Assignment 12

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The link to the solution is

https://github.com/Adarsh1310/EE5609

***Abstract*—This documents solves a problem based on Row Echelon form.**

1 Problem

Suppose $\mathbf{R}$ and $\mathbf{R}'$ are 2×3 row-reduced echelon matrices and that the system $\mathbf{RX}$=0 and $\mathbf{R}'\mathbf{X}$=0 have exactly the same solutions. Prove that $\mathbf{R}$=$\mathbf{R}'$.

2 Solution

Since $\mathbf{R}$ and $\mathbf{R}'$ are 2×3 row-reduced echelon matrices they can be of following three types:-

1) Suppose matrix $\mathbf{R}$ has one non-zero row then $\mathbf{RX}$=0 will have two free variables.Since $R'\mathbf{X}$=0 will have the exact same solution as $\mathbf{RX}$=0, $R'\mathbf{X}$=0 will also have two free variables. Thus $\mathbf{R}'$ have one non zero row.Now let's consider a matrix $\mathbf{A}$ with the first row as the non-zero row $\mathbf{R}$ and second row as the second row of $\mathbf{R}'$.

$$\mathbf{R} = \begin{pmatrix} 1 & a & b \\ 0 & 0 & 0 \end{pmatrix} \tag{2.0.1}$$

$$\mathbf{R}' = \begin{pmatrix} 1 & c & d \\ 0 & 0 & 0 \end{pmatrix} \tag{2.0.2}$$

$$\tag{2.0.3}$$

Let $\mathbf{X}$ satisfy

$$\mathbf{RX} = 0 \tag{2.0.4}$$

$$\begin{pmatrix} 1 & \mathbf{a}^T \end{pmatrix} \begin{pmatrix} x \\ \mathbf{y} \end{pmatrix} = 0 \tag{2.0.5}$$

$$x + \mathbf{a}^T \mathbf{y} = 0 \tag{2.0.6}$$

where

$$\mathbf{a} = \begin{pmatrix} a \\ b \end{pmatrix} \tag{2.0.7}$$

$$\mathbf{R}'\mathbf{X} = 0 \tag{2.0.8}$$

$$\begin{pmatrix} 1 & \mathbf{b}^T \end{pmatrix} \begin{pmatrix} x \\ \mathbf{y} \end{pmatrix} = 0 \tag{2.0.9}$$

$$x + \mathbf{b}^T \mathbf{y} = 0 \tag{2.0.10}$$

where

$$\mathbf{b} = \begin{pmatrix} c \\ d \end{pmatrix} \tag{2.0.11}$$

Subtracting (2.0.10) from (2.0.6),

$$x + \mathbf{a}^T \mathbf{y} - x - \mathbf{b}^T \mathbf{y} = 0 \tag{2.0.12}$$

$$(\mathbf{a}^T - \mathbf{b}^T)\mathbf{y} = 0 \tag{2.0.13}$$

Since $\mathbf{y}$ is a 2×1 vector,

$$\implies y_1\mathbf{a} - y_2\mathbf{b} = 0 \tag{2.0.14}$$

Which can be written as,

$$\mathbf{a} = k\mathbf{b} \tag{2.0.15}$$

where,k=$\frac{\mathrm{y}_2}{\mathrm{y}_1}$ assuming $\mathrm{y}_1 \neq 0$.Now,Substituting (2.0.15) in (2.0.6)

$$x + k\mathbf{b}^T \mathbf{y} = 0 \tag{2.0.16}$$

Comparing (2.0.16) with (2.0.10)

$$x + \mathbf{b}^T \mathbf{y} = 0 \tag{2.0.17}$$

$$x + k\mathbf{b}^T \mathbf{y} = 0 \tag{2.0.18}$$

Hence k=1 which means y_1=y_2 and from this we can say that $\mathbf{a}$=$\mathbf{b}$.If in the above case we take y_1=0 then

$$y_1\mathbf{a} - y_2\mathbf{b} = 0 \tag{2.0.19}$$

$$y_2\mathbf{b} = 0 \tag{2.0.20}$$

Hence for the (2.0.20) to be always true $\mathbf{b}$ should be zero.Now from (2.0.15) we will see that $\mathbf{a}$ will also be 0. Hence, $\mathbf{R}$=$\mathbf{R}'$

2) Let $\mathbf{R}$ and $\mathbf{R}'$ have all rows as non zero.

$$\mathbf{R} = \begin{pmatrix} 1 & 0 & b \\ 0 & 1 & c \end{pmatrix} \tag{2.0.21}$$

$$\mathbf{R}' = \begin{pmatrix} 1 & 0 & e \\ 0 & 1 & f \end{pmatrix} \tag{2.0.22}$$

Let $\mathbf{X}$ satisfy

$$\mathbf{RX} = 0 \tag{2.0.23}$$

$$\mathbf{X}^T\mathbf{R}^T = 0 \tag{2.0.24}$$

Here,

$$\mathbf{R} = \begin{pmatrix} \mathbf{I} & \mathbf{a} \end{pmatrix} \tag{2.0.25}$$

$$\mathbf{a} = \begin{pmatrix} b \\ c \end{pmatrix} \tag{2.0.26}$$

$$\mathbf{R}^T = \begin{pmatrix} \mathbf{I} \\ \mathbf{a}^T \end{pmatrix} \tag{2.0.27}$$

Let,

$$\mathbf{X}^T = \begin{pmatrix} \mathbf{y}^T & z \end{pmatrix} \tag{2.0.28}$$

where z is a scalar constant. Now,substituting (2.0.28) and (2.0.25) in (2.0.24)

$$\begin{pmatrix} \mathbf{y}^T & z \end{pmatrix} \begin{pmatrix} \mathbf{I} \\ \mathbf{a}^T \end{pmatrix} = 0 \tag{2.0.29}$$

$$\mathbf{y}^T + z\mathbf{a}^T = 0 \tag{2.0.30}$$

Now for,

$$\mathbf{R}'\mathbf{X} = 0 \tag{2.0.31}$$

$$\mathbf{X}^T\mathbf{R}'^T = 0 \tag{2.0.32}$$

Here,

$$\mathbf{R}' = \begin{pmatrix} \mathbf{I} & \mathbf{b} \end{pmatrix} \tag{2.0.33}$$

$$\mathbf{b} = \begin{pmatrix} e \\ f \end{pmatrix} \tag{2.0.34}$$

Let,

$$\mathbf{X}^T = \begin{pmatrix} \mathbf{y}^T & z \end{pmatrix} \tag{2.0.35}$$

where z is a scalar constant.Now,substituting (2.0.35) and (2.0.33) in (2.0.32)

$$\begin{pmatrix} \mathbf{y}^T & z \end{pmatrix} \begin{pmatrix} \mathbf{I} \\ \mathbf{b}^T \end{pmatrix} = 0 \tag{2.0.36}$$

$$\mathbf{y}^T + z\mathbf{b}^T = 0 \tag{2.0.37}$$

Subtracting (2.0.37) from (2.0.30)

$$\mathbf{y}^T + z\mathbf{a}^T - \mathbf{y}^T - z\mathbf{b}^T = 0 \tag{2.0.38}$$

$$(\mathbf{a}^T - \mathbf{b}^T)z = 0 \tag{2.0.39}$$

$$\mathbf{a}^T = \mathbf{b}^T \tag{2.0.40}$$

3) Suppose matrix $\mathbf{R}$ have all the rows as zero then $\mathbf{RX}$=0 will be satisfied for all values of $\mathbf{X}$. We know that $\mathbf{R}'\mathbf{X}$=0 will have the exact same solution as $\mathbf{RX}$=0 then we can say that for all values of $\mathbf{X}$=0 equation $\mathbf{R}'\mathbf{X}$=0 will be satisfied.Hence, $\mathbf{R}'$=$\mathbf{R}$=0.