Assignment 23

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The link to the solution is

https://github.com/Adarsh1310/EE5609

***Abstract*—This documents solves a problem based on spectral theorem.**

1 Problem

Let $\mathbf{A}$ be an n × n self-adjoint matrix with eigenvalues $\lambda_1, \cdots, \lambda_2$. Let,

$$\|\mathbf{X}\|_2 = \sqrt{|\mathbf{X}_1^2| + \cdots + |\mathbf{X}_n^2|}$$

for $\mathbf{X}$=$(\mathbf{X}_1, \cdots, \mathbf{X}_n) \in \mathbb{C}^n$. If

$$p(\mathbf{A}) = a_0\mathbf{I} + a_1\mathbf{A} + \cdots + a_n\mathbf{A}^n$$

then $sup_{\|\mathbf{X}\|_2=1} \|p(\mathbf{A})\mathbf{X}\|_2$ is equal to

2 Solution

We know that $\mathbf{A}$ is a self adjoint matrix and hence $\mathbf{A} = \mathbf{A}^*$ with eigen values $\lambda_1, \lambda_2, \cdots, \lambda_n$.Now as we are given,

$$p(\mathbf{A}) = a_0\mathbf{I} + a_1\mathbf{A} + \cdots + a_n\mathbf{A}^n \tag{2.0.1}$$

then,

$$(p(\mathbf{A}))^* = a_0\mathbf{I}^* + a_1\mathbf{A}^* + \cdots + a_n(\mathbf{A}^*)^n \tag{2.0.2}$$

Since, $\mathbf{A} = \mathbf{A}^*$ we can state that,

$$p(\mathbf{A})(p(\mathbf{A}))^* = (p(\mathbf{A}))^* p(\mathbf{A}) \tag{2.0.3}$$

Hence p($\mathbf{A}$) is a normal matrix. Now using spectral theorem for a normal matrix,

$$\|p(\mathbf{A})\|_2 = \rho(p(\mathbf{A})) \tag{2.0.4}$$

sup refers to the smallest element that is greater than or equal to every number in the set.Hence, sup of $\|p(\mathbf{A})\|_2$ will be,

$$= max\{|\alpha| : \alpha \text{ is the eigen value of p(A)}\} \tag{2.0.5}$$

$$= max\{|p(\lambda_j)| : j = 1, 2, \cdots n\} \tag{2.0.6}$$

$$= max\{|a_0 + a_1\lambda_j + \cdots + a_n\lambda_j{}^n| : j = 1, 2, \cdots n\} \tag{2.0.7}$$

Now, to find $sup \|p(\mathbf{A})\mathbf{X}\|_2$,

$$= max\{|a_0 + a_1\lambda_j + \cdots + a_n\lambda_j{}^n| : j = 1, 2, \cdots n\} \|\mathbf{X}\|_2 \tag{2.0.8}$$

Since, we have to find $sup_{\|\mathbf{X}\|_2=1}$ i.e,

$$\|\mathbf{X}\|_2 = \sqrt{|\mathbf{X}_1^2| + \cdots + |\mathbf{X}_n^2|} = 1 \tag{2.0.9}$$

Hence the final answer will be,

$$= max\{|a_0 + a_1\lambda_j + \cdots + a_n\lambda_j{}^n| : j = 1, 2, \cdots n\} \tag{2.0.10}$$