Theorems, Lemmas, Properties: Real Analysis and Measure Theory

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Ref: Real Analysis 3: Stein-Shakarachi

Chapter 1

MEASURE THEORY

1.1 Basic Definitions

Definition 1 (Open Ball). *The open ball in $\mathbb{R}^d$ centered at x and of radius r is defined by*

$$B_r(x) = \left\{ y \in \mathbb{R}^d : |y - x| < r \right\}$$

Definition 2 (Open Set and Closed Set). *A subset $E \subset \mathbb{R}^d$ is open if for every $x \in E$ there exists $r > 0$ with $B_r(x) \subset E$. By definition, a set is closed if its complement is open.*

- *Any (not necessarily countable) union of open sets is open*
- *The intersection of finitely many open sets is open*
- *Any (not necessarily countable) intersection of close sets is closed*
- *The union of finitely many close sets is close*

Definition 3 (Bounded Set and Compact Set). *A set E is bounded if it is contained in some ball of finite radius. A bounded set is compact if it is also closed.*

Property 1 (Heine-Borel covering property). *Any covering of a compact set by a collection of open sets contains a finite subcovering.*

Definition 4 (Limit Point). *A point $x \in \mathbb{R}^d$ is a limit point of the set E if for every $r > 0$, the ball $B_r(x)$ contains points of E.*

Definition 5 (Isolated Point). *An isolated point of E is a point $x \in E$ such that there exists an $r > 0$ where $B_r(x) \backslash E$ is equal to x.*

Definition 6 (Interior). *A point $x \in E$ is an interior point of E if there exists $r > 0$ such that $B_r(x) \subset E$. The set of all interior points of E is called the interior of E.*

Definition 7 (Closure). *The closure $\bar{E}$ of the E consists of the union of E and all its limit points.*

Definition 8 (Boundary). *The boundary of a set E, denoted by δE, is the set of points which are in the closure of E but not in the interior of E.*

Property 2.

- *The closure of a set is a closed set.*
- *Every point in E is a limit point of E.*
- *A set is closed if and only if it contains all its limit points.*

Definition 9 (Perfect Set). *A closed set E is perfect if E does not have any isolated points.*

Definition 10 (Rectangle). *A (closed) rectangle R in $\mathbb{R}^d$ is given by the product of d one-dimensional closed and bounded intervals,*

$$R = [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_d, b_d]$$

where a_j, b_j are real numbers, $j = 1, 2, \ldots, d$. If all intervals are equal then it is a cube.

Lemma 1. *If a rectangle is the almost disjoint (interior disjoint) union of finitely many other rectangles, say* $R = \bigcup_{k=1}^{N} R_k$ *then* $|R| = \sum_{k=1}^{N} |R_k|$.

Lemma 2. *If* $R, R_1, \ldots, R_N$ *are rectangles, and* $R \subset \bigcup_{k=1}^{N} R_k$ *then*$|R| \leq \sum_{k=1}^{N} |R_k|$.

Theorem 1. *Every open subset* O *of* $\mathbb{R}$ *can be written uniquely as a countable union of disjoint open intervals. In general, this is not true for* $\mathbb{R}^d, d > 1$.

Theorem 2. *Every open subset* O *of* $\mathbb{R}^d, d \geq 1$, *can be written as a countable union of almost disjoint closed rectangles (cubes).*

Definition 11 (The Cantor Set). *Let* $C_0 = [0, 1]$ *and we get* C_k *by dividing each disjoint interval of* C_{k-1} *in three equal parts and deleting the middle part (end points are included in the set), then the Cantor set is defined as,*

$$\mathcal{C} = \bigcap_{k=0}^{\infty} C_k$$

Property 3.

- $C_0 \supset C_1 \supset C_2 \cdots \supset C_k \supset C_{k+1} \ldots.$
- $\mathcal{C}$ *is closed and bounded, hence compact.*
- $\mathcal{C}$ *is totally disconnected: given any* $x, y \in \mathcal{C}$ *there exists* $z \notin \mathcal{C}$ *that lies between* x *and* y.
- $\mathcal{C}$ *is perfect: it has no isolated points and it is closed.*
- $\mathcal{C}$ *is not countable: maps to power set of* $\mathbb{N}$
- $\mathcal{C}$ *has measure 0.*
- $x \in \mathcal{C} \iff x = \sum_{k=0}^{\infty} a_k 3^{-k}, a_k \in \{0, 2\}$

1.2 The Outer (Exterior) Measure

Definition 12 (Outer Measure)**.** *If* E *is any subset of* $\mathbb{R}^d$, *the outer measure of* E *is,*

$$m_*(E) = \inf \sum_{k=1}^{\infty} \Big| R_k \Big|$$

where $\inf$ *is taken over all countable coverings* $E \subset \bigcup_{k=1}^{\infty} R_k$ *by closed rectangles (cubes). Note that* $0 \leq m_*(E) \leq \infty$.

Property 4. *The outer measure of a point is zero.*

Property 5. *The outer measure of a closed rectangle (open rectangle) is equal to its volume.*

Property 6. *The outer measure of* $\mathbb{R}^d$ *is infinite.*

Property 7. *For every* $\epsilon > 0$, *there exists a covering* $E \subset \bigcup_{j=1}^{\infty} R_j$ *with*

$$\sum_{j=1}^{\infty} m_*(R_j) \leq m_*(E) + \epsilon$$

Property 8 (Monotonicity). *If* $E_1 \subset E_2$, *then* $m_*(E1) \leq m_*(E_2)$.

Property 9 (Countable Sub-additivity). *If* $E = \bigcup_{j=1}^{\infty} E_j$, *then* $m_*(E) \leq \sum_{j=1}^{\infty} m_*(E_j)$.

Property 10. *If* $E \in \mathbb{R}^d$, *then* $m_*(E) = \inf m_*(\mathcal{O})$, *where the* $\inf$ *is taken over all open sets* $\mathcal{O}$ *containing* E.

Property 11. *If* $E = E_1 \bigcup E_2$ *and* $d(E_1, E_2) > 0$ *then,* $m_*(E) = m_*(E_1) + m_*(E_2)$ *where* $d(E_1, E_2) = \inf_{x \in E_1, y \in E_2} \Big| x - y \Big|$.

Property 12. *If a set* E *is the countable union of almost disjoint rectangles* $E = \bigcup_{k=1}^{\infty} R_k$, *then* $m_*(E) = \sum_{k=1}^{\infty} \Big| R_k \Big|$

Need to add more

1.3 Measurable Sets and Lebesgue Measure

Definition 13 (σ-algebra). *A σ-algebra of sets is a collection of subsets of $\mathbb{R}^d$ that is closed under countable unions, countable intersections, and complements. Example: Collection of all subsets of $\mathbb{R}^d$, collection of all measurable sets of $\mathbb{R}^d$*

Definition 14 (Borel σ-algebra and Borel sets). *A set $E \subseteq \mathbb{R}^d$ is an F_σ set provided that it is the countable union of closed sets and is a G_δ set if it is the countable intersection of open sets.The smallest σ-algebra that contains all open sets. Elements of this σ-algebra are called Borel sets.*

1.4 Measurable Functions

Definition 15 (Characteristic Function). *A characteristic function of a set E is defined by*

$$\chi_E(x) = \begin{cases} 1, & if\ x \in E \\ 0, & if\ x \notin E. \end{cases}$$

Definition 16 (Step Function). *Step functions are defined as finite sum, $f = \sum_{k=1}^{N} a_k \chi_{R_k}$, where each R_k is a rectangle and each a_k is a constant. These are used in Reimann integral.*

Definition 17 (Simple Function). *A simple function is a finite sum, $f = \sum_{k=1}^{N} a_k \chi_{E_k}$, where each E_k is a measurable set of finite measure, and the a_k are constants. These are used in Lebesgue integral.*

Definition 18 (Measurable Function). *A function f defined on a measurable subset E of $\mathbb{R}^d$ is measurable, if for all $a \in \mathbb{R}$, the set*

$$f^{-1}\big([-\infty, a)\big) = \big\{x \in E : f(x) < a\big\} = \big\{f < a\big\}$$

is measurable. Equivalently, $\{f \leq a\}, \{f > a\}, \{f \geq a\}$ are measurable. If f is finite valued then $\{a < f < b\}$ is measurable (with any combinations of $\leq, \geq$).

Property 13. *The finite-valued function f is measurable if and only if $f^{-1}(O)$ is measurable for every open set O, and if and only if $f^{-1}(F)$ is measurable for every closed set F.*

Property 14. *If f is continuous on $\mathbb{R}^d$, then f is measurable. If f is measurable and finite-valued, and ϕ is continuous, then $\phi \circ f$ is measurable. But $f \circ \phi$ may not.*

Property 15. *Suppose $\{f_n\}_{n=1}^{\infty}$ is a sequence of measurable functions. Then*

$$\sup_n f_n(x), \inf_n f_n(x), \limsup_{n\to\infty} f_n(x), \liminf_{n\to\infty} f_n(x)$$

are measurable.

Property 16. *Suppose $\{f_n\}_{n=1}^{\infty}$ is a sequence of measurable functions. Then*

$$\lim_{n\to\infty} f_n(x) = f(x)$$

then f is measurable.

Property 17. *If f and g are measurable, then,*

- *The integer powers $f^k, k \geq 1$ are measurable.*
- *$f + g$ and fg are measurable if both f and g are finite-valued.*

Definition 19 (Almost Everywhere). *We shall say that two functions f and g defined on a set E are equal almost everywhere, and write,*

$$f(x) = g(x)\ a.e.\ x \in E,$$

if the set $\{x \in E : f(x) \neq g(x)\}$ has measure zero. All the properties above can be relaxed to conditions holding almost everywhere.

Property 18. *Suppose f is measurable, and $f(x) = g(x)$ for a.e. x. Then g is measurable.*

Definition 20 (Pointwise Convergence of a Function)**.** *Let $E \subset \mathbb{R}^d$ and let $\{f_n\}_{n=1}^{\infty}$ be a sequence of real valued functions defined on E. Then $\{f_n\}_{n=1}^{\infty}$ converges pointwise to f if given any x in E and given any $\epsilon > 0$, there exists a natural number $N(x, \epsilon)$ such that $|f_n(x) - f(x)| < \epsilon$ for every $n > N(x, \epsilon)$.*

Definition 21 (Uniform Convergence of a Function)**.** *Let $E \subset \mathbb{R}^d$ and let $\{f_n\}_{n=1}^{\infty}$ be a sequence of real valued functions defined on E. Then $\{f_n\}_{n=1}^{\infty}$ converges uniformly to f if given any $\epsilon > 0$, there exists a natural number $N(\epsilon)$ such that $|f_n(x) - f(x)| < \epsilon$ for every $n > N(\epsilon)$ for every $x \in E$.*

Theorem 3. *Suppose f is a non-negative measurable function on $\mathbb{R}^d$. Then there exists an increasing sequence of non-negative simple functions $\{\phi\}_{k=1}^{\infty}$ that converges pointwise to f, namely,*

$$\phi_k(x) \leq \phi_{k+1}(x) \text{ and } \lim_{k\to\infty} = f(x) \ \forall\ x.$$

Theorem 4. *Suppose f is a measurable function on $\mathbb{R}^d$. Then there exists a sequence of simple functions $\{\phi\}_{k=1}^{\infty}$ that satisfies,*

$$|\phi_k(x)| \leq |\phi_{k+1}(x)| \text{ and } \lim_{k\to\infty} = f(x) \ \forall\ x.$$

Theorem 5. *Suppose f is measurable on $\mathbb{R}^d$. Then there exists a sequence of step functions $\{\psi\}_{k=0}^{\infty}$ that converges pointwise to $f(x)$ for almost every x.*

Theorem 6 (Egorov)**.** *Suppose $\{f_k\}_{k=1}^{\infty}$ is a sequence of measurable functions defined on a measurable set E with $m(E) < \infty$, and assume that $f_k \to f$ a.e on E. Given $\epsilon > 0$, we can find a closed set $A_\epsilon \subset E$ such that $m(E - A_\epsilon) \leq \epsilon$ and $f_k \to f$ uniformly on A_ϵ.*

Theorem 7 (Lusin)**.** *Suppose f is measurable and finite valued on E with E of finite measure. Then for every $\epsilon > 0$ there exists a closed set F_ϵ, with*

$$F_\epsilon \subset E \text{ and } m(E - F_\epsilon) \leq \epsilon$$

and such that $f|_{F_\epsilon}$ is continuous.

Chapter 2

INTEGRATION THEORY

Definition 22 (Canonical Form of Simple Function)**.** *The canonical form of a simple function ϕ is the unique decomposition as below,*

$$\phi = \sum_{k=1}^{M} c_k \chi_{F_k}$$

where the numbers c_k are distinct and non-zero, and the sets F_k are disjoint.

Property 19. *If ϕ is a simple function with canonical form $\phi = \sum_{k=1}^{M} c_k \chi_{F_k}$, then we define the Lebesgue integral of ϕ by $\int_{\mathbb{R}^d} \phi(x)dx = \sum_{k=1}^{M} c_k m(Fk)$.*

Property 20. *If E is a measurable subset of $\mathbb{R}^d$ with finite measure, then $\phi(x)\chi_E(x)$ is also a simple function, and we define,*

$$\int_E \phi(x)dx = \int \phi(x)\chi_E(x)dx$$

where second integral is over $\mathbb{R}^d$.

Proposition 1. *The integral of simple functions defined above satisfies the following properties:*

- *Independence of the representation. If $\phi = \sum_{k=1}^{N} a_k \chi_{E_k}$ is any representation of ϕ, then*

$$\int \phi = \sum_{k=1}^{N} a_k m(E_k)$$

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- *Linearity. If ϕ and ψ are simple, and $a, b \in \mathbb{R}$, then*

$$\int (a\phi + b\psi) = a\int \phi + b\int \psi$$

- *Additivity. If E and F are disjoint subsets of $\mathbb{R}^d$ with finite measure, then*

$$\int_{E \bigcup F} \phi = \int_E \phi + \int_F \phi$$

- *Monotonicity. If $\phi \leq \psi$ are simple, then*

$$\int \phi \leq \int \psi$$

- *Triangle inequality. If ϕ is a simple function, then so is $|\phi|$, and*

$$\Big|\int \phi\Big| \leq \int \big|\phi\big|$$