi}(\xi)\right|\\
&\cdot\left|\partial_y^{\gamma-\gamma_1}\partial_\eta^{\delta-\delta_1}\partial_x^{\alpha-\alpha_1}\partial_\xi^{\beta-\beta_1-\beta_2}\left(q-q^{(\nu)}\right)(x-y,\xi)\right|.
\end{aligned}$$

Again we look at both factors and get that

$$\left|\partial_\xi^{\beta_2}\lambda^{-2m-s,\psi}(\xi)\right|\leq C_2\lambda^{-2m-s-|\beta_2|,\psi}(\xi)\leq C_2\lambda^{-2m-s,\psi}(\xi), \tag{3.8}$$

as well as

$$\begin{aligned}
&\left|\partial_y^{\gamma-\gamma_1}\partial_\eta^{\delta-\delta_1}\partial_x^{\alpha-\alpha_1}\partial_\xi^{\beta-\beta_1-\beta_2}\left(q-q^{(\nu)}\right)(x-y,\xi)\right|\\
&\leq \sum_{\beta_3\leq\beta-\beta_1-\beta_2}\;\sum_{\alpha_2\leq\alpha-\alpha_1}\;\sum_{\delta_2\leq\delta-\delta_1}\;\sum_{\gamma_2\leq\gamma-\gamma_1}\binom{\beta-\beta_1-\beta_2}{\beta_3}\binom{\alpha-\alpha_1}{\alpha_2}\binom{\delta-\delta_1}{\delta_2}\\
&\quad\cdot\binom{\gamma-\gamma_1}{\gamma_2}\left|\partial_y^{\gamma_2}\partial_\eta^{\delta_2}\partial_x^{\alpha_2}\partial_\xi^{\beta_3}q(x-y,\xi)^2\right|\\
&\quad\cdot\left|\partial_y^{\gamma-\gamma_1-\gamma_2}\partial_\eta^{\delta-\delta_1-\delta_2}\partial_x^{\alpha-\alpha_1-\alpha_2}\partial_\xi^{\beta-\beta_1-\beta_2-\beta_3}\frac{1}{\nu+q(x-y,\xi)}\right|.
\end{aligned}$$

Note further that

$$\left|\partial_y^{\gamma_2}\partial_\eta^{\delta_2}\partial_x^{\alpha_2}\partial_\xi^{\beta_3}q(x-y,\xi)^2\right|\leq C_3\lambda^{2m,\psi}(\xi) \tag{3.9}$$

as $q^2\in S_0^{2m,\psi}$ for $q\in S_0^{m,\psi}$. Now define the function $f^{(\nu)}:]0,\infty[\to\mathbb{R}$ by

$$f^{(\nu)}(x)=\frac{1}{\nu+x},$$

and note that $f^{(\nu)}$ is a Bernstein function. Thus we may use estimate (3.285) in [9] to find

$$\left|\left(f^{(\nu)}\right)^{(j)}(x)\right|\leq\frac{j!}{x^j}f(x). \tag{3.10}$$

Then for $\vartheta = (\gamma-\gamma_1-\gamma_2, \delta-\delta_1-\delta_2, \alpha-\alpha_1-\alpha_2, \beta-\beta_1-\beta_2) \in \mathbb{N}_0^{4n}$, we get using (3.3) and (3.10),

$$
\begin{aligned}
&\left|\partial_y^{\gamma-\gamma_1-\gamma_2}\partial_\eta^{\delta-\delta_1-\delta_2}\partial_x^{\alpha-\alpha_1-\alpha_2}\partial_\xi^{\beta-\beta_1-\beta_2}\frac{1}{\nu+q(x-y,\xi)}\right| \\
&\le\ C_4\sum_{j=1}^{|\vartheta|}\left|\left(f^{(\nu)}\right)^{(j)}\left(q(x-y,\xi)\right)\right| \\
&\cdot \sum_{\substack{A_1+\ldots+A_j=\gamma-\gamma_1-\gamma_2\\ B_1+\ldots+B_j=\delta-\delta_1-\delta_2\\ C_1+\ldots+C_j=\alpha-\alpha_1-\alpha_2\\ D_1+\ldots D_j=\beta-\beta_1-\beta_2-\beta_3}}\prod_{l=1}^{j}\left|\partial_y^{A_l}\partial_\eta^{B_l}\partial_x^{C_l}\partial_\xi^{D_l}q(x-y,\xi)\right| \\
&\le\ C_4\sum_{j=1}^{|\vartheta|}\frac{j!}{q(x-y,\xi)^j}\cdot\frac{1}{\nu+q(x-y,\xi)} \\
&\cdot \sum_{\substack{A_1+\ldots+A_j=\gamma-\gamma_1-\gamma_2\\ B_1+\ldots+B_j=\delta-\delta_1-\delta_2\\ C_1+\ldots+C_j=\alpha-\alpha_1-\alpha_2\\ D_1+\ldots D_j=\beta-\beta_1-\beta_2-\beta_3}}\prod_{l=1}^{j}\left|\partial_y^{A_l}\partial_\eta^{B_l}\partial_x^{C_l}\partial_\xi^{D_l}q(x-y,\xi)\right| \qquad (3.11)\\
&\le\ C_4\sum_{j=1}^{|\vartheta|}j!\frac{1}{\nu+q(x-y,\xi)}\sum_{\substack{A_1+\ldots+A_j=\gamma-\gamma_1-\gamma_2\\ B_1+\ldots+B_j=\delta-\delta_1-\delta_2\\ C_1+\ldots+C_j=\alpha-\alpha_1-\alpha_2\\ D_1+\ldots D_j=\beta-\beta_1-\beta_2-\beta_3}}\prod_{l=1}^{j}\frac{\left|\partial_y^{A_l}\partial_\eta^{B_l}\partial_x^{C_l}\partial_\xi^{D_l}q(x-y,\xi)\right|}{q(x-y,\xi)} \\
&\le\ C_4\sum_{j=1}^{|\vartheta|}j!\frac{1}{\nu}\sum_{\substack{A_1+\ldots+A_j=\gamma-\gamma_1-\gamma_2\\ B_1+\ldots+B_j=\delta-\delta_1-\delta_2\\ C_1+\ldots+C_j=\alpha-\alpha_1-\alpha_2\\ D_1+\ldots D_j=\beta-\beta_1-\beta_2-\beta_3}}\prod_{l=1}^{j}\frac{C_{B_lD_l}\lambda^{m,\psi}(\xi)}{K\lambda^{m,\psi}(\xi)} \\
&\le\ \frac{1}{\nu}C_5.
\end{aligned}
$$

Collecting estimates (3.7), (3.8), (3.9), (3.11), and using Peetre's inequality, we finally end up with

$$
\begin{aligned}
\left|\partial_y^\gamma\partial_\eta^\delta c_{x,\xi}(y,\eta)\right| &\le \frac{1}{\nu}C_6\,\lambda^{s,\psi}(\xi-\eta)\lambda^{-2m-s,\psi}(\xi)\lambda^{2m,\psi}(\xi) \\
&\le \frac{1}{\nu}C_6\,2^{|s|}\lambda^{|s|,\psi}(\eta) \\
&\le \frac{1}{\nu}C_6\,2^{|s|}\left(1+|\eta|^2\right)^{\frac{|s|}{2}}\left(1+|y|^2\right)^{\frac{|s|}{2}}.
\end{aligned}
$$

Now, following Theorem 2.5 we get

$$
\begin{aligned}
|||c_{x,\xi}|||_{|s|+2n+1} &= \max_{|y+\delta|\leq|s|+2n+1}\ \sup_{y,\eta\in\mathbb{R}^n} \left| \left(1+|\eta|^2\right)^{-\frac{|s|}{2}} \left(1+|y|^2\right)^{-\frac{|s|}{2}} \partial_y^\gamma \partial_\eta^\delta c_{x,\xi}(y,\eta) \right| \\
&\leq \frac{1}{\nu}\, C_7.
\end{aligned}
$$

By estimate (2.10) this yields

$$
\begin{aligned}
&\left|\partial_x^\alpha \partial_\xi^\beta r^{(\nu)}(x,\xi)\right| \\
&\leq \sum_{\beta_1\leq\beta}\sum_{\alpha_1\leq\alpha} \binom{\beta}{\beta_1}\binom{\alpha}{\alpha_1}(2\pi)^{-n} \left| \mathrm{Os}\!-\!\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{-i\langle y,\eta\rangle} c_{x,\xi}(y,\eta)\,\mathrm{d}y\,\mathrm{d}\eta \right| \\
&\leq \sum_{\beta_1\leq\beta}\sum_{\alpha_1\leq\alpha} \binom{\beta}{\beta_1}\binom{\alpha}{\alpha_1} C_8\, |||c_{x,\xi}|||_{|s|+2n+1} \\
&\leq \frac{1}{\nu}\, C'_{\alpha\beta}.
\end{aligned}
$$

Now we use this result together with (2.11), and conclude for the operator $r^{(\nu)}(x,D)$ that

$$
\begin{aligned}
\left\| r^{(\nu)}(x,D)u \right\|_{L^2} &\leq 2^{-\frac{3}{4}n}\,\pi^{\frac{3}{4}n} \sum_{\alpha,\beta,\gamma\in\{0,1\}^n} \binom{\gamma}{\alpha} \frac{1}{\nu}\, C'_{\alpha\beta}\, \|u\|_{L^2} \\
&\leq \frac{1}{\nu}\, C_s\, \|u\|_{L^2}.
\end{aligned}
$$

This proves the theorem. □

Remark 3.2. With some more effort, compare [14], one can prove that the constant C'_s in (3.6) can be chosen independent of ν.

For our main result we need the following Theorem which can be found in [10].

Theorem 3.3. *Let ψ be a continuous negative definite function satisfying* (3.1), (3.2) *as well as* $\lim_{|\xi|\to\infty}\psi(\xi)=\infty$. *Moreover assume $q\in S_\rho^{m,\psi}$, $m>0$, satisfying* (3.3) *is given and real-valued. Then for $s>-m$ we have*

$$
\frac{K^2}{4}\|u\|^2_{H^{m+s,\psi}} \leq \|q(x,D)u\|^2_{H^{s,\psi}} + C\|u\|^2_{H^{m+s-\frac{1}{2},\psi}}
$$

for all $u\in H^{m+s,\psi}$.

By standard methods we deduce from Theorem 3.3

Lemma 3.4. *Let ψ be a continuous negative definite function satisfying* (3.1) *and* (3.2). *Moreover assume $q\in S_\rho^{m,\psi}$, $m>0$, satisfying* (3.3) *is given. Then for all $l>\frac{1}{2}$ and $k\in\mathbb{N}$ with $l-km\leq 0$ we have for $u\in H^{l,\psi}$*

$$
\|u\|_{H^{l,\psi}} \leq C_l \left\| q(x,D)^k u \right\|_{L^2} + C'_l\|u\|_{L^2}.
$$

Finally we arrive at our main results:

Theorem 3.5. *Let ψ be a continuous negative definite function satisfying* (3.1) *and* (3.2)*. Moreover assume $q \in S_\rho^{m,\psi}$, $m > 0$, satisfying* (3.3) *is given. Assume also that the pseudodifferential operator $-q(x,D)$ extends to the generator of a Feller semigroup $(T_t)_{t\geq 0}$. Then if $s > \max\{\frac{n}{r}, \frac{1}{2}\}$, $k \in \mathbb{N}$ such that $2m + s - km \leq 0$ we have for $u \in H^{km,\psi}$*

$$\left\| e^{-tq^{(\nu)}(x,D)} u - e^{-tq(x,D)} u \right\|_\infty \leq \frac{1}{\nu} C \|u\|_{H^{km,\psi}}.$$

(Here and in the following we use the formal notation $e^{-tq(x,D)}$ for T_t.)

Proof. Let us note that by our assumptions it follows from Theorem 2.1 that $H^{s,\psi} \hookrightarrow C_\infty$ and further we have $H^{km,\psi} \subset H^{s,\psi}$. For $u \in H^{km,\psi}$ we may write

$$\begin{aligned}
&\left\| e^{-tq^{(\nu)}(x,D)} u - e^{-tq(x,D)} u \right\|_\infty \\
&\leq \int_0^t \left\| e^{-tq^{(\nu)}(x,D)} \right\|_{L^\infty \to L^\infty} \left\| \left(q^{(\nu)}(x,D) - q(x,D) \right) e^{-(t-r)q(x,D)} u \right\|_\infty \mathrm{d}r \\
&\leq \int_0^t C_1 \left\| \left(q^{(\nu)}(x,D) - q(x,D) \right) e^{-(t-r)q(x,D)} u \right\|_\infty \mathrm{d}r.
\end{aligned}$$

Then, using Theorem 3.1 and Lemma 3.4, we find

$$\begin{aligned}
&\left\| \left(q^{(\nu)}(x,D) - q(x,D) \right) e^{(t-r)q(x,D)} u \right\|_\infty \\
&\leq C_2 \left\| \left(q^{\nu}(x,D) - q(x,D) \right) e^{-(t-r)q(x,D)} u \right\|_{H^{s,\psi}} \\
&\leq C_2 \frac{1}{\nu} C_3 \left\| e^{-(t-r)q(x,D)} u \right\|_{H^{2m+s,\psi}} \\
&\leq \frac{1}{\nu} C_4 \left(\left\| q(x,D)^k e^{-(t-r)q(x,D)} u \right\|_{L^2} + \left\| e^{-(t-r)q(x,D)} u \right\|_{L^2} \right) \\
&= \frac{1}{\nu} C_4 \left(\left\| e^{-(t-r)q(x,D)} q(x,D)^k u \right\|_{L^2} + \left\| e^{-(t-r)q(x,D)} u \right\|_{L^2} \right).
\end{aligned}$$

As $H^{km,\psi} \subset L^2$ it follows

$$\begin{aligned}
&\frac{1}{\nu} C_4 \left(\left\| e^{-(t-r)q(x,D)} q(x,D)^k u \right\|_{L^2} + \left\| e^{-(t-r)q(x,D)} u \right\|_{L^2} \right) \\
&\leq \frac{1}{\nu} C_4 \left\| e^{-(t-r)q(x,D)} \right\|_{L^2 \to L^2} \left(\|q(x,D)^k u\|_{L^2} + \|u\|_{L^2} \right) \\
&\leq \frac{1}{\nu} C_5 \left(\left\| q(x,D)^k u \right\|_{L^2} + \|u\|_{L^2} \right) \\
&\leq \frac{1}{\nu} C \|u\|_{H^{km,\psi}}.
\end{aligned}$$

□

Corollary 3.6. *In the situation of Theorem* 3.5 *we get for $u \in C_\infty$*

$$\lim_{\nu \to \infty} \left\| e^{-tq^{(\nu)}(x,D)} u - e^{-tq(x,D)} u \right\|_\infty = 0.$$

Proof. As $H^{km,\psi} \subset C_\infty$ is dense, we find for every $u \in C_\infty$ a sequence $(u_\mu)_{\mu\in\mathbb{N}}$, $u_\mu \in H^{km,\psi}$, such that $\lim_{\mu\to\infty} \|u - u_\mu\|_\infty = 0$. Then it follows from Theorem 3.5

$$\begin{aligned}
&\left\| e^{-tq^{(\nu)}(x,D)}u - e^{-tq(x,D)}u \right\|_\infty \\
&= \left\| e^{-tq^{(\nu)}(x,D)}u - e^{-tq^{(\nu)}(x,D)}u_\mu + e^{-tq^{(\nu)}(x,D)}u_\mu - e^{-tq(x,D)}u_\mu \right. \\
&\quad \left. + e^{-tq(x,D)}u_\mu - e^{-tq(x,D)}u \right\|_\infty \\
&\le \left\| e^{-q^{(\nu)}(x,D)}u - e^{-tq^{(\nu)}(x,D)}u_\mu \right\|_\infty + \left\| e^{-tq^{(\nu)}(x,D)}u_\mu - e^{-tq(x,D)}u_\mu \right\|_\infty \\
&\quad + \left\| e^{-tq(x,D)}u_\mu - e^{-tq(x,D)}u \right\|_\infty \\
&\le \left\| e^{-tq^{(\nu)}(x,D)} \right\|_{L^\infty\to L^\infty} \|u - u_\mu\|_\infty + \frac{1}{\nu} C \|u_\mu\|_{H^{km,\psi}} \\
&\quad + \left\| e^{-tq(x,D)} \right\|_{L^\infty\to L^\infty} \|u - u_\mu\|_\infty \\
&\le \|u - u_\mu\|_\infty + \frac{1}{\nu} C \|u_\mu\|_{H^{km,\psi}} + \|u - u_\mu\|_\infty.
\end{aligned}$$

When taking the limit we get for $\mu \in \mathbb{N}$ fixed

$$\limsup_{\nu\to\infty} \left\| e^{-tq^{(\nu)}(x,D)}u - e^{-tq(x,D)}u \right\|_\infty \le 2\|u - u_\mu\|_\infty.$$

and we conclude

$$\lim_{\nu\to\infty} \left\| e^{-tq^{(\nu)}(x,D)}u - e^{-tq(x,D)}u \right\|_\infty \le \lim_{\mu\to\infty} 2\|u - u_\mu\|_\infty \le 0. \qquad \square$$

Remark 3.7. The calculations leading to Theorem 3.5 and Corollary 3.6 are essentially taken from A. Potrykus [14], where in addition a detailed proof of Lemma 3.4 is given.

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