Assignment 5

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The link to the solution is

https://github.com/Adarsh1310/EE5609

***Abstract*—This documents solves a problem based on circles.**

1 Problem

Find the area of the region bounded by the circle $\mathbf{x}^\mathbf{T}\ \mathbf{x}= 4$ and $\left\|\mathbf{x}-\begin{pmatrix}2\\0\end{pmatrix}\right\| = 2.$

2 Solution

$\|\mathbf{x}\|^2 + 2\mathbf{u}^T\mathbf{x} + f = 0$
$\mathbf{x}^T\mathbf{x} + 2\mathbf{u}^T\mathbf{x} + f = 0$
So from above equation we can say that,

2.1 Circle 1

Taking equation of the first circle to be,

$$\|\mathbf{x}\|^2 + 2\mathbf{u}_1^T\mathbf{x} + f_1 = 0 \tag{2.1.1}$$
$$\mathbf{x}^T\mathbf{x} - 4 = 0(\text{given}) \tag{2.1.2}$$
$$\mathbf{u_1} = \begin{pmatrix}0\\0\end{pmatrix} \tag{2.1.3}$$
$$f_1 = -4 \tag{2.1.4}$$
$$\mathbf{O_1} = \begin{pmatrix}0\\0\end{pmatrix} \tag{2.1.5}$$

2.2 Circle 2

Taking equation of the second circle to be,

$$\left\|\mathbf{x}-\begin{pmatrix}2\\0\end{pmatrix}\right\|^2 = 2^2(\text{given}) \tag{2.2.1}$$
$$\mathbf{x}^T\mathbf{x} + 2\mathbf{u_2}^T\mathbf{x} = 0 \tag{2.2.2}$$
$$\mathbf{u_2} = \begin{pmatrix}-2\\0\end{pmatrix} \tag{2.2.3}$$
$$f_2 = 0 \tag{2.2.4}$$
$$\mathbf{O_2} = \begin{pmatrix}2\\0\end{pmatrix} \tag{2.2.5}$$

Now,Subtracting equation (2.2.2) from (2.1.2) We get,

$$\mathbf{x}^T\mathbf{x} - 2\mathbf{u_2}^T\mathbf{x} + f_1 - \mathbf{x}^T\mathbf{x} = 0 \tag{2.2.6}$$
$$2\mathbf{u}^T\mathbf{x} = -4 \tag{2.2.7}$$
$$\begin{pmatrix}-4 & 0\end{pmatrix}\mathbf{x} = -4 \tag{2.2.8}$$

Which can be written as:-

$$\begin{pmatrix}1 & 0\end{pmatrix}\mathbf{x} = 1 \tag{2.2.9}$$
$$\mathbf{x} = \begin{pmatrix}1\\0\end{pmatrix} + \lambda\begin{pmatrix}0\\1\end{pmatrix} \tag{2.2.10}$$
$$\mathbf{x} = \mathbf{q} + \lambda\mathbf{m} \tag{2.2.11}$$
$$\mathbf{q} = \begin{pmatrix}1\\0\end{pmatrix} \tag{2.2.12}$$
$$\mathbf{m} = \begin{pmatrix}0\\1\end{pmatrix} \tag{2.2.13}$$

Substituting (2.2.11) in (2.1.1)

$$\|\mathbf{x}\|^2 + 2\mathbf{u}_1^T\mathbf{x} + f_1 = 0 \tag{2.2.14}$$
$$\|\mathbf{q} + \lambda\mathbf{m}\|^2 + f_1 = 0 \tag{2.2.15}$$
$$\left\|\begin{pmatrix}1\\\lambda\end{pmatrix}\right\|^2 - 4 = 0 \tag{2.2.16}$$
$$\lambda^2 + 1 - 4 = 0 \tag{2.2.17}$$
$$\lambda = +\sqrt{3}, -\sqrt{3} \tag{2.2.18}$$
$$\tag{2.2.19}$$

Substituting the value of λ in(2.2.11)

$$\mathbf{x} = \mathbf{q} + \lambda\mathbf{m} \tag{2.2.20}$$
$$For\lambda = \sqrt{3}, \mathbf{A} = \begin{pmatrix}1\\\sqrt{3}\end{pmatrix} \tag{2.2.21}$$
$$For\lambda = -\sqrt{3}, \mathbf{B} = \begin{pmatrix}1\\-\sqrt{3}\end{pmatrix} \tag{2.2.22}$$

Now finding the direction vector $\mathbf{O_1A}$,$\mathbf{O_1B}$,$\mathbf{O_2A}$ and $\mathbf{O_2B}$.

$$\mathbf{O_1A} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 1 \\ \sqrt{3} \end{pmatrix} = k_1 \begin{pmatrix} -1 \\ \sqrt{3} \end{pmatrix} \tag{2.2.23}$$

$$\mathbf{O_1B} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 1 \\ -\sqrt{3} \end{pmatrix} = k_2 \begin{pmatrix} -1 \\ -\sqrt{3} \end{pmatrix} \tag{2.2.24}$$

$$\mathbf{O_2A} = \begin{pmatrix} 2 \\ 0 \end{pmatrix} - \begin{pmatrix} 1 \\ \sqrt{3} \end{pmatrix} = k_3 \begin{pmatrix} 1 \\ \sqrt{3} \end{pmatrix} \tag{2.2.25}$$

$$\mathbf{O_2B} = \begin{pmatrix} 2 \\ 0 \end{pmatrix} - \begin{pmatrix} 1 \\ -\sqrt{3} \end{pmatrix} = k_4 \begin{pmatrix} 1 \\ -\sqrt{3} \end{pmatrix} \tag{2.2.26}$$

Now finding the angle $\angle O_1AB$.

$$\mathbf{O_1AO_1B} = \|O_1A\|\,\|O_1B\| \cos\theta_1 \tag{2.2.27}$$

$$\frac{\mathbf{O_1AO_1B}}{\|O_1A\|\,\|O_1B\|} = \cos\theta_1 \tag{2.2.28}$$

$$\frac{1+3}{1+3} = \cos\theta_1 \tag{2.2.29}$$

$$\theta = 90^\circ \tag{2.2.30}$$

Now finding the angle $\angle O_2AB$.

$$\mathbf{O_2AO_2B} = \|O_2A\|\,\|O_2B\| \cos\theta_2 \tag{2.2.31}$$

$$\frac{\mathbf{O_2AO_2B}}{\|O_2A\|\,\|O_2B\|} = \cos\theta_2 \tag{2.2.32}$$

$$\frac{1+3}{1+3} = \cos\theta_2 \tag{2.2.33}$$

$$\theta = 90^\circ \tag{2.2.34}$$

Finding area of $\mathbf{O_1AB}$ and $\mathbf{O_2AB}$.

$$Area_1 = \frac{\theta_1}{360} r^2 - \frac{1}{2} 2\sqrt{3} \tag{2.2.35}$$

$$Area_1 = \frac{90}{360} 4\pi - \frac{1}{2} 2\sqrt{3} \tag{2.2.36}$$

$$Area_2 = \frac{\pi\theta_2}{360} r^2 - \frac{1}{2} 2\sqrt{3} \tag{2.2.37}$$

$$Area_2 = \frac{90}{360} 4\pi - \frac{1}{2} 2\sqrt{3} \tag{2.2.38}$$

$$TotalArea = 2\pi + 2\sqrt{3} (\because Area_1 + Area_2) \tag{2.2.39}$$

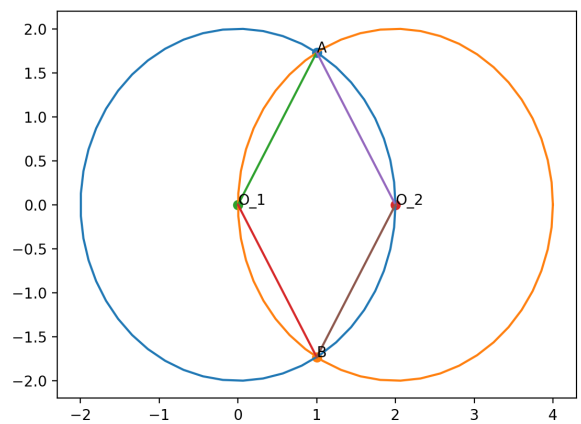

Fig. 0: Figure depicting intersection points of circle