Assignment 17

Adarsh Srivastava

The link to the solution is

https://github.com/Adarsh1310/EE5609

***Abstract*—This documents solves a problem based on coordinates.**

1 Problem

Let **V** be the real vector space of all polynomial functions from $\mathbb{R}$ to $\mathbb{R}$ of degree 2 or less, i.e, the space of all functions f of the form,

$$f(x) = c_0 + c_1x + c_2x^2$$

Let t be a fixed real number and define

$$g_1(x) = 1, g_2(x) = x + t, g_3(x) = (x + t)^2$$

Prove that β={g1,g2,g3} is a basis for V. If

$$f(x) = c_0 + c_1x + c_2x^2$$

what are the coordinates of f in the ordered basis β

2 Solution

Let's start by proving that $\{g_1, g_2, g_3\}$ are linearly independent,

$$\mathbf{v}\begin{pmatrix} g_1 \\ g_2 \\ g_3 \end{pmatrix} = 0 \tag{2.0.1}$$

$$\mathbf{v}\begin{pmatrix} 1 & 0 & 0 \\ t & 1 & 0 \\ t^2 & 2t & 1 \end{pmatrix}\begin{pmatrix} 1 \\ x \\ x^2 \end{pmatrix} = 0 \tag{2.0.2}$$

$$\begin{pmatrix} 1 & x & x^2 \end{pmatrix}\begin{pmatrix} 1 & t & t^2 \\ 0 & 1 & 2t \\ 0 & 0 & 1 \end{pmatrix}\mathbf{v}^T = 0 \tag{2.0.3}$$

$$\mathbf{ABv^T} = 0 \tag{2.0.4}$$

Now,taking **B** and applying row reduce operations,

$$\begin{pmatrix} 1 & t & t^2 \\ 0 & 1 & 2t \\ 0 & 0 & 1 \end{pmatrix} \tag{2.0.5}$$

$$\xleftrightarrow{R_1=R_1-tR_2}\begin{pmatrix} 1 & 0 & t^2 \\ 0 & 1 & 2t \\ 0 & 0 & 1 \end{pmatrix} \tag{2.0.6}$$

$$\xleftrightarrow{R_1=R_1-t^2R_3}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 2t \\ 0 & 0 & 1 \end{pmatrix} \tag{2.0.7}$$

$$\xleftrightarrow{R_2=R_2-2tR_3}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{2.0.8}$$

Hence,

$$\mathbf{Bv^T} = 0 \tag{2.0.9}$$

will have only trivial solution. We know that **v** is in the nullspace of **B** and hence it will also be in the nullspace of **AB**.So,$\{g1, g2, g3\}$ are linearly independent. Now, to find the coordinates,

$$f(x) = ag_1 + bg_2 + cg_3 \tag{2.0.10}$$

$$\begin{pmatrix} c_0 & c_1 & c_2 \end{pmatrix}\begin{pmatrix} 1 \\ x \\ x^2 \end{pmatrix} = \begin{pmatrix} a & 0 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ x \\ x^2 \end{pmatrix} + \begin{pmatrix} bt & b & 0 \end{pmatrix}\begin{pmatrix} 1 \\ x \\ x^2 \end{pmatrix} + \begin{pmatrix} ct^2 & 2ct & c \end{pmatrix}\begin{pmatrix} 1 \\ x \\ x^2 \end{pmatrix} \tag{2.0.11}$$

$$\begin{pmatrix} c_0 & c_1 & c_2 \end{pmatrix}\begin{pmatrix} 1 \\ x \\ x^2 \end{pmatrix} = \begin{pmatrix} a + bt + ct^2 & b + 2ct & c \end{pmatrix}\begin{pmatrix} 1 \\ x \\ x^2 \end{pmatrix} \tag{2.0.12}$$

From above we can see that,

$$c_2 = c \tag{2.0.13}$$

$$c_1 = b + 2ct \tag{2.0.14}$$

$$c_0 = a + bt + ct^2 \tag{2.0.15}$$

Solving the above equation we get,

$$c = c_2 \tag{2.0.16}$$

$$b = c_1 - 2c_2 t \tag{2.0.17}$$

$$a = c_0 - c_1 t + c_2 t^2 \tag{2.0.18}$$

So, finally the coordinates of f in ordered basis of β,

$$\begin{pmatrix} c_0 - c_1 t + c_2 t^2 \\ c_1 - 2c_2 t \\ c_2 \end{pmatrix} \tag{2.0.19}$$