Assignment 12

Adarsh Srivastava

The link to the solution is

https://github.com/Adarsh1310/EE5609

***Abstract*—This documents solves a problem based on Row Echelon form.**

1 Problem

Suppose $\mathbf{R}$ and $\mathbf{R}'$ are 2×3 row-reduced echelon matrices and that the system $\mathbf{RX}$=0 and $\mathbf{R}'\mathbf{X}$=0 have exactly the same solutions. Prove that $\mathbf{R}$=$\mathbf{R}'$.

2 Solution

Since $\mathbf{R}$ and $\mathbf{R}'$ are 2×3 row-reduced echelon matrices they can be of following three types:-

1) Suppose matrix $\mathbf{R}$ has one non-zero row then $\mathbf{RX}$=0 will have two free variables.Since R$'\mathbf{X}$=0 will have the exact same solution as $\mathbf{RX}$=0, R$'\mathbf{X}$=0 will also have two free variables. Thus $\mathbf{R}'$ have one non zero row.Now let's consider a matrix $\mathbf{A}$ with the first row as the non-zero row $\mathbf{R}$ and second row as the second row of $\mathbf{R}'$.

$$\mathbf{R} = \begin{pmatrix} 1 & a & b \\ 0 & 0 & 0 \end{pmatrix} \tag{2.0.1}$$

$$\mathbf{R}' = \begin{pmatrix} 1 & c & d \\ 0 & 0 & 0 \end{pmatrix} \tag{2.0.2}$$

$$\mathbf{A} = \mathbf{R} + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \mathbf{R}' \tag{2.0.3}$$

$$\mathbf{A} = \begin{pmatrix} 1 & a & b \\ 1 & c & d \end{pmatrix} \tag{2.0.4}$$

Let $\mathbf{X}$ satisfy

$$\mathbf{RX} = 0 \tag{2.0.5}$$

$$\mathbf{R}'\mathbf{X} = 0 \tag{2.0.6}$$

Now multiplying $\mathbf{A}$ and $\mathbf{X}$ and substituting (2.0.3).

$$= \mathbf{AX} \tag{2.0.7}$$

$$= \left(\mathbf{R} + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \mathbf{R}'\right)\mathbf{X} \tag{2.0.8}$$

$$= \mathbf{RX} + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \mathbf{R}'\mathbf{X} \tag{2.0.9}$$

From (2.0.5) and (2.0.6)

$$\mathbf{AX} = 0 \tag{2.0.10}$$

Now, considering the augmented matrix,

$$\left(\begin{array}{ccc|c} 1 & a & b & 0 \\ 1 & c & d & 0 \end{array}\right) \tag{2.0.11}$$

$$\xleftrightarrow{R_2=R_2-R_1} \left(\begin{array}{ccc|c} 1 & a & b & 0 \\ 0 & c-a & d-b & 0 \end{array}\right) \tag{2.0.12}$$

Now, Assuming c-a $\neq$0 and reducing (2.0.12) to row echelon form.

$$\left(\begin{array}{ccc|c} 1 & a & b & 0 \\ 0 & c-a & d-b & 0 \end{array}\right) \tag{2.0.13}$$

$$\xleftrightarrow{R_2=\frac{R_2}{R_1}} \left(\begin{array}{ccc|c} 1 & a & b & 0 \\ 0 & 1 & \frac{d-b}{c-a} & 0 \end{array}\right) \tag{2.0.14}$$

We can see that if c-a $\neq$0 then second row wouldn't come out to be zero which isn't possible because rows will be linear combination of each other as $\mathbf{R}$ and $\mathbf{R}'$ have the same solution. So, c-a=0,

$$c - a = 0 \tag{2.0.15}$$

$$c = a \tag{2.0.16}$$

$$d - b = 0 \tag{2.0.17}$$

$$d = b \tag{2.0.18}$$

Hence, $\mathbf{R}$=$\mathbf{R}'$

2) Let $\mathbf{R}$ and $\mathbf{R}^{'}$ have all rows as non zero.

$$\mathbf{R} = \begin{pmatrix} 1 & 0 & b \\ 0 & 1 & c \end{pmatrix} \tag{2.0.19}$$

$$\mathbf{R}^{'} = \begin{pmatrix} 1 & 0 & e \\ 0 & 1 & f \end{pmatrix} \tag{2.0.20}$$

Now let's consider another matrix $\mathbf{A}$ whose first two rows are from $\mathbf{R}$ and last two rows are from $\mathbf{R}^{'}$

$$\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{R} + \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{R}^{'} \tag{2.0.21}$$

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & b \\ 0 & 1 & c \\ 1 & 0 & e \\ 0 & 1 & f \end{pmatrix} \tag{2.0.22}$$

Let $\mathbf{X}$ satisfy

$$\mathbf{RX} = 0 \tag{2.0.23}$$

$$\mathbf{X}^T\mathbf{R}^T = 0 \tag{2.0.24}$$

Here,

$$\mathbf{R} = \begin{pmatrix} \mathbf{I} & \mathbf{a} \end{pmatrix} \tag{2.0.25}$$

$$\mathbf{a} = \begin{pmatrix} b \\ c \end{pmatrix} \tag{2.0.26}$$

$$\mathbf{R}^T = \begin{pmatrix} \mathbf{I} \\ \mathbf{a}^T \end{pmatrix} \tag{2.0.27}$$

Let,

$$\mathbf{X} = \begin{pmatrix} \mathbf{y}^T \\ z \end{pmatrix} \tag{2.0.28}$$

where z is a scalar constant. Now,substituting (2.0.28) and (2.0.25) in (2.0.24)

$$\begin{pmatrix} \mathbf{y}^T & z \end{pmatrix} \begin{pmatrix} \mathbf{I} \\ \mathbf{a}^T \end{pmatrix} = 0 \tag{2.0.29}$$

$$\mathbf{y}^T + za^T = 0 \tag{2.0.30}$$

Now for,

$$\mathbf{R}^{'}\mathbf{X} = 0 \tag{2.0.31}$$

$$\mathbf{X}^T\mathbf{R}^{'} = 0 \tag{2.0.32}$$

Here,

$$\mathbf{R}^{'} = \begin{pmatrix} \mathbf{I} & \mathbf{b} \end{pmatrix} \tag{2.0.33}$$

$$\mathbf{b} = \begin{pmatrix} e \\ f \end{pmatrix} \tag{2.0.34}$$

Let,

$$\mathbf{X} = \begin{pmatrix} \mathbf{y}^T \\ z \end{pmatrix} \tag{2.0.35}$$

where z is a scalar constant.Now,substituting (2.0.35) and (2.0.33) in (2.0.32)

$$\begin{pmatrix} \mathbf{y}^T & z \end{pmatrix} \begin{pmatrix} \mathbf{I} \\ \mathbf{b}^T \end{pmatrix} = 0 \tag{2.0.36}$$

$$\mathbf{y}^T + zb^T = 0 \tag{2.0.37}$$

Now multiplying $\mathbf{A}$ and $\mathbf{X}$ and substituting (2.0.21).

$$= \mathbf{AX} \tag{2.0.38}$$

$$= \left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{R} + \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{R}^{'} \right) \mathbf{X} \tag{2.0.39}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{RX} + \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{R}^{'}\mathbf{X} \tag{2.0.40}$$

$$= \begin{pmatrix} \mathbf{I} & \mathbf{a} \\ \mathbf{I} & \mathbf{b} \end{pmatrix} \mathbf{X} \tag{2.0.41}$$

Now using, $\mathbf{X}^T\mathbf{A}^T$

$$= \begin{pmatrix} \mathbf{y}^T & z \end{pmatrix} \begin{pmatrix} \mathbf{I} & \mathbf{I} \\ \mathbf{a}^T & \mathbf{b}^T \end{pmatrix} \tag{2.0.42}$$

$$= \begin{pmatrix} \mathbf{y}^T + z\mathbf{a}^T & \mathbf{y} + \mathbf{b}^T \end{pmatrix} \tag{2.0.43}$$

Since (2.0.43) is coming out to be a zero vector. We can say that $\mathbf{R}$ and $\mathbf{R}^{'}$ are linear combination of each other. Also from (2.0.30) and (2.0.37)

$$\mathbf{y}^T + za^T = \mathbf{y}^T + zb^T \tag{2.0.44}$$

$$a^T = b^T \tag{2.0.45}$$

$$b = e \tag{2.0.46}$$

$$c = f \tag{2.0.47}$$

Hence, $\mathbf{R}$=$\mathbf{R'}$

3) Suppose matrix $\mathbf{R}$ have all the rows as zero then $\mathbf{RX}$=0 will be satisfied for all values of

$\mathbf{X}$. We know that $\mathbf{R}'\mathbf{X}$=0 will have the exact same solution as $\mathbf{RX}$=0 then we can say that for all values of $\mathbf{X}$=0 equation $\mathbf{R}'\mathbf{X}$=0 will be satisfied Hence, $\mathbf{R}'$=$\mathbf{R}$=0.