MSM 707 Number Systems for Middle School Teachers
Semester Project

During the course of the semester, we will discuss some important concepts of Number Theory. The following projects are designed to give you a taste of mathematical research while expanding your knowledge beyond the homework problems in the text. The structure of the projects is to give you a set of questions to answer, and success in answering will lead you to the big picture. Some of the projects will be easy to understand what is being asked and will require some thought as to how to proceed whereas others are harder to understand what to do but will be able to be tackled directly once the question is absorbed. Spend some time with the topics before deciding which you want to be yours and remember - sometimes the ones that seem the easiest to tackle involve the most involved solutions.

1. Rings of Factors

This project involves factors and multiples of a sequence of numbers. The sequence of numbers $a_1, a_2, a_3, \ldots, a_k, a_{k+1}, \ldots, a_n$ is a **ring of factors** if $a_k | a_{k+1}$ or $a_{k+1} | a_k$ for $1 \leq k < n$ and $a_n | a_1$ or $a_1 | a_n$. In other words, for every pair of adjacent members in the sequence one is a multiple of the other. For example, $2, 10, 5, 1, 8, 4$ is a ring of factors and its length is 6. We often represent the rings in a circle, which makes sense since we pair the last term of the sequence with the first term in the divisibility relationship.

Our exploration involves building rings; rings as long as possible on a finite set of numbers. A **maximal ring** will use as many of the numbers from the set as possible. The ring above is not maximal on the set of numbers $n \leq 10$. A longer one would be $8, 2, 6, 3, 9, 1, 4$. It has length 7. We are unaware of any formula that will give you an exact answer to these problems. If you find one, that would be a real achievement. What you are being asked to do is develop a strategy that can be used to find maximal rings.

Denote the length of a maximal ring on the set of numbers $n \leq N$ by $M(N)$.

1. (a) Find a maximal ring within the set of numbers ≤ 25. What is $M(25)$?
   (b) How many different rings of length $M(25)$ can you find? Write them out.
2. (a) Find a maximal ring within the set of numbers ≤ 50. What is $M(50)$?
   (b) How many different rings of length $M(50)$ can you find? Write them out.
3. (a) Find a maximal ring within the set of numbers ≤ 100. What is $M(100)$?
   (b) How many different rings of length $M(100)$ can you find? Write them out.
4. In each of your answers above, what fraction of the available numbers did you use; that is, what is $M(N)/N$?
5. Talk about the strategy you used to find maximal rings. Characterize those numbers that could not be included in these rings.

6. Using your strategy, find (as accurately as you can) the length of the maximal ring you can build with the set of natural numbers ≤ 1000. Explain your reasoning.

7. Find the maximum $M(n)/n$ for

   (a) $n \leq 25$

   (b) $25 < n \leq 50$

   (c) $50 < n \leq 100$

8. What conclusions can you draw from the results you obtained?

2. Sums of Consecutive Numbers

This project examines the different ways of expressing a number as the sum of consecutive integers. We discussed in Chapter 1 how representing numbers as sums of as products of other numbers has been a time-honored activity of mathematicians of all centuries and all cultures.

Let us look at a couple of examples.

$12 = 3 + 4 + 5$
$15 = 7 + 8$; $15 = 4 + 5 + 6$; $15 = 1 + 2 + 3 + 4 + 5$

We see that 12 can be expressed as the sum of consecutive integers in just one way while 15 can be represented in three different ways. Let us introduce a definition and notation to make the conversation easier.

Definition The **consecutive index** of a number n is the number of ways that n can be written as the sum of consecutive integers.

Notation: The consecutive index of n will be denoted by $c(n)$.

1. Gather data. For example, for $2 \leq n \leq 50$ list the sequences of consecutive numbers whose sum is n. Note the beginning number of each sequence and the length of the sequence. Find $c(n)$ for each n.

2. Describe the nature of numbers whose consecutive index is

   (a) 0

   (b) 1

   (c) 2

   (d) 3

3. For all natural numbers $\leq m$, which number(s) has (have) the maximum consecutive index where $m =$

   (a) 100?

(b) 1000?

(c) 10000?

4. Find a formula for $c(n)$ for all natural numbers, n.

5. List the lengths of the sequences of consecutive integers that make up $c(n)$. For each length tell where the starting point of the sequence begins. Be as general as you can.

6. Find the smallest number n such that $c(n) \geq$

   (a) 100

   (b) 1,000

   (c) 10,000

3. Inside the Fibonacci Numbers

This project examines the prime form of Fibonacci numbers. It is noted that many surprising and beautiful relationships hold among the terms of the sequence. We hope that these give an indication of why these numbers hold endless interest to both lay people and mathematicians. As with the relationships suggested in the exercises, there are many elegant results concerning the prime forms of Fibonacci numbers. We shall denote the sequence like this: $F_1, F_2, F_3, \ldots, F_n, \ldots$ So, for example, $F_1 = 1$, $F_5 = 5$, $F_{11} = 89$, $F_{15} = 610$, and so on.

1. Answer the following: Where, in the Fibonacci sequence, do the following types of numbers occur?

   (a) even numbers

   (b) multiples of 3

   (c) multiples of 4

   (d) multiples of 5

2. Test the validity of the following and substantiate your answers.

   (a) If $m|F_n$, then $m|F_{kn}$

   (b) If $m|n$, then $F_m|F_n$

   (c) If $F_m|F_n$, then $m|n$

3. Test the validity of the following and substantiate your answers.

   (a) If p is prime, then F_p is prime

   (b) If F_p is prime, then p is prime

4. Find the smallest Fibonacci number F_n such that $p|F_n$ where p is

   (a) 2

(b) 3

(c) 5

(d) 7

(e) 11

(f) 13

(g) 17

(h) 19

(i) 23

(j) 29

5. Find the smallest Fibonacci number F_n such that

   (a) $4|Fn$

   (b) $8|F_n$

   (c) $16|F_n$

   (d) $9|F_n$

   (e) $27|F_n$

   (f) $81|F_n$

6. Find the smallest Fibonacci number F_n such that

   (a) $6|F_n$

   (b) $10|F_n$

   (c) $15|F_n$

   (d) $35|F_n$

   (e) $30|F_n$

7. (a) Suppose that p is a prime and F_n is the smallest number that p divides. Suppose also that q is a prime and F_m is the smallest number that q divides. What can you say about the smallest Fibonacci number that pq divides?

   (b) Extend your observations to three primes p, q, and r.

   (c) Extend your observations to more than three primes.

8. Find the prime decomposition of F_n, where n is

   (a) 120

   (b) 240

   (c) 600

   (d) 1200

4. Writing Fractions the Egyptian Way

This project examines how we can write any fraction as the sum of distinct unit fractions. A unit fraction is a fraction with numerator 1. Writing fractions as the sum of distinct unit fractions dates back to ancient Egypt. The oldest mathematical document in existence, the Rhind Papyrus (2000 - 1788 B.C.) shows these representations of fractions as sums of unit fractions:

$\frac{2}{7} = \frac{1}{4} + \frac{1}{28}$, $\frac{2}{11} = \frac{1}{6} + \frac{1}{66}$, $\frac{2}{97} = \frac{1}{56} + \frac{1}{679} + \frac{1}{776}$.

Apparently, with the exception of $\frac{2}{3}$, the Egyptians dealt with fractions as halves, thirds, fourths, and so on, and all other fractions were built from these. Of course, the Egyptians could have expressed $\frac{2}{97}$ as $\frac{1}{97} + \frac{1}{97}$, but repeating the denominator was not allowed. The equality showing $\frac{2}{97}$ as the sum of three unit fractions shows that Egyptians has considerable facility with arithmetic. It turns out that the fraction $\frac{2}{97}$ can be written as the sum of just two distinct unit fractions: $\frac{2}{97} = \frac{1}{49} + \frac{1}{4753}$, and they might have expressed it this way if they had the ability to write fractions as small as $\frac{1}{4753}$. But with a limit of size on the denominator the ability to write arbitrary fractions as the sum of unit fractions is quite a challenge. We will take the challenge into mathematical directions that probably didnt occur to the Egyptian mathematics.

Let us consider the example of $\frac{9}{20}$. It can be expressed in many different ways. Here are a few with three summands:

$\frac{9}{20} = \frac{1}{3} + \frac{1}{15} + \frac{1}{20} = \frac{1}{4} + \frac{1}{6} + \frac{1}{30} = \frac{1}{3} + \frac{1}{12} + \frac{1}{30} = \frac{1}{3} + \frac{1}{10} + \frac{1}{60} = \frac{1}{5} + \frac{1}{6} + \frac{1}{12} = \frac{1}{3} + \frac{1}{9} + \frac{1}{180}$.

Also it can be written with two summands: $\frac{9}{20} = \frac{1}{4} + \frac{1}{5}$. Here are three methods of generating three representations.

Method 1
Subtract the largest unit fraction less than $\frac{9}{20}$ and work with the difference. The largest such fraction is $\frac{1}{3}$, so we have $\frac{9}{20} - \frac{1}{3} = \frac{7}{60}$. Now the largest unit fraction less than $\frac{7}{60}$ is $\frac{1}{9}$, so we take $\frac{7}{60} - \frac{1}{9} = \frac{1}{180}$. Since the difference is a unit fraction, we are finished. Had it not been we would have applied the principle again. Thus the representation is $\frac{9}{20} = \frac{1}{3} + \frac{1}{9} + \frac{1}{180}$. This method uses a well-defined algorithm; that is, at each step we know how to proceed: Simply take the largest unit fraction less than the remainder. The following methods do not employ well-defined algorithms.

Method 2
Proceed as in Method 1 except successively subtract off unit fractions that need not be the largest unit fractions. For example, $\frac{9}{20} - \frac{1}{6} = \frac{17}{60}$; $\frac{17}{60} - \frac{1}{5} = \frac{1}{12}$. So $\frac{9}{20} = \frac{1}{6} + \frac{1}{5} + \frac{1}{12}$. This method can generate many different answers.

Method 3
Write $\frac{9}{20}$ in various no reduced forms. For example, $\frac{9}{20} = \frac{18}{40} = \frac{27}{60}$. Now find ways of writing the numerator as the sum of numbers that are also factors of the denominator. For example, looking at the version $\frac{18}{40}$, we see that $18 = 10 + 5 + 4 + 1$. Note that $10, 5, 4$, and 1 are all factors of 40. So $\frac{9}{20} = \frac{1}{4} + \frac{1}{8} + \frac{1}{10} + \frac{1}{40}$. Here is another alternative: Consider $\frac{27}{60}$.Note that

$27 = 20 + 6 + 1 = 20 + 4 + 3 = 15 + 12$. These yield the following sums: $\frac{1}{3} + \frac{1}{10} + \frac{1}{60}$; $\frac{1}{3} + \frac{1}{15} + \frac{1}{20}$; and $\frac{1}{4} + \frac{1}{5}$.

As we can see from the example of $\frac{9}{20}$, fractions can be written as the sums of distinct unit fractions in a variety of ways. It would be difficult to say there is a best way, but let us make some definitions with this in mind. A representation for $\frac{a}{b}$ will be called algorithmic if it is obtained by Method 1. A representation will be called frugal if it employs the fewest number of unit fractions. A representation will be called economical if it's largest denominator is the smallest of the largest denominators for all representations. The algorithmic representation will be denoted by $A(\frac{a}{b})$. A frugal representation will be denoted by $F(\frac{a}{b})$; it need not be unique. The economical representation will be denoted by $E(\frac{a}{b})$. We shall assume that $\frac{a}{b}$ and all representations of it are in reduced form. So, for $\frac{9}{20}$ have $A(\frac{9}{20}) = \frac{1}{3} + \frac{1}{9} + \frac{1}{180}$, $F(\frac{9}{20}) = E(\frac{9}{20}) = \frac{1}{4} + \frac{1}{5}$.

1. Find the algorithmic representation for the following fractions.
   (a) $\frac{4}{41}$
   (b) $\frac{21}{68}$
   (c) $\frac{7}{11}$
   (d) $\frac{12}{17}$
   (e) $\frac{35}{37}$
   (f) $\frac{47}{72}$

2. Prove or disprove the following statement: The reduced proper fraction $\frac{a}{b}$ can be written in the algorithmic representation as the sum of at most a different unit fractions.

3. (a) What can you say about $A(\frac{a}{b})$ if $\frac{a}{b} = \frac{2}{n}$? Prove your statements.
   (b) What can you say about $A(\frac{a}{b})$ if $\frac{a}{b} = \frac{3}{n}$? Prove your statements.
   (c) What can you say about $A(\frac{a}{b})$ if $\frac{a}{b} = \frac{4}{n}$? Prove your statements.

4. Find $A(\frac{a}{b})$, $F(\frac{a}{b})$, and $E(\frac{a}{b})$ for the following $\frac{a}{b}$.
   (a) $\frac{7}{12}$
   (b) $\frac{8}{15}$
   (c) $\frac{11}{49}$
   (d) $\frac{43}{105}$
   (e) $\frac{151}{273}$
   (f) $\frac{275}{504}$

5. (a) Find the fraction of smallest denominator, d, such that $F(\frac{n}{d})$ needs four summands. What is the fraction? If there are more than one with the same denominator, give the one wit the smallest numerator.
   (b) Find the fraction of smallest denominator, d, such that $F(\frac{n}{d})$ needs five summands. What is the fraction? If there are more than one with the same denominator, give the one with the smallest numerator.

6. This problem concerns writing unit fractions as sums of two unit fractions; for example, $\frac{1}{6}$ can be represented in four different ways: $\frac{1}{6} = \frac{1}{7} + \frac{1}{42} = \frac{1}{8} + \frac{1}{24} = \frac{1}{9} + \frac{1}{18} = \frac{1}{10} + \frac{1}{15}$.

   (a) Find all the different ways that $\frac{1}{n}$ can be expressed as sum of two unit fractions where n is

      i. 7

      ii. 8

      iii. 10

      iv. 12

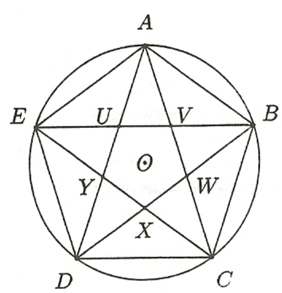
   (b) Find a formula for the general case $\frac{1}{n}$.

   (c) For what denominator(s) n for $n \leq 1000$ does the unit fraction $\frac{1}{n}$ allow the most ways of being written as the sum of two different unit fractions? How many ways is it?

5. The Making of a Star

This project examines the famous five point star and regular pentagon. First we construct these figures with straightedge and compass; then we examine the numbers that represent the lengths and areas.

The following diagram shows the pentagram, a figure that has been attributed mystical properties since ancient times. One reason for this is its close connection with the golden mean, which, in turn, is related to the famous Fibonacci sequence of numbers. The figure shows the two pentagrams, $ABCDE$ and $UVWXY$, along with the five-pointed star, $ACEBDA$. Let r denote the radius of the circumscribed circle and s the inscribed circle of $UVWXY$; let R denote the radius of the circumscribed circle; and let S denote the inscribed circle of $ABCDE$.



1. Construct the figure of the pentagrams with straightedge and compass.

2. Suppose that UV is of length 1. Write the following constructible lengths in terms of square roots.

   (a) EU

   (b) EV

   (c) EB

   (d) R

(e) r
(f) s
(g) S

3. Find the perimeters of
   (a) $ABCDE$
   (b) $UVWXY$
   (c) $VBWCXDYEU$

4. (a) Find the angle measures for triangles AUV, AUE, ADE, and ACE.
   (b) Find the cosines of those angles. Express them as constructible numbers.

5. Find the areas of
   (a) $ABCDE$
   (b) $UVWXY$
   (c) $AVBWCXDYEU$

6. Express the following ratios as constructible numbers.
   (a) $\frac{R}{r}$
   (b) $\frac{R}{S}$
   (c) $\frac{P}{p}$, where P is the perimeter of $ABCDE$ and p is the perimeter of $UVWXY$
   (d) $\frac{P}{Q}$, where P is as above and Q is the perimeter of $AVBWCXDYEU$
   (e) $\frac{M}{m}$, where M is the area of $ABCDE$ and m is the area of $UVWXY$
   (f) $\frac{M}{N}$, where N is the area of $AVBWCXDYEU$

7. Make a five-sided pyramid with base $UVWXY$ by folding up the triangles of the star (that is, UAV, VBW, WCX, XDY, and YEU) and creating the sides. Express the following as constructible numbers.
   (a) The height, h, of this pyramid
   (b) The ratio $\frac{h}{r}$

8. The golden mean numerically it is the positive solution to the polynomial equation $x^2-x-1=0$. This number is labelled Φ. In reviewing your answers to questions 2, 3, 4, 5, 6, and 7, express as many of them as possible in terms of Φ.