Problem 4 – Nonlinear SVM [10 pts bonus]

$$K(x, y) = \frac{x^t y}{\|x\| \cdot \|y\|}$$

1. Find the feature vector mapping, i.e., the φ that maps each sample to its feature space.

We will find φ:

As we learned: $K(x, y) = \varphi^t(x) * \varphi(y)$

$$= \frac{x^t}{||x||} * \frac{y}{||y||}$$

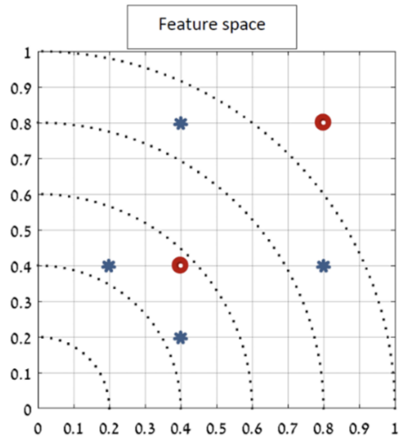
$$= \frac{x^t}{||x||} * \frac{y}{||y||} = \varphi^t(x) * \varphi(y)$$

$$\varphi(x) = \frac{x}{||x||}$$

This Because:

$\varphi^t(x) = \frac{x^t}{||x||}$, $\varphi(y) = \frac{y}{||y||}$

2. Given the following data, map the points to their new feature representations using the figure as the feature space. Draw the 6 new points (with colors) on the figure.



We will map the points to their new feature according to the $\varphi(x)$ that we find at section 1:

$$\varphi(0.4{,}0.2) = \frac{(0.4{,}0.2)}{\sqrt{(0.4^2+0.2^2)}} = \frac{(0.4{,}0.2)}{\frac{1}{\sqrt{5}}} = \left(\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right)$$

$$\varphi(0.2{,}0.4) = \frac{(0.2{,}0.4)}{\sqrt{(0.2^2+0.4^2)}} = \frac{(0.2{,}0.4)}{\frac{1}{\sqrt{5}}} = \left(\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}\right)$$

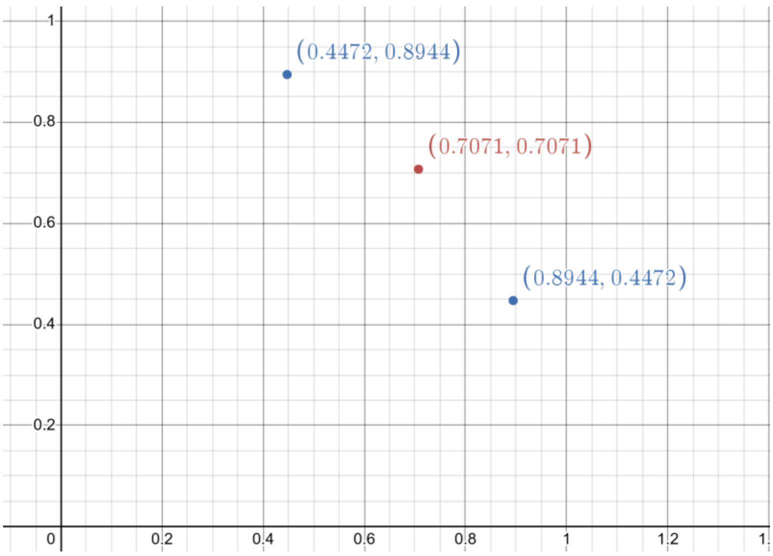
$$\varphi(0.4{,}0.8) = \frac{(0.4{,}0.8)}{\sqrt{(0.4^2+0.8^2)}} = \frac{(0.4{,}0.8)}{\frac{2}{\sqrt{5}}} = \left(\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right)$$

$$\varphi(0.4{,}0.4) = \frac{(0.4{,}0.4)}{\sqrt{(0.4^2+0.4^2)}} = \frac{(0.4{,}0.4)}{\frac{2*\sqrt{2}}{\sqrt{5}}} = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$$

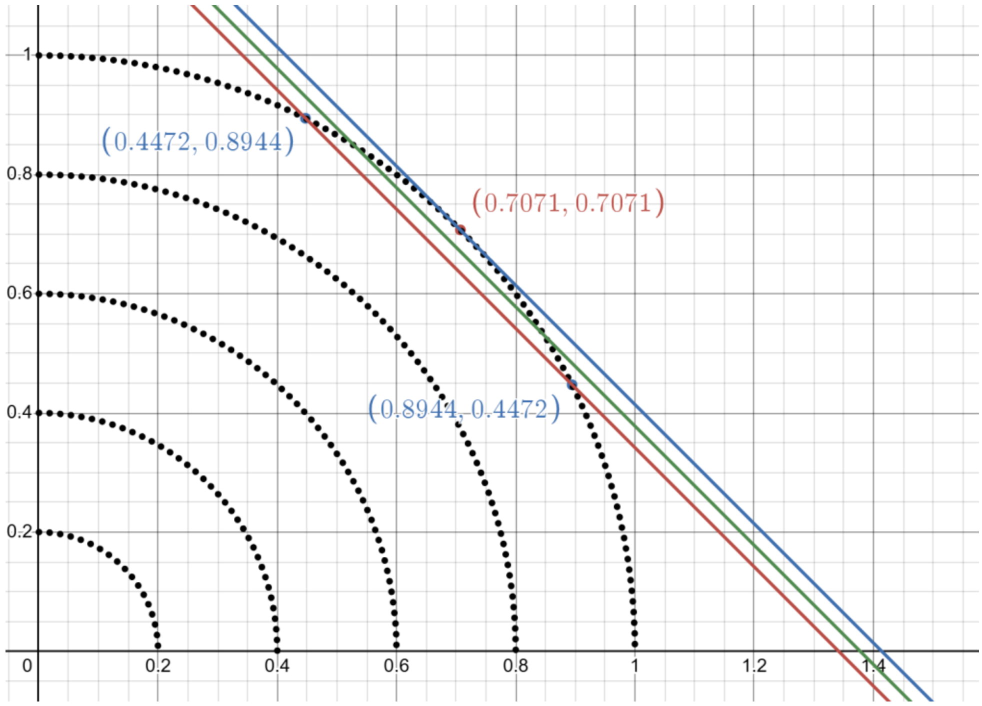
$$\varphi(0.8{,}0.4) = \frac{(0.8{,}0.4)}{\sqrt{(0.8^2+0.4^2)}} = \frac{(0.8{,}0.4)}{\frac{2}{\sqrt{5}}} = \left(\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}\right)$$

$$\varphi(0.8{,}0.8) = \frac{(0.8{,}0.8)}{\sqrt{(0.8^2+0.8^2)}} = \frac{(0.8{,}0.8)}{\frac{4*\sqrt{2}}{\sqrt{5}}} = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$$

We get 3 points: $\left(\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right)$, $\left(\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}\right)$, $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$



3. Draw the resulting margin decision boundary in the feature space. Also draw the separating line.



The total margin is the distance between the blue line and the red line, which equals 0.0513.

4. For this section, work only with the new feature space. Given that the separating hyperplane is defined by ℓ: $-x-y+1.378=0$, so w=(-1,-1), b=1.378, find the alphas for each sample.

Substitute: $w=(-1,-1)$

$$\sum_{i=1}^{3} y_i\alpha_i x_i = 1*\alpha_1*\left(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right)-1*\alpha_2*\left(\frac{1}{\sqrt{5}},\frac{2}{\sqrt{5}}\right)-1*\alpha_3*\left(\frac{2}{\sqrt{5}},\frac{1}{\sqrt{5}}\right)$$
$$=(-1,-1)$$

Then we get:

$$1)\ -1=\frac{\alpha_1}{\sqrt{2}}-\frac{\alpha_2}{\sqrt{5}}-\frac{2\alpha_3}{\sqrt{5}}$$

$2) - 1 = \frac{\alpha_1}{\sqrt{2}} - \frac{2\alpha_2}{\sqrt{5}} - \frac{\alpha_3}{\sqrt{5}}$

Substitute: $\sum_{i=1}^{3} y_i \alpha_i = 0$

$3) \alpha_1 - \alpha_2 - \alpha_3 = 0$

Then from 1,2 and 3 we get:

$\alpha_1 = \alpha_2 + \alpha_3$

Multiply 1 and 2 by $\sqrt{10}$

$1) - \sqrt{10} = \sqrt{5} * (\alpha_1) - \sqrt{2} * \alpha_2 - 2\sqrt{2} * \alpha_3$

$2) - \sqrt{10} = \sqrt{5} * (\alpha_1) - 2\sqrt{2} * \alpha_2 - \sqrt{2} * \alpha_3$

Subtract Equation 2 from Equation 1:

$0 = \alpha_2 * \left(-\sqrt{2} + 2\sqrt{2}\right) + \alpha_3 * \left(-2\sqrt{2} + \sqrt{2}\right)$

0 = $\alpha_2 * \left(\sqrt{2}\right) + \alpha_3 * \left(-\sqrt{2}\right)$

0= $\left(\sqrt{2}\right) * (\alpha_2 - \alpha_3)$

we can simplify to:

$\alpha_2 - \alpha_3 = 0$

Which means:

$\alpha_2 = \alpha_3$

$\alpha_1 = \alpha_2 + \alpha_3 = 2\alpha_2 = 2\alpha_3$

Substitute to (2)

$-1 = \frac{2\alpha_2}{\sqrt{2}} - \frac{\alpha_2}{\sqrt{5}} - \frac{2\alpha_2}{\sqrt{5}}$

Then we get:

$$\sqrt{5} * 2\alpha_2 - \sqrt{2}\alpha_2 - 2\sqrt{2}\alpha_2 = -\sqrt{10}$$

$$\alpha_2 = \alpha_3 = -5\sqrt{2} - 3\sqrt{5} = -13.779$$

$$\alpha_1 = 2\alpha_2 = -10\sqrt{2} - 6\sqrt{5} = -27.55854$$

5. Draw inside the Desmos link the decision boundary in the original input space, resulting from the kernel.

the nonlinear hyperplane given by:

$$\sum_{i=3}^{3} y_i \alpha_i K(x, x_i) + b = 0$$

$$x = (x_1, y_1)$$

$$1 * -27.558 * \left(\frac{\frac{x_1}{\sqrt{2}} + \frac{y_1}{\sqrt{2}}}{\sqrt{{x_1}^2 + {y_1}^2} * \sqrt{\frac{1}{\sqrt{2}}^2 + \frac{1}{\sqrt{2}}^2}} \right) - 1 * -13.779 * \frac{\frac{x_1}{\sqrt{5}} + \frac{2y_1}{\sqrt{5}}}{\sqrt{{x_1}^2 + {y_1}^2} * \sqrt{\frac{1}{\sqrt{5}}^2 + \frac{2}{\sqrt{5}}^2}}$$

$$- 1 * -13.779 * \frac{\frac{2x_1}{\sqrt{5}} + \frac{y_1}{\sqrt{5}}}{\sqrt{{x_1}^2 + {y_1}^2} * \sqrt{\frac{2}{\sqrt{5}}^2 + \frac{1}{\sqrt{5}}^2}} + 1.378 = 0$$

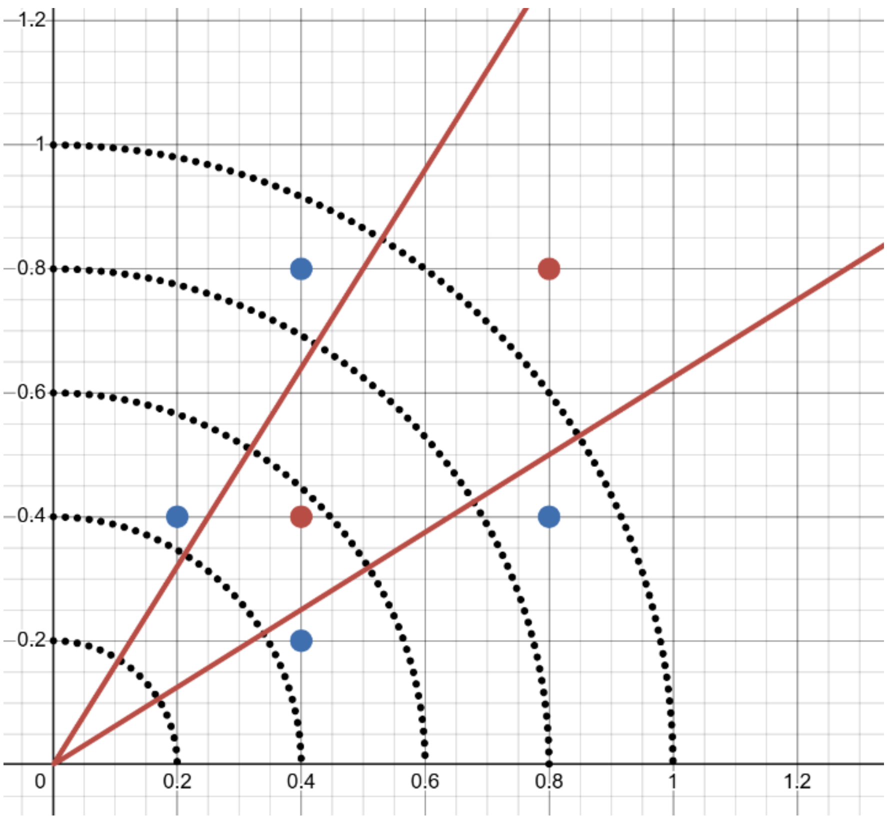
$$-27.558 * \left(\frac{x_1}{\sqrt{2}} + \frac{y_1}{\sqrt{2}}\right) + 13.779 * \left(\frac{x_1}{\sqrt{5}} + \frac{2y_1}{\sqrt{5}}\right) + 13.779 * \left(\frac{2x_1}{\sqrt{5}} + \frac{y_1}{\sqrt{5}}\right) + 1.378 * \sqrt{{x_1}^2 + {y_1}^2} = 0$$

$$x_1 * \left(\frac{-27.558}{\sqrt{2}} + \frac{13.779}{\sqrt{5}} + \frac{27.558}{\sqrt{5}}\right) + y_1 * \left(\frac{-27.558}{\sqrt{2}} + \frac{13.779}{\sqrt{5}} + \frac{27.558}{\sqrt{5}}\right) + 1.378 * \sqrt{{x_1}^2 + {y_1}^2} = 0$$

$$\frac{-27.558}{\sqrt{2}} + \frac{13.779}{\sqrt{5}} + \frac{27.558}{\sqrt{5}} = -1$$

Then we get:

$$-x - y + 1.378 * \sqrt{x^2 + y^2} = 0$$



6. Consider two distinct points $x1, x2 \in \mathbb{R}d$ with labels $y1 = 1, y2 = -1$. Compute the hyperplane that Hard SVM will return on this data, i.e., give explicit expression for w and b as functions of x1, x2.

נמצא את ה- w וה- b.

$$\begin{cases} w^T x_1 + b \geq 1 \\ w^T x_2 + b \leq -1 \end{cases}$$

נפעיל את מה שלמדנו:

$$= \frac{1}{2}\|w\|^2 - \left(\alpha_1\left(y_1(x_1 w + b) - 1\right) + \alpha_2\left(y_2(x_2 w + b) - 1\right)\right)$$

$$= \frac{1}{2}\|w\|^2 - \alpha_1(x_1 w + b - 1) + \alpha_2(x_2 w + b - 1)$$

כעת נגזור לפי w ואז לפי b. ונקבל.

$$w = \alpha_1 x_1 - \alpha_2 x_2$$

$$\alpha_1 = \alpha_2$$

$\alpha = \alpha_1 = \alpha_2$

כעת נציב ב- w. ונקבל, $w = \alpha(x_1 - x_2)$

$$L_D = 2\alpha - \frac{\alpha^2}{2}\left(\|x_1\|^2 + \|x_2\|^2 - 2(x_1^T x_2)\right)$$

נגזור שווה לאפס ונמצא את α:

$$\alpha = \frac{2}{\|x_1\|^2 + \|x_2\|^2 - 2(x_1^T x_2)}$$

כדי למצוא b נשתמש ב- $w^T x_1 + b = 1 \Leftarrow$ $b = 1 - w^T x_1$

$$\Downarrow$$

כעת נוכל למצוא את w :-

$$w = \frac{2(x_1 - x_2)}{\|x_1\|^2 + \|x_2\|^2 - 2(x_1^t x_2)}$$

נציב ל- b :-

$$b = 1 - \left(\frac{2(x_1 - x_2)}{\|x_1\|^2 + \|x_2\|^2 - 2(x_1^t x_2)} \right)^T x_1$$