Assignment 9

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The link to the solution is

Abstract—This documents solves a Singular Value decomposition problem.

1 PROBLEM

Find the shortest distance between the lines

$$\mathbf{x} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \lambda_1 \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} \tag{1.0.1}$$

$$\mathbf{x} = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} + \lambda_2 \begin{pmatrix} 3 \\ -5 \\ 2 \end{pmatrix} \tag{1.0.2}$$

2 SOLUTION

The lines will intersect if

$$\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \lambda_1 \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} + \lambda_2 \begin{pmatrix} 3 \\ -5 \\ 2 \end{pmatrix} \tag{2.0.1}$$

$$\begin{pmatrix} 2 & 3 \\ -1 & -5 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \tag{2.0.2}$$

$$\mathbf{Mx} = \mathbf{b} \tag{2.0.3}$$

Since the rank of augmented matrix will be 3. We can say that lines do not intersect.

$$\mathbf{M} = \mathbf{USV}^T \tag{2.0.4}$$

Where the columns of $\mathbf{V}$ are the eigenvectors of $\mathbf{A}^T\mathbf{A}$,the columns of $\mathbf{U}$ are the eigenvectors of $\mathbf{AA}^T$ and $\mathbf{S}$ is diagonal matrix of singular value of eigenvalues of $\mathbf{A}^T\mathbf{A}$.

$$\mathbf{M}^T\mathbf{M} = \begin{pmatrix} 6 & 13 \\ 13 & 38 \end{pmatrix} \tag{2.0.5}$$

$$\mathbf{MM}^T = \begin{pmatrix} 13 & -17 & 8 \\ -17 & 26 & -11 \\ 8 & -11 & 5 \end{pmatrix} \tag{2.0.6}$$

Calculating eigen value of $\mathbf{M}^T\mathbf{M}$.

$$\begin{vmatrix} 6 - \lambda & 13 \\ 13 & 38 - \lambda \end{vmatrix} \lambda^2 - 44\lambda + 59 = 0 \tag{2.0.7}$$

$$\lambda_2 = -5\sqrt{17} + 22, \lambda_1 = 5\sqrt{17} + 22 \tag{2.0.8}$$

Eigen vectors of $\mathbf{MM}^T$.

$$\begin{vmatrix} 13 - \lambda & -17 & 8 \\ 17 & 26 - \lambda & -11 \\ 8 & -11 & 5 - \lambda \end{vmatrix} - \lambda^3 + 44\lambda^2 - 59\lambda = 0 \tag{2.0.9}$$

$$\lambda_4 = -5\sqrt{17} + 22, \lambda_3 = 5\sqrt{17} + 22, \lambda_5 = 0, \tag{2.0.10}$$

Hence,The eigenvectors will be

$$\mathbf{u}_2 = \begin{pmatrix} \frac{\sqrt{17}+12}{5} \\ \frac{3\sqrt{17}+1}{5} \\ 1 \end{pmatrix} \mathbf{u}_1 = \begin{pmatrix} \frac{-\sqrt{17}+12}{5} \\ \frac{-3\sqrt{17}+1}{5} \\ 1 \end{pmatrix} \mathbf{u}_3 = \begin{pmatrix} \frac{-3}{7} \\ \frac{1}{7} \\ 1 \end{pmatrix} \tag{2.0.11}$$

Normalising the eigenvectors

$$l_1 = \sqrt{\left(\frac{12 - \sqrt{17}}{5} \right)^2 + \left(\frac{1 - 3\sqrt{17}}{5} \right)^2 + 1^2} \tag{2.0.12}$$

$$\mathbf{u}_1 = \begin{pmatrix} \frac{-\sqrt{17}+12}{\sqrt{340-30\sqrt{17}}} \\ \frac{-3\sqrt{17}+1}{\sqrt{340-30\sqrt{17}}} \\ \frac{5}{\sqrt{340-30\sqrt{17}}} \end{pmatrix} \tag{2.0.13}$$

$$\tag{2.0.14}$$

$$l_2 = \sqrt{\left(\frac{\sqrt{17} + 12}{5} \right)^2 + \left(\frac{3\sqrt{17} + 1}{5} \right)^2 + 1^2} \tag{2.0.15}$$

$$\mathbf{u}_2 = \frac{5}{\sqrt{340 + 30\sqrt{7}}} \begin{pmatrix} \frac{\sqrt{17}+12}{5} \\ \frac{3\sqrt{17}+1}{5} \\ 1 \end{pmatrix} \tag{2.0.16}$$

$$\mathbf{u}_2 = \begin{pmatrix} \frac{\sqrt{17}+12}{\sqrt{340+30\sqrt{17}}} \\ \frac{3\sqrt{17}+1}{\sqrt{340+30\sqrt{17}}} \\ \frac{5}{\sqrt{340+30\sqrt{17}}} \end{pmatrix} \tag{2.0.17}$$

$$l_3 = \sqrt{\left(\frac{-3}{7}\right)^2 + \left(\frac{1}{7}\right)^2 + 1^2} \qquad (2.0.18)$$

$$\mathbf{u}_3 = \frac{7}{\sqrt{59}}\begin{pmatrix} \frac{-3}{7} \\ \frac{1}{7} \\ 1 \end{pmatrix} \qquad (2.0.19)$$

$$\mathbf{u}_3 = \begin{pmatrix} \frac{-3}{\sqrt{59}} \\ \frac{1}{\sqrt{59}} \\ \frac{7}{\sqrt{59}} \\ \frac{7}{\sqrt{59}} \end{pmatrix} \qquad (2.0.20)$$

$$\mathbf{U} = \begin{pmatrix} \frac{-\sqrt{17}+12}{\sqrt{340-30\sqrt{17}}} & \frac{\sqrt{17}+12}{\sqrt{340+30\sqrt{17}}} & \frac{-3}{\sqrt{59}} \\ \frac{-3\sqrt{17}+1}{\sqrt{340-30\sqrt{17}}} & \frac{3\sqrt{17}+1}{\sqrt{340+30\sqrt{17}}} & \frac{1}{\sqrt{59}} \\ \frac{5}{\sqrt{340-30\sqrt{17}}} & \frac{5}{\sqrt{340+30\sqrt{17}}} & \frac{7}{\sqrt{59}} \end{pmatrix} \qquad (2.0.21)$$

Now,

$$\mathbf{S} = \begin{pmatrix} \sqrt{5\sqrt{17}+22} & 0 \\ 0 & \sqrt{-5\sqrt{17}+22} \\ 0 & 0 \end{pmatrix} \qquad (2.0.22)$$

Now, $V=\mathbf{M}^T\frac{\mathbf{u_i}}{\sqrt{\lambda_i}}$

$$\mathbf{V} = \begin{pmatrix} \frac{\sqrt{17}+28}{\sqrt{340-30\sqrt{17}}\sqrt{5\sqrt{17}+22}} & \frac{-\sqrt{17}+28}{\sqrt{340+30\sqrt{17}}\sqrt{-5\sqrt{17}+22}} \\ \frac{12\sqrt{17}+41}{\sqrt{340-30\sqrt{17}}\sqrt{5\sqrt{17}+22}} & \frac{-12\sqrt{17}+41}{\sqrt{340+30\sqrt{17}}\sqrt{-5\sqrt{17}+22}} \end{pmatrix} \qquad (2.0.23)$$

So,from equation (2.0.4)

$$\begin{pmatrix} 2 & 3 \\ -1 & -5 \\ 1 & 2 \end{pmatrix} =$$

$$\begin{pmatrix} \frac{-\sqrt{17}+12}{\sqrt{340-30\sqrt{17}}} & \frac{\sqrt{17}+12}{\sqrt{340+30\sqrt{17}}} & \frac{-3}{\sqrt{59}} \\ \frac{-3\sqrt{17}+1}{\sqrt{340-30\sqrt{17}}} & \frac{3\sqrt{17}+1}{\sqrt{340+30\sqrt{17}}} & \frac{1}{\sqrt{59}} \\ \frac{5}{\sqrt{340-30\sqrt{17}}} & \frac{5}{\sqrt{340+30\sqrt{17}}} & \frac{7}{\sqrt{59}} \end{pmatrix}$$

$$\begin{pmatrix} \sqrt{5\sqrt{17}+22} & 0 \\ 0 & \sqrt{-5\sqrt{17}+22} \\ 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} \frac{\sqrt{17}+28}{\sqrt{340-30\sqrt{17}}\sqrt{5\sqrt{17}+22}} & \frac{-\sqrt{17}+28}{\sqrt{340+30\sqrt{17}}\sqrt{-5\sqrt{17}+22}} \\ \frac{12\sqrt{17}+41}{\sqrt{340-30\sqrt{17}}\sqrt{5\sqrt{17}+22}} & \frac{-12\sqrt{17}+41}{\sqrt{340+30\sqrt{17}}\sqrt{-5\sqrt{17}+22}} \end{pmatrix}^T \qquad (2.0.24)$$

Now, Finding Moore-Penrose Pseudo inverse of S

$$\mathbf{S}_+ = \begin{pmatrix} \frac{1}{\sqrt{5\sqrt{17}+22}} & 0 & 0 \\ 0 & \frac{1}{\sqrt{-5\sqrt{17}+22}} & 0 \end{pmatrix} \qquad (2.0.25)$$

We,know that, $x=\mathbf{V}(\mathbf{S}_+(\mathbf{U}^T\mathbf{b}))$

$$\mathbf{U}^T\mathbf{b} = \begin{pmatrix} \frac{-\sqrt{17}+7}{\sqrt{340-30\sqrt{17}}} \\ \frac{\sqrt{17}+7}{\sqrt{340+0\sqrt{17}}} \\ \frac{-10}{\sqrt{59}} \end{pmatrix}$$

$$(2.0.26)$$

$$\mathbf{S}_+(\mathbf{U}^T\mathbf{b}) = \begin{pmatrix} \frac{-\sqrt{17}+7}{\sqrt{340-30\sqrt{17}}\sqrt{5\sqrt{17}+22}} \\ \frac{\sqrt{17}+7}{\sqrt{340+30\sqrt{17}}\sqrt{-5\sqrt{17}+22}} \end{pmatrix} \qquad (2.0.27)$$

$$\mathbf{x} = \begin{pmatrix} \frac{\sqrt{17}+28}{\sqrt{340-30\sqrt{17}}\sqrt{5\sqrt{17}+22}} & \frac{-\sqrt{17}+28}{\sqrt{340+30\sqrt{17}}\sqrt{-5\sqrt{17}+22}} \\ \frac{12\sqrt{17}+41}{\sqrt{340-30\sqrt{17}}\sqrt{5\sqrt{17}+22}} & \frac{-12\sqrt{17}+41}{\sqrt{340+30\sqrt{17}}\sqrt{-5\sqrt{17}+22}} \end{pmatrix}$$

$$\begin{pmatrix} \frac{-\sqrt{17}+7}{\sqrt{340-30\sqrt{17}}\sqrt{5\sqrt{17}+22}} \\ \frac{\sqrt{17}+7}{\sqrt{340+30\sqrt{17}}\sqrt{-5\sqrt{17}+22}} \end{pmatrix} \qquad (2.0.28)$$

$$\mathbf{x} = \begin{pmatrix} \frac{2507500}{(4930-1040\sqrt{17})(4930+1040\sqrt{17})} \\ \frac{-702100}{(4930-1040\sqrt{17})(4930+1040\sqrt{17})} \end{pmatrix} \qquad (2.0.29)$$

Simplifying the values of x_1 and x_2

$$x_2 = \frac{-702100}{(4930-1040\sqrt{17})(4930+1040\sqrt{17})} \qquad (2.0.30)$$

$$= \frac{-702100}{591700} \qquad (2.0.31)$$

$$= -\frac{7}{59} \qquad (2.0.32)$$

$$x_1 = \frac{2507500}{(4930 - 1040\sqrt{17})(4930 + 1040\sqrt{17})} \tag{2.0.33}$$

$$= \frac{2507500}{591700} \tag{2.0.34}$$

$$= \frac{25}{59} \tag{2.0.35}$$

Now, Verifying the values using

$$\mathbf{M}^T\mathbf{M}\mathbf{x} = \mathbf{M}^T\mathbf{b} \tag{2.0.36}$$

Solving R.H.S

$$\mathbf{M}^T\mathbf{M}\mathbf{x} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{2.0.37}$$

Now using equation (2.0.5) in (2.0.37)

$$\begin{pmatrix} 6 & 13 \\ 13 & 38 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{2.0.38}$$

Solving the augmented matrix.

$$\begin{pmatrix} 6 & 13 & 1 \\ 13 & 38 & 1 \end{pmatrix} \xrightarrow{R_2 - \frac{13}{6}R_1} \begin{pmatrix} 6 & 13 & 1 \\ 0 & \frac{59}{6} & -\frac{7}{6} \end{pmatrix} \tag{2.0.39}$$

$$\frac{59}{6}x_2 = -\frac{7}{6} \tag{2.0.40}$$

$$6x_1 + 13x_2 = 1 \tag{2.0.41}$$

$$x_1 = \frac{25}{59}, x_2 = -\frac{7}{59} \tag{2.0.42}$$

$$\mathbf{x} = \begin{pmatrix} \frac{25}{59} \\ -\frac{7}{59} \end{pmatrix} \tag{2.0.43}$$