PROVABLE SAMPLE COMPLEXITY GUARANTEES FOR LEARNING OF CONTINUOUS-ACTION GRAPHICAL GAMES WITH NONPARAMETRIC UTILITIES

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ABSTRACT

In this paper, we study the problem of learning the exact structure of continuous-action games with non-parametric utility functions. We propose an ℓ_1-regularized method which encourages sparsity of the coefficients of the Fourier transform of the recovered utilities. Our method works by accessing very few Nash equilibria and their noisy utilities. Under certain technical conditions, our method also recovers the exact structure of these utility functions, and thus, the exact structure of the game. Furthermore, our method only needs a logarithmic number of samples in terms of the number of players and runs in polynomial time. We follow the primal-dual witness framework to provide provable theoretical guarantees. *A full version of this paper is accessible at:* `https://www.cs.purdue.edu/homes/abarik/abarik_nonpara_icassp_full.pdf`

Index Terms— Graphical Games, Machine Learning, Continuous Action Games, Non-Parametric Payoffs

1. INTRODUCTION

Game theory has been extensively used as a framework to model and study the strategic interactions amongst rational but selfish individual players who are trying to maximize their payoffs. Game theory has been applied in many fields including but not limited to social and political science, economics, communication, system design and computer science. In non-cooperative games, each player decides its action based on the actions of others players. These games are characterized by the equilibrium solution concept such as *Nash equilibrium (NE)* [1] which serves a descriptive role of the stable outcome of the overall behavior of self-interested players (e.g., people, companies, governments, groups or autonomous systems) interacting strategically with each other in distributed settings.

Graphical games [2], introduced within the AI community about two decades ago, are a representation of multiplayer games which capture and exploit locality or sparsity of direct influences. They are most appropriate for large-scale population games in which the payoffs of each player are determined by the actions of only a small number of other players. Indeed, graphical games played a prominent role in establishing the computational complexity of computing NE in normal-form games as well as in succinctly representable multiplayer games [3–5]. Graphical games have been studied for both discrete and continuous actions.

Inference in Graphical Games. There has been a large body of work on computing classical equilibrium solution concepts. The Nash equilibria and *correlated equilibria* [6] in graphical games have been studied by [2, 7–16]. The computation of the *price of anarchy* was studied in [17]. In addition, [18] identified the most influential players, i.e., a small set of players whose collective behavior forces every other player to a unique choice of action. All the works above focus on inference problems for graphical games, and fall in the field of algorithmic game theory. All of these methods assume access to graphical game network and payoffs of the games under consideration.

Learning Graphical Games. In order to answer the *inference* problems discussed above, we would need to recover the structure of graphical game. In particular we ask: Given that we have access to a noisy observation of the player's revenue and few joint action vectors at equilibria, can we learn the neighbors of a player? Of course, once we learn the local structure for each individual player then we can combine them in order to obtain the entire structure of the graphical game. Learning the structure of a game is essential to the development, potential use and success of game-theoretic models in large-scale practical applications. It is also interesting to study the relationship between the pure strategy Nash equilibria (PSNE) set of the recovered game and the true game. In this paper, we study the problem of learning the graph structure in a continuous-action graphical game with non-parametric utility functions. We also provide the complete characterization and comparison of the payoff functions and the PSNE set of the recovered game and the true game.

A Motivating Real-World Case. To understand the difference between learning and inference, we provide a real world example of potato trade among various countries between 1973 to 2019. In this case, it is necessary to learn the structure of the graphical game before we can infer global efficiency quantities such as the price of anarchy (See subsection A.8 for details). We divided dataset into two time periods: 1973-1995 and 1996-2019. The structure of the graphical game is learnt using our method (See Appendix C for details). Once the structure is learnt, we computed the price of anarchy which is the ratio between the maximum welfare across all strategy profiles and the minimum welfare across all strategy profiles in the ϵ-PSNE set, where welfare was defined as the sum of payoffs of all players. The price of anarchy for the duration 1973-1995 and 1996-2019 were 1.1793 and 1.0685 respectively. We observe that the price of anarchy in period 1973-1995 is greater than that of period 1996-2019. From this, we infer that in the past, the total welfare across European countries could have been increased by deviating away from Nash equilibria, however now the gain in total welfare by deviating away from Nash equilibria is not too much. We speculate that this correlates with the formation of the European union in 1993.

Related Work. None of the prior literature has dealt with either continuous actions or non-parametric utilities. In discrete-action games, [19] proposed a maximum-likelihood approach to learn linear influence games - a class of parametric graphical games with binary actions and linear payoffs. However, their method runs in exponential time and the authors assumed a specific observation model for the strategy profiles. For the same specific observation model, [20] proposed a polynomial time algorithm, based on ℓ_1-regularized logistic regression, for learning linear influence games. Their strategy profiles (or joint actions) were drawn from a mixture of uniform distributions: one over the pure-strategy Nash equilibria (PSNE) set, and the other over its complement. [21] obtained necessary and sufficient conditions for learning linear influence games under arbitrary observation model. [22] use a discriminative, max-margin based approach, to learn tree structured polymatrix games[1]. Their method runs in exponential time and the authors show that learning polymatrix games is NP-hard under this max-margin setting, even when the class of graphs is restricted to trees. Finally, [23] proposed a polynomial time algorithm for learning sparse polymatrix games in the discrete-action setting.

Contribution. Our goal is to come up with a provably correct method which has a polynomial time and sample complexity. We propose a novel yet simple method to learn graphical games with non-parametric payoffs. Our proposed mathematical model, in its final form, resembles a constrained lasso problem. But before reaching this simple formulation, it carefully handles non-parametric payoffs by modeling them using an infinite weighted sum of orthonormal basis function and then truncating it for estimation. We use ℓ_1-norm regularizer to encourage sparsity of the coefficients of the Fourier transform of the recovered payoff functions. We go beyond solving the optimization problem and fully characterize a payoff function which is at most $\frac{\epsilon}{2}$ away from the true payoff function. Using these payoff functions, we recover a game whose ϵ-PSNE is contained in the PSNE of the true game (and vice versa). If the true game satisfies some technical conditions, then the PSNE of our recovered game matches exactly with the PSNE of the true game. For n players and at most d in-neighbors per player, we show that $\Omega(d^3 \log(n))$ *samples* are sufficient to recover the exact structure of the true game.

[1] Polymatrix games are graphical games where each player's utility is a sum of unary (single player) and pairwise (two players) potential functions.

2. PRELIMINARIES

In this section, we collect notations and describe our problem.

2.1. Notation

Consider a directed graph $\mathcal{G}(\mathcal{V}, \mathcal{E})$, where $\mathcal{V}$ and $\mathcal{E}$ are set of vertices and edges respectively. We define $\mathcal{V} \triangleq \{1, \ldots, n\}$, where each vertex corresponds to one player. We denote the set of in-neighbors of a player i by S_i, i.e., $\mathrm{S}_i = \{j \mid (j,i) \in \mathcal{E}\}$. An in-neighbor of player i is a node that points to node i in the graph. All the other players are denoted by S_i^c, i.e., $\mathrm{S}_i^c = \{1, \ldots, n\} \backslash (\mathrm{S}_i \cup \{i\})$. Let $|\mathrm{S}_i| \leqslant d$ and $|\mathrm{S}_i^c| \geqslant n-1-d$. For each player $i \in \mathcal{V}$, there is a set of actions or *pure-strategies* $\mathcal{A}_i$. That is, player i can take action $x_i \in \mathcal{A}_i$. Each action x_i consists of making k decisions on a limited budget $\mathrm{b} \in \mathbb{R}$. We consider games with continuous actions. Mathematically, $x_i \in \mathcal{A}_i = \mathbb{R}^k$ and $\|x_i\|_2 \leqslant \mathrm{b}$. We use $x_{\text{-}i}$ to denote the collection of actions of all players but i. For each player i, there is also a local payoff function $u_i : \mathcal{A}_i \times (\times_{j \in \mathrm{S}_i} \mathcal{A}_j) \to \mathbb{R}$ mapping the joint action of player i and its in-neighbors S_i, to a real number. A joint action $\mathbf{x}^* \in \times_{i \in \mathcal{V}} \mathcal{A}_i$ is a *pure-strategy Nash equilibrium (PSNE)* of a graphical game iff, no player i has any incentive to unilaterally deviate from the prescribed action $x_i^* \in \mathcal{A}_i$, given the joint action of its in-neighbors $x_{\mathrm{S}_i}^* \in \times_{j \in \mathrm{S}_i} \mathcal{A}_j$ in the equilibrium. We denote a game by $\mathscr{G}$, and the set of all PSNE and ϵ-PSNE are denoted by $\mathrm{NE}(\mathscr{G})$ and $\mathrm{NE}_\epsilon(\mathscr{G})$ respectively, for a constant $\epsilon > 0$. Mathematically, $\mathrm{NE}(\mathscr{G}) \triangleq \{x^* \in \times_{i \in \mathcal{V}} \mathcal{A}_i \mid x_i^* \in \arg\max_{x_i \in \mathcal{A}_i} u_i(x_i, x_{\text{-}i}^*), \forall i \in \mathcal{V}\}$ and $\mathrm{NE}_\epsilon(\mathscr{G}) \triangleq \{x^* \in \times_{i \in \mathcal{V}} \mathcal{A}_i \mid u_i(x_i^*, x_{\text{-}i}^*) \geqslant -\epsilon + \max_{x_i \in \mathcal{A}_i} u_i(x_i, x_{\text{-}i}^*), \forall i \in \mathcal{V}\}$.

2.2. Model

Now we describe the basic setup of our problem. Our first challenge arises due to not having access to parametric payoffs. To tackle this, we decompose our non-parametric payoffs as a weighted sum of orthonormal basis functions. We consider a graphical game with the property that utility function of each player i is decomposable into sum of pairwise functions which only depend on the in-neighbors of i, i.e.,

$$u_i(x) = \sum_{j \in \mathrm{S}_i} u_{ij}(x_i, x_j) \ , \qquad (1)$$

where S_i is the set of neighbors of player i. For two real valued functions $f(x)$ and $g(x)$ on the domain $x \in \mathcal{X}$, we define their inner product as $\langle f, g\rangle = \int_{x\in\mathcal{X}} f(x)g(x)dx$. Let $\psi_k(.,.), \forall k = \{1,\ldots,\infty\}$ be a set of uniformly bounded, orthonormal basis functions such that $u_{ij}(x_i, x_j) = \sum_{k=0}^{\infty} \beta^*_{ijk}$ $\psi_k(x_i, x_j)$, where $\beta^*_{ijk} = \langle u_{ij}, \psi_k\rangle$. We assume that for all $i, j \in \{1,\ldots,n\}, i \neq j$, the weight magnitudes $|\beta^*_{ijk}|, \forall k \in \{r+1,\ldots,\infty\}$ form a convergent series for a sufficiently large r. For example, for large enough r, Fourier coefficients of a periodic and twice continuously differentiable function form a convergent series [24]. For all $i, j \in \{1,\ldots,n\}$ we assume that $\sup_{x_i\in\mathcal{A}_i, x_j\in\mathcal{A}_j} |\psi_k(x_i, x_j)| \leqslant \overline{\psi}$. Outside the context of graphical games, a setup involving orthonormal basis functions was used by [25]. However, their setup considers additive combination of univariate functions, while we use additive combination of bivariate functions. Our setup also fundamentally differs in the generative model and the use of ℓ_1 regularizer. Even contextually, we go beyond the proof techniques and provide provable insights about the learned game.

2.3. Sampling Mechanism

Treating the outcomes of the game as "samples" observed across multiple "plays" of the same game is a recurring theme in the literature for learning games [19, 21, 23]. All of these works assume access to Nash equilibria and a noise mechanism. Noise could be added to each player's strategy at a local level or by mixing the sample with a non-Nash equilibria set at a global level. Since none of the prior literature has dealt with either continuous actions or non-parametric utilities, we propose a novel sampling mechanism. We assume that we have access to a set of joint actions $\mathbf{x}$ along with noisy black-box access to their payoffs. For a joint action x, the blackbox outputs a noisy payoff $\widetilde{u}_i(x)$. The blackbox computes this noisy payoff by first computing noisy basis function values $\widetilde{\psi}_k(x_i, x_j)$ and then finally taking their weighted sum to get the final noisy payoff. Mathematically, $\widetilde{\psi}_k(x_i, x_j) = \psi_k(x_i, x_j) + \gamma_k(x_i, x_j)$. where $\gamma_k(x_i, x_j)$ are independent zero mean sub-Gaussian noise with variance proxy σ^2. The vectors $\boldsymbol{\psi}(x)$ and $\boldsymbol{\gamma}(x)$ collect $\psi_k(x_i, x_j)$ and $\gamma_k(x_i, x_j)$ indexed by $j \in \{1,\ldots,n\}, j \neq i$ and $k \in \{1,\ldots,r\}$. The class of sub-Gaussian random variables includes for instance Gaussian random variables, any bounded random variable (e.g. Bernoulli, multinomial, uniform), any random variable with strictly log-concave density, and any finite mixture of sub-Gaussian variables. Thus, using sub-Gaussian noise makes our setup quite general in nature. Correspondingly, the resulting noisy payoff function that we observe can be written as $\widetilde{u}_i(x) = \sum_{j\in \mathrm{S}_i} \widetilde{u}_{ij}(x_i, x_j) = \sum_{j\in\mathrm{S}_i}\sum_{k=0}^{\infty} \beta^*_{ijk}\widetilde{\psi}_k(x_i, x_j)$, where $\widetilde{u}_{ij}(x_i, x_j) = \sum_{k=0}^{\infty} \beta^*_{ijk}\widetilde{\psi}_k(x_i, x_j)$.

Estimation. Let $\overline{u}_{ij}(x_i, x_j)$ be our estimation of $u_{ij}(x_i, x_j)$. Ideally, we would like to estimate an infinitely long vector with entries indexed by (j, k) for $j \in \{1,\ldots,n\}, j \neq i$ and $k \in \{1,\ldots,\infty\}$ for every player i. However, this is impractical with finite computing resources. Rather, we estimate an $(n-1)\times r$ dimensional vector $\boldsymbol{\beta}$ with entries indexed by (j, k) for $j \in \{1,\ldots,n\}, j \neq i$ and $k \in \{1,\ldots,r\}$ for every player i. Using these finite number of coefficients, we estimate pairwise utility in the following manner, $\overline{u}_{ij}(x_i, x_j) = \sum_{k=0}^{r} \beta_{ijk}\psi_k(x_i, x_j)$ and thus, the resulting estimation of the utility function is $\overline{u}_i(x) = \sum_{j\in\mathrm{S}_i}\sum_{k=0}^{r} \beta_{ijk}\psi_k(x_i, x_j)$.

3. THEORETICAL RESULTS

In this section, we setup our estimation problem. We also mention some technical assumptions for our setup. Let D be a collection of N samples. We estimate $\boldsymbol{\beta}^*$ by solving the following optimization problem for some $\lambda > 0$:

$$\begin{aligned} &\min_{\beta} \frac{1}{N}\sum_{x\in D}(\widetilde{u}_i(x) - \overline{u}_i(x))^2 + \lambda\sum_{j\neq i}\sum_{k=0}^{r}|\beta_{ijk}| \\ &\text{such that } \sum_{j\neq i}\sum_{k=0}^{r}|\beta_{ijk}| \leqslant C\ , \end{aligned} \tag{2}$$

where $C > 0$ is a constant that acts as a budget on the coefficients β_{ijk} to ensure that they are not unbounded. We would like to prove that $\beta_{ijk} = 0, \forall j \notin \mathrm{S}_i$ and $\beta_{ijk} \neq 0, \forall j \in \mathrm{S}_i$. This gives us a straight forward way of picking in-neighbors of player i by solving optimization problem (2).

3.1. Assumptions

In this subsection, we will discuss the key technical assumptions which are required for our theoretical results. For notational clarity, we define the following two quantities: $\mathbf{H} \triangleq \frac{1}{N}\sum_{x\in D}\boldsymbol{\psi}(x)\boldsymbol{\psi}(x)^\intercal$ and $\widehat{\mathbf{H}} \triangleq \frac{1}{N}\sum_{x\in D}\big(\boldsymbol{\psi}(x)\boldsymbol{\psi}(x)^\intercal + \boldsymbol{\psi}(x)\boldsymbol{\gamma}(x)^\intercal\big)$. It should be noted that $\mathbf{H} = \mathrm{E}(\widehat{\mathbf{H}})$. For our first assumption, we want $\mathbf{H}$ restricted to in-neighbors to be an invertible matrix. Formally,

Assumption 1. *We assume that* $\mathbf{H}$ *is positive definite matrix, i.e.,* $\Lambda_{\min}([\mathbf{H}]_{\mathrm{S}_i\mathrm{S}_i}) = C_{\min} > 0$ *where* $\Lambda_{\min}$ *denotes the minimum eigenvalue.*

This assumption ensures that our optimization problem (2) has a unique solution and all our theoretical guarantees hold for this particular solution. As for our second assumption, we require that non-in-neighbors of a player do not affect the player's action too much. We achieve this by proposing a mutual incoherence assumption.

Assumption 2. *We assume that* $\||[\mathbf{H}]_{\mathrm{S}_i^c\mathrm{S}_i}[\mathbf{H}]^{-1}_{\mathrm{S}_i\mathrm{S}_i}\||_\infty \leqslant 1 - \alpha$ *for some* $0 < \alpha < 1$.

While mutual incoherence is new to graphical games, it has been a standard assumption in various estimation problems such as compressed sensing [26], Markov random fields [27], non-parametric regression [25], diffusion networks [28], among others. It should be noted that Assumptions 1 and 2 also hold in the sample setting with sufficient number of samples (See Lemma 2 and 3 in Appendix).

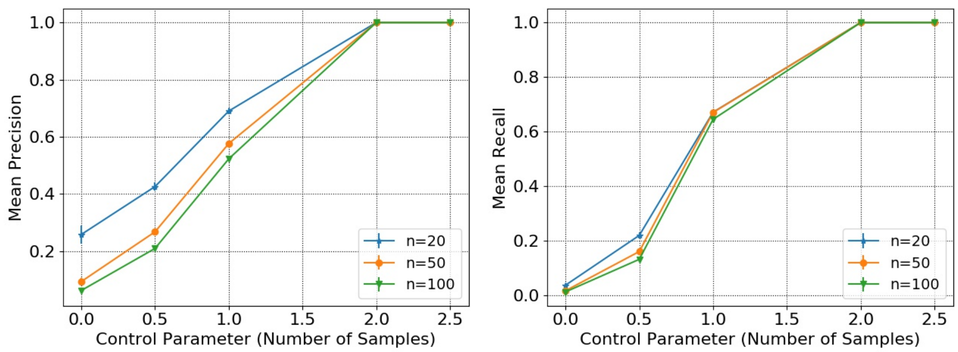

Fig. 1: Exact Structure Recovery in Non-parametric Games. Precision (Left) and Recall (Right) for recovered games across 30 runs against varying number of samples N. N is varied as $10^C \log(rn)$ where $r = 2$ and C is a control parameter.

Assumption 3. *We assume that* $\min_{j \in \mathrm{S}_i} |\beta^*_{ijk}| > \Delta(\lambda, C_{\min}, d, r, \alpha)$ *where* $\Delta(\lambda, C_{\min}, d, r, \alpha) = \frac{2}{C_{\min}}(\frac{\alpha}{24} + \sqrt{rd})\lambda$.

The minimum weight assumption is a standard practice in the literature employing the primal-dual witness technique [26, 27]. It ensures that the coefficients β^*_{ijk} are not arbitrarily close to zero which can make inference very difficult for any method.

3.2. Main Theorem

Here, we state our main result. We use primal-dual witness framework [26] to prove it (Details in Appendix A).

Theorem 1. *Consider a continuous-action graphical game* $\mathcal{G}$ *such that Assumptions 1, 2 and 3 are satisfied for each player. Furthermore, assume that the payoff function for each player is decomposable according to Equation* (1)*. Let* $\lambda > \frac{48\sqrt{2\bar{\psi}}\max(C, \sqrt{\delta})}{\alpha}\sqrt{\frac{d^2 \log(2n)}{N}}$ *and* $N = \Omega(\frac{\bar{\psi}^2\sigma^2 C^2 \delta r^3 d^3}{\epsilon^2}\log(rn))$, *if we solve the optimization problem* (2) *then with probability at least* $1 - \exp(-c\lambda^2 N)$ *for some positive constant c,*

1. *We recover the correct non-neighbors for each player.*
2. *We recover the exact structure of* $\mathcal{G}$.
3. *We estimate a payoff function which is* $\frac{\epsilon(\lambda, C_{\min}, d, \bar{\psi}, \delta, r, \alpha)}{2}$ *close to the true payoff by estimating* $\boldsymbol{\beta}$ *for each player* i *by solving the optimization problem* (2) *where* $\epsilon(\lambda, C_{\min}, d, \bar{\psi}, \delta, r, \alpha) = \frac{2}{C_{\min}}(\frac{\alpha}{24} + \sqrt{rd})\lambda\sqrt{d}\bar{\psi} + \bar{\psi}\delta$.
4. *Let* $\widehat{\mathcal{G}}$ *be the estimated game which uses estimated payoff functions then* $\mathrm{NE}(\widehat{\mathcal{G}}) \subseteq \mathrm{NE}_\epsilon(\mathcal{G})$ *and* $\mathrm{NE}(\mathcal{G}) \subseteq \mathrm{NE}_\epsilon(\widehat{\mathcal{G}})$.
5. *If* $\forall i \in \mathcal{V}, \forall (x_i, x_{\text{-}i}), (x'_i, x_{\text{-}i}) \in \mathcal{A}$ *such that* $(x_i, x_{\text{-}i}) \in \mathrm{NE}(\mathcal{G})$ *and* $(x'_i, x_{\text{-}i}) \notin \mathrm{NE}(\mathcal{G})$ *implies that* $u^*_i(x_i, x_{\text{-}i}) > u^*_i(x'_i, x_{\text{-}i}) + \epsilon$ *then* $\mathrm{NE}(\mathcal{G}) \subseteq \mathrm{NE}(\widehat{\mathcal{G}})$*. Furthermore, if* $\mathrm{NE}_\epsilon(\mathcal{G}) = \mathrm{NE}(\mathcal{G})$ *then* $\mathrm{NE}(\mathcal{G}) = \mathrm{NE}(\widehat{\mathcal{G}})$.

where $C_{\min} = \Lambda([\mathbf{H}]_{\mathrm{S}_i\mathrm{S}_i})$ *and* δ *is an arbitrary parameter which depends on* r.

In simple words, we show that by choosing an appropriate regularizer and with access to sufficient number of samples we can not only learn the structure of the graphical game but also estimate a game which is close to the true game.

4. EXPERIMENTAL RESULTS

In this section, we validate our theorem by running computational experiments on synthetic data. Our goal is to recover the correct in-neighbors of all the players, by using noisy utilities and basis functions. The performance of our method is measured using precision and recall. We defer the details of experiments to Appendix B. We provide results of our experiments in Figure 1. We can see that both precision and recall go to 1 as we increase the number of samples, obtaining perfect recovery with enough samples. Notice that the different curves for different number of players ($n = 20, 50$ and 100) line up with one another quite well. This matches with our theoretical results and shows that with a constant number of in-neighbors $\Omega(\log(rn))$ samples are sufficient to recover the exact structure of the graphical games.

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A. PROOF OF THEOREM 1

In this section, we provide detailed proof of our main result. We start by proving an auxiliary lemma which is important for our final result.

A.1. Proof of Lemma 1

We use an auxiliary variable $w = \sum_{j \neq i} \sum_{k=0}^{r} |\beta_{ijk}|$ in optimization problem (2) and prove the following lemma to get Karush-Kuhn-Tucker (KKT) conditions at the optimum.

Lemma 1 (KKT conditions)**.** *The following Karush-Kuhn-Tucker (KKT) conditions hold at the optimal solution of the optimization problem* (2)*:*

$$\begin{aligned}
&\textit{Stationarity: } \frac{\partial}{\partial \boldsymbol{\beta}} \Big[\frac{1}{N} \sum_{x \in D} (\widetilde{u}(x) - \overline{u}(x))^2 \Big] + \lambda \mathbf{z} = 0 \ , \\
&\textit{Primal Feasibility: } w \leqslant C, \quad w = \sum_{j \neq i} \sum_{k=0}^{r} |\beta_{ijk}| \ , \\
&\textit{Dual Feasibility: } \lambda \geqslant 0 \ ,
\end{aligned}$$

where $\mathbf{z} \in \mathbb{R}^{r(n-1)\times 1}$ *indexed by* $(j,k), j \in \{1,\ldots,n\}, j \neq i, k \in \{1,\ldots,r\}$. *It is defined as follows:*

$$z_{ijk} = \begin{cases} \text{sign}(\beta_{ijk}), \ \textit{if } \beta_{ijk} \neq 0 \\ [-1,1], \ \textit{otherwise} \end{cases}$$

Proof. Equivalently,

$$\begin{aligned}
&\min_{\beta} \frac{1}{N} \sum_{x \in D} (\widetilde{u}(x) - \overline{u}(x))^2 + \lambda w \\
&\text{such that } w \leqslant C \\
&\qquad w = \sum_{j \neq i} \sum_{k=0}^{r} |\beta_{ijk}|
\end{aligned}$$

The Lagrangian for the above can be written as,

$$L(\boldsymbol{\beta}, w; \mu, \eta) = \frac{1}{N} \sum_{x \in D} (\widetilde{u}(x) - \overline{u}(x))^2 + \lambda w + \mu(w - C) + \eta(w - \sum_{j \neq i} \sum_{k=0}^{r} |\beta_{ijk}|) \ , \tag{3}$$

where $\mu \geqslant 0$. Alternatively,

$$L(\boldsymbol{\beta}, w; \mu, \eta) = \frac{1}{N} \sum_{x \in D} (\widetilde{u}(x) - \overline{u}(x))^2 + \lambda w + \mu(w - C) + \eta(w - \mathbf{z}^{\mathsf{T}} \boldsymbol{\beta}) \ . \tag{4}$$

Here, $\boldsymbol{\beta} \in \mathbb{R}^{r(n-1)\times 1}, \mathbf{z} \in \mathbb{R}^{r(n-1)\times 1}$ with all of them indexed by $(j,k), j \in \{1,\ldots,n\}, j \neq i, k \in \{1,\ldots,r\}$. Vector $\mathbf{z} \in \mathbb{R}^{r(n-1)\times 1}$ is defined as follows:

$$z_{ijk} = \begin{cases} \text{sign}(\beta_{ijk}), \ \text{if } \beta_{ijk} \neq 0 \\ [-1,1], \ \text{otherwise} \end{cases}$$

By writing the KKT conditions from the Lagrangian in Equation (4), a solution is optimal if and only if:

$$
\begin{aligned}
\text{Stationarity: } & \frac{\partial}{\partial \boldsymbol{\beta}}\Big[\frac{1}{N}\sum_{x\in D}(\widetilde{u}(x)-\overline{u}(x))^2\Big]-\eta\mathbf{z}=0 \\
& \lambda+\mu+\eta=0 \\
\text{Complimentarity: } & \mu(w-C)=0 \\
\text{Primal Feasibility: } & w\leqslant C \\
& w=\sum_{k=0}^{r}|\beta_{ijk}| \\
\text{Dual Feasibility: } & \mu\geqslant 0
\end{aligned} \tag{5}
$$

If we choose $\mu=0$, then the following KKT conditions must hold at the optimal solution:

$$
\begin{aligned}
\text{Stationarity: } & \frac{\partial}{\partial \boldsymbol{\beta}}\Big[\frac{1}{N}\sum_{x\in D}(\widetilde{u}(x)-\overline{u}(x))^2\Big]+\lambda\mathbf{z}=0 \\
\text{Primal Feasibility: } & w\leqslant C \\
& w=\sum_{j\neq i}\sum_{k=0}^{r}|\beta_{ijk}|
\end{aligned}
$$

□

Next we will prove that with sufficient number of samples, Assumptions 1 and 2 hold in the sample setting.

A.2. Proof of Lemma 2

Lemma 2. *If* $\Lambda_{\min}(\big[\mathbf{H}\big]_{\mathrm{S}_i\mathrm{S}_i})=C_{\min}>0$ *then* $\Lambda_{\min}(\big[\widehat{\mathbf{H}}\big]_{\mathrm{S}_i\mathrm{S}_i})\geqslant C_{\min}-\epsilon$ *for some* $\epsilon>0$ *with probability at least* $\exp(\frac{-N\epsilon^2}{2\mathrm{b}^2\sigma^2}+\mathcal{O}(rd))$.

Proof. Let $\hat{C}_{\min}$ is minimum eigenvalue of $\widehat{\mathbf{H}}_{\mathrm{S}_i\mathrm{S}_i}$. Then,

$$
\begin{aligned}
\hat{C}_{\min} &= \min_{y,\|y\|_2=1} y^\intercal\widehat{\mathbf{H}}_{\mathrm{S}_i\mathrm{S}_i}y \\
&= \min_{y,\|y\|_2=1} y^\intercal\mathbf{H}y + y^\intercal\frac{1}{N}\sum_{x\in D}[\psi(x)\gamma(x)^\intercal]_{\mathrm{S}_i\mathrm{S}_i}y \\
&\geqslant C_{\min} + \min_{y,\|y\|_2=1} y^\intercal\frac{1}{N}\sum_{x\in D}[\psi(x)\gamma(x)^\intercal]_{\mathrm{S}_i\mathrm{S}_i}y \ .
\end{aligned}
$$

For some y such that $\|y\|_2=1$, we have a random variable $R\triangleq y^\intercal\frac{1}{N}\sum_{x\in D}[\psi(x)\gamma(x)^\intercal]_{\mathrm{S}_i\mathrm{S}_i}y$, then R is a zero mean sub-Gaussian random variable with variance proxy $\frac{\sum_{s=1}^{N}a_s^2\sigma^2}{N^2}$. Then using the tail bound for sub-Gaussian random variables, we can write

$$
\Pr(R\leqslant-\epsilon)\leqslant\exp\Big(\frac{-\epsilon^2}{2\frac{\sum_{s=1}^{N}a_s^2\sigma^2}{N^2}}\Big)\ .
$$

We assume that $\|\psi\|_2\leqslant \mathrm{b}$, then $\max_{y,\|y\|_2=1}a_s\leqslant\mathrm{b}$. Thus,

$$
\Pr(R\leqslant-\epsilon)\leqslant\exp\Big(\frac{-N\epsilon^2}{2\mathrm{b}^2\sigma^2}\Big)\ .
$$

Using the ϵ-nets argument and taking a union bound across $\exp(\mathcal{O}(rd))$ y of the net, we can write

$$
\Pr(R\leqslant-\epsilon)\leqslant\exp\Big(\frac{-N\epsilon^2}{2\mathrm{b}^2\sigma^2}+\mathcal{O}(rd)\Big)\ .
$$

Clearly, if $N=\mathcal{O}(\frac{8\mathrm{b}^2\sigma^2}{C_{\min}^2}rd)$ we have $\hat{C}_{\min}\geqslant\frac{C_{\min}}{2}$ with high probability. □

A.3. Proof of Lemma 3

Lemma 3. *If* $|||[\mathbf{H}]_{S_i^c S_i}[\mathbf{H}]_{S_i S_i}^{-1}|||_\infty \leqslant 1-\alpha$ *for some* $0<\alpha<1$ *then* $|||[\widehat{\mathbf{H}}]_{S_i^c S_i}[\widehat{\mathbf{H}}]_{S_i S_i}^{-1}|||_\infty \leqslant 1-\frac{\alpha}{2}$ *with probability at least* $1-\exp(\frac{-KNC_{\min}^2\alpha^2}{(1-\alpha)^2 r^3 d^3 \overline{\psi}^2\sigma^2}+\log(2r^2(n-d)d))-\exp(\frac{-N\alpha}{6r^2d^2\overline{\psi}^2\sigma^2}+\log(2r^2(n-d)d))-\exp(\frac{-2N}{\mathsf{b}^2\sigma^2 C_{\min}^2}+\mathcal{O}(rd))-2\exp(\frac{-N\alpha^2}{24r^3d^3\overline{\psi}^2\sigma^2}+\log(r^2d^2))$ *for a constant* $K>0$*.*

Proof. We start the proof by first proving one auxiliary lemma.

Lemma 4. *For any* $\delta>0$*, the following holds:*

$$\Pr(|||[\widehat{\mathbf{H}}]_{S_i^c S_i}-[\mathbf{H}]_{S_i^c S_i}|||_\infty \geqslant \delta) \leqslant \exp(\frac{-N\epsilon^2}{r^2d^2\overline{\psi}^2\sigma^2}+\log(2r^2(n-d)d)) \tag{6}$$

$$\Pr(|||[\widehat{\mathbf{H}}]_{S_i S_i}-[\mathbf{H}]_{S_i S_i}|||_\infty \geqslant \delta) \leqslant \exp(-\frac{-N\epsilon^2}{r^2d^2\overline{\psi}^2\sigma^2}+\log(2r^2d^2)) \tag{7}$$

$$\Pr(|||[\widehat{\mathbf{H}}]_{S_i S_i}^{-1}-[\mathbf{H}]_{S_i S_i}^{-1}|||_\infty \geqslant \delta) \leqslant \exp(\frac{-2N}{\mathsf{b}^2\sigma^2C_{\min}^2}+\mathcal{O}(rd))+2\exp(\frac{-N\delta^2}{4r^3d^3\overline{\psi}^2\sigma^2}+\log(r^2d^2)) \tag{8}$$

Proof. Note that,

$$[[\widehat{\mathbf{H}}]_{S_i^c S_i}-[\mathbf{H}]_{S_i^c S_i}]_{jk}=[\frac{1}{N}\sum_{x\in D}[\psi(x)\gamma(x)^\intercal]_{S_i^c S_i}]_{jk}\ .$$

We further note that the random variable $[\frac{1}{N}\sum_{x\in D}[\psi(x)\gamma(x)^\intercal]_{S_i^c S_i}]_{jk}$ is a zero mean sub-Gaussian random variable with variance proxy $\frac{\sigma^2\sum_{x\in D}\psi(x)_j^2}{N^2}$. Thus,

$$\Pr(|[\frac{1}{N}\sum_{x\in D}[\psi(x)\gamma(x)^\intercal]_{S_i^c S_i}]_{jk}|\geqslant\epsilon)\leqslant 2\exp(\frac{-\epsilon^2}{\frac{\sigma^2\sum_{x\in D}\psi(x)_j^2}{N^2}})\leqslant 2\exp(-\frac{-N\epsilon^2}{\overline{\psi}^2\sigma^2})\ .$$

Taking $\epsilon=\frac{\epsilon}{rd}$, we have

$$\Pr(|[\frac{1}{N}\sum_{x\in D}[\psi(x)\gamma(x)^\intercal]_{S_i^c S_i}]_{jk}|\geqslant\frac{\epsilon}{rd})\leqslant 2\exp(-\frac{-N\epsilon^2}{r^2d^2\overline{\psi}^2\sigma^2})\ .$$

We observe that,

$$|||[\widehat{\mathbf{H}}]_{S_i^c S_i}-[\mathbf{H}]_{S_i^c S_i}|||_\infty=\max_{j\in S_i^c}\sum_{k\in S_i}|[\frac{1}{N}\sum_{x\in D}[\psi(x)\gamma(x)^\intercal]_{S_i^c S_i}]_{jk}\ .$$

Thus, taking a union bound across $j\in S_i^c$ and $k\in S_i$, we get

$$\begin{aligned}\Pr(|||[\widehat{\mathbf{H}}]_{S_i^c S_i}-[\mathbf{H}]_{S_i^c S_i}|||_\infty\geqslant\epsilon)&\leqslant 2r^2(n-d)d\exp(-\frac{-N\epsilon^2}{r^2d^2\overline{\psi}^2\sigma^2})\\&=\exp(\frac{-N\epsilon^2}{r^2d^2\overline{\psi}^2\sigma^2}+\log(2r^2(n-d)d))\ .\end{aligned}$$

Similarly,

$$|||[\widehat{\mathbf{H}}]_{S_i S_i}-[\mathbf{H}]_{S_i S_i}|||_\infty=\max_{j\in S_i}\sum_{k\in S_i}|[\frac{1}{N}\sum_{x\in D}[\psi(x)\gamma(x)^\intercal]_{S_i S_i}]_{jk}\ .$$

It follows that,

$$\begin{aligned}\Pr(\||[\widehat{\mathbf{H}}]_{\mathrm{S}_i\mathrm{S}_i} - [\mathbf{H}]_{\mathrm{S}_i\mathrm{S}_i}\||_\infty \geqslant \epsilon) &\leqslant 2r^2d^2\exp(\frac{-N\epsilon^2}{r^2d^2\overline{\psi}^2\sigma^2}) \\ &= \exp(-\frac{-N\epsilon^2}{r^2d^2\overline{\psi}^2\sigma^2} + \log(2r^2d^2)) \ .\end{aligned}$$

Now,

$$\begin{aligned}\||[\widehat{\mathbf{H}}]^{-1}_{\mathrm{S}_i\mathrm{S}_i} - [\mathbf{H}]^{-1}_{\mathrm{S}_i\mathrm{S}_i}\||_\infty &= \||[\mathbf{H}]^{-1}_{\mathrm{S}_i\mathrm{S}_i}[[\mathbf{H}]_{\mathrm{S}_i\mathrm{S}_i} - [\widehat{\mathbf{H}}]_{\mathrm{S}_i\mathrm{S}_i}][\widehat{\mathbf{H}}]^{-1}_{\mathrm{S}_i\mathrm{S}_i}\||_\infty \\ &\leqslant \sqrt{dr}\||[\mathbf{H}]^{-1}_{\mathrm{S}_i\mathrm{S}_i}[[\mathbf{H}]_{\mathrm{S}_i\mathrm{S}_i} - [\widehat{\mathbf{H}}]_{\mathrm{S}_i\mathrm{S}_i}][\widehat{\mathbf{H}}]^{-1}_{\mathrm{S}_i\mathrm{S}_i}\||_2 \\ &\leqslant \sqrt{dr}\||[\mathbf{H}]^{-1}_{\mathrm{S}_i\mathrm{S}_i}\||_2\||[[\mathbf{H}]_{\mathrm{S}_i\mathrm{S}_i} - [\widehat{\mathbf{H}}]_{\mathrm{S}_i\mathrm{S}_i}]\||_2\||[\widehat{\mathbf{H}}]^{-1}_{\mathrm{S}_i\mathrm{S}_i}\||_2 \\ &\leqslant \frac{\sqrt{dr}}{C_{\min}}\||[[\mathbf{H}]_{\mathrm{S}_i\mathrm{S}_i} - [\widehat{\mathbf{H}}]_{\mathrm{S}_i\mathrm{S}_i}]\||_2\||[\widehat{\mathbf{H}}]^{-1}_{\mathrm{S}_i\mathrm{S}_i}\||_2 \ .\end{aligned}$$

We have shown that,

$$\Pr(\||[\widehat{\mathbf{H}}]^{-1}_{\mathrm{S}_i\mathrm{S}_i}\||_2 \geqslant \frac{2}{C_{\min}}) \leqslant \exp(\frac{-\epsilon^2}{2\mathrm{b}^2\sigma^2} + \mathcal{O}(rd)) \ .$$

Moreover,

$$\Pr(\||[[\mathbf{H}]_{\mathrm{S}_i\mathrm{S}_i} - [\widehat{\mathbf{H}}]_{\mathrm{S}_i\mathrm{S}_i}]\||_2 \geqslant \epsilon) \leqslant 2\exp(\frac{-N\epsilon^2}{r^2d^2\overline{\psi}^2\sigma^2} + \log(r^2d^2)) \ .$$

Taking $\epsilon = \frac{\delta}{2\sqrt{dr}}$, we have

$$\Pr(\||[[\mathbf{H}]_{\mathrm{S}_i\mathrm{S}_i} - [\widehat{\mathbf{H}}]_{\mathrm{S}_i\mathrm{S}_i}]\||_2 \geqslant \frac{\delta}{2\sqrt{dr}}) \leqslant 2\exp(\frac{-N\delta^2}{4r^3d^3\overline{\psi}^2\sigma^2} + \log(r^2d^2)) \ .$$

and furthermore

$$\Pr(\||[\widehat{\mathbf{H}}]^{-1}_{\mathrm{S}_i\mathrm{S}_i} - [\mathbf{H}]^{-1}_{\mathrm{S}_i\mathrm{S}_i}\||_\infty \geqslant \delta) \leqslant \exp(\frac{-2N}{\mathrm{b}^2\sigma^2C_{\min}^2} + \mathcal{O}(rd)) + 2\exp(\frac{-N\delta^2}{4r^3d^3\overline{\psi}^2\sigma^2} + \log(r^2d^2)) \ .$$

□

Now we are ready to prove the main lemma.

$$[\widehat{\mathbf{H}}]_{\mathrm{S}_i^c\mathrm{S}_i}[\widehat{\mathbf{H}}]^{-1}_{\mathrm{S}_i\mathrm{S}_i} = T_1 + T_2 + T_3 + T_4 \ ,$$

where

$$\begin{aligned}T_1 &\triangleq [\mathbf{H}]_{\mathrm{S}_i^c\mathrm{S}_i}[[\widehat{\mathbf{H}}]^{-1}_{\mathrm{S}_i\mathrm{S}_i} - [\mathbf{H}]^{-1}_{\mathrm{S}_i\mathrm{S}_i}] \ , \\ T_2 &\triangleq [[\widehat{\mathbf{H}}]_{\mathrm{S}_i^c\mathrm{S}_i} - [\mathbf{H}]_{\mathrm{S}_i^c\mathrm{S}_i}][\mathbf{H}]^{-1}_{\mathrm{S}_i\mathrm{S}_i} \ , \\ T_3 &\triangleq [[\widehat{\mathbf{H}}]_{\mathrm{S}_i^c\mathrm{S}_i} - [\mathbf{H}]_{\mathrm{S}_i^c\mathrm{S}_i}][[\widehat{\mathbf{H}}]^{-1}_{\mathrm{S}_i\mathrm{S}_i} - [\mathbf{H}]^{-1}_{\mathrm{S}_i\mathrm{S}_i}] \ , \\ T_4 &\triangleq [\mathbf{H}]_{\mathrm{S}_i^c\mathrm{S}_i}[\mathbf{H}]^{-1}_{\mathrm{S}_i\mathrm{S}_i} \ .\end{aligned}$$

We know that $\||T_4\||_\infty \leqslant 1 - \alpha$. Now,

$$\begin{aligned}\||T_1\||_\infty &= \||[\mathbf{H}]_{\mathrm{S}_i^c\mathrm{S}_i}[[\widehat{\mathbf{H}}]^{-1}_{\mathrm{S}_i\mathrm{S}_i} - [\mathbf{H}]^{-1}_{\mathrm{S}_i\mathrm{S}_i}]\||_\infty \\ &\leqslant \||[\mathbf{H}]_{\mathrm{S}_i^c\mathrm{S}_i}[\mathbf{H}]^{-1}_{\mathrm{S}_i\mathrm{S}_i}\||_\infty\||[\widehat{\mathbf{H}}]_{\mathrm{S}_i^c\mathrm{S}_i} - [\mathbf{H}]_{\mathrm{S}_i^c\mathrm{S}_i}\||_\infty\||[\widehat{\mathbf{H}}]^{-1}_{\mathrm{S}_i\mathrm{S}_i}\||_\infty \\ &\leqslant (1-\alpha)\||[\widehat{\mathbf{H}}]_{\mathrm{S}_i^c\mathrm{S}_i} - [\mathbf{H}]_{\mathrm{S}_i^c\mathrm{S}_i}\||_\infty\sqrt{rd}\|[\widehat{\mathbf{H}}]_{\mathrm{S}_i\mathrm{S}_i}\|_2 \\ &\leqslant (1-\alpha)\||[\widehat{\mathbf{H}}]_{\mathrm{S}_i^c\mathrm{S}_i} - [\mathbf{H}]_{\mathrm{S}_i^c\mathrm{S}_i}\||_\infty\sqrt{rd}\frac{2}{C_{\min}} \\ &\leqslant \frac{\alpha}{6} \ .\end{aligned}$$

The last step holds with probability at least $1 - \exp(\frac{-NC_{\min}^2\alpha^2}{144(1-\alpha)^2 r^3 d^3 \psi^2 \sigma^2} + \log(2r^2(n-d)d))$ by choosing $\delta = \frac{C_{\min}\alpha}{12(1-\alpha)\sqrt{rd}}$ in equation (6).

For the second term,

$$\begin{aligned} |||T_2|||_\infty &\leqslant \sqrt{rd}|||[\mathbf{H}]_{\mathrm{S}_i\mathrm{S}_i}^{-1}|||_2|||[[\widehat{\mathbf{H}}]_{\mathrm{S}_i^c\mathrm{S}_i} - [\mathbf{H}]_{\mathrm{S}_i^c\mathrm{S}_i}]|||_\infty \\ &\leqslant \frac{\sqrt{rd}}{C_{\min}}|||[[\widehat{\mathbf{H}}]_{\mathrm{S}_i^c\mathrm{S}_i} - [\mathbf{H}]_{\mathrm{S}_i^c\mathrm{S}_i}]|||_\infty \\ &\leqslant \frac{\alpha}{6} \ . \end{aligned}$$

The last step holds with probability at least $1 - \exp(\frac{-NC_{\min}^2\alpha^2}{36 r^3 d^3 \psi^2 \sigma^2} + \log(2r^2(n-d)d))$ by choosing $\delta = \frac{C_{\min}\alpha}{6\sqrt{rd}}$.

For the third term,

$$\begin{aligned} |||T_3|||_\infty &\leqslant |||[[\widehat{\mathbf{H}}]_{\mathrm{S}_i^c\mathrm{S}_i} - [\mathbf{H}]_{\mathrm{S}_i^c\mathrm{S}_i}]|||_\infty |||[[\widehat{\mathbf{H}}]_{\mathrm{S}_i\mathrm{S}_i}^{-1} - [\mathbf{H}]_{\mathrm{S}_i\mathrm{S}_i}^{-1}]|||_\infty \\ &\leqslant \frac{\alpha}{6} \ . \end{aligned}$$

The last step holds with probability at least $1 - \exp(\frac{-N\alpha}{6r^2d^2\psi^2\sigma^2} + \log(2r^2(n-d)d)) - \exp(\frac{-2N}{\mathrm{b}^2\sigma^2 C_{\min}^2} + \mathcal{O}(rd)) - 2\exp(\frac{-N\alpha^2}{24r^3d^3\psi^2\sigma^2} + \log(r^2d^2))$ by choosing $\delta = \sqrt{\frac{\alpha}{6}}$ in equations (6) and (8).

Combining all the terms together, we get the final result. $\square$

Now we are ready to begin the proof of our main results. We note that $\beta_{ijk}^* = 0, \forall j \neq \mathrm{S}_i$. We also assume that $\beta_{ijk} = 0, \forall j \neq \mathrm{S}_i$. As first step of our proof, we will justify this choice. Then we bound the Euclidean norm distance between the estimated $\boldsymbol{\beta}$ and the true $\boldsymbol{\beta}^*$. Subsequently, we show that if the non-zero entries in $\boldsymbol{\beta}^*$ satisfy a minimum weight criteria then the recovered $\boldsymbol{\beta}$ matches $\boldsymbol{\beta}^*$ up to its sign. This allows us to identify the in-neighbors for each player.

By doing some simple algebraic manipulation it is easy to see that,

$$\frac{1}{N}\sum_{x\in D}(\widetilde{u}_i(x) - \overline{u}_i(x))^2 = \frac{1}{N}\sum_{x\in D}((\boldsymbol{\beta}^* - \boldsymbol{\beta})^\intercal \widetilde{\boldsymbol{\psi}} + \sum_{j\neq i}\sum_{k=r+1}^{\infty}\beta_{ijk}^*\widetilde{\psi}_k(x_i,x_j) + \boldsymbol{\beta}^\intercal\boldsymbol{\gamma}(x))^2 \ .$$

Here, $\boldsymbol{\beta}^* \in \mathbb{R}^{r(n-1)\times 1}, \widetilde{\boldsymbol{\psi}} \in \mathbb{R}^{r(n-1)\times 1}$ and $\boldsymbol{\gamma} \in \mathbb{R}^{r(n-1)\times 1}$ are indexed by $(j,k), j \in \{1,\dots,n\}, j \neq i, k \in \{1,\dots,r\}$. Thus Stationarity KKT condition of Equation (2) becomes:

$$\frac{2}{N}(\sum_{x\in D}(\boldsymbol{\gamma} - \widetilde{\boldsymbol{\psi}}(x))(\widetilde{\boldsymbol{\psi}}(x)^\intercal(\boldsymbol{\beta}^* - \boldsymbol{\beta}) + \sum_{j\neq i}\sum_{k=r+1}^{\infty}\beta_{ijk}^*\widetilde{\psi}_k(x_i,x_j) + \boldsymbol{\gamma}^\intercal(x)\boldsymbol{\beta})) + \lambda\mathbf{z} = 0 \ .$$

Let $\boldsymbol{\beta} = \begin{bmatrix}\boldsymbol{\beta}_{\mathrm{S}_i}\\ \boldsymbol{\beta}_{\mathrm{S}_i^c}\end{bmatrix}$, where $\boldsymbol{\beta}_{\mathrm{S}_i}$ contains the entries $\beta_{ijk} \neq 0$ and $\boldsymbol{\beta}_{\mathrm{S}_i^c}$ contains the entries $\beta_{ijk} = 0$. From the above, we have

$$\frac{2}{N}(\sum_{x\in D}(\boldsymbol{\gamma} - \widetilde{\boldsymbol{\psi}}(x))(\widetilde{\boldsymbol{\psi}}_{\mathrm{S}_i}(x)^\intercal(\boldsymbol{\beta}_{\mathrm{S}_i}^* - \boldsymbol{\beta}_{\mathrm{S}_i}) + \sum_{j\neq i}\sum_{k=r+1}^{\infty}\beta_{ijk}^*\widetilde{\psi}_k(x_i,x_j) + \boldsymbol{\gamma}_{\mathrm{S}_i}^\intercal(x)\boldsymbol{\beta}_{\mathrm{S}_i})) + \lambda\mathbf{z} = 0 \ .$$

Separating the indices for the support (in-neighbors) and the non-support, we can write the above equation in two parts. The first for the indices in the support,

$$\frac{2}{N}(\sum_{x\in D} -\boldsymbol{\psi}_{\mathrm{S}_i}(x)(\widetilde{\boldsymbol{\psi}}_{\mathrm{S}_i}(x)^\intercal(\boldsymbol{\beta}_{\mathrm{S}_i}^* - \boldsymbol{\beta}_{\mathrm{S}_i}) + \sum_{j\neq i}\sum_{k=r+1}^{\infty}\beta_{ijk}^*\widetilde{\psi}_k(x_i,x_j) + \boldsymbol{\gamma}_{\mathrm{S}_i}^\intercal(x)\boldsymbol{\beta}_{\mathrm{S}_i})) + \lambda\mathbf{z}_{\mathrm{S}_i} = 0 \tag{9}$$

and the second for the indices in the non-support,

$$\frac{2}{N}(\sum_{x\in D} -\boldsymbol{\psi}_{\mathrm{S}_i^c}(x)(\widetilde{\boldsymbol{\psi}}_{\mathrm{S}_i}(x)^\intercal(\boldsymbol{\beta}_{\mathrm{S}_i}^* - \boldsymbol{\beta}_{\mathrm{S}_i}) + \sum_{j\neq i}\sum_{k=r+1}^{\infty}\beta_{ijk}^*\widetilde{\psi}_k(x_i,x_j) + \boldsymbol{\gamma}_{\mathrm{S}_i}^\intercal(x)\boldsymbol{\beta}_{\mathrm{S}_i})) + \lambda\mathbf{z}_{\mathrm{S}_i^c} = 0 \ . \tag{10}$$

We make use of Assumption 1 and rearrange Equation (9) to get,

$$
\begin{aligned}
\boldsymbol{\beta}^*_{\mathrm{S}_i} - \boldsymbol{\beta}_{\mathrm{S}_i} = &-(\frac{1}{N}\sum_{x\in D}\boldsymbol{\psi}_{\mathrm{S}_i}(x)\widetilde{\boldsymbol{\psi}}_{\mathrm{S}_i}(x)^\intercal)^{-1}(\frac{1}{N}\sum_{x\in D}\boldsymbol{\psi}_{\mathrm{S}_i}(x)\sum_{j\neq i}\sum_{k=r+1}^{\infty}\beta^*_{ijk}\widetilde{\psi}_k(x_i,x_j))- \\
&(\sum_{x\in D}\boldsymbol{\psi}_{\mathrm{S}_i}(x)\widetilde{\boldsymbol{\psi}}_{\mathrm{S}_i}(x)^\intercal)^{-1}(\sum_{x\in D}\boldsymbol{\psi}_{\mathrm{S}_i}(x)\boldsymbol{\gamma}^\intercal_{\mathrm{S}_i}(x)\boldsymbol{\beta}_{\mathrm{S}_i}) + \frac{\lambda}{2}(\frac{1}{N}\sum_{x\in D}\boldsymbol{\psi}_{\mathrm{S}_i}(x)\widetilde{\boldsymbol{\psi}}_{\mathrm{S}_i}(x)^\intercal)^{-1}\mathbf{z}_{\mathrm{S}_i}\ .
\end{aligned} \tag{11}
$$

Substituting Equation (11) in Equation (10) and rearranging the terms we get,

$$
\begin{aligned}
\frac{\lambda}{2}\mathbf{z}_{\mathrm{S}^c_i} = &-[\widehat{\mathbf{H}}]_{\mathrm{S}^c_i\mathrm{S}_i}[\widehat{\mathbf{H}}]^{-1}_{\mathrm{S}_i\mathrm{S}_i}(\frac{1}{N}\sum_{y\in D}\boldsymbol{\psi}_{\mathrm{S}_i}(y)\sum_{j\neq i}\sum_{k=r+1}^{\infty}\beta^*_{ijk}\widetilde{\psi}(y_i,y_j)) - [\widehat{\mathbf{H}}]_{\mathrm{S}^c_i\mathrm{S}_i}[\widehat{\mathbf{H}}]^{-1}_{\mathrm{S}_i\mathrm{S}_i}(\frac{1}{N}\sum_{y\in D}\boldsymbol{\psi}_{\mathrm{S}_i}(y) \\
&\boldsymbol{\gamma}^\intercal_{\mathrm{S}_i}(y)\boldsymbol{\beta}_{\mathrm{S}_i}) + \frac{\lambda}{2}[\widehat{\mathbf{H}}]_{\mathrm{S}^c_i\mathrm{S}_i}[\widehat{\mathbf{H}}]^{-1}_{\mathrm{S}_i\mathrm{S}_i}\mathbf{z}_{\mathrm{S}_i} + \frac{1}{N}\sum_{x\in D}\boldsymbol{\psi}_{\mathrm{S}^c_i}(x)\sum_{j\neq i}\sum_{k=r+1}^{\infty}\beta^*_{ijk}\widetilde{\psi}_k(x_i,x_j)+ \\
&\frac{1}{N}\sum_{x\in D}\boldsymbol{\psi}_{\mathrm{S}^c_i}(x)\boldsymbol{\gamma}^\intercal_{\mathrm{S}_i}(x)\boldsymbol{\beta}_{\mathrm{S}_i}\ .
\end{aligned}
$$

Using the triangle norm inequality and noting that $\|Ab\|_\infty \leqslant |||A|||_\infty\|b\|_\infty$, we obtain

$$
\begin{aligned}
\frac{\lambda}{2}\|\mathbf{z}_{\mathrm{S}^c_i}\|_\infty \leqslant &|||[\widehat{\mathbf{H}}]_{\mathrm{S}^c_i\mathrm{S}_i}[\widehat{\mathbf{H}}]^{-1}_{\mathrm{S}_i\mathrm{S}_i}|||_\infty\|(\frac{1}{N}\sum_{y\in D}\boldsymbol{\psi}_{\mathrm{S}_i}(y)\sum_{j\neq i}\sum_{k=r+1}^{\infty}\beta^*_{ijk}\widetilde{\psi}(y_i,y_j))\|_\infty + |||[\widehat{\mathbf{H}}]_{\mathrm{S}^c_i\mathrm{S}_i}[\widehat{\mathbf{H}}]^{-1}_{\mathrm{S}_i\mathrm{S}_i}|||_\infty \\
&\|\frac{1}{N}(\sum_{y\in D}\boldsymbol{\psi}_{\mathrm{S}_i}(y)\boldsymbol{\gamma}^\intercal_{\mathrm{S}_i}(y)\boldsymbol{\beta}_{\mathrm{S}_i})\|_\infty + \frac{\lambda}{2}|||[\widehat{\mathbf{H}}]_{\mathrm{S}^c_i\mathrm{S}_i}[\widehat{\mathbf{H}}]^{-1}_{\mathrm{S}_i\mathrm{S}_i}|||_\infty\|\mathbf{z}_{\mathrm{S}_i}\|_\infty + \frac{1}{N}\|\sum_{x\in D}\boldsymbol{\psi}_{\mathrm{S}^c_i}(x) \\
&\sum_{j\neq i}\sum_{k=r+1}^{\infty}\beta^*_{ijk}\widetilde{\psi}_k(x_i,x_j)\|_\infty + \frac{1}{N}\|\sum_{x\in D}\boldsymbol{\psi}_{\mathrm{S}^c_i}(x)\boldsymbol{\gamma}^\intercal_{\mathrm{S}_i}(x)\boldsymbol{\beta}_{\mathrm{S}_i}\|_\infty\ .
\end{aligned}
$$

Using Assumption 2 and noting that $\|\mathbf{z}_{\mathrm{S}_i}\|_\infty \leqslant 1$, we have

$$
\begin{aligned}
\|\mathbf{z}_{\mathrm{S}^c_i}\|_\infty \leqslant &(1-\frac{\alpha}{2})\|\frac{2}{\lambda}\frac{1}{N}(\sum_{y\in D}\boldsymbol{\psi}_{\mathrm{S}_i}(y)\sum_{j\neq i}\sum_{k=r+1}^{\infty}\beta^*_{ijk}\widetilde{\psi}(y_i,y_j))\|_\infty + (1-\frac{\alpha}{2})\|\frac{2}{\lambda}\frac{1}{N}(\sum_{y\in D}\boldsymbol{\psi}_{\mathrm{S}_i}(y)\boldsymbol{\gamma}^\intercal_{\mathrm{S}_i}(y) \\
&\boldsymbol{\beta}_{\mathrm{S}_i})\|_\infty + (1-\frac{\alpha}{2}) + \|\frac{2}{\lambda}\frac{1}{N}\sum_{x\in D}\boldsymbol{\psi}_{\mathrm{S}^c_i}(x)\sum_{j\neq i}\sum_{k=r+1}^{\infty}\beta^*_{ijk}\widetilde{\psi}_k(x_i,x_j)\|_\infty+ \\
&\|\frac{2}{\lambda}\frac{1}{N}\sum_{x\in D}\boldsymbol{\psi}_{\mathrm{S}^c_i}(x)\boldsymbol{\gamma}^\intercal_{\mathrm{S}_i}(x)\boldsymbol{\beta}_{\mathrm{S}_i}\|_\infty\ .
\end{aligned} \tag{12}
$$

We want to show that $\|\mathbf{z}_{\mathrm{S}^c_i}\|_\infty < 1$, which ensures that $\beta_{ijk} = 0, \forall j \notin \mathrm{S}_i$. We do this by bounding each term in Equation (12) using the following lemmas.

Lemma 5. *For some $\epsilon > 0$, the following statements are true:*

1. *If $\lambda > \frac{2\sqrt{2}\sigma\overline{\psi}C}{\epsilon}\sqrt{\frac{\log(2(n-d))}{N}}$, then for some constant $c_1 > 0$,*

$$
\Pr(\frac{2}{\lambda}\frac{1}{N}\|\sum_{x\in D}\boldsymbol{\psi}_{\mathrm{S}^c_i}(x)\boldsymbol{\gamma}^\intercal_{\mathrm{S}_i}(x)\boldsymbol{\beta}_{\mathrm{S}_i}\|_\infty \leqslant \epsilon) \geqslant 1-\exp(-c_1\lambda^2 N)\ .
$$

2. *If $\lambda > \frac{2\sqrt{2}\sigma\overline{\psi}C}{\epsilon}\sqrt{\frac{\log(2d)}{N}}$, then for some constant $c_2 > 0$,*

$$
\Pr(\frac{2}{\lambda}\frac{1}{N}\|\sum_{x\in D}\boldsymbol{\psi}_{\mathrm{S}^c_i}(x)\boldsymbol{\gamma}^\intercal_{\mathrm{S}_i}(x)\boldsymbol{\beta}_{\mathrm{S}_i}\|_\infty \leqslant \epsilon) \geqslant 1-\exp(-c_2\lambda^2 N)\ .
$$

Lemma 6. *For sufficiently large r and $\epsilon > 0$, the following statements are true:*

1. *If $\lambda > \frac{2\sqrt{2}\overline{\psi}\sqrt{\delta}}{\epsilon}\sqrt{\frac{d\log(2d)}{N}}$, then for some constant $c_3 > 0$,*

$$\Pr(\frac{2}{\lambda}\frac{1}{N}\|(\sum_{y\in D}\boldsymbol{\psi}_{\mathrm{S}_i}(y)\sum_{j\neq i}\sum_{k=r+1}^{\infty}\beta^*_{ijk}\widetilde{\psi}(y_i, y_j))\|_\infty \leqslant 2\epsilon) \geqslant 1 - \exp(-c_3\lambda^2 N) \ .$$

2. *If $\lambda > \frac{2\sqrt{2}\overline{\psi}\sqrt{\delta}}{\epsilon}\sqrt{\frac{d\log(2(n-d))}{N}}$, then for some constant $c_4 > 0$,*

$$\Pr(\|\frac{2}{\lambda}\frac{1}{N}(\sum_{y\in D}\boldsymbol{\psi}_{\mathrm{S}_i^c}(y)\sum_{j\neq i}\sum_{k=r+1}^{\infty}\beta^*_{ijk}\widetilde{\psi}(y_i, y_j))\|_\infty \leqslant 2\epsilon) \geqslant 1 - \exp(-c_4\lambda^2 N) \ ,$$

where $\delta > 0$ is an arbitrary constant which depends on r.

Combining all the results together and taking $\epsilon = \frac{\alpha}{24}$ in Lemma 5 and 6, $\|\mathbf{z}_{\mathrm{S}_i^c}\|_\infty \leqslant 1 - \frac{\alpha}{4}$, which justifies our choice that $\beta_{ijk} = 0, \forall j \neq \mathrm{S}_i$.

A.4. Proof of Lemma 5

Lemma 5 *For some $\epsilon > 0$, the following statements are true:*

1. *If $\lambda > \frac{2\sqrt{2}\sigma\overline{\psi}C}{\epsilon}\sqrt{\frac{\log(2(n-d))}{N}}$, then for some constant $c_1 > 0$,*

$$\Pr(\frac{2}{\lambda}\frac{1}{N}\|\sum_{x\in D}\boldsymbol{\psi}_{\mathrm{S}_i^c}(x)\boldsymbol{\gamma}^\intercal_{\mathrm{S}_i}(x)\boldsymbol{\beta}_{\mathrm{S}_i}\|_\infty \leqslant \epsilon) \geqslant 1 - \exp(-c_1\lambda^2 N) \ .$$

2. *If $\lambda > \frac{2\sqrt{2}\sigma\overline{\psi}C}{\epsilon}\sqrt{\frac{\log(2d)}{N}}$, then for some constant $c_2 > 0$,*

$$\Pr(\frac{2}{\lambda}\frac{1}{N}\|\sum_{x\in D}\boldsymbol{\psi}_{\mathrm{S}_i^c}(x)\boldsymbol{\gamma}^\intercal_{\mathrm{S}_i}(x)\boldsymbol{\beta}_{\mathrm{S}_i}\|_\infty \leqslant \epsilon) \geqslant 1 - \exp(-c_2\lambda^2 N) \ .$$

Proof. Let $R(x) \triangleq \boldsymbol{\psi}_{\mathrm{S}_i^c}(x)\boldsymbol{\gamma}^\intercal_{\mathrm{S}_i}(x)\boldsymbol{\beta}_{\mathrm{S}_i}$ be a random variable and $R_j(x)$ be the jth entry of $R(x)$. Then $R_j(x) = \boldsymbol{\psi}_{\mathrm{S}_i^c}(x)_j\boldsymbol{\gamma}^\intercal_{\mathrm{S}_i}(x)\boldsymbol{\beta}_{\mathrm{S}_i}$. Now note that $\boldsymbol{\gamma}_{\mathrm{S}_i}(x)$ is a zero mean sub-Gaussian random variable with variance proxy $\sigma^2\mathbf{I}_{d\times d}$. Thus, $\boldsymbol{\gamma}^\intercal_{\mathrm{S}_i}(x)\boldsymbol{\beta}_{\mathrm{S}_i}$ is also a zero mean sub-Gaussian random variable with variance proxy $\sigma^2\|\boldsymbol{\beta}_{\mathrm{S}_i}\|_2^2$ and $R_j(x)$ is a zero mean sub-Gaussian random variable with variance proxy $\sigma^2\boldsymbol{\psi}_{\mathrm{S}_i^c}(x)_j^2\|\boldsymbol{\beta}_{\mathrm{S}_i}\|_2^2$. For a given $\boldsymbol{\beta}$, using a concentration bound for sub-Gaussian random variables, we can write the following:

$$\Pr(|R_j(x)| \geqslant \epsilon|\boldsymbol{\beta}) \leqslant 2\exp(-\frac{\epsilon^2}{2\sigma^2\boldsymbol{\psi}_{\mathrm{S}_i^c}(x)_j^2\|\boldsymbol{\beta}_{\mathrm{S}_i}\|_2^2}) \ .$$

Note that from Equation (5), $\|\boldsymbol{\beta}\|_1 \leqslant C \implies \|\boldsymbol{\beta}\|_2^2 \leqslant C^2$. Thus,

$$\Pr(|R_j(x)| \geqslant \epsilon|\boldsymbol{\beta}) \leqslant 2\exp(-\frac{\epsilon^2}{2\sigma^2\boldsymbol{\psi}_{\mathrm{S}_i^c}(x)_j^2 C^2}) \ .$$

Let $\max_{x,j}\boldsymbol{\psi}(x)_j = \overline{\boldsymbol{\psi}}$, then

$$\Pr(|R_j(x)| \geqslant \epsilon|\boldsymbol{\beta}) \leqslant 2\exp(-\frac{\epsilon^2}{2\sigma^2\overline{\boldsymbol{\psi}}^2 C^2}) \ .$$

Note that given $\boldsymbol{\beta}$, $R_j(x)|\boldsymbol{\beta}, \forall x \in D$ are mutually independent. Also note that $\Pr(\frac{1}{N}\sum_{x\in D}|R_j(x)| \geqslant \epsilon) \geqslant \Pr(\frac{1}{N}|\sum_{x\in D}R_j(x)| \geqslant \epsilon)$, we get:

$$\Pr(\frac{1}{N}|\sum_{x\in D}R_j(x)| \geqslant \epsilon|\boldsymbol{\beta}) \leqslant 2\exp(-\frac{N\epsilon^2}{2\sigma^2\overline{\boldsymbol{\psi}}^2 C^2}) \ .$$

Again taking a union bound across $j \in \mathrm{S}_i^c$, we get

$$\Pr(\frac{1}{N}\|\sum_{x\in D}\boldsymbol{\psi}_{\mathrm{S}_i^c}(x)\boldsymbol{\gamma}_{\mathrm{S}_i}^{\intercal}(x)\boldsymbol{\beta}_{\mathrm{S}_i}\|_\infty \geqslant \epsilon|\boldsymbol{\beta}) \leqslant 2(n-d)\exp(-\frac{N\epsilon^2}{2\sigma^2\overline{\boldsymbol{\psi}}^2C^2}) \ .$$

Computing the expectation with respect to $\boldsymbol{\beta}$, we get

$$\Pr(\frac{1}{N}\|\sum_{x\in D}\boldsymbol{\psi}_{\mathrm{S}_i^c}(x)\boldsymbol{\gamma}_{\mathrm{S}_i}^{\intercal}(x)\boldsymbol{\beta}_{\mathrm{S}_i}\|_\infty \geqslant \epsilon) \leqslant 2(n-d)\exp(-\frac{N\epsilon^2}{2\sigma^2\overline{\boldsymbol{\psi}}^2C^2}) \ .$$

Therefore,

$$\Pr(\frac{2}{\lambda}\frac{1}{N}\|\sum_{x\in D}\boldsymbol{\psi}_{\mathrm{S}_i^c}(x)\boldsymbol{\gamma}_{\mathrm{S}_i}^{\intercal}(x)\boldsymbol{\beta}_{\mathrm{S}_i}\|_\infty \geqslant \epsilon) \leqslant 2(n-d)\exp(-\frac{N\lambda^2\epsilon^2}{8\sigma^2\overline{\boldsymbol{\psi}}^2C^2}) \ .$$

If $\lambda > \frac{2\sqrt{2}\sigma\overline{\boldsymbol{\psi}}C}{\epsilon}\sqrt{\frac{\log(2(n-d))}{N}}$, then for some constant $c_1 > 0$,

$$\Pr(\frac{2}{\lambda}\frac{1}{N}\|\sum_{x\in D}\boldsymbol{\psi}_{\mathrm{S}_i^c}(x)\boldsymbol{\gamma}_{\mathrm{S}_i}^{\intercal}(x)\boldsymbol{\beta}_{\mathrm{S}_i}\|_\infty \leqslant \epsilon) \geqslant 1-\exp(-c_1\lambda^2N) \ .$$

Following the arguments similar to the previous proof, we can write that

$$\Pr(\frac{2}{\lambda}\frac{1}{N}\|\sum_{y\in D}\boldsymbol{\psi}_{\mathrm{S}_i}(y)\boldsymbol{\gamma}_{\mathrm{S}_i}^{\intercal}(y)\boldsymbol{\beta}_{\mathrm{S}_i}\|_\infty \geqslant \epsilon) \leqslant 2d\exp(-\frac{N\lambda^2\epsilon^2}{8\sigma^2\overline{\boldsymbol{\psi}}^2C^2}) \ .$$

If $\lambda > \frac{2\sqrt{2}\sigma\overline{\boldsymbol{\psi}}C}{\epsilon}\sqrt{\frac{\log(2d)}{N}}$, then for some constant $c_2 > 0$,

$$\Pr(\frac{2}{\lambda}\frac{1}{N}\|\sum_{x\in D}\boldsymbol{\psi}_{\mathrm{S}_i^c}(x)\boldsymbol{\gamma}_{\mathrm{S}_i}^{\intercal}(x)\boldsymbol{\beta}_{\mathrm{S}_i}\|_\infty \leqslant \epsilon) \geqslant 1-\exp(-c_2\lambda^2N) \ .$$

□

A.5. Proof of Lemma 6

Lemma 6 *For sufficiently large r and $\epsilon > 0$, the following statements are true:*

1. *If $\lambda > \frac{2\sqrt{2}\overline{\boldsymbol{\psi}}\sqrt{\delta}}{\epsilon}\sqrt{\frac{d\log(2d)}{N}}$, then for some constant $c_3 > 0$,*

$$\Pr(\frac{2}{\lambda}\frac{1}{N}\|(\sum_{y\in D}\boldsymbol{\psi}_{\mathrm{S}_i}(y)\sum_{j\neq i}\sum_{k=r+1}^{\infty}\beta_{ijk}^*\widetilde{\psi}(y_i,y_j))\|_\infty \leqslant 2\epsilon) \geqslant 1-\exp(-c_3\lambda^2N) \ .$$

2. *If $\lambda > \frac{2\sqrt{2}\overline{\boldsymbol{\psi}}\sqrt{\delta}}{\epsilon}\sqrt{\frac{d\log(2(n-d))}{N}}$, then for some constant $c_4 > 0$,*

$$\Pr(\|\frac{2}{\lambda}\frac{1}{N}(\sum_{y\in D}\boldsymbol{\psi}_{\mathrm{S}_i^c}(y)\sum_{j\neq i}\sum_{k=r+1}^{\infty}\beta_{ijk}^*\widetilde{\psi}(y_i,y_j))\|_\infty \leqslant 2\epsilon) \geqslant 1-\exp(-c_4\lambda^2N) \ ,$$

where $\delta > 0$ is an arbitrary constant which depends on r.

Proof. Note that,

$$\|(\sum_{y\in D}\boldsymbol{\psi}_{\mathrm{S}_i}(y)\sum_{j\neq i}\sum_{k=r+1}^{\infty}\beta_{ijk}^*\widetilde{\psi}(y_i,y_j))\|_\infty \leqslant \|(\sum_{y\in D}\boldsymbol{\psi}_{\mathrm{S}_i}(y)\sum_{j\neq i}\sum_{k=r+1}^{\infty}\beta_{ijk}^*\psi_k(y_i,y_j))\|_\infty +$$
$$\|(\sum_{y\in D}\boldsymbol{\psi}_{\mathrm{S}_i}(y)\sum_{j\neq i}\sum_{k=r+1}^{\infty}\beta_{ijk}^*\gamma_k(y_i,y_j))\|_\infty \ .$$

We will bound both the terms separately.

Bound on $\|\frac{2}{\lambda}\frac{1}{N}(\sum_{y\in D}\boldsymbol{\psi}_{\mathrm{S}_i}(y)\sum_{j\neq i}\sum_{k=r+1}^{\infty}\beta^*_{ijk}\psi_k(y_i,y_j))\|_\infty$**.** Let $R \triangleq (\sum_{y\in D}\boldsymbol{\psi}_{\mathrm{S}_i}(y)\sum_{j\neq i}\sum_{k=r+1}^{\infty}\beta^*_{ijk}\psi_k(y_i,y_j))$. Let lth entry of R be R_l, i.e., $R_l = (\sum_{y\in D}\boldsymbol{\psi}_{\mathrm{S}_i}(y)_l\sum_{j\neq i}\sum_{k=r+1}^{\infty}\beta^*_{ijk}\psi_k(y_i,y_j))$.

$$
\begin{aligned}
|(\sum_{y\in D}\boldsymbol{\psi}_{\mathrm{S}_i}(y)_l\sum_{j\neq i}\sum_{k=r+1}^{\infty}\beta^*_{ijk}\psi_k(y_i,y_j))| &\leqslant (\sum_{y\in D}|\boldsymbol{\psi}_{\mathrm{S}_i}(y)_l|\sum_{j\neq i}\sum_{k=r+1}^{\infty}|\beta^*_{ijk}||\psi_k(y_i,y_j)|) \\
&\leqslant (\sum_{y\in D}|\overline{\boldsymbol{\psi}}|\sum_{j\neq i}\sum_{k=r+1}^{\infty}|\beta^*_{ijk}||\overline{\psi}|) \\
&\leqslant N|\overline{\psi}|^2\sum_{j\neq i}\sum_{k=r+1}^{\infty}|\beta^*_{ijk}|\ .
\end{aligned}
$$

Let $[\alpha_{ijk}]_{k=1}^{\infty}$ be a convergent series with positive entries such that $|\boldsymbol{\beta}^*_{ijk}| \leqslant \alpha_{ijk}, \forall k \geqslant r+1$. To give an example, let β^*_{ijk} be the Fourier coefficients, then we can choose $\alpha_{ijk} = \frac{D}{k^2}$ for some $D > 0$. Then it holds that $|\beta_{ijk}| \leqslant \frac{D}{k^2}, \forall k \geqslant r+1$ for sufficiently large r.

Lemma 7. *For sufficiently large r, $\sum_{k=r+1}^{\infty}|\beta^*_{ijk}| \leqslant \delta$ for any $\delta > 0$.*

Proof. The tail sum of a convergent series goes to 0, i.e., for sufficiently large r,

$$
\sum_{k=r+1}^{\infty}\alpha^*_{ijk} \leqslant \delta\ .
$$

It readily follows that $\sum_{k=r+1}^{\infty}|\beta^*_{ijk}| \leqslant \delta$. □

Using Lemma 7 and taking $\delta = \frac{\lambda\epsilon}{2d|\overline{\psi}|^2}$, it follows that:

$$
|(\sum_{y\in D}\boldsymbol{\psi}_{\mathrm{S}_i}(y)_l\sum_{j\neq i}\sum_{k=r+1}^{\infty}\beta^*_{ijk}\psi_k(y_i,y_j))| \leqslant \frac{\lambda}{2}N\epsilon\ .
$$

This ensures that for large enough r, we have $\|\frac{2}{\lambda}\frac{1}{N}(\sum_{y\in D}\boldsymbol{\psi}_{\mathrm{S}_i}(y)\sum_{j\neq i}\sum_{k=r+1}^{\infty}\beta^*_{ijk}\psi_k(y_i,y_j))\|_\infty \leqslant \epsilon$.

Bound on $\|\frac{2}{\lambda}\frac{1}{N}(\sum_{y\in D}\boldsymbol{\psi}_{\mathrm{S}_i}(y)\sum_{j\neq i}\sum_{k=r+1}^{\infty}\beta^*_{ijk}\gamma_k(y_i,y_j))\|_\infty$**.** Let R be a random variable such that $R \triangleq \boldsymbol{\psi}_{\mathrm{S}_i}(y)\sum_{j\neq i}\sum_{k=r+1}^{\infty}\beta^*_{ijk}\gamma_k$ Let the l-th entry of R be $R_l = \boldsymbol{\psi}_{\mathrm{S}_i}(y)_l\sum_{j\neq i}\sum_{k=r+1}^{\infty}\beta^*_{ijk}\gamma_k(y_i,y_j)$. For given $\boldsymbol{\beta}^*$, R_l is a zero mean sub-Gaussian random variable with variance proxy $\boldsymbol{\psi}_{\mathrm{S}_i}(y)^2\sum_{j\neq i}\sum_{k=r+1}^{\infty}(\beta^*_{ijk})^2$. Note that since α_{ijk} is convergent series with positive entries, α^2_{ijk} is also convergent. This means that using the tail sum of a convergent series

$$
\sum_{k=r+1}^{\infty}\alpha^2_{ijk} \leqslant \delta\ ,
$$

for sufficiently large r and any $\delta > 0$. It follows that

$$
\sum_{k=r+1}^{\infty}(\beta^*_{ijk})^2 \leqslant \delta\ .
$$

Also, note that $R_l|\boldsymbol{\beta}^*$ are mutually independent for $y \in D$. Thus

$$
\Pr(\frac{1}{N}|R_l| \geqslant \epsilon|\boldsymbol{\beta}^*) \leqslant 2\exp(-\frac{N\epsilon^2}{2\overline{\boldsymbol{\psi}}^2 d\delta})\ .
$$

Taking a union bound across $l \in \mathrm{S}_i$, we get

$$
\Pr(\frac{1}{N}\|(\sum_{y\in D}\boldsymbol{\psi}_{\mathrm{S}_i}(y)\sum_{j\neq i}\sum_{k=r+1}^{\infty}\beta^*_{ijk}\gamma_k(y_i,y_j))\|_\infty \geqslant \epsilon|\boldsymbol{\beta}^*) \leqslant 2d\exp(-\frac{N\epsilon^2}{2\overline{\boldsymbol{\psi}}^2 d\delta})\ .
$$

Computing the expectation with respect to $\boldsymbol{\beta}^*$, we get

$$\Pr(\|\frac{2}{\lambda}\frac{1}{N}(\sum_{y\in D}\boldsymbol{\psi}_{\mathrm{S}_i}(y)\sum_{j\neq i}\sum_{k=r+1}^{\infty}\beta^*_{ijk}\gamma_k(y_i,y_j))\|_\infty \geqslant \epsilon) \leqslant 2d\exp(-\frac{N\lambda^2\epsilon^2}{8\overline{\boldsymbol{\psi}}^2 d\delta})\ ,$$

where δ is an arbitrary constant which depends on r.

Combining above results,

$$\Pr(\frac{2}{\lambda}\frac{1}{N}\|(\sum_{y\in D}\boldsymbol{\psi}_{\mathrm{S}_i}(y)\sum_{j\neq i}\sum_{k=r+1}^{\infty}\beta^*_{ijk}\widetilde{\psi}(y_i,y_j))\|_\infty \geqslant 2\epsilon) \leqslant 2d\exp(-\frac{\lambda^2 N\epsilon^2}{8\overline{\boldsymbol{\psi}}^2 d\delta})\ .$$

If $\lambda > \frac{2\sqrt{2}\overline{\boldsymbol{\psi}}\sqrt{\delta}}{\epsilon}\sqrt{\frac{d\log(2d)}{N}}$, then for some constant $c_3 > 0$,

$$\Pr(\frac{2}{\lambda}\frac{1}{N}\|(\sum_{y\in D}\boldsymbol{\psi}_{\mathrm{S}_i}(y)\sum_{j\neq i}\sum_{k=r+1}^{\infty}\beta^*_{ijk}\widetilde{\psi}(y_i,y_j))\|_\infty \leqslant 2\epsilon) \geqslant 1-\exp(-c_3\lambda^2 N)\ .$$

Using the similar argument as the previous proof, we can write

$$\Pr(\|\frac{2}{\lambda}\frac{1}{N}(\sum_{y\in D}\boldsymbol{\psi}_{\mathrm{S}_i^c}(y)\sum_{j\neq i}\sum_{k=r+1}^{\infty}\beta^*_{ijk}\widetilde{\psi}(y_i,y_j))\|_\infty \geqslant 2\epsilon) \leqslant 2(n-d)\exp(-\frac{N\lambda^2\epsilon^2}{8\overline{\boldsymbol{\psi}}^2 d\delta})\ .$$

If $\lambda > \frac{2\sqrt{2}\overline{\boldsymbol{\psi}}\sqrt{\delta}}{\epsilon}\sqrt{\frac{d\log(2(n-d))}{N}}$, then for some constant $c_4 > 0$,

$$\Pr(\|\frac{2}{\lambda}\frac{1}{N}(\sum_{y\in D}\boldsymbol{\psi}_{\mathrm{S}_i^c}(y)\sum_{j\neq i}\sum_{k=r+1}^{\infty}\beta^*_{ijk}\widetilde{\psi}(y_i,y_j))\|_\infty \leqslant 2\epsilon) \geqslant 1-\exp(-c_4\lambda^2 N)\ .$$

Here δ could be arbitrarily small depending on r. $\square$

A.6. Estimation Error

Now that we have a criteria for choosing the regularization parameter λ, we can put an upper bound on the estimation error between $\boldsymbol{\beta}$ and $\boldsymbol{\beta}^*$. In particular, we provide the following lemma to provide bound on the estimation error.

Lemma 8. *If $\lambda > \frac{48\sqrt{2}\overline{\boldsymbol{\psi}}\max(C,\sqrt{\delta})}{\alpha}\sqrt{\frac{d^2\log(2n)}{N}}$, then for some constant $c > 0$, we have $\|\boldsymbol{\beta}^*_{\mathrm{S}_i} - \boldsymbol{\beta}_{\mathrm{S}_i}\|_2 \leqslant \frac{1}{C_{\min}}(\frac{\alpha}{24} + \sqrt{rd})\lambda$ with probability at least $1-\exp(-c\lambda^2 N)$.*

Note that $\|\boldsymbol{\beta}^*_{\mathrm{S}_i} - \boldsymbol{\beta}_{\mathrm{S}_i}\|_\infty \leqslant \|\boldsymbol{\beta}^*_{\mathrm{S}_i} - \boldsymbol{\beta}_{\mathrm{S}_i}\|_2$, thus

$$\|\boldsymbol{\beta}^*_{\mathrm{S}_i} - \boldsymbol{\beta}_{\mathrm{S}_i}\|_\infty \leqslant \frac{1}{C_{\min}}(\frac{\alpha}{24} + \sqrt{rd})\lambda\ . \tag{13}$$

Equation (13) provides a *minimum weight criteria*, i.e., if $\min_{j\in\mathrm{S}_i}|\beta^*_j| \geqslant \Delta(\lambda, C_{\min}, d, r, \alpha) = \frac{2}{C_{\min}}(\frac{\alpha}{24} + \sqrt{rd})\lambda$ then we recover $\boldsymbol{\beta}^*$ up to correct sign. This ensures that we recover the exact structure of the true game.

A.7. Recovering Payoffs and ϵ-Nash Equilibria.

In this subsection, we show that the recovered payoff function $\widehat{u}_i(x)$ is not far away from the true payoff function $u_i^*(x)$. To that end we provide a bound between the recovered payoff function and the true payoff function.

$$
\begin{aligned}
|u_i^*(x) - \widehat{u}_i(x)| &= |\sum_{j\in \mathrm{S}_i}\sum_{k=0}^{\infty} \beta_{ijk}^*\psi_k(x_i,x_j) - \sum_{j\in \mathrm{S}_i}\sum_{k=0}^{r} \beta_{ijk}\psi_k(x_i,x_j)| \\
&= |\sum_{j\in \mathrm{S}_i}\sum_{k=0}^{r} (\beta_{ijk}^* - \beta_{ijk})\psi_k(x_i,x_j) + \sum_{j\in \mathrm{S}_i}\sum_{k=r+1}^{\infty} \beta_{ijk}^*\psi_k(x_i,x_j)| \\
&\leqslant \|\boldsymbol{\beta}_{\mathrm{S}_i}^* - \boldsymbol{\beta}_{\mathrm{S}_i}\|_2 \|\boldsymbol{\psi}_{\mathrm{S}_i}\|_2 + \bar{\psi}\delta \\
&\leqslant \frac{1}{C_{\min}}(\frac{\alpha}{24} + \sqrt{rd})\lambda\sqrt{d}\bar{\psi} + \bar{\psi}\delta \\
&= \frac{\epsilon(\lambda, C_{\min}, d, \bar{\psi}, \delta, r, \alpha)}{2} \ .
\end{aligned}
$$

It follows that the recovered payoff function is $\epsilon(\lambda, C_{\min}, d, \bar{\psi}, \delta, r, \alpha)$ away from the true payoff function. Furthermore, their distance decreases as we choose smaller λ and smaller δ. This can be done by increasing the number of samples and the parameter r respectively. Now, let $\mathrm{NE}(\widehat{\mathscr{G}}) = \{\widehat{x} \in \times_{i\in\mathcal{V}}\mathcal{A}_i \mid \widehat{x}_i \in \arg\max \widehat{u}_i(x_i, \widehat{x_{\bar{i}}}), \forall i \in \mathcal{V}\}$. Using Equation (**??**), $\forall i \in \mathcal{V}, \widehat{x} \in \mathrm{NE}(\widehat{\mathscr{G}})$ and $\forall x_i \in \mathcal{A}_i$, we have

$$
\begin{aligned}
&u_i^*(\widehat{x}_i, \widehat{x}_{\text{-}i}) + \frac{\epsilon}{2} \geqslant \widehat{u}_i(\widehat{x}_i, \widehat{x}_{\text{-}i}) \geqslant \widehat{u}_i(x_i, \widehat{x}_{\text{-}i}) \\
&u_i^*(\widehat{x}_i, \widehat{x}_{\text{-}i}) \geqslant \widehat{u}_i(x_i, \widehat{x}_{\text{-}i}) - \frac{\epsilon}{2} \\
&u_i^*(\widehat{x}_i, \widehat{x}_{\text{-}i}) \geqslant u_i^*(x_i, \widehat{x}_{\text{-}i}) - \epsilon \ .
\end{aligned}
$$

This proves that $\mathrm{NE}(\widehat{\mathscr{G}}) \subseteq \mathrm{NE}_\epsilon(\mathscr{G})$. A similar argument can be made to show that $\mathrm{NE}(\mathscr{G}) \subseteq \mathrm{NE}_\epsilon(\widehat{\mathscr{G}})$. Now, if $\forall i \in \mathcal{V}, \forall (x_i, x_{\text{-}i}), (x_i', x_{\text{-}i}) \in \mathcal{A}$ such that $(x_i, x_{\text{-}i}) \in \mathrm{NE}(\mathscr{G})$ and $(x_i', x_{\text{-}i}) \notin \mathrm{NE}(\mathscr{G})$ implies that $u_i^*(x_i, x_{\text{-}i}) > u_i^*(x_i', x_{\text{-}i}) + \epsilon$, then $\forall i \in \mathcal{V}$

$$
\begin{aligned}
u_i^*(x_i, x_{\text{-}i}) &> u_i^*(x_i', x_{\text{-}i}) + \epsilon \\
\widehat{u}_i(x_i, x_{\text{-}i}) + \frac{\epsilon}{2} &> \widehat{u}_i(x_i', x_{\text{-}i}) - \frac{\epsilon}{2} + \epsilon \\
\widehat{u}_i(x_i, x_{\text{-}i}) &> \widehat{u}_i(x_i', x_{\text{-}i}) \ .
\end{aligned}
$$

Thus we have that $\mathrm{NE}(\mathscr{G}) \subseteq \mathrm{NE}(\widehat{\mathscr{G}})$. Furthermore, if $\mathrm{NE}_\epsilon(\mathscr{G}) = \mathrm{NE}(\mathscr{G})$ then $\mathrm{NE}(\mathscr{G}) = \mathrm{NE}(\widehat{\mathscr{G}})$ as we have already shown that $\mathrm{NE}(\widehat{\mathscr{G}}) \subseteq \mathrm{NE}_\epsilon(\mathscr{G}) = \mathrm{NE}(\mathscr{G})$.

A.8. Computation of Global Efficiency Quantities

Now that we have recovered payoffs and $\mathrm{NE}(\widehat{\mathscr{G}})$, we can compute global efficiency quantities such as the Price of Anarchy (PoA), the Price of Stability (PoS) and the volume of Nash equilibria (Vol) for the recovered game. To do this, we define a welfare function as follows: $\mathrm{Wel}(x) \triangleq \sum_{i\in\mathcal{V}} \widehat{u}_i(x)$. Let $\mathcal{A}$ denote the action set $\times_{i\in\mathcal{V}}\mathcal{A}_i$. Then,

$$
\begin{aligned}
\mathrm{PoA}_\epsilon(\widehat{\mathscr{G}}) &= \frac{\sup_{x\in\mathcal{A}} \mathrm{Wel}(x)}{\inf_{x\in\mathrm{NE}_\epsilon(\widehat{\mathscr{G}})} \mathrm{Wel}(x)} \ , \\
\mathrm{PoS}_\epsilon(\widehat{\mathscr{G}}) &= \frac{\sup_{x\in\mathcal{A}} \mathrm{Wel}(x)}{\sup_{x\in\mathrm{NE}_\epsilon(\widehat{\mathscr{G}})} \mathrm{Wel}(x)} \ , \\
\mathrm{Vol}_\epsilon(\widehat{\mathscr{G}}) &= \frac{\tau(\mathrm{NE}_\epsilon(\widehat{\mathscr{G}}))}{\tau(\mathcal{A})} \ ,
\end{aligned}
$$

where $\tau(A)$ is the Lebesgue measure of set A. The subscript ϵ in the global efficiency quantities denotes that ϵ-Nash equilibria is used in their computation.

A.9. Proof of Lemma 8

Lemma 8 *If* $\lambda > \frac{48\sqrt{2}\overline{\psi}\max(C,\sqrt{\delta})}{\alpha}\sqrt{\frac{d^2\log(2n)}{N}}$*, then for some constant* $c > 0$*, we have* $\|\boldsymbol{\beta}^*_{\mathrm{S}_i} - \boldsymbol{\beta}_{\mathrm{S}_i}\|_2 \leqslant \frac{1}{C_{\min}}(\frac{\alpha}{24} + \sqrt{rd})\lambda$ *with probability at least* $1 - \exp(-c\lambda^2 N)$.

Proof. We know from equation (11) that,

$$\begin{aligned}\|\boldsymbol{\beta}^*_{\mathrm{S}_i} - \boldsymbol{\beta}_{\mathrm{S}_i}\|_2 =& \| - [\widehat{\mathbf{H}}]^{-1}_{\mathrm{S}_i\mathrm{S}_i}\frac{1}{N}(\sum_{x\in D}\boldsymbol{\psi}_{\mathrm{S}_i}(x)\sum_{j\neq i}\sum_{k=r+1}^{\infty}\beta^*_{ijk}\widetilde{\psi}_k(x_i,x_j)) - \\ & [\widehat{\mathbf{H}}]^{-1}_{\mathrm{S}_i\mathrm{S}_i}(\frac{1}{N}\sum_{x\in D}\boldsymbol{\psi}_{\mathrm{S}_i}(x)\boldsymbol{\gamma}^{\intercal}_{\mathrm{S}_i}(x)\boldsymbol{\beta}_{\mathrm{S}_i}) + \lambda[\widehat{\mathbf{H}}]^{-1}_{\mathrm{S}_i\mathrm{S}_i}\frac{1}{N}\mathbf{z}_{\mathrm{S}_i}\|_2 \ .\end{aligned}$$

Using triangle norm inequality and noticing that $\|Ax\|_2 \leqslant \|A\|_2\|x\|_2$, where $\|A\|_2$ is the spectral norm of matrix A, we can write

$$\begin{aligned}\|\boldsymbol{\beta}^*_{\mathrm{S}_i} - \boldsymbol{\beta}_{\mathrm{S}_i}\|_2 \leqslant& \|[\widehat{\mathbf{H}}]^{-1}_{\mathrm{S}_i\mathrm{S}_i}\|_2\|\frac{1}{N}(\sum_{x\in D}\boldsymbol{\psi}_{\mathrm{S}_i}(x)\sum_{j\neq i}\sum_{k=r+1}^{\infty}\beta^*_{ijk}\widetilde{\psi}_k(x_i,x_j))\|_2 + \\ & \|[\widehat{\mathbf{H}}]^{-1}_{\mathrm{S}_i\mathrm{S}_i}\|_2\|\frac{1}{N}(\sum_{x\in D}\boldsymbol{\psi}_{\mathrm{S}_i}(x)\boldsymbol{\gamma}^{\intercal}_{\mathrm{S}_i}(x)\boldsymbol{\beta}_{\mathrm{S}_i})\|_2 + \frac{\lambda}{2}\|[\widehat{\mathbf{H}}]^{-1}_{\mathrm{S}_i\mathrm{S}_i}\|_2\|\mathbf{z}_{\mathrm{S}_i}\|_2 \ .\end{aligned}$$

We have already shown that $\Lambda_{\min}([\widehat{\mathbf{H}}]_{\mathrm{S}_i\mathrm{S}_i}) \geqslant \frac{C_{\min}}{2}$. It follows that $\Lambda_{\min}([\widehat{\mathbf{H}}]^{-1}_{\mathrm{S}_i\mathrm{S}_i}) \leqslant \frac{2}{C_{\min}}$. We also note that $\|\mathbf{z}_{\mathrm{S}_i}\|_2 \leqslant \sqrt{rd}\|\mathbf{z}_{\mathrm{S}_i}\|_\infty \leqslant \sqrt{rd}$. Thus,

$$\begin{aligned}\|\boldsymbol{\beta}^*_{\mathrm{S}_i} - \boldsymbol{\beta}_{\mathrm{S}_i}\|_2 \leqslant& \frac{2}{C_{\min}}(\|\frac{1}{N}(\sum_{x\in D}\boldsymbol{\psi}_{\mathrm{S}_i}(x)\sum_{j\neq i}\sum_{k=r+1}^{\infty}\beta^*_{ijk}\widetilde{\psi}_k(x_i,x_j))\|_2 + \|\frac{1}{N}(\sum_{x\in D}\boldsymbol{\psi}_{\mathrm{S}_i}(x)\boldsymbol{\gamma}^{\intercal}_{\mathrm{S}_i}(x)\boldsymbol{\beta}_{\mathrm{S}_i})\|_2 \\ & + \frac{\lambda}{2}\sqrt{rd}) \ .\end{aligned} \tag{14}$$

It remains to show that $\|\frac{1}{N}(\sum_{x\in D}\boldsymbol{\psi}_{\mathrm{S}_i}(x)\sum_{j\neq i}\sum_{k=r+1}^{\infty}\beta^*_{ijk}\widetilde{\psi}_k(x_i,x_j))\|_2$ and $\|\frac{1}{N}(\sum_{x\in D}\boldsymbol{\psi}_{\mathrm{S}_i}(x)\boldsymbol{\gamma}^{\intercal}_{\mathrm{S}_i}(x)$ $\boldsymbol{\beta}_{\mathrm{S}_i})\|_2$ are bounded which we will do in the following lemma.

Lemma 9. *For some* $\epsilon > 0$*, the following statements are true:*

1. *If* $\lambda > \frac{2\sqrt{2}\sigma\overline{\psi}C}{\epsilon}\sqrt{\frac{d\log(2d)}{N}}$*, then for some constant* $c_5 > 0$*,*

$$\Pr(\|\frac{1}{N}(\sum_{x\in D}\boldsymbol{\psi}_{\mathrm{S}_i}(x)\boldsymbol{\gamma}^{\intercal}_{\mathrm{S}_i}(x)\boldsymbol{\beta}_{\mathrm{S}_i})\|_2 \leqslant \frac{\lambda}{2}\epsilon) \geqslant 1 - \exp(-c_5\lambda^2 N) \ .$$

2. *If* $\lambda > \frac{2\sqrt{2}\sigma\overline{\psi}\sqrt{\delta}}{\epsilon}\sqrt{\frac{d^2\log(2d)}{N}}$*, then for some constant* $c_6 > 0$*,*

$$\Pr(\|\frac{1}{N}(\sum_{x\in D}\boldsymbol{\psi}_{\mathrm{S}_i}(x)\sum_{j\neq i}\sum_{k=r+1}^{\infty}\beta^*_{ijk}\widetilde{\psi}_k(x_i,x_j))\|_2 \leqslant 2\frac{\lambda}{2}\epsilon) \geqslant 1 - \exp(-c_6\lambda^2 N) \ ,$$

where $\delta > 0$ *could be an arbitrary small constant which depends on* r*.*

Using results from Lemma 9 with $\epsilon = \frac{\alpha}{24}$, we can rewrite Equation (14) as $\|\boldsymbol{\beta}^*_{\mathrm{S}_i} - \boldsymbol{\beta}_{\mathrm{S}_i}\|_2 \leqslant \frac{1}{C_{\min}}(\frac{\alpha}{24} + \sqrt{rd})\lambda$. □

A.10. Proof of Lemma 9

Lemma 9 *For some* $\epsilon > 0$*, the following statements are true:*

1. *If* $\lambda > \frac{2\sqrt{2}\sigma\overline{\psi}C}{\epsilon}\sqrt{\frac{d\log(2d)}{N}}$*, then for some constant* $c_5 > 0$*,*

$$\Pr(\|\frac{1}{N}(\sum_{x\in D}\boldsymbol{\psi}_{\mathrm{S}_i}(x)\boldsymbol{\gamma}^{\intercal}_{\mathrm{S}_i}(x)\boldsymbol{\beta}_{\mathrm{S}_i})\|_2 \leqslant \frac{\lambda}{2}\epsilon) \geqslant 1 - \exp(-c_5\lambda^2 N) \ .$$

2. *If $\lambda > \frac{2\sqrt{2}\sigma\overline{\psi}\sqrt{\delta}}{\epsilon}\sqrt{\frac{d^2\log(2d)}{N}}$, then for some constant $c_6 > 0$,*

$$\Pr(\|\frac{1}{N}(\sum_{x\in D}\boldsymbol{\psi}_{\mathrm{S}_i}(x)\sum_{j\neq i}\sum_{k=r+1}^{\infty}\beta^*_{ijk}\widetilde{\psi}_k(x_i,x_j))\|_2 \leqslant 2\frac{\lambda}{2}\epsilon) \geqslant 1-\exp(-c_6\lambda^2 N)\ ,$$

where $\delta > 0$ could be an arbitrary small constant which depends on r.

Proof. Following the notations used in Lemma 5, we know that

$$\Pr(|\frac{1}{N}\sum_{x\in D}R_j(x)| \geqslant \epsilon \mid \boldsymbol{\beta}) \leqslant 2\exp(\frac{-N\epsilon^2}{2\sigma^2\bar{\psi}^2C^2})\ .$$

We take $\epsilon = \frac{\epsilon}{\sqrt{d}}$ and apply a union bound over $j \in \mathrm{S}_i$, we get

$$\Pr(\|\frac{1}{N}(\sum_{x\in D}\boldsymbol{\psi}_{\mathrm{S}_i}(x)\boldsymbol{\gamma}^{\intercal}_{\mathrm{S}_i}(x)\boldsymbol{\beta}_{\mathrm{S}_i})\|_2 \geqslant \epsilon \mid \boldsymbol{\beta}) \leqslant 2d\exp(\frac{-N\epsilon^2}{2\sigma^2 d\bar{\psi}^2C^2})\ .$$

Computing the expectation with respect to $\boldsymbol{\beta}$,

$$\Pr(\|\frac{1}{N}(\sum_{x\in D}\boldsymbol{\psi}_{\mathrm{S}_i}(x)\boldsymbol{\gamma}^{\intercal}_{\mathrm{S}_i}(x)\boldsymbol{\beta}_{\mathrm{S}_i})\|_2 \geqslant \frac{\lambda}{2}\epsilon) \leqslant 2d\exp(\frac{-N\lambda^2\epsilon^2}{8\sigma^2 d\bar{\psi}^2C^2})\ .$$

If $\lambda > \frac{2\sqrt{2}\sigma\overline{\psi}C}{\epsilon}\sqrt{\frac{d\log(2d)}{N}}$, then for some constant $c_5 > 0$,

$$\Pr(\|\frac{1}{N}(\sum_{x\in D}\boldsymbol{\psi}_{\mathrm{S}_i}(x)\boldsymbol{\gamma}^{\intercal}_{\mathrm{S}_i}(x)\boldsymbol{\beta}_{\mathrm{S}_i})\|_2 \leqslant \frac{\lambda}{2}\epsilon) \geqslant 1-\exp(-c_5\lambda^2 N)\ .$$

Similarly, using notations used in Lemma 6, we know that

$$\Pr(\frac{1}{N}|R_l| \geqslant \epsilon \mid \boldsymbol{\beta}^*) \leqslant 2\exp(\frac{-N\epsilon^2}{2\bar{\psi}^2 d\delta})\ ,$$

where $\delta > 0$ is an arbitrary parameter which depends on r. We take $\epsilon = \frac{\epsilon}{\sqrt{d}}$ and union bound over $l \in \mathrm{S}_i$ and get

$$\Pr(\|\frac{1}{N}(\sum_{x\in D}\boldsymbol{\psi}_{\mathrm{S}_i}(x)\sum_{j\neq i}\sum_{k=r+1}^{\infty}\beta^*_{ijk}\widetilde{\psi}_k(x_i,x_j))\|_2 \geqslant 2\epsilon|\boldsymbol{\beta}^*) \leqslant 2d\exp(\frac{-N\epsilon^2}{2d^2\bar{\psi}^2\delta})\ .$$

Taking expectation with respect to $\boldsymbol{\beta}^*$, we get

$$\Pr(\|\frac{1}{N}(\sum_{x\in D}\boldsymbol{\psi}_{\mathrm{S}_i}(x)\sum_{j\neq i}\sum_{k=r+1}^{\infty}\beta^*_{ijk}\widetilde{\psi}_k(x_i,x_j))\|_2 \geqslant 2\frac{\lambda}{2}\epsilon) \leqslant 2d\exp(\frac{-N\lambda^2\epsilon^2}{8d^2\bar{\psi}^2\delta})\ .$$

If $\lambda > \frac{2\sqrt{2}\sigma\overline{\psi}\sqrt{\delta}}{\epsilon}\sqrt{\frac{d^2\log(2d)}{N}}$, then for some constant $c_6 > 0$,

$$\Pr(\|\frac{1}{N}(\sum_{x\in D}\boldsymbol{\psi}_{\mathrm{S}_i}(x)\sum_{j\neq i}\sum_{k=r+1}^{\infty}\beta^*_{ijk}\widetilde{\psi}_k(x_i,x_j))\|_2 \leqslant 2\frac{\lambda}{2}\epsilon) \geqslant 1-\exp(-c_6\lambda^2 N)\ .$$

$\square$

B. EXPERIMENTAL RESULTS

In this section, we validate our theorem by running computational experiments on synthetic data. Our goal in this experiment is to recover the exact structure of a non-parametric continuous game, i.e., to recover the correct in-neighbors of all the players, by using noisy utilities and basis functions. The performance of our method is measured using precision and recall. Let $\hat{\pi}_i$ be the set of recovered in-neighbors for player i and π_i be the set of true in-neighbors for player i, then the performance measures are defined as follows:

$$\text{Precision} = \frac{\sum_{i=1}^{n_b} |\pi_i \cap \hat{\pi}_i|}{\sum_{i=1}^{n_b} |\hat{\pi}_i|}, \quad \text{Recall} = \frac{\sum_{i=1}^{n_b} |\pi_i \cap \hat{\pi}_i|}{\sum_{i=1}^{n_b} |\pi_i|} \ . \tag{15}$$

Note that both these measures are important as it is possible to have high recall (or precision) by recovering all (or very few) edges in the graph. The experiments were run 30 times independently and the reported values are averaged across 30 runs.

Constructing Graphical Games. To generate synthetic data for our experiments, we considered continuous games played on DAGs (Directed Acyclic Graphs). Each player was restricted to have up to 3 in-neighbors. Each player i has a parameter vector $\boldsymbol{\beta}^* \in \mathbb{R}^{(n-1)\times r}$ associated with it. The entries of $\boldsymbol{\beta}^*$ were generated uniformly at random from $[0.25, 1]$. In theory, this parameter vector could be of infinite length but to use them in practice, we truncated it to have finite length for our experiments using a truncation parameter r. The players at the root nodes (with no parents) of the graph were free to choose their actions between $[0, 1]$. The remaining players chose their actions by maximizing their utilities. The actions were perturbed by a small additive noise to generate noisy utilities. The features for our experiments were generated using the basis functions and the target was to predict the noisy utility. The basis functions for utilities were chosen from the following set:

$$\begin{aligned}\psi_k(x_i, x_j) \in \bigcup_{l,m \in \{1,2\}} &\{\cos(2\pi l x_i)\cos(2\pi m x_j), \cos(2\pi l x_i)\sin(2\pi m x_j), \sin(2\pi l x_i)\cos(2\pi m x_j), \\ &\sin(2\pi l x_i)\sin(2\pi m x_j)\} \ .\end{aligned}$$

Experimental Setup. We conducted experiments for games with $n = 20, 50$ and 100 players. As per our theoretical results, the number of samples was varied as $10^C \log(rn)$ where C is a control parameter. The regularization parameter was set to $\Omega(\sqrt{\frac{\log n}{N}})$ as per our theorem.

Results.

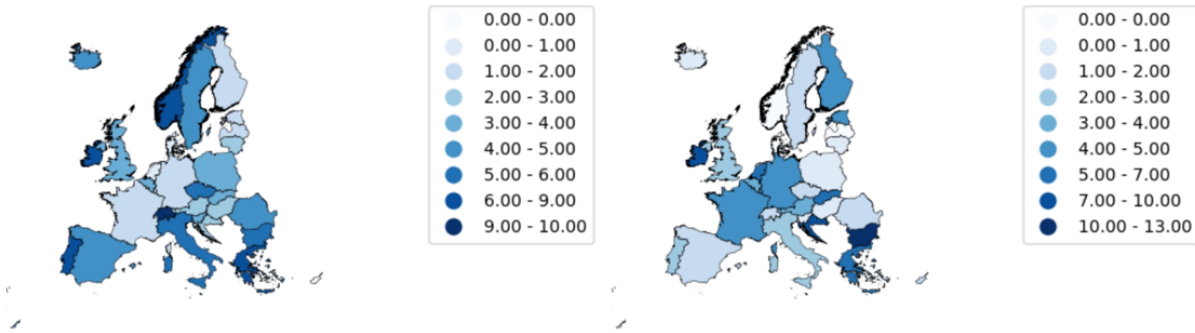
We provide results of our experiments in Figure 1. Left figure shows how the precision for games with various number of players varies with number of samples and right figure shows the recall for the same setup. All results are averaged across 30 independent runs. We can see that both precision and recall go to 1 as we increase the number of samples, obtaining perfect recovery with enough samples. Notice that the different curves for different number of players ($n = 20, 50$ and 100) line up with one another quite well. This matches with our theoretical results and shows that with a constant number of in-neighbors $\Omega(\log(rn))$ samples are sufficient to recover the exact structure of the graphical games.

C. APPLICATION - IDENTIFYING COUNTRIES INFLUENCING POTATO TRADE IN EUROPE

In this section, we performed a computational experiment on real world data to demonstrate the effectiveness of our method. We used our method to identify a set of European countries which have influenced potato trade within Europe between $1973 - 2019$. The trading interactions among various countries can be modeled as a non-parametric strategic game among self-interested countries.

Real World Dataset. Our experiments were conducted on the publicly available potato trade data from `https://ec.europa.eu/eurostat/web/agriculture/data/database`. We extracted the potato trade data among $n = 31$ European countries between 1973 to 2019. Each training sample corresponds to a year, and thus, we had $T = 47$ samples for our experiments. The dataset contains the country-wise production volume of potato for 47 years which we treated as the action x of players. The dataset also contains country-wise trade prices of potato for 47 years which we treated as payoff $\widetilde{u}_i(x)$ for each country. We chose $r = 16$ basis functions for the Fourier series expansion of each pairwise utility function to run our experiments. In particular, for each pairwise utility function $\bar{u}_{ij}(x_i, x_j), \forall i, j \in \{1, \ldots, n\}, i \neq j$, we chose the basis functions from the following set:

$$\begin{aligned}\psi_k(x_i, x_j) \in \bigcup_{l,m \in \{1,2\}} &\{\cos(2\pi l x_i)\cos(2\pi m x_j), \cos(2\pi l x_i)\sin(2\pi m x_j), \sin(2\pi l x_i)\cos(2\pi m x_j), \\ &\sin(2\pi l x_i)\sin(2\pi m x_j)\}\end{aligned}$$



(a) 1973-1995

(b) 1996-2019

Fig. 2: Influence of various European countries on potato trade between $1973 - 2019$. Left image shows the result for the duration 1973-1995 while the right image shows the result for the duration 1996-2019. The influence of a country is measured by the number of its out-neighbors in the learnt game.

Learning the Structure of Graphical Game. To study the change in influence of European countries over the time period, we divided dataset into two parts. The first part contained data from 1973 to 1995 and the second part contained data from 1996 to 2019. We computed the influence of the countries by first learning the global graphical game and then computing the number of out-neighbors for each country in the learnt game for both durations. The results are shown in Figure 2. We observe that countries in Eurozone (formed in 1999) have gained in influence during the period of 1996-2019.

Inference in Learnt Graphical Game.

	1973-1995	1996-2019
Price of Anarchy	1.1793	1.0685
Price of Stability	1	1
Volume of ϵ-Nash equilibria	44.2%	42.0%

Table 1: Inference of global efficiency quantities in the graphical game for potato trade across Europe between 1973-2019

Once the structure was learnt, the global efficiency quantities such as the Price of Anarchy (PoA), the Price of Stability (PoS) and the volume of Nash equilibria (Vol) were computed. Recall from subsection A.8 that we can define a welfare function as follows: $\text{Wel}(x) \triangleq \sum_{i \in \mathcal{V}} \widehat{u}_i(x)$. Let $\mathcal{A}$ denote the action set $\times_{i \in \mathcal{V}} \mathcal{A}_i$. Then,

$$\text{PoA}_\epsilon(\widehat{\mathscr{G}}) = \frac{\sup_{x \in \mathcal{A}} \text{Wel}(x)}{\inf_{x \in \text{NE}_\epsilon(\widehat{\mathscr{G}})} \text{Wel}(x)} \, ,$$

$$\text{PoS}_\epsilon(\widehat{\mathscr{G}}) = \frac{\sup_{x \in \mathcal{A}} \text{Wel}(x)}{\sup_{x \in \text{NE}_\epsilon(\widehat{\mathscr{G}})} \text{Wel}(x)} \, ,$$

$$\text{Vol}_\epsilon(\widehat{\mathscr{G}}) = \frac{\tau(\text{NE}_\epsilon(\widehat{\mathscr{G}}))}{\tau(\mathcal{A})} \, ,$$

where $\tau(A)$ is the Lebesgue measure of set A. The subscript ϵ in the global efficiency quantities denotes that ϵ-Nash equilibria is used in their computation.

To compute these quantities for our experiment, we first defined welfare of a strategy profile as the sum of payoffs of all players for that particular strategy profile. Then, 1000 strategy profiles $\mathbf{x} \in \times_{i \in \mathcal{V}} \mathcal{A}_i$ were generated uniformly at random such that $x_i \in \mathbb{R}^2$ and $\|x_i\|_2 \leqslant 1$. A strategy profile $\mathbf{x}$ was considered to be an ϵ-Nash equilibria if for all the players, the payoff does not increase by more than ϵ after perturbing their actions by a small $\delta > 0$. The proportion of all 1000 strategy profiles which were part of ϵ-Nash equilibria, termed as the volume of ϵ-Nash equilibria, were 44.2% and 42.0% for 1973-1995 and 1996-2019 respectively. To compute the global efficiency quantities, we shifted all the payoffs by a constant to make the payoffs non-negative. Note that this does not change the ϵ-Nash equilibria set of the game.

The price of anarchy was computed as the ratio between the maximum welfare across all strategy profiles and the minimum welfare across all strategy profiles in the ϵ-Nash equilibria set. The price of anarchy was computed to be the ratio between the maximum welfare across all strategy profiles and the minimum welfare across all strategy profiles in the ϵ-Nash equilibria set. The price of anarchy for the duration 1973-1995 and 1996-2019 were, respectively, 1.1793 and 1.0685. We observe that the price of anarchy in period 1973-1995 is greater than that of period 1996-2019. From this, we infer that in the past, the total welfare across European countries could have been increased by deviating away from Nash equilibria, however now the gain in total welfare by deviating away from Nash equilibria is not too much. This might correlate with the fact that the European union was created in 1993. Similarly, the price of stability was computed as the ratio between the maximum welfare across all strategy profiles and the maximum welfare across all strategy profiles in the ϵ-Nash equilibria set. We found the price of stability to be 1 for both time periods. This implies that the maximum total welfare was achieved by an ϵ-Nash equilibria during both time periods.