Theorems, Lemmas, Properties: Real Analysis and Measure Theory

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Ref: Real Analysis 3: Stein-Shakarachi

Chapter 1

MEASURE THEORY

1.1 Basic Definitions

Definition 1 (Open Ball): The open ball in $\mathbb{R}^d$ centered at x and of radius r is defined by

$$B_r(x) = \{y \in \mathbb{R}^d : |y - x| < r\}$$

Definition 2 (Open Set and Closed Set): A subset $E \subset \mathbb{R}^d$ is open if for every $x \in E$ there exists $r > 0$ with $B_r(x) \subset E$. By definition, a set is closed if its complement is open.

- Any (not necessarily countable) union of open sets is open
- The intersection of finitely many open sets is open
- Any (not necessarily countable) intersection of close sets is closed
- The union of finitely many close sets is close

Remark 1: A set can be neither closed nor open. Eg. $\Big[0, 1\Big)$ is neither closed nor open.

Definition 3 (Bounded Set and Compact Set): A set E is bounded if it is contained in some ball of finite radius. A bounded set is compact if it is also closed.

Property 1 (Heine-Borel covering property): Any covering of a compact set by a collection of open sets contains a finite subcovering.

Definition 4 (Limit Point): A point $x \in \mathbb{R}^d$ is a limit point of the set E if for every $r > 0$, the ball $B_r(x)$ contains points of E.

Definition 5 (Isolated Point): An isolated point of E is a point $x \in E$ such that there exists an $r > 0$ where $B_r(x) \bigcap E$ is equal to x.

Definition 6 (Interior): A point $x \in E$ is an interior point of E if there exists $r > 0$ such that $B_r(x) \subset E$. The set of all interior points of E is called the interior of E.

Definition 7 (Closure): The closure $\bar{E}$ of the E consists of the union of E and all its limit points.

Definition 8 (Boundary): The boundary of a set E, denoted by δE, is the set of points which are in the closure of E but not in the interior of E.

Property 2:

- The closure of a set is a closed set.
- Every point in E is a limit point of E.
- A set is closed if and only if it contains all its limit points.

Definition 9 (Perfect Set): A closed set E is perfect if E does not have any isolated points.

Definition 10 (Rectangle): A (closed) rectangle R in $\mathbb{R}^d$ is given by the product of d one-dimensional closed and bounded intervals,

$$R = [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_d, b_d]$$

where a_j, b_j are real numbers, $j = 1, 2, \ldots, d$. If all intervals are equal then it is a cube.

Lemma 1: If a rectangle is the almost disjoint (interior disjoint) union of finitely many other rectangles, say $R = \bigcup_{k=1}^{N} R_k$ then $|R| = \sum_{k=1}^{N} |R_k|$.

Lemma 2: If $R, R_1, \ldots, R_N$ are rectangles, and $R \subset \bigcup_{k=1}^{N} R_k$ then$|R| \leq \sum_{k=1}^{N} |R_k|$.

Theorem 1: Every open subset O of $\mathbb{R}$ can be written uniquely as a countable union of disjoint open intervals. In general, this is not true for $\mathbb{R}^d, d > 1$.

Theorem 2: Every open subset O of $\mathbb{R}^d, d \geq 1$, can be written as a countable union of almost disjoint closed rectangles (cubes).

Definition 11 (The Cantor Set): Let $C_0 = [0,1]$ and we get C_k by dividing each disjoint interval of C_{k-1} in three equal parts and deleting the middle part (end points are included in the set), then the Cantor set is defined as,

$$\mathcal{C} = \bigcap_{k=0}^{\infty} C_k$$

Property 3:

- $C_0 \supset C_1 \supset C_2 \cdots \supset C_k \supset C_{k+1} \ldots$.
- $\mathcal{C}$ is closed and bounded, hence compact.
- $\mathcal{C}$ is totally disconnected: given any $x, y \in \mathcal{C}$ there exists $z \notin \mathcal{C}$ that lies between x and y.
- $\mathcal{C}$ is perfect: it has no isolated points and it is closed.
- $\mathcal{C}$ is not countable: maps to power set of $\mathbb{N}$
- $\mathcal{C}$ has measure 0.
- $x \in \mathcal{C} \iff x = \sum_{k=0}^{\infty} a_k 3^{-k}, a_k \in \{0,2\}$

1.2 The Outer (Exterior) Measure

Definition 12 (Outer Measure): If E is any subset of $\mathbb{R}^d$, the outer measure of E is,

$$m_*(E) = \inf \sum_{k=1}^{\infty} \Big|R_k\Big|$$

where inf is taken over all countable coverings $E \subset \bigcup_{k=1}^{\infty} R_k$ by closed rectangles (cubes). Note that $0 \leq m_*(E) \leq \infty$.

Property 4: The outer measure of a point is zero.

Property 5: The outer measure of a closed rectangle (open rectangle) is equal to its volume.

Property 6: The outer measure of $\mathbb{R}^d$ is infinite.

Property 7: For every $\epsilon > 0$, there exists a covering $E \subset \bigcup_{j=1}^{\infty} R_j$ with

$$\sum_{j=1}^{\infty} m_*(R_j) \leq m_*(E) + \epsilon$$

Property 8 (Monotonicity): If $E_1 \subset E_2$, then $m_*(E1) \leq m_*(E_2)$.

Property 9 (Countable Sub-additivity): If $E = \bigcup_{j=1}^{\infty} E_j$, then $m_*(E) \leq \sum_{j=1}^{\infty} m_*(E_j)$.

Property 10: If $E \in \mathbb{R}^d$, then $m_*(E) = \inf m_*(\mathcal{O})$, where the inf is taken over all open sets $\mathcal{O}$ containing E.

Property 11: If $E = E_1 \bigcup E_2$ and $d(E_1, E_2) > 0$ then, $m_*(E) = m_*(E_1) + m_*(E_2)$ where $d(E_1, E_2) = \inf_{x \in E_1, y \in E_2} \Big|x - y\Big|$.

Property 12: If a set E is the countable union of almost disjoint rectangles $E = \bigcup_{k=1}^{\infty} R_k$, then $m_*(E) = \sum_{k=1}^{\infty} \Big|R_k\Big|$

Remark 2: Despite above two properties, in general it is **NOT TRUE** that if $E_1 \bigcup E_2$ is a disjoint union of subsets of $\mathbb{R}^d$, then

$$m_*(E_1 \bigcup E_2) = m_*(E_1) + m_*(E_2).$$

1.3 Measurable Sets and Lebesgue Measure

Definition 13 (Lebesgue Measurable): A subset E of $\mathbb{R}^d$ is Lebesgue measurable, or simply measurable, if for any $\epsilon > 0$ there exists an open set $\mathcal{O}$ with $E \subset \mathcal{O}$ and $m_*(\mathcal{O} - E) \leq \epsilon$. If E is measurable, we define its Lebesgue measure (or measure) $m(E)$ by $m(E) = m_*(E)$.

Property 13: Every open set in $\mathbb{R}^d$ is measurable.

Property 14: If $m_*(E) = 0$, then E is measurable. In particular, if F is a subset of a set of exterior measure 0, then F is measurable.

Property 15: A countable union of measurable sets is measurable.

Property 16: Closed sets are measurable.

Lemma 3: If F is closed, K is compact, and these sets are disjoint, then $d(F, K) > 0$.

Property 17: The complement of a measurable set is measurable.

Property 18: A countable intersection of measurable sets is measurable.

Remark 3: The operations of uncountable unions or intersections are not permissible when dealing with measurable sets!

Theorem 3: If $E_1, E_2, \dots$ are disjoint measurable sets, and $E = \bigcup_{j=1}^{\infty} E_j$, then $m(E) = \sum_{j=1}^{\infty} m(E_j)$.

Definition 14 ($\mathbf{E_k} \nearrow \mathbf{E}$): If $E_1, E_2, \dots$ is a countable collection of subsets of $S \in \mathbb{R}^d$ that increases to E in the sense that $E_1 \subseteq E_2 \subseteq \cdots \subseteq E_k \subseteq E_{k+1} \subseteq \dots$ and $E = \bigcup_{k=1}^{\infty} E_k$, then we write $E_k \nearrow E$.

Property 19: If $E_k \nearrow E$, then $m(E) = \lim_{k\to\infty} m(E_k)$

Definition 15 ($\mathbf{E_k} \searrow \mathbf{E}$): If $E_1, E_2, \dots$ is a countable collection of subsets of $S \in \mathbb{R}^d$ that decreases to E in the sense that $E_1 \supseteq E_2 \supseteq \cdots \supseteq E_k \supseteq E_{k+1} \supseteq \dots$ and $E = \bigcap_{k=1}^{\infty} E_k$, then we write $E_k \searrow E$.

Property 20: If $E_k \searrow E$ and $m(E_k) < \infty$ for some k, then $m(E) = \lim_{k\to\infty} m(E_k)$

Theorem 4: Suppose E is a measurable subset of $\mathbb{R}^d$. Then, for every $\epsilon > 0$:

- There exists an open set $\mathcal{O}$ with $E \subset \mathcal{O}$ and $m(\mathcal{O} - E) \leq \epsilon$.
- There exists a closed set F with $F \subset E$ and $m(E - F) \leq \epsilon$.
- If $m(E)$ is finite, there exists a compact set K with $K \subset E$ and $m(E - K) \leq \epsilon$.
- If $m(E)$ is finite, there exists a finite union $F = \bigcup_{j=1}^{N} R_j$ of closed rectangles (cubes) such that $m(E \triangle F) \leq \epsilon$, where symmetric difference $E \triangle F = (E - F) \bigcup (F - E)$.

Property 21 (Invariance):

- Translation Invariance. If E is a measurable set and $h \in \mathbb{R}^d$, then the set $E_h = E + h = \{x + h : x \in E\}$ is also measurable, and $m(E + h) = m(E)$.
- Dilation Invariance. If E is a measurable set and $\delta > 0 \in \mathbb{R}$, then the set $\delta E = \{\delta x : x \in E\}$ is also measurable, and $m(\delta E) = \delta^d m(E)$.
- Reflection Invariance. Whenever E is measurable, so is $-E = \{-x : x \in E\}$ and $m(-E) = m(E)$.

Definition 16 ($\mathbf{G_\delta}$ set): A G_δ set is, an intersection of a countable family of open sets.

Property 22: A subset E of $\mathbb{R}^d$ is measurable if and only if E differs from a G_δ by a set of measure zero.

Definition 17 ($\mathbf{F_\sigma}$ set): A F_σ set is, a union of a countable family of closed sets.

Property 23: A subset E of $\mathbb{R}^d$ is measurable if and only if E differs from a F_σ by a set of measure zero.

Definition 18 (σ-algebra): A σ-algebra of sets is a collection of subsets of $\mathbb{R}^d$ that is closed under countable unions, countable intersections, and complements. Example: Collection of all subsets of $\mathbb{R}^d$, collection of all measurable sets of $\mathbb{R}^d$

Definition 19 (Borel σ-algebra and Borel sets): A set $E \subseteq \mathbb{R}^d$ is an F_σ set provided that it is the countable union of closed sets and is a G_δ set if it is the countable intersection of open sets.The smallest σ-algebra that contains all open sets. Elements of this σ-algebra are called Borel sets.

Property 24: G_δ and F_σ are examples of Borel set.

Remark 4: Remember that it is possible to construct subsets of $\mathbb{R}^d$ which are not measurbale. Check reference book for construction of a non-measurable set $\mathcal{N} \subset \mathbb{R}$.

1.4 Measurable Functions

Definition 20 (Characteristic Function): A characteristic function of a set E is defined by

$$\chi_E(x) = \begin{cases} 1, & if\ x \in E \\ 0, & if\ x \notin E. \end{cases}$$

Definition 21 (Step Function): Step functions are defined as finite sum, $f = \sum_{k=1}^{N} a_k \chi_{R_k}$, where each R_k is a rectangle and each a_k is a constant. These are used in Reimann integral.

Definition 22 (Simple Function): A simple function is a finite sum, $f = \sum_{k=1}^{N} a_k \chi_{E_k}$, where each E_k is a measurable set of finite measure, and the a_k are constants. These are used in Lebesgue integral.

Definition 23 (Measurable Function): A function f defined on a measurable subset E of $\mathbb{R}^d$ is measurable, if for all $a \in \mathbb{R}$, the set

$$f^{-1}\big([-\infty, a)\big) = \big\{x \in E : f(x) < a\big\} = \big\{f < a\big\}$$

is measurable. Equivalently, $\{f \leq a\}, \{f > a\}, \{f \geq a\}$ are measurable. If f is finite valued then $\{a < f < b\}$ is measurable (with any combinations of $\leq, \geq$).

Property 25: The finite-valued function f is measurable if and only if $f^{-1}(O)$ is measurable for every open set O, and if and only if $f^{-1}(F)$ is measurable for every closed set F.

Property 26: If f is continuous on $\mathbb{R}^d$, then f is measurable. If f is measurable and finite-valued, and ϕ is continuous, then $\phi \circ f$ is measurable. But $f \circ \phi$ may not.

Property 27: Suppose $\{f_n\}_{n=1}^{\infty}$ is a sequence of measurable functions. Then

$$\sup_n f_n(x), \inf_n f_n(x), \limsup_{n\to\infty} f_n(x), \liminf_{n\to\infty} f_n(x)$$

are measurable.

Remark 5: lim inf is basically inf of limit points and likewise lim sup is sup of limit points. If there is just one limit point and limit exists then lim sup and lim inf are equal. Mathematically, $\liminf_{n\to\infty} f_n = \lim_{n\to\infty} \inf_{k\geq n} f_k$ and $\limsup_{n\to\infty} f_n = \lim_{n\to\infty} \sup_{k\geq n} f_k$

Property 28: Suppose $\{f_n\}_{n=1}^{\infty}$ is a sequence of measurable functions. Then

$$\lim_{n\to\infty} f_n(x) = f(x)$$

then f is measurable.

Property 29: If f and g are measurable, then,

- The integer powers $f^k, k \geq 1$ are measurable.
- $f + g$ and fg are measurable if both f and g are finite-valued.

Definition 24 (Almost Everywhere): We shall say that two functions f and g defined on a set E are equal almost everywhere, and write,

$$f(x) = g(x)\ a.e.\ x \in E,$$

if the set $\{x \in E : f(x) \neq g(x)\}$ has measure zero. All the properties above can be relaxed to conditions holding almost everywhere.

Property 30: Suppose f is measurable, and $f(x) = g(x)$ for a.e. x. Then g is measurable.

Definition 25 (Pointwise Convergence of a Function): Let $E \subset \mathbb{R}^d$ and let $\{f_n\}_{n=1}^{\infty}$ be a sequence of real valued functions defined on E. Then $\{f_n\}_{n=1}^{\infty}$ converges pointwise to f if given any x in E and given any $\epsilon > 0$, there exists a natural number $N(x, \epsilon)$ such that $|f_n(x) - f(x)| < \epsilon$ for every $n > N(x, \epsilon)$.

Definition 26 (Uniform Convergence of a Function): Let $E \subset \mathbb{R}^d$ and let $\{f_n\}_{n=1}^{\infty}$ be a sequence of real valued functions defined on E. Then $\{f_n\}_{n=1}^{\infty}$ converges uniformly to f if given any $\epsilon > 0$, there exists a natural number $N(\epsilon)$ such that $|f_n(x) - f(x)| < \epsilon$ for every $n > N(\epsilon)$ for every $x \in E$.

Theorem 5: Suppose f is a non-negative measurable function on $\mathbb{R}^d$. Then there exists an increasing sequence of non-negative simple functions $\{\varphi\}_{k=1}^{\infty}$ that converges pointwise to f, namely,

$$\varphi_k(x) \leq \varphi_{k+1}(x) \text{ and } \lim_{k \to \infty} = f(x) \; \forall \; x.$$

Theorem 6: Suppose f is a measurable function on $\mathbb{R}^d$. Then there exists a sequence of simple functions $\{\varphi\}_{k=1}^{\infty}$ that satisfies,

$$|\varphi_k(x)| \leq |\varphi_{k+1}(x)| \text{ and } \lim_{k \to \infty} = f(x) \; \forall \; x.$$

Theorem 7: Suppose f is measurable on $\mathbb{R}^d$. Then there exists a sequence of step functions $\{\psi\}_{k=0}^{\infty}$ that converges pointwise to $f(x)$ for almost every x.

Theorem 8 (Egorov): Suppose $\{f_k\}_{k=1}^{\infty}$ is a sequence of measurable functions defined on a measurable set E with $m(E) < \infty$, and assume that $f_k \to f$ a.e on E. Given $\epsilon > 0$, we can find a closed set $A_\epsilon \subset E$ such that $m(E - A_\epsilon) \leq \epsilon$ and $f_k \to f$ uniformly on A_ϵ.

Theorem 9 (Lusin): Suppose f is measurable and finite valued on E with E of finite measure. Then for every $\epsilon > 0$ there exists a closed set F_ϵ, with

$$F_\epsilon \subset E \text{ and } m(E - F_\epsilon) \leq \epsilon$$

and such that $f|_{F_\epsilon}$ is continuous.

Chapter 2

INTEGRATION THEORY

2.1 The Lebesgue Integral: Basic Properties and Convergence Theorems

Property 31: By definition, all integrable functions are measurable.

Definition 27 (Canonical Form of Simple Function): The canonical form of a simple function φ is the unique decomposition as below,

$$\varphi = \sum_{k=1}^{M} c_k \chi_{F_k}$$

where the numbers c_k are distinct and non-zero, and the sets F_k are disjoint.

Property 32: If φ is a simple function with canonical form $\varphi = \sum_{k=1}^{M} c_k \chi_{F_k}$, then we define the Lebesgue integral of φ by $\int_{\mathbb{R}^d} \varphi(x)dx = \sum_{k=1}^{M} c_k m(Fk)$.

Property 33: If E is a measurable subset of $\mathbb{R}^d$ with finite measure, then $\varphi(x)\chi_E(x)$ is also a simple function, and we define,

$$\int_E \varphi(x)dx = \int \varphi(x)\chi_E(x)dx$$

where second integral is over $\mathbb{R}^d$.

Proposition 1: The integral of simple functions defined above satisfies the following properties:

- Independence of the representation. If $\varphi = \sum_{k=1}^{N} a_k \chi_{E_k}$ is any representation of φ, then $\int \varphi = \sum_{k=1}^{N} a_k m(E_k)$.
- Linearity. If φ and ψ are simple, and $a, b \in \mathbb{R}$, then $\int (a\varphi + b\psi) = a\int \varphi + b\int \psi$
- Additivity. If E and F are disjoint subsets of $\mathbb{R}^d$ with finite measure, then $\int_{E\bigcup F} \varphi = \int_E \varphi + \int_F \varphi$
- Monotonicity. If $\varphi \leq \psi$ are simple, then $\int \varphi \leq \int \psi$
- Triangle inequality. If φ is a simple function, then so is $|\varphi|$, and $\left|\int \varphi\right| \leq \int |\varphi|$

Definition 28 (Support of a Function): Support is defined as $\mathrm{supp}(f) = \{x : f(x) \neq 0\}$. We shall say that f is supported on a set E, if $f(x) = 0$ whenever $x \notin E$.

Lemma 4: Let f be a bounded function supported on a set E of finite measure. If $\{\varphi_n\}_{n=1}^{\infty}$ is any sequence of simple functions bounded by M, supported on E, and with $\varphi_n(x) \to f(x)$ for a.e. x, then:

- The limit $\lim_{n\to\infty} \int \varphi_n$ exists.
- If $f = 0$ a.e., then the limit $\lim_{n\to\infty} \int \varphi_n$ equals to 0.

Definition 29 (Lebesgue Integral of Bounded Functions Supported on Sets of Finite Measure): For such a function f, we define its Lebesgue Integral by,

$$\int f(x)dx = \lim_{n\to\infty} \int \varphi_n(x)dx$$

where $\{\varphi_n\}$ is **any** sequence of simple functions satisfying: $|\varphi_n| \leq M$, each φ_n is supported on the support of f, and $\varphi_n(x) \to f(x)$ for a.e. x as n tends to infinity.

Property 34: If E is a subset of $\mathbb{R}^d$ with finite measure, and f is bounded with $m(supp(f)) < \infty$, then:

$$\int_E f(x)dx = \int f(x)\chi_E(x)dx$$

Proposition 2: Suppose f and g are bounded functions supported on sets of finite measure. Then the following properties hold.

- Linearity. If $a, b \in \mathbb{R}$, then $\int (af + bg) = a\int f + b\int g$.
- Additivity. If E and F are disjoint subsets of $\mathbb{R}^d$, then $\int_{E\bigcup F} f = \int_E f + \int_F f$.
- Monotonicity. If $f \leq g$, then $\int f \leq \int g$.
- Triangle Inequality. $|f|$ is also bounded, supported on a set of finite measure, and $\left|\int f\right| \leq \int |f|$.

Theorem 10 (Bounded convergence theorem): Suppose that $\{f_n\}$ is a sequence of measurable functions that are all bounded by M, are supported on a set E of finite measure, and $f_n(x) \to f(x)$ a.e. x as $n \to \infty$. Then f is measurable, bounded, supported on E for a.e. x, and

$$\int |f_n - f| \to 0, \ as\ n \to \infty.$$

Consequently,

$$\int f_n \to \int f, \ as\ n \to \infty.$$

Theorem 11: Suppose f is Riemann integrable on the closed interval $[a, b]$. Then f is measurable, and

$$\int_{[a,b]}^{\mathcal{R}} f(x)dx = \int_{[a,b]}^{\mathcal{L}} f(x)dx,$$

where the integral on the left-hand side is the standard Riemann integral, and that on the right-hand side is the Lebesgue integral. Note that if f is Riemann integrable, then f is bounded.

Definition 30 (Lebesgue Integral for Non-negative Functions): For non-negative functions f we define its (extended) Lebesgue integral by

$$\int f(x)dx = \sup_g \int g(x)dx$$

where the supremum is taken over all measurable functions g such that $0 \leq g \leq f$, and where g is bounded and supported on a set of finite measure. If $\int f(x)dx < \infty$ then f is said to be Lebesgue integrable or simply integrable.

Property 35: The integral of non-negative measurable functions enjoys the following properties:

- Linearity. If $f, g \geq 0$, and a, b are positive real numbers, then $\int (af + bg) = a\int f + b\int g$.
- Additivity. If E and F are disjoint subsets of $\mathbb{R}^d$, and $f \geq 0$, then $\int_{E\bigcup F} f = \int_E f + \int_F f$.
- Monotonicity. If $0 \leq f \leq g$, then $\int f \leq \int g$.
- If g is integrable and $0 \leq f \leq g$, then f is integrable.
- If f is integrable, then $f(x) < \infty$ for almost every x.
- If $f = 0$, then $f(x) = 0$ for almost every x.

Lemma 5 (Fatou): Suppose $\{f_n\}$ is a sequence of measurable functions with $f_n \geq 0$. If $\lim_{n\to\infty} f_n(x) = f(x)$ for a.e. x, then

$$\int f \leq \liminf_{n\to\infty} \int f_n$$

Corollary 1: Suppose f is a non-negative measurable function, and $\{f_n\}$ a sequence of non-negative measurable functions with $f_n(x) \leq f(x)$ and $f_n(x) \to f(x)$ for almost every x. Then $\lim_{n\to\infty} \int f_n = \int f$.

Definition 31 ($\mathbf{f_n \nearrow f}$ and $\mathbf{f_n \searrow f}$): we shall write $fn \nearrow f$ whenever $\{f_n\}_{n=1}^{\infty}$ is a sequence of measurable functions that satisfies $f_n(x) \leq f_{n+1}(x)$ a.e x, all $n \geq 1$ and $\lim_{n\to\infty} f_n(x) = f(x)$ a.e x. Similarly, we write $fn \searrow f$ whenever $\{f_n\}_{n=1}^{\infty}$ is a sequence of measurable functions that satisfies $f_n(x) \geq f_{n+1}(x)$ a.e x, all $n \geq 1$ and $\lim_{n\to\infty} f_n(x) = f(x)$ a.e x.

Theorem 12 (Monotone Convergence Theorem): Suppose $\{f_n\}$ is a sequence of non-negative measurable functions with $f_n \nearrow f$. Then $\lim_{n\to\infty} \int f_n = \int f$.

Corollary 2: Consider a series $\sum_{k=1}^{\infty} a_k(x)$, where $a_k(x) \geq 0$ is measurable for every $k \geq 1$. Then, $\int \sum_{k=1}^{\infty} a_k(x)dx = \sum_{k=1}^{\infty} \int a_k(x)dx$. If $\sum_{k=1}^{\infty} \int a_k(x)dx$ is finite, then the series $\sum_{k=1}^{\infty} a_k(x)$ converges for a.e. x.

Definition 32 (Lebesgue Integral for Any Real Valued Function): If f is any real-valued measurable function on $\mathbb{R}^d$, we say that f is Lebesgue integrable (or just integrable) if the non-negative measurable function $|f|$ is integrable in the sense of the previous definitions. Let f be Lebesgue integral and $f^+(x) = \max(f(x), 0)$, $f^-(x) = \max(-f(x), 0)$, so that f^+ and f^- are non-negative and $f = f^+ - f^-$. Since $f^{\pm} \leq |f|$, both functions f^+ and f^- are integrable whenever f is, and we then define the Lebesgue integral of f by, $\int f = \int f^+ - \int f^-$.

Property 36: If $f = f_1 - f_2$ such that $f_1, f_2 \geq 0$ then $\int f = \int f_1 - \int f_2$.

Proposition 3: The integral of Lebesgue integrable functions is linear, additive, monotonic, and satisfies the triangle inequality.

Proposition 4: Suppose f is integrable on $\mathbb{R}^d$. Then for every $\epsilon > 0$:

1. There exists a set of finite measure B (a ball, for example) such that $\int_{B^c} |f| < \epsilon$.
2. There is a $\delta > 0$ such that $\int_E |f| < \epsilon$ whenever $m(E) < \delta$. This is known as absolute continuity.

Theorem 13 (Dominated Convergence Theorem): Suppose $\{f_n\}$ is a sequence of measurable functions such that $f_n(x) \to f(x)$ a.e. x, as n tends to infinity. If $|f_n(x)| \leq g(x)$, where g is integrable, then $|f_n - f| \to 0$ as $n \to \infty$ and consequently $\int f_n \to \int f$ as $n \to \infty$.

Definition 33 (Complex Valued Function): If f is a complex-valued function on $\mathbb{R}^d$, we may write it as $f(x) = u(x)+iv(x)$ where u and v are real-valued functions called the real and imaginary parts of f, respectively.

Property 37: The function f is measurable if and only if both u and v are measurable.

Definition 34 (Lebesgue Integrable Complex Valued Functions): We say that f is Lebesgue integrable if the function $|f(x)| = (u(x)^2 + v(x)^2)^{\frac{1}{2}}$ (which is non-negative) is Lebesgue integrable in the sense defined previously. The Lebesgue integral of f is defined by $\int f(x)dx = \int u(x)dx + i \int v(x)dx$.

Property 38: $|u(x)| \leq |f(x)|$, $|v(x)| \leq |f(x)|$, $|f(x)| \leq |u(x)| + |v(x)|$

Property 39: If f and g are integrable then so is $f + g$. Also, if $a \in \mathbb{C}$ and if f is integrable then so is af.

2.2 The Space L^1 of Integrable Functions

Definition 35 (Norm ($\mathbf{L^1}$-Norm) of f): For any integrable function f on $\mathbb{R}^d$ we define the norm of f, $\|f\| = \|f\|_{L^1} = \|f\|_{L^1(\mathbb{R}^d)} = \int_{\mathbb{R}^d} |f(x)|dx$.

Definition 36 ($\mathbf{L^1(\mathbb{R}^d)}$): $L^1(\mathbb{R}^d)$ is the space of equivalence classes of integrable functions, where we define two functions to be equivalent if they agree almost everywhere.

Property 40: $L^1(\mathbb{R}^d)$ is a vector space.

Proposition 5: Suppose f and g are two functions in $L^1(\mathbb{R}^d)$,

- $\|af\|_{L^1(\mathbb{R}^d)} = |a|\|f\|_{L^1(\mathbb{R}^d)}, \forall a \in \mathbb{C}$
- $\|f+g\|_{L^1(\mathbb{R}^d)} \leq \|f\|_{L^1(\mathbb{R}^d)} + \|g\|_{L^1(\mathbb{R}^d)}$
- $\|f\|_{L^1(\mathbb{R}^d)} = 0$ if and only if $f = 0$, a.e.
- $d(f,g) = \|f-g\|_{L^1(\mathbb{R}^d)}$ defines a metric on $L^1(\mathbb{R}^d)$ i.e. $d(f,g) \geq 0$, $d(f,g) = d(g,f)$ and $d(f,g) \leq d(f,h) + d(h,g) \forall f,g,h \in L^1(\mathbb{R}^d)$

Definition 37 (Complete Space): A space V with a metric d is said to be complete if for every Cauchy sequence $\{x_k\}$ in V (that is, $\forall \epsilon > 0, \exists K_\epsilon$ such that $d(x_r, x_s) < \epsilon \forall r, s \geq K_\epsilon$) there exists $x \in V$ such that $\lim_{k\to\infty} x_k = x$ in the sense that $d(x_k, x) \to 0$ as $k \to \infty$.

Theorem 14 (Riesz-Fischer Theorem): The vector space L^1 is complete in its metric. Suppose $\{f_n\}$ is a Cauchy sequence in the norm, so that $\|f_n - f_m\| \to 0$ as $n, m \to \infty$. The plan of the proof is to extract a subsequence of $\{f_n\}$ that converges to f, both pointwise almost everywhere and in the norm.

Corollary 3: If $\{f_n\}_{n=1}^{\infty}$ converges to f in L^1, then there exists a subsequence $\{f_{n_k}\}$ such that $f_{n_k} \to f(x)$ a.e. x.

Definition 38 (Dense in $\mathbf{L^1}$): We say that a family $\mathcal{G}$ of integrable functions is dense in L^1 if for any $f \in L^1$ and $\epsilon > 0$, there exists $g \in \mathcal{G}$ so that $\|f - g\|_{L^1} < \epsilon$.

Theorem 15: The following families of functions are dense in $L^1(\mathbb{R}^d)$:

- The simple functions.
- The step functions.
- The continuous functions of compact support.
- The continuous smooth functions of compact support.

Property 41 (Invariance Properties):

- Translation. Let $f_h(x) = f(x-h)$ then $\int f_h = \int f$
- Dilation. $\delta^d \int f(\delta x) = \int f(x)$
- Reflection. $\int f(-x) = \int f(x)$
- $\int f(x-y)g(y)dy = \int f(y)g(x-y)dy$

Proposition 6: Suppose $f \in L^1(\mathbb{R}^d)$. Then $\|f_h - f\|_{L^1} \to 0$ as $h \to 0$.

2.3 Fubini's Theorem

Property 42: We may write $\mathbb{R}^d$ as a product $\mathbb{R}^d = \mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$ where $d_1 + d_2 = d$ and $d_1, d_2 \geq 1$.

Definition 39 (Slice of a Function): If f is a function in $\mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$, the slice of f corresponding to $y \in \mathbb{R}^{d_2}$ is the function f^y of the $x \in \mathbb{R}^{d_1}$ variable, given by $f^y(x) = f(x,y)$. Similarly, the slice of f for a fixed $x \in \mathbb{R}^{d_1}$ is $f_x(y) = f(x,y)$.

Definition 40 (Slice of a Set): In the case of a set $E \in \mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$ we define its slices by $E^y = \{x \in \mathbb{R}^{d_1} : (x,y) \in E\}$ and $E_x = \{y \in \mathbb{R}^{d_2} : (x,y) \in E\}$.

Theorem 16 (Fubini's Theorem): Suppose $f(x,y)$ is integrable on $\mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$. Then for almost every $y \in \mathbb{R}^{d_2}$:

1. The slice f^y is integrable on $\mathbb{R}^{d_1}$.
2. The function defined by $\int_{\mathbb{R}^{d_1}} f^y(x)dx$ is integrable on $\mathbb{R}^{d_2}$. Moreover,
3. $\int_{\mathbb{R}^{d_2}} \left(\int_{\mathbb{R}^{d_1}} f(x,y)dx \right) dy = \int_{\mathbb{R}^d} f$.

Property 43: The theorem is symmetric in x and y, hence the order can be interchanged.

Proof of Fubini's theorem. We begin by letting $\mathcal{F}$ denote the set of integrable functions on $\mathbb{R}^d$ which satisfy all three conclusions in the theorem, and set out to prove that $L^1(\mathbb{R}^d) \in \mathcal{F}$.

Property 44: Any finite linear combination of functions in $\mathcal{F}$ also belongs to $\mathcal{F}$.

Property 45: Suppose $\{f_k\}$ is a sequence of measurable functions in $\mathcal{F}$ so that $f_k \nearrow f$ or $f_k \searrow f$, where f is integrable (on $\mathbb{R}^d$). Then $f \in \mathcal{F}$.

Property 46: Any characteristic function of a set E that is a G_δ and of finite measure belongs to $\mathcal{F}$.

Property 47: If E has measure 0, then χ_E belongs to $\mathcal{F}$.

Property 48: If E is any measurable subset of $\mathbb{R}^d$ with finite measure, then χ_E belongs to $\mathcal{F}$.

Property 49: If f is integrable, then $f \in F$.

□

Theorem 17: Suppose $f(x, y)$ is a non-negative measurable function on $\mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$. Then for almost every $y \in \mathbb{R}^{d_2}$:

1. The slice f^y is measurable on $\mathbb{R}^{d_1}$.
2. The function defined by $\int_{\mathbb{R}^{d_1}} f^y(x)dx$ is measurable on $\mathbb{R}^{d_2}$.
3. $\int_{\mathbb{R}^{d_2}} \left(\int_{\mathbb{R}^{d_1}} f(x, y)dx\right)dy = \int_{\mathbb{R}^d} f(x, y)dxdy$ in the extended sense.

Corollary 4: If E is a measurable set in $\mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$, then for almost every $y \in \mathbb{R}^{d_2}$ the slice $E^y = \{x \in \mathbb{R}^{d_1} : (x, y) \in E\}$ is a measurable subset of $\mathbb{R}^{d_1}$. Moreover $m(E^y)$ is a measurable function of y and $m(E) = \int_{\mathbb{R}^{d_2}} m(E^y)dy$.

Proposition 7: If $E = E_1 \times E_2$ is a measurable subset of $\mathbb{R}^d$, and $m_*(E2) > 0$, then E_1 is measurable.

Lemma 6: If $E_1 \subset \mathbb{R}^{d_1}$ and $E_2 \subset \mathbb{R}^{d_2}$, then $m_*(E_1 \times E_2) \le m_*(E_1)m_*(E_2)$, with the understanding that if one of the sets E_j has exterior measure zero, then $m_*(E_1 \times E_2) = 0$.

Proposition 8: Suppose E_1 and E_2 are measurable subsets of $\mathbb{R}^{d_1}$ and $\mathbb{R}^{d_2}$, respectively. Then $E = E_1 \times E_2$ is a measurable subset of $\mathbb{R}^d$. Moreover, $m(E) = m(E_1)m(E_2)$ with the understanding that if one of the sets E_j has measure zero, then $m(E) = 0$.

Corollary 5: Suppose f is a measurable function on $\mathbb{R}^{d_1}$. Then the function $\tilde{f}$ defined by $\tilde{f}(x, y) = f(x)$ is measurable on $\mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$.

Corollary 6: Suppose $f(x)$ is a non-negative function on $\mathbb{R}^d$, and let $\mathcal{A} = \{(x, y) \in \mathbb{R}^d \times \mathbb{R} : 0 \le y \le f(x)\}$. Then:

1. f is measurable on $\mathbb{R}^d$ if and only if $\mathcal{A}$ is measurable in $\mathbb{R}^{d+1}$.
2. If the conditions in 1 hold, then $\int_{\mathbb{R}^d} f(x)dx = m(\mathcal{A})$

Proposition 9: If f is a measurable function on $\mathbb{R}^d$, then the function $\tilde{f}(x, y) = f(x - y)$ is measurable on $\mathbb{R}^d \times \mathbb{R}^d$.

2.4 A Fourier Inversion Formula

Definition 41 (Fourier Inversion):

$$\hat{f}(\xi) = \int_{\mathbb{R}^d} f(x)e^{-2\pi i x.\xi}dx \tag{2.1}$$

$$f(x) = \int_{\mathbb{R}^d} \hat{f}(\xi)e^{2\pi i x.\xi}d\xi \tag{2.2}$$

Proposition 10: Suppose $f \in L^1(\mathbb{R}^d)$. Then $\hat{f}$ defined by (2.1) is continuous and bounded on $\mathbb{R}^d$.

Theorem 18: Suppose $f \in L^1(\mathbb{R}^d)$ and assume also that $\hat{f} \in L^1(\mathbb{R}^d)$. Then the inversion formula (2.1) holds for almost every x.

Corollary 7: Suppose $\hat{f}(\xi) = 0$ for all ξ. Then $f = 0$ a.e.

Lemma 7: Suppose f and g belong to $L^1(\mathbb{R}^d)$. Then $\int_{\mathbb{R}^d} \hat{f}(\xi)g(\xi)d\xi = \int_{\mathbb{R}^d} f(y)\hat{g}(y)dy$.