## 1.7 Exercises

1. Show that no integer in the following sequence can be a perfect square:

$$99,\ 999,\ 9999,\ 99999,\ \ldots$$

2. Prove that a perfect square must be of the form of $5n$ or $5n \pm 1$.

3. Prove that $7k+3$ can not be a perfect square for any integer $a$.

4. Prove that if an integer is simultaneously a square and a cube, then it must be of the form $7n$ or $7n+1$.

5. Prove that if $a$ and $b$ are integers with $b > 0$, then there exist unique integers $q$ and $r$ satisfying $a = qb + r$, where $5b \leq r < 6b$.

6. Find $gcd(174, 204)$ using Euclid's algorithm.

7. Show that the gcd of $(n+1)!+1$ and $n!+1$ is 1.

8. (A) Let $a$, $b$ and $c$ be three integers such that $abc \neq 0$ and $a \mid bc$. Show that

$$a \mid gcd(a,b)gcd(a,c).$$

   (B) Now let $a \mid b_1 b_2 \cdots b_n$ where $ab_1 \cdots b_n \neq 0$. Show that

$$a \mid gcd(a,b_1) \cdots gcd(a,b_n).$$

9. Show that $5n+3$ and $7n+4$ are coprime for any natural number $n$.

10. If $a$ and $b$ are coprime integers, show that $a+b$ and $a^2+ab+b^2$ are also coprime.

11. If $a$ and $b$ are coprime integers and $3 \mid a$, show that $a+b$ and $a^2-ab+b^2$ are also coprime.

12. Find $gcd(2002+2,\ 2002^2+2,\ 2002^3+2,\ \ldots)$.

13. Let $a > 1$ be an integer, and $m$ and $n$ $(m > n)$ are any two distinct positive integers. Prove the following.

   (A) $gcd(a^m-1, a^n-1) = gcd(a^{m-n}-1, a^n-1)$.

   (B) $gcd(a^m-1, a^n-1) = a^{(m,n)}-1$.

   (C) $gcd(a^n+1, a^m+1) \ \mid\ a^{gcd(m,n)}+1$.

14. Find all the integral solutions of $174x + 204y = 18$.

15. Show that for any integer $n$, the rational number $\frac{55n+7}{33n+4}$ is irreducible (i.e., the numerator and the denominator are coprime).

16. Show that $101x + 257y = n$ has solutions in integers for any integer $n$.

17. If $d = gcd(a, b)$, then what is the gcd of $a^n$ and $b^n$? What will be the gcd of $a^m$ and $b^n$?

18. If $p$ is a prime and $k$ is an integer such that $1 \leq k < p$, then show that
$$p \mid \binom{p}{k}.$$

19. A *Pythagorean triple* consists of three positive integers $a,\ b,\ c$ such that $a^2 + b^2 = c^2$. Such a triple is called *primitive* if $gcd(a, b, c) = 1$. Show that any primitive Pythagorean triple can be expressed as $m^2 - n^2,\ 2mn,\ m^2 + n^2$ where $m$ and $n$ are two coprime integers. (For example, for $(3, 4, 5)$ $[3^2 + 4^2 = 5^2]$, we can take $m = 2$ and $n = 1$.)

20. Show that there are infinitely many positive integers $A$ such that $2A$ is a square, $3A$ is a cube and $5A$ is a fifth power.

21. If $a,\ b,\ c$ are three natural numbers with $gcd(a, b, c) = 1$ such that
$$\frac{1}{a} + \frac{1}{b} = \frac{1}{c},$$
then show that $a + b$ is a perfect square.

22. Show that
$$1! + 2! + 3! + \ldots + n!$$
is a perfect power if and only if $n = 3$.

23. Show that
$$(1!)^3 + (2!)^3 + (3!)^3 + \ldots + (n!)^3$$
is a perfect square if and only if $n = 3$.

24. Show that
$$(1!)^3 + (2!)^3 + (3!)^3 + \ldots + (n!)^3$$
is a perfect power if and only if $n \leq 3$.

25. Are there any three consecutive odd integers which are all primes? How many such triples exist?

26. Show that there are infinitely many primes of the form $6k + 5$.

27. Prove that $\sqrt{p}$ is irrational for any prime number $p$.

28. Find all prime numbers $p$ such that $p^2 + 2$ is also a prime.

29. Let $p_n$ denote the $n$-th prime number. Show that $p_1 p_2. \cdots .p_n + 1$ is never a square.

30. Let $R_n$ be an integer consisting only of $n$ number of 1's in its decimal expansion. If $R_n$ is prime, show that $n$ must be prime.

31. Show that there are infinitely many primes which do not belong to any pair of twin primes. (You may assume Dirichlet's theorem that there are infinitely many primes in any arithmetic progression whose first term and common difference are coprime.)

32. Show that the sum of any pair of twin primes except $(3, 5)$ is divisible by 12.