Assignment 24

Adarsh Srivastava

The link to the solution is

https://github.com/Adarsh1310/EE5609

***Abstract*—This documents solves a problem based on Jordan Form.**

1 Problem

If $\mathbf{N}$ is a nilpotent 3×3 matrix over C, prove that $\mathbf{A} = \mathbf{I} + \frac{1}{2}\mathbf{N} - \frac{1}{8}\mathbf{N}^2$ satisfies $\mathbf{A}^2 = \mathbf{I} + \mathbf{N}$, i.e., $\mathbf{A}$ is a square root of $\mathbf{I} + \mathbf{N}$. Use the binomial series for $(1+t)^{\frac{1}{2}}$ to obtain a similar formula for a square root of $\mathbf{I} + \mathbf{N}$, where $\mathbf{N}$ is any nilpotent n $\times$ n matrix over C.

2 Solution

We know that $\mathbf{N}^3$=0 since the minimal polynomial of $\mathbf{N}$ is x^3,So,

$$\mathbf{A}^2 = \left(\mathbf{I} + \frac{1}{2}\mathbf{N} - \frac{1}{8}\mathbf{N}^2\right)\left(\mathbf{I} + \frac{1}{2}\mathbf{N} - \frac{1}{8}\mathbf{N}^2\right) \tag{2.0.1}$$

$$= \mathbf{I} + \frac{1}{2}\mathbf{N} - \frac{1}{8}\mathbf{N}^2 + \frac{1}{4}\mathbf{N}^2 - \frac{1}{8}\mathbf{N}^2 \tag{2.0.2}$$

$$= \mathbf{I} + \mathbf{N} \tag{2.0.3}$$

Expanding $(1+t)^{1/2}$,

$$(1+t)^{1/2} = \sum_{k=0}^{\infty} \binom{1/2}{k} t^k \tag{2.0.4}$$

Here,

$$\binom{1/2}{k} = \frac{\frac{1}{2}(\frac{1}{2}-1)(\frac{1}{2}-2)\cdots(\frac{1}{2}-k+1)}{k!} \tag{2.0.5}$$

$$= \frac{(-1)^{k-1}}{2^k k!} 1 \cdot 3 \cdot 5 \cdots (2k-3) \tag{2.0.6}$$

$$= \frac{(-1)^{k-1}}{2^k k!} \frac{(2k-2)!}{2^{k-1}(k-1)!} \tag{2.0.7}$$

$$= \frac{(-1)^{k-1}}{k 2^{2k-1}} \binom{2k-2}{k-1} \tag{2.0.8}$$

Thus,

$$(1+t)^{1/2} = 1 - \sum_{k=1}^{\infty} \frac{2}{k} \binom{2k-2}{k-1} \left(-\frac{t}{4}\right)^k \tag{2.0.9}$$

$$= 1 - \sum_{k=0}^{\infty} \frac{2}{k+1} \binom{2k}{k} \left(-\frac{t}{4}\right)^{k+1} \tag{2.0.10}$$

So a square root for $\mathbf{I}$+$\mathbf{N}$ where $\mathbf{N}$ is a n$\times$ n nilpotent matrix can be,

$$= \mathbf{I} - \sum_{k=0}^{n-1} \frac{2}{k+1} \binom{2k}{k} \left(-\frac{\mathbf{N}}{4}\right)^{k+1} \tag{2.0.11}$$