Assignment 17

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The link to the solution is

https://github.com/Adarsh1310/EE5609

***Abstract*—This documents solves a problem based on coordinates.**

1 Problem

Let **V** be the real vector space of all polynomial functions from $\mathbb{R}$ to $\mathbb{R}$ of degree 2 or less, i.e, the space of all functions f of the form,

$$f(x) = c_0 + c_1 x + c_2 x^2$$

Let t be a fixed real number and define

$$g_1(x) = 1, g_2(x) = x + t, g_3(x) = (x + t)^2$$

Prove that β={g1,g2,g3} is a basis for V. If

$$f(x) = c_0 + c_1 x + c_2 x^2$$

what are the coordinates of f in the ordered basis β

2 Solution

We start by taking,

$$\mathbf{f} = \begin{pmatrix} 1 \\ x \\ x^2 \end{pmatrix} \tag{2.0.1}$$

Let's start by proving that **g** is linearly independent.

$$\mathbf{g} = \mathbf{B}\mathbf{f} \tag{2.0.2}$$

where,

$$\mathbf{B} = \begin{pmatrix} 1 & 0 & 0 \\ t & 1 & 0 \\ t^2 & 2t & 1 \end{pmatrix} \tag{2.0.3}$$

Now,

$$\mathbf{v}^T\mathbf{g} = 0 \tag{2.0.4}$$

$$\implies \mathbf{v}^T\mathbf{B}\mathbf{f} = 0 \tag{2.0.5}$$

Since **f** is linearly independent,

$$\mathbf{v}^T\mathbf{B} = 0 \tag{2.0.6}$$

$$\mathbf{B}^T\mathbf{v} = 0 \tag{2.0.7}$$

Since $\mathbf{B}^T$ is an upper triangular matrix with non zero values in principal diagonal , it is invertible matrix and hence **v** will be zero vector.Now, Finding the inverse of $\mathbf{B}^T$

$$\left(\begin{array}{ccc|ccc} 1 & t & t^2 & 1 & 0 & 0 \\ 0 & 1 & 2t & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right) \tag{2.0.8}$$

$$\xleftrightarrow{R_1=R_1-tR_2} \left(\begin{array}{ccc|ccc} 1 & 0 & -t^2 & 1 & -t & 0 \\ 0 & 1 & 2t & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right) \tag{2.0.9}$$

$$\xleftrightarrow{R_1=R_1+t^2R_3} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -t & t^2 \\ 0 & 1 & 2t & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right) \tag{2.0.10}$$

$$\xleftrightarrow{R_2=R_2-2tR_3} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -t & t^2 \\ 0 & 1 & 0 & 0 & 1 & -2t \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right) \tag{2.0.11}$$

So,

$$(\mathbf{B}^T)^{-1} = \begin{pmatrix} 1 & -t & t^2 \\ 0 & 1 & -2t \\ 0 & 0 & 1 \end{pmatrix} \tag{2.0.12}$$

Now, to find the coordinates,

$$f(x) = \mathbf{w}^T\mathbf{g} \tag{2.0.13}$$

So,

$$\mathbf{c}^T\mathbf{f} = \mathbf{w}^T\mathbf{g} \tag{2.0.14}$$

$$\mathbf{c}^T\mathbf{f} = \mathbf{w}^T\mathbf{B}\mathbf{f} \tag{2.0.15}$$

$$(\mathbf{c}^T - \mathbf{w}^T\mathbf{B})\mathbf{f} = 0 \tag{2.0.16}$$

Since, **f** is linearly independent,

$$\mathbf{c}^T - \mathbf{w}^T\mathbf{B} = 0 \tag{2.0.17}$$

$$\mathbf{c}^T = \mathbf{w}^T\mathbf{B} \tag{2.0.18}$$

$$\mathbf{c}^T\mathbf{B}^{-1} = \mathbf{w}^T \tag{2.0.19}$$

$$(\mathbf{B}^{-1})^T\mathbf{c} = \mathbf{w} \tag{2.0.20}$$

Using (2.0.12) in (2.0.20)

$$\mathbf{w} = \begin{pmatrix} 1 & -t & t^2 \\ 0 & 1 & -2t \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \\ c_2 \end{pmatrix} \tag{2.0.21}$$