① Let: $A = \{\theta_1 x_1 + \theta_2 x_2 + \dots + \theta_k x_k \mid x_1, x_2, \dots, x_k \in C,\ \theta_1 + \theta_2 + \dots + \theta_k = 1\}$.

First of all note that $\forall x_i \in C$, $x_i = \underbrace{1}_{\theta_1} \cdot x_i + 0 + 0 = 1 \cdot x_i$, so $x_i \in A$ (since x_i is combination of x_i with $\theta_1 = 1$ with all other $\theta_2, \dots = 0$, so x_i is in A). Since $\forall x_i \in C \Rightarrow x_i \in A \Rightarrow C \subseteq A$.

Next, let $x_1', \dots, x_k' \in A$, and $\theta_1 + \dots + \theta_k = 1$.

Then:

$$\theta_1 x_1' + \theta_2 x_2' + \dots + \theta_k x_k' = \theta_1 \overbrace{\left(\alpha_1 x_1^1 + \alpha_2 x_2^1 + \dots + \alpha_{j^1} x_{j^1}\right)}^{x_1'} +$$

$$\dots + \theta_k \underbrace{\left(\gamma_1 x_1^k + \gamma_2 x_2^k + \dots + \gamma_{j^k} x_{j^k}\right)}_{x_k'} = \theta_1 \alpha_1 x_1^1 + \theta_1 \alpha_2 x_2^1 + \dots + \theta_1 \alpha_{j^1} x_{j^1}$$

$$+ \dots + \theta_k \gamma_1 x_1^k + \theta_k \gamma_2 x_2^k + \dots + \theta_k \gamma_{j^k} x_{j^k},$$ where

$x_1^1, x_2^1, \dots, x_1^k, \dots, x_{j^k} \in C$ and $\alpha_1 + \alpha_2 + \dots + \alpha_{j^1} = 1$ and ... and $\gamma_1 + \dots + \gamma_{j^k} = 1$.

Since we know that $x_1^1, x_2^1, \dots, x_{j^k} \in C$, and:

$$\theta_1 \alpha_1 + \theta_1 \alpha_2 + \dots + \theta_1 \alpha_{j^1} + \dots + \theta_k \gamma_1 + \dots + \theta_k \gamma_{j^k} = \theta_1 \overbrace{(\alpha_1 + \dots + \alpha_{j^1})}^{1} + (\dots + \theta_k \overbrace{(\gamma_1 + \dots + \gamma_{j^k})}^{1}$$

$= \theta_1 + \dots + \theta_k = 1$ (by assumption), then by definition, the combination

$$\theta_1 x_1' + \dots + \theta_k x_k' \in A$$

(is in A). Since this is true for any combination of points in A for $\theta_1 + \dots + \theta_k = 1$, then by definition A is an affine set. ?

So I showed that A is affine and $C \subseteq A$. Since, from definition, $\mathrm{aff}(C) :=$ the smallest affine set that contains C, then $\mathrm{aff}(C) \subseteq A$ (since A is an affine set containing C).

Next, let $y \in A$. Then we know that $y = \theta_1 x_1 + \ldots + \theta_k x_k$, $x_1, \ldots, x_k \in C$, $\theta_1 + \ldots + \theta_k = 1$. Now, note that this is an affine combination of points in C, so this combination must belong to an affine hull of C (since the points are in C and y is an affine combination). So therefore $y \in \mathrm{aff}(C)$, so $y \in A \Rightarrow y \in \mathrm{aff}(C)$ $\forall y \in A$ (since it was arbitrary). Therefore $A \subseteq \mathrm{aff}(C)$.

And since $A \subseteq \mathrm{aff}(C)$ and $\mathrm{aff}(C) \subseteq A$ (from before) $\Rightarrow A = \mathrm{aff}(C)$, which shows that

$$\mathrm{aff}(C) = \{\theta_1 x_1 + \theta_2 x_2 + \ldots + \theta_k x_k \mid x_1, x_2, \ldots, x_k \in C, \theta_1 + \theta_2 + \ldots + \theta_k = 1\}$$

$\square$

2) Using θ's:

$$\mathrm{aff}(C) = \{y \mid y = (\theta_1, 2\theta_2, 3\theta_3), \theta_1 + \theta_2 + \theta_3 = 1\}$$

$$\mathrm{conv}(C) = \{y \mid y = (\theta_1, 2\theta_2, 3\theta_3), \theta_1 + \theta_2 + \theta_3 = 1, 0 \leq \theta_1, \theta_2, \theta_3 \leq 1\}$$

Based on geometry

$$\mathrm{aff}(C) = \{(x, y, z) \mid \frac{x}{1} + \frac{y}{2} + \frac{z}{3} = 1\} \rightarrow \text{a plane}$$

$$\mathrm{conv}(C) = \{(x, y, z) \mid \frac{x}{1} + \frac{y}{2} + \frac{z}{3} = 1, x, y, z \geq 0\} \rightarrow \text{a triangle}$$

③ Let A, B be convex sets.

We know that:

$\forall x_1, \dots, x_k \in A : \theta_1 x_1 + \dots + \theta_k x_k \in A \quad \forall \theta_1 + \dots + \theta_k = 1, \; 0 \leq \theta_1, \dots, \theta_k \leq 1$

and

$\forall y_1, \dots, y_k \in B : \theta_1 y_1 + \dots + \theta_k y_k \in B \quad \forall \theta_1 + \dots + \theta_k = 1, \; 0 \leq \theta_1, \dots, \theta_k \leq 1.$

Then:

$A \cap B = \{z \mid z \in A \wedge z \in B\}$

Now let $z_1, \dots, z_k \in A \cap B$, so then $z_1, \dots, z_k \in A$ and $z_1, \dots, z_k \in B$

Since $z_1, \dots, z_k \in A$, then by definition, $\forall \theta_1 + \dots + \theta_k = 1, \; 0 \leq \theta_1, \dots, \theta_k \leq 1$,

$\theta_1 z_1 + \dots + \theta_k z_k \in A$

Also, since $z_1, \dots, z_k \in B$, then again, $\forall \theta_1 + \dots + \theta_k = 1, 0 \leq \theta_1, \dots, \theta_k \leq 1$

$\theta_1 z_1 + \dots + \theta_k z_k \in B$

Since $\theta_1 z_1 + \dots + \theta_k z_k \in A$ and $\theta_1 z_1 + \dots + \theta_k z_k \in B \; \forall \theta_1 + \dots + \theta_k = 1$, $0 \leq \theta_1, \dots, \theta_k \leq 1$, then $\theta_1 z_1 + \dots \theta_k z_k \in A \cap B$ (by definition).

This means that for any $z_1, \dots, z_k \in A \cap B$, $\forall \theta_1 + \dots + \theta_k = 1, \; 0 \leq \theta_1, \dots, \theta_k \leq 1$,

$\theta_1 z_1 + \dots + \theta_k z_k \in A \cap B$

and this, by definition, means that $A \cap B$ is a convex set $\square$.

Also for a case: $A_1, A_2, \dots, A_k$, where A_i is convex $\forall i \in \{1, \dots, k\}$, any intersection $\bigcap_{i \in \{1, \dots, k\}} A_i$ can be thought as: $A_{i_1} \cap A_{i_2} \cap \dots \cap A_{i_j}$. Then, they are pairwise convex $\rightarrow$ apply proof above until last pair. $\square$

(4) let $A = \{x \in \mathbb{R}^2_+ \mid x_1 x_2 \geq 1\}$. Then let $x_1, x_2, \ldots, x_k \in A$.
Also, let: $\theta_1 + \theta_2 + \ldots + \theta_k = 1, 0 \leq \theta_1, \ldots, \theta_k \leq 1$.
First, let:

$$x_{new} = \theta_1 x_1 + \theta_2 x_2 + \ldots + \theta_k x_k.$$

Then note that: $x_{new} = \begin{bmatrix} \theta_1 x_1^1 + \theta_2 x_2^1 + \ldots + \theta_k x_k^1 \\ \theta_1 x_1^2 + \theta_2 x_2^2 + \ldots + \theta_k x_k^2 \end{bmatrix}$.

Since $0 \leq \theta_1, \ldots, \theta_k \leq 1$ and $0 \leq x_1^1, x_2^1, \ldots, x_k^1, x_1^2, x_2^2, \ldots, x_k^2$ (by definition of A), then $\theta_1 x_1^1 + \theta_2 x_2^1 + \ldots + \theta_k x_k^1 \geq 0$ and $\theta_1 x_1^2 + \ldots + \theta_k x_k^2 \geq 0$, so $x_{new} \in \mathbb{R}^2_+$.

Next, let $x, y \in A$, and $\theta \in [0,1]$. Then:

$$\theta x + (1-\theta) y = \begin{bmatrix} \theta x_1 \\ \theta x_2 \end{bmatrix} + \begin{bmatrix} (1-\theta) y_1 \\ (1-\theta) y_2 \end{bmatrix} = \begin{bmatrix} \theta x_1 + (1-\theta) y_1 \\ \theta x_2 + (1-\theta) y_2 \end{bmatrix}$$

From before we know that $\theta x + (1-\theta) y \in \mathbb{R}^2_+$. Next:

$$(\theta x_1 + (1-\theta) y_1)(\theta x_2 + (1-\theta) y_2) \underset{\text{by hint}}{\geq} x_1^\theta y_1^{(1-\theta)} \cdot x_2^\theta y_2^{(1-\theta)} = (x_1 x_2)^\theta (y_1 y_2)^{(1-\theta)} \geq$$

$1^\theta \cdot 1^{(1-\theta)} = 1 \Rightarrow$ so $\theta x + (1-\theta) y \in A$. Since a convex combination
(since $x_1 x_2 \geq 1$ and $y_1 y_2 \geq 1$ since $x, y \in A$)
of any 2 points $x, y \in A$ is in A, by extension (follows from induction), a convex combination $x_{new} = \theta_1 x_1 + \ldots + \theta_k x_k$, $\theta_1 + \ldots + \theta_k = 1, 0 \leq \theta_1, \ldots, \theta_k \leq 1$ of any k points is also in A, and therefore A is a convex set □.

Finally, if $A = \{x \in \mathbb{R}^n_+ \mid \prod_{i=1}^n x_i \geq 1\}$, let $x, y \in A$, $\theta \in [0,1]$. Then $\theta x + (1-\theta) y = \begin{bmatrix} \theta x_1 + (1-\theta) y_1 \\ \vdots \\ \theta x_n + (1-\theta) y_n \end{bmatrix}$ is in $\mathbb{R}^n_+$ (since $x_i, y_i, \theta \geq 0 \ \forall i \in \{1, \ldots, n\}$). Also: $\prod_{i=1}^n \theta x_i + (1-\theta) y_i \geq$

$\underset{\text{by hint}}{\geq} \prod_{i=1}^n x_i^\theta y_i^{(1-\theta)} = \left(\prod_{i=1}^n x_i\right)^\theta \left(\prod_{i=1}^n y_i\right)^{1-\theta} \underset{\text{definition}}{\geq} 1^\theta \cdot 1^{(1-\theta)} = 1$. So $\theta x + (1-\theta) y \in A$, and by the same argument, A is a convex set. □ -

⑤ a) It is convex. Let $x_1, \dots, x_k \in \{x \in \mathbb{R}^n \mid \alpha \le a^T x \le \beta\} = A$.
Let $\theta_1 + \dots + \theta_k = 1$, $0 \le \theta_1, \dots, \theta_k \le 1$. Then:

$$\theta_1 x_1 + \theta_2 x_2 + \dots + \theta_k x_k \in \mathbb{R}^n \text{ and:}$$

$$a^T(\theta_1 x_1 + \theta_2 x_2 + \dots + \theta_k x_k) \underset{\substack{\text{since}\\ \text{vector multiplication}\\ \text{is distributive}}}{=} \theta_1 a^T x_1 + \theta_2 a^T x_2 + \dots + \theta_k a^T x_k.$$

Since $x_1, \dots, x_k \in A$, then
$$\left.\begin{array}{l} \alpha \le a^T x_1 \le \beta \\ \alpha \le a^T x_2 \le \beta \\ \vdots \\ \alpha \le a^T x_k \le \beta \end{array}\right. \Rightarrow + \left\{\begin{array}{l} \theta_1 \alpha \le \theta_1 a^T x_1 \le \theta_1 \beta \\ \vdots \\ \theta_k \alpha \le \theta_k a^T x_k \le \theta_k \beta \end{array}\right.$$

$$\Rightarrow \overbrace{\theta_1 \alpha + \theta_2 \alpha + \dots + \theta_k \alpha}^{= \alpha(\theta_1 + \dots + \theta_k) = \alpha} \le \theta_1 a^T x_1 + \theta_2 a^T x_2 + \dots + \theta_k a^T x_k \le \overbrace{\theta_1 \beta + \theta_2 \beta + \dots + \theta_k \beta}^{= \beta(\theta_1 + \dots + \theta_k) = \beta}$$

$$\Leftrightarrow$$

$$\alpha \le \theta_1 a^T x_1 + \theta_2 a^T x_2 + \dots + \theta_k a^T x_k \le \beta$$

So therefore $\theta_1 x_1 + \theta_2 x_2 + \dots + \theta_k x_k \in A$, and since this is true for arbitrary $\theta_1 + \dots + \theta_k = 1$, $0 \le \theta_1, \dots, \theta_k \le 1$ and any $x_1, \dots, x_2 \in A$, then A is a convex set.

c) It is convex. First, note that $\{x \in \mathbb{R}^n \mid a_1^T x \le b_1, a_2^T x \le b_2\} =$
$= \{x \in \mathbb{R}^n \mid a_1^T x \le b_1\} \cap \{x \in \mathbb{R}^n \mid a_2^T x \le b_2\}$, i.e. it is an intersection of 2 sets. Then let $A = \{x \in \mathbb{R}^n \mid a^T x \le b\}$ be a generic set. Let $x_1, \dots, x_k \in A$, $\theta_1 + \dots + \theta_k = 1$, $0 \le \theta_1, \dots, \theta_k \le 1$. Then:

$$\theta_1 x_1 + \dots + \theta_k x_k \in \mathbb{R}^n, \text{ and}$$

$a^T(\theta_1 x_1 + \dots + \theta_k x_k) = \theta_1 a^T x_1 + \dots + \theta_k a^T x_k$. Since $x_1, \dots, x_k \in A$,

then: $\begin{matrix} a^T x_1 \leq b \\ \vdots \\ a^T x_k \leq b \end{matrix} \Rightarrow \left\{ \begin{matrix} \theta_1 a^T x_1 \leq \theta_1 b \\ \vdots \\ + \; \theta_k a^T x_k \leq \theta_k b \end{matrix} \right. \Rightarrow \theta_1 a^T x_1 + \dots + \theta_k a^T x_k \leq \theta_1 b + \theta_2 b + \dots + \theta_k b$

$\Rightarrow \theta_1 a^T x_1 + \dots + \theta_k a^T x_k \leq b(\theta_1 + \theta_2 + \dots + \theta_k) = b$

So: $a^T(\theta_1 x_1 + \dots + \theta_k x_k) \leq b$. Since this is true for any $x_1, \dots, x_k$, $\forall \theta_1 + \dots + \theta_k = 1$, $0 \leq \theta_1, \dots, \theta_k \leq 1$, A is a convex set.

The wedge set is then intersection of 2 sets A_1 and A_2 (both convex), and therefore as proven before, is also convex.

e) Not convex. Counterexample: $S = \{(-1,0), (1,0)\}$, $T = \{(0,0)\}$.

Let $A = \{x \mid dist(x, S) \leq dist(x, T)\}$ where $dist(x, S) = \inf\{\|x - z\|_2 \mid z \in S\}$

Then, let $x_1 = (2,0)$ and $x_2 = (-2,0)$. Then $x_1 \in A$ and $x_2 \in A$ (since $dist(x_1, S) = 1$, $dist(x_2, S) = 1$, $dist(x_1, T) = 2$, $dist(x_2, T) = 2$), but if $\theta_1 = 0.5$, $\theta_2 = 0.5$ then: $x_1\theta_1 + x_2\theta_2 = (\frac{2}{2}, 0) + (-\frac{2}{2}, 0) = (0,0)$ and $dist(\theta_1 x_1 + \theta_2 x_2, S) = 1$ but $dist(\theta_1 x_1 + \theta_2 x_2, T) = 0$, so $\theta_1 x_1 + \theta_2 x_2 \notin A$, so A is not a convex set.

5 cont

g) let $A = \{x \mid \|x-a\|_2 \le \theta \|x-b\|_2\}$, $a \ne b$, $0 \le \theta \le 1$

A is convex. Proof:

If $\theta = 0 \Rightarrow A = \{x \mid \|x-a\|_2 \le 0\} = \{a\}$. Then a is convex, since

just a

$\forall \theta_1 + \dots + \theta_k = 1, 0 \le \theta_1, \dots, \theta_k \le 1$, $\theta_1 a + \dots + \theta_k a = a \in A$.

If $\theta = 1 \Rightarrow A = \{x \mid \|x-a\|_2 \le \|x-b\|_2\} = \{x \mid (x-a)^T(x-a) \le (x-b)^T(x-b)\} =$

since both sides are positive

$= \{x \mid 2(b-a)^T x \le b^T b - a^T a\}$.

Then let $x_1, \dots, x_k \in A$, $\theta_1 + \dots + \theta_k = 1, 0 \le \theta_1, \dots, \theta_k \le 1 \Rightarrow x' = \theta_1 x_1 + \dots + \theta_k x_k$

$$2(b-a)^T x' = 2(b-a)^T(\theta_1 x_1 + \dots + \theta_k x_k) = \underbrace{2\theta_1 (b-a)^T x_1}_{\le \theta_1 (b^T b - a^T a)} + \dots + 2\theta_k (b-a)^T x_k \le$$

$\le (b^T b - a^T a)(\theta_1 + \dots + \theta_k) = b^T b - a^T a$. So then, A is convex.

Finally, for

$0 < \theta < 1$:

In this case we have: $A = \{x \mid f(x) \le 0\}$ where

$f(x) = (1-\theta^2) x^T x - 2(a - \theta^2 b)^T x + \|a\|^2 - \theta^2 \|b\|^2 \le 0$. Now, this is a power function with power equal to 2, and in class, we said that power functions are convex, so $f(x)$ is a convex function. Then, since $A = \{x \in dom(f) \mid f(x) \le 0\}$, and since $f(x)$ is a convex function, then from class we know that A is convex set, since it's a sub-level set of a convex function f.

So therefore, A is a convex set $\forall\, 0 \le \theta \le 1$. □

Convex Functions:

1. First of all, we know that $\mathbb{R}^n$ is a convex set.

Then, let $f(x) = \|x\|$ be a norm (any norm). It follows that:

$$f(\theta x+(1-\theta)y) = \|\theta x + (1-\theta)y\| \underset{\text{by (iv)}}{\leq} \|\theta x\| + \|(1-\theta)y\| \underset{\text{by (iii)}}{=} |\theta|\,\|x\| + |1-\theta|\,\|y\|$$

$$\underset{\substack{\text{since}\\ \theta\in[0,1]}}{=} \theta\|x\| + (1-\theta)\|y\| = \theta f(x) + (1-\theta)f(y) \quad \forall x,y \in \mathbb{R}^n,\ 0\leq\theta\leq 1,$$

and by definition (of a convex function), $f(x) = \|x\|$ is a convex function (for any norm)

2. Firstly, assume f is differentiable in $dom(f)$. Also f is convex iff $dom(f)$ is a convex set, so that has to hold. Next:

f is convex $\Rightarrow f(y) \geq f(x) + f'(x)(y-x)$. Let $t \in (0,1]$. Then by convexity:

$$f(\underbrace{(1-t)x+ty}_{=x+t(y-x)}) \leq (1-t)f(x) + tf(y) \Leftrightarrow f(x+t(y-x)) \leq f(x) + t(f(y)-f(x)) \Leftrightarrow$$

$$\Leftrightarrow \frac{f(x+t(y-x))-f(x)}{t(y-x)} \leq \frac{f(y)-f(x)}{y-x} \overset{\lim_{t\to 0}}{\Longrightarrow} \underbrace{\lim_{t\to 0}\frac{f(x+t(y-x))-f(x)}{t(y-x)}}_{=f'(x)} \leq \frac{f(y)-f(x)}{y-x} \Leftrightarrow$$

(assume, $x,y \in dom(f)$, $y>x$ without the loss of generality) (if $x>y$, just flip the sign, and then flip again when multiplying by $(y-x)<0$)

$$f'(x) \leq \frac{f(y)-f(x)}{y-x} \Leftrightarrow f(y) \geq f(x) + f'(x)(y-x),$$ so this proves $\Rightarrow$ direction.

Then

f is convex $\Leftarrow f(y) \geq f(x) + f'(x)(y-x)$. For any x,y, $y>x$, let $z = tx+(1-t)y$. Then:

$t\in[0,1]$, $x,y \in dom(f)$

next page.

③ a) The set is: $\{\lambda \mid \frac{1}{\mu-\lambda} \leq d\}$ which is equivalent to $\{\lambda \mid \frac{1}{\mu-\lambda} - d \leq 0\}$. obviously $\frac{1}{\mu-\lambda} > 0$, so $\mu > \lambda$.

Then $f(\lambda) = \frac{1}{\mu-\lambda} - d$ is a convex function for $\mu > \lambda$ (which it is)

As a result, $\{\lambda \mid \frac{1}{\mu-\lambda} - d \leq 0\}$ is a sublevel set of a convex function $f(\lambda)$, so $\{\lambda \mid \frac{1}{\mu-\lambda} - d \leq 0\}$ is a convex set.

If d is a variable:

$$\{(\lambda, d) \mid \frac{1}{\mu-\lambda} - d \leq 0\} \Rightarrow f(\lambda, d) = \frac{1}{\mu-\lambda} - d.$$

$\frac{1}{\mu-\lambda}$ is convex in λ (assuming $\mu > \lambda$, which holds), and $-d$ is linear in d, so $f(\lambda, d) = \frac{1}{\mu-\lambda} - d$ is convex, so the sublevel set $\{(\lambda, d) \mid \frac{1}{\mu-\lambda} - d \leq 0\}$ is a convex set.

b) $A = \{[P_1, \dots, P_I] \mid W \log(1 + \frac{P_0}{\sum_{i \in \{1,\dots,I\}} P_i + N}) > r\}$. The condition is:

$$W \log\left(1 + \frac{P_0}{\sum_i P_i + N}\right) > r \Leftrightarrow \log\left(1 + \frac{P_0}{\sum_i P_i + N}\right) > \frac{r}{W} \Leftrightarrow 1 + \frac{P_0}{\sum_i P_i + N} > e^{\frac{r}{W}}$$

$$\Leftrightarrow e^{\frac{r}{W}} - 1 - \frac{P_0}{\underbrace{\sum_i P_i + N}_{\text{positive}}} < 0 \Leftrightarrow \left(\sum_i P_i + N\right)\left(e^{\frac{r}{W}} - 1\right) - P_0 < 0$$

Then: Since this is linear in P_i (and so convex), A is a convex set.

If r is a variable, $e^{\frac{r}{W}} \cdot \sum_i P_i$ is NOT convex, so the set is also not convex.

↷

We have $z = tx + (1-t)y$, $x, y \in \mathbb{R}$. Using the assumption:

We know that:

$$f(y) \geq f(z) + f'(z)(y-z) \quad / \cdot (1-t)$$
$$f(x) \geq f(z) + f'(z)(x-z) \quad / \cdot t$$

$$+ \Downarrow$$

$$t f(x) + (1-t) f(y) \geq \overbrace{t f(z) + (1-t) f(z)}^{f(z)} + t f'(z)(x-z) + (1-t) f'(z)(y-z)$$

$$\Leftrightarrow$$

$$t f(x) + (1-t) f(y) \geq f(\overbrace{tx + (1-t)y}^{z}) + t f'(z)(x-z) + (1-t) f'(z)(y-z).$$

Finally:

$$t f'(z)(x-z) + (1-t) f'(z)(y-z) = t f'(z) x - \cancel{t f'(z) z} + f'(z) y - f'(z) z$$
$$+ \cancel{t f'(z) z} - t f'(z) y = \cancel{t f'(z) x} + f'(z) y - t f'(z) y \overbrace{- \cancel{t f'(z) x} - f'(z)(1-t) y} =$$
$$= (1-t) f'(z) y - f'(z)(1-t) y = 0.$$ So we get

$$t f(x) + (1-t) f(y) \geq f(tx + (1-t)y) \quad \text{for } x, y \in \mathbb{R} \ (x \neq y),$$
for $t \in [0,1]$.

Since this is the definition of convexity, this proves the $\Leftarrow$ direction. Since both directions are proven, f is convex iff $dom(f)$ is convex and $f(y) \geq f(x) + f'(x)(y-x)$ $\forall x, y \in dom(f)$ $\square$.

3 cont

c) The set is $A = \{(x,y) \mid x^2+y^2 \leq 4r^2\} = \{(x,y) \mid x^2+y^2-4r^2 \leq 0\}$.

If r is constant $f(x,y) = x^2+y^2-4r^2$ is convex in x and y and therefore is convex in (x,y) (this is a power function). So A is a convex set for set r.

If r is a variable:

$$B = \{(x,y,r) \mid x^2+y^2-4r^2 \leq 0, r>0\}$$

Note that this is a set of points above a surface defined as $x^2+y^2 = 4r^2$ (if r is a 3rd dimension, like z) for positive r. $r = f(x,y) = \frac{1}{2}\sqrt{x^2+y^2} = \frac{1}{2}\|(x,y)\|_2$. Since this is a norm, $f(x,y)$ is convex (as proven before).

Then B is the epigraph of $f(x,y)$, so from lecture, we know B must be a convex set.

④ First note that: for a constant s, $f(sx)$ is convex in x:

Let $z = \theta x + (1-\theta) y$, $0 \le \theta \le 1$, then:

$$f(sz) = f(s\theta x + s(1-\theta)y) = f(\theta \underbrace{x'}_{sx} + (1-\theta)\underbrace{y'}_{sy}) \le \theta f(x') + (1-\theta) f(y') =$$

$$= \theta f(sx) + (1-\theta) f(sy), \text{ so:}$$

$$f(\theta sx + (1-\theta)sy) \le \theta f(sx) + (1-\theta) f(sy) \quad \forall x, y \in \mathrm{dom}(f),$$

$0 \le \theta \le 1$, (also $sx, sy \in \mathrm{dom}(f)$ still holds, since if $s \in \mathbb{R} \Rightarrow sx, sy \in \mathbb{R}$).

Next, note that:

The function: $\int_0^1 f(sx)\,ds$ is also convex since this is a non-negative weighted sum operation, so it preserves convexity.

Finally, note that:

$$F(x) = \frac{1}{x}\int_0^x f(t)\,dt = \frac{1}{x}\int_0^1 f(xs)\,x \cdot ds = \frac{x}{x}\int_0^1 f(xs)\,ds = \int_0^1 f(sx)\,ds.$$

change of variables:
$t = xs$
$dt = x\,ds$ from 0 to 1

Therefore, $F(x) = \int_0^1 f(sx)\,ds$. Finally, $\mathrm{dom}\, F = \mathbb{R}_{++}$ is a convex set, and since I showed earlier that $\int_0^1 f(sx)\,ds$ is a convex function, $F(x)$ is convex. □

5 a) $\mathbb{R}$ is a convex set.

$f(x) = e^x - 1 \Rightarrow f'(x) = e^x \Rightarrow f''(x) = e^x > 0 \ \forall x$, so by the second order condition, f is convex.

c) $\mathbb{R}^2_{++}$ is a convex set

$f(x_1, x_2) = \frac{1}{x_1 x_2}$

$$\nabla f = \begin{bmatrix} -\frac{1}{x_1^2 x_2} \\ -\frac{1}{x_1 x_2^2} \end{bmatrix} \Rightarrow \nabla^2 f = \begin{bmatrix} \frac{2}{x_1^3 x_2} & \frac{1}{x_1^2 x_2^2} \\ \frac{1}{x_1^2 x_2^2} & \frac{2}{x_1 x_2^3} \end{bmatrix}$$

The leading principal minor $\frac{2}{x_1^3 x_2} > 0 \ \forall x_1, x_2 \in \mathbb{R}_{++}$,

and $\det(\nabla^2 f) = \begin{vmatrix} \frac{2}{x_1^3 x_2} & \frac{1}{x_1^2 x_2^2} \\ \frac{1}{x_1^2 x_2^2} & \frac{2}{x_1 x_2^3} \end{vmatrix} = \frac{4}{x_1^4 x_2^4} - \frac{1}{x_1^4 x_2^4} = \frac{3}{x_1^4 x_2^4} > 0$

$\forall x_1, x_2 \in \mathbb{R}_{++}$ are both positive, so the Hessian is positive definite, so $f(x_1, x_2) = \frac{1}{x_1 x_2}$ is convex.

e) $\mathbb{R} \times \mathbb{R}_{++}$ is a convex set.

$f(x_1, x_2) = \frac{x_1^2}{x_2} \Rightarrow \nabla f = \begin{bmatrix} \frac{2x_1}{x_2} \\ \frac{-x_1^2}{x_2^2} \end{bmatrix} \Rightarrow \nabla^2 f = \begin{bmatrix} \frac{2}{x_2} & -\frac{2x_1}{x_2^2} \\ -\frac{2x_1}{x_2^2} & \frac{2x_1^2}{x_2^3} \end{bmatrix}$

$\frac{2}{x_2} > 0 \ \forall x_2 \in \mathbb{R}_{++}$ and $\det(\nabla^2 f) = \frac{4x_1^2}{x_2^4} - \frac{4x_1^2}{x_2^4} = 0 \geq 0 \ \forall x_1, x_2 \in \mathbb{R} \times \mathbb{R}_{++}$, so both the leading principal minor and the determinant of the Hessian are non-negative, so the Hessian is positive semi-definite, so $f(x_1, x_2) = \frac{x_1^2}{x_2}$ is convex.

⑥ Firstly, let's define a function:

$$g_i(x) = \|A^{(i)}x - b^{(i)}\|$$

From ① in the homework, we know that a norm $\|x\|$ is convex. Furthermore, from class we know that an affine mapping of the argument preserves the convexity. So for a matrix $A^{(i)} \in \mathbb{R}^{m \times n}$ and vector $-b^{(i)} \in \mathbb{R}^m$ (sign doesn't matter, can flip since the domain is $\mathbb{R}^m$), we know that since $\|x\|$ is convex, then $\|A^{(i)}x - b^{(i)}\|$ must be convex.

So $\forall i$, $g_i(x) = \|A^{(i)}x - b^{(i)}\|$ are all convex.

Then: $f(x) = \max\limits_{i=1,\dots,k} g_i(x)$. Let $x, y \in \mathbb{R}^n$, then:

$$f(\theta x + (1-\theta)y) = \max_{i=1,\dots,k} g_i(\theta x + (1-\theta)y) \underset{\substack{\uparrow \\ \text{since all} \\ g_i \text{ are convex}}}{\le} \max_{i=1,\dots,k}\{\theta g_i(x) + (1-\theta)g_i(y)\} \underset{\substack{\uparrow \\ \text{triangle inequality} \\ \text{for max, works since} \\ \max \text{ is an } L\text{-}\infty \text{ norm}}}{\le}$$

$$\le \max_{i=1,\dots,k} \theta g_i(x) + \max_{i=1,\dots,k}(1-\theta)g_i(y) = \theta \max_{i=1,\dots,k} g_i(x) + (1-\theta)\max_{i=1,\dots,k} g_i(y) =$$

$$= \theta f(x) + (1-\theta)f(y).$$

And since $\mathbb{R}^n$ is a convex set, and $f(\theta x + (1-\theta)y) \le \theta f(x) + (1-\theta)f(y)$, then f is convex. $\square$