Subtraction By The Nine's Complement Addition

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Special Case of Subtraction by Addition:

This section will develop an algorithm to accomplish the operation of subtraction by addition. For many students, subtraction is difficult to learn. A second level of difficulty occurs when the subtrahend digit is greater than the matching minuend digit and borrowing is required.

Acronym: **9C** = 9's complement of the subtrahend, that is, **subtrahend + 9's complement = 9**

Subtraction Table: subtrahend digit greater than the corresponding minuend digit.

	10	11	12	13	14	15	16	17	18
– 1	9								
– 2	8	9							
– 3	7	8	9						
– 4	6	7	8	9					
– 5	5	6	7	8	9				
– 6	4	5	6	7	8	9			
– 7	3	4	5	6	7	8	9		
– 8	2	3	4	5	6	7	8	9	
– 9	1	2	3	4	5	6	7	8	9

To accomplish subtraction by addition, start with an equation: 15 – 7 = 8 and note that **1 + 5 is the sum of the minuend digits** and **2 is the 9's complement of subtrahend 7**, that is, 7 + **2** = 9.

Algorithm (– by +): 15 – 7 = **1** + **5** + **2** → 15 – 7 = **6** + **2** → 15 – 7 = 8
Algorithm verified: 15 – 7 = (9 + **6**) – (9 – **2**) → 15 – 7 = 9 + **6** – 9 + **2** → 15 – 7 = 8

(**9C**) **9's Complement table** for the **digits 0** to **9** are: **(9 + 0), (8 + 1), (7 + 2), (6 + 3), (5 + 4) = 9**
Examples of Subtraction by Addition using the above algorithm: (See **Note-1)**

18 1 + 8 =+ 9 ← Add the minuend digits 1 + 8
–9 + **0** ← **0** is the number that added to subtrahend 9 equals 9, or use the **9C** table
? + 9 ← Difference by Addition; verified by: 18 – 9 = (9 + **9**) – (9 – **0**) = **9** + **0** = 9

16 1 + 6 =+ 7 ← Add the minuend digits 1 + 6
–8 + **1** ← **1** is the number that added to subtrahend 8 equals 9, or use the **9C** table
? + 8 ← Difference by Addition; verified by: 16 – 8 = (9 + **7**) – (9 – **1**) = **7** + **1** = 8

13 1 + 3 =+ 4 ← Add the minuend digits 1 + 3
–7 + **2** ← **2** is the number that added to subtrahend 7 equals 9, or use the **9C** table
? + 6 ← Difference by Addition; verified by: 13 – 7 = (9 + **4**) – (9 – **2**) = **4** + **2** = 6

11 1 + 1 =+ 2 ← Add the minuend digits 1 + 1
–6 + **3** ← **3** is the number that added to subtrahend 6 equals 9, or use the **9C** table
? + 5 ← Difference by Addition, verified by: 11 – 6 = (9 + **2**) – (9 – **3**) = **2** + **3** = 5

12 1 + 2 =+ 3 ← Add the minuend digits 1 + 2
–5 + **4** ← **4** is the number that added to subtrahend 5 equals 9, or use the **9C** table
? + 7 ← Difference by Addition; verified by: 12 – 5 = (9 + **3**) – (9 – **4**) = **3** + **4** = 7

Note-1: The Algorithm is valid in the ranges, **Minuend = 10 to 19** and **Subtrahend = 0 to 9**, otherwise the procedure will fail. The students will memorize the basic facts of subtraction, when borrowing is required, by the repetitive use of this algorithm.

<u>Subtraction by Addition: Invent an Algorithm that is valid for all numbers</u>:

Acronyms: SA: Subtraction by Addition, **9C:** 9's complement of the subtrahend

Consider the identity Equation (A) with a 2-digit minuend:

(A) Minuend – Subtrahend = Minuend – Subtrahend

Adding a zero: 0 = 99 – 100 + 1 to the right hand side of (A) does not change the identity:

(A) Minuend – Subtrahend = Minuend – Subtrahend + 99 – 100 + 1

Reformulate (A) by re-arranging and grouping terms:

Equation (A): Minuend – Subtrahend = (99 – Subtrahend) + Minuend – 100 + 1

Example (1) Subtract 7 from 15 using Equation (A):

15	99	← 99; the 1st term in Equation (A)
– <u>7</u>	– <u>7</u>	← Subtract the subtrahend; the 2nd term in Equation (A)
	92	← This term is the 9's complement, **9C**, of the subtrahend (see **9C Table**)
	<u>15</u>	← Addition of the minuend; the 3rd term in Equation (A)
	107	← Intermediate result
	<u>– 100</u>	← Subtract 100
	+7	← Intermediate result
	<u>+1</u>	← Addition of 1
	8	← Difference

These 4 steps will be done mentally hereafter. Transposing the left-most digit 1 in 107 and adding 1 to the right digit 7 is equivalent to subtracting 100 and adding 1. The number to be transposed and added will always equal 1.

9C Table: the nine's complement of the Subtrahend with:

digits	are	because
0 & 9	9 & 0	0 + 9 = 9 + 0 = 9
1 & 8	8 & 1	1 + 8 = 8 + 1 = 9
2 & 7	7 & 2	2 + 7 = 7 + 2 = 9
3 & 6	6 & 3	3 + 6 = 6 + 3 = 9
4 & 5	5 & 4	4 + 5 = 5 + 4 = 9

Reformulate Equation (A) by re-arranging terms to obtain Algorithm (**SA**):

Algorithm (SA): Minuend – Subtrahend = Minuend + (99 – Subtrahend) – 100 + 1

Example (2) Subtract 7 from 15; using steps in Algorithm (**SA**) and the **9C** table above

15	15	← The subtrahend must equal the minuend digits; subtract 07
–<u>07</u>	+<u>92</u>	← The 9's complement, **9C**, of the subtrahend 07 is: (0 + **<u>9</u>**), (7 + **<u>2</u>**) = 92
	107	← Intermediate result; transpose and add the left-most digit 1, to the right digit 7
	8	← Difference obtained by 1 Addition, the 9's complement of the subtrahend

The above is the **Final Form** of **Algorithm (SA)**. The invention of Algorithm (**SA**) was obtained by defining the 9's complement of the subtrahend, evaluating the term (99 – subtrahend) by addition and adding a zero to the identity equation (A).

Acronyms: **min**: minuend, **sub**: subtrahend, **9C**: 9's complement of the subtrahend

Minuend **D**igits	**Algorithm (SA) Table**
1	**min – sub = min** + (9 – **sub)** – 10 + 1
2	**min – sub = min** + (99 – **sub)** – 100 + 1
3	**min – sub = min** + (999 – **sub)** – 1000 + 1
4	**min – sub = min** + (9999 – **sub)** – 10000 + 1

Example (3): Subtract 2365 from 4053, using the **Final Form** of Algorithm (**SA**)

```
 4053    4053 ← Subtrahend digits are equal to the Minuend digits; subtract 2365
–2365    7634 ← The 9's complement, 9C of 2365 is: (2 + 7),(3 + 6),(6 + 3),(5 + 4) = 7634
        11687 ← Addition of 9C; transpose and add the left-most digit 1, to the right digit 7
         1688 ← Difference obtained by 1 Addition, the 9's complement of the subtrahend
```

Example (4) Using Equation (A) simplified such that the last 4 steps are done mentally
Subtract 2365 from 4053

```
   9999  ← Subtract from Minuend 9999 first; the 9's equal to the (4) minuend digits
 – 2365  ← Subtract the Subtrahend from 9999, not from the original Minuend 4053
   7634  ← Result is the 9's complement of the subtrahend (see Example (3) above)
   4053  ← Add the original Minuend
  11687  ← Result; transpose and add the left-most digit 1, to the right digit 7
   1688  ← Difference obtained by (4) Subtractions without borrowing and 1 Addition
```

Comparing Examples (3) and (4); in (3), the 9's complement is found by addition not subtraction, that is, what number added to 2 = 9, what number added to 3 = 9, and so on.

Example (5) Invent Algorithm (SWB) such that the Subtraction is Without Borrowing
Subtract 2365 from 4053; Express the minuend as the sum of 2 numbers and add a 0 = + 1 – 1

```
54 + 3999  ← Expand & Add 0 to the Minuend: 4053 = 53 + 1 – 1 + 4000 = 54 + 3999
   – 2365  ← Subtract the Subtrahend
     1634  ← Result
       54  ← Add 54; amount that must be added to obtain the original Minuend 4053
     1688  ← Difference obtained by (4) Subtractions without borrowing and 1 Addition
```

Example (6) Invent Algorithm (RMS) to obtain a Reduced Minuend and Subtrahend
Subtract 2365 from 4053

```
4   0   5   3    ← Write the Minuend digits in expanded form
2   3   6   5    ← Write the Subtrahend digits in expanded form
2  (–3)(–1)(–2)  ← Subtract the smaller from the larger digit
```

If the Subtrahend digit is greater than the Minuend digit, the sign is (–) negative

```
1 + 1999  ← Write the thousands place digit (+2) as 0 + 2000 = 1 – 1 + 2000 = 1 + 1999
   – 312  ← Subtract the hundreds, tens and ones place digits: (–3)(–1)(–2) = – 312
    1687  ← Add 1 to the right digit 7, the amount subtracted from the Minuend = 2000
    1688  ← Difference obtained by (7) Subtractions without borrowing
```

Example (7): Exercise from *Basic Mathematical Skills*, by Streeter et al, 4[th] Ed, p 62.
Subtract 2365 from 4053, using the Textbook Algorithm (TA)

Step 1
$$\begin{array}{r} 41 \\ 40\,\cancel{5}3 \\ \underline{-\,2365} \\ 8 \end{array}$$
In the first step we borrow 1 ten. This is written as ten ones and combined with the original 3 ones. We can then subtract the ones column.

Step 2
$$\begin{array}{r} 3141 \\ \cancel{4}0\cancel{5}3 \\ \underline{-\,2365} \\ 8 \end{array}$$
We must borrow again to subtract in the tens column. There are no hundreds, and so we move to the thousands column.

Step 3
$$\begin{array}{r} 1 \\ 3941 \\ \cancel{4}\cancel{0}\cancel{5}3 \\ \underline{-\,2365} \\ 8 \end{array}$$
The minuend is now renamed as 3 thousands, 9 hundreds, 14 tens and 13 ones.

Step 4
$$\begin{array}{r} 1 \\ 3941 \\ \cancel{4}\cancel{0}\cancel{5}3 \\ \underline{-\,2365} \\ 1688 \end{array}$$
The subtraction can now be completed.

To check our subtraction: 1688 + 2365 = 4053

Comparing the two algorithms: (**SA**) in Example (3) and the Textbook Algorithm (**TA**).

The textbook algorithm (**TA**), Example (7), is the most difficult way possible to do subtraction. The method requires extensive minuend modification, 3 borrows and 6 number transformations: 3 to 13, 5 to 4 to 14, 0 to 10 to 9 and 4 to 3, and the subtraction of 5 from 13, 6 from 14, 3 from 9 and 2 from 3. These are much more difficult operations than transforming the subtrahend into its 9’s complement and adding that to the minuend.

Subtraction by Addition, Algorithm (**SA**) also applies to mixed numbers and long division.

Example (7) Subtract $5\frac{3}{8}$ from $7\frac{1}{4}$ Nine's compliment of : $5\frac{3}{8}$ *is* Þ $3\frac{5}{8}$ *because*, $3\frac{5}{8}+5\frac{3}{8}=9$

$$7\frac{1}{4}+3\frac{5}{8}=7\frac{1}{4}\left(\frac{2}{2}\right)+3\frac{5}{8} \quad \text{Þ} \quad 10\frac{7}{8}$$ Add the left digit 1, to the right digit 0, (mental step)

$$7\frac{1}{4}-5\frac{3}{8}=1\frac{7}{8}$$ Difference obtained by Addition

Example (8) Divide 9456 by 321, using Algorithm (**SA**) for the subtractions.

20 + 9 ← Multiply by a multiple of ten such that the product ≤ dividend
$321\overline{)9456}$
3579 → 20 * 321 = 6420 < 9456; **9C** of 6420: (6 + **3**),(4 + **5**),(2 + **7**),(0 + **9**) = 3579
13035 ← Addition of the 9C; transpose left-most digit 1 and add to the right digit 5
3036 ← Result
7110 → 9 * 321 = 2889 < 3036; **9C** of 2889: (2 + **7**),(8 + **1**),(8 + **1**),(9 + **0**) = 7110

10146 ← Result; transpose left-most digit 1 and add to the right digit 6
147 ← Remainder is 147; Quotient is 29, Remainder 147

Conclusion:

Subtraction by Algorithms (SA), (SWB) and (RMS), Examples (3), (5) and (6), appears to be the easiest methods to do the operation of subtraction. Algorithms (SA), (SWB) and (RMS) avoid all the borrowing, the carries and the extensive minuend modifications. Algorithms (SA), Example (3) is the easiest method, in my opinion; Algorithm (SWB), Example (5), by expressing the minuend as the sum of two numbers is the method, most likely, within the procedural and conceptual understanding of a majority of students.

Which method of subtraction should be taught to elementary school students? All known methods and those that will be invented in the future; the students should be permitted to choose which method is within their procedural and conceptual understanding.