1.6 Lecture 6

Preamble: In this lecture we will discuss how primes are distributed amongst the set of natural numbers.

Keywords: prime number theorem, Goldbach conjecture, twin prime conjecture

1.6.1 Prime Number Theorem

One way to look at how prime numbers are distributed is to count the number of integers not exceeding a real number x. This gives us a function, which we denote by $\pi(x)$. By definition,

$$\pi(x) = \#\{p \leq x \mid p \text{ is a prime}\} \qquad x \in \mathbb{R}.$$

For example, $\pi(9.5) = 4$. The *Prime Number Theorem* tells us about the *asymptotic* behavior of $\pi(x)$, i.e., the behavior of $\pi(x)$ as a function when x is very large. Gauss, one of the most influential mathematician, conjectured in 1793 that the value of $\pi(x)$ is very close to the value of the more familiar function

$$li(x) = \int_2^x \frac{dt}{\log t}$$

in the sense that

$$\frac{\pi(x)}{li(x)} \to 1 \text{ as } x \to \infty.$$

This was later proved by by Hadamard and de la Valleé Poussin in 1896, and is known as the *Prime Number Theorem*. The proof is beyond the scope of these notes. By applying l'Hospital's rule, one can observe that

$$\lim_{x\to\infty} \frac{li(x)}{x/\log x} = 1.$$

The Prime number theorem can be restated as

$$\frac{\pi(x)}{x/\log x} \to 1 \text{ as } x \to \infty,$$

and interpreted as showing that the proportion $\frac{\pi(x)}{x}$ of primes amongst the positive integers $n \leq x$ is approximately $\frac{1}{\log x}$ for large x. Since $\frac{1}{\log x} \to 0$ as $x \to \infty$, the theorem says that the primes occur less and less frequently among larger integers.

1.6.2 Conjectures about Primes

There are many open questions and "*conjectures*" involving primes. A *conjecture* is a statement for which there is enough evidence but which is not still proved mathematically. Twin prime conjecture is one such famous conjectures involving primes.

DEFINITION 1.30. *If p is a prime number such that $p+2$ is also a prime, then p and $p+2$ are called twin primes.*

The Twin Prime Conjecture: There are infinitely many twin primes.I.e., there are infinitely many pairs of primes such as $(3,5),\ (5,7),\ (11,13),\ \ldots$.

A more general conjecture is **Polignac's conjecture**, which states that for any positive integer n there are infinitely many primes p such that $p+2n$ is also a prime. The twin-prime conjecture is a special case of Polignac's conjecture with $n=1$.

Another unanswered question involves the Fibonacci sequence F_n, which is defined as $F_0 = 1,\ F_1 = 1,\ F_2 = 1+1 = 2,\ F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$. The question whether there are infinitely many primes in the Fibonacci sequence is still open.

Bertrand Conjecture: For each natural number $n \geq 2$, there exists a prime number p such that p lies between n and $2n$.

The above conjecture has been verified up to large values by Bertrand, who formulated it in 1845. It was proved by Tchebysheff in 1852. As a consequence of this conjecture, one can show that the n-th prime does not exceed 2^n for $n \geq 2$.

THEOREM 1.31. *If p_n denotes the n-th prime, then $p_n < 2^n$ for $n \geq 2$.*

Proof: We use induction on n. The assertion is clear for $n=2$. Assume it is true for $n = k \geq 2$. Then, by Bertrand's conjecture there is a prime p such that $2^k < p < 2^{k+1}$. Then, $p_k < p$. Therefore, we have found a prime bigger than the first k primes which is less than 2^{k+1}. $\square$

1.6.3 Goldbach Conjecture

Goldbach Conjecture: Any even number greater than 4 can be expressed as a sum of two odd primes.

For example, $6 = 3+3,\ 8 = 3+5,\ 10 = 3+7 = 5+5$ etc. One can take as large an even number as possible and find two primes of which the given number is a sum. However, a proof has been elusive for more than 240 years, since Goldbach first put this question to Euler, who was one of the leading mathematicians of that era. A similar open question is that *any even number can be written as the difference of two primes*, for example, $2 = 5-3,\ \ 4 = 7-3,\ \ 6 = 11-5,\ \ \cdots$ etc.

Note that if Goldbach conjecture is true, then any odd integer $n > 7$ will be a sum of three odd primes: $n-3$ is an even integer bigger than 4, hence $n-3 = p_1 + p_2$ for two odd primes p_1 and p_2 by Goldbach conjecture and $n = 3 + p_1 + p_2$. In a major progress, Hardy and Littlewood showed that under another conjecture (known as the Riemann Hypothesis , which we will allude to towards the final lectures) every sufficiently large odd integer can be expressed as the sum of three odd primes. It has also been proved that 'almost all' even integers satisfy Goldbach conjecture in the following sense: if $g(x)$ is the number of even integers n not exceeding a real number x such that n does not satisfy the Goldbach Conjecture, then

$$\lim_{x\to\infty} \frac{g(x)}{x} = 0.$$

However, note that the above result does not rule out the possibility that there still may be infinitely many even integers which are not expressible as sum of two primes. The vanishing of the above limit merely says that such occurrence will be very rare.