Context-Adaptive Inference: A Unified Statistical and Foundation-Model View

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Abstract

context-adaptive inference

c $\theta(c)$ $f(x; \theta(c))$

Introduction

context→parameters

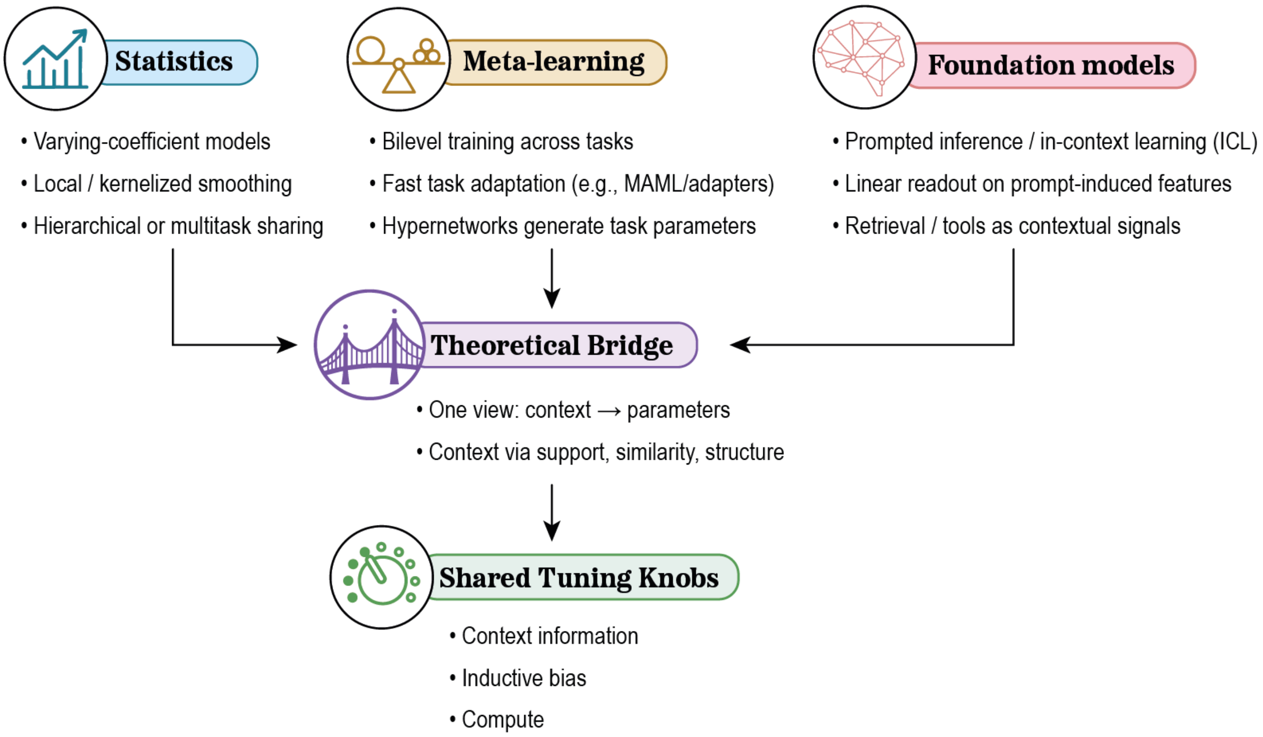

Statistics
• Varying-coefficient models
• Local / kernelized smoothing
• Hierarchical or multitask sharing
Meta-learning
• Bilevel training across tasks
• Fast task adaptation (e.g., MAML/adapters)
• Hypernetworks generate task parameters
Foundation models
• Prompted inference / in-context learning (ICL)
• Linear readout on prompt-induced features
• Retrieval / tools as contextual signals
Theoretical Bridge
• One view: context ⟶ parameters
• Context via support, similarity, structure
Shared Tuning Knobs
• Context information
• Inductive bias
• Compute

Figure 1:

x_i

θ_i

$$x_i \sim P(x;\, \theta_i).$$

$\theta_i = \theta$ i

$$\theta_i = f(c_i) \quad \text{or} \quad \theta_i \sim P(\theta \mid c_i),$$

c_i i

f

θ_i

Problem Setup and Notation

$i = 1, \ldots, n$ **context** $c_i \in \mathcal{C}$

$\mathcal{D}_i = \{(x_{ij}, y_{ij})\}_{j=1}^{m_i}$ $x_{ij} \in \mathcal{X}$ $y_{ij} \in \mathcal{Y}$

$\mathcal{H} = \{h_\theta : \mathcal{X} \to \mathcal{Y} \mid \theta \in \Theta\}$

global $\theta_i \equiv \theta^\star$ **context-adaptive**

$\theta_i = f(c_i)$ $\theta_i \sim P(\theta \mid c_i)$

c

$$\hat{\theta}(c) \in \arg\min_{\theta\in\Theta} \underbrace{\sum_{(i,j)\in S(c)} \ell\big(h_\theta(x_{ij}), y_{ij}\big)}_{\text{context-dependent support}} + \underbrace{\mathcal{R}(\theta;\, c)}_{\text{context-structured regularization}} , \qquad (\star)$$

ℓ $S(c) \subseteq \{1,\dots,n\} \times \mathbb{N}$ **support set**
c $\mathcal{R}(\theta; c)$

How context enters.
Explicit parameterization: $f : \mathcal{C} \to \Theta$ $\theta_i = f(c_i)$
$\mathcal{R}(\theta; c)$ f $\mathcal{C}$
Implicit parameterization:
θ $g(x, c)$ $S(c)$
$R(c)$ $P(c)$

context encoder $\phi : \mathcal{C} \to \mathbb{R}^d$ $K(c, c')$

$$\sum_{i,j} w_{ij}(c)\, \ell\big(h_\theta(x_{ij}), y_{ij}\big) \;+\; \mathcal{R}(\theta), \qquad w_{ij}(c) \propto K\big(\phi(c), \phi(c_i)\big) \cdot \mathbf{1}\big[(i,j) \in S(c)\big].$$

Granularity. $g \in \{\text{group}, \text{unit}, \text{example}\}$

Information $S(c)$ $P(c)$
Inductive bias $\mathcal{R}(\theta; c)$
Compute

Standing assumptions (used as needed).
(θ_i, c_i) (x_{ij}, y_{ij})
$\theta = f(c)$ f
$w_{ij}(c)$
ℓ $\mathcal{R}$
$|S(c)|$ **adaptation efficiency**

Theoretical Bridge

Proposition 1 (Explicit varying-coefficients and linear ICL coincide with kernel ridge on joint features in the linear squared-loss setting).
$y = \langle \theta(c), x \rangle + \varepsilon$ $\mathbb{E}[\varepsilon] = 0$
$\phi : \mathcal{C} \to \mathbb{R}^{d_c}$ $\psi(x, c) := x \otimes \phi(c) \in \mathbb{R}^{d_x d_c}$
$S(c)$ $w_{ij}(c)$

- **(A) Explicit varying-coefficients.** $\theta(c) = B\,\phi(c)$ $B \in \mathbb{R}^{d_x \times d_c}$ $\lambda\|B\|_F^2$

$$\hat{y}(x,c) = k_{(x,c)}^{\top}\big(K + \lambda I\big)^{-1} y, \quad K_{ab} = \langle \psi_a, \psi_b \rangle = \langle x_a, x_b \rangle \cdot \langle \phi(c_a), \phi(c_b) \rangle,$$

kernel ridge regression (KRR)

- **(B) Implicit adaptation via linear ICL.** $S(c)$ $q = Q\psi$ $k = K\psi$ $v = V\psi$ $w_{ij}(c) \cdot \langle q, k_{ij} \rangle$

$$k\big((x,c),(x',c')\big) = \langle q(x,c), k(x',c') \rangle,$$

dot-product kernel

ψ

Corollary 1 (Retrieval, gating, and weighting are kernel/measure choices). $S(c)$ $R(c)$ $w_{ij}(c)$ ψ

context-aware $S(c)$ $\mathcal{R}(\theta; c)$

Scope of Review and Relation to Prior Work

Related Surveys and Reviews

Survey	Topic Focus	Scope	Coverage of Adaptivity	Gap Relative to This Work

Survey	Topic Focus	Scope	Coverage of Adaptivity	Gap Relative to This Work

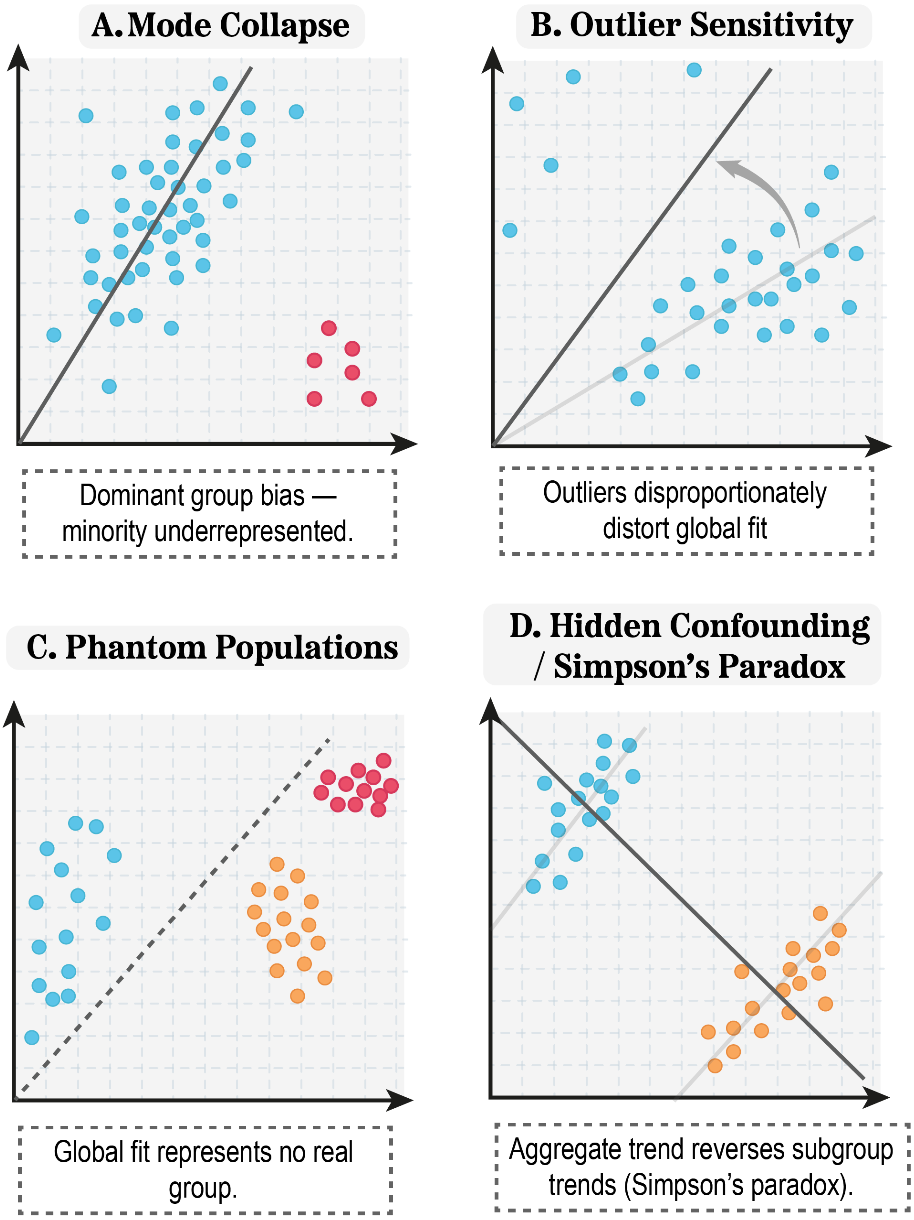
From Population Assumptions to Context-Adaptive Inference

Failure Modes of Population Models

Mode Collapse

Outlier Sensitivity

Phantom Populations



A. Mode Collapse

Dominant group bias — minority underrepresented.

B. Outlier Sensitivity

Outliers disproportionately distort global fit

C. Phantom Populations

Global fit represents no real group.

D. Hidden Confounding / Simpson's Paradox

Aggregate trend reverses subgroup trends (Simpson's paradox).

Figure 2:

Mode Collapse

Outlier Sensitivity

Phantom Populations **Hidden Confounding**

/ Simpson's Paradox

Toward Context-Aware Models

$$x_i \sim P(x;\theta_i),$$

N N θ_i c_i

$$\theta_i = f(c_i) \quad \text{or} \quad \theta_i \sim P(\theta \mid c_i).$$

f

$Y(1)$ $Y(0)$ $E[Y(1) - Y(0)]$

X C

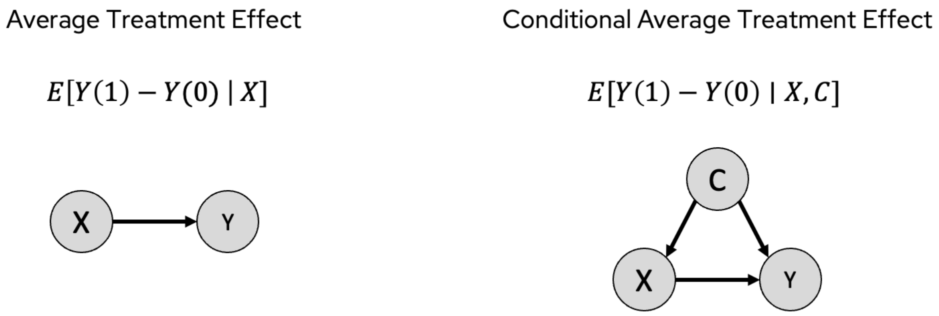

Average Treatment Effect
$E[Y(1) - Y(0) \mid X]$
X
Y
Conditional Average Treatment Effect
$E[Y(1) - Y(0) \mid X, C]$
C
X
Y

Figure 3: X C

$f(c)$

Classical Remedies: Grouped and Distance-Based Models

$f(c)$

Conditional and Clustered Models

$$\{\hat{\theta}_0, \ldots, \hat{\theta}_C\} = \arg \max_{\theta_0, \ldots, \theta_C} \sum_{c \in \mathcal{C}} \ell(X_c; \theta_c),$$

$\ell(X;\theta)$ θ X c

Distance-Regularized Estimation

θ_i

c_i

$$\{\hat{\theta}_0, \ldots, \hat{\theta}_N\} = \arg \max_{\theta_0, \ldots, \theta_N} \left(\sum_i \ell(x_i; \theta_i) - \sum_{i,j} \frac{\|\theta_i - \theta_j\|}{D(c_i, c_j)} \right),$$

$D(c_i, c_j)$

D λ

Parametric and Semi-parametric Varying-Coefficient Models

$$\widehat{A} = \arg\max_A \sum_i \ell(x_i; Ac_i).$$

Contextualized Models

$f(c)$ f f

$$\widehat{f} = \arg\max_{f \in \mathcal{F}} \sum_i \ell(x_i; f(c_i)).$$

Partition and Latent-Structure Models

$$\{\hat{\theta}_0, \ldots, \hat{\theta}_N\} = \arg\max_{\theta_0, \ldots, \theta_N} \left(\sum_i \ell(x_i; \theta_i) + \lambda \sum_{i=2}^{N} \|\theta_i - \theta_{i-1}\| \right).$$

Fine-tuned Models and Transfer Learning

Models for Explicit Subgroup Separation

A Spectrum of Context-Awareness

- **Global models** $\theta_i = \theta$ i
- **Grouped models** $\theta_i = \theta_c$
- **Smooth models** $\theta_i = f(c_i)$ f
- **Latent models** $\theta_i \sim P(\theta \mid c_i)$ f

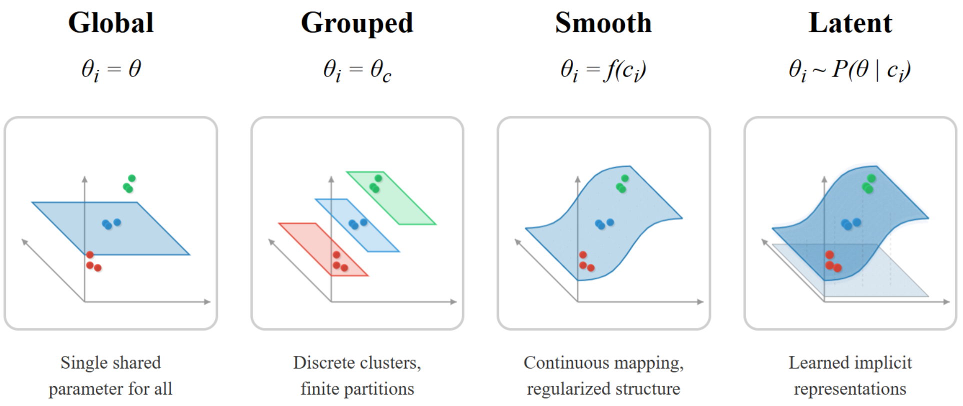

Global
$\theta_i = \theta$
Single shared parameter for all
Grouped
$\theta_i = \theta_c$
Discrete clusters, finite partitions
Smooth
$\theta_i = f(c_i)$
Continuous mapping, regularized structure
Latent
$\theta_i \sim P(\theta \mid c_i)$
Learned implicit representations

Figure 4:

Independent and identically distributed samples

$$\hat{\beta} = \arg\min_{\beta} \sum_{i=1}^{n} \left(y_i - x_i^\top \beta\right)^2$$

$$\hat{\beta} = (X^\top X)^{-1} X^\top y$$

$$g(\mu_i) = \eta_i$$

$$\log \frac{p_i}{1 - p_i} = x_i^\top \beta$$

$$\log(\mu_i) = x_i^\top \beta$$

$$F = \frac{MS_{\text{Between}}}{MS_{\text{Within}}}$$

Hierarchical data

$$y_{ij} = \mu + u_j + \varepsilon_{ij}, \quad u_j \sim N(0, \sigma_u^2), \quad \varepsilon_{ij} \sim N(0, \sigma^2)$$

$$y = X\beta + Zu + \varepsilon, \quad u \sim N(0, G), \quad \varepsilon \sim N(0, R)$$

$$g(\mu_i) = x_i^\top \beta + z_i^\top u$$

β u

$$y_{ij} \mid \theta_j \sim p(y_{ij} \mid \theta_j), \quad \theta_j \sim p(\theta_j \mid \phi), \quad \phi \sim p(\phi)$$

Functional types and high-dimensional data

$x_i(t)$

$x_i(t)$

$$y_i = \alpha + \sum_{j=1}^{p} f_j(x_{ij}) + \varepsilon_i$$

f_j

$h_\phi(x)$ $g_\theta(z)$

$$(\theta^{ast}, \phi^{ast}) = \arg\min_{\theta,\phi} \sum_{i=1}^{n} |x_i - g_\theta(h_\phi(x_i))|^2$$

$z_i = h_\phi(x_i)$ x_i

Heterogeneous tasks and sparse data

T (X^t, Y^t) $t = 1, \ldots, T$ w^t

$$\min_W \sum_{t=1}^{T} \sum_{i=1}^{n_t} \ell(y_i^t, f(x_i^t; w^t)) + \lambda\, \Omega(W)$$

$W = [w^1, \ldots, w^T]$ $\Omega(W)$

$p(x)$ $p(y|x)$

$p_{\text{train}}(x)$ $p_{\text{test}}(x)$

$$\mathbb{E}_{x\sim p_{\text{test}}}[\ell(f(x), y)] = \mathbb{E}_{x\sim p_{\text{train}}}\left[\frac{p_{\text{test}}(x)}{p_{\text{train}}(x)}\, \ell(f(x), y)\right], \quad w(x) = \frac{p_{\text{test}}(x)}{p_{\text{train}}(x)}$$

$p(x)$ $p(y|x)$

$\phi(x)$

Online and interactive data

$t = 1, \ldots, T$

$x_t \in F$ $F \subset \mathbb{R}^n$

$c_t : F \to \mathbb{R}$ $c_t(x_t)$

$$R(T) = \sum_{t=1}^{T} c_t(x_t) - \min_{x \in F} \sum_{t=1}^{T} c_t(x)$$

$g_t \in \partial c_t(x_t)$

$$x_{t+1} = \Pi_F(x_t - \eta_t g_t)$$

η_t Π_F F

$R(T) = O(\sqrt{T})$

x y

$\hat{p}\_i$ i s_i

$(p\_\text{min}, s\_\text{min})$

$$\hat{p}_i + s_i \geq p_{\min} + \alpha \cdot s_{\min}$$

α

k_w k_d $(p_{\min}, s_{\min})$

$c_t(\cdot)$

ϵ $1-\epsilon$

ϵ

t a

$$\mathrm{UCB}_a(t) = \hat{\mu}_a + \sqrt{\frac{2\ln t}{n_a}}$$

$\arg\max_a \mathrm{UCB}_a(t)$

t x_t $\pi : X \to A$

t

$$a_t = \arg\max_{a\in\mathcal{A}} \left(x_t^\top \hat{\theta}_a + \alpha\sqrt{x_t^\top A_a^{-1} x_t} \right)$$

$\hat{\theta}_a$ A_a

S A

$P(s' \mid s, a)$ $r(s,a)$ $\gamma \in [0,1)$ t

s_t $a_t \sim \pi(\cdot \mid s_t)$ r_t

s_{t+1} π

$$J(\pi) = \mathbb{E}_\pi\left[\sum_{t=0}^{\infty} \gamma^t r_t\right]$$

Multimodal data

$$f_\theta : \mathcal{X} \to \mathcal{Z}$$

$\mathcal{X}$ $\mathcal{Z}$

$p(z|x)$

$p_\theta(x|z)$

$q(z|x)$

$q_\phi(z|x)$

$p_\theta(x|z)$ $q_\phi(z|x)$

$$\mathcal{L}(\theta, \phi; x) = \mathbb{E}_{q_\phi(z|x)}[\log p_\theta(x|z)] - D_{KL}[q_\phi(z|x) \,||\, p(z)]$$

$q_\phi(z|x)$

$$\min_\theta \sum_{T_i \sim p(T)} \mathcal{L}_{T_i}\big(U(\theta, T_i)\big)$$

θ T_i

$p(T)$ $U(\theta, T_i)$

θ

Large-scale pre-trained data

$\mathcal{D} = \{x_i\}_{i=1}^{N}$

f_θ

$$\theta^* = \arg\min_{\theta} \ \mathbb{E}_{x \sim \mathcal{D}} \, \ell(f_\theta(x))$$

ℓ

$\{(x_i, y_i)\}_{i=1}^{k}$ x_{k+1}

$\hat{y}_{k+1}$

$$\hat{y}_{k+1} = f_\theta(x_{k+1} \mid x_1, y_1, \ldots, x_k, y_k)$$

Principles of Context-Adaptive Inference

1. Adaptivity requires flexibility

2. Adaptivity requires a signal of heterogeneity

3. Modularity improves adaptivity

4. Adaptivity implies selectivity

5. Adaptivity is bounded by data efficiency

6. Adaptivity is not a free lunch

When Adaptivity Fails: Common Failure Modes

Spurious adaptation.

Overfitting in low-data contexts.

Modularity mis-specification.

Feedback loops.

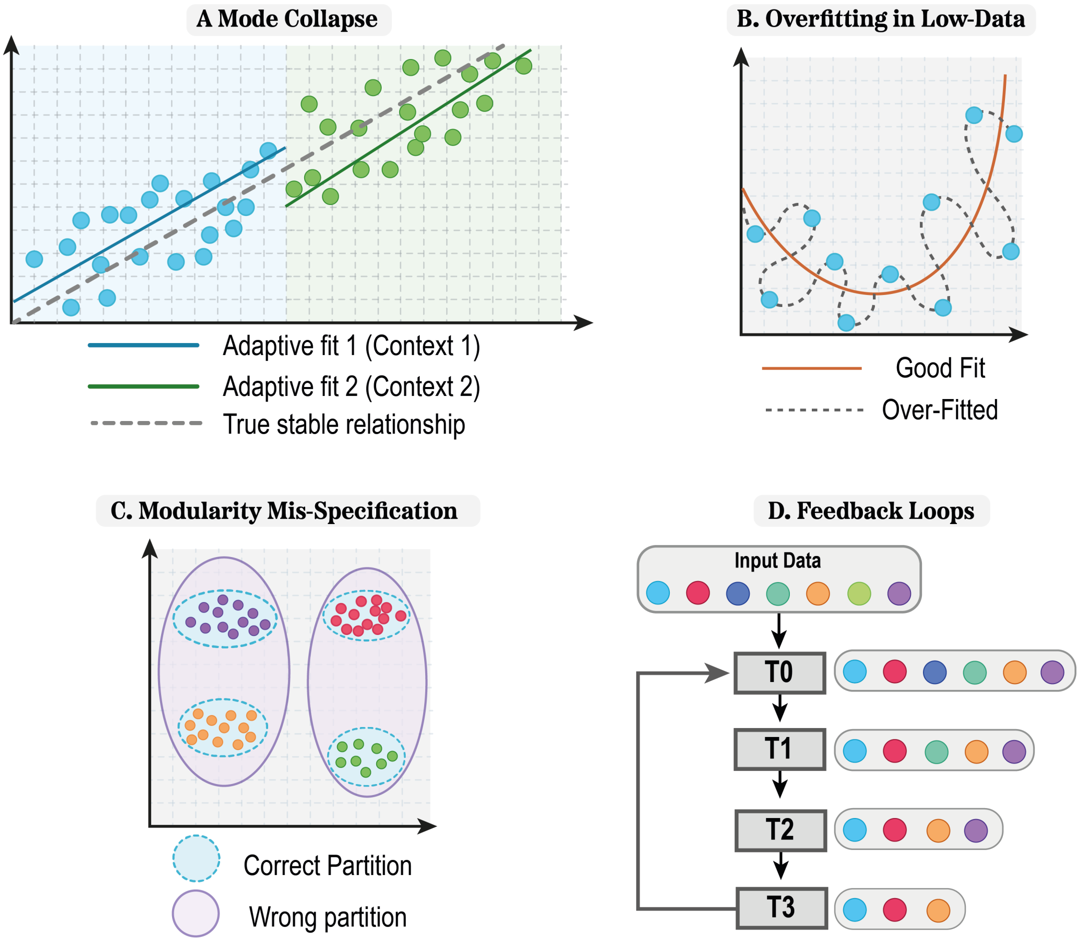


Figure 5: **Mode Collapse** **Overfitting in Low-Data Contexts** **Modularity Mis-Specification** **Feedback Loops**

Synthesis and Implications

Context-Aware Efficiency Principles and Design

Adaptivity is bounded by data efficiency

Formalization: data-efficiency constraints on adaptivity

$\mathcal{C}$

$\theta(c) \in \Theta$

(x, y, c) $p_\theta(y \mid x, c)$ $\ell(\theta; x, y, c)$

$\mathcal{N}_\delta(c) = \{c' : d(c, c') \leq \delta\}$ d

$\theta(c)$

$$N_{\text{eff}}(c, \delta) = \sum_{i=1}^{n} w_\delta(c_i, c), \quad w_\delta(c_i, c) \propto K\left(\frac{d(c_i, c)}{\delta}\right), \quad \sum_i w_\delta(c_i, c) = 1,$$

K

$$\mathcal{R}(\theta) = \int \|\nabla_c \theta(c)\|^2 \, \mathrm{d}c$$

$$\hat{\theta} = \arg\min_{\theta \in \Theta} \frac{1}{n} \sum_{i=1}^{n} \ell(\theta; x_i, y_i, c_i) + \lambda \mathcal{R}(\theta).$$

c L μ θ

j

$$\mathbb{E}\Big[\|\hat{\theta} j(c) - \theta_j(c)\|^2\Big] \lesssim \underbrace{\frac{\sigma^2}{N_{\text{eff}}(c, \delta)}}_{\text{variance}} + \underbrace{\delta^{2\alpha}}_{\text{approx. bias}} + \underbrace{\lambda^2}_{\text{reg. bias}}, \quad \alpha > 0,$$

δ

N_{eff}

δ λ

$\theta(c)$ $f_\phi(c)$ ϕ N_{eff}

η $T(c)$

$$\mathcal{L}\big(\theta^{(T(c))}\big) - \mathcal{L}(\theta^\star) \;\leq\; (1 - \eta\mu)^{T(c)} \Big(\mathcal{L}\big(\theta^{(0)}\big) - \mathcal{L}(\theta^\star)\Big) + \frac{\eta L \sigma^2}{2\mu \, N_{\text{eff}}(c, \delta)}.$$

$T(c)$ $N_{\text{eff}}(c, \delta)$

c

Formal optimization view of context-aware efficiency

$f_\phi : \mathcal{X} \times \mathcal{C} \to \mathcal{Y}$ ϕ

$T(c)$ $\Omega(\phi)$

$$\min_{\phi} \; \mathbb{E}_{(x,y,c)\sim\mathcal{D}} \ell\big(f_\phi(x, c), y\big) + \lambda \, \Omega(\phi) \quad \text{s.t.} \quad \mathbb{E}_c \, \mathcal{C}\big(f_\phi; T(c), c\big) \leq B,$$

$\mathcal{C}(\cdot)$

$$\min_{\phi}\ \mathbb{E}_{(x,y,c)}\ell\big(f_\phi(x,c),y\big)+\lambda\,\Omega(\phi)+\gamma\,\mathbb{E}_c\,\mathcal{C}\big(f_\phi;T(c),c\big),$$

γ

$\phi=(\phi_1,\dots,\phi_M)$

$\pi_\phi(m\mid c)$

$$\Omega(\phi)=\sum_{m=1}^{M}\alpha_m\,\|\phi_m\|_2^2+\tau\,\mathbb{E}_c\sum_{m=1}^{M}\pi_\phi(m\mid c),$$

$$\nabla_\phi\Big(\mathbb{E}\,\ell+\lambda\,\Omega+\gamma\,\mathbb{E}_c\,\mathcal{C}\Big)=0,\quad \gamma\,\big(\mathbb{E}_c\,\mathcal{C}-B\big)=0,\quad \gamma\geq 0.$$

$T(c)$

Explicit Adaptivity: Structured Estimation of $f(c)$

$\theta_i=f(c_i)$ f

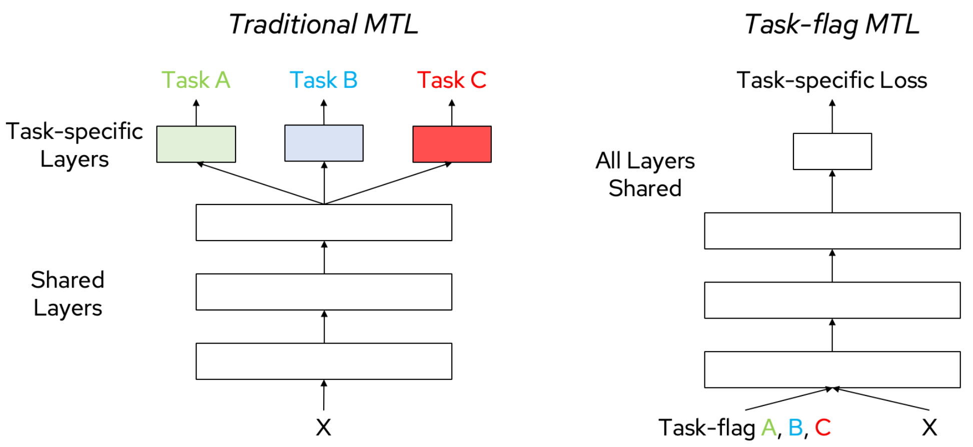


Figure 6:

$f(c)$

$f(c)$

Classical Varying-Coefficient Models: A Foundation

$$y_i = \sum_{j=1}^{p} \beta_j(c_i) x_{ij} + \varepsilon_i$$

$\beta_j(c)$

Advances in Modeling $f(c)$

$f(c)$

Smooth Non-parametric Models

$f(c)$

c

Structured Regularization for Graphical and Network Models

$f(c)$

c

$f(c)$

Piecewise-Constant and Partition-Based Models.

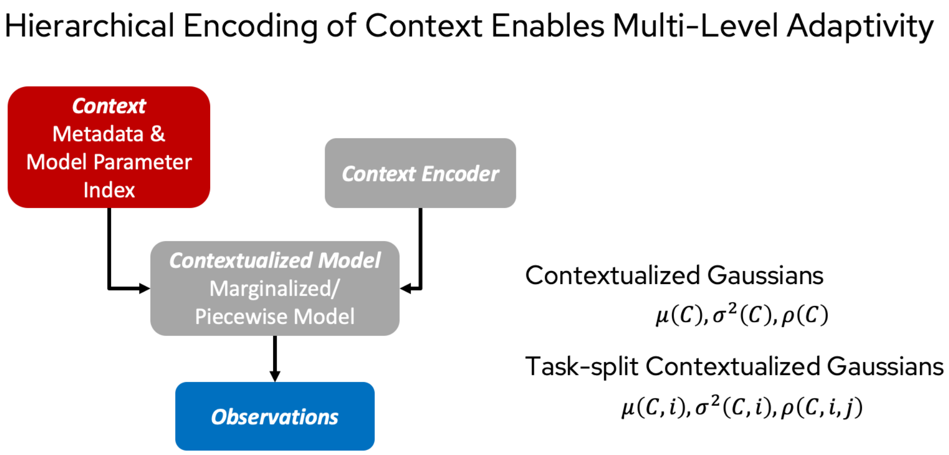

Hierarchical Encoding of Context Enables Multi-Level Adaptivity
Context
Metadata &
Model Parameter
Index
Context Encoder
Contextualized Model
Marginalized/
Piecewise Model
Observations
Contextualized Gaussians
$\mu(C), \sigma^2(C), \rho(C)$
Task-split Contextualized Gaussians
$\mu(C,i), \sigma^2(C,i), \rho(C,i,j)$

Figure 7: c (i,j)

simple parametric models within each context c Z **aggregate across contexts**

$$P(Y \mid X, C) \; = \int P(Y \mid X, C, Z) \, dP(Z \mid C)$$

global flexibility can emerge from compositional, context-specific parametrics

c

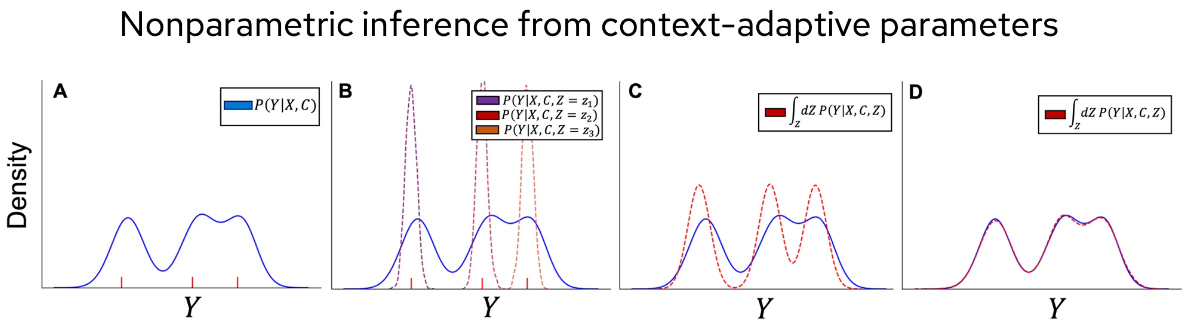


Figure 8: $P(Y \mid X, C)$ $P(Y \mid X, C, Z = z_i)$ Z

$\int_Z P(Y \mid X, C, Z)$

context-to-mixture weights **local parametric maps**

Structured Regularization for Spatial, Graph, and Network Data.

c

Learned Function Approximators

$f(c)$

$f(c)$

Tree-Based Ensembles.

Deep Neural Networks.

$f(c)$

Key Theoretical Advances

Theory for Smooth Non-parametric Models.

Theory for Structurally Constrained Models.

Theory for High-Capacity and Learned Models.

Sparsity and Incomplete Measurements as Context

measurement sparsity itself as context.

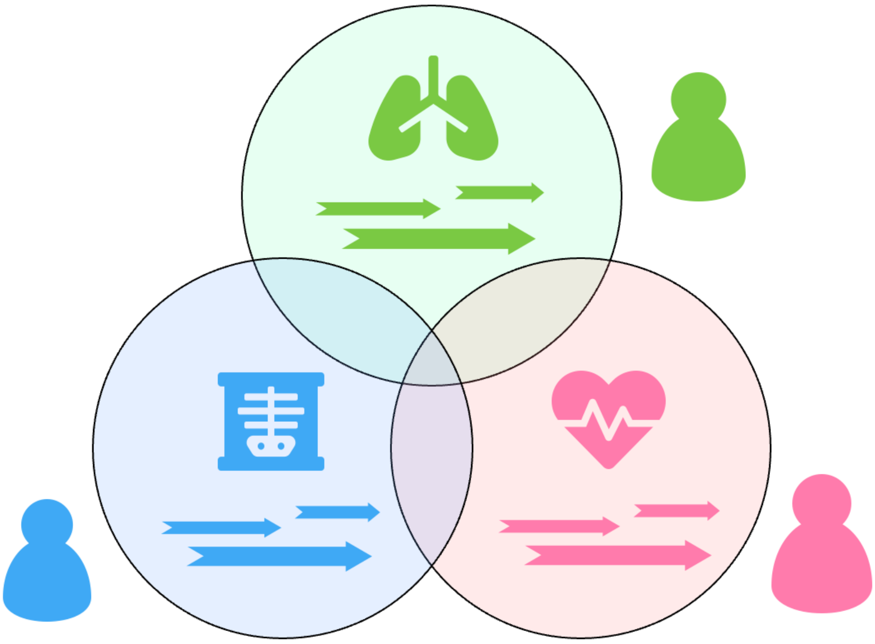


Figure 9:

$p(x_{\text{missing}} \mid x_{\text{observed}})$

GRU-D **BRITS**
GAIN **VAEAC**
XGBoost

Context-Aware Efficiency Principles and Design

Synthesis and Future Directions

$f(c)$

c_i

$f(c)$

Implicit Adaptivity: Emergent Contextualization in Complex Models

Introduction: From Explicit to Implicit Adaptivity.

$\theta_i = f(c_i)$ c_i

θ_i

Foundations of Implicit Adaptation

Architectural Conditioning via Context Inputs

$y_i = g([x_i, c_i]; \Phi)$

c_i x_i g

Φ

x_i y_i c_i

Interaction Effects and Attention Mechanisms

Amortized Inference and Meta-Learning

Amortized Inference

Meta-Learning: Learning to Learn

In-Context Learning in Foundation Models

The Phenomenon of Few-Shot In-Context Learning

Deconstructing ICL: Key Influencing Factors

The Role of Scale.

Prompt Engineering and Example Selection.

Hypothesized Mechanisms: How Does ICL Work?

ICL as Implicit Meta-Learning.

ICL as Implicit Bayesian Inference.

The Role of Induction Heads.

Limitations and Open Questions

Theoretical Bridges Between Varying-Coefficient Models and In-Context Learning

Varying-Coefficient Models as Kernel Regression

θ_i c_i c^*

$$\hat{\theta}(c^*) = \arg\max_{\theta} \sum_{i=1}^{n} K_\lambda(c_i, c^*)\, \ell(x_i; \theta),$$

K_λ ℓ

$y = (y_1, \ldots, y_n)^\top$ $K \in \mathbb{R}^{n \times n}$ $K_{ij} = k(c_i, c_j)$ c^*

$$\hat{y}(c^*) = k(c^*)^\top (K + \lambda I)^{-1} y,$$

$k(c^*) = (k(c^*, c_1), \ldots, k(c^*, c_n))^\top$

c^*

$$\widehat{f}\,(x^*, c^*) = \sum_{i=1}^{n} \alpha_i(c^*)\, y_i,$$

$\alpha_i(c^*)$

λ

Transformers as Ridge and Kernel Regressors In-Context

$$\widehat{w} = (X^\top X + \lambda I)^{-1} X^\top y$$

(x_i, y_i)

x^*

$k(c_i, c_j)$

Synthesis: Two Paths to the Same Estimators

$$\widehat{f}\,(x^*, c^*) = \sum_{i=1}^{n} \alpha_i(c^*)\, y_i,$$

$\alpha_i(c^*)$ c^*
$\{c_i\}$

- $\alpha_i(c^*)$ K_λ
- $\alpha_i(c^*)$

Comparative Synthesis: Implicit versus Explicit Adaptivity

Implicit Adaptivity.

$f(c)$

Explicit Adaptivity. $f(c)$

Open Challenges and the Motivation for Interpretability

Toward Explicit Modeling of Implicit Adaptivity: Local Models, Surrogates and Post Hoc Approximations

Motivation

From Implicit to Explicit Adaptivity

Fidelity vs. Interpretability **Local vs. Global Scope** **Approximation vs. Control**

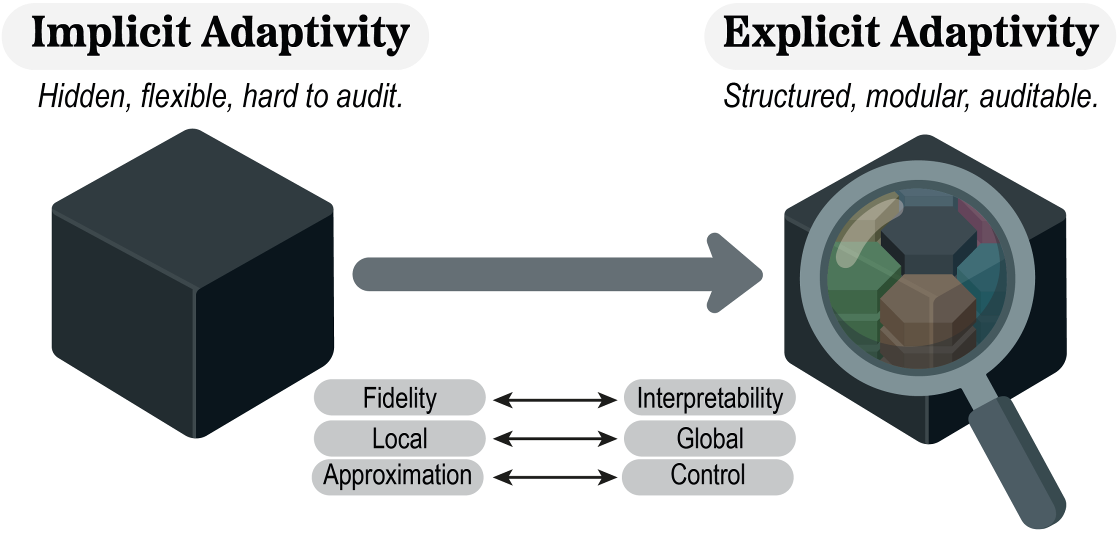


Figure 10:

Approaches

Surrogate Modeling

$h(x, c)$

$f(c)$

$$\hat{g}_{x_0,c_0} = \arg\min_{g \in \mathcal{G}} \mathbb{E}_{(x,c)\sim\mathcal{N}_{x_0,c_0}} \left[\ell\big(h(x,c), g(x,c)\big)\right] + \Omega(g),$$

$\mathcal{N}_{x_0,c_0}$ ℓ Ω

$\mathcal{G}$

$$R^2_{\text{local}} = 1 - \frac{\sum_i w_i \left(h_i - g_i\right)^2}{\sum_i w_i \left(h_i - \bar{h}\right)^2}, \qquad w_i \propto \kappa\big((x_i, c_i), (x_0, c_0)\big).$$

Prototype and Nearest-Neighbor Methods

k

Amortization Diagnostics

$q_\phi(\theta \mid x)$ $f(c)$

Disentangled and Bottlenecked Representations

β

Parameter Extraction and Probing

LLMs as Post-hoc Explainers

Trade-offs

Fidelity vs. Interpretability

$$\min_{g\in\mathcal{G}} \underbrace{\phi_{\text{fid}}(g;U)}_{\text{faithfulness on use set }U} + \lambda$$

$$underbrace\psi_{\text{simplicity}}(g)_{\text{sparsity / size / semantic load}},$$

ϕ_{fid} R^2 h $\psi_{\text{simplicity}}$

Local vs. Global Scope

$g_{x_0,c_0} \approx h$ $\mathcal{N}_{x_0,c_0}$ $g_{\text{global}} \approx h$

$$g(x,c) = \sum_{k=1}^{K} w_k(x,c)\, g_k(x,c), \qquad \sum_k w_k(x,c) = 1, \quad w_k \geq 0,$$

g_k w_k

Approximation vs. Control

Open Research Directions

Reusable Modules

Performance Gains

Abstraction Level

Evaluation and Reporting Standards for Classical Post-hoc Methods

Scope and locality

Attribution methods in practice

Faithfulness and robustness

R^2

$\widetilde{\mathrm{AUC}}_S$ $\widetilde{F}_{1,S}$

Minimal reporting checklist

Item	Description
Data slice and context definition	
Surrogate specification and regularization details	
Faithfulness and robustness metrics	R^2
Sensitivity and uncertainty analysis	
Computational constraints	
Observed limitations and failure modes	

Table 2. Minimal Reporting Checklist for Post-hoc Explanations

From post hoc analysis to design

Implications for classical models

Context-Invariant Training: A View from the Converse

Φ w

$R^e(\cdot)$ e

$w = 1$

$P(X)$ $P(Y|X)$

(Y, ID)

Adversarial Robustness as Context-Invariant Training

$x' = x + \delta$ $\|\delta\|_p \leq \varepsilon$

Perception Robustness

Training methods for Context-Invariant Models

$$\min_{f} \frac{1}{n} \sum_{i=1}^{n} L(f(x_i), y_i)$$

$$\min_{f} \max_{g \in \mathcal{G}} \mathbb{E}_{(x,y) \sim P_g} [L(f(x), y)]$$

$\mathcal{G}$ P_g

g

strong regularization

Applications, Case Studies, Evaluation Metrics, and Tools

Implementation Across Sectors

Context-Aware Efficiency in Practice

Formal Metrics for Evaluating Context-Aware Performance

$\mathcal{C}$ $\mathcal{D}_{\text{test}}$ (x, y, c)

$\hat{f}$

$$\mathcal{R}(\hat{f} \mid c) = \mathbb{E}\Big[\,\ell\big(\hat{f}(x,c), y\big) \,\Big|\, c\Big], \qquad \mathcal{R}(\hat{f}) = \mathbb{E}_{c\sim\mathcal{D}_{\text{test}}}\Big[\mathcal{R}(\hat{f} \mid c)\Big].$$

$\mathcal{R}(\hat{f} \mid c)$

$\int \mathcal{R}(\hat{f} \mid c)\, \mathrm{d}\Pi(c)$ Π

Adaptation Efficiency

$S_k(c) = \{(x_j, y_j, c)\}_{j=1}^{k}$ k c

$$\mathrm{AE}_k(c) = \mathcal{R}(\hat{f}_0 \mid c) - \mathcal{R}(\hat{f}_{S_k} \mid c), \qquad \mathrm{AE}_k = \mathbb{E}_c\big[\,\mathrm{AE}_k(c)\,\big],$$

$\hat{f}_0$ $\hat{f}_{S_k}$

$k \mapsto \mathrm{AE}_k$

Transfer Performance

$\mathcal{C}_{\mathrm{src}} \to \mathcal{C}_{\mathrm{tgt}}$

ϕ

$$\mathrm{TP}(\phi) = \mathcal{R}_{\mathcal{C}_{\mathrm{tgt}}}(\hat{f}_{\phi}) - \mathcal{R}_{\mathcal{C}_{\mathrm{tgt}}}(\hat{f}_{\mathrm{scratch}}),$$

ϕ

Robustness to Context Shift

Q f

$$\mathrm{RS}(\hat{f}\,; Q) = \sup_{\widetilde{\mathcal{D}} \in Q} \left[\mathcal{R}_{\widetilde{\mathcal{D}}}(\hat{f}\,) - \mathcal{R}_{\mathcal{D}_{\mathrm{test}}}(\hat{f}\,) \right],$$

Context-Aware Efficiency in Practice

Contextualized Network Inference

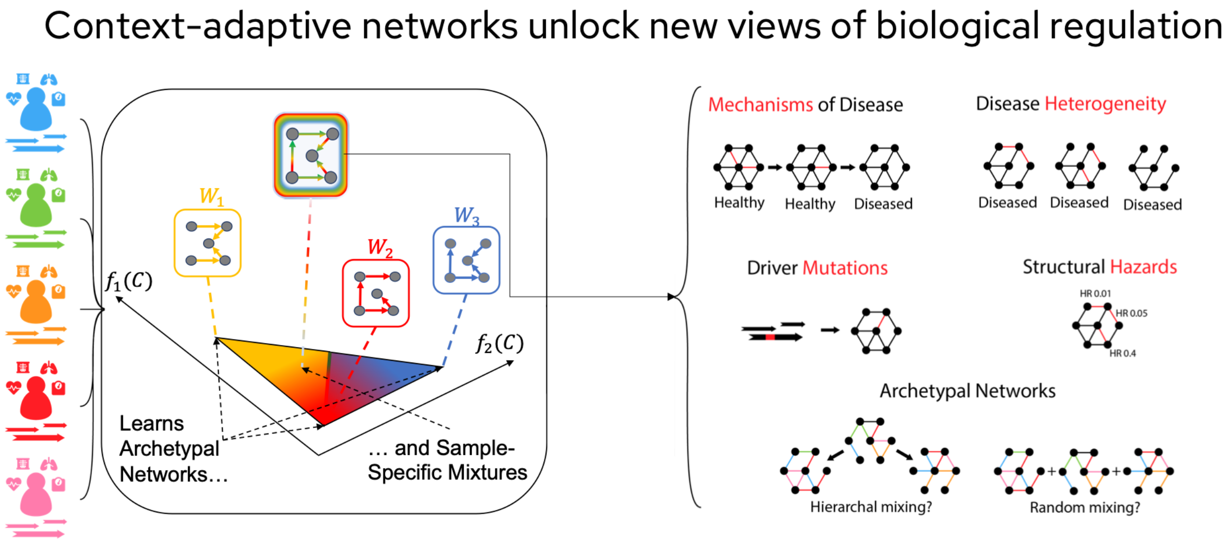

Context-adaptive networks unlock new views of biological regulation

Figure 11:

Performance Evaluation

Survey of Tools

Selection and Usage Guidance

Future Trends and Opportunities with Foundation Models

A New Paradigm for Context-Adaptive Inference

Universal Context Encoders

Dynamic Adaptation Mechanisms

Bridging with Statistical and Causal Reasoning

Next-Generation Methods for Contextualized Adaptive Inference

Modular Fine-Tuning and Compositional Adaptation

In-Context Learning and Mechanistic Insights

Reliability, Calibration, and Context-Sensitive Evaluation

Expanding Frameworks with Foundation Models

Foundation Models as Context

Feature Extraction and Interpretation:

Contextualized Representations for Downstream Modeling:

Post-hoc Interpretability:

Recent Innovations and Outlook

FLAN-MoE

LMPriors

Mixture of In-Context Experts

Open Problems

Open Research Questions

Can Reusable Modules Enable Portability Across Tasks?

What Are the Theoretical and Practical Benefits of Explicit Structure?

At What Level of Abstraction Should Explicit Structure Be Imposed?

What Theoretical and Practical Barriers Remain?

Interpretable-by-Design vs Post-hoc Interpretability: What Is the Right Path Forward?

Broader Challenges and Future Outlook

Conclusion

Overview of Insights

Context-Aware Efficiency: A Unifying Framework

Future Directions

Theoretical Foundations

Modular and Compositional Methods

Evaluation and Reliability

Responsible and Sustainable Deployment

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Generalized Linear Models

Generalized Linear Models

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Appendix A

A.0 Preliminaries and identities

- **Joint features.** (x, c)

$$\psi(x, c) := x \otimes \phi(c) \in \mathbb{R}^{d_x d_c}.$$

a (i, j) $\psi_a := \psi(x_a, c_a)$

- **Design/labels/weights.** $N = \sum_i m_i$

$$Z \in \mathbb{R}^{N \times d_x d_c} \text{ with rows } Z_a = \psi_a^T, \qquad y \in \mathbb{R}^N, \qquad W = \mathrm{diag}(w_a) \in \mathbb{R}^{N \times N},\ w_a \geq 0.$$

$$K := ZZ^\top$$

$$K_W := W^{1/2} K\, W^{1/2} \ = \ W^{1/2} Z Z^\top W^{1/2}.$$

(x, c) $k(\cdot, (x, c)) := Z\,\psi(x, c) \in \mathbb{R}^N$ $k_{(x,c)} := W^{1/2} k(\cdot, (x, c))$

- **Vectorization identity.** A, B, C

$$\mathrm{vec}(ABC) = \big(C^\top \otimes A\big)\mathrm{vec}(B), \quad \langle \mathrm{vec}(B),\, x \otimes z\rangle = x^\top B z.$$

- **Weighted ridge solution.** $X \in \mathbb{R}^{N \times p}$

$$\min_\beta \ \|W^{1/2}(y - X\beta)\|_2^2 + \lambda \|\beta\|_2^2$$

$$\widehat{\beta} = (X^\top W X + \lambda I)^{-1} X^\top W y$$

$$\widehat{\beta} = X^\top W^{1/2} \big(W^{1/2} X X^\top W^{1/2} + \lambda I\big)^{-1} W^{1/2} y.$$

$x_\star$

$$\widehat{f}\,(x_\star) = x_\star^\top \widehat{\beta} \ = \ \underbrace{\big(W^{1/2} X x_\star\big)^\top}_{k_\star^\top} \big(W^{1/2} X X^\top W^{1/2} + \lambda I\big)^{-1} W^{1/2} y.$$

kernel ridge regression $K_W = W^{1/2} X X^\top W^{1/2}$

$k_\star = W^{1/2} X x_\star$

A.1 Proof of Proposition 1(A): explicit varying-coefficients ⇔ weighted KRR on joint features

$y = \langle \theta(c), x\rangle + \varepsilon$ $\mathbb{E}[\varepsilon] = 0$

$\theta(c) = B\,\phi(c)$ $B \in \mathbb{R}^{d_x \times d_c}$ $\lambda \|B\|_F^2$

Step 1 (reduce to ridge in joint-feature space).

B $\beta = \mathrm{vec}(B) \in \mathbb{R}^{d_x d_c}$

$$x_a^T B\,\phi(c_a) = \langle \beta,\, x_a \otimes \phi(c_a)\rangle = \langle \beta,\, \psi_a\rangle.$$

$$\min_{\beta\in\mathbb{R}^{d_x d_c}} \;\|W^{1/2}\big(y - Z\beta\big)\|_2^2 + \lambda\|\beta\|_2^2,$$

$$X \equiv Z$$

Step 2 (closed form and prediction).

$$\widehat{\beta} = (Z^T W Z + \lambda I)^{-1} Z^T W y,$$

(x, c) $\psi(x, c)$

$$\hat{y}(x,c) = \psi(x,c)^T\widehat{\beta} = \underbrace{\big(W^{1/2} Z\,\psi(x,c)\big)}_{k_{(x,c)}}^{\top}\big(W^{1/2}ZZ^{\top}W^{1/2} + \lambda I\big)^{-1} W^{1/2} y.$$

Step 3 (kernel form).

$K := ZZ^T$ $K_W := W^{1/2} K W^{1/2}$

$$\boxed{\hat{y}(x,c) \;=\; k_{(x,c)}^T\big(K_W + \lambda I\big)^{-1} W^{1/2} y}.$$

(a, b) K

$$K_{ab} = \langle\psi_a, \psi_b\rangle = \big\langle x_a \otimes \phi(c_a),\, x_b \otimes \phi(c_b)\big\rangle = \langle x_a, x_b\rangle \cdot \langle\phi(c_a), \phi(c_b)\rangle,$$

KRR on joint features W

A.2 Proof of Proposition 1(B): linear ICL ⇒ kernel regression

$S(c)$ **linear**

$$q(x,c) = Q\,\psi(x,c), \qquad k_a = K\,\psi_a, \qquad v_a = V\,\psi_a,$$

$Q \in \mathbb{R}^{d_q\times d_\psi}$ $K \in \mathbb{R}^{d_k\times d_\psi}$ $V \in \mathbb{R}^{d_v\times d_\psi}$ $d_\psi = d_x d_c$ **unnormalized**

a

$$s_a(x,c) := w_a\,\langle q(x,c), k_a\rangle \;=\; w_a\,\psi(x,c)^T Q^T K\,\psi_a.$$

$\alpha_a(x,c) := s_a(x,c)/\sum_b s_b(x,c)$

$\{\alpha_a\}$

$$z(x,c) = \sum_a \alpha_a(x,c)\,v_a, \qquad \hat{y}(x,c) = u^T z(x,c).$$

(B1)

(B2)

A.2.1 (B1) Fixed attention, trained linear head ⇒ exact KRR

Q, K, V $\alpha_a(x,c)$ **deterministic**

(x, c) **feature map**

$$\varphi(x,c) := \sum_a \alpha_a(x,c)\,v_a \;\in\; \mathbb{R}^{d_v}.$$

$\varphi_a := \varphi(x_a, c_a)$ $\Phi \in \mathbb{R}^{N\times d_v}$ u

$$\widehat{u} \in \arg\min_u \;\|W^{1/2}(y - \Phi u)\|_2^2 + \lambda\|u\|_2^2$$

$\widehat{u} = (\Phi^T W\Phi + \lambda I)^{-1}\Phi^T W y$

$$\hat{y}(x,c) = \varphi(x,c)^T\hat{u} = \underbrace{\big(W^{1/2}\Phi\,\varphi(x,c)\big)}_{k_{(x,c)}}^{T}\big(W^{1/2}\Phi\Phi^T W^{1/2} + \lambda I\big)^{-1}W^{1/2}y.$$

$$\boxed{\hat{y}(x,c) = k_{(x,c)}^T\big(K_W + \lambda I\big)^{-1}W^{1/2}y}, \quad K_W := W^{1/2}\underbrace{(\Phi\Phi^T)}_{=:K}W^{1/2},$$

kernel ridge regression

$$k\big((x,c),(x',c')\big) = \langle\varphi(x,c),\varphi(x',c')\rangle.$$

$v_a = V\psi_a$ $\alpha_a(x,c) \propto w_a\,\psi(x,c)^T Q^T K\psi_a$ φ **weighted average of joint features** $\{\psi_a\}$

A.2.2 (B2) Training attention in the linearized/NTK regime ⇒ kernel regression with NTK

$\theta = (Q,K,V,u)$

θ_0 **linearized model** θ_0

$$\hat{y}_\theta(x,c) \;\approx\; \hat{y}_{\theta_0}(x,c) + \nabla_\theta\hat{y}_{\theta_0}(x,c)^T(\theta-\theta_0) =: \hat{y}_{\theta_0}(x,c) + \phi_{\text{NTK}}(x,c)^T(\theta-\theta_0),$$

$\phi_{\text{NTK}}(x,c) := \nabla_\theta\hat{y}_{\theta_0}(x,c)$ **tangent features**

kernel regression with the NTK

$$k_{\text{NTK}}\big((x,c),(x',c')\big) := \big\langle\phi_{\text{NTK}}(x,c),\,\phi_{\text{NTK}}(x',c')\big\rangle,$$

K_{NTK}

ϕ_{NTK} **linear attention**

linear transforms of joint features $\hat{y}(x,c) = u^T\sum_a \alpha_a(x,c)\,V\psi_a$ θ_0

- **Readout path (u).** $\partial\hat{y}/\partial u = \sum_a \alpha_a(x,c)\,V\psi_a = \varphi_0(x,c)$ $\{\psi_a\}$
- **Value path (V).** $\partial\hat{y}/\partial V = \sum_a \alpha_a(x,c)\,u\,\psi_a^T$ $(u\otimes I)\sum_a \alpha_a(x,c)\psi_a$ $\{\psi_a\}$
- **Query/key paths (Q,K).** $s_a = w_a\,\psi(x,c)^T Q^T K\psi_a$ $\alpha_a = s_a/\sum_b s_b$ α_a Q K $\psi(x,c)$ $\{\psi_a\}$

$$\frac{\partial\alpha_a}{\partial Q} \propto \sum_b \big[\delta_{ab} - \alpha_b(x,c)\big]\,w_a w_b\big(K\psi_a\,\psi(x,c)^T\big),$$

$$\frac{\partial\alpha_a}{\partial K} \propto \sum_b \big[\delta_{ab} - \alpha_b(x,c)\big]\,w_a w_b\big(\psi(x,c)\,\psi_a^T Q^T\big),$$

$\partial\hat{y}/\partial Q$ $\partial\hat{y}/\partial K$ $\psi(x,c)$ ψ_a u V $\psi(x,c)$ $\{\psi_a\}$

$$\phi_{\mathrm{NTK}}(x,c) = \mathcal{L}\big(\psi(x,c), \{\psi_a\}\big),$$

$\mathcal{L}$ θ_0 W

dot-product

$$k_{\mathrm{NTK}}\big((x,c),(x',c')\big) = \Psi(x,c)^T\,\mathcal{M}\,\Psi(x',c'),$$

$\mathcal{M}$ Ψ

$\psi(x,c)$ $\{\psi_a\}$ k_{NTK}

linear transforms of the joint features

Assumptions for A.2.2.

A.3 Proof of Corollary 1: retrieval/gating/weighting as kernel/measure choices

$$\hat{y}(x,c) = k^T_{(x,c)}\big(K^\sharp + \lambda I\big)^{-1}\mu,$$

$K^\sharp$

$K_W = W^{1/2}ZZ^TW^{1/2}$ $W^{1/2}\Phi\Phi^TW^{1/2}$ K_{NTK} $k_{(x,c)}$

$\mu = W^{1/2}y$

- **Retrieval $R(c)$ / gating.** $S(c)$ **removes or adds rows/columns** $K^\sharp$ $k_{(x,c)}$ **empirical measure**
- **Weights $w_{ij}(c)$.** W K $K_W = W^{1/2}KW^{1/2}$ k $k_{(x,c)} = W^{1/2}k$ **importance weighting**
- **Induced kernels.** **feature map** $\psi \mapsto V\psi$ $\psi \mapsto Q\psi$ $k((x,c),(x',c')) = \langle \Phi(x,c), \Phi(x',c')\rangle$

neighborhood selection

kernel choice

A.4 Remarks

- **No Gaussianity is required.** $y = f(x,c) + \varepsilon$ $\mathbb{E}[\varepsilon] = 0$
- **Early stopping vs. explicit ridge.** λ
- **Multiple layers / nonlinear value stacks.**