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Yosida approximation equations technique for system of generalized set-valued variational inclusions

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Abstract

In this paper, under the assumption with no continuousness, a new system of generalized variational inclusions in the Banach space is introduced. By using the Yosida approximation operator technique, the existence and uniqueness theorems for solving this kind of variational inclusion are established.

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Keywords: system of generalized variational inclusions; *m*-accretive mapping; resolvent operators; approximation operators

1 Introduction

Variational inclusions are useful and important extensions and generalizations of the variational inequalities with a wide range of applications in industry, mathematical finance, economics, decisions sciences, ecology, mathematical and engineering sciences. In general, the method based on the resolvent operator technique has been widely used to solve variational inclusions.

In this paper, under the assumption with no continuousness, we first introduce a new system of generalized variational inclusions in the Banach space. By using the Yosida approximation technique for m-accretive operator, we prove some existence and uniqueness theorems of solutions for this kind of system of generalized variational inclusions. Our results generalize and improve main results in [1–7].

For $i = 1, 2$, let E_i be a real Banach space, let $T_i : E_i \to 2^{E_i}$, $M_i : E_1 \times E_2 \to 2^{E_i}$ be set-valued mappings, let $h_i, g_i : E_i \to E_i$, $F_i : E_1 \times E_2 \to E_i$ be single-valued mappings, and let $(f_1, f_2) \in E_1 \times E_2$. We consider the following problem: finding $(x, y) \in E_1 \times E_2$ such that

$$\begin{cases} f_1 \in T_1x + F_1(x,y) + M_1(h_1(x), g_1(x)); \\ f_2 \in T_2y + F_2(x,y) + M_2(h_2(y), g_2(y)). \end{cases} \tag{1.1}$$

This problem is called the system of generalized set-valued variational inclusions.

There are some special cases in literature.

(1) If $T_1 = 0$, $T_2 = 0$, $f_1 = 0$, $f_2 = 0$, then (1.1) reduces to the problem of finding $(x,y) \in E_1 \times E_2$ such that

$$\begin{cases} 0 \in F_1(x,y) + M_1(h_1(x), g_1(x)); \\ 0 \in F_2(x,y) + M_2(h_2(y), g_2(y)). \end{cases} \tag{1.2}$$

Problem (1.2) was introduced and studied by Kazmi and Khan [1, 2] ($g_1 = g_2 = I$ in [2]).

(2) If $h_i = g_i = I$ is the identity operator, $M_i(\cdot,\cdot) = 0$, $f_1 = f_2 = 0$, then (1.1) reduces to the problem of finding $(x,y) \in E_1 \times E_2$ such that

$$\begin{cases} 0 \in T_1 x + F_1(x,y); \\ 0 \in T_2 y + F_2(x,y). \end{cases} \tag{1.3}$$

Problem (1.3) was introduced and studied by Verma [3], Fang and Huang [5].

(3) If $E_1 = E_2 = H$ is a Hilbert space, $F_1 = F_2 = F(x)$, $M(\cdot,\cdot) = M(\cdot)$, $f_1 = f_2 = 0$, then (1.1) reduces to the problem of finding $x \in H$ such that

$$0 \in F(x) + M(x). \tag{1.4}$$

Problem (1.4) was introduced and studied by Zeng *et al.* [6]. If $F(x) = S(x) - T(x) - f$, $f \neq 0$, (1.4) becomes $f \in S(x) - T(x) + M(x)$ considered by Verma [4].

Let E be a real Banach space with dual E^*, $J : E \to 2^{E^*}$ is the normalized duality mapping defined by

$$Jx = \{ f \in E^* : \langle x, f \rangle = \|x\|^2 = \|f\|^2 \},$$

where $\langle \cdot,\cdot \rangle$ denotes the generalized duality paring. In the sequel, we shall denote the single-valued normalized duality map by j. It is well known that if E is smooth, then J is single-valued, and E^* is uniformly convex, then j is uniformly continuous on bounded set.

We assume that E, E_1, E_2 are smooth Banach spaces. For convenience, the norms of E, E_1 and E_2 are all denoted by $\|\cdot\|$. The norm of $E_1 \times E_2$ is defined by $\|\cdot\| + \|\cdot\|$, *i.e.*, if $(x,y) \in E_1 \times E_2$, then $\|(x,y)\| = \|x\| + \|y\|$.

Definition 1.1 Let $T : E \to 2^E$ be a set-valued mapping.

(i) T is said to be accretive, if $\forall x, y \in E$, $u \in Tx$, $v \in Ty$,

$$\langle u - v, j(x-y) \rangle \geq 0, \quad \forall x, y \in E, u \in Tx, v \in Ty.$$

(ii) T is said to be α-strongly-accretive if there exists $\alpha > 0$ such that $\forall x, y \in E$, $u \in Tx$, $v \in Ty$,

$$\langle u - v, j(x-y) \rangle \geq \alpha \|x - y\|^2.$$

(iii) T is said to be m-accretive if T is accretive and $(I + \lambda T)(E) = E$, $\forall \lambda > 0$.

Definition 1.2 Let $N : E_1 \times E_2 \to 2^{E_1}$ be a set-valued mapping.

(i) The mapping $x \mapsto N(x,y)$ is said to be accretive if $\forall x_1, x_2 \in E_1$, $u \in N(x_1,y)$, $v \in N(x_2,y)$, $y \in E_2$,

$$\langle u - v, j(x_1 - x_2)\rangle \geq 0.$$

(ii) The mapping $x \mapsto N(x,y)$ is said to be α-strongly-accretive if there exists $\alpha > 0$ such that $\forall x_1, x_2 \in E_1$, $u_1 \in N(x_1,y)$, $u_2 \in N(x_2,y)$, $y \in E_2$,

$$\langle u_1 - u_2, j(x_1 - x_2)\rangle \geq \alpha\|x_1 - x_2\|^2.$$

(iii) The mapping $x \mapsto N(x,y)$ is said to be m-α-strongly-accretive if $N(\cdot,y)$ is α-strongly-accretive and $(I + N(\cdot,y))(E) = E$, $\forall y \in E_2$, $\lambda > 0$.

In a similar way, we can define the strong accretiveness of the mapping $N : E_1 \times E_2 \to 2^{E_2}$ with respect to the second argument.

Definition 1.3 Let $T : E \to 2^E$ be m-accretive mapping.

(i) The resolvent operator of T is defined by $R_\lambda^T x = (I + \lambda T)^{-1}x$, $\forall x \in E, \lambda > 0$.

(ii) The Yosida approximation of T is defined by $J_\lambda^T x = \frac{1}{\lambda}(I - R_\lambda^T)x$, $\forall x \in E$, $\lambda > 0$.

Definition 1.4 The mapping $F : E_1 \times E_2 \to E$ is said to be (r,s)-mixed Lipschitz continuous if there exist $r > 0$, $s > 0$ such that $\forall (x_1,y_1), (x_2,y_2) \in E_1 \times E_2$,

$$\|F(x_1,y_1) - F(x_2,y_2)\| \leq r\|x_1 - x_2\| + s\|y_1 - y_2\|.$$

In the sequel, we use the notation $\to$ and $\rightharpoonup$ to denote strong and weak convergence, respectively.

Proposition 1.1 [8–10] *If $T : E \to 2^E$ is m-accretive, then*

(1) *R_λ^T is single-valued and $\|R_\lambda^T x - R_\lambda^T y\| \leq \|x - y\|$, $\forall x, y \in E$;*

(2) *$\|J_\lambda^T x\| \leq |Tx| = \inf\{\|y\| : y \in Tx\}$, $\forall x \in D(T)$;*

(3) *J_λ^T is m-accretive on E, and $\|J_\lambda^T x - J_\lambda^T y\| \leq \frac{2}{\lambda}\|x - y\|$, $\forall x, y \in E$, $\lambda > 0$;*

(4) *$J_\lambda^T x \in TR_\lambda^T x$;*

(5) *If E^* is uniformly convex Banach space, then T is demiclosed, i.e., $[x_n, y_n] \in \mathrm{Graph}(T)$, $x_n \to x$, $y_n \rightharpoonup y$ implies that $[x,y] \in \mathrm{Graph}(T)$.*

Lemma 1.1 *If $T : E \to 2^E$ is m-α-strongly-accretive, then*

(i) *R_λ^T is $\frac{1}{1+\lambda\alpha}$-Lipschitz continuous;*

(ii) *J_λ^T is $\frac{\alpha}{1+\lambda\alpha}$-strongly-accretive.*

Proof (i) Let $u = R_\lambda^T x$, $v = R_\lambda^T y$. Then $x - u \in \lambda Tu$, $y - v \in \lambda Tv$. Since T is α-strongly-accretive, λT is $\lambda\alpha$-strongly-accretive, $\lambda\alpha\|u - v\|^2 \leq \langle x - u - y + v, j(u - v)\rangle \leq \|x - y\|\|u - v\| - \|u - v\|^2$. Therefore, $\|u - v\| \leq \frac{1}{1+\lambda\alpha}\|x - y\|$. This completes the proof of (i).

(ii) By definition of J_λ^T and (i), we have

$$\begin{aligned}\langle J_\lambda^T x - J_\lambda^T y, j(x-y)\rangle &= \frac{1}{\lambda}\langle x - y - (R_\lambda^T x - R_\lambda^T y), j(x-y)\rangle \\ &\geq \frac{1}{\lambda}\left(\|x-y\|^2 - \|R_\lambda^T x - R_\lambda^T y\| \|x-y\|\right) \geq \frac{\alpha}{1+\lambda\alpha}\|x-y\|^2.\end{aligned}$$

This completes the proof of (ii). □

Remark 1.1 Let $N_i : E_1 \times E_2 \to 2^{E_i}$ be set-valued mapping, let $x \mapsto N_1(x,y)$ and $y \mapsto N_2(x,y)$ be m-accretive. Then the resolvent operator and Yosida approximation of N_i can be rewritten as

$$\begin{aligned}R_\lambda^{N_1(\cdot,y)} x &= \left(I + \lambda N_1(\cdot,y)\right)^{-1} x, \qquad J_\lambda^{N_1(\cdot,y)} x = \frac{1}{\lambda}\left(I - R_\lambda^{N_1(\cdot,y)}\right)x, \\ R_\lambda^{N_2(x,\cdot)} y &= \left(I + \lambda N_2(x,\cdot)\right)^{-1} y, \qquad J_\lambda^{N_2(x,\cdot)} y = \frac{1}{\lambda}\left(I - R_\lambda^{N_2(x,\cdot)}\right)y,\end{aligned}$$

respectively.

Lemma 1.2 *Let $N_1(x,y) = T_1 x + F_1(x,y)$ and $N_2(x,y) = T_2 y + F_2(x,y)$. If $T_i : E_i \to 2^{E_i}$ is m-accretive, $F_i : E_1 \times E_2 \to E_i$ is α_i-strongly-accretive in the ith argument, and (r_i, s_i)-mixed Lipschitz continuous, then*

(i) *N_i is m-α_i-strongly-accretive in the ith argument* $(i = 1,2)$;
(ii) $\|R_\lambda^{N_1(\cdot,y_1)} x - R_\lambda^{N_1(\cdot,y_2)} x\| \leq \lambda s_1 \|y_1 - y_2\|$;
(iii) $\|R_\lambda^{N_2(x_1,\cdot)} y - R_\lambda^{N_2(x_2,\cdot)} y\| \leq \lambda r_2 \|x_1 - x_2\|$.

Proof (i) The fact directly follows from Kobayashi [11] (Theorem 5.3).
(ii) Let $u = R_\lambda^{N_1(\cdot,y_1)} x$, $v = R_\lambda^{N_1(\cdot,y_2)} x$. Then

$$x - u - \lambda F_1(u,y_1) \in \lambda T_1 u, \qquad x - v - \lambda F_1(v,y_2) \in \lambda T_1 v.$$

By accretiveness of T_i and α_i-strong accretiveness of F_i, we have that

$$\begin{aligned}0 &\leq \langle -u - \lambda F_1(u,y_1) + v + \lambda F_1(v,y_2), j(u-v)\rangle \\ &= -\|u-v\|^2 + \lambda\langle F_1(v,y_2) - F_1(u,y_1), j(u-v)\rangle \\ &= -\|u-v\|^2 + \lambda\langle F_1(v,y_2) - F_1(u,y_2), j(u-v)\rangle + \lambda\langle F_1(u,y_2) - F_1(u,y_1), j(u-v)\rangle \\ &\leq -\|u-v\|^2 - \lambda\alpha_1\|u-v\|^2 + \lambda\|F_1(u,y_2) - F_1(u,y_1)\| \|u-v\| \\ &\leq -(1+\lambda\alpha_1)\|u-v\|^2 + \lambda s_1\|y_1 - y_2\|\|u-v\|,\end{aligned}$$

Therefore, $\|u-v\| \leq \frac{\lambda s_1}{1+\lambda\alpha_1}\|y_1 - y_2\| \leq \lambda s_1 \|y_1 - y_2\|$. This completes the proof of (ii).
(iii) The proof is similar. We omit it. □

2 Main results

We assume that $\mathrm{CB}(E)$ in the family of all nonempty closed and bounded subset of E.

Lemma 2.1 [12] *Let $T_1 : E_1 \times E_2 \to E_1$ and $T_2 = E_1 \times E_2 \to E_2$ be two continuous mappings. If there exist θ_1, θ_2, $0 < \theta_1, \theta_2 < 1$ such that*

$$\begin{aligned} &\left\| T_1(x_1, y_1) - T_1(x_2, y_2) \right\| + \left\| T_2(x_1, y_1) - T_2(x_2, y_2) \right\| \\ &\quad \le \theta_1 \|x_1 - x_2\| + \theta_2 \|y_1 - y_2\|, \end{aligned}$$

then there exists $(x, y) \in E_1 \times E_2$ such that $x = T_1(x, y)$, $y = T_2(x, y)$.

Theorem 2.1 *For $i = 1, 2$, let E_i be a real Banach space with uniformly convex dual E_i^*, and let $F_i : E_1 \times E_2 \to E_i$, $h_i, g_i : E_i \to E_i$ be three single-valued mappings, let $T_i : E_i \to 2^{E_i}$, $M_i : E_i \times E_i \to 2^{E_i}$ be two set-valued mappings satisfying the following conditions that*

(1) *$M_i(h_i(\cdot), g_i(\cdot)) : E_i \to \mathrm{CB}(E_i)$ is m-accretive*;
(2) *T_i is m-accretive.*
(3) *F_i is α_i-strongly-accretive in the ith argument and (r_i, s_i)-mixed Lipschitz continuous, $N_1(x, y) = T_1(x) + F_1(x, y)$, $N_2(x, y) = T_2(y) + F_2(x, y)$.*

If λ satisfies that

$$0 < \lambda < \min\left\{ \frac{\alpha_1 - r_2}{t_2\alpha_1}, \frac{\alpha_2 - s_1}{s_1\alpha_2} \right\}, \quad r_2 < \alpha_1, s_1 < \alpha_2, \frac{s_1 r_2}{\alpha_1\alpha_2} < 1, \tag{2.1}$$

and $(f_1, f_2) \in E_1 \times E_2$, then

(i) *for any λ in* (2.1), *there exists $(x_\lambda, y_\lambda) \in E_1 \times E_2$ such that*

$$\begin{cases} f_1 \in J_\lambda^{N_1(\cdot, y_\lambda)} x_\lambda + M_1(h_1(x_\lambda), g_1(x_\lambda)), \\ f_2 \in J_\lambda^{N_2(x_\lambda, \cdot)} y_\lambda + M_2(h_2(y_\lambda), g_2(y_\lambda)), \end{cases} \tag{2.2}$$

and $\{x_\lambda\}_{\lambda\to 0}$ and $\{y_\lambda\}_{\lambda\to 0}$ are bounded;

(ii) *if $\{J_\lambda^{N_1(\cdot, y)} x_\lambda\}_{\lambda\to 0}, \{J_\lambda^{N_2(x, \cdot)} y_\lambda\}_{\lambda\to 0}$ are bounded, then there exists unique $(x, y) \in E_1 \times E_2$, which is a solution of Problem* (1.1), *such that $x_\lambda \to x$, $y_\lambda \to y$ as $\lambda \to 0$.*

Remark 2.1 Equation (2.2) is called the system of Yosida approximation inclusions (equations).

Proof of Theorem 2.1 (i) By Definition 1.3, we can easily show that (x_λ, y_λ) satisfies (2.2), if and only if (x_λ, y_λ) satisfies the relation that

$$\begin{aligned} x &= R_\lambda^{M_1(h_1(\cdot), g_1(\cdot))} \left[\lambda f_1 + R_\lambda^{N_1(\cdot, y)} x \right] \triangleq B_1(x, y), \\ y &= R_\lambda^{M_2(h_2(\cdot), g_2(\cdot))} \left[\lambda f_2 + R_\lambda^{N_2(x, \cdot)} y \right] \triangleq B_2(x, y). \end{aligned} \tag{2.3}$$

Now, we study the mapping $B_i : E_1 \times E_2 \to E_i$ ($i = 1, 2$) defined by (2.3). By Proposition 1.1(1), Lemma 1.1 and Lemma 1.2, and Eq. (2.3), for any $x_1, x_2 \in E_1$, $y_1, y_2 \in E_2$, we have that

$$\begin{aligned} &\left\| B_1(x_1, y_1) - B_1(x_2, y_2) \right\| \\ &\quad = \left\| R_\lambda^{M_1(h_1(\cdot), g_1(\cdot))} \left[\lambda f_1 + R_\lambda^{N_1(\cdot, y_1)} x_1 \right] - R_\lambda^{M_1(h_1(\cdot), g_1(\cdot))} \left[\lambda f_1 + R_\lambda^{N_1(\cdot, y_2)} x_2 \right] \right\| \end{aligned}$$

$$\begin{aligned}
&\leq \left\| R_\lambda^{N_1(\cdot,y_1)} x_1 - R_\lambda^{N_1(\cdot,y_2)} x_2 \right\| \\
&\leq \left\| R_\lambda^{N_1(\cdot,y_1)} x_1 - R_\lambda^{N_1(\cdot,y_2)} x_1 \right\| + \left\| R_\lambda^{N_1(\cdot,y_2)} x_1 - R_\lambda^{N_1(\cdot,y_2)} x_2 \right\| \\
&\leq \lambda s_1 \|y_1 - y_2\| + \frac{1}{1+\lambda\alpha_1} \|x_1 - x_2\|. \qquad (2.4)
\end{aligned}$$

Similarly, by Proposition 1.1(1), Lemma 1.1 and Lemma 1.2, we can prove that

$$\left\| B_2(x_1,y_1) - B_2(x_2,y_2) \right\| \leq \lambda r_2 \|x_1 - x_2\| + \frac{1}{1+\lambda\alpha_2} \|y_1 - y_2\|. \qquad (2.5)$$

Equations (2.4) and (2.5) imply that

$$\left\| B_1(x_1,y_1) - B_1(x_2,y_2) \right\| + \left\| B_2(x_1,y_1) - B_2(x_2,y_2) \right\| \leq \theta_1 \|x_1 - x_2\| + \theta_2 \|x_1 - x_2\|,$$

where $\theta_1 = \frac{1}{1+\lambda\alpha_1} + \lambda r_2$, $\theta_2 = \frac{1}{1+\lambda\alpha_2} + \lambda s_1$. By (2.1), $0 < \theta_1, \theta_2 < 1$. Therefore, by Lemma 2.1, for λ in (2.1), there exists $(x_\lambda, y_\lambda) \in E_1 \times E_2$ such that $x_\lambda = B_1(x_\lambda, y_\lambda)$, $y_\lambda = B_2(x_\lambda, y_\lambda)$, *i.e.*, (x_λ, y_λ) satisfies (2.3), and hence (2.2) hold.

Now, we show that $\{x_\lambda\}_{\lambda\to 0}$ and $\{y_\lambda\}_{\lambda\to 0}$ are bounded. For $(x_1, y_1) \in E_1 \times E_2$, and λ in (2.1), let

$$z_\lambda \in J_\lambda^{N_1(\cdot,y_\lambda)} x_1 + M_1\big(h_1(x_1), g_1(x_1)\big); \qquad (2.6)$$

$$w_\lambda \in J_\lambda^{N_2(x_\lambda,\cdot)} y_1 + M_2\big(h_2(y_1), g_2(y_1)\big). \qquad (2.7)$$

Equations (2.6) plus (2.2) indicates that

$$z_\lambda - J_\lambda^{N_1(\cdot,y_\lambda)} x_1 - f_1 + J_\lambda^{N_1(\cdot,y_\lambda)} x_\lambda \in M_1\big(h_1(x_1), g_1(x_1)\big) - M_1\big(h_1(x_\lambda), g_1(x_\lambda)\big).$$

By Lemma 1.1 and condition (1) in Theorem 2.1, we obtain that

$$\begin{aligned}
0 &\leq \left\langle z_\lambda - f_1 - J_\lambda^{N_1(\cdot,y_\lambda)} x_1 + J_\lambda^{N_1(\cdot,y_\lambda)} x_\lambda, j(x_1 - x_\lambda) \right\rangle \\
&= \left\langle z_\lambda - f_1, j(x_1 - x_\lambda) \right\rangle - \left\langle J_\lambda^{N_1(\cdot,y_\lambda)} x_1 - J_\lambda^{N_1(\cdot,y_\lambda)} x_\lambda, j(x_1 - x_\lambda) \right\rangle \\
&\leq \|z_\lambda - f_1\| \|x_1 - x_\lambda\| - \frac{\alpha_1}{1+\lambda\alpha_1} \|x_1 - x_\lambda\|^2. \qquad (2.8)
\end{aligned}$$

By Definition 1.3(ii), Proposition 1.1(2) and Lemma 1.2, we get that

$$\begin{aligned}
\left\| J_\lambda^{N_1(\cdot,y_\lambda)} x_1 \right\| &\leq \left\| J_\lambda^{N_1(\cdot,y_\lambda)} x_1 - J_\lambda^{N_1(\cdot,y_1)} x_1 \right\| + \left\| J_\lambda^{N_1(\cdot,y_1)} x_1 \right\| \\
&= \left\| \frac{1}{\lambda}\big(x_1 - R_\lambda^{N_1(\cdot,y_\lambda)} x_1 - x_1 + R_\lambda^{N_1(\cdot,y_1)} x_1\big) \right\| + \left| N_1(\cdot, y_1) x_1 \right| \\
&\leq \frac{s_1}{1+\lambda\alpha_1} \|y_1 - y_\lambda\| + \left| N_1(x_1, y_1) \right|. \qquad (2.9)
\end{aligned}$$

For any λ in (2.1), take $u_\lambda \in M_1(h_1(x_1), g_1(x_1))$, $v_\lambda \in M_2(h_2(y_1), g_2(y_1))$ such that $z_\lambda = J_\lambda^{N_1(\cdot,y_\lambda)} x_1 + u_\lambda$, $w_\lambda = J_\lambda^{N_2(x_\lambda,\cdot)} y_1 + v_\lambda$. Since $\{u_\lambda\} \subset M_1(h_1(x_1), g_1(x_1))$ and $\{v_\lambda\} \subset M_2(h_2(y_1), g_2(y_1))$, by condition (1), $\{u_\lambda\}$ and $\{v_\lambda\}$ are bounded. Combining (2.6), (2.8) and (2.9) yields that

$$\begin{aligned}\|x_1 - x_\lambda\| &\le \frac{1+\lambda\alpha_1}{\alpha_1}\|z_\lambda - f_1\| \le \frac{1+\lambda\alpha_1}{\alpha_1}\big(\|z_\lambda\| + \|f_1\|\big)\\ &\le \frac{1+\lambda\alpha_1}{\alpha_1}\big(\big\|J_\lambda^{N_1(\cdot,y_\lambda)}x_1\big\| + \|u_\lambda\| + \|f_1\|\big)\\ &\le \frac{1+\lambda\alpha_1}{\alpha_1}\left(\frac{s_1}{1+\lambda\alpha_1}\|y_1 - y_\lambda\| + \big|N_1(x_1,y_1)\big| + \|u_\lambda\| + \|f_1\|\right).\end{aligned} \tag{2.10}$$

By using similar methods, we obtain that

$$\|y_1 - y_\lambda\| \le \frac{1+\lambda\alpha_2}{\alpha_2}\left(\frac{r_2}{1+\lambda\alpha_2}\|x_1 - x_\lambda\| + \big|N_2(x_1,y_1)\big| + \|v_\lambda\| + \|f_2\|\right). \tag{2.11}$$

It follows from (2.10) and (2.11) that $\{x_\lambda\}_{\lambda\to 0}$ and $\{y_\lambda\}_{\lambda\to 0}$ are bounded since $0 < \frac{s_1 r_2}{\alpha_1\alpha_2} < 1$.

(ii) Note that for $\lambda, \mu > 0$

$$f_1 - J_\lambda^{N_1(\cdot,y_\lambda)}x_\lambda \in M_1\big(h_1(\cdot), g_1(\cdot)\big)x_\lambda \quad \text{and} \quad f_1 - J_\mu^{N_1(\cdot,y_\mu)}x_\mu \in M_1\big(h_1(\cdot), g_1(\cdot)\big)x_\mu.$$

By Proposition 1.1(4), we have that

$$\begin{aligned}0 &\le \big\langle -J_\lambda^{N_1(\cdot,y_\lambda)}x_\lambda + J_\mu^{N_1(\cdot,y_\mu)}x_\mu, j(x_\lambda - x_u)\big\rangle\\ &= \big\langle J_\lambda^{N_1(\cdot,y_\lambda)}x_\lambda - J_\mu^{N_1(\cdot,y_\mu)}x_\mu, j\big(R_\lambda^{N_1(\cdot,y_\lambda)}x_\lambda - R_\mu^{N_1(\cdot,y_\mu)}x_\mu\big) - j(x_\lambda - x_\mu)\big\rangle\\ &\quad - \big\langle J_\lambda^{N_1(\cdot,y_\lambda)}x_\lambda - J_\mu^{N_1(\cdot,y_\mu)}x_\mu, j\big(R_\lambda^{N_1(\cdot,y_\lambda)}x_\lambda - R_\mu^{N_1(\cdot,y_\mu)}x_\mu\big)\big\rangle\\ &\le \big\langle J_\lambda^{N_1(\cdot,y_\lambda)}x_\lambda - J_\mu^{N_1(\cdot,y_\mu)}x_\mu, j\big(R_\lambda^{N_1(\cdot,y_\lambda)}x_\lambda - R_\mu^{N_1(\cdot,y_\mu)}x_\mu\big) - j(x_\lambda - x_\mu)\big\rangle\\ &\quad - \alpha_1\big\|R_\lambda^{N_1(\cdot,y_\lambda)}x_\lambda - R_\mu^{N_1(\cdot,y_\mu)}x_\mu\big\|^2,\end{aligned}$$

and hence,

$$\begin{aligned}&\alpha_1\big\|R_\lambda^{N_1(\cdot,y_\lambda)}x_\lambda - R_\mu^{N_1(\cdot,y_\mu)}x_\mu\big\|^2\\ &\quad\le \big\langle J_\lambda^{N_1(\cdot,y_\lambda)}x_\lambda - J_\mu^{N_1(\cdot,y_\mu)}x_\mu, j\big(R_\lambda^{N_1(\cdot,y_\lambda)}x_\lambda - R_\mu^{N_1(\cdot,y_\mu)}x_\mu\big) - j(x_\lambda - x_\mu)\big\rangle.\end{aligned} \tag{2.12}$$

Since $x_\lambda - R_\lambda^{N_1(\cdot,y_\lambda)}x_\lambda = \lambda J_\lambda^{N_1(\cdot,y_\lambda)}x_\lambda \to 0$ (as $\lambda \to 0$), $\{J_\lambda^{N_1(\cdot,y_\lambda)}x_\lambda\}_{\lambda\to 0}$ is bounded. The j is uniformly continuous on bounded set, and (2.12) reduces to that

$$\begin{aligned}&\alpha_1\big\|R_\lambda^{N_1(\cdot,y_\lambda)}x_\lambda - R_\mu^{N_1(\cdot,y_\mu)}x_\mu\big\|^2\\ &\quad\le O\big(\big\|x_\lambda - x_\mu - R_\lambda^{N_1(\cdot,y_\lambda)}x_\lambda + R_\mu^{N_1(\cdot,y_\mu)}x_\mu\big\|\big) \le O(\lambda + \mu).\end{aligned}$$

Similarly, we have that

$$\alpha_2\big\|R_\lambda^{N_2(x_\lambda,\cdot)}y_\lambda - R_\mu^{N_2(x_\mu,\cdot)}y_\mu\big\| \le O(\lambda + \mu).$$

Consequently, $\{R_\lambda^{N_1(\cdot,y_\lambda)}x_\lambda\}_{\lambda\to 0}$ and $\{R_\lambda^{N_2(x_\lambda,\cdot)}y_\lambda\}_{\lambda\to 0}$ are the Cauchy net. There exists $(x,y) \in E_1 \times E_2$ such that $R_\lambda^{N_1(\cdot,y_\lambda)}x_\lambda \to x$, $R_\lambda^{N_2(x_\lambda,\cdot)}y_\lambda \to y$ as $\lambda \to 0$ from which and $x_\lambda - R_\lambda^{N_1(\cdot,y_\lambda)}x_\lambda = \lambda J_\lambda^{N_1(\cdot,y_\lambda)}x_\lambda$ and $y_\lambda - R_\lambda^{N_2(x_\lambda,\cdot)}y_\lambda = \lambda J_\lambda^{N_2(x_\lambda,\cdot)}y_\lambda$, it follows that $x_\lambda \to x$ and $y_\lambda \to y$ as $\lambda \to 0$.

Now, we show that (x,y) is a solution of (1.1). Since E_i is reflexive and $\{J_\lambda^{N_1(\cdot,y_\lambda)}x_\lambda\}_{\lambda\to 0}$ and $\{J_\lambda^{N_2(x_\lambda,\cdot)}y_\lambda\}_{\lambda\to 0}$ are bounded, there exist $\lambda_n > 0$ $(n = 1,2,\ldots)$ such that $\lambda_n \to 0$ and $J_{\lambda_n}^{N_1(\cdot,y_{\lambda_n})}x_{\lambda_n} \rightharpoonup z_1$, $J_{\lambda_n}^{N_2(x_{\lambda_n},\cdot)}y_{\lambda_n} \rightharpoonup z_2$ for some $(z_1,z_2) \in E_1 \times E_2$. Let $w'_{\lambda_n} = f_1 - J_{\lambda_n}^{N_1(\cdot,y_{\lambda_n})}x_{\lambda_n} \in M_1(h_1(x_{\lambda_n}),g_1(x_{\lambda_n}))$, $w''_{\lambda_n} = f_2 - J_{\lambda_n}^{N_2(x_{\lambda_n},\cdot)}y_{\lambda_n} \in M_2(h_2(x_{\lambda_n}),g_2(y_{\lambda_n}))$. Then $w'_{\lambda_n} \rightharpoonup w_1$, $w''_{\lambda_n} \rightharpoonup w_2$ for some $(w_1,w_2) \in E_1 \times E_2$. Since $N_1(\cdot,y)$, $N_2(x,\cdot)$ $(\forall (x,y) \in E_1 \times E_2)$, T_i and $M_i(h_i(\cdot), g_1(\cdot))$ $(i = 1,2)$ are demiclosed (see Proposition 1.1(5)), we have that

$$\begin{aligned}
&x \in E_1 = D\big(N_1(\cdot,y)\big) \cap D\big(M_1\big(h_1(\cdot),g_1(\cdot)\big)\big),\\
&y \in E_2 = D\big(N_2(x,\cdot)\big) \cap D\big(M_2\big(h_2(\cdot),g_2(\cdot)\big)\big),\\
&z_1 \in N_1(\cdot,y)x = N_1(x,y) = T_1x + F_1(x,y),\\
&z_2 \in N_2(x,\cdot)y = N_2(x,y) = T_2y + F_2(x,y),\\
&w_1 \in M_1\big(h_1(x),g_1(x)\big) \quad \text{and} \quad w_2 \in M_2\big(h_2(y),g_2(y)\big).
\end{aligned}$$

Therefore,

$$\begin{aligned}
&f_1 = z_1 + w_1 \in T_1x + F_1(x,y) + M_1\big(h_1(x),g_1(x)\big),\\
&f_2 = z_2 + w_2 \in T_2y + F_2(x,y) + M_2\big(h_2(y),g_2(y)\big).
\end{aligned}$$

Finally, we show the uniqueness of solutions. Let (x,y) and (x_1,y_1) be two solutions of Problem (1.1). Let $u \in T_1x$, $u_1 \in T_1x_1$, $w \in M_1(h_1(x),g_1(x))$, $w_1 \in M_1(h_1(x_1),g_1(x_1))$ such that

$$f_1 = u + F_1(x,y) + w, \qquad f_1 = u_1 + F_1(x_1,y_1) + w_1.$$

Then by accretiveness of M_i and T_i, we have that

$$\begin{aligned}
0 &= \big\langle f_1 - f_1, j(x-x_1)\big\rangle\\
&= \big\langle u + F_1(x,y) + w - u_1 - F_1(x_1,y_1) - w_1, j(x-x_1)\big\rangle\\
&\geq \big\langle F_1(x,y) - F_1(x_1,y_1), j(x-x_1)\big\rangle\\
&= \big\langle F_1(x,y) - F_1(x_1,y), j(x-x_1)\big\rangle + \big\langle F_1(x_1,y) - F_1(x_1,y_1), j(x-x_1)\big\rangle\\
&\geq \alpha_1\|x-x_1\|^2 - \big\|F_1(x_1,y) - F_1(x_1,y_1)\big\|\|x-x_1\|\\
&\geq \alpha_1\|x-x_1\|^2 - s_1\|y-y_1\|\|x-x_1\|.
\end{aligned}$$

That is,

$$\|x-x_1\| \leq \frac{s_1}{\alpha_1}\|y-y_1\|. \tag{2.13}$$

Let $v \in T_2y$, $v_1 \in T_2y_1$, $z \in M_2(h_2(y),g_2(y))$, $z_1 \in M_2(h_2(y_1),g_2(y_1))$ such that

$$\begin{aligned}
&f_2 = v + F_2(x,y) + w,\\
&f_2 = v_1 + F_2(x_1,y_1) + w_1.
\end{aligned}$$

The by the similar discussion, we have that

$$\|y - y_1\| \leq \frac{r_2}{\alpha_2}\|x - x_1\|. \tag{2.14}$$

Equations (2.1), (2.13) and (2.14) imply that $x = x_1$, $y = y_1$. □

Theorem 2.2 *Suppose that E_i, T_i, M_i, F_i, f_i and h_i ($i = 1, 2$) are the same as in Theorem* 2.1. *If for any $R_i > 0$, there exist $L_i > 0$, $a_i > 0$ and $0 < L_i < 1$ such that*

$$|T_1x| \leq L_1\big|M_1\big(h_1(x), g_1(x)\big)\big| + a_1, \quad \|x\| \leq R_1, \tag{2.15}$$

$$|T_2y| \leq L_2\big|M_2\big(h_2(y), g_2(y)\big)\big| + a_2, \quad \|y\| \leq R_2, \tag{2.16}$$

then Problem (1.1) *has a unique solution.*

Proof It suffices to show that $\{J_\lambda^{N_1(\cdot,y_\lambda)}x_\lambda\}_{\lambda\to 0}$ and $\{J_\lambda^{N_2(x_\lambda,\cdot)}y_\lambda\}_{\lambda\to 0}$ in Theorem 2.1 are bounded. Because $\{x_\lambda\}$ and $\{y_\lambda\}$ are bounded, therefore, there exists $R_i > 0$ ($i = 1, 2$) such that for λ in (2.1), $\|x_\lambda\| \leq R_1$ and $\|y_\lambda\| \leq R_2$. By Proposition 1.1(2) and (2.15),

$$\begin{aligned}
\big\|J_\lambda^{N_1(\cdot,y_\lambda)}x_\lambda\big\| &\leq \big|N_1(x_\lambda, y_\lambda)\big| = \big|T_1x_\lambda + F_1(x_\lambda, y_\lambda)\big| \\
&= \inf\big\{\|u\| : u = u_0 + F_1(x_\lambda, y_\lambda) \in T_1x_\lambda + F_1(x_\lambda, y_\lambda)\big\} \\
&= \inf\big\{\big\|u_0 + F_1(x_\lambda, y_\lambda)\big\| : u_0 \in T_1x_\lambda\big\} \\
&\leq \inf\big\{\|u_0\| : u_0 \in T_1x_\lambda\big\} + \big\|F_1(x_\lambda, y_\lambda)\big\| \\
&= |T_1x_\lambda| + \big\|F_1(x_\lambda, y_\lambda)\big\| \\
&\leq L_1\big|M_1\big(h_1(x_\lambda), g_1(x_\lambda)\big)\big| + \big\|F_1(x_\lambda, y_\lambda)\big\| + a_1;
\end{aligned} \tag{2.17}$$

Similarly, by Proposition 1.1(2) and (2.16), we get that

$$\big\|J_\lambda^{N_2(x_\lambda,\cdot)}y_\lambda\big\| \leq L_2\big|M_2\big(h_2(y_\lambda), g_2(y_\lambda)\big)\big| + \big\|F_2(x_\lambda, y_\lambda)\big\| + a_2. \tag{2.18}$$

By (2.2),

$$\big|M_1\big(h_1(x_\lambda), g_1(x_\lambda)\big)\big| \leq \|f_1\| + \big\|J_\lambda^{N_1(\cdot,y_\lambda)}x_\lambda\big\|, \tag{2.19}$$

$$\big|M_2\big(h_2(y_\lambda), g_2(y_\lambda)\big)\big| \leq \|f_2\| + \big\|J_\lambda^{N_2(x_\lambda,\cdot)}y_\lambda\big\|. \tag{2.20}$$

Therefore, from (2.17)-(2.20), it follows that

$$\big\|J_\lambda^{N_1(\cdot,y_\lambda)}x_\lambda\big\| \leq \frac{L_1}{1 - L_1}\|f_1\| + \frac{1}{1 - L_1}\big\|F_1(x_\lambda, y_\lambda)\big\| + \frac{a_1}{1 - L_1}, \tag{2.21}$$

$$\big\|J_\lambda^{N_2(x_\lambda,\cdot)}y_\lambda\big\| \leq \frac{L_2}{1 - L_2}\|f_2\| + \frac{1}{1 - L_2}\big\|F_2(x_\lambda, y_\lambda)\big\| + \frac{a_2}{1 - L_2}. \tag{2.22}$$

Since F_i ($i = 1, 2$) is uniformly continuous, F_i map bounded set in $E_1 \times E_2$ to bounded set. Hence, (2.21) and (2.22) imply that $\{J_\lambda^{N_1(\cdot,y_\lambda)}x_\lambda\}_{\lambda\to 0}$ and $\{J_\lambda^{N_2(x_\lambda,\cdot)}y_\lambda\}_{\lambda\to 0}$ are bounded. □

Theorem 2.3 *Suppose that E_i, T_i, M_i, h_i, g_i, F_i and f_i $(i = 1, 2)$ are the same as in Theorem 2.1. If for any $R_i > 0$, there exists bounded functional $B_i : E_1 \times E_2 \to \Re_+$ (i.e., B_i map a bounded set in $E_1 \times E_2$ to a bounded set in $\Re_+$) such that for $[x, z] \in \mathrm{Graph}(M_1(h_1(\cdot), g_1(\cdot)))$, $[y, w] \in \mathrm{Graph}(M_2(h_2(\cdot), g_2(\cdot)))$ and $\lambda > 0$,*

$$\langle z, jJ_\lambda^{N_1(\cdot,y)} x\rangle \geq -B_1(x, y), \tag{2.23}$$

$$\langle w, jJ_\lambda^{N_2(x,\cdot)} y\rangle \geq -B_2(x, y), \quad \|x\| \leq R_1, \|y\| \leq R_2 \tag{2.24}$$

for $x \in E_1$, $\|x\| \leq R_1$, $y \in E$, $\|y\| \leq R_2$, then Problem (1.1) has a unique solution.

Proof It suffices to show that $\{J_\lambda^{N_1(\cdot,y_\lambda)} x_\lambda\}_{\lambda\to 0}$ and $\{J_\lambda^{N_2(x_\lambda,\cdot)} y_\lambda\}_{\lambda\to 0}$ are bounded. Since $\{x_\lambda\}_{\lambda\to 0}$ and $\{y_\lambda\}_{\lambda\to 0}$ are bounded, then by (2.23), for $u_\lambda \in M_1(h_1(\cdot), g_2(\cdot))x_\lambda$,

$$\begin{aligned}\|f_1\| \left\|J_\lambda^{N_1(\cdot,y_\lambda)} x_\lambda\right\| &\geq \langle f_1, jJ_\lambda^{N_1(\cdot,y_\lambda)} x_\lambda\rangle = \langle J_\lambda^{N_1(\cdot,y_\lambda)} x_\lambda + u_\lambda, jJ_\lambda^{N_1(\cdot,y_\lambda)} x_\lambda\rangle \\ &= \left\|J_\lambda^{N_1(\cdot,y_\lambda)} x_\lambda\right\|^2 + \langle u_\lambda, jJ_\lambda^{N_1(\cdot,y_\lambda)} x_\lambda\rangle \geq \left\|J_\lambda^{N_1(\cdot,y_\lambda)} x_\lambda\right\|^2 - B_1(x_\lambda, y_\lambda),\end{aligned}$$

which implies that $\|J_\lambda^{N_1(\cdot,y_\lambda)} x_\lambda\| \leq (B_1(x_\lambda, y_\lambda) + \frac{1}{4}\|f_1\|^2)^{\frac{1}{2}} + \frac{\|f_1\|}{2}$. Similarly, $\|J_\lambda^{N_2(x_\lambda,\cdot)} y_\lambda\| \leq (B_2(x_\lambda, y_\lambda))^{\frac{1}{2}} + \frac{1}{2}\|f_2\|$. This completes the proof of Theorem 2.3. □

3 Conclusion and future perspective

Two of the most difficult and important problems in variation inclusions are the establishment of system of variational inclusions and the development of an efficient numerical methods. A new system of generalized variational inclusions in the Banach space under the assumption with no continuousness is introduced, and some existence and uniqueness theorems of solutions for this kind of system of generalized variational inclusions are proved by using the Yosida approximation technique for m-accretive operator.

More approaches [13–15], which have been applied in variational inequalities, could be manipulated in variational inclusions. We will make further research to solve this kind of system of generalized variational inclusions by using extragradient method and implicit iterative methods.

Competing interests

The author declares that they have no competing interests.

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