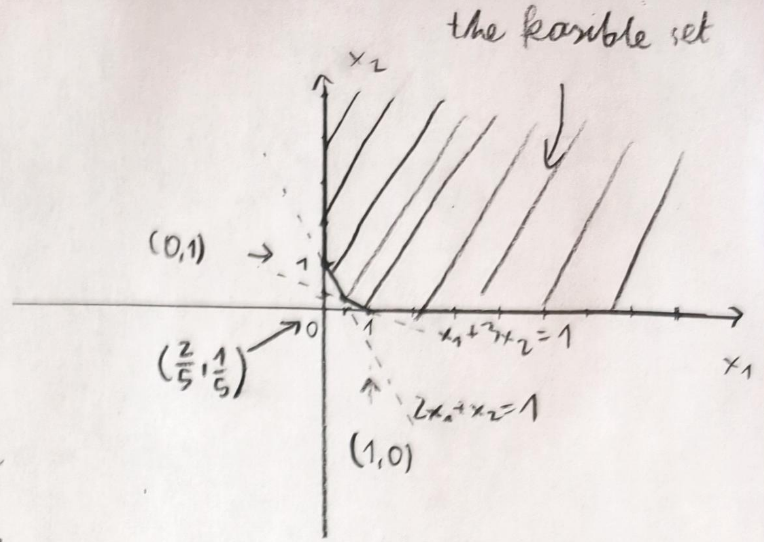
① min $f_0(x_1,x_2)$
s.t. $2x_1+x_2 \geq 1$
$x_1+3x_2 \geq 1$
$x_1 \geq 0, x_2 \geq 0$

The feasible set sketch:



$1-2x_1=x_2$
$x_1+3(1-2x_1)=1$
$x_1+3-6x_1=1$
$-5x_1=-2 \Rightarrow x_1=\frac{2}{5}$
$x_2=\frac{1}{5}$

a) $-\nabla f_0(x_1,x_2)=\begin{bmatrix}-1\\-1\end{bmatrix}$, so the optimal point is at the boundary (if not, we can find a more optimal point).

The vertices:

$f_0(0,1)=1=f_0(1,0)$

$f_0(\frac{2}{5},\frac{1}{5})=\frac{3}{5}$

$f_0(0,\alpha)>1\ \forall \alpha>1$ (the $x_2=0$ boundary)

$f_0(\alpha,0)>1\ \forall \alpha>1$ (the $x_1=0$ boundary)

Finally:

If $x_1 \in (\frac{2}{5},1) \Rightarrow x_1+x_2=x_1+\frac{1-x_1}{3}=\frac{2x_1+1}{3}$

$\frac{2}{5}<x_1<1 \iff \frac{3}{5}<\frac{2x_1+1}{5}<1$, so $f_0(x_1,x_2)>\frac{3}{5}$

If $x_2 \in (\frac{1}{5},1) \Rightarrow x_1+x_2=\frac{1-x_2}{2}+x_2=\frac{x_2+1}{2}$

$\frac{3}{5}<\frac{x_2+1}{2}<1 \Rightarrow$ so $f(x_1,x_2)=\frac{3}{5}$.

So, the optimal value is $\frac{3}{5}$, and the optimal set is $\left\{(\frac{2}{5},\frac{1}{5})\right\}$ (just one point $(\frac{2}{3},\frac{1}{5})$)

b) $-\nabla f_0(x_1,x_2)=\begin{bmatrix}1\\1\end{bmatrix}$, so $\forall x_1,x_2$, $\exists x_1',x_2'$ s.t. $f_0(x_1',x_2')=f_0(x_1,x_2)-\alpha\nabla f(x_1,x_2)<f_0(x_1,x_2)$ in the feasible set. So, then: $x_1,x_2 \to \infty$, the optimal value of $f_0(x_1,x_2)=-\infty$, and the optimal set does not exist.

c) $-\nabla f_0(x_1,x_2) = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$, so once again the optimal point is on the boundary. Also note:

$f_0(x_1,x_2) = x_1 \geq 0$ by the constraint, and:

$\forall x_2 \in$ feasible set

$f_0(0,x_2) = 0 \leq f_0(x_1,x_2) \; \forall x_1,x_2 \in$ feasible set.

Since 0 is the lowest possible value of $f_0(\cdot,\cdot)$, attained at any $(0,x_2)$, the optimal value is 0, and the optimal set is:

$\{(0,x_2) \mid x_2 \in [1,\infty)\}$

d) $f_0(x_1,x_2) = \max\{x_1,x_2\}$, the gradient depends on different regions, but this can be solved just by looking at the regions and constraints:

$x_1 > x_2 \Rightarrow f_0(x_1,x_2) = x_1$

then, we know:

$x_1 + 2x_1 > x_1 + 2x_2 \geq 1 \Leftrightarrow x_1 > \frac{1}{3} \Leftrightarrow f_0(x_1,x_2) > \frac{1}{3}$.

$x_1 = x_2 \Rightarrow f_0(x_1,x_2) = x_1 \Rightarrow x_1 + 2x_2 = 3x_1 \geq \frac{1}{3} \Rightarrow x_1 \geq \frac{1}{3} \Rightarrow f_0(x_1,x_2) \geq \frac{1}{3} \Rightarrow$ f_0 attains $f_0(x_1,x_2) = \frac{1}{3}$ at $(\frac{1}{3},\frac{1}{3}) \in$ feasible set

$x_1 < x_2 \Rightarrow f_0(x_1,x_2) = x_2 \Rightarrow x_2 + 2x_2 > x_1 + 2x_2 \geq 1 \Rightarrow x_2 > \frac{1}{3} \Rightarrow f_0(x_1,x_2) > \frac{1}{3}$.

So $\forall x_1,x_2 \in$ feasible set $f_0(x_1,x_2) \geq \frac{1}{3}$, so $\frac{1}{3}$ is the optimal value, and the optimal set is $\{(\frac{1}{3},\frac{1}{3})\}$

e) $-\nabla f_0(x_1,x_2) = \begin{bmatrix} 2x_1 \\ 18x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ at $(0,0) \notin$ feasible set, so the optimum point is at the boundary.

Then: $f_0(0,1) = 9$, $f_0(1,0) = 1$, $f_0(\frac{2}{5},\frac{1}{5}) = \frac{13}{25}$, $f_0(x_1,\frac{1-x_1}{3}) \Rightarrow x_1^2 + x_1^2 - 2x_1 + 1 \overset{\frac{d}{dx_1}}{\Rightarrow} 4x_1 - 2 = 0 \Rightarrow x_1 = \frac{1}{2}$, $x_2 = \frac{1}{6}$

$\Rightarrow f_0(\frac{1}{2},\frac{1}{6}) = \frac{1}{2}$, and finally: $f_0(\frac{1-x_2}{2},x_2) \Rightarrow \frac{x_2^2 + 1 - 2x_2}{4} + 9x_2^2 \overset{\frac{d}{dx}}{\Rightarrow} x_2 = \frac{1}{37}, x_1 = \frac{18}{37} \rightarrow$ not in feasible set, and $f_0(\frac{1-x_2}{2},x_2) \geq f_0(\frac{18}{37},\frac{1}{37}) \; \forall x_2$. Out of all these, clearly, the optimal value is $\frac{1}{2}$, and the optimal set is $\{(\frac{1}{2},\frac{1}{6})\}$.

② Let $F = [-1,1]^3$. We know that $x^* \in F$ since $\begin{bmatrix} 1 \\ \frac{1}{2} \\ -1 \end{bmatrix} \in F$.

Next: Notice that $P = P^T$, so:

$$\nabla \tfrac{1}{2}x^T P x + q^T x + r = \tfrac{1}{2} 2Px + q = Px + q$$

$$\Downarrow$$

$$\text{Hessian}\left\{ \nabla^2 \tfrac{1}{2}x^T P x + q^T x + r = \nabla Px + q = P. \right.$$

By checking the eigenvalues (which I did using a calculator, the approximate values are: $\lambda_1 \approx 27.9$, $\lambda_2 \approx 13.8$, $\lambda_3 \approx 0.26$), we see that P is positive definite so:

$$f(x) = \tfrac{1}{2}x^T P x + q^T x + r$$

is a convex function.

Then:

$$\nabla f(x^*) = Px^* + q = \begin{bmatrix} 13 & 12 & -2 \\ 12 & 17 & 6 \\ -2 & 6 & 12 \end{bmatrix} \begin{bmatrix} 1 \\ \frac{1}{2} \\ -1 \end{bmatrix} + \begin{bmatrix} -22.0 \\ -14.5 \\ 13.0 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}$$

Then, $\forall x \in F$:

$$\nabla f(x^*)^T (x - x^*) = [-1 \; 0 \; 2] \begin{bmatrix} x_1 - 1 \\ x_2 - \frac{1}{2} \\ x_3 + 1 \end{bmatrix} =$$

$$= -x_1 + 1 + 2x_3 + 2 = -x_1 + 2x_3 + 3 \geq 0 \quad \forall x \in F$$

(since $-x_1 + 2x_3 \geq -1-2 \geq -3 \Rightarrow -x_1 + 2x_3 + 3 \geq 0$).

This satisfies the necessary condition (F is a convex set, and $f'(x^*; x - x^*) \geq 0$, and so x^* is a local minimizer of f, and since f is convex it is also a global minimizer of f.)

So this shows that x^* is optimal for $\min f(x)$ s.t. $x \in F$ $\square$

③ $g_0(y,t) = t f_0(y/t)$ by definition. Similarly: $g_i(y,t) = t f_i(\frac{y}{t})$.

let: $y = \frac{x}{c^T x + d}$, $t = \frac{1}{c^T x + d}$, $t > 0$ because $\{x \in \mathrm{dom}\, f_0 \mid \underbrace{c^T x + d > 0}_{\frac{1}{t} > 0 \Rightarrow t > 0}\}$

Then, the new objective becomes:

$$\min f_0(x) / (c^T x + d) = \min \underbrace{\frac{1}{c^T x + d}}_{t} f_0\left(\frac{\frac{x}{c^T x + d}}{\frac{1}{c^T x + d}}\right) =$$

$$= \min t f_0\left(\frac{y}{t}\right) = \min g_0(y,t).$$

The constraints then are:

$$f_i(x) \leq 0 \Leftrightarrow t f_i(x) \leq 0 \Leftrightarrow t f_i\left(\frac{y}{t}\right) \leq 0 \Leftrightarrow g_i(y,t) \leq 0, \; i = 1, 2, \ldots, m.$$

(because $t > 0$ always)

$$Ax = b \Leftrightarrow A\frac{y}{t} = b \Leftrightarrow Ay = bt$$

(since $t > 0$).

And finally note that: $c^T y + dt = c^T \frac{x}{c^T x + d} + d \frac{1}{c^T x + d} = \frac{c^T x + d}{c^T x + d} = 1$.

So by equivalence:

$$\begin{array}{l} \min f_0(x) / c^T x + d \\ \text{s.t. } f_i(x) \leq 0 \quad i = 1, \ldots, m \\ \quad Ax = b \\ \{x \in \mathrm{dom}\, f_0 \mid c^T x + d > 0\} \end{array} \quad \Longleftrightarrow \quad \begin{array}{l} \min g_0(y,t) \\ \text{s.t. } g_i(y,t) \leq 0 \quad i = 1, \ldots, m \\ \quad Ay = bt \\ \quad c^T y + dt = 1 \\ t > 0, \; \{y \in \mathrm{dom}\, g_0(\cdot, t) \mid t > 0\}. \end{array}$$

$\square$

4. $\min c^T x$
s.t. $1^T x = 1$, $x \succeq 0$. The constraint is equivalent to:

$$\sum_{i=1}^{n} x_i = 1, \; x_i \geq 0.$$

The goal is to minimize:

$$c^T x = \sum_{i=1}^{n} c_i x_i \quad \text{where } x_i \in [0,1] \text{ and } \sum_{i=1}^{n} x_i = 1.$$

Now, notice that this is a convex combination of c_i's. Also:

$$c^T x = \sum_{i=1}^{n} c_i x_i \geq \sum_{i=1}^{n} \underbrace{(\min_j c_j)}_{\text{smallest } c_i} x_i = \min_j c_j \sum_{i=1}^{n} x_i = \min_j c_j$$

So $c^T x$ is minimized when $c^T x = \min_j c_j$. Let $c_k = \min_j c_j$

Then $c^T x = c_k$ iff $\sum_{i=1}^{n} c_i x_i = c_k \iff c_k \cdot 1 + \sum_{i=1 \setminus \{k\}}^{n} c_i \cdot 0 = c_k$.

So the solution is: Let $c_k = \min_j c_j$, then $x = [0, \dots, 1, \dots, 0]^T$ or $\underset{\text{idx}}{} \; [1, \dots, \underset{k}{}, \dots, n]$

$x_k = 1$ and $x_i = 0 \; \forall i \in \{0, 1, \dots, n\} \setminus \{k\}$, where $x \in \mathbb{R}^n$.

If the problem becomes:

$\min c^T x$
s.t. $1^T x \leq 1$, $x \succeq 0$ then this is equivalent to: $\min c^T x$ s.t. $\sum_{i=1}^{n} x_i \leq 1$, $x_i \geq 0$.

This is still true:

$c^T x \geq \min_j c_j \sum_{i=1}^{n} x_i$, BUT: if $\min_j c_j > 0$, then $\min_j c_j \sum_{j=1}^{n} x_i \geq 0$ iff $\sum_{j=1}^{n} x_i = 0$, so the optimal solution is $x = \vec{0}$.

If $c_k \leq 0$ ($c_k = \min_j c_j$) $c^T x = c_k \underbrace{x_k}_{1}$ is the optimal solution. So the new solution is:

$$x = \begin{cases} \vec{0}, & c_k = \min_j c_j \geq 0 \\ [0, \dots, \underset{k}{1}, \dots, 0]^T, & c_k = \min_j c_j < 0 \end{cases}$$

(5) The problem is:

$$\min c^T x$$
$$\text{s.t. } Ax \leq b \quad \forall A \in \mathcal{A}$$

We can rewrite the constraint as:

$$\sum_{j=1}^{n} A_{ij} x_j \leq b_i \quad \text{or} \quad A_i x \leq b_i$$ where A_{ij} is the i-th row, j-th column element of A and A_i is the i-th row of A.

Then, we know that:

≥ 0 for this to make sense.

$$\bar{A}_{ij} - V_{ij} \leq A_{ij} \leq \bar{A}_{ij} + V_{ij}$$ from the definition of $\mathcal{A}$.

since $\sum_{j=1}^{n} A_{ij} x_j \leq b_i$ has to hold $\forall A_{ij} \in [\bar{A}_{ij} - V_{ij}, \bar{A}_{ij} + V_{ij}]$ it is equivalent to: $$\sum_{j=1}^{n} \max_{A_{ij}} (A_{ij} x_j) \leq b_i$$

choose A_{ij} that makes $A_{ij} x_j$ maximized.

Now, notice that for a fixed j, if $x_j \geq 0$, the A_{ij} that maximizes $A_{ij} x_j$ is $(\bar{A}_{ij} + V_{ij})$, and if $x < 0$, we choose smallest $A_{ij} = \bar{A}_{ij} - V_{ij}$ to maximize $A_{ij} x_j$.
(largest val)

(assuming $V_{ij} \geq 0$) so $\Rightarrow$ $$\max_{A_{ij}} (A_{ij} x_j) = \begin{cases} (\bar{A}_{ij} + V_{ij}) x_j, & x_j \geq 0 \\ (\bar{A}_{ij} - V_{ij}) x_j, & x_j < 0 \end{cases} = \bar{A}_{ij} x_j + V_{ij} |x_j|.$$

So the problem becomes: $$\min c^T x$$
$$\text{s.t. } \bar{A}_i x + \sum_{i=1}^{n} V_{ij} |x_j| \leq b_i, \quad i = 1, \dots, m.$$

↻

5 cont. This still isn't an LP problem since $\sum_{i=1}^{n} V_{ij} |x_j|$ is non-linear.
But: let $y_j \geq |x_j|$, $j = 1, \dots, n$ (a new constraint).
This can be rewritten as:

$$y_j \geq x_j \quad \text{and} \quad y_j \geq -x_j$$

$$\Leftrightarrow$$

$$-y_j \leq -x_j \quad \text{and} \quad -y_j \leq x_j$$

Given that, we arrive at the new formulation in the LP form:

$$\begin{aligned} & \min c^T x \\ \text{linear constraints} \quad & \begin{cases} \text{s.t. } \bar{A}_i x + \sum V_{ij} \overbrace{y_j}^{\text{replaced } |x_j|} \leq b_i, & i = 1, \dots, m \\ -y_j \leq -x_j, & j = 1, \dots, n \\ -y_j \leq x_j, & j = 1, \dots, n \end{cases} \end{aligned}$$

Or in a vector form

$$\begin{aligned} & \min c^T x \\ & \text{s.t. } \bar{A} x + V y \preceq b \\ & -y \preceq x \\ & -y \preceq -x \\ & x \in \mathbb{R}^n, y \in \mathbb{R}^n \end{aligned}$$

Both are forms of an LP problem.

⑥ Firstly, $c^T x$ is linear, so the solution will be on the boundary of the feasible set (which exists, because the constraint set is compact, because A is positive definite).

Since A is symmetric $A = A^T$. Then, we can use change of variables:

$y = x - x_c \Rightarrow x = y + x_c$. The new problem is:

$$\min c^T y + \underbrace{c^T x_c}_{\text{const, so drop}} \quad \text{s.t. } y^T A y \le 1 \iff \min c^T y \quad \text{s.t. } (A^{\frac{1}{2}} y)^T (A^{\frac{1}{2}} y) \le 1 \iff \min (A^{-\frac{1}{2}} c)^T z \quad \text{s.t. } z^T z \le 1$$

let $z = (A^{\frac{1}{2}} y) \Rightarrow y = A^{-\frac{1}{2}} z$

$$y^T A y = y^T A^{\frac{1}{2}} A^{\frac{1}{2}} y = (\underbrace{A^{\frac{1}{2} T}}_{= A^{\frac{1}{2}} \text{ since symmetric}} y)^T A^{\frac{1}{2}} y = (A^{\frac{1}{2}} y)^T (A^{\frac{1}{2}} y)$$

$$c^T y = c^T A^{-\frac{1}{2}} z = (A^{-\frac{1}{2} T} c)^T z = (A^{-\frac{1}{2}} c)^T z$$

symmetric.

Since the solution is still on the boundary (and linear) $z^T z = 1$

$\Rightarrow \|z\|^2 = 1 \Rightarrow \underbrace{\|z\| = 1}_{\text{2-norm}}$. So z is a unit vector and $(A^{-\frac{1}{2}} c)$ is some vector. From the properties of the dot product:

$$(A^{-\frac{1}{2}} c)^T z = \|A^{-\frac{1}{2}} c\| \|z\| \cos \underbrace{\theta}_{\text{angle between}} \ge \underbrace{-\|A^{-\frac{1}{2}} c\| \|z\|}_{\text{min possible value}}$$

Note that the minimum value is achieved when $\cos\theta = -1 \Rightarrow \theta = \pi$, i.e. the vectors are in opposite directions! So, by that, the z that minimizes $(A^{-\frac{1}{2}} c)^T z$ is: $z^* = -\frac{A^{-\frac{1}{2}} c}{\|A^{-\frac{1}{2}} c\|}$ } has to be a unit vector. So:

$$y^* = -\frac{\overbrace{A^{-\frac{1}{2}} A^{-\frac{1}{2}}}^{A^{-1}} c}{\|A^{-\frac{1}{2}} c\|} \Rightarrow x^* = x_c - \frac{A^{-1} c}{\|A^{-\frac{1}{2}} c\|} \text{ and } c^T x^* = c^T x_c - \frac{c^T A^{-1} c}{\|A^{-\frac{1}{2}} c\|}$$

⑦ The original problem is:

$$\max \sum_{t=0}^{T} \beta^t u(c_t) \quad \} \text{ concave, increasing}$$

$$\text{s.t. } k_{t+1} = k_t + f(k_t) - c_t,\ t = 0, \dots, T \quad \} \text{ concave in } k_t$$

$$k_t \geq 0,\ t = 1, \dots, T+1 \quad \} \text{ concave (affine)}$$

this is equivalent to:

$$\max \sum_{t=0}^{T} \beta^t u(c_t)$$

$$\text{s.t. } k_{t+1} - k_t - f(k_t) + c_t = 0,\ t = 0, \dots, T$$

$$-k_t \leq 0,\ t = 1, \dots, T+1$$

} both are convex $\forall t$ since $-k_t$ is linear, i.e. convex, and $-f(k_t)$ is convex, so $\underbrace{k_{t+1} - k_t + c_t}_{\text{linear}} \underbrace{- f(k_t)}_{\text{convex}}$ is convex.

Now, I claim that the following convex optimization problem is equivalent:

$$\max \sum_{t=0}^{T} \beta^t u(c_t) \quad \} \text{ concave.}$$

$$\text{s.t. } k_{t+1} - k_t - f(k_t) + c_t \leq 0,\ t = 0, \dots, T$$

$$-k_t \leq 0,\ t = 1, \dots, T+1$$

} convex ineq.

That's because for optimal k^*, c^*, this: $k^*_{t+1} - k^*_t - f(k^*_t) + c^*_t$ has to be equal to 0 (i.e. the equality will hold anyway). Why? Suppose for some (any) t:

$$k^*_{t+1} < k^*_t + f(k^*_t) - c^*_t$$

this means $\exists\, c'_t > c^*_t$ s.t. $k^*_{t+1} = k^*_t + f(k^*_t) - c'_t$. But, then

$$U' = \sum_{i \in \{0, \dots, T\} \setminus \{t\}} \beta^i u(c^*_i) + \beta^t u(c'_t) > \sum_{i=0}^{T} \beta^i u(c^*_i) = U \ ↯ \text{ (contradicts optimality of } c^*)$$

So the equality has to hold in $k_{t+1} - k_t - f(k_t) + c_t = 0$ for the optimal k^*, c^*.

So the problem of maximizing U can be formulated as the following problem:

$$\max \sum_{t=0}^{T} \beta^t u(c_t)$$

$$\text{s.t. } k_{t+1} - k_t - f(k_t) + c_t \leq 0, \quad t = 0, \dots, T$$

$$-k_t \leq 0 \quad , t = 1, \dots, T+1$$

which is a convex optimization problem (concave maximization to be precise).