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MOREAU–YOSIDA APPROXIMATION OF CONDITIONAL MINIMIZATION PROBLEMS AND ITS LIMIT PROPERTIES

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For a family of conditional minimization problems $\{\langle \inf_{x\in X_\alpha} F^\alpha(x)\rangle, \alpha\in A\}$ *we obtain a representation of its variational S-limits in terms of pointwise limits of Moreau–Yosida approximations. Bibliography: 4 titles.*

Let (X,τ) be a topological vector space with a countable local base, i.e., it has an invariant metric d_τ that agrees with the τ-topology. Consider in (X,τ) a net of conditional minimization problems,

$$\left\{\left\langle \inf_{x\in X_\alpha} F^\alpha(x)\right\rangle, \alpha\in A\right\}, \tag{1}$$

and the corresponding net of functions

$$\{F^\alpha: X_\alpha \to \overline{R}\}_{\alpha\in A}. \tag{2}$$

Here $\{X_\alpha\}_{\alpha\in A}$ is an arbitrary family of subsets of the space (X,τ) and A is a partially ordered set of indices with increasing order. In the sequel, any local optimization problem of the form $\langle \inf_{x\in E} F(x)\rangle$ is understood as an object defined by the pair $\langle F, E\rangle$, whereas we denote by $\inf_{x\in E} F(x)$ the least value of the function $F: X\to\overline{R}$ on the set E.

It was shown in [1] that the net of problems (1) can be put into correspondence with the so-called lower, $\mathcal{P}_s$, and upper, $\mathcal{P}^s$, S-limits that are also conditional optimization problems with the following structure:

$$(\mathcal{P}_s): \left\langle \inf_{x\in(\tau-Ls\,X_\alpha)} (\tau-li_s F^\alpha)(x)\right\rangle, \qquad (\mathcal{P}^s): \left\langle \inf_{x\in(\tau-Li\,X_\alpha)} (\tau-ls_s F^\alpha)(x)\right\rangle. \tag{3}$$

Here $\tau-Li\,X_\alpha$ and $\tau-Ls\,X_\alpha$ are, respectively, the lower and upper topological limit of the net of sets $\{X_\alpha\}_{\alpha\in A}$ and the function $\tau-li_s F^\alpha: (\tau-Ls\,X_\alpha)\to\overline{R}$ is the lower (and $\tau-ls_s F^\alpha: (\tau-Li\,X_\alpha)\to\overline{R}$ is the upper) S-limit of the functional net (2).

We say that the problem $\langle \inf_{x\in(\tau-Li\,X_\alpha)} (\tau-lm_s F^\alpha)(x)\rangle$ is the strong variational S-limit of a family of problems (1) if for every value $x\in(\tau-Li\,X_\alpha)$ we have

$$\tau-li_s F^\alpha(x) = \tau-ls_s F^\alpha(x) \triangleq \tau-lm_s F^\alpha(x). \tag{4}$$

Correspondingly, the problem $\langle \inf_{x\in(\tau-Lm\,X_\alpha)} (\tau-lm_s F^\alpha)(x)\rangle$ is called the absolute variational S-limit if condition (4) holds and there exists a topological limit $\tau-Lm\,X_\alpha$ of the net of sets $\{X_\alpha\}_{\alpha\in A}$, i.e.,

$$\tau-Li\,X_\alpha = \tau-Ls\,X_\alpha \triangleq \tau-Lm\,X_\alpha.$$

The aim of this article is to show that variational S-limits of such nets, as well as topological limits of the sets $\{X_\alpha\}_{\alpha\in A}$, can be represented in terms of pointwise limits of nets of τ-continuous approximations to functions of the form (2). Assuming that each function $F^\alpha: X_\alpha\to\overline{R}$ may be undefined outside the corresponding set X_α, let us introduce a natural generalization of the notion of the Moreau–Yosida approximation [2, 3].

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Definition 1. For every value of $\alpha \in A$ and constants $\lambda > 0$ and $\beta > 0$, the function $F^\alpha_{\lambda,\beta}: X \to \overline{R}$ is called the Moreau–Yosida approximation of the function $F^\alpha: X_\alpha \to \overline{R}$ with degree β and index λ if

$$F^\alpha_{\lambda,\beta}(x) = \inf_{y \in X_\alpha} \left\{ F_\alpha(y) + \lambda^{-1} d^\beta_\tau(x,y) \right\} \quad \text{for all } x \in X. \tag{5}$$

We give the main properties of the functions $F^\alpha_{\lambda,\beta}$ without proof; one can check these properties by using the schemes of proofs of Theorems 9.13 and 9.15 of [4].

Propositions 1. *Let (X, d_τ) be a metric space. Then for any function $F^\alpha: X_\alpha \to \overline{R}$, its Moreau–Yosida approximation $F^\alpha_{\lambda,\beta}: X \to \overline{R}$, for $\lambda > 0$ and $\beta > 0$, is the greatest function from $Q: X \to \overline{R}$ satisfying the following conditions:*

(a) *$Q(x) \le F^\alpha(x)$ for all $x \in X_\alpha$;*
(b) *Q is Hölder continuous with degree β and factor λ^{-1}, i.e., for all $x, y \in X$, we have*

$$Q(x) \le Q(y) + \lambda^{-1} d^\beta_\tau(x,y).$$

Proposition 2. *Let (X, d_τ) be a metric space and $F^\alpha: X_\alpha \to [0,\infty]$ be an arbitrary nonnegative function. Let x_0 be any point of X such that $F^\alpha_{\lambda,\beta}(x_0) \le M$, where $M \ge 0$, $\lambda > 0$, and $\beta > 0$. Then there exists a constant $c = c(M, \lambda, \beta, r)$ such that*

$$F^\alpha_{\lambda,\beta}(x) - F^\alpha_{\lambda,\beta}(y) \le c \cdot d_\tau(x,y)$$

for all $x, y \in X$ that satisfy the conditions $d_\tau(x, x_0) \le r$, $d_\tau(y, x_0) \le r$.

The following results reflect the possibility of a unique representation of lower semicontinuous functions $F^\alpha: X_\alpha \to \overline{R}$ in terms of their Moreau–Yosida approximations.

Lemma 1. *Let (X, τ) be a metrizable topological space and $F^\alpha: X_\alpha \to [0,\infty]$ be an arbitrary function satisfying the condition $F^\alpha \not\equiv \infty$. Then, for all $\beta > 0$, we have*

$$\sup_{\lambda > 0} F^\alpha_{\lambda,\beta}(x) = \lim_{\lambda \downarrow 0} F^\alpha_{\lambda,\beta}(x) = \mathrm{sc}^- F^\alpha(x) \quad \textit{for all } x \in \mathrm{cl}_\tau X_\alpha, \tag{6}$$

where $\mathrm{sc}^- F^\alpha: \mathrm{cl}_\tau X_\alpha \to [0,\infty]$ is a lower semicontinuous regularization of the function $F^\alpha: X_\alpha \to [0,\infty]$ and $\mathrm{cl}_\tau X_\alpha$ is the closure of its domain.

Proof. Using identity (5), we have for all $y \in X_\alpha$

$$F^\alpha_{\lambda,\beta}(x) \le F_\alpha(y) + \lambda^{-1} d^\beta_\tau(x,y).$$

By setting $x = y$, we get

$$F^\alpha_{\lambda,\beta}(x) \le F^\alpha(x) \quad \text{for all } x \in X_\alpha. \tag{7}$$

Since this identity holds for all $\lambda > 0$, we get, by passing in (17) to a τ-lower semicontinuous regularization,

$$\mathrm{sc}^- \sup_{\lambda > 0} F^\alpha_{\lambda,\beta}(x) \le \mathrm{sc}^- F^\alpha(x) \quad \text{for all } x \in \mathrm{cl}_\tau X_\alpha.$$

The functions $F^\alpha_{\lambda,\beta}$ are τ-continuous, hence the function $\sup_{\lambda>0} F^\alpha_{\lambda,\beta}(x)$ is τ-lower semicontinuous. Consequently, we get from the above that $\sup_{\lambda>0} F^\alpha_{\lambda,\beta}(x) \le \mathrm{sc}^- F^\alpha(x)$ for all $x \in \mathrm{cl}_\tau X_\alpha$. Let us prove the converse inequality, i.e., that

$$\sup_{\lambda > 0} F^\alpha_{\lambda,\beta}(x) \ge \mathrm{sc}^- F^\alpha(x) \quad \text{for all } x \in \mathrm{cl}_\tau X_\alpha. \tag{8}$$

If $\sup_{\lambda>0} F^{\alpha}_{\lambda,\beta}(x) = +\infty$ at a chosen point $x \in \mathrm{cl}_\tau X_\alpha$, then relation (18) is obvious. Let $\sup_{\lambda>0} F^{\alpha}_{\lambda,\beta}(x) < +\infty$. Since the function F^α is bounded on X_α from below, for every $\lambda > 0$ one can find an element $x_\lambda \in X_\alpha$ such that $F^{\alpha}_{\lambda,\beta}(x) \le F^\alpha(x_\lambda) + \lambda^{-1} d^\beta_\tau(x, x_\lambda) \le F^{\alpha}_{\lambda,\beta}(x) + \lambda$. Hence,

$$F^\alpha(x_\lambda) + \lambda^{-1} d^\beta_\tau(x, x_\lambda) \le \sup_{\lambda>0} F^{\alpha}_{\lambda,\beta}(x) + \lambda, \tag{9}$$

and thus $d^\beta_\tau(x, x_\lambda) \le \lambda \sup_{\lambda>0} F^{\alpha}_{\lambda,\beta}(x) + \lambda^2$, i.e., $d_\tau(x, x_\lambda) \to 0$ as $\lambda \to 0$.

Since in a metrizable space (X,τ), the τ-closure of the set X_α coincides with its d_τ-closure, it follows from the above that $x_\lambda \to x$, where $x \in \mathrm{cl}_\tau X_\alpha$. Since $\sup_{\lambda>0} F^{\alpha}_{\lambda,\beta}(x) + \lambda \ge F^\alpha(x_\lambda)$, the relation (8) is obtained by passing to limit as $\lambda \downarrow 0$. This proves that the identity $\sup_{\lambda>0} F^{\alpha}_{\lambda,\beta}(x) = \mathrm{sc}^- F^\alpha(x)$ holds on the set $\mathrm{cl}_\tau X_\alpha$. Note that the mapping $\lambda \to F^{\alpha}_{\lambda,\beta}(x)$ is monotone increasing for every fixed value of x. Hence, $\sup_{\lambda>0} F^{\alpha}_{\lambda,\beta}(x) = \lim_{\lambda\downarrow 0} F^{\alpha}_{\lambda,\beta}(x)$. □

Corollary 1. *Let X_α be a subset of a metrizable topological space (X,τ) and $F^\alpha\colon X_\alpha \to [0,\infty]$ be an arbitrary bounded function. Then for any $\beta > 0$, we have* $\mathrm{Dom}(\sup_{\lambda>0} F^{\alpha}_{\lambda,\beta}) = \mathrm{cl}_\tau X_\alpha$, *where* $\mathrm{Dom}(f) = \{x \in X\colon f(x) < +\infty\}$.

Proof. By Lemma 1, $\sup_{\lambda>0} F^{\alpha}_{\lambda,\beta}(x) = \mathrm{sc}^- F^\alpha(x)$ for all $x \in \mathrm{cl}_\tau X\alpha$. Since the function $F^\alpha\colon X_\alpha \to [0,\infty]$ is bounded, its τ-lower semicontinuous regularization $\mathrm{sc}^- F^\alpha(x)$ is also continuous. Consequently,

$$\sup_{\lambda>0} F^{\alpha}_{\lambda,\beta}(x) < +\infty$$

for all $x \in \mathrm{cl}_\tau X_\alpha$, and hence, $\mathrm{cl}_\tau X_\alpha \subseteq \mathrm{Dom}(\sup_{\lambda>0} F^{\alpha}_{\lambda,\beta})$. Consider an arbitrary element x such that $x \notin \mathrm{cl}_\tau X_\alpha$. The function F^α is nonnegative, hence for every value of $\lambda > 0$ there exists an element $x_\lambda \in X_\alpha$ such that $F^{\alpha}_{\lambda,\beta}(x) \le F^\alpha(x_\lambda) + \lambda^{-1} d^\beta_\tau(x, x_\lambda) \le F^{\alpha}_{\lambda,\beta}(x) + \lambda$. Consequently, $F^{\alpha}_{\lambda,\beta}(x) \ge \lambda^{-1} d^\beta_\tau(x, x_\lambda) - \lambda$. Since $\liminf_{\lambda\downarrow 0} d^\beta_\tau(x, x_\lambda) > 0$, we find that $\sup_{\lambda>0} F^{\alpha}_{\lambda,\beta}(x) = +\infty$. Hence, for the chosen element x, we have the inclusion $x \notin \mathrm{Dom}(\sup_{\lambda>0} F^{\alpha}_{\lambda,\beta})$. Thus the equality $\mathrm{cl}_\tau X_\alpha = \mathrm{Dom}(\sup_{\lambda>0} F^{\alpha}_{\lambda,\beta})$ is proved. □

The following result shows that it is possible to represent S-limits of nets of nonnegative functions in terms of the corresponding Moreau–Yosida approximations.

Theorem 1. *Let (X,τ) be a metrizable topological space, $\{X_\alpha\}_{\alpha\in A}$ be a family of its τ-open subsets such that $\tau{-}Li\, X_\alpha \ne \varnothing$, and $\{F^\alpha\colon X_\alpha \to [0,+\infty]\}_{\alpha\in A}$ be an arbitrary net of functions each of which admits a continuation to the τ-closure of the corresponding subset X_α. Then for every value of $\beta > 0$, we have*

$$(\tau{-}li_s F^\alpha)(x) = \sup_{\lambda>0} \liminf_{\alpha\in A} F^{\alpha}_{\lambda,\beta}(x), \tag{10}$$

$$(\tau{-}ls_s F^\alpha)(x) = \sup_{\lambda>0} \limsup_{\alpha\in A} F^{\alpha}_{\lambda,\beta}(x). \tag{11}$$

We prove only identity (10), since the proof of the second relation is analogous. Let x be an arbitrary element from $\tau{-}Ls\, X_\alpha$. Introduce the following notation:

$$F_s(x) = (\tau{-}li_s F^\alpha)(x), \qquad H_s(x) = \sup_{\lambda>0} \liminf_{\alpha\in A} \inf_{y\in X_\alpha} \left\{F_\alpha(y) + \lambda^{-1} d^\beta_\tau(x,y)\right\}.$$

Let a number $t \in R$ be such that $t < F_s(x)$. Using properties of S-limits, we see that the function $F_s(x)$ is the lower S-limit of the net

$$\{F^\alpha\colon \mathrm{cl}_\tau X_\alpha \to [0,+\infty]\}_{\alpha\in A}.$$

Let $\mathcal{N}_\tau(x)$ be a filter of all τ-open neighborhoods of the point x. Therefore, one can find a neighborhood $U \in \mathcal{N}_\tau(x)$ such that

$$t < \liminf_{\substack{\alpha\in A\\ \mathrm{cl}_\tau X_\alpha \cap U \ne \varnothing}} \inf_{y\in \mathrm{cl}_\tau X_\alpha \cap U} F^\alpha(y).$$

Let the number $\lambda > 0$ be such that $d_\tau^\beta(x,y) > t \cdot \lambda$ for all $y \in \mathrm{cl}_\tau X_\alpha \setminus U$. Then

$$t < \liminf_{\substack{\alpha\in A\\ \mathrm{cl}_\tau X_\alpha\cap U\neq\varnothing}} \inf_{y\in \mathrm{cl}_\tau X_\alpha\cap U} \left(F_\alpha(y) + \lambda^{-1} d_\tau^\beta(x,y)\right) \liminf_{\substack{\alpha\in A\\ \mathrm{cl}_\tau X_\alpha\cap U\neq\varnothing}} \inf_{y\in \mathrm{cl}_\tau X_\alpha} \left(F_\alpha(y) + \lambda^{-1} d_\tau^\beta(x,y)\right). \tag{12}$$

Consider those values of $\alpha \in A' \subset A$ for which $\mathrm{cl}_\tau X_\alpha \cap U = \varnothing$. It is clear that for these α, the chosen point $x \in \tau{-}Ls\, X_\alpha$ does not belong to the set X_α, and hence,

$$\lim_{\lambda\downarrow 0} \inf_{y\in \mathrm{cl}_\tau X_\alpha} \left\{F_\alpha(y) + \lambda^{-1} d_\tau^\beta(x,y)\right\} = +\infty$$

for all $\alpha \in A$. Thus, there exists $\lambda^0 > 0$ such that

$$\liminf_{\substack{\alpha\in A\\ \mathrm{cl}_\tau X_\alpha\cap U\neq\varnothing}} \inf_{y\in \mathrm{cl}_\tau X_\alpha} \left\{F_\alpha(y) + \lambda^{-1} d_\tau^\beta(x,y)\right\} = \liminf_{\alpha\in A} \inf_{y\in \mathrm{cl}_\tau X_\alpha} \left\{F_\alpha(y) + \lambda^{-1} d_\tau^\beta(x,y)\right\}$$

for all $0 < \lambda < \lambda^0$. Since

$$\liminf_{\alpha\in A} \inf_{y\in \mathrm{cl}_\tau X_\alpha} \left\{F_\alpha(y) + \lambda^{-1} d_\tau^\beta(x,y)\right\} \le \liminf_{\alpha\in A} \inf_{y\in X_\alpha} \left\{F_\alpha(y) + \lambda^{-1} d_\tau^\beta(x,y)\right\} = \liminf_{\alpha\in A} F^\alpha_{\lambda,\beta}(x),$$

using (12) we get that $t < \liminf_{\alpha\in A} F^\alpha_{\lambda,\beta}(x) \le H_s(x)$. Since the latter relation holds for all $t < F_s(x)$, the inequality

$$F_s(x) \le H_s(x) \tag{13}$$

is proved.

Let us now prove an inequality converse to (13). Let $\varepsilon > 0$ be an arbitrary fixed number. Since the metric d_τ is continuous on X, for a chosen point x there exists a neighborhood $U \in \mathcal{N}_\tau(x)$ such that $d_\tau^\beta(x,y) < \varepsilon$ for all $y \in U$. Hence, for every $\alpha \in A$ satisfying $X_\alpha \cap U \neq \varnothing$, we can write

$$\inf_{y\in X_\alpha} \left\{F_\alpha(y) + \lambda^{-1} d_\tau^\beta(x,y)\right\} \le \inf_{y\in X_\alpha\cap U} \left\{F_\alpha(y) + \lambda^{-1} d_\tau^\beta(x,y)\right\} \le \inf_{y\in X_\alpha\cap U} F^\alpha(y) + \varepsilon.$$

Then

$$\liminf_{\alpha\in A} F^\alpha_{\lambda,\beta}(x) \le \liminf_{\substack{\alpha\in A\\ X_\alpha\cap U\neq\varnothing}} \inf_{y\in X_\alpha\cap U} F^\alpha(y) \le F_s(x) + \varepsilon.$$

This inequality holds for any $\lambda > 0$ and $\varepsilon > 0$, so that we get

$$H_s(x) \le F_s(x). \tag{14}$$

Thus, since the point $x \in \tau{-}Ls\, X_\alpha$ was chosen arbitrarily, relations (13)–(14) yield equality (10).

The following theorem gives an analytical representation for topological limits of an arbitrary net of subsets of a metrizable topological space.

Theorem 2. *Let (X,τ) be a metrizable topological space, $\{X_\alpha\}_{\alpha\in A}$ be a family of its subsets, and*

$$\{F^\alpha : X_\alpha \to [0,+\infty)\}_{\alpha\in A}$$

be an arbitrary equibounded (i.e., uniformly bounded) net of functions. Then

$$\tau{-}Li\, X_\alpha = \mathrm{Dom}\left(\sup_{\lambda>0} \limsup_{\alpha\in A} F^\alpha_{\lambda,\beta}\right), \tag{15}$$

$$\tau{-}Ls\, X_\alpha = \mathrm{Dom}\left(\sup_{\lambda>0} \liminf_{\alpha\in A} F^\alpha_{\lambda,\beta}\right), \tag{16}$$

where $\beta > 0$ is an arbitrary constant and $\mathrm{Dom}(f)$ *is an effective set of functions $f : X \to \overline{R}$.*

Proof. Since the net of functions,

$$\{F^\alpha : X_\alpha \to [0,+\infty)\}_{\alpha\in A}$$

is equibounded, there exists a constant $c > 0$ such that $F^\alpha(x) \leq c$ for all $x \in X_\alpha$ and for all $\alpha \in A$. Thus the lower and upper S-limits of the net will also be bounded functions. Hence, using (10)–(11) we get

$$\tau{-}Li\, X_\alpha \subseteq \mathrm{Dom}\left(\sup_{\lambda>0} \limsup_{\alpha\in A} F^\alpha_{\lambda,\beta}\right), \tag{17}$$

$$\tau{-}Ls\, X_\alpha \subseteq \mathrm{Dom}\left(\sup_{\lambda>0} \liminf_{\alpha\in A} F^\alpha_{\lambda,\beta}\right). \tag{18}$$

Therefore, in order to prove (15), it is sufficient to show that for all $x \in X \setminus \tau{-}Li\, X_\alpha$, the following condition holds:

$$x \notin \mathrm{Dom}\left(\sup_{\lambda>0} \liminf_{\alpha\in A} F^\alpha_{\lambda,\beta}\right). \tag{19}$$

Let x be an arbitrary element of $x \setminus \tau{-}Li\, X_\alpha$. Then for every neighborhood $U \in \mathcal{N}_\tau(x)$ there exists a subnet $\{Y_\gamma\}_{\gamma\in\Lambda}$ of the net $\{X_\alpha\}_{\alpha\in A}$ such that $U \cap Y_\gamma = \varnothing$ for all $\gamma \in \Lambda$. Since the functions F^α are nonnegative and bounded, for every fixed $\lambda > 0$ there exist elements $x^\alpha_\lambda \in X_\alpha$ satisfying the relation $F^\alpha_{\lambda,\beta}(x) \leq F^\alpha(x^\alpha_\lambda) + \lambda^{-1} d^\beta_\tau(x, x^\alpha_\lambda) \leq F^\alpha_{\lambda,\beta}(x) + \lambda$, whence we find that

$$F^\alpha_{\lambda,\beta}(x) \geq \lambda^{-1} d^\beta_\tau(x, x^\alpha_\lambda) - \lambda. \tag{20}$$

Let $\{y^\gamma_\lambda\}_{\gamma\in\Lambda}$ be a subnet of the net $\{x^\alpha_\lambda\}_{\alpha\in A}$ corresponding to the choice of $\{Y_\gamma\}_{\gamma\in\Lambda}$. Then (20) implies the obvious inequality

$$\limsup_{\alpha\in A} F^\alpha_{\lambda,\beta}(x) \geq \lambda^{-1} \limsup_{\alpha\in A} d^\beta_\tau(x, x^\alpha_\lambda) - \lambda \geq \lambda^{-1} \limsup_{\gamma\in\Lambda} d^\beta_\tau(x, y^\gamma_\lambda) - \lambda.$$

Since $U \cap Y_\gamma = \varnothing$, one can find a constant $c > 0$ such that $d^\beta_\tau(y, y^\gamma_\lambda)$ for all $y \in U$ and $\gamma \in \Lambda$. Consequently,

$$\limsup_{\alpha\in A} F^\alpha_{\lambda,\beta}(x) \geq \lambda^{-1} c - \lambda,$$

whence we find that $\sup_{\lambda>0} \limsup_{\alpha\in A} F^\alpha_{\lambda,\beta}(x) = +\infty$, i.e., inclusion (19) holds. Thus, inequality (15) is proved.

Let us now prove relation (16). Let x be an arbitrary element of $X \setminus \tau{-}Ls\, X_\alpha$. Then there exists a neighborhood $U \in \mathcal{N}_\tau(x)$ and $\mu \in A$ such that $U \cap X_\alpha = \varnothing$ for all $\alpha \geq \mu$. Thus it follows from (20) that $\liminf_{\alpha\in A} F^\alpha_{\lambda,\beta}(x) \geq \lambda^{-1} \liminf_{\alpha\in A} d^\beta_\tau(x, x^\alpha_\lambda) - \lambda$, where elements of the net $\{x^\alpha_\lambda\}_{\alpha\in A}$ satisfy the condition $x^\alpha_\lambda \in X_\alpha$ for all $\alpha \in A$. Since $U \cap X_\alpha = \varnothing$ for all $\alpha \geq \mu$, one can find a constant $c > 0$ such that $d_\tau(y, x^\alpha_\lambda) \geq c > 0$ for all $y \in U$ and $\alpha \geq \mu$. Thus, $\liminf_{\alpha\in A} F^\alpha_{\lambda,\beta}(x) \geq \lambda^{-1} c - \lambda$, whence we have the obvious relation

$$\sup_{\lambda>0} \liminf_{\alpha\in A} F^\alpha_{\lambda,\beta}(x) = +\infty.$$

Thus, for all elements $x \notin X \setminus \tau{-}Ls\, X_\alpha$, the condition $x \notin \mathrm{Dom}\big(\sup_{\lambda>0} \liminf_{\alpha\in A} F^\alpha_{\lambda,\beta}\big)$ holds. Consequently, using inclusion (18), we get identity (16). □

Remark 1. As follows from the above theorem, relations (15)–(16) hold for an arbitrary equibounded net of functions

$$\{F^\alpha : X_\alpha \to [0, +\infty)\}_{\alpha\in A}.$$

Therefore, in order to find the lower and upper topological limits of an arbitrary family of subspaces $\{X_\alpha\}_{\alpha\in A}$ of a metrizable topological space, one can use the following representation:

$$\begin{aligned} \tau{-}Li\, X_\alpha &= \mathrm{Dom}\left(\sup_{\lambda>0} \limsup_{\alpha\in A} \inf_{y\in X_\alpha} \left(c + \lambda^{-1} d^\beta_\tau(x,y)\right)\right), \\ \tau{-}Ls\, X_\alpha &= \mathrm{Dom}\left(\sup_{\lambda>0} \liminf_{\alpha\in A} \inf_{y\in X_\alpha} \left(c + \lambda^{-1} d^\beta_\tau(x,y)\right)\right), \end{aligned} \tag{21}$$

where $c > 0$ and $\beta > 0$ are arbitrary constants.

Thus, a net of sets $\{X_\alpha\}_{\alpha\in A}$ has a topological limit if for an arbitrary $\lambda > 0$ and some $c > 0$ and $\beta > 0$, the sequence

$$\left\{\inf_{y\in X_\alpha} \left(c+\lambda^{-1} d_\tau^\beta(x,y)\right)\right\}_{\alpha\in A}$$

of real numbers has a limit at every point of X and there exists at least one point $x \in X$ such that

$$\sup_{\lambda>0} \lim_{\alpha\in A} \inf_{y\in X_\alpha} \left(c+\lambda^{-1} d_\tau^\beta(x,y)\right) < +\infty.$$

In this case, the topological limit can be represented as

$$\tau\text{--}Lm\, X_\alpha = \mathrm{Dom}\left(\sup_{\lambda>0} \lim_{\alpha\in A} \inf_{y\in X_\alpha} \left(c+\lambda^{-1} d_\tau^\beta(x,y)\right)\right).$$

Remark 2. According to the above results, the following representations hold for the lower and upper variational S-limits of a net of minimization problems (1) in a metrizable space (X,τ):

$$\mathcal{P}_s\colon \left\langle \inf_{x\in(\tau-Ls\, X_\alpha)} F_s(x)\right\rangle$$
$$= \left\langle \inf_{\sup_{\lambda>0}\liminf_{\alpha\in A}\inf_{y\in X_\alpha}\left(c+\lambda^{-1}d_\tau^\beta(x,y)\right)} \left(\sup_{\lambda>0}\liminf_{\alpha\in A}\inf_{y\in X_\alpha}\left\{F_\alpha(y)+\lambda^{-1}d_\tau^\beta(x,y)\right\}\right)\right\rangle,$$

$$\mathcal{P}^s\colon \left\langle \inf_{x\in(\tau-Li\, X_\alpha)} F^s(x)\right\rangle$$
$$= \left\langle \inf_{\sup_{\lambda>0}\limsup_{\alpha\in A}\inf_{y\in X_\alpha}\left(c+\lambda^{-1}d_\tau^\beta(x,y)\right)} \left(\sup_{\lambda>0}\limsup_{\alpha\in A}\inf_{y\in X_\alpha}\left\{F_\alpha(y)+\lambda^{-1}d_\tau^\beta(x,y)\right\}\right)\right\rangle,$$

where $c > 0$ and $\beta > 0$ are arbitrary constants.

Theorem 3. *Let (X,τ) be a metrizable topological space,*

$$\{F^\alpha\colon X_\alpha \to [0,+\infty)\}_{\alpha\in A}$$

be an equibounded net of functions, and $F\colon E \to [0,+\infty)$ be a lower semicontinuous function with the nonempty domain

$$E = \mathrm{Dom}\left(\sup_{\lambda>0}\limsup_{\alpha\in A}\inf_{y\in X_\alpha}\left(c+\lambda^{-1}d_\tau^\beta(x,y)\right)\right)$$

(here $c > 0$ and $\beta > 0$ are certain constants). Then the following conditions are equivalent:

(a) *$\langle \inf_{x\in E} F(x)\rangle$ is a strong variational S-limit of the net of conditional minimization problems* (1);
(b) *for every $j \in N$ and all $x \in E$, the following relation holds:*

$$\inf_{y\in E}\left(F(y)+\lambda_j^{-1}d_\tau^\beta(x,y)\right) = \lim_{\alpha\in A}\inf_{y\in X_\alpha}\left(F^\alpha(y)+\lambda_j^{-1}d_\tau^\beta(x,y)\right),$$

where $\{\lambda_j\}_{j\in N}$ is a monotone decreasing sequence of positive numbers.

Proof. Note that, by Remark 1, the set E coincides with the lower topological limit $\tau\text{--}Li\, X_\alpha$. Let us prove the implication (a)$\Rightarrow$(b). Suppose condition (a) holds. Then the function $F\colon E \to [0,+\infty)$ is the S-limit of the net of functions $\{F^\alpha\colon X_\alpha \to [0,+\infty)\}_{\alpha\in A}$. Since the S-limit is stable with respect to τ-continuous perturbations (see, for example, [4] Proposition 6.20), for fixed values of $\lambda_j > 0$ and $\beta > 0$ the function $F(y)+\lambda_j^{-1}d_\tau^\beta(x,y)$ is the S-limit of the net

$$\left\{F^\alpha(y)+\lambda_j^{-1}d_\tau^\beta(x,y)\right\}_{\alpha\in A}; \tag{22}$$

here $y \in E$. On the other hand, net (22) is τ-equicoercive for every value of $x \in X$. Consequently, condition (b) is a direct consequence of Theorem 4 of [1]. Thus, the implication (a)$\Rightarrow$(b) is proved.

Let us prove the converse. Suppose that condition (b) holds. By Theorem 1, the relation

$$(\tau{-}li_s F^\alpha)(x) = (\tau{-}ls_s F^\alpha)(x) = \sup_{j\in N} \inf_{y\in(\tau-Li\,X_\alpha)} \left(F(y) + \lambda_j^{-1} d_\tau^\beta(x,y)\right)$$

holds on the set $\tau{-}Li\,X_\alpha$. By the conditions of the theorem, the function $F\colon (\tau{-}Li\,X_\alpha) \to [0,\infty)$ is τ-lower semicontinuous. Hence, according to Lemma 1, we can write

$$F(x) = \sup_{j\in N} \inf_{y\in(\tau-Li\,X_\alpha)} \left(F(y) + \lambda_j^{-1} d_\tau^\beta(x,y)\right).$$

Therefore, for all $x \in (\tau{-}Li\,X_\alpha)$ the relation

$$F(x) = (\tau{-}li_s F^\alpha)(x) = (\tau{-}ls_s F^\alpha)(x)$$

holds.

Thus the function F is the S-limit of the net

$$\{F^\alpha\colon X_\alpha \to [0,+\infty)\}_{\alpha\in A},$$

whence $\langle \inf_{x\in E} F(x)\rangle$ is the strong variational S-limit of the family of optimization problems (1). This proves the converse implication, (b)$\Rightarrow$(a). $\square$

Corollary. *If the set E in the theorem can be represented as*

$$E = \mathrm{Dom}\left(\sup_{\lambda>0} \lim_{\alpha\in A} \inf_{y\in X_\alpha} \left(c + \lambda^{-1} d_\tau^\beta(x,y)\right)\right),$$

then, by Remark 1, the following statements are equivalent:

(a) *for a net of conditional optimization problems* (1), *there exists an absolute variational S-limit, and it can be represented as $\langle \inf_{x\in E} F(x)\rangle$;*

(b) *for every $j \in N$ and all $x \in E$, the following relation holds:*

$$\inf_{y\in E} \left(F(y) + \lambda_j^{-1} d_\tau^\beta(x,y)\right) = \lim_{\alpha\in A} \inf_{y\in X_\alpha} \left(F^\alpha(y) + \lambda_j^{-1} d_\tau^\beta(x,y)\right),$$

where $\{\lambda_j\}_{j\in N}$ is a decreasing to zero monotone sequence of positive numbers.

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