Assignment 7

Adarsh Srivastava

The link to the solution is

https://github.com/Adarsh1310/EE5609

***Abstract*—This documents solves a problem based on curve and tangent.**

1 Problem

Find the slope of the tangent to the curve y=$\frac{x-1}{x-2}$, x≠2 at x=10.

2 Solution

$$y = \frac{x-1}{x-2} \tag{2.0.1}$$

Equation (2.0.1) can be expressed as

$$y(x-2) = x-1 \tag{2.0.2}$$

$$yx - 2y - x + 1 = 0 \tag{2.0.3}$$

From above we can say,

$$\mathbf{V} = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{2.0.4}$$

$$\mathbf{u} = \begin{pmatrix} -\frac{1}{2} & -1 \end{pmatrix} \tag{2.0.5}$$

$$f = 1 \tag{2.0.6}$$

Now,

$$\because |V| = \begin{vmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{vmatrix} < 0, \tag{2.0.7}$$

(2.0.1) is the equation of a hyperbola. To verify that this we will find the the characteristic equation of $\mathbf{V}$.

$$\left|\lambda\mathbf{I} - \mathbf{V}\right| = \begin{vmatrix} \lambda & \frac{1}{2} \\ \frac{1}{2} & \lambda \end{vmatrix} = 0 \tag{2.0.8}$$

$$\implies \lambda^2 - 2\lambda + \frac{3}{4} = 0 \tag{2.0.9}$$

The eigenvalues are the roots of (2.0.9) given by

$$\lambda_1 = \frac{1}{2}, \lambda_2 = -\frac{1}{2} \tag{2.0.10}$$

The eigenvector $\mathbf{p}$ is defined as

$$\mathbf{V}\mathbf{p} = \lambda\mathbf{p} \tag{2.0.11}$$

$$\implies (\lambda\mathbf{I} - \mathbf{V})\,\mathbf{p} = 0 \tag{2.0.12}$$

where λ is the eigenvalue. For $\lambda_1 = \frac{1}{2}$,

$$(\lambda_1\mathbf{I} - \mathbf{V}) = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \xleftrightarrow[R_1 \leftarrow 2R_1]{R_2 \leftarrow R_2 - R_1} \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \tag{2.0.13}$$

$$\implies \mathbf{p}_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix} \tag{2.0.14}$$

Now,λ is the eigenvalue. For $\lambda_2 = -\frac{1}{2}$,

$$(\lambda_2\mathbf{I} - \mathbf{V}) = \begin{pmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix} \xleftrightarrow[R_1 \leftarrow 2R_1]{R_2 \leftarrow R_2 + R_1} \begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix} \tag{2.0.15}$$

$$\implies \mathbf{p}_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{2.0.16}$$

From Equations,

$$\mathbf{V} = \mathbf{P}\mathbf{D}\mathbf{P}^{-1} = \mathbf{P}\mathbf{D}\mathbf{P}^{T} \quad \because \mathbf{P}^{-1} = \mathbf{P}^{T} \tag{2.0.17}$$

$$\text{or, } \mathbf{D} = \mathbf{P}^{T}\mathbf{V}\mathbf{P} \tag{2.0.18}$$

We can say that

$$\mathbf{P} = \begin{pmatrix} \mathbf{p}_1 & \mathbf{p}_2 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \tag{2.0.19}$$

$$\mathbf{D} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix} \tag{2.0.20}$$

$\because \mathbf{u}^T\mathbf{V}^{-1}\mathbf{u} - f > 0$,there isn't a need to swap axes.In hyperbola,

$$\mathbf{c} = -\mathbf{V}^{-}1\mathbf{u} \tag{2.0.21}$$

$$axes = \begin{cases} \sqrt{\frac{\mathbf{u}^T\mathbf{V}^{-1}\mathbf{u} - f}{\lambda_1}} \\ \sqrt{\frac{f - \mathbf{u}^T\mathbf{V}^{-1}\mathbf{u}}{\lambda_2}} \end{cases} \tag{2.0.22}$$

From above equations we can say that,

$$\mathbf{c} = \begin{pmatrix} -2 \\ -1 \end{pmatrix} \tag{2.0.23}$$

$$\sqrt{\frac{\mathbf{u}^T\mathbf{V}^{-1}\mathbf{u} - f}{\lambda_1}} = \sqrt{2} \tag{2.0.24}$$

$$\sqrt{\frac{f - \mathbf{u}^T\mathbf{V}^{-1}\mathbf{u}}{\lambda_2}} = \sqrt{2} \tag{2.0.25}$$

with the standard hyperbola equation becoming

$$\frac{x^2}{2} - \frac{y^2}{2} = 1, \tag{2.0.26}$$

Let us assume slope to be l,now finding the direction vector and normal vector of the tangent with slope l.

$$\mathbf{m} = \begin{pmatrix} 1 \\ l \end{pmatrix} \tag{2.0.27}$$

$$\mathbf{n} = \begin{pmatrix} l \\ -1 \end{pmatrix} \tag{2.0.28}$$

Now considering the equations to find point of contact

$$\mathbf{q} = \mathbf{V}^{-1}\left(\kappa\mathbf{n} - \mathbf{u}\right) \tag{2.0.29}$$

$$\kappa = \pm\sqrt{\frac{\mathbf{u}^T\mathbf{V}^{-1}\mathbf{u} - f}{\mathbf{n}^T\mathbf{V}^{-1}\mathbf{n}}} \tag{2.0.30}$$

By using (2.0.30)

$$\kappa = \sqrt{-\frac{1}{4l}} \tag{2.0.31}$$

Now substituting this κ in (2.0.29)

$$\mathbf{q} = \begin{pmatrix} -2\sqrt{-\frac{1}{4l}} + 2 \\ 2\sqrt{\frac{-l}{4}} + 1 \end{pmatrix} \tag{2.0.32}$$

We know that x=10.

$$-2\sqrt{-\frac{1}{4l}} + 2 = 10 \tag{2.0.33}$$

$$-2\sqrt{-\frac{1}{4l}} = 8 \tag{2.0.34}$$

$$\sqrt{-\frac{1}{4l}} = 4 \tag{2.0.35}$$

$$-\frac{1}{4l} = 16 \tag{2.0.36}$$

$$l = -\frac{1}{64} \tag{2.0.37}$$

The slope of the tangent to the curve y=$\frac{x-1}{x-2}$, x≠2 at x=10 is $\frac{1}{64}$. So,from the above we can say that κ=4,-4 and from equation (2.0.27) and (2.0.28) direction and normal vectors will come out to be

$$\mathbf{m} = \begin{pmatrix} 1 \\ -\frac{1}{64} \end{pmatrix} \tag{2.0.38}$$

$$\mathbf{n} = \begin{pmatrix} -\frac{1}{64} \\ -1 \end{pmatrix} \tag{2.0.39}$$

Now using equation (2.0.29)

$$\mathbf{q}_1 = \mathbf{V}^{-1}\left(\kappa_1\mathbf{n} - \mathbf{u}\right) \tag{2.0.40}$$

$$\mathbf{q}_1 = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}\left(-4\begin{pmatrix} -\frac{1}{64} \\ -1 \end{pmatrix} - \begin{pmatrix} -\frac{1}{2} \\ -1 \end{pmatrix}\right) \tag{2.0.41}$$

$$\mathbf{q}_1 = \begin{pmatrix} 10 \\ \frac{9}{8} \end{pmatrix} \tag{2.0.42}$$

$$\mathbf{q}_2 = \mathbf{V}^{-1}\left(\kappa_2\mathbf{n} - \mathbf{u}\right) \tag{2.0.43}$$

$$\mathbf{q}_2 = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}\left(4\begin{pmatrix} -\frac{1}{64} \\ -1 \end{pmatrix} - \begin{pmatrix} -\frac{1}{2} \\ -1 \end{pmatrix}\right) \tag{2.0.44}$$

$$\mathbf{q}_2 = \begin{pmatrix} -6 \\ \frac{7}{8} \end{pmatrix} \tag{2.0.45}$$

$$\tag{2.0.46}$$

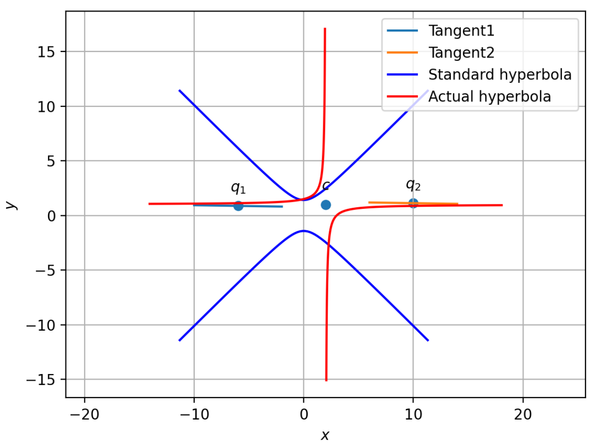

Fig. 0: Tangent 2 shows the tangent