PROVABLE COMPUTATIONAL AND STATISTICAL GUARANTEES FOR EFFICIENT LEARNING OF CONTINUOUS-ACTION GRAPHICAL GAMES

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ABSTRACT

In this paper, we study the problem of learning the set of pure strategy Nash equilibria and the exact structure of a continuous-action graphical game with parametric payoffs by observing a small set of perturbed equilibria. A continuous-action graphical game can possibly have an uncountable set of Nash equilibria. We propose an ℓ_{12} block regularized method which recovers a graphical game, whose Nash equilibria are contained in the ϵ-Nash equilibria of the game from which the data was generated (true game). Under a slightly stringent condition on the parameters of the true game, our method recovers the exact structure of the graphical game. Our method has a logarithmic sample complexity with respect to the number of players. It also runs in polynomial time. *A full version of this paper is accessible at:* `https://www.cs.purdue.edu/homes/abarik/icassp_para_games.pdf`

Index Terms— Graphical Games, Machine Learning, Continuous Action Games, Parametric Payoffs

1. INTRODUCTION

The real world is filled with scenarios which arise due to competitive actions by selfish individual players who are trying to maximize their own utilities or payoffs. Non-cooperative game theory has been considered as the appropriate mathematical framework to formally study *strategic* behavior in such multi-agent scenarios. In such scenarios, each agent decides its action based on the actions of other players. The core solution concept of *Nash equilibrium (NE)* [1] serves a descriptive role of the stable outcome of the overall behavior of self-interested agents (e.g., people, companies, governments, groups or autonomous systems) interacting strategically with each other in distributed settings.

1.1. Graphical Games

The introduction of compact representations to game theory over the last two decades have extended algorithmic game theory's potential for large-scale, practical applications often encountered in the real world. Introduced within the AI community about two decades ago, *graphical games* [2] constitute an example of one of the first and arguably one of the most influential graphical models for game theory. Indeed, graphical games played a prominent role in establishing the computational complexity of computing NE in normal-form games as well as in succinctly representable multiplayer games (see, e.g., [3, 4, 5] and the references therein). Players can take actions in either a discrete space (for example in voting) or in a continuous space (for example in simultaneous auctions in online advertising). Correspondingly, graphical games can be studied in both domains. In this paper, we focus on continuous-action graphical games. There has been considerable progress on *computing* classical equilibrium solution concepts such as NE and *correlated equilibria* [6] in graphical games . In addition, [7] identified the most influential players, i.e., a small set of players whose collective behavior forces every other player to a unique choice of action. All the work above focus on inference problems for graphical games, and fall in the field of algorithmic game theory.

1.2. Beyond Inference: Learning Graphical Games

While studying graphical games one often assumes that the structure (i.e., neighbors for all players) and payoffs of the games under consideration are already available. Relatively less attention has been paid to the problem of *learning* the structure of graphical games from data. Addressing this problem is essential to the development, potential use and success of game-theoretic models in practical applications. In this paper, we study the problem of learning the characterization of pure strategy Nash equilibrium and structure of the graph in a continuous-action graphical game.

1.3. A Motivating Real-World Example

We shed more light on the difference between learning and inference by computing the price of anarchy for $k = 2$ products (crude petroleum and cars) among different countries of the world between 1962 to 2017. We see that some of world's most powerful economies (such as Germany, Japan and Saudi

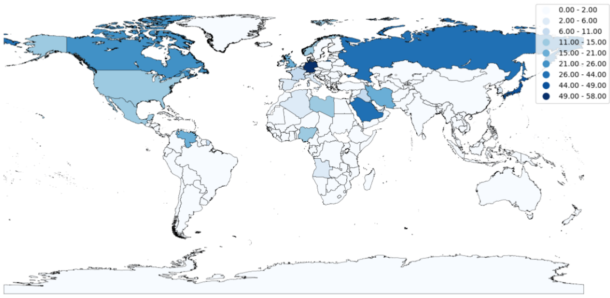


Fig. 1: Most "locally" influential trading countries for the top 2 most exported products ($1962 - 2017$). "Local" influence is measured by the number of out-neighbors in the learnt game.

Arabia) had large influence on the crude oil and automobile market. The price of anarchy was computed to be the ratio between the maximum welfare across all strategy profiles and the minimum welfare across all strategy profiles in the ϵ-Nash equilibria set, where welfare is defined as the sum of payoffs of all players (See subsection 3.1 for details). In this case, it was necessary to learn the structure of the graphical game before we could compute price of anarchy. The structure of the graphical game was learnt using our method (See Appendix E for details). Once the structure was learnt, the price of anarchy could be computed. We found the price of anarchy to be 1.2575. Our finding implies that the total welfare across countries can be increased by more than 25% by deviating away from Nash equilibria.

1.4. Related Work

There has been considerable amount of work done for learning games in the discrete-action setting. [8] proposed a maximum-likelihood approach to learn "linear influence games" - a class of parametric graphical games with binary actions and linear payoffs. However, their method runs in exponential time and the authors assumed a specific observation model for the strategy profiles. For the same specific observation model, [9] proposed a polynomial time algorithm, based on ℓ_1-regularized logistic regression, for learning linear influence games. Their strategy profiles (or joint actions) were drawn from a mixture of uniform distributions: one over the pure-strategy Nash equilibria (PSNE) set, and the other over its complement. [10] obtained necessary and sufficient conditions for learning linear influence games under arbitrary observation model. [11] use a discriminative, max-margin based approach, to learn tree structured polymatrix games . Their method runs in exponential time and the authors show that learning polymatrix games is NP-hard under this max-margin setting, even when the class of graphs is restricted to trees. Finally, [12] proposed a polynomial time algorithm for learning sparse polymatrix games in the discrete-action setting. Continuous-action games with quadratic payoffs have been used extensively in the game theory literature [13, 14, 15, 16]. [17] proposed algorithms to learn games with quadratic payoffs, in a simplified setting. However, the authors do not provide any theoretical guarantees. In this work, we focus on provable guarantees for a far more general class of games with parametric payoffs in the high-dimensional regime. Our work directly extends to quadratic payoff setting by simply taking square of the payoff function in our analysis.

1.5. Our Contribution

We aim to propose a novel method to learn graphical games with parametric payoffs, with the following provable guarantees in mind: 1. **Correctness -** We want to develop a method which correctly recovers the set of Nash equilibria and the structure of the graphical games. 2. **Computational efficiency -** Our method must run fast enough to handle the high dimensional cases. Ideally, we want to have polynomial time complexity with respect to the number of players. 3. **Sample complexity -** We would like to use as few samples as possible for recovering the set of Nash equilibria. We want to achieve logarithmic sample complexity with respect to the number of players. To this end, we propose a block-norm regularized method to learn graphical games with parametric payoff functions. The novelty of this formulation lies in its applicability to samples generated through a novel sampling mechanism while still being amenable to rigorous technical analysis. Furthermore, we go beyond solving the optimization problem by providing guarantees for recovery of a subset of the true ϵ-Nash equilibria. We also compute global efficiency quantities such as price of anarchy. For n players, at most d in-neighbors per player, and k-dimensional action vectors, we show that $Rk^5 d^3 \log(dnk)$ *samples* are sufficient to recover the complete characterization of a set which is contained in ϵ-Nash equilibria of the true game where R is a positive constant independent of n, d and k. Under slightly more stringent conditions, we also recover the true structure of the game. Our method also runs in polynomial time complexity.

2. PRELIMINARIES

In this section, we introduce our notation and formally define the problem of learning graphical games with parametric utility functions.

Notation. For a matrix $\mathbf{A} \in \mathbb{R}^{p \times q}$ and two sets $S \subseteq \{1, \dots, p\}$ and $T \subseteq \{1, \dots, q\}$, $\mathbf{A}_{ST}$ denotes $\mathbf{A}$ restricted to rows in S and columns in T. Similarly, $\mathbf{A}_{S.}$ and $\mathbf{A}_{.T}$ are row and column restricted matrices respectively. For a vector $\mathbf{m} \in \mathbb{R}^q$, the ℓ_∞-norm is defined as $\|\mathbf{m}\|_\infty = \max_{i \in \{1,\dots,p\}} |\mathbf{m}_i|$. The Frobenius norm for a matrix $\mathbf{A} \in \mathbb{R}^{p \times q}$ is defined as $\|\mathbf{A}\|_{\text{F}} = \sqrt{\sum_{i=1}^p \sum_{j=1}^q |\mathbf{A}_{ij}|^2}$. The ℓ_∞-operator norm for $\mathbf{A}$ is defined as $\|\mathbf{A}\|_{\infty,\infty} =$

$\max_{i\in\{1,\ldots,p\}} \sum_{j=1}^{q} |\mathbf{A}_{ij}|$. The spectral norm of $\mathbf{A}$ is defined as $\|\mathbf{A}\|_{2,2} = \sup_{\|\mathbf{x}\|_2=1} \|\mathbf{A}\mathbf{x}\|_2$. We also define a block matrix norm for row-partitioned block matrices. Let $\mathbf{A} \in \mathbb{R}^{\sum_{i=1}^{k} p_i \times q}, \forall i \in \{1,\ldots,k\}$ be a row-partitioned block matrix defined as follows: $\mathbf{A} = \begin{bmatrix}\mathbf{A}_1 & \cdots & \mathbf{A}_k\end{bmatrix}^\intercal$ where each $\mathbf{A}_i \in \mathbb{R}^{p_i \times q}$. We define two block matrix norms as $\|\mathbf{A}\|_{\mathrm{B},\infty,\mathrm{F}} = \max_{i\in\{1,\ldots,k\}} \|\mathbf{A}_i\|_{\mathrm{F}}$ and $\|\mathbf{A}\|_{\mathrm{B},\infty,1} = \max_{i\in\{1,\ldots,k\}} \sum_{l=1,m=1}^{l=p_i,m=q} |[\mathbf{A}_i]_{lm}|$.

Setup. Consider a directed graph $\mathcal{G}(\mathcal{V},\mathcal{E})$, where $\mathcal{V}$ and $\mathcal{E}$ are set of vertices and edges respectively. We define $\mathcal{V} \triangleq \{1,\ldots,n\}$, where each vertex corresponds to one player. We denote the in-neighbors of a player i by S_i, i.e., $\mathrm{S}_i = \{j \mid (j,i) \in \mathcal{E}\}$. All the other players are denoted by S_i^c, i.e., $\mathrm{S}_i^c = \{1,\ldots,n\}\backslash(\mathrm{S}_i \cup i)$. Let $|\mathrm{S}_i| = d$ and $|\mathrm{S}_i^c| = n-d$. For each player $i \in \mathcal{V}$, there is a set of actions or *pure-strategies* $\mathcal{A}_i$, i.e., player i can take action $x_i \in \mathcal{A}_i$. Each action x_i consists of making k decisions on a limited budget $\mathrm{b} \in \mathbb{R}_+$. We consider games with continuous actions. Mathematically, $x_i \in \mathbb{R}^k$ and $\|x_i\|_2 \leqslant \mathrm{b}$. For each player i, there is also a local payoff function $u_i : \mathcal{A}_i \times (\times_{j\in\mathrm{S}_i} \mathcal{A}_j) \to \mathbb{R}$ mapping the joint action of player i and its in-neighbors S_i, to a real number. Later, we will define a particular kind of local payoff function which is of our interest. A joint action $\mathbf{x}^* \in \times_{i\in\mathcal{V}} \mathcal{A}_i$ is a *pure-strategy Nash equilibrium (PSNE)* of a graphical game iff, no player i has any incentive to unilaterally deviate from the prescribed action $x_i^* \in \mathcal{A}_i$, given the joint action of its in-neighbors $x_{\mathrm{S}_i}^* \in \times_{j\in\mathrm{S}_i} \mathcal{A}_j$ in the equilibrium. We denote a game by $\mathscr{G}$, and the set of all PSNE and ϵ-PSNE of $\mathscr{G}$, by NE($\mathscr{G}$) and NE$_\epsilon$($\mathscr{G}$) respectively, for a constant $\epsilon > 0$. Formally, NE($\mathscr{G}$) $\triangleq \{\mathbf{x}^* \in \times_{i\in\mathcal{V}} \mathcal{A}_i \mid x_i^* \in \arg\max_{x_i\in\mathcal{A}_i} u_i(x_i, x_{\mathrm{S}_i}^*), \forall i \in \mathcal{V}\}$ and NE$_\epsilon$($\mathscr{G}$) $\triangleq \{\mathbf{x}^* \in \times_{i\in\mathcal{V}} \mathcal{A}_i \mid u_i(x_i^*, x_{\mathrm{S}_i}^*) \geqslant -\epsilon + \max_{x_i\in\mathcal{A}_i} u_i(x_i, x_{\mathrm{S}_i}^*), \forall i \in \mathcal{V}\}$.

Parametric Payoffs. We are interested in solving a parametrized version of the problem where payoffs are a parametric function of the joint action. In that, given the weights $W_{ij}^* \in \mathbb{R}^{k\times k}, \forall i,j \in \mathcal{V}$, for each player i, we define the set of in-neighbors of player i as $\mathrm{S}_i = \{j \mid W_{ij}^* \neq 0\}$ and the payoff function $u_i(x_i, x_{\mathrm{S}_i}) = -\|x_i - \sum_{j\in\mathrm{S}_i} W_{ij}^* x_j\|_2$. We consider $\max_{x_i} u_i(x_i, x_{\mathrm{S}_i}^*) = 0, \forall i \in \{1,\ldots,n\}$, such that in a PSNE, each player i matches its action x_i to the weighted actions of its neighbors, i.e., $\sum_{j\in\mathrm{S}_i} W_{ij}^* x_j^*$. Let $\epsilon > 0$ be a constant. The set of all ϵ-PSNE of $\mathscr{G}$ is NE$_\epsilon$($\mathscr{G}$) $= \{\mathbf{x}^* \in \times_{i=1}^{n} \mathcal{A}_i \mid \|x_i^* - \sum_{j\in\mathrm{S}_i} W_{ij}^* x_j^*\|_2 \leqslant \epsilon, \forall i \in \mathcal{V}\}$.

2.1. Sampling

Given the above characterization, the set of ϵ-PSNE is a convex polytope. We observe samples from the set of noisy PSNE. Treating the outcomes of the game as "samples" observed across multiple "plays" of the same game is a recurring theme in the literature for learning games [8, 10, 12]. All of these works assume access to *noisy* Nash equilibria. *Learning* from a training set of *noisy* Nash equilibria is not the same as computing Nash equilibria (i.e., *inference*). Noise could be added to each player's strategy at a local level or by mixing the sample with a non-Nash equilibria set at a global level. Our method is similar to adding noise at a local level. We observe the noisy joint actions $\mathbf{x} = \mathbf{x}^* + \mathbf{e}$ where $\mathbf{x}^*$ is a Nash equilibrium, that is $\mathbf{x}* \in$ NE($\mathscr{G}$) and $\mathbf{e}$ is independent zero mean bounded sub-Gaussian noise with variance proxy σ^2. Boundedness is required to keep the norm of the noisy action within budget b. Using sub-Gaussian noise (possibly with truncation) makes our setup quite general in nature as it includes for instance truncated Gaussian random variables, any bounded random variable (e.g. Bernoulli, multinomial, uniform), any random variable with strictly log-concave density, and any finite mixture of sub-Gaussian variables.

3. MAIN RESULT

In this section, we describe our main theoretical results. But before we do that, we discuss some technical assumptions which are needed for our proofs.

Assumption 1. *For all* $x_i \in \mathcal{A}_i, \|x_i\|_2 \leqslant \mathrm{b}, \forall i \in \{1,\ldots,n\}$ *for some* $\mathrm{b} > 0$.

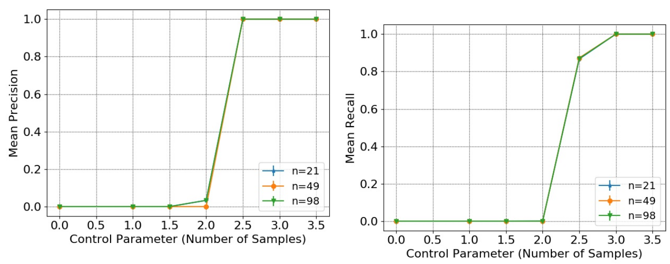
Assumption 2. *At PSNE,* $u_i(x_i^*, x_{\text{-}i}^*) = 0, \forall i \in \{1,\ldots,n\}$.

Assumption 3. *Consider* $\mathbf{H} = \frac{1}{T}\sum_{t=1}^{T} \left(\mathbf{x}_{\text{-}i}^{*t} \mathbf{x}_{\text{-}i}^{*t\intercal} + \sigma^2 \mathbf{I}\right)$ *where* $\mathbf{I}$ *is the identity matrix, then* $\|\left[\mathbf{H}\right]_{\mathrm{S}_i^c \mathrm{S}_i} \left[\mathbf{H}\right]_{\mathrm{S}_i \mathrm{S}_i}^{-1}\|_{\mathrm{B},\infty,1} \leqslant 1-\alpha$ *for some* $\alpha \in (0,1]$.

Assumption 1 simply states that each player has a limited budget to allocate for its actions. For instance, consider simultaneous auctions in an online advertising, where a company chooses how to allocate its budget into several options. For a sufficiently large budget b, Assumption 2 is not difficult to fulfill for our parametric payoffs. We propose a mutual incoherence assumption (Assumption 3) for games. While mutual incoherence is new to graphical games, it has been a standard assumption in various estimation problems such as compressed sensing [18], Markov random fields [19], non-parametric regression [20], diffusion networks [21], among others. Now that all our assumptions are in place, we are ready to setup our estimation problem. Consider that we have access to T perturbed equilibria, i.e., we have access to $x_i^t = x_i^{*t} + e_i^t$ where superscript t denotes the t-th sample and $e_i^t \in \mathbb{R}^k$ is a vector of zero-mean mutually independent sub-Gaussian noises with variance proxy σ^2. We define the following loss function: $\ell(\mathbf{x}, \mathbf{W}_{i\cdot}) = \frac{1}{T}\sum_{t=1}^{T} \|x_i^t - \sum_{j=1,j\neq i}^{n} W_{ij} x_j^t\|_2^2$ where $\mathbf{x}$ denotes a collection of T samples and $\mathbf{W}_{i\cdot}$ denotes the collection of all $W_{ij}, \forall j \in \{1,\ldots,n\}, j \neq i$. We estimate the parameters W_{ij} for each $i,j \in \{1,\ldots,n\}$ by solving the following optimization problem:

$$\widehat{\mathbf{W}}_{i\cdot} = \arg\min_{W_{i\cdot}} \ell(\mathbf{x}, \mathbf{W}_{i\cdot}) + \lambda_T \sum_{\substack{j=1 \\ j\neq i}}^{n} \|W_{ij}\|_{\mathrm{F}} \,. \qquad (1)$$

Note that if one were to vectorize the matrices W_{ij}, then $\sum_{\substack{j=1 \\ j \neq i}}^{n} \|W_{ij}\|_{\mathrm{F}}$ could be interpreted as an ℓ_{12} norm. Our next theorem states that the recovered $\widehat{\mathbf{W}}_{i\cdot}$ completely characterizes a set of solutions which are contained in ϵ-Nash equilibria of the true game.



(a) Precision against number of samples

(b) Recall against number of clients

Fig. 2: Exact Structure Recovery in Parametric Games for $n_b = 21, 49$, and 98. The number of samples is varied as $10^C \log(n_b)$ for both figures where C is a control parameter.

Theorem 1. *Consider a continuous-action graphical game $\mathscr{G}$ such that Assumptions 1, 2 and 3 are satisfied for each player. Let* $\lambda_T \geqslant \max(\frac{640\sigma \max(\mathrm{b}, \sigma, \overline{W}\mathrm{b}, \overline{W}\sigma)}{\alpha}\sqrt{\frac{k^3 d \log(2k^2 n)}{T}}, \frac{80\sigma^2}{\alpha} \|[\mathbf{W}_{i\cdot}^*]_{\mathrm{S}_i\cdot}\|_{\infty,2})$ *and* $T = \Omega(k^5 d^3 \log(kn))$*, then the following claims hold with probability at least* $1 - \exp(-cT\lambda_T^2)$ *for a positive constant* c*.*

1. *We recover the correct set of non-neighbors for each player* i*.*
2. *For each player* i*, if* $\min_{j \in \mathrm{S}_i} \|W_{ij}^*\|_F > \frac{k\sqrt{kd}\lambda_T(10\alpha+1)}{5C}$*, then we recover the exact structure of the graphical game* $\mathscr{G}$*.*
3. *Furthermore, we recover a set* $\overline{\mathrm{NE}}_\epsilon(\mathscr{G}) \subseteq \mathrm{NE}_\epsilon(\mathscr{G})$ *by estimating* $\mathbf{W}_{i\cdot}$ *for each player* i *by solving the optimization problem* (1)*.*

where C *is the minimum eigenvalue of* $[\mathbf{H}]_{\mathrm{S}_i\mathrm{S}_i}$*,* $\overline{W}$ *is the maximum entry in absolute value of* $\mathbf{W}_{i\cdot}^*$*,* $\underline{W}$ *is the minimum non-zero entry in absolute value of* $\mathbf{W}_{i\cdot}^*$ *and* $\epsilon = d\frac{k\sqrt{kd}\lambda_T(10\alpha+1)}{5C}\mathrm{b}$.

Our theorem states that there is a value of λ_T which allows us to correctly identify the non-neighbors of a player. Depending on the value of λ_T, if parameters $[\mathbf{W}_{i\cdot}]_{\mathrm{S}_i\cdot}$ satisfy a minimum weight requirement then we discover the true structure of the game. We also provide a characterization of a set which is fully contained in the ϵ-Nash equilibria of the true game. Note that we do not put any constraints on the choice of Nash equilibria samples. However, this comes at a price of compromising the solution quality. We discuss this in detail in Appendix C.

3.1. Computation of Global Efficiency Quantities

The recovered $\overline{\mathrm{NE}}_\epsilon(\mathscr{G})$ enables us to compute global efficiency quantities such as the Price of Anarchy (PoA), the Price of Stability (PoS) and the volume of Nash equilibria (Vol) for the recovered game. To do this, we first define the recovered utility function as follows: $\widehat{u}_i(x) = -\|x_i - \sum_{j \in \mathrm{S}_i} W_{ij} x_j\|_2$. Using these utility functions, we define the following welfare function: $\mathrm{Wel}(x) \triangleq \sum_{i \in \mathcal{V}} \widehat{u}_i(x)$. Let $\mathcal{A}$ denote the action set $\times_{i \in \mathcal{V}} \mathcal{A}_i$. Then,

$$\overline{\mathrm{PoA}}_\epsilon(\mathscr{G}) = \frac{\sup_{x \in \mathcal{A}} \mathrm{Wel}(x)}{\inf_{x \in \overline{\mathrm{NE}}_\epsilon(\mathscr{G})} \mathrm{Wel}(x)}, \overline{\mathrm{Vol}}_\epsilon(\mathscr{G}) = \frac{\tau(\overline{\mathrm{NE}}_\epsilon(\mathscr{G}))}{\tau(\mathcal{A})},$$

$$\overline{\mathrm{PoS}}_\epsilon(\mathscr{G}) = \frac{\sup_{x \in \mathcal{A}} \mathrm{Wel}(x)}{\sup_{x \in \overline{\mathrm{NE}}_\epsilon(\mathscr{G})} \mathrm{Wel}(x)},$$

where $\tau(A)$ is the Lebesgue measure of set A. The subscript ϵ and bar in the global efficiency quantities denote that $\overline{\mathrm{NE}}_\epsilon(\mathscr{G})$ is used in their computation.

3.2. Sample and Time Complexity

If we have $T = \Omega(k^5 d^3 \log(k(n-d)d))$ and all other conditions mentioned in Theorem 1 are satisfied for every player then all our high probability statements are valid for every player i. Taking a union bound over n players only adds a factor of $\log n$. Thus the sample complexity for our method is $\Omega(k^5 d^3 \log(k(n-d)d))$. As for the time complexity, we can formulate the block-regularized multi-variate regression problem as a second order cone programing problem [22] which can be solved in polynomial time by interior point methods [23].

3.3. Experimental Validation

We validate results of Theorem 1 by running computational experiments on synthetic data. Figure 2a and 2b show how the precision and recall for games with various number of players vary with number of samples. All results are averaged across 30 independent runs. We can see that both precision and recall go to 1 as we increase the number of samples, obtaining perfect recovery with enough samples. Notice that the different curves for different number of players ($n_b = 21, 49$ and 98) line up with one another quite well. This matches with our theoretical results and shows that with a constant number of in-neighbors $\mathcal{O}(\log(n))$ samples are sufficient to recover the exact structure of the graphical games (Check Appendix D for details).

Concluding Remarks. We have shown that the ℓ_{12}−block regularized estimation method can be used to recover a non-trivial subset of epsilon Nash equilibria $\mathrm{NE}_\epsilon(\mathscr{G})$ of a continuous-action graphical game with parametric payoffs. We also provided theoretical guarantees for our method and showed that it runs in polynomial time and logarithmic sample complexity. We have also shown the effectiveness of our method by running it on a trading dataset and computing the most "locally" influential trading countries (See Appendix E).

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A. PROOFS OF THEOREMS AND LEMMAS

A.1. Proof of Theorem 1

Theorem 1 *Consider a continuous-action graphical game $\mathcal{G}$ such that Assumptions 1, 2 and 3 are satisfied for each player. Let* $\lambda_T \geqslant \max(\frac{640\sigma \max(\mathrm{b},\sigma,\overline{W}\mathrm{b},\overline{W}\sigma)}{\alpha}\sqrt{\frac{k^3 d\log(2k^2n)}{T}}, \frac{80\sigma^2}{\alpha}\|[\mathbf{W}^*_{i.}]_{\mathrm{S}_i\cdot}\|_{\infty,2})$ *and* $T = \Omega(k^5 d^3 \log(kn))$, *then the following claims hold with probability at least* $1 - \exp(-cT\lambda_T^2)$ *for a positive constant c.*

1. *We recover the correct set of non-neighbors for each player i.*
2. *For each player i, if* $\min_{j\in\mathrm{S}_i}\|W^*_{ij}\|_F > \frac{k\sqrt{kd}\lambda_T(10\alpha+1)}{5C}$, *then we recover the exact structure of the graphical game $\mathcal{G}$.*
3. *Furthermore, we recover a set* $\overline{\mathrm{NE}}_\epsilon(\mathcal{G}) \subseteq \mathrm{NE}_\epsilon(\mathcal{G})$ *by estimating* $\mathbf{W}_{i\cdot}$ *for each player i by solving the optimization problem* (1).

where C is the minimum eigenvalue of $[\mathbf{H}]_{\mathrm{S}_i\mathrm{S}_i}$, $\overline{W}$ *is the maximum entry in absolute value of* $\mathbf{W}^*_{i\cdot}$, $\underline{W}$ *is the minimum non-zero entry in absolute value of* $\mathbf{W}^*_{i\cdot}$ *and* $\epsilon = d\frac{k\sqrt{kd}\lambda_T(10\alpha+1)}{5C}\mathrm{b}$.

Proof. We will make use of the *primal-dual witness* method to prove Theorem 1. By using the definition of Frobenious norm, optimization problem (1) can be equivalently written as

$$\widehat{\mathbf{W}}_{i\cdot} = \arg\min_{\mathbf{W}_{i\cdot}} \ell(\mathbf{x}, \mathbf{W}_{i\cdot}) + \lambda_T \sum_{j=1,j\neq i}^{n} \sup_{\|Z_{ij}\|_{\mathrm{F}}\leqslant 1} \langle Z_{ij}\,, W_{ij}\rangle\,. \tag{2}$$

Consider the term $\sup_{\|Z_{ij}\|_{\mathrm{F}}\leqslant 1}\langle Z_{ij}\,, W_{ij}\rangle$. We can assign specific values to Z_{ij} to get the maximum possible value of $\langle Z_{ij}\,, W_{ij}\rangle$. In particular, we can take if $W_{ij} \neq \mathbf{0}$ then $Z_{ij} = \frac{W_{ij}}{\|W_{ij}\|_{\mathrm{F}}}$, and if $W_{ij} = \mathbf{0}$ then $\|Z_{ij}\|_{\mathrm{F}} \leqslant 1$. Note that in the first case $\|Z_{ij}\|_{\mathrm{F}} = 1$ and thus it gives the maximum value for $\langle Z_{ij}\,, W_{ij}\rangle$ and no further improvement is possible. In the second case, since $W_{ij} = \mathbf{0}$, Z_{ij} can take any value such that $\|Z_{ij}\|_{\mathrm{F}} \leqslant 1$ without affecting $\langle Z_{ij}\,, W_{ij}\rangle$. We fix Z_{ij} to one such value and rewrite equation (2) as

$$\widehat{\mathbf{W}}_{i\cdot} = \arg\min_{\mathbf{W}_{i\cdot}} \ell(\mathbf{x}, \mathbf{W}_{i\cdot}) + \lambda_T \sum_{j=1,j\neq i}^{n} \langle Z_{ij}\,, W_{ij}\rangle\,, \tag{3}$$

where the last equality comes by keeping in mind that Z_{ij} are chosen as described above. Using the stationarity Karush-Kuhn-Tucker condition at the optimum, for each W_{ij} we can write,

$$\frac{\partial \ell(\mathbf{x}, \mathbf{W}_{i\cdot})}{\partial W_{ij}} + \lambda_T Z_{ij} = 0\,.$$

Note that $x_i = x_i^* + e_i, \forall i \in \{1, \dots, n\}$, where $e_i \in \mathbb{R}^k$ is zero mean bounded sub-Gaussian noise with variance proxy σ^2. Furthermore, $x_i^* = \sum_{j=1,j\neq i}^n W_{ij}^* x_j^*$. Making these substitutions and denoting $\Delta\mathbf{W}_{i\cdot} \triangleq \mathbf{W}_{i\cdot} - \mathbf{W}^*_{i\cdot}$, we get

$$2\widehat{\mathrm{E}}\big(\mathbf{x}_{\text{-}i}\mathbf{x}_{\text{-}i}{}^{\mathsf{T}}\big)\Delta\mathbf{W}_{i\cdot} - 2\widehat{\mathrm{E}}\big(\mathbf{x}_{\text{-}i}\mathbf{e}_{\text{-}i}{}^{\mathsf{T}}\big)\mathbf{W}^*_{i\cdot} - 2\widehat{\mathrm{E}}\big(\mathbf{x}_{\text{-}i}\mathbf{e}_i{}^{\mathsf{T}}\big) + \lambda_T\mathbf{Z}_{i\cdot} = \mathbf{0}\,, \tag{4}$$

where $\mathbf{W}^*_{i\cdot}, \mathbf{W}_{i\cdot}, \mathbf{Z}_{i\cdot} \in \mathbb{R}^{(n-1)k\times k}$ are block matrices with blocks ${W^*_{ij}}^{\mathsf{T}}, W_{ij}^{\mathsf{T}}$ and $Z_{ij}^{\mathsf{T}}, \forall j \in \{1, \dots, n\}, j \neq i$ respectively. Similarly, $\mathbf{x}^t_{\text{-}i}, \mathbf{e}^t_{\text{-}i} \in \mathbb{R}^{(n-1)k\times 1}$ are block matrices with blocks x_j^t and e_j^t respectively. Note that ${W^*_{ij}}^{\mathsf{T}} = \mathbf{0}$ for all $j \notin \mathrm{S}_i$. We also take ${W_{ij}}^{\mathsf{T}} = \mathbf{0}$ for all $j \notin \mathrm{S}_i$ and later show in our proofs that this indeed holds true. Thus, the stationarity condition can be written as,

$$2\big[\widehat{\mathrm{E}}\big(\mathbf{x}_{\text{-}i}\mathbf{x}_{\text{-}i}{}^{\mathsf{T}}\big)\big]_{\cdot\mathrm{S}_i}\big[\Delta\mathbf{W}_{i\cdot}\big]_{\mathrm{S}_i\cdot} - 2\big[\widehat{\mathrm{E}}\big(\mathbf{x}_{\text{-}i}\mathbf{e}_{\text{-}i}{}^{\mathsf{T}}\big)\big]_{\cdot\mathrm{S}_i}\big[\mathbf{W}^*_{i\cdot}\big]_{\mathrm{S}_i\cdot} - 2\widehat{\mathrm{E}}\big(\mathbf{x}_{\text{-}i}\mathbf{e}_i{}^{\mathsf{T}}\big) + \lambda_T\mathbf{Z}_{i\cdot} = \mathbf{0}\,. \tag{5}$$

Equation (5) can be decomposed in two separate equations. One for the players in S_i and other for players not in S_i which we denote by S_i^c.

$$2\big[\widehat{\mathrm{E}}\big(\mathbf{x}_{\text{-}i}\mathbf{x}_{\text{-}i}{}^{\mathsf{T}}\big)\big]_{\mathrm{S}_i\mathrm{S}_i}\big[\Delta\mathbf{W}_{i\cdot}\big]_{\mathrm{S}_i\cdot} - 2\big[\widehat{\mathrm{E}}\big(\mathbf{x}_{\text{-}i}\mathbf{e}_{\text{-}i}{}^{\mathsf{T}}\big)\big]_{\mathrm{S}_i\mathrm{S}_i}\big[\mathbf{W}^*_{i\cdot}\big]_{\mathrm{S}_i\cdot} - 2\big[\widehat{\mathrm{E}}\big(\mathbf{x}_{\text{-}i}\big)\big]_{\mathrm{S}_i\cdot}\mathbf{e}_i{}^{\mathsf{T}} + \lambda_T\big[\mathbf{Z}_{i\cdot}\big]_{\mathrm{S}_i\cdot} = \mathbf{0}, \tag{6}$$

and

$$2\big[\widehat{\mathrm{E}}\big(\mathbf{x}_{\text{-}i}\mathbf{x}_{\text{-}i}{}^{\intercal}\big)\big]_{\mathrm{S}_i^c\mathrm{S}_i}\big[\Delta\mathbf{W}_{i\cdot}\big]_{\mathrm{S}_i\cdot} - 2\big[\widehat{\mathrm{E}}\big(\mathbf{x}_{\text{-}i}\mathbf{e}_{\text{-}i}{}^{\intercal}\big)\big]_{\mathrm{S}_i^c\mathrm{S}_i}\big[\mathbf{W}^*_{i\cdot}\big]_{\mathrm{S}_i\cdot} - 2\big[\widehat{\mathrm{E}}\big(\mathbf{x}_{\text{-}i}\big)\big]_{\mathrm{S}_i^c\cdot}\mathbf{e}_i{}^{\intercal} + \lambda_T\big[\mathbf{Z}_{i\cdot}\big]_{\mathrm{S}_i^c\cdot} = \mathbf{0}. \tag{7}$$

Before we move ahead, we provide some properties of the finite-sample regime which hold with high probability. We define $\widehat{\mathbf{H}} \triangleq \widehat{\mathrm{E}}\big(\mathbf{x}_{\text{-}i}\mathbf{x}_{\text{-}i}{}^{\intercal}\big)$ – a finite sample analogue to $\mathbf{H}$. The detailed proofs of these lemmas are available in Appendix A.

Lemma 1 (Positive minimum eigenvalue)**.** $\Lambda_{\min}(\big[\widehat{\mathbf{H}}\big]_{\mathrm{S}_i\mathrm{S}_i}) > 0$ *with probability at least* $1 - \exp(-cT\sigma^4 + \mathcal{O}(kd)) - \exp(-\frac{\sigma^2 T}{128\mathrm{b}^2} + \mathcal{O}(kd))$ *for some constant* $c > 0$ *where* $\Lambda_{\min}$ *denotes the minimum eigenvalue.*

Next, we show that the mutual incoherence condition also holds in the finite-sample regime with high probability.

Lemma 2 (Mutual incoherence in sample)**.** *If* $\|\big[\mathbf{H}\big]_{\mathrm{S}_i^c\mathrm{S}_i}\big[\mathbf{H}\big]^{-1}_{\mathrm{S}_i\mathrm{S}_i}\|_{\mathrm{B},\infty,1} \leqslant 1-\alpha$ *for* $\alpha \in (0,1]$*, then*

$$\Pr[\|\big[\widehat{\mathbf{H}}\big]_{\mathrm{S}_i^c\mathrm{S}_i}\big[\widehat{\mathbf{H}}\big]^{-1}_{\mathrm{S}_i\mathrm{S}_i}\|_{\mathrm{B},\infty,1} \leqslant 1-\frac{\alpha}{2}] \geqslant 1 - \mathcal{O}(\exp(\frac{-KT}{k^5d^3\max(b^2\sigma^2, 64\sigma^4)} + \log k^2 d(n-d))).$$

Now we can use Lemma 1 and 2 to prove our main result. We can rewrite equation (6) as,

$$\begin{aligned}\big[\Delta\mathbf{W}_{i\cdot}\big]_{\mathrm{S}_i\cdot} =& \big[\widehat{\mathrm{E}}\big(\mathbf{x}_{\text{-}i}\mathbf{x}_{\text{-}i}{}^{\intercal}\big)\big]_{\mathrm{S}_i\mathrm{S}_i}{}^{\text{-}1}\big[\widehat{\mathrm{E}}\big(\mathbf{x}_{\text{-}i}\mathbf{e}_{\text{-}i}{}^{\intercal}\big)\big]_{\mathrm{S}_i\mathrm{S}_i}\big[\mathbf{W}^*_{i\cdot}\big]_{\mathrm{S}_i\cdot} + \big[\widehat{\mathrm{E}}\big(\mathbf{x}_{\text{-}i}\mathbf{x}_{\text{-}i}{}^{\intercal}\big)\big]_{\mathrm{S}_i\mathrm{S}_i}{}^{\text{-}1}\widehat{\mathrm{E}}\big(\big[\mathbf{x}_{\text{-}i}\big]_{\mathrm{S}_i\cdot}\mathbf{e}_i{}^{\intercal}\big) - \\ & \frac{\lambda_T}{2}\big[\widehat{\mathrm{E}}\big(\mathbf{x}_{\text{-}i}\mathbf{x}_{\text{-}i}{}^{\intercal}\big)\big]_{\mathrm{S}_i\mathrm{S}_i}{}^{\text{-}1}\big[\mathbf{Z}_{i\cdot}\big]_{\mathrm{S}_i\cdot}\ .\end{aligned} \tag{8}$$

This is possible because $\Lambda_{\min}(\big[\widehat{\mathrm{E}}\big(\mathbf{x}_{\text{-}i}\mathbf{x}_{\text{-}i}{}^{\intercal}\big)\big]_{\mathrm{S}_i\mathrm{S}_i}) > 0$ from Lemma 1. Using equation (8), we can write equation (7) as,

$$\begin{aligned}&\big[\widehat{\mathrm{E}}\big(\mathbf{x}_{\text{-}i}\mathbf{x}_{\text{-}i}{}^{\intercal}\big)\big]_{\mathrm{S}_i^c\mathrm{S}_i}\big(\big[\widehat{\mathrm{E}}\big(\mathbf{x}_{\text{-}i}\mathbf{x}_{\text{-}i}{}^{\intercal}\big)\big]_{\mathrm{S}_i\mathrm{S}_i}{}^{\text{-}1}\big[\widehat{\mathrm{E}}\big(\mathbf{x}_{\text{-}i}\mathbf{e}_{\text{-}i}{}^{\intercal}\big)\big]_{\mathrm{S}_i\mathrm{S}_i}\big[\mathbf{W}^*_{i\cdot}\big]_{\mathrm{S}_i\cdot} + \big[\widehat{\mathrm{E}}\big(\mathbf{x}_{\text{-}i}\mathbf{x}_{\text{-}i}{}^{\intercal}\big)\big]_{\mathrm{S}_i\mathrm{S}_i}{}^{\text{-}1}\widehat{\mathrm{E}}\big(\big[\mathbf{x}_{\text{-}i}\big]_{\mathrm{S}_i\cdot}\mathbf{e}_i{}^{\intercal}\big) - \\ &\frac{\lambda_T}{2}\big[\widehat{\mathrm{E}}\big(\mathbf{x}_{\text{-}i}\mathbf{x}_{\text{-}i}{}^{\intercal}\big)\big]_{\mathrm{S}_i\mathrm{S}_i}{}^{\text{-}1}\big[\mathbf{Z}_{i\cdot}\big]_{\mathrm{S}_i\cdot}\big) - \widehat{\mathrm{E}}\big(\big[\mathbf{x}_{\text{-}i}\mathbf{e}_{\text{-}i}{}^{\intercal}\big]_{\mathrm{S}_i^c\mathrm{S}_i}\big[\mathbf{W}^*_{i\cdot}\big]_{\mathrm{S}_i\cdot}\big) - \widehat{\mathrm{E}}\big(\big[\mathbf{x}_{\text{-}i}\big]_{\mathrm{S}_i^c\cdot}\mathbf{e}_i{}^{\intercal}\big) + \frac{\lambda_T}{2}\big[\mathbf{Z}_{i\cdot}\big]_{\mathrm{S}_i^c\cdot} = \mathbf{0}\ .\end{aligned} \tag{9}$$

Let $\mathbf{M} = \big[\widehat{\mathrm{E}}\big(\mathbf{x}_{\text{-}i}\mathbf{x}_{\text{-}i}{}^{\intercal}\big)\big]_{\mathrm{S}_i^c\mathrm{S}_i}\big(\big[\widehat{\mathrm{E}}\big(\mathbf{x}_{\text{-}i}\mathbf{x}_{\text{-}i}{}^{\intercal}\big)\big]_{\mathrm{S}_i\mathrm{S}_i}{}^{\text{-}1}$, then

$$\begin{aligned}\frac{\lambda_T}{2}\big[\mathbf{Z}_{i\cdot}\big]_{\mathrm{S}_i^c\cdot} =& -\mathbf{M}\big[\widehat{\mathrm{E}}\big(\mathbf{x}_{\text{-}i}\mathbf{e}_{\text{-}i}{}^{\intercal}\big)\big]_{\mathrm{S}_i\mathrm{S}_i}\big[\mathbf{W}^*_{i\cdot}\big]_{\mathrm{S}_i\cdot} - \mathbf{M}\widehat{\mathrm{E}}\big(\big[\mathbf{x}_{\text{-}i}\big]_{\mathrm{S}_i\cdot}\mathbf{e}_i{}^{\intercal}\big) + \frac{\lambda_T}{2}\mathbf{M}\big[\mathbf{Z}_{i\cdot}\big]_{\mathrm{S}_i\cdot} + \widehat{\mathrm{E}}\big(\big[\mathbf{x}_{\text{-}i}\mathbf{e}_{\text{-}i}{}^{\intercal}\big]_{\mathrm{S}_i^c\mathrm{S}_i} \\ & \big[\mathbf{W}^*_{i\cdot}\big]_{\mathrm{S}_i\cdot}\big) + \widehat{\mathrm{E}}\big(\big[\mathbf{x}_{\text{-}i}\big]_{\mathrm{S}_i^c\cdot}\mathbf{e}_i{}^{\intercal}\big)\ .\end{aligned}$$

By taking the $\mathrm{B},\infty,\mathtt{F}$-norm on both sides and using the norm triangle inequality,

$$\begin{aligned}\frac{\lambda_T}{2}\|\big[\mathbf{Z}_{i\cdot}\big]_{\mathrm{S}_i^c\cdot}\|_{\mathrm{B},\infty,\mathtt{F}} \leqslant& \|\mathbf{M}\big[\widehat{\mathrm{E}}\big(\mathbf{x}_{\text{-}i}\mathbf{e}_{\text{-}i}{}^{\intercal}\big)\big]_{\mathrm{S}_i\mathrm{S}_i}\big[\mathbf{W}^*_{i\cdot}\big]_{\mathrm{S}_i\cdot} - \mathbf{M}\widehat{\mathrm{E}}\big(\big[\mathbf{x}_{\text{-}i}\big]_{\mathrm{S}_i\cdot}\mathbf{e}_i{}^{\intercal}\big) + \frac{\lambda_T}{2}\mathbf{M}\big[\mathbf{Z}_{i\cdot}\big]_{\mathrm{S}_i\cdot}\|_{\mathrm{B},\infty,\mathtt{F}} + \\ & \|\widehat{\mathrm{E}}\big(\big[\mathbf{x}_{\text{-}i}\mathbf{e}_{\text{-}i}{}^{\intercal}\big]_{\mathrm{S}_i^c\mathrm{S}_i}\big)\big[\mathbf{W}^*_{i\cdot}\big]_{\mathrm{S}_i\cdot}\|_{\mathrm{B},\infty,\mathtt{F}} + \|\widehat{\mathrm{E}}\big(\big[\mathbf{x}_{\text{-}i}\big]_{\mathrm{S}_i^c\cdot}\mathbf{e}_i{}^{\intercal}\big)\|_{\mathrm{B},\infty,\mathtt{F}}\ .\end{aligned}$$

Using the inequality $\|AB\|_{\mathrm{B},\infty,\mathtt{F}} \leqslant \|A\|_{\mathrm{B},\infty,1}\|B\|_{\infty,2}$ form Lemma 12, we get

$$\begin{aligned}\frac{\lambda_T}{2}\|\big[\mathbf{Z}_{i\cdot}\big]_{\mathrm{S}_i^c\cdot}\|_{\mathrm{B},\infty,\mathtt{F}} \leqslant& \|\mathbf{M}\|_{\mathrm{B},\infty,1}\big(\|\big[\widehat{\mathrm{E}}\big(\mathbf{x}_{\text{-}i}\mathbf{e}_{\text{-}i}{}^{\intercal}\big)\big]_{\mathrm{S}_i\mathrm{S}_i}\big[\mathbf{W}^*_{i\cdot}\big]_{\mathrm{S}_i\cdot} - \widehat{\mathrm{E}}\big(\big[\mathbf{x}_{\text{-}i}\big]_{\mathrm{S}_i\cdot}\mathbf{e}_i{}^{\intercal}\big) + \frac{\lambda_T}{2}\big[\mathbf{Z}_{i\cdot}\big]_{\mathrm{S}_i\cdot}\|_{\infty,2}\big) + \\ & \|\widehat{\mathrm{E}}\big(\big[\mathbf{x}_{\text{-}i}\mathbf{e}_{\text{-}i}{}^{\intercal}\big]_{\mathrm{S}_i^c\mathrm{S}_i}\big)\big[\mathbf{W}^*_{i\cdot}\big]_{\mathrm{S}_i\cdot}\|_{\mathrm{B},\infty,\mathtt{F}} + \|\widehat{\mathrm{E}}\big(\big[\mathbf{x}_{\text{-}i}\big]_{\mathrm{S}_i^c\cdot}\mathbf{e}_i{}^{\intercal}\big)\|_{\mathrm{B},\infty,\mathtt{F}}\ .\end{aligned}$$

Again using the norm triangle inequality,

$$\begin{aligned}\frac{\lambda_T}{2}\|\big[\mathbf{Z}_{i\cdot}\big]_{\mathrm{S}_i^c\cdot}\|_{\mathrm{B},\infty,\mathtt{F}} \leqslant& \|\mathbf{M}\|_{\mathrm{B},\infty,1}\big(\|\big[\widehat{\mathrm{E}}\big(\mathbf{x}_{\text{-}i}\mathbf{e}_{\text{-}i}{}^{\intercal}\big)\big]_{\mathrm{S}_i\mathrm{S}_i}\big[\mathbf{W}^*_{i\cdot}\big]_{\mathrm{S}_i\cdot}\|_{\infty,2} + \|\widehat{\mathrm{E}}\big(\big[\mathbf{x}_{\text{-}i}\big]_{\mathrm{S}_i\cdot}\mathbf{e}_i{}^{\intercal}\big)\|_{\infty,2} + \frac{\lambda_T}{2} \\ & \|\big[\mathbf{Z}_{i\cdot}\big]_{\mathrm{S}_i\cdot}\|_{\infty,2}\big) + \|\widehat{\mathrm{E}}\big(\big[\mathbf{x}_{\text{-}i}\mathbf{e}_{\text{-}i}{}^{\intercal}\big]_{\mathrm{S}_i^c\mathrm{S}_i}\big)\big[\mathbf{W}^*_{i\cdot}\big]_{\mathrm{S}_i\cdot}\|_{\mathrm{B},\infty,\mathtt{F}} + \|\widehat{\mathrm{E}}\big(\big[\mathbf{x}_{\text{-}i}\big]_{\mathrm{S}_i^c\cdot}\mathbf{e}_i{}^{\intercal}\big)\|_{\mathrm{B},\infty,\mathtt{F}}\ .\end{aligned} \tag{10}$$

Therefore,

$$
\begin{aligned}
\|\big[\mathbf{Z}_{i\cdot}\big]_{\mathrm{S}_i^c\cdot}\|_{\mathrm{B},\infty,\mathrm{F}} \leqslant \|\mathbf{M}\|_{\mathrm{B},\infty,1}\big(\|\frac{2}{\lambda_T}\big[\widehat{\mathrm{E}}\big(\mathbf{x}^*_{\text{-}i}\mathbf{e}_{\text{-}i}{}^{\intercal}\big)\big]_{\mathrm{S}_i\mathrm{S}_i}\big[\mathbf{W}^*_{i.}\big]_{\mathrm{S}_i\cdot}\|_{\infty,2} + \|\frac{2}{\lambda_T}(\big[\widehat{\mathrm{E}}\big(\mathbf{e}_{\text{-}i}\mathbf{e}_{\text{-}i}{}^{\intercal}\big)\big]_{\mathrm{S}_i\mathrm{S}_i}\big[\mathbf{W}^*_{i.}\big]_{\mathrm{S}_i\cdot} \\
- \sigma^2\big[\mathbf{W}^*_{i.}\big]_{\mathrm{S}_i\cdot})\|_{\infty,2} + \|\frac{2}{\lambda_T}\sigma^2\big[\mathbf{W}^*_{i.}\big]_{\mathrm{S}_i\cdot}\|_{\infty,2} + \|\frac{2}{\lambda_T}\widehat{\mathrm{E}}\big(\big[\mathbf{x}^*{}_{\text{-}i}\big]_{\mathrm{S}_i\cdot}\mathbf{e}_i{}^{\intercal}\big)\|_{\infty,2} + \\
\|\frac{2}{\lambda_T}\widehat{\mathrm{E}}\big(\big[\mathbf{e}_{\text{-}i}\big]_{\mathrm{S}_i\cdot}\mathbf{e}_i{}^{\intercal}\big)\|_{\infty,2} + \|\big[\mathbf{Z}_{i\cdot}\big]_{\mathrm{S}_i\cdot}\|_{\infty,2}\big) + \|\frac{2}{\lambda_T}\widehat{\mathrm{E}}\big(\big[\mathbf{x}^*{}_{\text{-}i}\mathbf{e}_{\text{-}i}{}^{\intercal}\big]_{\mathrm{S}_i^c\mathrm{S}_i}\big)\big[\mathbf{W}^*_{i.}\big]_{\mathrm{S}_i\cdot}\|_{\mathrm{B},\infty,\mathrm{F}} + \\
\|\frac{2}{\lambda_T}\widehat{\mathrm{E}}\big(\big[\mathbf{e}_{\text{-}i}\mathbf{e}_{\text{-}i}{}^{\intercal}\big]_{\mathrm{S}_i^c\mathrm{S}_i}\big)\big[\mathbf{W}^*_{i.}\big]_{\mathrm{S}_i\cdot}\|_{\mathrm{B},\infty,\mathrm{F}} + \|\frac{2}{\lambda_T}\widehat{\mathrm{E}}\big(\big[\mathbf{x}^*{}_{\text{-}i}\big]_{\mathrm{S}_i^c\cdot}\mathbf{e}_i{}^{\intercal}\big)\|_{\mathrm{B},\infty,\mathrm{F}} + \\
\|\frac{2}{\lambda_T}\widehat{\mathrm{E}}\big(\big[\mathbf{e}_{\text{-}i}\big]_{\mathrm{S}_i^c\cdot}\mathbf{e}_i{}^{\intercal}\big)\|_{\mathrm{B},\infty,\mathrm{F}} \ .
\end{aligned}
\tag{11}
$$

We denote the right hand side of the above equation by a function $f(\mathbf{x}, \mathbf{e}, \mathbf{W}_{i.}, \mathrm{b}, \sigma, \alpha, \lambda, T, n, k)$. Next, we provide a technical lemma (detailed proof in Appendix A) to bound
$f(\mathbf{x}, \mathbf{e}, \mathbf{W}_{i.}, \mathrm{b}, \sigma, \alpha, \lambda, T, n, k)$.

Lemma 3. *If $T = \Omega(k^5 d^3 \log(kn))$ and $\lambda_T \geqslant \max(\frac{640\sigma \max(\mathrm{b},\sigma,\overline{W}\mathrm{b},\overline{W}\sigma)}{\alpha}\sqrt{\frac{k^3 d\log(2k^2 n)}{T}}, \frac{80}{\alpha}\sigma^2\|\big[\mathbf{W}^*_{i.}\big]_{\mathrm{S}_i\cdot}\|_{\infty,2})$, then for some constant $c > 0$, $f(\mathbf{x}, \mathbf{e}, \mathbf{W}_{i.}, \mathrm{b}, \sigma, \alpha, \lambda, T, n, k) \leqslant 1 - \frac{\alpha}{4}$ with probability at least $1 - \exp(-cT\lambda_T^2)$.*

The above result ensures that $\|W_{ij}\|_F = 0$ for all $j \in \mathrm{S}_i^c$ with high probability, i.e., for each player i, we do not recover any player which is not an in-neighbor. The next lemma provides bound on the difference between estimated solution and the true solution in $\mathrm{B}, \infty, \mathtt{F}$-norm.

Lemma 4. *If $T = \Omega(k^5 d^3 \log(kn))$ and $\lambda_T \geqslant \max(\frac{640\sigma \max(\mathrm{b},\sigma,\overline{W}\mathrm{b},\overline{W}\sigma)}{\alpha}\sqrt{\frac{k^3 d\log(2k^2 n)}{T}}, \frac{80}{\alpha}\sigma^2$
$\|\big[\mathbf{W}^*_{i.}\big]_{\mathrm{S}_i\cdot}\|_{\infty,2})$, then $\|\big[\Delta\mathbf{W}_{i\cdot}\big]_{\mathrm{S}_i\cdot}\|_{\mathrm{B},\infty,F} \leqslant \frac{k\sqrt{kd}\lambda_T(10\alpha+1)}{10C}$ with probability at least $1 - \exp(-cT\lambda_T^2)$ for some positive constant c.*

Next, we will characterize $\mathrm{NE}_\epsilon(\mathscr{G})$ by $\mathbf{W}_{i\cdot}$. In particular, we define

$$
\overline{\mathrm{NE}}_\epsilon(\mathscr{G}) = \{\mathbf{x}^* \in \bigtimes_{i\in\mathcal{V}} \mathcal{A}_i \mid x_i^* = \sum_{j\in\mathrm{S}_i} W_{ij}x_j^*, \forall i \in \{1,\ldots,n\}\} \ . \tag{12}
$$

We explicitly compute the payoffs to prove that every element of set (12) is indeed in the indeed in the $\mathrm{NE}_\epsilon(\mathscr{G})$, i.e., for all $\mathbf{x}^* \in \overline{\mathrm{NE}}_\epsilon(\mathscr{G})$

$$
\begin{aligned}
u_i(x_i^*, x_{\text{-}i}^*) = -\|\sum_{j\in\mathrm{S}_i} W_{ij}x_j^* - \sum_{j\in\mathrm{S}_i} W_{ij}^*x_j^*\|_2 \geqslant -\sum_{j\in\mathrm{S}_i}\|(W_{ij} - W_{ij}^*)x_j^*\|_2 \geqslant -\sum_{j\in\mathrm{S}_i}\|W_{ij} - W_{ij}^*\|_F\|x_j\|_2 \\
\geqslant -d\frac{k\sqrt{kd}\lambda_T(10\alpha+1)}{10C}\mathrm{b} \ .
\end{aligned}
$$

Thus the set defined in equation (12) is completely contained in ϵ-PSNE for $\epsilon = d\frac{k\sqrt{kd}\lambda_T(10\alpha+1)}{10C}\mathrm{b}$. It follows that if for each player $i \in \{1,\ldots,n\}$, if $\min_{j\in\mathrm{S}_i}\|W_{ij}^*\|_F > 2\frac{k\sqrt{kd}\lambda_T(10\alpha+1)}{10C}$ then we recover the exact structure of the graphical game. Note that if $\min_{j\in\mathrm{S}_i}\|W_{ij}^*\|_F > 2\frac{k\sqrt{kd}\lambda_T(10\alpha+1)}{10C}$ then $\|W_{ij}^*\|_F > 0$ implies that $\|W_{ij}\|_F > 0$. We have already shown that we do not recover any extra player in the set of in-neighbors S_i and this added condition ensures that for all the players in S_i, $\|W_{ij}\|_F > 0$. Thus, we recover the exact set of players in S_i for each player $i \in \{1,\ldots,n\}$. We recover the exact graphical game by combining the results for all the players. $\square$

A.2. Proof of Lemma 1

Lemma 1[Positive minimum eigenvalue] $\Lambda_{\min}(\big[\widehat{\mathbf{H}}\big]_{\mathrm{S}_i\mathrm{S}_i}) > 0$ *with probability at least* $1 - \exp(-cT\sigma^4 + \mathcal{O}(kd)) - \exp(-\frac{\sigma^2 T}{128\mathrm{b}^2} + \mathcal{O}(kd))$ *for some constant $c > 0$ where $\Lambda_{\min}$ denotes the minimum eigenvalue.*

Proof. We prove the lemma in two steps. First, recall that $\mathbf{H} = \frac{1}{T}\sum_{t=1}^{T}\left(\mathbf{x}^{*t}_{\text{-}i}\mathbf{x}^{*t\intercal}_{\text{-}i} + \sigma^2\mathbf{I}\right)$ where $\mathbf{I}$ is identity matrix. Then,

$$\Lambda_{\min}([\mathbf{H}]_{\mathrm{S}_i\mathrm{S}_i}) = \Lambda_{\min}(\frac{1}{T}\sum_{t=1}^{T}([\mathbf{x}^{*t}_{\text{-}i}\mathbf{x}^{*t\intercal}_{\text{-}i}]_{\mathrm{S}_i\mathrm{S}_i}) + \sigma^2\mathbf{I}_{\mathrm{S}_i\mathrm{S}_i})$$

Using the inequality

$$\Lambda_{\min}(A+B) \geqslant \Lambda_{\min}(A) + \Lambda_{\min}(B)$$

$$\geqslant \Lambda_{\min}(\frac{1}{T}\sum_{t=1}^{T}[\mathbf{x}^{*t}_{\text{-}i}\mathbf{x}^{*t\intercal}_{\text{-}i}]_{\mathrm{S}_i\mathrm{S}_i}) + \sigma^2$$

$$\geqslant \sigma^2 > 0\ .$$

Last inequality follows by noting that $\frac{1}{T}\sum_{t=1}^{T}([\mathbf{x}^{*t}_{\text{-}i}\mathbf{x}^{*t\intercal}_{\text{-}i}]_{\mathrm{S}_i\mathrm{S}_i})$ is a positive semi-definite matrix with non-negative eigenvalues. Next, we prove that if $T > \mathcal{O}(\frac{1}{\sigma^2}\max(\mathrm{b}^2, \frac{1}{\sigma^2})kd)$, then $\Lambda_{\min}([\widehat{\mathbf{H}}]_{\mathrm{S}_i\mathrm{S}_i}) > 0$ with high probability.

$$\begin{aligned}\Lambda_{\min}([\widehat{\mathbf{H}}]_{\mathrm{S}_i\mathrm{S}_i}) &= \min_{\|\mathbf{y}\|_2=1}\mathbf{y}^\intercal(\frac{1}{T}\sum_{t=1}^{T}[\mathbf{x}^{*t}\mathbf{x}^{*t\intercal}]_{\mathrm{S}_i\mathrm{S}_i} + [\mathbf{x}^{*t}\mathbf{e}^{t\intercal}]_{\mathrm{S}_i\mathrm{S}_i} + [\mathbf{e}^{t}\mathbf{x}^{*t\intercal}]_{\mathrm{S}_i\mathrm{S}_i} + [\mathbf{e}^{t}\mathbf{e}^{t\intercal}]_{\mathrm{S}_i\mathrm{S}_i})\mathbf{y} \\ &\geqslant \min_{\|\mathbf{y}\|_2=1}\mathbf{y}^\intercal\frac{1}{T}\sum_{t=1}^{T}[\mathbf{x}^{*t}\mathbf{x}^{*t\intercal}]_{\mathrm{S}_i\mathrm{S}_i}\mathbf{y} + \min_{\|\mathbf{y}\|_2=1}\mathbf{y}^\intercal\frac{1}{T}\sum_{t=1}^{T}([\mathbf{x}^{*t}\mathbf{e}^{t\intercal}]_{\mathrm{S}_i\mathrm{S}_i} + [\mathbf{e}^{t}\mathbf{x}^{*t\intercal}]_{\mathrm{S}_i\mathrm{S}_i})\mathbf{y} + \\ &\min_{\|\mathbf{y}\|_2=1}\mathbf{y}^\intercal\frac{1}{T}\sum_{t=1}^{T}[\mathbf{e}^{t}\mathbf{e}^{t\intercal}]_{\mathrm{S}_i\mathrm{S}_i}\mathbf{y}\end{aligned}$$

Noting that $\frac{1}{T}\sum_{t=1}^{T}([\mathbf{x}^{*t}_{\text{-}i}\mathbf{x}^{*t\intercal}_{\text{-}i}]_{\mathrm{S}_i\mathrm{S}_i})$ is a positive semidefinite matrix

$$\geqslant \min_{\|\mathbf{y}\|_2=1}\mathbf{y}^\intercal\frac{1}{T}\sum_{t=1}^{T}([\mathbf{x}^{*t}\mathbf{e}^{t\intercal}]_{\mathrm{S}_i\mathrm{S}_i} + [\mathbf{e}^{t}\mathbf{x}^{*t\intercal}]_{\mathrm{S}_i\mathrm{S}_i})\mathbf{y} + \min_{\|\mathbf{y}\|_2=1}\mathbf{y}^\intercal\frac{1}{T}\sum_{t=1}^{T}[\mathbf{e}^{t}\mathbf{e}^{t\intercal}]_{\mathrm{S}_i\mathrm{S}_i}\mathbf{y}\ .$$

We define a random variable $R \triangleq \frac{1}{T}\sum_{t=1}^{T}\mathbf{y}^\intercal([\mathbf{x}^{*t}\mathbf{e}^{t\intercal}]_{\mathrm{S}_i\mathrm{S}_i} + [\mathbf{e}^{t}\mathbf{x}^{*t\intercal}]_{\mathrm{S}_i\mathrm{S}_i})\mathbf{y}$. Notice that R is a sub-Gaussian random variable with mean 0 and parameter $\frac{4\sum_{t=1}^{T}a_t^2\sigma^2}{T^2}$, where $a_t = \mathbf{y}^\intercal[\mathbf{x}^*]_{\mathrm{S}_i.} \leqslant \mathrm{b}$. Thus,

$$\Pr(R \leqslant -\epsilon) \leqslant \exp(-\frac{\epsilon^2}{2\frac{4\sum_{t=1}^{T}a_t^2\sigma^2}{T^2}}) \leqslant \exp(-\frac{\epsilon^2}{2\frac{4\sum_{t=1}^{T}\mathrm{b}^2\sigma^2}{T^2}}) = \exp(-\frac{T\epsilon^2}{8\mathrm{b}^2\sigma^2})\ .$$

Following ϵ-*nets* argument from [24] and covariance matrix concentration for sub-Gaussian random variables, we can write

$$\Pr(\min_{\|\mathbf{y}\|_2=1}\mathbf{y}^\intercal\frac{1}{T}\sum_{t=1}^{T}[\mathbf{e}^{t}\mathbf{e}^{t\intercal}]_{\mathrm{S}_i\mathrm{S}_i}\mathbf{y} > \sigma^2 - \epsilon) \geqslant 1 - \exp(-c\epsilon^2T + \mathcal{O}(kd))\ ,$$

and

$$\Pr(\min_{\|\mathbf{y}\|_2=1}\mathbf{y}^\intercal\frac{1}{T}\sum_{t=1}^{T}([\mathbf{x}^{*t}\mathbf{e}^{t\intercal}]_{\mathrm{S}_i\mathrm{S}_i} + [\mathbf{e}^{t}\mathbf{x}^{*t\intercal}]_{\mathrm{S}_i\mathrm{S}_i})\mathbf{y} \geqslant -\epsilon) \leqslant \exp(-\frac{T\epsilon^2}{8\mathrm{b}^2\sigma^2} + \mathcal{O}(kd))\ .$$

Thus, choosing $\epsilon = \frac{\sigma^2}{4}$ and choosing $T = \mathcal{O}(\frac{1}{\sigma^2}\max(\mathrm{b}^2, \frac{1}{\sigma^2})kd)$, we get $\Lambda_{\min}([\widehat{\mathbf{H}}]_{\mathrm{S}_i\mathrm{S}_i}) \geqslant \frac{\sigma^2}{2}$ with high probability. □

A.3. Proof of Lemma 2

First, we prove a technical lemma that will be used in Lemma 2.

Lemma 5. *For any $\delta > 0$, the following holds:*

$$\Pr(\|[\widehat{\mathbf{H}}]_{\mathrm{S}_i^c\mathrm{S}_i} - [\mathbf{H}]_{\mathrm{S}_i^c\mathrm{S}_i}\|_{\mathrm{B},\infty,1} \geqslant \delta) \leqslant 6\exp(\frac{-\delta^2T}{k^4d^2\max(b^2\sigma^2, 64\sigma^4)} + \log(k^2(n-d)d))\ , \quad (13)$$

$$\Pr(\|[\widehat{\mathbf{H}}]_{\mathrm{S}_i\mathrm{S}_i} - [\mathbf{H}]_{\mathrm{S}_i\mathrm{S}_i}\|_{\infty,\infty} \geqslant \delta) \leqslant 6\exp(\frac{-\delta^2 T}{k^2 d^2 \max(b^2\sigma^2, 64\sigma^4)} + 2\log kd) \ , \tag{14}$$

$$\begin{aligned} \Pr(\|([\widehat{\mathbf{H}}]_{\mathrm{S}_i\mathrm{S}_i})^{-1} - ([\mathbf{H}]_{\mathrm{S}_i\mathrm{S}_i})^{-1}\|_{\infty,\infty} \geqslant \delta) \leqslant 6\exp(-\frac{\delta^2 C^4 T}{4k^3 \max(b^2\sigma^2, 64\sigma^4) d^3} + 2\log(kd)) \\ - 2\exp(-\frac{C^2 T}{32k^2 d^2} + 2\log kd) \ . \end{aligned} \tag{15}$$

Proof. Note that,

$$\begin{aligned} [[\widehat{\mathbf{H}}]_{\mathrm{S}_i^c\mathrm{S}_i} - [\mathbf{H}]_{\mathrm{S}_i^c\mathrm{S}_i}]_{jk} &= Z_{jk} \\ &= [\frac{1}{T}\sum_{t=1}^{T} \left(\mathbf{x}^{*}_{\mathrm{S}_i^c}{}^t \mathbf{e}_{\mathrm{S}_i}{}^{t\intercal} + \mathbf{e}_{\mathrm{S}_i^c}{}^t \mathbf{x}^{*}_{\mathrm{S}_i}{}^{t\intercal} + \mathbf{e}_{\mathrm{S}_i^c}{}^t \mathbf{e}_{\mathrm{S}_i}{}^{t\intercal}\right)]_{jk} \\ &= \frac{1}{T}\sum_{t=1}^{T} Z_{jk}^t \ . \end{aligned}$$

We define three random variables $R_1 \triangleq [\frac{1}{T}\sum_{t=1}^{T} \left(\mathbf{x}^{*}_{\mathrm{S}_i^c}{}^t \mathbf{e}_{\mathrm{S}_i}{}^{t\intercal}\right)]_{jk}$, $R_2 = [\frac{1}{T}\sum_{t=1}^{T} \left(\mathbf{e}_{\mathrm{S}_i^c}{}^t \mathbf{x}^{*}_{\mathrm{S}_i}{}^{t\intercal}\right)]_{jk}$ and $R_3 \triangleq [\frac{1}{T}\sum_{t=1}^{T} \left(\mathbf{e}_{\mathrm{S}_i^c}{}^t \mathbf{e}_{\mathrm{S}_i}{}^{t\intercal}\right)]_{jk}$. We will provide a separate bound on these random variables. Note that R_1 is a sub-Gaussian random variable with 0 mean and parameter $\frac{\sigma^2 \sum_{t=1}^{T} \mathbf{x}_{\cdot i}{}^{*t2}_{j}}{T^2}$ and R_2 is a sub-Gaussian random variable with 0 mean and parameter $\frac{\sigma^2 \sum_{t=1}^{T} \mathbf{x}_{\cdot i}{}^{*t2}_{k}}{T^2}$. Thus,

$$\Pr(|R_1| \geqslant \epsilon) \leqslant 2\exp(-\frac{\epsilon^2}{\frac{\sigma^2 \sum_{t=1}^{T} \mathbf{x}_{\cdot i}{}^{*t2}_{j}}{T^2}}) \leqslant 2\exp(-\frac{T\epsilon^2}{\mathrm{b}^2\sigma^2}) \ ,$$

and

$$\Pr(|R_2| \geqslant \epsilon) \leqslant 2\exp(-\frac{\epsilon^2}{\frac{\sigma^2 \sum_{t=1}^{T} \mathbf{x}_{\cdot i}{}^{*t2}_{k}}{T^2}}) \leqslant 2\exp(-\frac{T\epsilon^2}{\mathrm{b}^2\sigma^2}) \ .$$

R_3 is a sub-exponential random variable (check Lemma 6 and 11). Thus, for $0 < \epsilon < 8\sigma^2$.

$$\Pr(|R_3| \geqslant \epsilon) \leqslant 2\exp(-\frac{T\epsilon^2}{64\sigma^4}) \ .$$

Now,

$$\|[\widehat{\mathbf{H}}]_{\mathrm{S}_i^c\mathrm{S}_i} - [\mathbf{H}]_{\mathrm{S}_i^c\mathrm{S}_i}\|_{\mathrm{B},\infty,1} = \max_{i\in\mathrm{S}_i^c}(\sum_{j\in\mathrm{ind}(i)}\sum_{l\in\mathrm{S}_i}\sum_{k\in\mathrm{ind}(l)} |Z_{jk}|) \ ,$$

where $\mathrm{ind}(i)$ returns set of row indices of block i in $\mathbf{W}^*_{i\cdot}$. Combining the results for random variables R_1, R_2 and R_3 and applying union bound, we get

$$\Pr[|Z_{jk}| \geqslant \epsilon] \leqslant 6\exp(\frac{-\epsilon^2 T}{\max(b^2\sigma^2, 64\sigma^4)}) \ .$$

Taking $\epsilon = \frac{\delta}{k^2 d}$ for any $i \in \mathrm{S}_i^c$ such that $0 < \delta \leqslant 8k^2 d\sigma^2$.

$$\Pr(|Z_{jk}| \geqslant \frac{\delta}{k^2 d}) \leqslant 6\exp(\frac{-\delta^2 T}{k^4 d^2 \max(b^2\sigma^2, 64\sigma^4)}) \ .$$

Using the union bound over $\forall i \in \mathrm{S}_i^c, j \in \mathrm{ind}(i), \forall l \in \mathrm{S}_i, k \in \mathrm{ind}(l)$ we can write,

$$\begin{aligned} \Pr(\|[\widehat{\mathbf{H}}]_{\mathrm{S}_i^c\mathrm{S}_i} - [\mathbf{H}]_{\mathrm{S}_i^c\mathrm{S}_i}\|_{\mathrm{B},\infty,1} \geqslant \delta) &\leqslant 6(n-d)dk^2 \exp(\frac{-\delta^2 T}{k^4 d^2 \max(b^2\sigma^2, 64\sigma^4)}) \\ &\leqslant 6\exp(\frac{-\delta^2 T}{k^4 d^2 \max(b^2\sigma^2, 64\sigma^4)} + \log(k^2(n-d)d)) \ . \end{aligned}$$

Similarly we can prove equation (14), for $0 < \delta \leqslant 8kd\sigma^2$

$$\Pr(\|\big[\widehat{\mathbf{H}}\big]_{\mathrm{S}_i\mathrm{S}_i} - \big[\mathbf{H}\big]_{\mathrm{S}_i\mathrm{S}_i}\|_{\infty,\infty} \geqslant \delta) \leqslant k^2d^2\Pr(|Z_{jk}| \geqslant \frac{\delta}{kd}) \leqslant 6\exp(\frac{-\delta^2 T}{k^2d^2\max(b^2\sigma^2, 64\sigma^4)} + 2\log kd) \ .$$

Now we prove equation (15). Note that,

$$\begin{aligned}
&\|(\big[\widehat{\mathbf{H}}\big]_{\mathrm{S}_i\mathrm{S}_i})^{-1} - (\big[\mathbf{H}\big]_{\mathrm{S}_i\mathrm{S}_i})^{-1}\|_{\infty,\infty} = \\
&\|(\big[\mathbf{H}\big]_{\mathrm{S}_i\mathrm{S}_i})^{-1}[\big[\mathbf{H}\big]_{\mathrm{S}_i\mathrm{S}_i} - \big[\widehat{\mathbf{H}}\big]_{\mathrm{S}_i\mathrm{S}_i}](\big[\widehat{\mathbf{H}}\big]_{\mathrm{S}_i\mathrm{S}_i})^{-1}\|_{\infty,\infty} \\
&\leqslant \sqrt{kd}\|(\big[\mathbf{H}\big]_{\mathrm{S}_i\mathrm{S}_i})^{-1}[\big[\mathbf{H}\big]_{\mathrm{S}_i\mathrm{S}_i} - \big[\widehat{\mathbf{H}}\big]_{\mathrm{S}_i\mathrm{S}_i}](\big[\widehat{\mathbf{H}}\big]_{\mathrm{S}_i\mathrm{S}_i})^{-1}\|_{2,2} \\
&\leqslant \sqrt{kd}\|(\big[\mathbf{H}\big]_{\mathrm{S}_i\mathrm{S}_i})^{-1}\|_{2,2}\|[\big[\mathbf{H}\big]_{\mathrm{S}_i\mathrm{S}_i} - \big[\widehat{\mathbf{H}}\big]_{\mathrm{S}_i\mathrm{S}_i}]\|_{2,2}\|(\big[\widehat{\mathbf{H}}\big]_{\mathrm{S}_i\mathrm{S}_i})^{-1}\|_{2,2} \\
&\leqslant \frac{\sqrt{kd}}{C}\|[\big[\mathbf{H}\big]_{\mathrm{S}_i\mathrm{S}_i} - \big[\widehat{\mathbf{H}}\big]_{\mathrm{S}_i\mathrm{S}_i}]\|_{2,2}\|(\big[\widehat{\mathbf{H}}\big]_{\mathrm{S}_i\mathrm{S}_i})^{-1}\|_{2,2} \ .
\end{aligned}$$

Note that, $\Pr(\Lambda_{\min}(\big[\widehat{\mathbf{H}}\big]_{\mathrm{S}_i\mathrm{S}_i}) \geqslant C - \delta] \geqslant 1 - 2\exp(-\frac{\delta^2 T}{8k^2d^2} + 2\log kd)$. Taking $\delta = \frac{C}{2}$, we get $\Pr(\Lambda_{\min}(\big[\widehat{\mathbf{H}}\big]_{\mathrm{S}_i\mathrm{S}_i}) \geqslant \frac{C}{2}) \geqslant 1 - 2\exp(-\frac{C^2 T}{32k^2d^2} + 2\log kd)$. This means that,

$$\Pr(\|(\big[\widehat{\mathbf{H}}\big]_{\mathrm{S}_i\mathrm{S}_i})^{-1}\|_{2,2} \leqslant \frac{2}{C}) \geqslant 1 - 2\exp(-\frac{C^2T}{32k^2d^2} + 2\log kd) \ . \tag{16}$$

Furthermore for $0 < \epsilon \leqslant 8kd\sigma^2$,

$$\Pr(\|\big[\mathbf{H}\big]_{\mathrm{S}_i\mathrm{S}_i} - \big[\widehat{\mathbf{H}}\big]_{\mathrm{S}_i\mathrm{S}_i}\|_{2,2} \geqslant \epsilon) \leqslant 6\exp(-\frac{\epsilon^2 T}{k^2d^2\max(b^2\sigma^2, 64\sigma^4)} + 2\log kd) \ .$$

Taking $\epsilon = \delta\frac{C^2}{2\sqrt{kd}}$ where $0 < \delta \leqslant \frac{16}{C^2}k^{\frac{3}{2}}d^{\frac{3}{2}}$, we get:

$$\Pr(\|\mathbf{H}_{\mathrm{S}_i\mathrm{S}_i} - \widehat{\mathbf{H}}_{\mathrm{S}_i\mathrm{S}_i}\|_{2,2} \geqslant \delta\frac{C^2}{2\sqrt{kd}}) \leqslant 6\exp(-\frac{\delta^2C^4T}{4k^3\max(b^2\sigma^2, 64\sigma^4)d^3} + 2\log kd) \ .$$

It follows that,

$$\begin{aligned}
\Pr(\|(\big[\widehat{\mathbf{H}}\big]_{\mathrm{S}_i\mathrm{S}_i})^{-1} - (\big[\mathbf{H}\big]_{\mathrm{S}_i\mathrm{S}_i})^{-1}\|_{\infty,\infty} \leqslant \delta) \geqslant & 1 - 6\exp(-\frac{\delta^2C^4T}{4k^3\max(b^2\sigma^2, 64\sigma^4)d^3} + 2\log(kd)) \\
& - 2\exp(-\frac{C^2T}{32k^2d^2} + 2\log kd) \ .
\end{aligned}$$

□

Now, we provide the detailed proof of Lemma 2.

Lemma 2[Mutual incoherence in sample] *If* $\|\big[\mathbf{H}\big]_{\mathrm{S}_i^c\mathrm{S}_i}\big[\mathbf{H}\big]^{-1}_{\mathrm{S}_i\mathrm{S}_i}\|_{\mathrm{B},\infty,1} \leqslant 1 - \alpha$ *for* $\alpha \in (0, 1]$ *then*

$$\Pr[\|\big[\widehat{\mathbf{H}}\big]_{\mathrm{S}_i^c\mathrm{S}_i}\big[\widehat{\mathbf{H}}\big]^{-1}_{\mathrm{S}_i\mathrm{S}_i}\|_{\mathrm{B},\infty,1} \leqslant 1 - \frac{\alpha}{2}] \geqslant 1 - \mathcal{O}(\exp(\frac{-KT}{k^5d^3\max(b^2\sigma^2, 64\sigma^4)} + \log k^2d(n-d)) \ .$$

Proof. We can rewrite $\widehat{\mathbf{H}}_{\mathrm{S}_i^c\mathrm{S}_i}(\widehat{\mathbf{H}}_{\mathrm{S}_i\mathrm{S}_i})^{-1}$ as the sum of four terms defined as:

$$\begin{aligned}
&\big[\widehat{\mathbf{H}}\big]_{\mathrm{S}_i^c\mathrm{S}_i}(\big[\widehat{\mathbf{H}}\big]_{\mathrm{S}_i\mathrm{S}_i})^{-1} = T_1 + T_2 + T_3 + T4 \\
&\|\big[\widehat{\mathbf{H}}\big]_{\mathrm{S}_i^c\mathrm{S}_i}(\big[\widehat{\mathbf{H}}\big]_{\mathrm{S}_i\mathrm{S}_i})^{-1}\|_{\mathrm{B},\infty,1} \leqslant \|T_1\|_{\mathrm{B},\infty,1} + \|T_2\|_{\mathrm{B},\infty,1} + \|T_3\|_{\mathrm{B},\infty,1} + \|T_4\|_{\mathrm{B},\infty,1} \ ,
\end{aligned} \tag{17}$$

where,

$$T_1 \triangleq [\mathbf{H}]_{\mathrm{S}_i^c\mathrm{S}_i}[([\widehat{\mathbf{H}}]_{\mathrm{S}_i\mathrm{S}_i})^{-1} - [\mathbf{H}]_{\mathrm{S}_i\mathrm{S}_i}^{-1}] ,$$
$$T_2 \triangleq [[\widehat{\mathbf{H}}]_{\mathrm{S}_i^c\mathrm{S}_i} - [\mathbf{H}]_{\mathrm{S}_i^c\mathrm{S}_i}][\mathbf{H}]_{\mathrm{S}_i\mathrm{S}_i}^{-1} ,$$
$$T_3 \triangleq [[\widehat{\mathbf{H}}]_{\mathrm{S}_i^c\mathrm{S}_i} - [\mathbf{H}]_{\mathrm{S}_i^c\mathrm{S}_i}][([\widehat{\mathbf{H}}]_{\mathrm{S}_i\mathrm{S}_i})^{-1} - [\mathbf{H}]_{\mathrm{S}_i\mathrm{S}_i}^{-1}] ,$$
$$T_4 \triangleq [\mathbf{H}]_{\mathrm{S}_i^c\mathrm{S}_i}[\mathbf{H}]_{\mathrm{S}_i\mathrm{S}_i}^{-1} ,$$

and each T_i is treated as a row-partitioned block matrix of $(n-d)$ blocks with each block containing k rows. From the mutual incoherence Assumption, it is clear that $\|T_4\|_{\mathrm{B},\infty,1} \leqslant 1-\alpha$. We control the other three terms by using results from Lemma 5.

Controlling the first term of equation (17)**.** We can write T_1 as,

$$T_1 = -[\mathbf{H}]_{\mathrm{S}_i^c\mathrm{S}_i}([\mathbf{H}]_{\mathrm{S}_i\mathrm{S}_i})^{-1}[[\widehat{\mathbf{H}}]_{\mathrm{S}_i\mathrm{S}_i} - [\mathbf{H}]_{\mathrm{S}_i\mathrm{S}_i}]([\widehat{\mathbf{H}}]_{\mathrm{S}_i\mathrm{S}_i})^{-1} .$$

Then,

$$\begin{aligned}\|T_1\|_{\mathrm{B},\infty,1} &= \|[\mathbf{H}]_{\mathrm{S}_i^c\mathrm{S}_i}([\mathbf{H}]_{\mathrm{S}_i\mathrm{S}_i})^{-1}[[\widehat{\mathbf{H}}]_{\mathrm{S}_i\mathrm{S}_i} - [\mathbf{H}]_{\mathrm{S}_i\mathrm{S}_i}]([\widehat{\mathbf{H}}]_{\mathrm{S}_i\mathrm{S}_i})^{-1}\|_{\mathrm{B},\infty,1} \\ &\leqslant \|[\mathbf{H}]_{\mathrm{S}_i^c\mathrm{S}_i}([\mathbf{H}]_{\mathrm{S}_i\mathrm{S}_i})^{-1}\|_{\mathrm{B},\infty,1}\|[[\widehat{\mathbf{H}}]_{\mathrm{S}_i\mathrm{S}_i} - [\mathbf{H}]_{\mathrm{S}_i\mathrm{S}_i}]\|_{\infty,\infty}\|([\widehat{\mathbf{H}}]_{\mathrm{S}_i\mathrm{S}_i})^{-1}\|_{\infty,\infty} \\ &\leqslant (1-\alpha)\|[[\widehat{\mathbf{H}}]_{\mathrm{S}_i\mathrm{S}_i} - [\mathbf{H}]_{\mathrm{S}_i\mathrm{S}_i}]\|_{\infty,\infty}\sqrt{kd}\|([\widehat{\mathbf{H}}]_{\mathrm{S}_i\mathrm{S}_i})^{-1}\|_{2,2} .\end{aligned}$$

Now using equation (16) and equation (14) with $\delta = \frac{\alpha C}{12\sqrt{kd}(1-\alpha)}$ we can say that,

$$\Pr[\|T_1\|_{\mathrm{B},\infty,1} \leqslant \frac{\alpha}{6}] \geqslant 1 - 2\exp(-\frac{C^2T}{32k^2d^2} + 2\log kd) - 6\exp(\frac{-\alpha^2C^2T}{144(1-\alpha)^2k^3d^3\max(b^2\sigma^2, 64\sigma^4)} + 2\log kd) .$$

Controlling the second term of equation (17)**.** We can write $\|T_2\|_{\mathrm{B},\infty,1}$ as,

$$\begin{aligned}\|T_2\|_{\mathrm{B},\infty,1} &= \|[[\widehat{\mathbf{H}}]_{\mathrm{S}_i^c\mathrm{S}_i} - [\mathbf{H}]_{\mathrm{S}_i^c\mathrm{S}_i}][\mathbf{H}]_{\mathrm{S}_i\mathrm{S}_i}^{-1}\|_{\mathrm{B},\infty,1} \\ &\leqslant \|[[\widehat{\mathbf{H}}]_{\mathrm{S}_i^c\mathrm{S}_i} - [\mathbf{H}]_{\mathrm{S}_i^c\mathrm{S}_i}]\|_{\mathrm{B},\infty,1}\|[\mathbf{H}]_{\mathrm{S}_i\mathrm{S}_i}^{-1}\|_{\infty,\infty} \\ &\leqslant \|[[\widehat{\mathbf{H}}]_{\mathrm{S}_i^c\mathrm{S}_i} - [\mathbf{H}]_{\mathrm{S}_i^c\mathrm{S}_i}]\|_{\mathrm{B},\infty,1}\sqrt{kd}\|[\mathbf{H}]_{\mathrm{S}_i\mathrm{S}_i}^{-1}\|_{2,2} \\ &\leqslant \frac{\sqrt{kd}}{C}\|[[\widehat{\mathbf{H}}]_{\mathrm{S}_i^c\mathrm{S}_i} - [\mathbf{H}]_{\mathrm{S}_i^c\mathrm{S}_i}]\|_{\mathrm{B},\infty,1} .\end{aligned}$$

Using equation (13) with $\delta = \frac{\alpha C}{6\sqrt{kd}}$ we get,

$$\Pr(\|T_2\|_{\mathrm{B},\infty,1} \leqslant \frac{\alpha}{6}) \geqslant 1 - 6\exp(\frac{-\alpha^2C^2T}{36k^5d^3\max(b^2\sigma^2, 64\sigma^4)} + \log(k^2(n-d)d)) .$$

Controlling the third term of equation (17)**.** We can write $\|T_3\|_{\mathrm{B},\infty,1}$ as,

$$\|T_3\|_{\mathrm{B},\infty,1} \leqslant \|[[\widehat{\mathbf{H}}]_{\mathrm{S}_i^c\mathrm{S}_i} - [\mathbf{H}]_{\mathrm{S}_i^c\mathrm{S}_i}]\|_{\mathrm{B},\infty,1}\|[([\widehat{\mathbf{H}}]_{\mathrm{S}_i\mathrm{S}_i})^{-1} - [\mathbf{H}]_{\mathrm{S}_i\mathrm{S}_i}^{-1}]\|_{\infty,\infty} .$$

Using equation (13) and (15) both with $\delta = \sqrt{\frac{\alpha}{6}}$, we get

$$\begin{aligned}\Pr(\|T_3\|_{\mathrm{B},\infty,1} \leqslant \frac{\alpha}{6}) \geqslant 1 - 6\exp(\frac{-\alpha T}{6k^4d^2\max(b^2\sigma^2, 64\sigma^4)} + \log(k^2(n-d)d)) - \\ 6\exp(-\frac{\alpha C^4T}{24k^3\max(b^2\sigma^2, 64\sigma^4)d^3} + 2\log(kd)) - 2\exp(-\frac{C^2T}{32k^2d^2} + 2\log kd) .\end{aligned}$$

Putting everything together we get,

$$\Pr[\|[\widehat{\mathbf{H}}]_{\mathrm{S}_i^c\mathrm{S}_i}[\widehat{\mathbf{H}}]_{\mathrm{S}_i\mathrm{S}_i}^{-1}\|_{\mathrm{B},\infty,1} \leqslant 1 - \frac{\alpha}{2}] \geqslant 1 - \mathcal{O}(\exp(\frac{-KT}{k^5d^3\max(b^2\sigma^2, 64\sigma^4)} + \log k(n-d) + \log kd) ,$$

which approaches 1 as long as we have $N > \mathcal{O}(k^5d^3\max(b^2\sigma^2, 64\sigma^4)\log nk)$. $\square$

A.4. Proof of Lemma 3

Lemma 3 *If* $T = \mathcal{O}(k^5 d^3 \log(kn))$ *and* $\lambda_T \geqslant \max(\frac{640\sigma \max(\mathrm{b},\sigma,\overline{W}\mathrm{b},\overline{W}\sigma)}{\alpha}\sqrt{\frac{k^3 d \log(2k^2 n)}{T}}, \frac{80}{\alpha}\sigma^2$ $\|[\mathbf{W}^*_{i.}]_{\mathrm{S}_i.}\|_{\infty,2})$*, then for some constant* $c > 0$*,* $f(\mathbf{x}, \mathbf{e}, \mathbf{W}_{i.}, \mathrm{b}, \sigma, \alpha, \lambda, T, n, k) \leqslant 1 - \frac{\alpha}{4}$ *with probability at least* $1 - \exp(-cT\lambda^2)$*.*

Proof. We will prove this statement by proving several helper lemmas, each bounding a separate term.

Lemma 6. *For some* $\epsilon_1 > 0, \epsilon_2 \in (0, \frac{16}{\lambda_T}\sqrt{k}\sigma^2\underline{W})$ *and* $\epsilon_3 \in (0, \frac{16}{\lambda_T}k\sigma^2\sqrt{d}\underline{W})$ *, the following statements are true:*

1. *If* $\lambda_T > \frac{2\sqrt{2}\sigma\mathrm{b}\overline{W}}{\epsilon_1}\sqrt{\frac{k^2 d \log(2k^2 d)}{T}}$ *then for some constant* $c_1 > 0$*,*

$$\Pr(\frac{2}{\lambda_T}\|[\widehat{E}(\mathbf{x}^*_{\text{-}i}\mathbf{e}_{\text{-}i}{}^\intercal)]_{\mathrm{S}_i\mathrm{S}_i}[\mathbf{W}^*_{i.}]_{\mathrm{S}_i.}\|_{\infty,2} \leqslant \epsilon_1) \geqslant 1 - \exp(-c_1 T\lambda_T^2)\ .$$

2. *If* $\lambda_T > \max(\frac{16\sigma^2\overline{W}}{\epsilon_2}\sqrt{\frac{\log(2k^2 d)}{T}}, \frac{16\sigma^2\overline{W}}{\epsilon_3}\sqrt{\frac{kd\log(2k^2 d)}{T}})$ *then for some constant* $c_2 > 0$*,*

$$\Pr(\frac{2}{\lambda_T}\|[\widehat{E}(\mathbf{e}_{\text{-}i}\mathbf{e}_{\text{-}i}{}^\intercal)]_{\mathrm{S}_i\mathrm{S}_i}[\mathbf{W}^*_{i.}]_{\mathrm{S}_i.} - \sigma^2[\mathbf{W}^*_{i.}]_{\mathrm{S}_i.}\|_{\infty,2} \leqslant \epsilon_2 + \epsilon_3) \geqslant 1 - \exp(c_2 T\lambda_T^2)\ .$$

Proof. We will bound both terms separately.

Bound on $\|\frac{2}{\lambda_T}[\widehat{\mathbf{E}}(\mathbf{x}^*_{\text{-}i}\mathbf{e}_{\text{-}i}{}^\intercal)]_{\mathbf{S}_i\mathbf{S}_i}[\mathbf{W}^*_{i.}]_{\mathbf{S}_i.}\|_{\infty,2}$**.** For simplicity, let $[\mathbf{x}^*_{\text{-}i}]_{\mathrm{S}_i} = \mathbf{y}^* \in \mathbb{R}^{dk\times 1}$, $[\mathbf{e}_{\text{-}i}]_{\mathrm{S}_i} = \mathbf{u} \in \mathbb{R}^{dk\times 1}$ and $[\mathbf{W}^*_{i.}]_{\mathrm{S}_i.} = \mathbf{W}^* \in \mathbb{R}^{dk\times k}$. We define a random variable R and then,

$$\begin{aligned} R &\triangleq \frac{2}{\lambda_T}\widehat{\mathrm{E}}(\mathbf{y}^*\mathbf{u}^\intercal\mathbf{W}^*)_{ij} \\ &= \frac{2}{\lambda_T}\widehat{\mathrm{E}}(\sum_{l=1}^{dk}\mathbf{y}^*_i\mathbf{u}_l\mathbf{W}^*_{lj}) \\ &= \frac{2}{\lambda_T}\frac{1}{T}\sum_{t=1}^{T}(\sum_{l=1}^{dk}\mathbf{y}^{*t}_i\mathbf{u}^t_l\mathbf{W}^*_{lj})\ . \end{aligned}$$

For a given $\mathbf{y}^{t*}_i, \forall t \in \{1,\ldots,T\}$, R is a sub-Gaussian random variable with 0 mean and parameter $\frac{4}{\lambda_T^2}\frac{\sigma^2}{T^2}\sum_{t=1}^{T}\sum_{l=1}^{dk}(\mathbf{y}^{*t}_i\mathbf{W}^*_{lj})^2$. Thus, for some $\epsilon_1 > 0$, we can use a tail bound on a sub-Gaussian random variable:

$$\begin{aligned} \Pr_{\cdot|\mathbf{y}^{t*}_i}(|R| > \epsilon_1) &\leqslant 2\exp(-\frac{\lambda_T^2\epsilon_1^2T^2}{8\sigma^2\sum_{t=1}^{T}\sum_{l=1}^{dk}(\mathbf{y}^{*t}_i\mathbf{W}^*_{lj})^2}) \\ &\leqslant 2\exp(-\frac{\lambda_T^2\epsilon_1^2T}{8\sigma^2\mathrm{b}^2\sum_{l=1}^{dk}\mathbf{W}^{*\,2}_{lj}})\ , \end{aligned}$$

where last inequality holds because $\mathbf{y}^{*t}_i \leqslant \mathrm{b}, \forall t \in \{1,\ldots,T\}$. Therefore,

$$\begin{aligned} \Pr(|R| > \epsilon_1) &= \mathrm{E}_{\mathbf{y}^{t*}_i}(\Pr_{\cdot|\mathbf{y}^{t*}_i}(|R| > \epsilon_1)) \\ &\leqslant 2\exp(-\frac{\lambda_T^2\epsilon_1^2T}{8\sigma^2\mathrm{b}^2\sum_{l=1}^{dk}\mathbf{W}^{*\,2}_{lj}})\ . \end{aligned}$$

Let $\overline{W}$ be the maximum entry of $[\mathbf{W}^*_{i.}]_{\mathrm{S}_i.}$ in absolute value. Taking union bound across $i \in \mathrm{ind}(l), \forall l \in \mathrm{S}_i$ and $j \in \{1,\ldots,k\}$, where $\mathrm{ind}(l)$ returns set of row indices of block l in $\mathbf{W}^*_{i.}$, we get

$$\begin{aligned} \Pr(\|\frac{2}{\lambda_T}[\widehat{\mathrm{E}}(\mathbf{x}^*_{\text{-}i}\mathbf{e}_{\text{-}i}{}^\intercal)]_{\mathrm{S}_i\mathrm{S}_i}[\mathbf{W}^*_{i.}]_{\mathrm{S}_i.}\|_{\infty,2} > \epsilon_1) &\leqslant 2k^2 d\exp(-\frac{\lambda_T^2\epsilon_1^2T}{8k\sigma^2\mathrm{b}^2\max_j\sum_{l=1}^{dk}\mathbf{W}^{*\,2}_{lj}}) \\ &= \exp(-\frac{\lambda_T^2\epsilon_1^2T}{8k^2\sigma^2\mathrm{b}^2 d\overline{W}^2} + \log(2k^2 d))\ . \end{aligned}$$

If $\lambda_T > \frac{2\sqrt{2}\sigma \mathrm{b}\overline{W}}{\epsilon_1}\sqrt{\frac{k^2 d \log(2k^2 d)}{T}}$ then for some constant $c_1 > 0$,

$$\Pr(\| \frac{2}{\lambda_T}\big[\widehat{\mathrm{E}}\big(\mathbf{x}^*_{\text{-}i}\mathbf{e}_{\text{-}i}{}^\intercal\big)\big]_{\mathrm{S}_i\mathrm{S}_i}\big[\mathbf{W}^*_{i.}\big]_{\mathrm{S}_i.}\|_{\infty,2} \leqslant \epsilon_1) \geqslant 1 - \exp(-c_1 T\lambda_T^2) \ .$$

Bound on $\|\frac{2}{\lambda_T}(\big[\widehat{\mathbf{E}}\big(\mathbf{e}_{\text{-}i}\mathbf{e}_{\text{-}i}{}^\intercal\big)\big]_{\mathbf{S}_i\mathbf{S}_i}\big[\mathbf{W}^*_{i.}\big]_{\mathbf{S}_i.} - \sigma^2\big[\mathbf{W}^*_{i.}\big]_{\mathbf{S}_i.})\|_{\infty,2}$**.** Again for simplicity, let $\big[\mathbf{e}_{\text{-}i}\big]_{\mathrm{S}_i} = \mathbf{u} \in \mathbb{R}^{dk\times 1}$ and $\big[\mathbf{W}^*_{i.}\big]_{\mathrm{S}_i.} = \mathbf{W}^* \in \mathbb{R}^{dk\times k}$.We define a random variable R and then,

$$\begin{aligned}
R &\triangleq \widehat{\mathrm{E}}\big(\frac{2}{\lambda_T}(\mathbf{u}\mathbf{u}^\intercal\mathbf{W}^* - \sigma^2\mathbf{W}^*)\big)_{ij} \\
&= \frac{2}{\lambda_T}\frac{1}{T}\sum_{t=1}^{T}\big(\sum_{l=1}^{dk}\mathbf{u}_i^t\mathbf{u}_l^t\mathbf{W}^*_{lj} - \sigma^2\mathbf{W}^*_{ij}\big) \\
&= \frac{2}{\lambda_T}\frac{1}{T}\sum_{t=1}^{T}(\mathbf{u}_i^t\mathbf{u}_i^t\mathbf{W}^*_{ij} - \sigma^2\mathbf{W}^*_{ij}) + \frac{2}{\lambda_T}\frac{1}{T}\sum_{t=1}^{T}\big(\sum_{\substack{l=1\\ l\neq i}}^{dk}\mathbf{u}_i^t\mathbf{u}_l^t\mathbf{W}^*_{lj}\big) \ .
\end{aligned}$$

We will bound the random variable R in two steps. First, we will bound the random variable $R_1 \triangleq \frac{2}{\lambda_T}\frac{1}{T}\sum_{t=1}^{T}(\mathbf{u}_i^t\mathbf{u}_i^t\mathbf{W}^*_{ij} - \sigma^2\mathbf{W}^*_{ij})$ and then we will bound the random variable $R_2 \triangleq \frac{2}{\lambda_T}\frac{1}{T}\sum_{t=1}^{T}\big(\sum_{\substack{l=1\\ l\neq i}}^{dk}\mathbf{u}_i^t\mathbf{u}_l^t\mathbf{W}^*_{lj}\big)$.

Bound on $R1$ We observe that

$$R_1 = \frac{2}{\lambda_T}\frac{1}{T}\sum_{t=1}^{T}(\sigma^2\mathbf{W}^*_{ij}\frac{\mathbf{u}_i^t}{\sigma}\frac{\mathbf{u}_i^t}{\sigma} - \sigma^2\mathbf{W}^*_{ij}) \ .$$

We define a random variable $y \triangleq \frac{\mathbf{u}_i^t}{\sigma}$. Note that y is a sub-Gaussian random variable with 0 mean and parameter 1. We prove in Lemma 10 that y^2 is a sub-exponential random variable with parameter $(4\sqrt{2}, 4)$. Therefore, we can use a Bernstein type bound on $(\frac{\mathbf{u}_i^t}{\sigma})^2$, thus for some $\epsilon_2 \in (0, 8)$,

$$\Pr(|\ \frac{1}{T}\sum_{t=1}^{T}(\frac{\mathbf{u}_i^t}{\sigma}\frac{\mathbf{u}_i^t}{\sigma} - \mathrm{E}\big(\frac{\mathbf{u}_i^t}{\sigma}\frac{\mathbf{u}_i^t}{\sigma}\big))\ | > \epsilon_2) \leqslant 2\exp(-\frac{T\epsilon_2^2}{64}) \ .$$

Thus, for $\epsilon_2 \in (0, \frac{16}{\lambda_T}\sigma^2\mathbf{W}^*_{ij})$

$$\Pr(|\ \frac{2}{\lambda_T}\frac{1}{T}\sum_{t=1}^{T}(\sigma^2\mathbf{W}^*_{ij}\frac{\mathbf{u}_i^t}{\sigma}\frac{\mathbf{u}_i^t}{\sigma} - \sigma^2\mathbf{W}^*_{ij})\ | > \epsilon_2) \leqslant 2\exp(-\frac{T\lambda_T^2\epsilon_2^2}{256\sigma^4{\mathbf{W}^*_{ij}}^2}) \ .$$

$$\Pr(|\ R_1\ | > \epsilon_2) \leqslant 2\exp(-\frac{T\lambda_T^2\epsilon_2^2}{256\sigma^4{\mathbf{W}^*_{ij}}^2}) \ .$$

Bound on $R2$ We observe that

$$R_2 = \frac{2}{\lambda_T}\sigma^2\sqrt{\sum_{\substack{l=1\\ l\neq i}}^{dk}{\mathbf{W}^*_{lj}}^2}\,\frac{1}{T}\sum_{t=1}^{T}\frac{\mathbf{u}_i^t}{\sigma}\frac{\sum_{\substack{l=1\\ l\neq i}}^{dk}\mathbf{u}_l^t\mathbf{W}^*_{lj}}{\sigma\sqrt{\sum_{\substack{l=1\\ l\neq i}}^{dk}{\mathbf{W}^*_{lj}}^2}} \ .$$

Here $\frac{\mathbf{u}_i^t}{\sigma}$ and $\frac{\sum_{\substack{l=1\\ l\neq i}}^{dk}\mathbf{u}_l^t\mathbf{W}^*_{lj}}{\sigma\sqrt{\sum_{\substack{l=1\\ l\neq i}}^{dk}{\mathbf{W}^*_{lj}}^2}}$ are independent sub-Gaussian random variables with 0 mean and parameter 1. Thus, similar to Lemma 7, we can use a Bernstein type tail bound for the sum of sub-exponential random variables. For some $\epsilon_3 < 8$,

$$\Pr(|\frac{1}{T}\sum_{t=1}^{T}\frac{\mathbf{u}_i^t}{\sigma}\frac{\sum_{\substack{l=1\\l\neq i}}^{dk}\mathbf{u}_l^t\mathbf{W}_{lj}^*}{\sigma\sqrt{\sum_{\substack{l=1\\l\neq i}}^{dk}\mathbf{W}_{lj}^{*\,2}}}| > \epsilon_3) \leqslant 2\exp(-\frac{T\epsilon_3^2}{64}) ,$$

or for some $\epsilon_3 < \frac{16}{\lambda_T}\sigma^2|\sqrt{\sum_{\substack{l=1\\l\neq i}}^{dk}\mathbf{W}_{lj}^{*\,2}}|$,

$$\Pr(|R_2| > \epsilon_3) \leqslant 2\exp(-\frac{\lambda_T^2 T\epsilon_3^2}{256\sigma^4\sum_{\substack{l=1\\l\neq i}}^{dk}\mathbf{W}_{lj}^{*\,2}}) .$$

Combining the bounds on R_1 and R_2 and taking a union bound, we get

$$\Pr(|R| \geqslant \epsilon_2 + \epsilon_3) \leqslant 2\exp(-\frac{\lambda_T^2 T\epsilon_2^2}{256\sigma^4\mathbf{W}_{ij}^{*\,2}}) + 2\exp(-\frac{\lambda_T^2 T\epsilon_3^2}{256\sigma^4\sum_{\substack{l=1\\l\neq i}}^{dk}\mathbf{W}_{lj}^{*\,2}}) .$$

Taking union bound across $i \in \mathrm{ind}(l), \forall l \in \mathrm{S}_i$ and $j \in \{1, \ldots, k\}$, where $\mathrm{ind}(l)$ returns set of row indices of block l in $\mathbf{W}_{i.}^*$, for $\epsilon_2 \in (0, \frac{16}{\lambda_T}\sqrt{k}\sigma^2\underline{W})$ and $\epsilon_3 \in (0, \frac{16}{\lambda_T}k\sigma^2\sqrt{d}\underline{W})$ we get

$$\Pr(\|\frac{2}{\lambda_T}\big[\widehat{\mathrm{E}}\big(\mathbf{e}_{\text{-}i}\mathbf{e}_{\text{-}i}{}^{\intercal}\big)\big]_{\mathrm{S}_i\mathrm{S}_i}\big[\mathbf{W}_{i.}^*\big]_{\mathrm{S}_i.} - \sigma^2\big[\mathbf{W}_{i.}^*\big]_{\mathrm{S}_i.}\|_{\infty,2} > \epsilon_2 + \epsilon_3) \leqslant k^2 d(2\exp(-\frac{\lambda_T^2 T\epsilon_2^2}{256\sigma^4\overline{W}^2}) + 2\exp(-\frac{\lambda_T^2 T\epsilon_3^2}{256\sigma^4 dk\overline{W}^2})) .$$

If $\lambda_T > \max(\frac{16\sigma^2\overline{W}}{\epsilon_2}\sqrt{\frac{\log(2k^2d)}{T}}, \frac{16\sigma^2\overline{W}}{\epsilon_3}\sqrt{\frac{kd\log(2k^2d)}{T}})$ then for some constant $c_2 > 0$,

$$\Pr(\|\frac{2}{\lambda_T}\big[\widehat{\mathrm{E}}\big(\mathbf{e}_{\text{-}i}\mathbf{e}_{\text{-}i}{}^{\intercal}\big)\big]_{\mathrm{S}_i\mathrm{S}_i}\big[\mathbf{W}_{i.}^*\big]_{\mathrm{S}_i.} - \sigma^2\big[\mathbf{W}_{i.}^*\big]_{\mathrm{S}_i.}\|_{\infty,2} \leqslant \epsilon_2 + \epsilon_3) \geqslant 1 - \exp(c_2 T\lambda_T^2) .$$

□

Lemma 7. *For some $\epsilon_4 > 0$ and $\epsilon_5 < \frac{16}{\lambda_T}\sqrt{k}\sigma^2$, the following statements are true:*

1. *If $\lambda_T > \frac{2\sqrt{2}\sigma \mathrm{b}}{\epsilon_4}\sqrt{\frac{k\log(2k^2d)}{T}}$, then for some $c_3 > 0$,*

$$\Pr(\|\frac{2}{\lambda_T}\widehat{E}\big(\big[\mathbf{x}_{\text{-}i}^*\big]_{\mathrm{S}_i.}\mathbf{e}_i{}^{\intercal}\big)\|_{\infty,2} \leqslant \epsilon_4) \geqslant 1 - \exp(c_4 T\lambda_T^2) .$$

2. *If $\lambda_T > \frac{16\sigma^2}{\epsilon_5}\sqrt{\frac{k\log(2k^2d)}{T}}$, then for some constant $c_4 > 0$,*

$$\Pr(\|\frac{2}{\lambda_T}\widehat{E}\big(\big[\mathbf{e}_{\text{-}i}\big]_{\mathrm{S}_i.}\mathbf{e}_i{}^{\intercal}\big)\|_{\infty,2} \leqslant \epsilon_5) \leqslant 1 - \exp(-c_4 T\lambda_T^2) .$$

Proof. We will bound both the terms separately.

Bound on $\|\frac{2}{\lambda_T}\widehat{\mathbf{E}}\big(\big[\mathbf{x}_{\text{-}i}^*\big]_{\mathbf{S}_i.}\mathbf{e}_i{}^{\intercal}\big)\|_{\infty,2}$. For simplicity, let $\big[\mathbf{x}_{\text{-}i}^*\big]_{\mathrm{S}_i.} = \mathbf{y}^* \in \mathbb{R}^{dk\times 1}$ and $\mathbf{e}_i = \mathbf{u} \in \mathbb{R}^k$ and we define a random variable R

$$\begin{aligned}
R &\triangleq \frac{2}{\lambda_T}\widehat{\mathrm{E}}\big(\mathbf{y}^*\mathbf{u}^{\intercal}\big)_{ij}\\
&= \frac{2}{\lambda_T}\widehat{\mathrm{E}}\big(\mathbf{y}_{i1}^*\mathbf{u}_j\big)\\
&= \frac{2}{\lambda_T}\frac{1}{T}\sum_{t=1}^{T}(\mathbf{y}_{i1}^{*\;t}\mathbf{u}_j^t)\\
&= \frac{2}{\lambda_T}\sum_{t=1}^{T}R^t .
\end{aligned}$$

Then for a given $\mathbf{y}_{i1}^t$, random variable R^t is a sub-Gaussian random variable with 0 mean and parameter $\frac{{\mathbf{y}_{i1}^t}^2\sigma^2}{T^2}$. Correspondingly R is a sub-Gaussian random variable with 0 mean and parameter $\sigma^2\frac{\sum_{t=1}^T {\mathbf{y}_{i1}^t}^2}{T^2}$. Using the tail bound for the sub-Gaussian variable for some $\epsilon_4 > 0$, we can write

$$\begin{aligned}\Pr_{.|\mathbf{y}_{i1}^t}(\mid R \mid > \epsilon_4) &\leqslant 2\exp(-\frac{\epsilon_4^2}{2\sigma^2\frac{\sum_{t=1}^T {\mathbf{y}_{i1}^t}^2}{T^2}}) \\ &\leqslant 2\exp(-\frac{\epsilon_4^2 T}{2\sigma^2 \mathrm{b}^2})\ ,\end{aligned}$$

where last inequality follows by noting that ${\mathbf{y}_{i1}^t}^2 \leqslant \mathrm{b}^2$. Thus,

$$\begin{aligned}\Pr(\mid \frac{2}{\lambda_T}\widehat{\mathrm{E}}\big(\mathbf{y}^*\mathbf{u}^\intercal\big)_{ij} \mid > \epsilon_4) &= \mathbb{E}_{\mathbf{y}_{i1}^t}\big(\Pr_{.|\mathbf{y}_{i1}^t}(\mid \frac{2}{\lambda_T}\widehat{\mathrm{E}}\big(\mathbf{y}^*\mathbf{u}^\intercal\big)_{ij} \mid > \epsilon_4)\big) \\ &\leqslant 2\exp(-\frac{\lambda_T^2\epsilon_4^2 T}{8\sigma^2\mathrm{b}^2})\ .\end{aligned}$$

Now,

$$\|\widehat{\mathrm{E}}\big(\mathbf{y}^*\mathbf{u}^\intercal\big)\|_{\infty,2} = \max_{l\in \mathrm{S}_i}\max_{i\in \mathrm{ind}(l)} \|\widehat{\mathrm{E}}\big(\mathbf{y}_{l1}^*\mathbf{u}^\intercal\big)\|_2\ .$$

Taking union bound across $i \in \mathrm{ind}(l), \forall l \in \mathrm{S}_i$ and $j \in \{1,\ldots,k\}$, where $\mathrm{ind}(l)$ returns set of row indices of block l in $\mathbf{W}_{i.}^*$, we get

$$\begin{aligned}\Pr(\|\frac{2}{\lambda_T}\widehat{\mathrm{E}}\big(\big[\mathbf{x}_{\text{-}i}^*\big]_{\mathrm{S}_i.}\mathbf{e}_i{}^\intercal\big)\|_{\infty,2} \geqslant \epsilon_4) &\leqslant 2k^2 d\exp(-\frac{\lambda_T^2\epsilon_4^2 T}{8k\sigma^2\mathrm{b}^2}) \\ &= \exp(-\frac{\lambda_T^2\epsilon_4^2 T}{8k\sigma^2\mathrm{b}^2} + \log(2k^2 d))\ .\end{aligned}$$

If $\lambda_T > \frac{2\sqrt{2}\sigma\mathrm{b}}{\epsilon_4}\sqrt{\frac{k\log(2k^2d)}{T}}$, then for some $c_3 > 0$,

$$\Pr(\|\frac{2}{\lambda_T}\widehat{\mathrm{E}}\big(\big[\mathbf{x}_{\text{-}i}^*\big]_{\mathrm{S}_i.}\mathbf{e}_i{}^\intercal\big)\|_{\infty,2} \leqslant \epsilon_4) \geqslant 1 - \exp(c_4 T\lambda_T^2)\ .$$

Bound on $\|\frac{2}{\lambda_T}\widehat{\mathbf{E}}\big(\big[\mathbf{e}_{\text{-}i}\big]_{\mathbf{S}_i.}\mathbf{e}_i{}^\intercal\big)\|_{\infty,2}$. Again for simplicity, let $\big[\mathbf{e}_{\text{-}i}\big]_{\mathrm{S}_i.} = \mathbf{v} \in \mathbb{R}^{dk\times 1}$ and $\mathbf{e}_i = \mathbf{u} \in \mathbb{R}^k$ and we define a random variable R

$$\begin{aligned}R &\triangleq \frac{2}{\lambda_T}\widehat{\mathrm{E}}\big(\mathbf{v}\mathbf{u}^\intercal\big)_{ij} \\ &= \frac{2}{\lambda_T}\frac{1}{T}\sum_{t=1}^T \mathbf{v}_{i1}^t\mathbf{u}_j^t\ .\end{aligned}$$

Note that $\mathbf{v}_{i1}^t$ and $\mathbf{u}_j^t$ are independent sub-Gaussian random variables with 0 mean and σ^2 parameter. We will use Lemma 11 to get a tail bound on the random variable R.

Now, $\mathrm{E}\big(R\big) = 0$ and for some $\epsilon_5 > 0$,

$$\Pr(\mid \frac{1}{T}\sum_{t=1}^T \mathbf{v}_{i1}^t\mathbf{u}_j^t \mid > \epsilon_5) = \Pr(\sigma^2 \mid \frac{1}{T}\sum_{t=1}^T \frac{\mathbf{v}_{i1}^t}{\sigma}\frac{\mathbf{u}_j^t}{\sigma} \mid > \epsilon_5)\ .$$

Here $\frac{\mathbf{v}_{i1}^t}{\sigma}$ and $\frac{\mathbf{u}_j^t}{\sigma}$ are sub-Gaussian random variables with parameter 1. Thus using results from Lemma 11, we can use a Bernstein tail bound for the sum of sub-exponential random variables and write,

$$\Pr(\mid \frac{2}{\lambda_T}\frac{1}{T}\sum_{t=1}^T \mathbf{v}_{i1}^t\mathbf{u}_j^t \mid > \epsilon_5) \leqslant 2\exp(-\frac{\lambda_T^2 T\epsilon_5^2}{256\sigma^4}), \forall \epsilon_5 < \frac{16}{\lambda_T}\sigma^2\ .$$

Again, taking union bound across $i \in \text{ind}(l), \forall l \in \mathrm{S}_i$ and $j \in \{1, \dots, k\}$, for all $\epsilon_5 < \frac{16}{\lambda_T}\sqrt{k}\sigma^2$ we get

$$\Pr(\|\frac{2}{\lambda_T}\widehat{\mathrm{E}}\big(\big[\mathbf{e}_{\text{-}i}\big]_{\mathrm{S}_i.}\mathbf{e}_i{}^{\intercal}\big)\|_{\infty,2} \geqslant \epsilon_5) \leqslant 2k^2 d \exp(-\frac{\lambda_T^2\epsilon_5^2 T}{256k\sigma^4})$$
$$= \exp(-\frac{\lambda_T^2\epsilon_5^2 T}{256k\sigma^4} + \log(2k^2 d)) \; .$$

If $\lambda_T > \frac{16\sigma^2}{\epsilon_5}\sqrt{\frac{k\log(2k^2 d)}{T}}$, then for some constant $c_4 > 0$,

$$\Pr(\|\frac{2}{\lambda_T}\widehat{\mathrm{E}}\big(\big[\mathbf{e}_{\text{-}i}\big]_{\mathrm{S}_i.}\mathbf{e}_i{}^{\intercal}\big)\|_{\infty,2} \leqslant \epsilon_5) \leqslant 1 - \exp(-c_4 T\lambda_T^2) \; .$$

□

Lemma 8. *For some $\epsilon_6 > 0$ and $\epsilon_7 \in \{0, \frac{16}{\lambda_T}\sigma^2 k^2 d\underline{W}\}$, the following statements are true:*

1. *If $\lambda_T > \frac{2\sqrt{2}\sigma \mathrm{b}\overline{W}}{\epsilon_6}\sqrt{\frac{k^3 d\log(2k^2(n-d))}{T}}$ then for some constant $c_5 > 0$,*

$$\Pr(\|\frac{2}{\lambda_T}\big[\widehat{E}\big(\mathbf{x}^*_{\text{-}i}\mathbf{e}_{\text{-}i}{}^{\intercal}\big)\big]_{\mathrm{S}_i^c\mathrm{S}_i}\big[\mathbf{W}^*_{i.}\big]_{\mathrm{S}_i.}\|_{\mathrm{B},\infty,F} \leqslant \epsilon_6) \geqslant 1 - \exp(-c_5\lambda_T^2 T) \; .$$

2. *If $\lambda_T > \frac{16\sigma^2\overline{W}}{\epsilon_7}\sqrt{\frac{k^3 d\log(2k^2(n-d))}{T}}$, then for some constant $c_6 > 0$,*

$$\Pr(\|\frac{2}{\lambda_T}\big[\widehat{E}\big(\mathbf{e}_{\text{-}i}\mathbf{e}_{\text{-}i}{}^{\intercal}\big)\big]_{\mathrm{S}_i^c\mathrm{S}_i}\big[\mathbf{W}^*_{i.}\big]_{\mathrm{S}_i.}\|_{\mathrm{B},\infty,F} \leqslant \epsilon_7) \geqslant 1 - \exp(-c_6 T\lambda_T^2) \; .$$

Proof. We can follow the exact same argument of Lemma 6 to bound the above two terms until we take the union bound. This time we will take union bound across $i \in \mathrm{S}_i^c$ and $j \in \{1, \dots, k \times k\}$, we get

$$\Pr(\|\frac{2}{\lambda_T}\big[\widehat{\mathrm{E}}\big(\mathbf{x}^*_{\text{-}i}\mathbf{e}_{\text{-}i}{}^{\intercal}\big)\big]_{\mathrm{S}_i^c\mathrm{S}_i}\big[\mathbf{W}^*_{i.}\big]_{\mathrm{S}_i.}\|_{\mathrm{B},\infty,\mathrm{F}} > \epsilon_6) \leqslant 2k^2(n-d)\exp(-\frac{\lambda_T^2\epsilon_6^2 T}{8k^3 d\sigma^2\mathrm{b}^2\overline{W}^2})$$
$$= \exp(-\frac{\lambda_T^2\epsilon_6^2 T}{8k^3 d\sigma^2\mathrm{b}^2\overline{W}^2} + \log(2k^2(n-d))) \; .$$

If $\lambda_T > \frac{2\sqrt{2}\sigma \mathrm{b}\overline{W}}{\epsilon_6}\sqrt{\frac{k^3 d\log(2k^2(n-d))}{T}}$ then for some constant $c_5 > 0$,

$$\Pr(\|\frac{2}{\lambda_T}\big[\widehat{\mathrm{E}}\big(\mathbf{x}^*_{\text{-}i}\mathbf{e}_{\text{-}i}{}^{\intercal}\big)\big]_{\mathrm{S}_i^c\mathrm{S}_i}\big[\mathbf{W}^*_{i.}\big]_{\mathrm{S}_i.}\|_{\mathrm{B},\infty,\mathrm{F}} \leqslant \epsilon_6) \geqslant 1 - \exp(-c_5\lambda_T^2 T) \; .$$

Using a similar bound as R_2 in Lemma 6, for $\epsilon_7 \in \{0, \frac{16}{\lambda_T}\sigma^2 k^2 d\underline{W}\}$

$$\Pr(\|\frac{2}{\lambda_T}\big[\widehat{\mathrm{E}}\big(\mathbf{e}_{\text{-}i}\mathbf{e}_{\text{-}i}{}^{\intercal}\big)\big]_{\mathrm{S}_i^c\mathrm{S}_i}\big[\mathbf{W}^*_{i.}\big]_{\mathrm{S}_i.}\|_{\mathrm{B},\infty,\mathrm{F}} > \epsilon_7) \leqslant 2k^2(n-d)\exp(-\frac{T\lambda_T^2\epsilon_7^2}{256\sigma^4 k^3 d\overline{W}^2}) \; .$$

If $\lambda_T > \frac{16\sigma^2\overline{W}}{\epsilon_7}\sqrt{\frac{k^3 d\log(2k^2(n-d))}{T}}$, then for some constant $c_6 > 0$,

$$\Pr(\|\frac{2}{\lambda_T}\big[\widehat{\mathrm{E}}\big(\mathbf{e}_{\text{-}i}\mathbf{e}_{\text{-}i}{}^{\intercal}\big)\big]_{\mathrm{S}_i^c\mathrm{S}_i}\big[\mathbf{W}^*_{i.}\big]_{\mathrm{S}_i.}\|_{\mathrm{B},\infty,\mathrm{F}} \leqslant \epsilon_7) \geqslant 1 - \exp(-c_6 T\lambda_T^2) \; .$$

□

Lemma 9. *For some $\epsilon_9 > 0$ and $\epsilon_{10} < \frac{16}{\lambda_T}k\sigma^2$, the following statements are true:*

1. *If $\lambda_T > \frac{2\sqrt{2}\sigma \mathrm{b}}{\epsilon_9}\sqrt{\frac{k^2\log(2k^2(n-d))}{T}}$, then for some constant $c_7 > 0$,*

$$\Pr(\|\frac{2}{\lambda_T}\widehat{E}\big(\big[\mathbf{x}^*_{\text{-}i}\big]_{\mathrm{S}_i^c.}\mathbf{e}_i{}^{\intercal}\big)\|_{\mathrm{B},\infty,F} \leqslant \epsilon_9) \geqslant 1 - \exp(-c_7\lambda_T^2 T) \; .$$

2. *If* $\lambda_T > \frac{16\sigma^2}{\epsilon_{10}}\sqrt{\frac{k^2\log(2k^2(n-d))}{T}}$, *then for some constant* $c_8 > 0$,

$$\Pr(\|\frac{2}{\lambda_T}\widehat{E}\big(\big[\mathbf{e}_{\text{-}i}\big]_{\mathrm{S}_i^c\cdot}\mathbf{e}_i{}^{\intercal}\big)\|_{\mathrm{B},\infty,F} \leqslant \epsilon_{10}) \geqslant 1-\exp(-c_8\lambda_T^2 T) \ .$$

Proof. Like in Lemma 7, we can bound both the terms separately using similar arguments. The only change would be that this time we will take union bound across $i \in \mathrm{S}_i^c$ and $j \in \{1,\ldots,k\times k\}$, we get

$$\begin{aligned}\Pr(\|\frac{2}{\lambda_T}\widehat{\mathrm{E}}\big(\big[\mathbf{x}^*_{\text{-}i}\big]_{\mathrm{S}_i^c\cdot}\mathbf{e}_i{}^{\intercal}\big)\|_{\mathrm{B},\infty,\mathrm{F}} \geqslant \epsilon_9) &\leqslant 2k^2 \mid \mathrm{S}_i^c \mid \exp(-\frac{\lambda_T^2\epsilon_9^2 T}{8k^2\sigma^2\mathrm{b}^2}) \\ &= \exp(-\frac{\lambda_T^2\epsilon_9^2 T}{8k^2\sigma^2\mathrm{b}^2} + \log(2k^2(n-d))) \ .\end{aligned}$$

If $\lambda_T > \frac{2\sqrt{2}\sigma\mathrm{b}}{\epsilon_9}\sqrt{\frac{k^2\log(2k^2(n-d))}{T}}$, then for some constant $c_7 > 0$,

$$\Pr(\|\frac{2}{\lambda_T}\widehat{\mathrm{E}}\big(\big[\mathbf{x}^*_{\text{-}i}\big]_{\mathrm{S}_i^c\cdot}\mathbf{e}_i{}^{\intercal}\big)\|_{\mathrm{B},\infty,\mathrm{F}} \leqslant \epsilon_9) \geqslant 1-\exp(-c_7\lambda_T^2 T) \ .$$

Similarly for $\epsilon_{10} < \frac{16}{\lambda_T}k\sigma^2$,

$$\begin{aligned}\Pr(\|\frac{2}{\lambda_T}\widehat{\mathrm{E}}\big(\big[\mathbf{e}_{\text{-}i}\big]_{\mathrm{S}_i^c\cdot}\mathbf{e}_i{}^{\intercal}\big)\|_{\mathrm{B},\infty,\mathrm{F}} \geqslant \epsilon_{10}) &\leqslant 2k^2 \mid \mathrm{S}_i^c \mid \exp(-\frac{\lambda_T^2\epsilon_{10}^2 T}{256k^2\sigma^4}) \\ &= \exp(-\frac{\lambda_T^2\epsilon_{10}^2 T}{256k^2\sigma^4} + \log(2k^2(n-d))) \ .\end{aligned}$$

If $\lambda_T > \frac{16\sigma^2}{\epsilon_{10}}\sqrt{\frac{k^2\log(2k^2(n-d))}{T}}$, then for some constant $c_8 > 0$,

$$\Pr(\|\frac{2}{\lambda_T}\widehat{\mathrm{E}}\big(\big[\mathbf{e}_{\text{-}i}\big]_{\mathrm{S}_i^c\cdot}\mathbf{e}_i{}^{\intercal}\big)\|_{\mathrm{B},\infty,\mathrm{F}} \leqslant \epsilon_{10}) \geqslant 1-\exp(-c_8\lambda_T^2 T) \ .$$

□

Note that if $T = \mathcal{O}(k^5 d^3\log(kn))$, then $\|\mathbf{M}\|_{\mathrm{B},\infty,1} \leqslant 1-\frac{\alpha}{2}$ for some $\alpha \in (0,1)$. Furthermore, we choose $\epsilon_i = \frac{\alpha}{40}, \forall i \in \{1,\cdots,10\}$, then it follows that if

$$\lambda_T \geqslant \max(\frac{640\sigma\max(\mathrm{b},\sigma,\overline{W}\mathrm{b},\overline{W}\sigma)}{\alpha}\sqrt{\frac{k^3 d\log(2k^2 n)}{T}}, \frac{80}{\alpha}\sigma^2\|\big[\mathbf{W}^*_{i.}\big]_{\mathrm{S}_i\cdot}\|_{\infty,2}) \ ,$$

then for some constant $c > 0$, $f(\mathbf{x},\mathbf{e},\mathbf{W}_{i.},\mathrm{b},\sigma,\alpha,\lambda,T,n,k) \leqslant 1-\frac{\alpha}{4}$ with probability at least $1-\exp(-cT\lambda^2)$. □

A.5. Proof of Lemma 4

Lemma 4 *If* $T = \Omega(k^5 d^3\log(kn))$ *and* $\lambda_T \geqslant \max(\frac{640\sigma\max(\mathrm{b},\sigma,\overline{W}\mathrm{b},\overline{W}\sigma)}{\alpha}\sqrt{\frac{k^3 d\log(2k^2 n)}{T}}, \frac{80}{\alpha}\sigma^2 \|\big[\mathbf{W}^*_{i.}\big]_{\mathrm{S}_i\cdot}\|_{\infty,2})$, *then* $\|\big[\Delta\mathbf{W}_{i\cdot}\big]_{\mathrm{S}_i\cdot}\|_{\mathrm{B},\infty,F} \leqslant \frac{k\sqrt{kd}\lambda_T(10\alpha+1)}{10C}$ *with probability at least* $1-\exp(-cT\lambda_T^2)$ *for some positive constant* c.

Proof. Note that,

$$\begin{aligned}\big[\Delta\mathbf{W}_{i\cdot}\big]_{\mathrm{S}_i\cdot} =& \big[\widehat{\mathrm{E}}\big(\mathbf{x}_{\text{-}i}\mathbf{x}_{\text{-}i}{}^{\intercal}\big)\big]_{\mathrm{S}_i\mathrm{S}_i}^{-1}\big[\widehat{\mathrm{E}}\big(\mathbf{x}_{\text{-}i}\mathbf{e}_{\text{-}i}{}^{\intercal}\big)\big]_{\mathrm{S}_i\mathrm{S}_i}\big[\mathbf{W}^*_{i.}\big]_{\mathrm{S}_i\cdot} + \big[\widehat{\mathrm{E}}\big(\mathbf{x}_{\text{-}i}\mathbf{x}_{\text{-}i}{}^{\intercal}\big)\big]_{\mathrm{S}_i\mathrm{S}_i}^{-1}\widehat{\mathrm{E}}\big(\big[\mathbf{x}_{\text{-}i}\big]_{\mathrm{S}_i\cdot}\mathbf{e}_i{}^{\intercal}\big) - \\ & \frac{\lambda_T}{2}\big[\widehat{\mathrm{E}}\big(\mathbf{x}_{\text{-}i}\mathbf{x}_{\text{-}i}{}^{\intercal}\big)\big]_{\mathrm{S}_i\mathrm{S}_i}^{-1}\big[\mathbf{Z}_{i\cdot}\big]_{\mathrm{S}_i\cdot} \ .\end{aligned}$$

By taking the B, ∞, F-norm on both sides,

$$\|\big[\Delta \mathbf{W}_{i\cdot}\big]_{\mathrm{S}_i\cdot}\|_{\mathrm{B},\infty,\mathrm{F}} = \|\big[\widehat{\mathrm{E}}\big(\mathbf{x}_{\text{-}i}\mathbf{x}_{\text{-}i}{}^{\intercal}\big)\big]_{\mathrm{S}_i\mathrm{S}_i}^{\text{-}1}\big[\widehat{\mathrm{E}}\big(\mathbf{x}_{\text{-}i}\mathbf{e}_{\text{-}i}{}^{\intercal}\big)\big]_{\mathrm{S}_i\mathrm{S}_i}\big[\mathbf{W}^*_{i\cdot}\big]_{\mathrm{S}_i\cdot} + \big[\widehat{\mathrm{E}}\big(\mathbf{x}_{\text{-}i}\mathbf{x}_{\text{-}i}{}^{\intercal}\big)\big]_{\mathrm{S}_i\mathrm{S}_i}^{\text{-}1}\widehat{\mathrm{E}}\big(\big[\mathbf{x}_{\text{-}i}\big]_{\mathrm{S}_i\cdot} \tag{18}$$

$$\mathbf{e}_i{}^{\intercal}\big) - \frac{\lambda_T}{2}\big[\widehat{\mathrm{E}}\big(\mathbf{x}_{\text{-}i}\mathbf{x}_{\text{-}i}{}^{\intercal}\big)\big]_{\mathrm{S}_i\mathrm{S}_i}^{\text{-}1}\big[\mathbf{Z}_{i\cdot}\big]_{\mathrm{S}_i\cdot}\|_{\mathrm{B},\infty,\mathrm{F}} \ . \tag{19}$$

Using the norm triangle inequality and noticing the inequality $\|AB\|_{\mathrm{B},\infty,\mathrm{F}} \leqslant \|A\|_{\mathrm{B},\infty,1}\|B\|_{\infty,2}$ from Lemma 12, we get

$$\begin{aligned}\|\big[\Delta \mathbf{W}_{i\cdot}\big]_{\mathrm{S}_i\cdot}\|_{\mathrm{B},\infty,\mathrm{F}} \leqslant &\|\big[\widehat{\mathrm{E}}\big(\mathbf{x}_{\text{-}i}\mathbf{x}_{\text{-}i}{}^{\intercal}\big)\big]_{\mathrm{S}_i\mathrm{S}_i}^{\text{-}1}\|_{\mathrm{B},\infty,1}\|\big[\widehat{\mathrm{E}}\big(\mathbf{x}_{\text{-}i}\mathbf{e}_{\text{-}i}{}^{\intercal}\big)\big]_{\mathrm{S}_i\mathrm{S}_i}\big[\mathbf{W}^*_{i\cdot}\big]_{\mathrm{S}_i\cdot}\|_{\infty,2} + \|\big[\widehat{\mathrm{E}}\big(\mathbf{x}_{\text{-}i}\mathbf{x}_{\text{-}i}{}^{\intercal}\big)\big]_{\mathrm{S}_i\mathrm{S}_i}^{\text{-}1}\|_{\mathrm{B},\infty,1}\\ &\|\widehat{\mathrm{E}}\big(\big[\mathbf{x}_{\text{-}i}\big]_{\mathrm{S}_i\cdot}\mathbf{e}_i{}^{\intercal}\big)\|_{\infty,2} + \|\frac{\lambda_T}{2}\big[\widehat{\mathrm{E}}\big(\mathbf{x}_{\text{-}i}\mathbf{x}_{\text{-}i}{}^{\intercal}\big)\big]_{\mathrm{S}_i\mathrm{S}_i}^{\text{-}1}\|_{\mathrm{B},\infty,\mathrm{F}}\|\big[\mathbf{Z}_{i\cdot}\big]_{\mathrm{S}_i\cdot}\|_{\infty,2} \ .\end{aligned}$$

Using the inequality $\|A\|_{\mathrm{B},\infty,1} \leqslant k\|A\|_{\infty,\infty}$, where k is the maximum number of rows in a block of A, we obtain

$$\begin{aligned}\|\big[\Delta \mathbf{W}_{i\cdot}\big]_{\mathrm{S}_i\cdot}\|_{\mathrm{B},\infty,\mathrm{F}} \leqslant &k\|\big[\widehat{\mathrm{E}}\big(\mathbf{x}_{\text{-}i}\mathbf{x}_{\text{-}i}{}^{\intercal}\big)\big]_{\mathrm{S}_i\mathrm{S}_i}^{\text{-}1}\|_{\infty,\infty}\|\big[\widehat{\mathrm{E}}\big(\mathbf{x}_{\text{-}i}\mathbf{e}_{\text{-}i}{}^{\intercal}\big)\big]_{\mathrm{S}_i\mathrm{S}_i}\big[\mathbf{W}^*_{i\cdot}\big]_{\mathrm{S}_i\cdot}\|_{\infty,2} + k\|\big[\widehat{\mathrm{E}}\big(\mathbf{x}_{\text{-}i}\mathbf{x}_{\text{-}i}{}^{\intercal}\big)\big]_{\mathrm{S}_i\mathrm{S}_i}^{\text{-}1}\\ &\|_{\infty,\infty}\|\widehat{\mathrm{E}}\big(\big[\mathbf{x}_{\text{-}i}\big]_{\mathrm{S}_i\cdot}\mathbf{e}_i{}^{\intercal}\big)\|_{\infty,2} + \frac{\lambda_T}{2}k\|\big[\widehat{\mathrm{E}}\big(\mathbf{x}_{\text{-}i}\mathbf{x}_{\text{-}i}{}^{\intercal}\big)\big]_{\mathrm{S}_i\mathrm{S}_i}^{\text{-}1}\|_{\infty,\infty} \ .\end{aligned}$$

Since $\|A\|_{\infty,\infty} \leqslant \sqrt{p}\|A\|_{2,2}$ for $A \in \mathbb{R}^{p\times p}$,

$$\begin{aligned}\|\big[\Delta \mathbf{W}_{i\cdot}\big]_{\mathrm{S}_i\cdot}\|_{\mathrm{B},\infty,\mathrm{F}} \leqslant &k\sqrt{kd}\|\big[\widehat{\mathrm{E}}\big(\mathbf{x}_{\text{-}i}\mathbf{x}_{\text{-}i}{}^{\intercal}\big)\big]_{\mathrm{S}_i\mathrm{S}_i}^{\text{-}1}\|_{2,2}\|\big[\widehat{\mathrm{E}}\big(\mathbf{x}_{\text{-}i}\mathbf{e}_{\text{-}i}{}^{\intercal}\big)\big]_{\mathrm{S}_i\mathrm{S}_i}\big[\mathbf{W}^*_{i\cdot}\big]_{\mathrm{S}_i\cdot}\|_{\infty,2} + k\sqrt{kd}\\ &\|\big[\widehat{\mathrm{E}}\big(\mathbf{x}_{\text{-}i}\mathbf{x}_{\text{-}i}{}^{\intercal}\big)\big]_{\mathrm{S}_i\mathrm{S}_i}^{\text{-}1}\|_{2,2}\|\widehat{\mathrm{E}}\big(\big[\mathbf{x}_{\text{-}i}\big]_{\mathrm{S}_i\cdot}\mathbf{e}_i{}^{\intercal}\big)\|_{\infty,2} + \frac{\lambda_T}{2}k\sqrt{kd}\|\big[\widehat{\mathrm{E}}\big(\mathbf{x}_{\text{-}i}\mathbf{x}_{\text{-}i}{}^{\intercal}\big)\big]_{\mathrm{S}_i\mathrm{S}_i}^{\text{-}1}\|_{2,2} \ .\end{aligned}$$

Using results from Lemmas 1, 6 and 7, with high probability,

$$\begin{aligned}\|\big[\Delta \mathbf{W}_{i\cdot}\big]_{\mathrm{S}_i\cdot}\|_{\mathrm{B},\infty,\mathrm{F}} &\leqslant k\sqrt{kd}\frac{2}{C}\frac{\lambda_T}{2}2\frac{\alpha}{40} + k\sqrt{kd}\frac{2}{C}\frac{\lambda_T}{2}2\frac{\alpha}{40} + \frac{\lambda_T}{2}k\sqrt{kd}\frac{2}{C}\\ &= \frac{k\sqrt{kd}\lambda_T(10\alpha+1)}{10C} \ .\end{aligned}$$

□

B. PROOFS OF AUXILIARY LEMMAS

B.1. Subexponentiality of square of sub-Gaussian random variables

Lemma 10. *If y is a sub-Gaussian random variable with 0 mean and parameter 1, then y^2 is a sub-exponential random variable with parameters $(4\sqrt{2}, 4)$.*

Proof. Since y is a 0 mean sub-Gaussian random variable with parameter 1, we can write

$$(\forall \lambda_T \in \mathbb{R})\mathrm{E}\big(\exp(\lambda_T y)\big) \leqslant \exp(\frac{\lambda_T^2}{2}) \ .$$

Let $\Gamma(r)$ be the Gamma function, then moments of the sub-Gaussian variable y are bounded as follows:

$$(\forall r \geqslant 0)\mathrm{E}\big(|y|^r\big) \leqslant r2^{\frac{r}{2}}\Gamma(\frac{r}{2}) \ .$$

Let $v \triangleq y^2$ and $\mu_v \triangleq \mathrm{E}\big(v\big)$. Using power series expansion and noting that $\Gamma(r) = (r-1)!$ for an integer r, we have:

$$\begin{aligned}
\mathrm{E}\big(\exp(\lambda_T(v-\mu_v))\big) &= 1 + \lambda_T \mathrm{E}\big(v-\mu_v\big) + \sum_{r=2}^{\infty} \frac{\lambda_T^r \mathrm{E}\big((v-\mu_v)^r\big)}{r!} \\
&\leqslant 1 + \sum_{r=2}^{\infty} \frac{\lambda_T^r \mathrm{E}\big(|y|^{2r}\big)}{r!} \\
&\leqslant 1 + \sum_{r=2}^{\infty} \frac{\lambda_T^r 2r 2^r \Gamma(r)}{r!} \\
&= 1 + \sum_{r=2}^{\infty} \lambda_T^r 2^{r+1} \\
&= 1 + \frac{8\lambda_T^2}{1-2\lambda_T} \ .
\end{aligned}$$

We take $\lambda_T \leqslant \frac{1}{4}$. Thus,

$$\begin{aligned}
\mathrm{E}\big(\exp(\lambda_T(v-\mu_v))\big) &\leqslant 1 + 16\lambda_T^2 \\
&\leqslant \exp(16\lambda_T^2) \\
&\leqslant \exp\Big(\frac{(4\sqrt{2})^2\lambda_T^2}{2}\Big) \ .
\end{aligned}$$

It follows that $v = y^2$ is a subexponential random variable with parameters $(4\sqrt{2}, 4)$. $\square$

B.2. Subexponentiality of product of independent sub-Gaussian random variables

Lemma 11. *Let p and q be two independent sub-Gaussian random variables with 0 mean and parameter 1, then pq is a sub-exponential random variable with parameters $(4\sqrt{2}, 4)$.*

Proof. Since p and q are both 0 mean sub-Gaussian random variable with parameter 1, we can write

$$\begin{aligned}
(\forall \lambda_T \in \mathbb{R})\mathrm{E}\big(\exp(\lambda_T p)\big) &\leqslant \exp\Big(\frac{\lambda_T^2}{2}\Big) \\
(\forall \lambda_T \in \mathbb{R})\mathrm{E}\big(\exp(\lambda_T q)\big) &\leqslant \exp\Big(\frac{\lambda_T^2}{2}\Big) \ .
\end{aligned}$$

Let $\Gamma(r)$ be the Gamma function, then moments of the sub-Gaussian variable p and q are bounded as follows:

$$\begin{aligned}
(\forall r \geqslant 0)\mathrm{E}\big(|p|^r\big) &\leqslant r 2^{\frac{r}{2}} \Gamma\Big(\frac{r}{2}\Big) \\
(\forall r \geqslant 0)\mathrm{E}\big(|q|^r\big) &\leqslant r 2^{\frac{r}{2}} \Gamma\Big(\frac{r}{2}\Big) \ .
\end{aligned}$$

Let $v \triangleq pq$. Note that $\mathrm{E}\big(v\big) = \mathrm{E}\big(pq\big) = \mathrm{E}\big(p\big)\mathrm{E}\big(q\big) = 0$ due to independence. Using power series expansion and noting that $\Gamma(r) = (r-1)!$ for an integer r, we have:

$$\begin{aligned}
\mathrm{E}\big(\exp(\lambda_T v)\big) &= 1 + \lambda_T \mathrm{E}\big(v\big) + \sum_{r=2}^{\infty} \frac{\lambda_T^r \mathrm{E}\big(v^r\big)}{r!} \\
&\leqslant 1 + \sum_{r=2}^{\infty} \frac{\lambda_T^r \mathrm{E}\big(|p|^r |q|^r\big)}{r!} \\
&\leqslant 1 + \sum_{r=2}^{\infty} \frac{\lambda_T^r \mathrm{E}\big(|p|^r\big)\mathrm{E}\big(|q|^r\big)}{r!} \\
&\leqslant 1 + \sum_{r=2}^{\infty} \frac{\lambda_T^r r^2 2^r \Gamma(\frac{r}{2})^2}{r!} \ .
\end{aligned}$$

Note that $\Gamma(\frac{r}{2})^2 \leqslant \Gamma(r)$. Thus,

$$\begin{aligned}
\mathrm{E}\big(\exp(\lambda_T v)\big) &\leqslant 1 + \sum_{r=2}^{\infty} \frac{\lambda_T^r r^2 2^r \Gamma(r)}{r!} \\
&= 1 - \frac{8(\lambda_T - 1)\lambda_T^2}{(1 - 2\lambda_T)^2} \\
&\leqslant \exp(16\lambda_T^2) \\
&\leqslant \exp(\frac{(4\sqrt{2})^2}{2}\lambda_T^2) \ ,
\end{aligned}$$

where last inequality holds for $|\lambda_T| \leqslant \frac{1}{4}$. Thus, pq is subexponential with parameters $(4\sqrt{2}, 2)$. □

B.3. Norm Inequalities

Here we will derive some norm inequalities which we will use in our proofs.

Lemma 12 (Norm Inequalities)**.** *Let* $\mathbf{A}$ *be a row-partitioned block matrix which consists of* p *blocks where block* $\mathbf{A}_i \in \mathbb{R}^{m_i \times n}$, $\forall i \in \{1, \ldots, p\}$ *and* $\mathbf{B} \in \mathbb{R}^{n \times o}$*. Then the following inequalities hold:*

$$\|\mathbf{AB}\|_{\mathrm{B},\infty,F} \leqslant \|\mathbf{A}\|_{\mathrm{B},\infty,1} \|\mathbf{B}\|_{\infty,2} \ ,$$

$$\|\mathbf{AB}\|_{\mathrm{B},\infty,1} \leqslant \|\mathbf{A}\|_{\mathrm{B},\infty,1} \|\mathbf{B}\|_{\infty,\infty} \ .$$

Proof. Let $\mathrm{vec}(.)$ be an operator which flattens the matrix and converts it to a vector. Let $\mathbf{Y}$ be a row-partitioned block matrix with same size and block structure as $\mathbf{A}$.

$$\begin{aligned}
\|\mathbf{AB}\|_{\mathrm{B},\infty,\mathrm{F}} &= \max_{i \in \{1,\ldots,p\}} \|\mathrm{vec}((\mathbf{AB})_i)\|_2 \\
&= \max_{i \in \{1,\ldots,p\}, \|\mathrm{vec}(\mathbf{Y}_i)\|_2 \leqslant 1} \mathrm{vec}((\mathbf{AB})_i)^\intercal \mathrm{vec}(\mathbf{Y}_i) \\
&= \max_{i \in \{1,\ldots,p\}, \|\mathrm{vec}(\mathbf{Y}_i)\|_2 \leqslant 1} [(\mathbf{A}_i)_{1.}\mathbf{B} \ldots (\mathbf{A}_i)_{m_i.}\mathbf{B}]\mathrm{vec}(\mathbf{Y}_i) \\
&= \max_{i \in \{1,\ldots,p\}, \|\mathrm{vec}(\mathbf{Y}_i)\|_2 \leqslant 1} [(\mathbf{A}_i)_{1.}\mathbf{B}(\mathbf{Y}_i)_{1.} + \cdots + (\mathbf{A}_i)_{m_i.}\mathbf{B}(\mathbf{Y}_i)_{m_i.}] \\
&\leqslant \max_{\substack{i \in \{1,\ldots,p\}, \\ \|\mathrm{vec}(\mathbf{Y}_i)\|_2 \leqslant 1}} \|(\mathbf{A}_i)_{1.}\|_1 \|\mathbf{B}(\mathbf{Y}_i)_{1.}\|_\infty + \cdots + \|(\mathbf{A}_i)_{m_i.}\|_1 \|\mathbf{B}(\mathbf{Y}_i)_{m_i.}\|_\infty \\
&\leqslant \|\mathbf{A}\|_{\mathrm{B},\infty,1} \|\mathbf{B}\|_{\infty,2} \ .
\end{aligned}$$

We follow a similar procedure for the last norm inequality.

$$\begin{aligned}
\|\mathbf{AB}\|_{\mathrm{B},\infty,1} &= \max_{i \in \{1,\ldots,p\}} \|\mathrm{vec}((\mathbf{AB})_i)\|_1 \\
&= \max_{i \in \{1,\ldots,p\}, \|\mathrm{vec}(\mathbf{Y}_i)\|_\infty \leqslant 1} \mathrm{vec}((\mathbf{AB})_i)^\intercal \mathrm{vec}(\mathbf{Y}_i) \\
&= \max_{i \in \{1,\ldots,p\}, \|\mathrm{vec}(\mathbf{Y}_i)\|_\infty \leqslant 1} [(\mathbf{A}_i)_{1.}\mathbf{B} \ldots (\mathbf{A}_i)_{m_i.}\mathbf{B}]\mathrm{vec}(\mathbf{Y}_i) \\
&= \max_{i \in \{1,\ldots,p\}, \|\mathrm{vec}(\mathbf{Y}_i)\|_\infty \leqslant 1} [(\mathbf{A}_i)_{1.}\mathbf{B}(\mathbf{Y}_i)_{1.} + \cdots + (\mathbf{A}_i)_{m_i.}\mathbf{B}(\mathbf{Y}_i)_{m_i.}] \\
&\leqslant \max_{i \in \{1,\ldots,p\}, \|\mathrm{vec}(\mathbf{Y}_i)\|_\infty \leqslant 1} \|(\mathbf{A}_i)_{1.}\|_1 \|\mathbf{B}(\mathbf{Y}_i)_{1.}\|_\infty + \ldots \|(\mathbf{A}_i)_{m_i.}\|_1 \|\mathbf{B}(\mathbf{Y}_i)_{m_i.}\|_\infty \\
&\leqslant \|\mathbf{A}\|_{\mathrm{B},\infty,1} \|\mathbf{B}\|_{\infty,\infty} \ .
\end{aligned}$$

□

C. DISCUSSION ON ACCESS TO NASH EQUILIBRIA AND SOLUTION QUALITY

Note that we do not put any constraints on the choice of Nash equilibria samples. This comes at a price of compromising the solution quality. Consider for instance a corner case, where we repeatedly sample the noisy version of a Nash equilibrium which is $\mathbf{x}^* = \mathbf{0}$. Since, regardless of the choice of $\mathbf{W}^*_{i.}$, $\mathbf{x}^* = \mathbf{0}$ is always in the Nash equilibria set of any game described in Section 2, it is impossible to recover the correct $\mathbf{W}^*_{i.}$ with only access to $\mathbf{x}^* = \mathbf{0}$. Since this choice satisfies Assumptions 1, 2 and 3, it would appear that our method could recover the true structure of the game, which should not be possible. However, while Statement 1 and Statement 3 of Theorem 1 still hold, the minimum weight requirement of Statement 2 fails to hold and we do not correctly recover the true neighbors. To see this, consider we have access to infinite samples. Since $\mathbf{x}^* = \mathbf{0}$, we the optimization problem (1) becomes:

$$\arg\min_{\mathbf{W}_{i\cdot}} \sum_j tr(\sigma^2 \mathbf{W}_{ij}^\intercal \mathbf{W}_{ij}) + \lambda \sum_j \|\mathbf{W_{ij}}\|_F \ .$$

Clearly, the solution $\mathbf{W}_{i\cdot} = \mathbf{0}$ is trivially the optimal solution. This solution, however trivial, recovers all the non-neighbors correctly. Thus, Statement 1 of Theorem 1 holds. Now note that the minimum weight condition of Statement 2 requires $\min_{j \in \mathrm{S}_i} \|W^*_{ij}\|_F > 2\|\big[\Delta\mathbf{W}_{i\cdot}\big]_{\mathrm{S}_i\cdot}\|_{\mathrm{B},\infty,\mathrm{F}}$ and we provide an upper bound on $\|\big[\Delta\mathbf{W}_{i\cdot}\big]_{\mathrm{S}_i\cdot}\|_{\mathrm{B},\infty,\mathrm{F}}$ in our proofs. In the case when we have access to only $\mathbf{x}^* = 0$, this minimum condition never holds. To see this, we rewrite Equation (18) for this particular case and it becomes

$$\|\big[\Delta\mathbf{W}_{i\cdot}\big]_{\mathrm{S}_i\cdot}\|_{\mathrm{B},\infty,\mathrm{F}} = \|\big[\mathbf{W}^*_{i.}\big]_{\mathrm{S}_i\cdot} - \frac{\lambda_T}{2\sigma^2}\big[\mathbf{Z}_{i\cdot}\big]_{\mathrm{S}_i\cdot}\|_{\mathrm{B},\infty,\mathrm{F}} \ .$$

As before, we use triangle inequality for norms and end up with a bound which is:

$$\|\big[\Delta\mathbf{W}_{i\cdot}\big]_{\mathrm{S}_i\cdot}\|_{\mathrm{B},\infty,\mathrm{F}} \leqslant \|\big[\mathbf{W}^*_{i.}\big]_{\mathrm{S}_i\cdot}\|_{\mathrm{B},\infty,\mathrm{F}} + \|\frac{\lambda_T}{2\sigma^2}\big[\mathbf{Z}_{i\cdot}\big]_{\mathrm{S}_i\cdot}\|_{\mathrm{B},\infty,\mathrm{F}} \ .$$

For Statement 2 of Theorem 1 to work, we need $\min_{j \in \mathrm{S}_i} \|W^*_{ij}\|_F$ to be greater than $2(\|\big[\mathbf{W}^*_{i.}\big]_{\mathrm{S}_i\cdot}\|_{\mathrm{B},\infty,\mathrm{F}} + \|\frac{\lambda_T}{2\sigma^2}\big[\mathbf{Z}_{i\cdot}\big]_{\mathrm{S}_i\cdot}\|_{\mathrm{B},\infty,\mathrm{F}})$ which is clearly impossible. Thus, the minimum weight criteria is not fulfilled and Statement 2 of our statement does not hold. By using equation (12), we recover $\mathbf{x} = \mathbf{0}$ as the only point in set $\overline{\mathrm{NE}}_\epsilon(\mathscr{G})$, thus Statement 3 holds.

In conclusion, while there is no constraint on the choice of Nash equilibria, the solution quality does depend on it and for corner cases, one or more technical assumptions may break.

D. EXPERIMENTAL VALIDATION

In this section, we validate results of Theorem 1 by running computational experiments on synthetic data. The experiments were conducted across 30 independent trials and we report the average result across these experiments.

Generating Graphical Games and Synthetic Data. For our experiments, we considered a setup where each player takes action $x_i \in \mathbb{R}$, i.e., we consider the case of $k = 1$. Note that in this scenario, the set of Nash equilibria can be calculated by computing the null space of $\mathbf{I}_s - \mathbf{W}^*_s$ matrix where $\mathbf{I}_s \in \mathbb{R}^{n_s \times n_s}$ is an identity matrix and $\mathbf{W}^*_s \in \mathbb{R}^{n_s \times n_s}$ is the weight matrix. The weights of $\mathbf{W}^*_s$ come from the set $\{-1, 0\}$. To ensure that the Hessian restricted on supports, i.e., $\big[\mathbf{H}\big]_{\mathrm{S}_i\mathrm{S}_i}$ is invertible we imposed the condition that the rank of $\mathbf{I}_s - \mathbf{W}^*_s$ is less that $n_s - d_s - 1$ where d_s is the number of in-neighbors for each player. For $n_s = 7$ and $d_s = 2$, we generated 100 instances of $\mathbf{W}^*_s$ and corresponding graphical games satisfying all our assumptions. These smaller graphical games were combined together to generate graphs for bigger graphical games. This process ensures that all our assumptions are valid for bigger graphical games. The final experiments were conducted for graphical games with $n_b = 21, 49$ and 98 players. These graphical games were created by combining multiple instances of smaller games chosen at random from the set of 100 games. The noisy version of Nash equilibria set was generated by taking a random combination of vectors from the null space of $\mathbf{I}_b - \mathbf{W}^*_b$ together with an additive Gaussian noise. Here $\mathbf{I}_b \in \mathbb{R}^{n_b \times n_b}$ is an identity matrix and $\mathbf{W}^*_b \in \mathbb{R}^{n_b \times n_b}$ is the weight matrix of the bigger game. We used our method on this noisy Nash equilibria set to recover the structure of the game in each of these cases.

Experimental Setup and Results. We conducted experiments for games with $n_b = 21, 49$ and 98 players. As per our theoretical results, the number of samples was varied as $10^C \log(n_b)$ where C is a "control parameter". We provide results of our experiments in Figure 2. Let $\hat{\pi}_i$ be the set of recovered in-neighbors for player i and π_i be the set of true in-neighbors for player i, then the performance measures are defined as follows:

$$\text{Precision} = \frac{\sum_{i=1}^{n_b} |\pi_i \cap \hat{\pi}_i|}{\sum_{i=1}^{n_b} |\hat{\pi}_i|}, \qquad \text{Recall} = \frac{\sum_{i=1}^{n_b} |\pi_i \cap \hat{\pi}_i|}{\sum_{i=1}^{n_b} |\pi_i|} \ . \tag{20}$$

Figure 2a shows how the precision for games with various number of players varies with number of samples and Figure 2b shows the recall for the same setup. All results are averaged across 30 independent runs. We can see that both precision and recall go to 1 as we increase the number of samples, obtaining perfect recovery with enough samples. Notice that the different curves for different number of players ($n_b = 21, 49$ and 98) line up with one another quite well. This matches with our theoretical results and shows that with a constant number of in-neighbors $\mathcal{O}(\log(n))$ samples are sufficient to recover the exact structure of the graphical games.

E. REAL WORLD APPLICATION - COMPUTING MOST INFLUENTIAL TRADING COUNTRIES

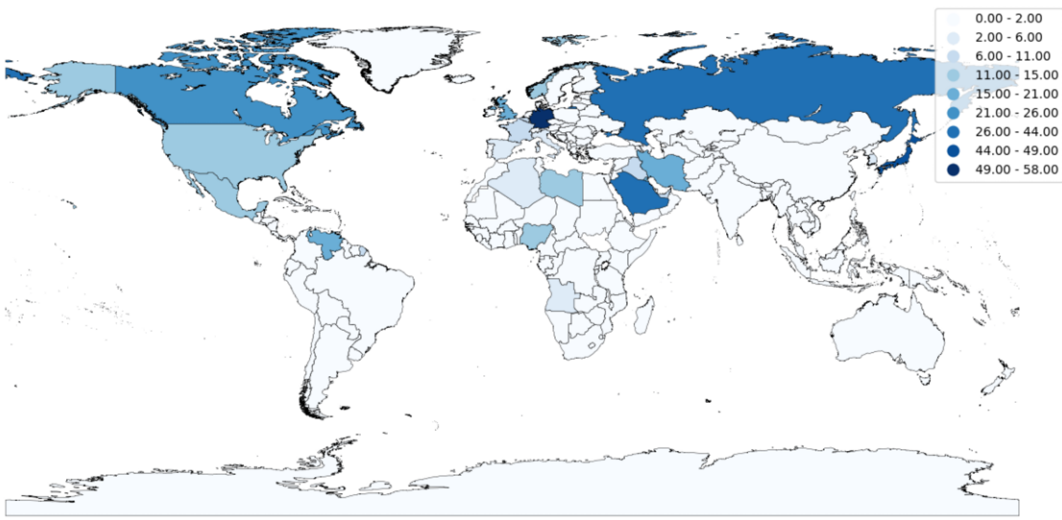

Fig. 3: Most "locally" influential trading countries for the top 2 most exported products between 1962 to 2017. A country's "local" influence is measured by the number of its out-neighbors in the learnt game.

In this section, we demonstrate the effectiveness of our method by using it to compute the most "locally" influential trading countries in the world. The trading interactions among various countries can be modeled as a strategic game among self-interested countries. We will figure out the most "locally" influential trading countries by learning the structure of the global trade network using our proposed method. This experimental setup is similar in nature to the work of [8] where games with *binary* actions were learnt for the U.S. Senate.

Real World Dataset. We conducted our experiments on the publicly available trade data from `https://oec.world/en/resources/data/`. This dataset contains trade data among $n = 261$ countries between 1962 to 2017. We considered this as $T = 56$ samples for our experiments. As the number of samples were very small, we decided to focus on the top $k = 2$ most traded products between 1962 to 2017 based on their net export. In the trading dataset, these products are categorized as 'Crude Petroleum' and 'Cars'.

Learning Structure of Graphical Game. Then our method was used to learn the global trade network using this trading data. An edge from country A to country B indicates that country A influences trade decision of country B. The most "locally" influential trading countries were identified by counting their number of out-neighbors. The results are shown in Figure 3. As expected, our method identifies the top 3 most "locally" influential countries as Germany, Japan and Saudi Arabia which are some of world's most powerful economies and have had large influence on the crude oil and automobile market between the time period of 1962 to 2017.

Inference in Learnt Graphical Game.

Once the structure was learnt, the global efficiency quantities such as the Price of Anarchy (PoA), the Price of Stability (PoS) and the volume of Nash equilibria (Vol) were computed. Recall from subsection 3.1 that we can define these global efficiency

	1962-2017
Price of Anarchy	1.2575
Price of Stability	1.0424
Volume of ϵ-Nash equilibria	38.7%

Table 1: Inference of global efficiency quantities in the graphical game for the trade of the top 2 most exported products across the world between 1962 to 2017

quantities as follows:

$$\overline{\text{PoA}}_\epsilon(\mathscr{G}) = \frac{\sup_{x \in \mathcal{A}} \text{Wel}(x)}{\inf_{x \in \overline{\text{NE}}_\epsilon(\mathscr{G})} \text{Wel}(x)}$$
$$\overline{\text{PoS}}_\epsilon(\mathscr{G}) = \frac{\sup_{x \in \mathcal{A}} \text{Wel}(x)}{\sup_{x \in \overline{\text{NE}}_\epsilon(\mathscr{G})} \text{Wel}(x)}$$
$$\overline{\text{Vol}}_\epsilon(\mathscr{G}) = \frac{\tau(\overline{\text{NE}}_\epsilon(\mathscr{G}))}{\tau(\mathcal{A})},$$

where $\mathcal{A}$ denotes the action set $\times_{i \in \mathcal{V}} \mathcal{A}_i$, and $\tau(A)$ is the Lebesgue measure of set A. The subscript ϵ and bar in the global efficiency quantities denote that $\overline{\text{NE}}_\epsilon(\mathscr{G})$ is used in their computation. The welfare function $\text{Wel}(x)$ is defined as $\text{Wel}(x) \triangleq \sum_{i \in \mathcal{V}} \widehat{u}_i(x)$, where $\widehat{u}_i(x) \triangleq -\|x_i - \sum_{j \in \text{S}_i} W_{ij} x_j\|_2$ is the recovered utility function.

To compute these quantities for our experiment, we first defined welfare of a strategy profile as the sum of payoffs of all players for that particular strategy profile. Then, 1000 strategy profiles $\mathbf{x} \in \times_{i \in \mathcal{V}} \mathcal{A}_i$ were generated uniformly at random such that $x_i \in \mathbb{R}^2$ and $\|x_i\|_2 \leqslant \sqrt{2}$. A strategy profile $\mathbf{x}$ was considered to be an ϵ-Nash equilibria if the payoff for it was less than a fixed ϵ across all players. The proportion of all 1000 strategy profiles which were part of ϵ-Nash equilibria, termed as the volume of ϵ-Nash equilibria, was 38.7%. To compute the global efficiency quantities, we shifted all the payoffs by a constant to make the payoffs non-negative. Note that this does not change the ϵ-Nash equilibria set of the game.

The price of anarchy was computed as the ratio between the maximum welfare across all strategy profiles and the minimum welfare across all strategy profiles in the ϵ-Nash equilibria set. We found the price of anarchy to be 1.2575. Our finding implies that the total welfare across countries can be increased by more than 25% by deviating away from ϵ-Nash equilibria. Similarly, the price of stability was computed as the ratio between the maximum welfare across all strategy profiles and the maximum welfare across all strategy profiles in the ϵ-Nash equilibria set. We found the price of stability to be 1.0424. This implies that the maximum total welfare does not improve too much when players deviate away from ϵ-Nash equilibria.