

Assignment 12

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The link to the solution is

<https://github.com/Adarsh1310/EE5609>

Abstract—This documents solves a problem based on Row Echelon form.

1 PROBLEM

Suppose \mathbf{R} and \mathbf{R}' are 2×3 row-reduced echelon matrices and that the system $\mathbf{RX}=0$ and $\mathbf{R}'\mathbf{X}=0$ have exactly the same solutions. Prove that $\mathbf{R}=\mathbf{R}'$.

2 SOLUTION

Since \mathbf{R} and \mathbf{R}' are 2×3 row-reduced echelon matrices they can be of following three types:-

- 1) Suppose matrix \mathbf{R} has one non-zero row then $\mathbf{RX}=0$ will have two free variables. Since $\mathbf{R}'\mathbf{X}=0$ will have the exact same solution as $\mathbf{RX}=0$, $\mathbf{R}'\mathbf{X}=0$ will also have two free variables. Thus \mathbf{R}' have one non zero row. Now let's consider a matrix \mathbf{A} with the first row as the non-zero row \mathbf{R} and second row as the second row of \mathbf{R}' .

$$\mathbf{R} = \begin{pmatrix} 1 & a & b \\ 0 & 0 & 0 \end{pmatrix} \quad (2.0.1)$$

$$\mathbf{R}' = \begin{pmatrix} 1 & c & d \\ 0 & 0 & 0 \end{pmatrix} \quad (2.0.2)$$

$$(2.0.3)$$

Let \mathbf{X} satisfy

$$\mathbf{RX} = 0 \quad (2.0.4)$$

$$\begin{pmatrix} 1 & \mathbf{a}^T \end{pmatrix} \begin{pmatrix} x \\ \mathbf{y} \end{pmatrix} = 0 \quad (2.0.5)$$

$$x + \mathbf{a}^T \mathbf{y} = 0 \quad (2.0.6)$$

where

$$\mathbf{a} = \begin{pmatrix} a \\ b \end{pmatrix} \quad (2.0.7)$$

$$\mathbf{R}'\mathbf{X} = 0 \quad (2.0.8)$$

$$\begin{pmatrix} 1 & \mathbf{b}^T \end{pmatrix} \begin{pmatrix} x \\ \mathbf{y} \end{pmatrix} = 0 \quad (2.0.9)$$

$$x + \mathbf{b}^T \mathbf{y} = 0 \quad (2.0.10)$$

where

$$\mathbf{b} = \begin{pmatrix} c \\ d \end{pmatrix} \quad (2.0.11)$$

Subtracting (2.0.10) from (2.0.6),

$$x + \mathbf{a}^T \mathbf{y} - x - \mathbf{b}^T \mathbf{y} = 0 \quad (2.0.12)$$

$$(\mathbf{a}^T - \mathbf{b}^T) \mathbf{y} = 0 \quad (2.0.13)$$

Hence for (2.0.13) to be always valid

$$\mathbf{a}^T - \mathbf{b}^T = 0 \quad (2.0.14)$$

$$\mathbf{a}^T = \mathbf{b}^T \quad (2.0.15)$$

Hence, $\mathbf{R}=\mathbf{R}'$

- 2) Let \mathbf{R} and \mathbf{R}' have all rows as non zero.

$$\mathbf{R} = \begin{pmatrix} 1 & 0 & b \\ 0 & 1 & c \end{pmatrix} \quad (2.0.16)$$

$$\mathbf{R}' = \begin{pmatrix} 1 & 0 & e \\ 0 & 1 & f \end{pmatrix} \quad (2.0.17)$$

Let \mathbf{X} satisfy

$$\mathbf{RX} = 0 \quad (2.0.18)$$

$$\mathbf{X}^T \mathbf{R}^T = 0 \quad (2.0.19)$$

Here,

$$\mathbf{R} = (\mathbf{I} \quad \mathbf{a}) \quad (2.0.20)$$

$$\mathbf{a} = \begin{pmatrix} b \\ c \end{pmatrix} \quad (2.0.21)$$

$$\mathbf{R}^T = \begin{pmatrix} \mathbf{I} \\ \mathbf{a}^T \end{pmatrix} \quad (2.0.22)$$

Let,

$$\mathbf{X}^T = (\mathbf{y}^T \quad z) \quad (2.0.23)$$

where z is a scalar constant. Now, substituting

(2.0.23) and (2.0.20) in (2.0.19)

$$\begin{pmatrix} \mathbf{y}^T & z \end{pmatrix} \begin{pmatrix} \mathbf{I} \\ \mathbf{a}^T \end{pmatrix} = 0 \quad (2.0.24)$$

$$\mathbf{y}^T + z\mathbf{a}^T = 0 \quad (2.0.25)$$

Now for,

$$\mathbf{R}'\mathbf{X} = 0 \quad (2.0.26)$$

$$\mathbf{X}^T\mathbf{R}'^T = 0 \quad (2.0.27)$$

Here,

$$\mathbf{R}' = \begin{pmatrix} \mathbf{I} & \mathbf{b} \end{pmatrix} \quad (2.0.28)$$

$$\mathbf{b} = \begin{pmatrix} e \\ f \end{pmatrix} \quad (2.0.29)$$

Let,

$$\mathbf{X}^T = \begin{pmatrix} \mathbf{y}^T & z \end{pmatrix} \quad (2.0.30)$$

where z is a scalar constant. Now, substituting (2.0.30) and (2.0.28) in (2.0.27)

$$\begin{pmatrix} \mathbf{y}^T & z \end{pmatrix} \begin{pmatrix} \mathbf{I} \\ \mathbf{b}^T \end{pmatrix} = 0 \quad (2.0.31)$$

$$\mathbf{y}^T + z\mathbf{b}^T = 0 \quad (2.0.32)$$

Subtracting (2.0.32) from (2.0.25)

$$\mathbf{y}^T + z\mathbf{a}^T - \mathbf{y}^T - z\mathbf{b}^T = 0 \quad (2.0.33)$$

$$(\mathbf{a}^T - \mathbf{b}^T)\mathbf{y} = 0 \quad (2.0.34)$$

Hence (2.0.34) to be always valid

$$\mathbf{a}^T - \mathbf{b}^T = 0 \quad (2.0.35)$$

$$\mathbf{a}^T = \mathbf{b}^T \quad (2.0.36)$$

Hence, $\mathbf{R} = \mathbf{R}'$

- 3) Suppose matrix \mathbf{R} have all the rows as zero then $\mathbf{RX} = 0$ will be satisfied for all values of \mathbf{X} . We know that $\mathbf{R}'\mathbf{X} = 0$ will have the exact same solution as $\mathbf{RX} = 0$ then we can say that for all values of $\mathbf{X} = 0$ equation $\mathbf{R}'\mathbf{X} = 0$ will be satisfied Hence, $\mathbf{R}' = \mathbf{R} = 0$.