Theorems, Lemmas, Properties: Real Analysis and Measure Theory

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Ref: Real Analysis 3: Stein-Shakarachi

Chapter 1

MEASURE THEORY

1.1 Basic Definitions

Definition 1 (Open Ball): The open ball in $\mathbb{R}^d$ centered at x and of radius r is defined by

$$B_r(x) = \left\{y \in \mathbb{R}^d : |y - x| < r\right\}$$

Definition 2 (Open Set and Closed Set): A subset $E \subset \mathbb{R}^d$ is open if for every $x \in E$ there exists $r > 0$ with $B_r(x) \subset E$. By definition, a set is closed if its complement is open.

- Any (not necessarily countable) union of open sets is open
- The intersection of finitely many open sets is open
- Any (not necessarily countable) intersection of close sets is closed
- The union of finitely many close sets is close

Definition 3 (Bounded Set and Compact Set): A set E is bounded if it is contained in some ball of finite radius. A bounded set is compact if it is also closed.

Property 1 (Heine-Borel covering property): Any covering of a compact set by a collection of open sets contains a finite subcovering.

Definition 4 (Limit Point): A point $x \in \mathbb{R}^d$ is a limit point of the set E if for every $r > 0$, the ball $B_r(x)$ contains points of E.

Definition 5 (Isolated Point): An isolated point of E is a point $x \in E$ such that there exists an $r > 0$ where $B_r(x) \bigcap E$ is equal to x.

Definition 6 (Interior): A point $x \in E$ is an interior point of E if there exists $r > 0$ such that $B_r(x) \subset E$. The set of all interior points of E is called the interior of E.

Definition 7 (Closure): The closure $\bar{E}$ of the E consists of the union of E and all its limit points.

Definition 8 (Boundary): The boundary of a set E, denoted by δE, is the set of points which are in the closure of E but not in the interior of E.

Property 2:

- The closure of a set is a closed set.
- Every point in E is a limit point of E.
- A set is closed if and only if it contains all its limit points.

Definition 9 (Perfect Set): A closed set E is perfect if E does not have any isolated points.

Definition 10 (Rectangle): A (closed) rectangle R in $\mathbb{R}^d$ is given by the product of d one-dimensional closed and bounded intervals,

$$R = [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_d, b_d]$$

where a_j, b_j are real numbers, $j = 1, 2, \ldots, d$. If all intervals are equal then it is a cube.

Lemma 1: If a rectangle is the almost disjoint (interior disjoint) union of finitely many other rectangles, say $R = \bigcup_{k=1}^{N} R_k$ then $|R| = \sum_{k=1}^{N} |R_k|$.

Lemma 2: If $R, R_1, \ldots, R_N$ are rectangles, and $R \subset \bigcup_{k=1}^{N} R_k$ then$|R| \leq \sum_{k=1}^{N} |R_k|$.

Theorem 1: Every open subset O of $\mathbb{R}$ can be written uniquely as a countable union of disjoint open intervals. In general, this is not true for $\mathbb{R}^d, d > 1$.

Theorem 2: Every open subset O of $\mathbb{R}^d, d \geq 1$, can be written as a countable union of almost disjoint closed rectangles (cubes).

Definition 11 (The Cantor Set): Let $C_0 = [0, 1]$ and we get C_k by dividing each disjoint interval of C_{k-1} in three equal parts and deleting the middle part (end points are included in the set), then the Cantor set is defined as,

$$\mathcal{C} = \bigcap_{k=0}^{\infty} C_k$$

Property 3:

- $C_0 \supset C_1 \supset C_2 \cdots \supset C_k \supset C_{k+1} \ldots$.
- $\mathcal{C}$ is closed and bounded, hence compact.
- $\mathcal{C}$ is totally disconnected: given any $x, y \in \mathcal{C}$ there exists $z \notin \mathcal{C}$ that lies between x and y.
- $\mathcal{C}$ is perfect: it has no isolated points and it is closed.
- $\mathcal{C}$ is not countable: maps to power set of $\mathbb{N}$
- $\mathcal{C}$ has measure 0.
- $x \in \mathcal{C} \iff x = \sum_{k=0}^{\infty} a_k 3^{-k}, a_k \in \{0, 2\}$

1.2 The Outer (Exterior) Measure

Definition 12 (Outer Measure): If E is any subset of $\mathbb{R}^d$, the outer measure of E is,

$$m_*(E) = \inf \sum_{k=1}^{\infty} \Big|R_k\Big|$$

where inf is taken over all countable coverings $E \subset \bigcup_{k=1}^{\infty} R_k$ by closed rectangles (cubes). Note that $0 \leq m_*(E) \leq \infty$.

Property 4: The outer measure of a point is zero.

Property 5: The outer measure of a closed rectangle (open rectangle) is equal to its volume.

Property 6: The outer measure of $\mathbb{R}^d$ is infinite.

Property 7: For every $\epsilon > 0$, there exists a covering $E \subset \bigcup_{j=1}^{\infty} R_j$ with

$$\sum_{j=1}^{\infty} m_*(R_j) \leq m_*(E) + \epsilon$$

Property 8 (Monotonicity): If $E_1 \subset E_2$, then $m_*(E1) \leq m_*(E_2)$.

Property 9 (Countable Sub-additivity): If $E = \bigcup_{j=1}^{\infty} E_j$, then $m_*(E) \leq \sum_{j=1}^{\infty} m_*(E_j)$.

Property 10: If $E \in \mathbb{R}^d$, then $m_*(E) = \inf m_*(\mathcal{O})$, where the inf is taken over all open sets $\mathcal{O}$ containing E.

Property 11: If $E = E_1 \bigcup E_2$ and $d(E_1, E_2) > 0$ then, $m_*(E) = m_*(E_1) + m_*(E_2)$ where $d(E_1, E_2) = \inf_{x \in E_1, y \in E_2} \Big|x - y\Big|$.

Property 12: If a set E is the countable union of almost disjoint rectangles $E = \bigcup_{k=1}^{\infty} R_k$, then $m_*(E) = \sum_{k=1}^{\infty} \Big|R_k\Big|$

Remark 1: Despite above two properties, in general it is **NOT TRUE** that if $E_1 \bigcup E_2$ is a disjoint union of subsets of $\mathbb{R}^d$, then

$$m_*(E_1 \bigcup E_2) = m_*(E_1) + m_*(E_2).$$

1.3 Measurable Sets and Lebesgue Measure

Definition 13 (Lebesgue Measurable): A subset E of $\mathbb{R}^d$ is Lebesgue measurable, or simply measurable, if for any $\epsilon > 0$ there exists an open set $\mathcal{O}$ with $E \subset \mathcal{O}$ and $m_*(\mathcal{O} - E) \leq \epsilon$. If E is measurable, we define its Lebesgue measure (or measure) $m(E)$ by $m(E) = m_*(E)$.

Property 13: Every open set in $\mathbb{R}^d$ is measurable.

Property 14: If $m_*(E) = 0$, then E is measurable. In particular, if F is a subset of a set of exterior measure 0, then F is measurable.

Property 15: A countable union of measurable sets is measurable.

Property 16: Closed sets are measurable.

Lemma 3: If F is closed, K is compact, and these sets are disjoint, then $d(F, K) > 0$.

Property 17: The complement of a measurable set is measurable.

Property 18: A countable intersection of measurable sets is measurable.

Remark 2: The operations of uncountable unions or intersections are not permissible when dealing with measurable sets!

Theorem 3: If $E_1, E_2, \dots$ are disjoint measurable sets, and $E = \bigcup_{j=1}^{\infty} E_j$, then $m(E) = \sum_{j=1}^{\infty} m(E_j)$.

Definition 14 ($\mathbf{E_k} \nearrow \mathbf{E}$): If $E_1, E_2, \dots$ is a countable collection of subsets of $S \in \mathbb{R}^d$ that increases to E in the sense that $E_1 \subseteq E_2 \subseteq \cdots \subseteq E_k \subseteq E_{k+1} \subseteq \dots$ and $E = \bigcup_{k=1}^{\infty} E_k$, then we write $E_k \nearrow E$.

Property 19: If $E_k \nearrow E$, then $m(E) = \lim_{k\to\infty} m(E_k)$

Definition 15 ($\mathbf{E_k} \searrow \mathbf{E}$): If $E_1, E_2, \dots$ is a countable collection of subsets of $S \in \mathbb{R}^d$ that decreases to E in the sense that $E_1 \supseteq E_2 \supseteq \cdots \supseteq E_k \supseteq E_{k+1} \supseteq \dots$ and $E = \bigcap_{k=1}^{\infty} E_k$, then we write $E_k \searrow E$.

Property 20: If $E_k \searrow E$ and $m(E_k) < \infty$ for some k, then $m(E) = \lim_{k\to\infty} m(E_k)$

Theorem 4: Suppose E is a measurable subset of $\mathbb{R}^d$. Then, for every $\epsilon > 0$:

- There exists an open set $\mathcal{O}$ with $E \subset \mathcal{O}$ and $m(\mathcal{O} - E) \leq \epsilon$.
- There exists a closed set F with $F \subset E$ and $m(E - F) \leq \epsilon$.
- If $m(E)$ is finite, there exists a compact set K with $K \subset E$ and $m(E - K) \leq \epsilon$.
- If $m(E)$ is finite, there exists a finite union $F = \bigcup_{j=1}^{N} R_j$ of closed rectangles (cubes) such that $m(E \triangle F) \leq \epsilon$, where symmetric difference $E \triangle F = (E - F) \bigcup (F - E)$.

Property 21 (Invariance):

- Translation Invariance. If E is a measurable set and $h \in \mathbb{R}^d$, then the set $E_h = E + h = \{x + h : x \in E\}$ is also measurable, and $m(E + h) = m(E)$.
- Dilation Invariance. If E is a measurable set and $\delta > 0 \in \mathbb{R}$, then the set $\delta E = \{\delta x : x \in E\}$ is also measurable, and $m(\delta E) = \delta^d m(E)$.
- Reflection Invariance. Whenever E is measurable, so is $-E = \{-x : x \in E\}$ and $m(-E) = m(E)$.

Definition 16 ($\mathbf{G}_\delta$ set): A G_δ set is, an intersection of a countable family of open sets.

Property 22: A subset E of $\mathbb{R}^d$ is measurable if and only if E differs from a G_δ by a set of measure zero.

Definition 17 ($\mathbf{F}_\sigma$ set): A F_σ set is, a union of a countable family of closed sets.

Property 23: A subset E of $\mathbb{R}^d$ is measurable if and only if E differs from a F_σ by a set of measure zero.

Definition 18 (σ-algebra): A σ-algebra of sets is a collection of subsets of $\mathbb{R}^d$ that is closed under countable unions, countable intersections, and complements. Example: Collection of all subsets of $\mathbb{R}^d$, collection of all measurable sets of $\mathbb{R}^d$

Definition 19 (Borel σ-algebra and Borel sets): A set $E \subseteq \mathbb{R}^d$ is an F_σ set provided that it is the countable union of closed sets and is a G_δ set if it is the countable intersection of open sets.The smallest σ-algebra that contains all open sets. Elements of this σ-algebra are called Borel sets.

Property 24: G_δ and F_σ are examples of Borel set.

Remark 3: Remember that it is possible to construct subsets of $\mathbb{R}^d$ which are not measurbale. Check reference book for construction of a non-measurable set $\mathcal{N} \subset \mathbb{R}$.

1.4 Measurable Functions

Definition 20 (Characteristic Function): A characteristic function of a set E is defined by

$$\chi_E(x) = \begin{cases} 1, & if\ x \in E \\ 0, & if\ x \notin E. \end{cases}$$

Definition 21 (Step Function): Step functions are defined as finite sum, $f = \sum_{k=1}^{N} a_k \chi_{R_k}$, where each R_k is a rectangle and each a_k is a constant. These are used in Reimann integral.

Definition 22 (Simple Function): A simple function is a finite sum, $f = \sum_{k=1}^{N} a_k \chi_{E_k}$, where each E_k is a measurable set of finite measure, and the a_k are constants. These are used in Lebesgue integral.

Definition 23 (Measurable Function): A function f defined on a measurable subset E of $\mathbb{R}^d$ is measurable, if for all $a \in \mathbb{R}$, the set

$$f^{-1}([-\infty, a)) = \{x \in E : f(x) < a\} = \{f < a\}$$

is measurable. Equivalently, $\{f \leq a\}, \{f > a\}, \{f \geq a\}$ are measurable. If f is finite valued then $\{a < f < b\}$ is measurable (with any combinations of $\leq, \geq$).

Property 25: The finite-valued function f is measurable if and only if $f^{-1}(O)$ is measurable for every open set O, and if and only if $f^{-1}(F)$ is measurable for every closed set F.

Property 26: If f is continuous on $\mathbb{R}^d$, then f is measurable. If f is measurable and finite-valued, and ϕ is continuous, then $\phi \circ f$ is measurable. But $f \circ \phi$ may not.

Property 27: Suppose $\{f_n\}_{n=1}^{\infty}$ is a sequence of measurable functions. Then

$$\sup_n f_n(x), \inf_n f_n(x), \limsup_{n\to\infty} f_n(x), \liminf_{n\to\infty} f_n(x)$$

are measurable.

Property 28: Suppose $\{f_n\}_{n=1}^{\infty}$ is a sequence of measurable functions. Then

$$\lim_{n\to\infty} f_n(x) = f(x)$$

then f is measurable.

Property 29: If f and g are measurable, then,

- The integer powers $f^k, k \geq 1$ are measurable.
- $f + g$ and fg are measurable if both f and g are finite-valued.

Definition 24 (Almost Everywhere): We shall say that two functions f and g defined on a set E are equal almost everywhere, and write,

$$f(x) = g(x)\ a.e.\ x \in E,$$

if the set $\{x \in E : f(x) \neq g(x)\}$ has measure zero. All the properties above can be relaxed to conditions holding almost everywhere.

Property 30: Suppose f is measurable, and $f(x) = g(x)$ for a.e. x. Then g is measurable.

Definition 25 (Pointwise Convergence of a Function): Let $E \subset \mathbb{R}^d$ and let $\{f_n\}_{n=1}^{\infty}$ be a sequence of real valued functions defined on E. Then $\{f_n\}_{n=1}^{\infty}$ converges pointwise to f if given any x in E and given any $\epsilon > 0$, there exists a natural number $N(x, \epsilon)$ such that $|f_n(x) - f(x)| < \epsilon$ for every $n > N(x, \epsilon)$.

Definition 26 (Uniform Convergence of a Function): Let $E \subset \mathbb{R}^d$ and let $\{f_n\}_{n=1}^{\infty}$ be a sequence of real valued functions defined on E. Then $\{f_n\}_{n=1}^{\infty}$ converges uniformly to f if given any $\epsilon > 0$, there exists a natural number $N(\epsilon)$ such that $|f_n(x) - f(x)| < \epsilon$ for every $n > N(\epsilon)$ for every $x \in E$.

Theorem 5: Suppose f is a non-negative measurable function on $\mathbb{R}^d$. Then there exists an increasing sequence of non-negative simple functions $\{\varphi\}_{k=1}^{\infty}$ that converges pointwise to f, namely,

$$\varphi_k(x) \leq \varphi_{k+1}(x) \text{ } and \lim_{k \to \infty} = f(x) \ \forall \ x.$$

Theorem 6: Suppose f is a measurable function on $\mathbb{R}^d$. Then there exists a sequence of simple functions $\{\varphi\}_{k=1}^{\infty}$ that satisfies,

$$|\varphi_k(x)| \leq |\varphi_{k+1}(x)| \text{ } and \lim_{k \to \infty} = f(x) \ \forall \ x.$$

Theorem 7: Suppose f is measurable on $\mathbb{R}^d$. Then there exists a sequence of step functions $\{\psi\}_{k=0}^{\infty}$ that converges pointwise to $f(x)$ for almost every x.

Theorem 8 (Egorov): Suppose $\{f_k\}_{k=1}^{\infty}$ is a sequence of measurable functions defined on a measurable set E with $m(E) < \infty$, and assume that $f_k \to f$ a.e on E. Given $\epsilon > 0$, we can find a closed set $A_\epsilon \subset E$ such that $m(E - A_\epsilon) \leq \epsilon$ and $f_k \to f$ uniformly on A_ϵ.

Theorem 9 (Lusin): Suppose f is measurable and finite valued on E with E of finite measure. Then for every $\epsilon > 0$ there exists a closed set F_ϵ, with

$$F_\epsilon \subset E \text{ } and \text{ } m(E - F_\epsilon) \leq \epsilon$$

and such that $f|_{F_\epsilon}$ is continuous.

Chapter 2

INTEGRATION THEORY

Definition 27 (Canonical Form of Simple Function): The canonical form of a simple function φ is the unique decomposition as below,

$$\varphi = \sum_{k=1}^{M} c_k \chi_{F_k}$$

where the numbers c_k are distinct and non-zero, and the sets F_k are disjoint.

Property 31: If φ is a simple function with canonical form $\varphi = \sum_{k=1}^{M} c_k \chi_{F_k}$, then we define the Lebesgue integral of φ by $\int_{\mathbb{R}^d} \varphi(x)dx = \sum_{k=1}^{M} c_k m(Fk)$.

Property 32: If E is a measurable subset of $\mathbb{R}^d$ with finite measure, then $\varphi(x)\chi_E(x)$ is also a simple function, and we define,

$$\int_E \varphi(x)dx = \int \varphi(x)\chi_E(x)dx$$

where second integral is over $\mathbb{R}^d$.

Proposition 1: The integral of simple functions defined above satisfies the following properties:

- Independence of the representation. If $\varphi = \sum_{k=1}^{N} a_k \chi_{E_k}$ is any representation of φ, then $\int \varphi = \sum_{k=1}^{N} a_k m(E_k)$.
- Linearity. If φ and ψ are simple, and $a, b \in \mathbb{R}$, then $\int (a\varphi + b\psi) = a\int \varphi + b\int \psi$
- Additivity. If E and F are disjoint subsets of $\mathbb{R}^d$ with finite measure, then $\int_{E\bigcup F} \varphi = \int_E \varphi + \int_F \varphi$
- Monotonicity. If $\varphi \leq \psi$ are simple, then $\int \varphi \leq \int \psi$
- Triangle inequality. If φ is a simple function, then so is $|\varphi|$, and $\left|\int \varphi\right| \leq \int |\varphi|$

Definition 28 (Support of a Function): Support is defined as $\text{supp}(f) = \{x : f(x) \neq 0\}$. We shall say that f is supported on a set E, if $f(x) = 0$ whenever $x \notin E$.

Lemma 4: Let f be a bounded function supported on a set E of finite measure. If $\{\varphi_n\}_{n=1}^{\infty}$ is any sequence of simple functions bounded by M, supported on E, and with $\varphi_n(x) \to f(x)$ for a.e. x, then:

- The limit $\lim_{n\to\infty} \int \varphi_n$ exists.
- If $f = 0$ a.e., then the limit $\lim_{n\to\infty} \int \varphi_n$ equals to 0.

Definition 29 (Lebesgue Integral of Bounded Functions Supported on Sets of Finite Measure): For such a function f, we define its Lebesgue Integral by,

$$\int f(x)dx = \lim_{n\to\infty} \int \varphi_n(x)dx$$

where $\{\varphi_n\}$ is **any** sequence of simple functions satisfying: $|\varphi_n| \leq M$, each φ_n is supported on the support of f, and $\varphi_n(x) \to f(x)$ for a.e. x as n tends to infinity.

Property 33: If E is a subset of $\mathbb{R}^d$ with finite measure, and f is bounded with $m(supp(f)) < \infty$, then:

$$\int_E f(x)dx = \int f(x)\chi_E(x)dx$$

Proposition 2: Suppose f and g are bounded functions supported on sets of finite measure. Then the following properties hold.

- Linearity. If $a, b \in \mathbb{R}$, then $\int(af + bg) = a\int f + b\int g$.
- Additivity. If E and F are disjoint subsets of $\mathbb{R}^d$, then $\int_{E\bigcup F} f = \int_E f + \int_F f$.
- Monotonicity. If $f \leq g$, then $\int f \leq \int g$.
- Triangle Inequality. $|f|$ is also bounded, supported on a set of finite measure, and $\left|\int f\right| \leq \int |f|$.

Theorem 10 (Bounded convergence theorem): Suppose that $\{f_n\}$ is a sequence of measurable functions that are all bounded by M, are supported on a set E of finite measure, and $f_n(x) \to f(x)$ a.e. x as $n \to \infty$. Then f is measurable, bounded, supported on E for a.e. x, and

$$\int |f_n - f| \to 0, \ as \ n \to \infty.$$

Consequently,

$$\int f_n \to \int f, \ as \ n \to \infty.$$

Theorem 11: Suppose f is Riemann integrable on the closed interval $[a, b]$. Then f is measurable, and

$$\int_{[a,b]}^{\mathcal{R}} f(x)dx = \int_{[a,b]}^{\mathcal{L}} f(x)dx,$$

where the integral on the left-hand side is the standard Riemann integral, and that on the right-hand side is the Lebesgue integral. Note that if f is Riemann integrable, then f is bounded.

Definition 30 (Lebesgue Integral for Non-negative Functions): For non-negative functions f we define its (extended) Lebesgue integral by

$$\int f(x)dx = \sup_g \int g(x)dx$$

where the supremum is taken over all measurable functions g such that $0 \leq g \leq f$, and where g is bounded and supported on a set of finite measure. If $\int f(x)dx < \infty$ then f is said to be Lebesgue integrable or simply integrable.