Theorems, Lemmas, Properties: Real Analysis and Measure Theory

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Ref: Real Analysis 3: Stein-Shakarachi

Chapter 1

MEASURE THEORY

1.1 Basic Definitions

Definition 1 (Open Ball): The open ball in $\mathbb{R}^d$ centered at x and of radius r is defined by

$$B_r(x) = \left\{y \in \mathbb{R}^d : |y - x| < r\right\}$$

Definition 2 (Open Set and Closed Set): A subset $E \subset \mathbb{R}^d$ is open if for every $x \in E$ there exists $r > 0$ with $B_r(x) \subset E$. By definition, a set is closed if its complement is open.

- Any (not necessarily countable) union of open sets is open
- The intersection of finitely many open sets is open
- Any (not necessarily countable) intersection of close sets is closed
- The union of finitely many close sets is close

Definition 3 (Bounded Set and Compact Set): A set E is bounded if it is contained in some ball of finite radius. A bounded set is compact if it is also closed.

Property 1 (Heine-Borel covering property): Any covering of a compact set by a collection of open sets contains a finite subcovering.

Definition 4 (Limit Point): A point $x \in \mathbb{R}^d$ is a limit point of the set E if for every $r > 0$, the ball $B_r(x)$ contains points of E.

Definition 5 (Isolated Point): An isolated point of E is a point $x \in E$ such that there exists an $r > 0$ where $B_r(x) \bigcap E$ is equal to x.

Definition 6 (Interior): A point $x \in E$ is an interior point of E if there exists $r > 0$ such that $B_r(x) \subset E$. The set of all interior points of E is called the interior of E.

Definition 7 (Closure): The closure $\bar{E}$ of the E consists of the union of E and all its limit points.

Definition 8 (Boundary): The boundary of a set E, denoted by δE, is the set of points which are in the closure of E but not in the interior of E.

Property 2:

- The closure of a set is a closed set.
- Every point in E is a limit point of E.
- A set is closed if and only if it contains all its limit points.

Definition 9 (Perfect Set): A closed set E is perfect if E does not have any isolated points.

Definition 10 (Rectangle): A (closed) rectangle R in $\mathbb{R}^d$ is given by the product of d one-dimensional closed and bounded intervals,

$$R = [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_d, b_d]$$

where a_j, b_j are real numbers, $j = 1, 2, \ldots, d$. If all intervals are equal then it is a cube.

Lemma 1: If a rectangle is the almost disjoint (interior disjoint) union of finitely many other rectangles, say $R = \bigcup_{k=1}^{N} R_k$ then $|R| = \sum_{k=1}^{N} |R_k|$.

Lemma 2: If $R, R_1, \ldots, R_N$ are rectangles, and $R \subset \bigcup_{k=1}^{N} R_k$ then$|R| \leq \sum_{k=1}^{N} |R_k|$.

Theorem 1: Every open subset O of $\mathbb{R}$ can be written uniquely as a countable union of disjoint open intervals. In general, this is not true for $\mathbb{R}^d, d > 1$.

Theorem 2: Every open subset O of $\mathbb{R}^d, d \geq 1$, can be written as a countable union of almost disjoint closed rectangles (cubes).

Definition 11 (The Cantor Set): Let $C_0 = [0, 1]$ and we get C_k by dividing each disjoint interval of C_{k-1} in three equal parts and deleting the middle part (end points are included in the set), then the Cantor set is defined as,

$$\mathcal{C} = \bigcap_{k=0}^{\infty} C_k$$

Property 3:

- $C_0 \supset C_1 \supset C_2 \cdots \supset C_k \supset C_{k+1} \ldots.$
- $\mathcal{C}$ is closed and bounded, hence compact.
- $\mathcal{C}$ is totally disconnected: given any $x, y \in \mathcal{C}$ there exists $z \notin \mathcal{C}$ that lies between x and y.
- $\mathcal{C}$ is perfect: it has no isolated points and it is closed.
- $\mathcal{C}$ is not countable: maps to power set of $\mathbb{N}$
- $\mathcal{C}$ has measure 0.
- $x \in \mathcal{C} \iff x = \sum_{k=0}^{\infty} a_k 3^{-k}, a_k \in \{0, 2\}$

1.2 The Outer (Exterior) Measure

Definition 12 (Outer Measure): If E is any subset of $\mathbb{R}^d$, the outer measure of E is,

$$m_*(E) = \inf \sum_{k=1}^{\infty} \Big|R_k\Big|$$

where inf is taken over all countable coverings $E \subset \bigcup_{k=1}^{\infty} R_k$ by closed rectangles (cubes). Note that $0 \leq m_*(E) \leq \infty$.

Property 4: The outer measure of a point is zero.

Property 5: The outer measure of a closed rectangle (open rectangle) is equal to its volume.

Property 6: The outer measure of $\mathbb{R}^d$ is infinite.

Property 7: For every $\epsilon > 0$, there exists a covering $E \subset \bigcup_{j=1}^{\infty} R_j$ with

$$\sum_{j=1}^{\infty} m_*(R_j) \leq m_*(E) + \epsilon$$

Property 8 (Monotonicity): If $E_1 \subset E_2$, then $m_*(E1) \leq m_*(E_2)$.

Property 9 (Countable Sub-additivity): If $E = \bigcup_{j=1}^{\infty} E_j$, then $m_*(E) \leq \sum_{j=1}^{\infty} m_*(E_j)$.

Property 10: If $E \in \mathbb{R}^d$, then $m_*(E) = \inf m_*(\mathcal{O})$, where the inf is taken over all open sets $\mathcal{O}$ containing E.

Property 11: If $E = E_1 \bigcup E_2$ and $d(E_1, E_2) > 0$ then, $m_*(E) = m_*(E_1) + m_*(E_2)$ where $d(E_1, E_2) = \inf_{x \in E_1, y \in E_2} \Big|x - y\Big|$.

Property 12: If a set E is the countable union of almost disjoint rectangles $E = \bigcup_{k=1}^{\infty} R_k$, then $m_*(E) = \sum_{k=1}^{\infty} \Big|R_k\Big|$

Need to add more

1.3 Measurable Sets and Lebesgue Measure

Definition 13 (σ-algebra): A σ-algebra of sets is a collection of subsets of $\mathbb{R}^d$ that is closed under countable unions, countable intersections, and complements. Example: Collection of all subsets of $\mathbb{R}^d$, collection of all measurable sets of $\mathbb{R}^d$

Definition 14 (Borel σ-algebra and Borel sets): A set $E \subseteq \mathbb{R}^d$ is an F_σ set provided that it is the countable union of closed sets and is a G_δ set if it is the countable intersection of open sets.The smallest σ-algebra that contains all open sets. Elements of this σ-algebra are called Borel sets.

1.4 Measurable Functions

Definition 15 (Characteristic Function): A characteristic function of a set E is defined by

$$\chi_E(x) = \begin{cases} 1, & if\ x \in E \\ 0, & if\ x \notin E. \end{cases}$$

Definition 16 (Step Function): Step functions are defined as finite sum, $f = \sum_{k=1}^{N} a_k \chi_{R_k}$, where each R_k is a rectangle and each a_k is a constant. These are used in Reimann integral.

Definition 17 (Simple Function): A simple function is a finite sum, $f = \sum_{k=1}^{N} a_k \chi_{E_k}$, where each E_k is a measurable set of finite measure, and the a_k are constants. These are used in Lebesgue integral.

Definition 18 (Measurable Function): A function f defined on a measurable subset E of $\mathbb{R}^d$ is measurable, if for all $a \in \mathbb{R}$, the set

$$f^{-1}\big([-\infty, a)\big) = \big\{x \in E : f(x) < a\big\} = \big\{f < a\big\}$$

is measurable. Equivalently, $\big\{f \leq a\big\}, \big\{f > a\big\}, \big\{f \geq a\big\}$ are measurable. If f is finite valued then $\big\{a < f < b\big\}$ is measurable (with any combinations of $\leq, \geq$).

Property 13: The finite-valued function f is measurable if and only if $f^{-1}(O)$ is measurable for every open set O, and if and only if $f^{-1}(F)$ is measurable for every closed set F.

Property 14: If f is continuous on $\mathbb{R}^d$, then f is measurable. If f is measurable and finite-valued, and ϕ is continuous, then $\phi \circ f$ is measurable. But $f \circ \phi$ may not.

Property 15: Suppose $\big\{f_n\big\}_{n=1}^{\infty}$ is a sequence of measurable functions. Then

$$\sup_n f_n(x), \inf_n f_n(x), \limsup_{n\to\infty} f_n(x), \liminf_{n\to\infty} f_n(x)$$

are measurable.

Property 16: Suppose $\big\{f_n\big\}_{n=1}^{\infty}$ is a sequence of measurable functions. Then

$$\lim_{n\to\infty} f_n(x) = f(x)$$

then f is measurable.

Property 17: If f and g are measurable, then,

- The integer powers $f^k, k \geq 1$ are measurable.
- $f + g$ and fg are measurable if both f and g are finite-valued.

Definition 19 (Almost Everywhere): We shall say that two functions f and g defined on a set E are equal almost everywhere, and write,

$$f(x) = g(x)\ a.e.\ x \in E,$$

if the set $\big\{x \in E : f(x) \neq g(x)\big\}$ has measure zero. All the properties above can be relaxed to conditions holding almost everywhere.

Property 18: Suppose f is measurable, and $f(x) = g(x)$ for a.e. x. Then g is measurable.

Definition 20 (Pointwise Convergence of a Function): Let $E \subset \mathbb{R}^d$ and let $\{f_n\}_{n=1}^{\infty}$ be a sequence of real valued functions defined on E. Then $\{f_n\}_{n=1}^{\infty}$ converges pointwise to f if given any x in E and given any $\epsilon > 0$, there exists a natural number $N(x, \epsilon)$ such that $|f_n(x) - f(x)| < \epsilon$ for every $n > N(x, \epsilon)$.

Definition 21 (Uniform Convergence of a Function): Let $E \subset \mathbb{R}^d$ and let $\{f_n\}_{n=1}^{\infty}$ be a sequence of real valued functions defined on E. Then $\{f_n\}_{n=1}^{\infty}$ converges uniformly to f if given any $\epsilon > 0$, there exists a natural number $N(\epsilon)$ such that $|f_n(x) - f(x)| < \epsilon$ for every $n > N(\epsilon)$ for every $x \in E$.

Theorem 3: Suppose f is a non-negative measurable function on $\mathbb{R}^d$. Then there exists an increasing sequence of non-negative simple functions $\{\phi\}_{k=1}^{\infty}$ that converges pointwise to f, namely,

$$\phi_k(x) \leq \phi_{k+1}(x) \text{ and } \lim_{k\to\infty} = f(x) \ \forall \ x.$$

Theorem 4: Suppose f is a measurable function on $\mathbb{R}^d$. Then there exists a sequence of simple functions $\{\phi\}_{k=1}^{\infty}$ that satisfies,

$$|\phi_k(x)| \leq |\phi_{k+1}(x)| \text{ and } \lim_{k\to\infty} = f(x) \ \forall \ x.$$

Theorem 5: Suppose f is measurable on $\mathbb{R}^d$. Then there exists a sequence of step functions $\{\psi\}_{k=0}^{\infty}$ that converges pointwise to $f(x)$ for almost every x.

Theorem 6 (Egorov): Suppose $\{f_k\}_{k=1}^{\infty}$ is a sequence of measurable functions defined on a measurable set E with $m(E) < \infty$, and assume that $f_k \to f$ a.e on E. Given $\epsilon > 0$, we can find a closed set $A_\epsilon \subset E$ such that $m(E - A_\epsilon) \leq \epsilon$ and $f_k \to f$ uniformly on A_ϵ.

Theorem 7 (Lusin): Suppose f is measurable and finite valued on E with E of finite measure. Then for every $\epsilon > 0$ there exists a closed set F_ϵ, with

$$F_\epsilon \subset E \text{ and } m(E - F_\epsilon) \leq \epsilon$$

and such that $f|_{F_\epsilon}$ is continuous.

Chapter 2

INTEGRATION THEORY

Definition 22 (Canonical Form of Simple Function): The canonical form of a simple function ϕ is the unique decomposition as below,

$$\phi = \sum_{k=1}^{M} c_k \chi_{F_k}$$

where the numbers c_k are distinct and non-zero, and the sets F_k are disjoint.
Property 19: If ϕ is a simple function with canonical form $\phi = \sum_{k=1}^{M} c_k \chi_{F_k}$, then we define the Lebesgue integral of ϕ by $\int_{\mathbb{R}^d} \phi(x)dx = \sum_{k=1}^{M} c_k m(Fk)$.
Property 20: If E is a measurable subset of $\mathbb{R}^d$ with finite measure, then $\phi(x)\chi_E(x)$ is also a simple function, and we define,

$$\int_E \phi(x)dx = \int \phi(x)\chi_E(x)dx$$

where second integral is over $\mathbb{R}^d$.
Proposition 1: The integral of simple functions defined above satisfies the following properties:

- Independence of the representation. If $\phi = \sum_{k=1}^{N} a_k \chi_{E_k}$ is any representation of ϕ, then

$$\int \phi = \sum_{k=1}^{N} a_k m(E_k)$$

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- Linearity. If ϕ and ψ are simple, and $a, b \in \mathbb{R}$, then

$$\int (a\phi + b\psi) = a \int \phi + b \int \psi$$

- Additivity. If E and F are disjoint subsets of $\mathbb{R}^d$ with finite measure, then

$$\int_{E \bigcup F} \phi = \int_E \phi + \int_F \phi$$

- Monotonicity. If $\phi \leq \psi$ are simple, then

$$\int \phi \leq \int \psi$$

- Triangle inequality. If ϕ is a simple function, then so is $|\phi|$, and

$$\left| \int \phi \right| \leq \int \left| \phi \right|$$