

Assignment 5

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The link to the solution is

<https://github.com/Adarsh1310/EE5609>

Abstract—This documents solves a problem based on circles.

1 PROBLEM

Find the area of the region bounded by the circle $\mathbf{x}^T \mathbf{x} = 4$ and $\left\| \mathbf{x} - \begin{pmatrix} 2 \\ 0 \end{pmatrix} \right\| = 2$.

2 SOLUTION

$$\|\mathbf{x}\|^2 + 2\mathbf{u}^T \mathbf{x} + f = 0$$

$$\mathbf{x}^T \mathbf{x} + 2\mathbf{u}^T \mathbf{x} + f = 0$$

So from above equation we can say that,

2.1 Circle 1

Taking equation of the first circle to be,

$$\|\mathbf{x}\|^2 + 2\mathbf{u}_1^T \mathbf{x} + f_1 = 0 \quad (2.1.1)$$

$$\mathbf{x}^T \mathbf{x} - 4 = 0 \text{ (given)} \quad (2.1.2)$$

$$\mathbf{u}_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (2.1.3)$$

$$f_1 = -4 \quad (2.1.4)$$

$$\mathbf{O}_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (2.1.5)$$

2.2 Circle 2

Taking equation of the second circle to be,

$$\left\| \mathbf{x} - \begin{pmatrix} 2 \\ 0 \end{pmatrix} \right\|^2 = 2^2 \text{ (given)} \quad (2.2.1)$$

$$\mathbf{x}^T \mathbf{x} + 2\mathbf{u}_2^T \mathbf{x} = 0 \quad (2.2.2)$$

$$\mathbf{u}_2 = \begin{pmatrix} -2 \\ 0 \end{pmatrix} \quad (2.2.3)$$

$$f_2 = 0 \quad (2.2.4)$$

$$\mathbf{O}_2 = \begin{pmatrix} 2 \\ 0 \end{pmatrix} \quad (2.2.5)$$

Now, Subtracting equation (2.2.2) from (2.1.2) We get,

$$\mathbf{x}^T \mathbf{x} - 2\mathbf{u}_2^T \mathbf{x} + f_1 - \mathbf{x}^T \mathbf{x} = 0 \quad (2.2.6)$$

$$2\mathbf{u}_2^T \mathbf{x} = -4 \quad (2.2.7)$$

$$\begin{pmatrix} -4 & 0 \end{pmatrix} \mathbf{x} = -4 \quad (2.2.8)$$

Which can be written as:-

$$\begin{pmatrix} 1 & 0 \end{pmatrix} \mathbf{x} = 1 \quad (2.2.9)$$

$$\mathbf{x} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad (2.2.10)$$

$$\mathbf{x} = \mathbf{q} + \lambda \mathbf{m} \quad (2.2.11)$$

$$\mathbf{q} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (2.2.12)$$

$$\mathbf{m} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad (2.2.13)$$

Substituting (2.2.11) in (2.1.1)

$$\|\mathbf{x}\|^2 + 2\mathbf{u}_1^T \mathbf{x} + f_1 = 0 \quad (2.2.14)$$

$$\|\mathbf{q} + \lambda \mathbf{m}\|^2 + f_1 = 0 \quad (2.2.15)$$

$$\left\| \begin{pmatrix} 1 \\ \lambda \end{pmatrix} \right\|^2 - 4 = 0 \quad (2.2.16)$$

$$\lambda^2 + 1 - 4 = 0 \quad (2.2.17)$$

$$\lambda = +\sqrt{3}, -\sqrt{3} \quad (2.2.18)$$

$$\lambda = +\sqrt{3}, -\sqrt{3} \quad (2.2.19)$$

Substituting the value of λ in (2.2.11)

$$\mathbf{x} = \mathbf{q} + \lambda \mathbf{m} \quad (2.2.20)$$

$$\text{For } \lambda = \sqrt{3}, \mathbf{A} = \begin{pmatrix} 1 \\ \sqrt{3} \end{pmatrix} \quad (2.2.21)$$

$$\text{For } \lambda = -\sqrt{3}, \mathbf{B} = \begin{pmatrix} 1 \\ -\sqrt{3} \end{pmatrix} \quad (2.2.22)$$

Now finding the direction vector $\mathbf{O}_1\mathbf{A}, \mathbf{O}_1\mathbf{B}, \mathbf{O}_2\mathbf{A}$ and $\mathbf{O}_2\mathbf{B}$.

$$\mathbf{O}_1\mathbf{A} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 1 \\ \sqrt{3} \end{pmatrix} = k_1 \begin{pmatrix} -1 \\ -\sqrt{3} \end{pmatrix} \quad (2.2.23)$$

$$\mathbf{O}_1\mathbf{B} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 1 \\ -\sqrt{3} \end{pmatrix} = k_2 \begin{pmatrix} -1 \\ \sqrt{3} \end{pmatrix} \quad (2.2.24)$$

$$\mathbf{O}_2\mathbf{A} = \begin{pmatrix} 2 \\ 0 \end{pmatrix} - \begin{pmatrix} 1 \\ \sqrt{3} \end{pmatrix} = k_3 \begin{pmatrix} 1 \\ -\sqrt{3} \end{pmatrix} \quad (2.2.25)$$

$$\mathbf{O}_2\mathbf{B} = \begin{pmatrix} 2 \\ 0 \end{pmatrix} - \begin{pmatrix} 1 \\ -\sqrt{3} \end{pmatrix} = k_4 \begin{pmatrix} 1 \\ \sqrt{3} \end{pmatrix} \quad (2.2.26)$$

Now finding the angle $\angle O_1AB$.

$$\mathbf{O}_1\mathbf{A} \cdot \mathbf{O}_1\mathbf{B} = \|\mathbf{O}_1\mathbf{A}\| \|\mathbf{O}_1\mathbf{B}\| \cos \theta_1 \quad (2.2.27)$$

$$\frac{\mathbf{O}_1\mathbf{A} \cdot \mathbf{O}_1\mathbf{B}}{\|\mathbf{O}_1\mathbf{A}\| \|\mathbf{O}_1\mathbf{B}\|} = \cos \theta_1 \quad (2.2.28)$$

$$\frac{-1}{2} = \cos \theta_1 \quad (2.2.29)$$

$$\theta_1 = 120^\circ \quad (2.2.30)$$

Now finding the angle $\angle O_2AB$.

$$\mathbf{O}_2\mathbf{A} \cdot \mathbf{O}_2\mathbf{B} = \|\mathbf{O}_2\mathbf{A}\| \|\mathbf{O}_2\mathbf{B}\| \cos \theta_2 \quad (2.2.31)$$

$$\frac{\mathbf{O}_2\mathbf{A} \cdot \mathbf{O}_2\mathbf{B}}{\|\mathbf{O}_2\mathbf{A}\| \|\mathbf{O}_2\mathbf{B}\|} = \cos \theta_2 \quad (2.2.32)$$

$$\frac{-1}{2} = \cos \theta_2 \quad (2.2.33)$$

$$\theta_2 = 120^\circ \quad (2.2.34)$$

Finding area of $\mathbf{O}_1\mathbf{AB}$ and $\mathbf{O}_2\mathbf{AB}$.

$$Area_1 = \frac{\theta_1}{360} r^2 - \frac{1}{2} 2 \sqrt{3} \quad (2.2.35)$$

$$Area_1 = \frac{120}{360} 4\pi - \frac{1}{2} 2 \sqrt{3} \quad (2.2.36)$$

$$Area_2 = \frac{\pi \theta_2}{360} r^2 - \frac{1}{2} 2 \sqrt{3} \quad (2.2.37)$$

$$Area_2 = \frac{120}{360} 4\pi - \frac{1}{2} 2 \sqrt{3} \quad (2.2.38)$$

$$TotalArea = \frac{8\pi}{3} - 2\sqrt{3} (\because Area_1 + Area_2) \quad (2.2.39)$$

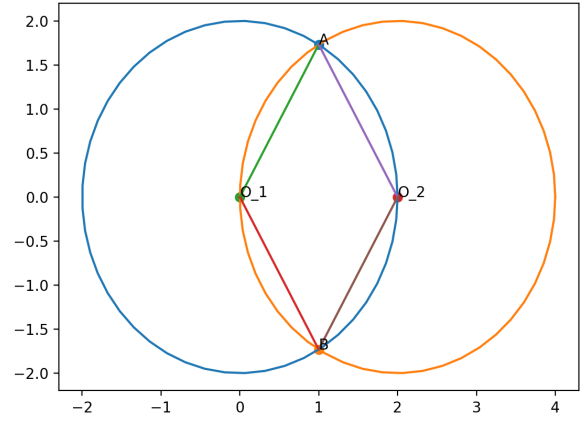


Fig. 0: Figure depicting intersection points of circle