Assignment 9

Adarsh Srivastava

The link to the solution is

https://github.com/Adarsh1310/EE5609

***Abstract*—This documents solves a Singular Value decomposition problem.**

1 Problem

Find the shortest distance between the lines

$$\mathbf{x} = \begin{pmatrix}1\\1\\0\end{pmatrix} + \lambda_1 \begin{pmatrix}2\\-1\\1\end{pmatrix} \tag{1.0.1}$$

$$\mathbf{x} = \begin{pmatrix}2\\1\\-1\end{pmatrix} + \lambda_2 \begin{pmatrix}3\\-5\\2\end{pmatrix} \tag{1.0.2}$$

2 Solution

The lines will intersect if

$$\begin{pmatrix}1\\1\\0\end{pmatrix} + \lambda_1 \begin{pmatrix}2\\-1\\1\end{pmatrix} = \begin{pmatrix}2\\1\\-1\end{pmatrix} + \lambda_2 \begin{pmatrix}3\\-5\\2\end{pmatrix} \tag{2.0.1}$$

$$\begin{pmatrix}2 & 3\\-1 & -5\\1 & 2\end{pmatrix}\begin{pmatrix}\lambda_1\\\lambda_2\end{pmatrix} = \begin{pmatrix}1\\0\\-1\end{pmatrix} \tag{2.0.2}$$

$$\mathbf{A}\lambda = \mathbf{b} \tag{2.0.3}$$

Since the rank of augmented matrix will be 3. We can say that lines do not intersect.

$$\mathbf{A} = \mathbf{U}\mathbf{S}\mathbf{V}^T \tag{2.0.4}$$

Where the columns of $\mathbf{V}$ are the eigenvectors of $\mathbf{A}^T\mathbf{A}$,the columns of $\mathbf{U}$ are the eigenvectors of $\mathbf{A}\mathbf{A}^T$ and $\mathbf{S}$ is diagonal matrix of singular value of eigenvalues of $\mathbf{A}^T\mathbf{A}$.

$$\mathbf{A}^T\mathbf{A} = \begin{pmatrix}6 & 13\\13 & 38\end{pmatrix} \tag{2.0.5}$$

$$\mathbf{A}\mathbf{A}^T = \begin{pmatrix}13 & -17 & 8\\-17 & 26 & -11\\8 & -11 & 5\end{pmatrix} \tag{2.0.6}$$

Eigen vectors of $\mathbf{A}^T$A.

$$\begin{vmatrix}6-\lambda & 13\\13 & 38-\lambda\end{vmatrix} \lambda^2 - 44\lambda + 59 = 0 \tag{2.0.7}$$

$$\lambda_1 = -5\sqrt{17} + 22, \lambda_2 = 5\sqrt{17} + 22 \tag{2.0.8}$$

Eigen vectors of $\mathbf{A}\mathbf{A}^T$.

$$\begin{vmatrix}13-\lambda & -17 & 8\\17 & 26-\lambda & -11\\8 & -11 & 5-\lambda\end{vmatrix} - \lambda^3 + 44\lambda^2 - 59\lambda = 0 \tag{2.0.9}$$

$$\lambda_2 = -5\sqrt{17} + 22, \lambda_1 = 5\sqrt{17} + 22, \lambda_3 = 0, \tag{2.0.10}$$

Hence,The eigenvectors will be

$$\mathbf{u}_2 = \begin{pmatrix}\frac{\sqrt{17}+12}{5}\\\frac{3\sqrt{17}+1}{5}\\1\end{pmatrix} \mathbf{u}_1 = \begin{pmatrix}\frac{-\sqrt{17}+12}{5}\\\frac{-3\sqrt{17}+1}{5}\\1\end{pmatrix} \mathbf{u}_3 = \begin{pmatrix}\frac{-3}{7}\\\frac{1}{7}\\1\end{pmatrix} \tag{2.0.11}$$

Normalising the eigenvectors

$$l_1 = \sqrt{\left(\frac{12-\sqrt{17}}{5}\right)^2 + \left(\frac{1-3\sqrt{17}}{5}\right)^2 + 1^2} \tag{2.0.12}$$

$$\mathbf{u}_1 = \begin{pmatrix}\frac{-\sqrt{17}+12}{\sqrt{340-20\sqrt{17}}}\\\frac{-3\sqrt{17}+1}{\sqrt{340-20\sqrt{17}}}\\\frac{5}{\sqrt{340-20\sqrt{17}}}\end{pmatrix} \tag{2.0.13}$$

$$\tag{2.0.14}$$

$$l_2 = \sqrt{\left(\frac{\sqrt{17}+12}{5}\right)^2 + \left(\frac{3\sqrt{17}+1}{5}\right)^2 + 1^2} \tag{2.0.15}$$

$$\mathbf{u}_2 = \frac{5}{\sqrt{340+20\sqrt{7}}}\begin{pmatrix}\frac{\sqrt{17}+12}{5}\\\frac{3\sqrt{17}+1}{5}\\1\end{pmatrix} \tag{2.0.16}$$

$$\mathbf{u}_2 = \begin{pmatrix}\frac{\sqrt{17}+12}{\sqrt{340+20\sqrt{17}}}\\\frac{3\sqrt{17}+1}{\sqrt{340+20\sqrt{17}}}\\\frac{5}{\sqrt{340+20\sqrt{17}}}\end{pmatrix} \tag{2.0.17}$$

$$l_3 = \sqrt{\left(\frac{-3}{7}\right)^2 + \left(\frac{1}{7}\right)^2 + 1^2} \tag{2.0.18}$$

$$\mathbf{u}_3 = \frac{7}{\sqrt{59}}\begin{pmatrix}\frac{-3}{7}\\ \frac{1}{7}\end{pmatrix} \tag{2.0.19}$$

$$\mathbf{u}_3 = \begin{pmatrix}\frac{-3}{\sqrt{59}}\\ \frac{1}{\sqrt{59}}\\ \frac{7}{\sqrt{59}}\\ \frac{7}{\sqrt{59}}\end{pmatrix} \tag{2.0.20}$$

$$\mathbf{U} = \begin{pmatrix}\frac{-\sqrt{17}+12}{\sqrt{340-20\sqrt{17}}} & \frac{\sqrt{17}+12}{\sqrt{340+20\sqrt{17}}} & \frac{-3}{\sqrt{59}}\\ \frac{-3\sqrt{17}+1}{\sqrt{340-20\sqrt{17}}} & \frac{3\sqrt{17}+1}{\sqrt{340+20\sqrt{17}}} & \frac{1}{\sqrt{59}}\\ \frac{7}{\sqrt{59}} & & \\ \frac{5}{\sqrt{340-20\sqrt{17}}} & \frac{5}{\sqrt{340+20\sqrt{17}}} & \frac{7}{\sqrt{59}}\end{pmatrix} \tag{2.0.21}$$

Now,

$$\mathbf{S} = \begin{pmatrix}\sqrt{5\sqrt{17}+22} & 0\\ 0 & \sqrt{-5\sqrt{17}+22}\\ 0 & 0\end{pmatrix} \tag{2.0.22}$$

So,from equation (2.0.4)

$$\begin{pmatrix}2 & 3\\ -1 & -5\\ 1 & 2\end{pmatrix} = \begin{pmatrix}\frac{-\sqrt{17}+12}{\sqrt{340-20\sqrt{17}}} & \frac{\sqrt{17}+12}{\sqrt{340+20\sqrt{17}}} & \frac{-3}{\sqrt{59}}\\ \frac{-3\sqrt{17}+1}{\sqrt{340-20\sqrt{17}}} & \frac{3\sqrt{17}+1}{\sqrt{340+20\sqrt{17}}} & \frac{1}{\sqrt{59}}\\ \frac{5}{\sqrt{340-20\sqrt{17}}} & \frac{5}{\sqrt{340+20\sqrt{17}}} & \frac{7}{\sqrt{59}}\end{pmatrix} \begin{pmatrix}\sqrt{5\sqrt{17}+22} & 0\\ 0 & \sqrt{-5\sqrt{17}+22}\\ 0 & 0\end{pmatrix} \begin{pmatrix}\frac{5\sqrt{17}-16}{\sqrt{850-160\sqrt{17}}} & \frac{-5\sqrt{17}-16}{\sqrt{850+160\sqrt{17}}}\\ \frac{13}{\sqrt{850-160\sqrt{17}}} & \frac{13}{\sqrt{850+160\sqrt{17}}}\end{pmatrix}^T \tag{2.0.23}$$

Now, Finding Moore-Penrose Pseudo inverse of S

$$\mathbf{S}_+ = \begin{pmatrix}\frac{1}{\sqrt{5\sqrt{17}+22}} & 0 & 0\\ 0 & \frac{1}{\sqrt{-5\sqrt{17}+22}} & 0\end{pmatrix} \tag{2.0.24}$$

Now, V=$\mathbf{A}^T \frac{\mathbf{u_i}}{\sqrt{\lambda_i}}$

$$\mathbf{V} = \begin{pmatrix}\frac{\sqrt{17}+28}{\sqrt{340-20\sqrt{17}}\sqrt{5\sqrt{17}+22}} & \frac{-\sqrt{17}+26}{\sqrt{340+20\sqrt{17}}\sqrt{-5\sqrt{17}+22}}\\ \frac{-\sqrt{17}+26}{\sqrt{340+20\sqrt{17}}\sqrt{-5\sqrt{17}+22}} & \frac{-\sqrt{17}-28}{\sqrt{340-20\sqrt{17}}\sqrt{5\sqrt{17}+22}}\end{pmatrix} \tag{2.0.25}$$

$$\mathbf{x} = \mathbf{V}(\mathbf{S}_+(\mathbf{U}^T\mathbf{b})) \tag{2.0.26}$$

$$\mathbf{U}^T\mathbf{b} = \begin{pmatrix}\frac{-\sqrt{17}+7}{\sqrt{340-20\sqrt{17}}}\\ \frac{\sqrt{17}+7}{\sqrt{340+20\sqrt{17}}}\\ \frac{-10}{\sqrt{59}}\end{pmatrix} \tag{2.0.27}$$

$$\mathbf{S}_+(\mathbf{U}^T\mathbf{b}) = \begin{pmatrix}\frac{-\sqrt{17}+7}{\sqrt{340-20\sqrt{17}}\sqrt{5\sqrt{17}+22}}\\ \frac{\sqrt{17}+7}{\sqrt{340+20\sqrt{17}}\sqrt{-5\sqrt{17}+22}}\end{pmatrix} \tag{2.0.28}$$

$$\mathbf{x} = \begin{pmatrix}\frac{\sqrt{17}+28}{\sqrt{340-20\sqrt{17}}\sqrt{5\sqrt{17}+22}} & \frac{-\sqrt{17}+26}{\sqrt{340+20\sqrt{17}}\sqrt{-5\sqrt{17}+22}}\\ \frac{-\sqrt{17}+26}{\sqrt{340+20\sqrt{17}}\sqrt{-5\sqrt{17}+22}} & \frac{-\sqrt{17}-28}{\sqrt{340-20\sqrt{17}}\sqrt{5\sqrt{17}+22}}\end{pmatrix} \begin{pmatrix}\frac{-\sqrt{17}+7}{\sqrt{340-20\sqrt{17}}\sqrt{5\sqrt{17}+22}}\\ \frac{\sqrt{17}+7}{\sqrt{340+20\sqrt{17}}\sqrt{-5\sqrt{17}+22}}\end{pmatrix} \tag{2.0.29}$$

Now, Verifying the values using

$$\mathbf{A}^T\mathbf{A}\mathbf{x} = \mathbf{A}^T\mathbf{b} \tag{2.0.30}$$

Taking R.H.S

$$= \begin{pmatrix}1\\ 1\end{pmatrix} \tag{2.0.31}$$

Taking L.H.S

$$\begin{pmatrix}6 & 13\\13 & 38\end{pmatrix}$$
$$\begin{pmatrix}\frac{\sqrt{17}+28}{\sqrt{340-20\sqrt{17}}\sqrt{5\sqrt{17}+22}} & \frac{-\sqrt{17}+26}{\sqrt{340+20\sqrt{17}}\sqrt{-5\sqrt{17}+22}}\\ \frac{-\sqrt{17}+26}{\sqrt{340+20\sqrt{17}}\sqrt{-5\sqrt{17}+22}} & \frac{-\sqrt{17}-28}{\sqrt{340-20\sqrt{17}}\sqrt{5\sqrt{17}+22}}\end{pmatrix}$$
$$\begin{pmatrix}\frac{-\sqrt{17}+7}{\sqrt{340-20\sqrt{17}}\sqrt{5\sqrt{17}+22}}\\ \frac{\sqrt{17}+7}{\sqrt{340+20\sqrt{17}}\sqrt{-5\sqrt{17}+22}}\end{pmatrix} \tag{2.0.32}$$

$$=\begin{pmatrix}1\\1\end{pmatrix} \tag{2.0.33}$$