

Assignment 7

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The link to the solution is

<https://github.com/Adarsh1310/EE5609>

Abstract—This documents solves a problem based on curve and tangent.

1 PROBLEM

Find the slope of the tangent to the curve $y = \frac{x-1}{x-2}$, $x \neq 2$ at $x=10$.

2 SOLUTION

$$y = \frac{x-1}{x-2} \quad (2.0.1)$$

Equation (2.0.1) can be expressed as

$$y(x-2) = x-1 \quad (2.0.2)$$

$$yx - 2y - x + 1 = 0 \quad (2.0.3)$$

From above we can say,

$$\mathbf{V} = \frac{1}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad (2.0.4)$$

$$\mathbf{u} = \begin{pmatrix} -\frac{1}{2} & -1 \end{pmatrix} \quad (2.0.5)$$

$$f = 1 \quad (2.0.6)$$

Now,

$$\because |V| = \begin{vmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{vmatrix} < 0, \quad (2.0.7)$$

(2.0.1) is the equation of a hyperbola. To verify that this we will find the the characteristic equation of \mathbf{V} .

$$|\lambda \mathbf{I} - \mathbf{V}| = \begin{vmatrix} \lambda & \frac{1}{2} \\ \frac{1}{2} & \lambda \end{vmatrix} = 0 \quad (2.0.8)$$

$$\Rightarrow \lambda^2 - 2\lambda + \frac{3}{4} = 0 \quad (2.0.9)$$

The eigenvalues are the roots of (2.0.9) given by

$$\lambda_1 = \frac{1}{2}, \lambda_2 = -\frac{1}{2} \quad (2.0.10)$$

The eigenvector \mathbf{p} is defined as

$$\mathbf{V}\mathbf{p} = \lambda\mathbf{p} \quad (2.0.11)$$

$$\Rightarrow (\lambda \mathbf{I} - \mathbf{V})\mathbf{p} = 0 \quad (2.0.12)$$

where λ is the eigenvalue. For $\lambda_1 = \frac{1}{2}$,

$$(\lambda_1 \mathbf{I} - \mathbf{V}) = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \xrightarrow[R_1 \leftarrow R_2 - R_1]{R_2 \leftarrow R_2 - R_1} \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \quad (2.0.13)$$

$$\Rightarrow \mathbf{p}_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad (2.0.14)$$

Now, λ is the eigenvalue. For $\lambda_2 = -\frac{1}{2}$,

$$(\lambda_2 \mathbf{I} - \mathbf{V}) = \begin{pmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix} \xrightarrow[R_1 \leftarrow -R_1]{R_2 \leftarrow R_2 + R_1} \begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix} \quad (2.0.15)$$

$$\Rightarrow \mathbf{p}_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad (2.0.16)$$

From Equations,

$$\mathbf{V} = \mathbf{P}\mathbf{D}\mathbf{P}^{-1} = \mathbf{P}\mathbf{D}\mathbf{P}^T \quad \because \mathbf{P}^{-1} = \mathbf{P}^T \quad (2.0.17)$$

$$\text{or, } \mathbf{D} = \mathbf{P}^T \mathbf{V} \mathbf{P} \quad (2.0.18)$$

We can say that

$$\mathbf{P} = (\mathbf{p}_1 \quad \mathbf{p}_2) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \quad (2.0.19)$$

$$\mathbf{D} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix} \quad (2.0.20)$$

$\because \mathbf{u}^T \mathbf{V}^{-1} \mathbf{u} - f > 0$, there isn't a need to swap axes. In hyperbola,

$$\mathbf{c} = -\mathbf{V}^{-1} \mathbf{u} \quad (2.0.21)$$

$$\text{axes} = \begin{cases} \sqrt{\frac{\mathbf{u}^T \mathbf{V}^{-1} \mathbf{u} - f}{\lambda_1}} \\ \sqrt{\frac{f - \mathbf{u}^T \mathbf{V}^{-1} \mathbf{u}}{\lambda_2}} \end{cases} \quad (2.0.22)$$

From above equations we can say that,

$$\mathbf{c} = \begin{pmatrix} -2 \\ -1 \end{pmatrix} \quad (2.0.23)$$

$$\sqrt{\frac{\mathbf{u}^T \mathbf{V}^{-1} \mathbf{u} - f}{\lambda_1}} = \sqrt{2} \quad (2.0.24)$$

$$\sqrt{\frac{f - \mathbf{u}^T \mathbf{V}^{-1} \mathbf{u}}{\lambda_2}} = \sqrt{2} \quad (2.0.25)$$

with the standard hyperbola equation becoming

$$\frac{x^2}{2} - \frac{y^2}{2} = 1, \quad (2.0.26)$$

Let us assume slope to be l , now finding the direction vector and normal vector of the tangent with slope l .

$$\mathbf{m} = \begin{pmatrix} 1 \\ l \end{pmatrix} \quad (2.0.27)$$

$$\mathbf{n} = \begin{pmatrix} l \\ -1 \end{pmatrix} \quad (2.0.28)$$

Now considering the equations to find point of contact

$$\mathbf{q} = \mathbf{V}^{-1} (\kappa \mathbf{n} - \mathbf{u}) \quad (2.0.29)$$

$$\kappa = \pm \sqrt{\frac{\mathbf{u}^T \mathbf{V}^{-1} \mathbf{u} - f}{\mathbf{n}^T \mathbf{V}^{-1} \mathbf{n}}} \quad (2.0.30)$$

By using (2.0.30)

$$\kappa = \sqrt{-\frac{1}{4l}} \quad (2.0.31)$$

Now substituting this κ in (2.0.29)

$$\mathbf{q} = \begin{pmatrix} -2\sqrt{-\frac{1}{4l}} + 2 \\ 2\sqrt{-\frac{l}{4}} + 1 \end{pmatrix} \quad (2.0.32)$$

We know that $x=10$.

$$-2\sqrt{-\frac{1}{4l}} + 2 = 10 \quad (2.0.33)$$

$$-2\sqrt{-\frac{1}{4l}} = 8 \quad (2.0.34)$$

$$\sqrt{-\frac{1}{4l}} = 4 \quad (2.0.35)$$

$$-\frac{1}{4l} = 16 \quad (2.0.36)$$

$$l = -\frac{1}{64} \quad (2.0.37)$$

The slope of the tangent to the curve $y = \frac{x-1}{x-2}$, $x \neq 2$ at $x=10$ is $\frac{1}{64}$. So, from the above we can say that $\kappa=4, -4$ and from equation (2.0.27) and (2.0.28) direction and normal vectors will come out to be

$$\mathbf{m} = \begin{pmatrix} 1 \\ -\frac{1}{64} \end{pmatrix} \quad (2.0.38)$$

$$\mathbf{n} = \begin{pmatrix} -\frac{1}{64} \\ -1 \end{pmatrix} \quad (2.0.39)$$

Now using equation (2.0.29)

$$\mathbf{q}_1 = \mathbf{V}^{-1} (\kappa_1 \mathbf{n} - \mathbf{u}) \quad (2.0.40)$$

$$\mathbf{q}_1 = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix} \left(-4 \begin{pmatrix} -\frac{1}{64} \\ -1 \end{pmatrix} - \begin{pmatrix} -\frac{1}{2} \\ -1 \end{pmatrix} \right) \quad (2.0.41)$$

$$\mathbf{q}_1 = \begin{pmatrix} 10 \\ \frac{9}{8} \end{pmatrix} \quad (2.0.42)$$

$$\mathbf{q}_2 = \mathbf{V}^{-1} (\kappa_2 \mathbf{n} - \mathbf{u}) \quad (2.0.43)$$

$$\mathbf{q}_2 = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix} \left(4 \begin{pmatrix} -\frac{1}{64} \\ -1 \end{pmatrix} - \begin{pmatrix} -\frac{1}{2} \\ -1 \end{pmatrix} \right) \quad (2.0.44)$$

$$\mathbf{q}_2 = \begin{pmatrix} -6 \\ \frac{7}{8} \end{pmatrix} \quad (2.0.45)$$

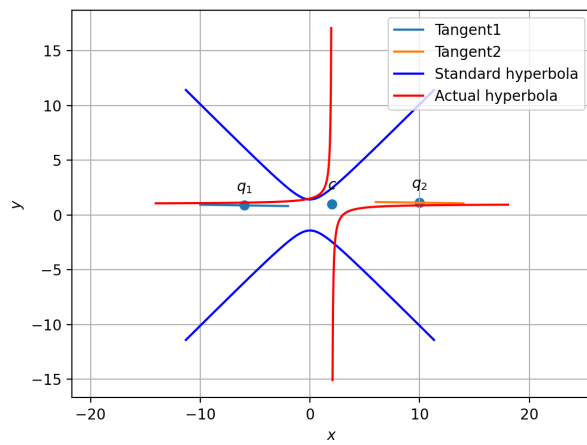


Fig. 0: Tangent 2 shows the tangent