

# Assignment 12

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The link to the solution is

<https://github.com/Adarsh1310/EE5609>

**Abstract**—This documents solves a problem based on Row Echelon form.

## 1 PROBLEM

Suppose  $\mathbf{R}$  and  $\mathbf{R}'$  are  $2 \times 3$  row-reduced echelon matrices and that the system  $\mathbf{R}\mathbf{X}=0$  and  $\mathbf{R}'\mathbf{X}=0$  have exactly the same solutions. Prove that  $\mathbf{R}=\mathbf{R}'$ .

## 2 SOLUTION

Since  $\mathbf{R}$  and  $\mathbf{R}'$  are  $2 \times 3$  row-reduced echelon matrices they can be of following three types:-

- Suppose matrix  $\mathbf{R}$  has one non-zero row then  $\mathbf{R}\mathbf{X}=0$  will have two free variables. Since  $\mathbf{R}'\mathbf{X}=0$  will have the exact same solution as  $\mathbf{R}\mathbf{X}=0$ ,  $\mathbf{R}'\mathbf{X}=0$  will also have two free variables. Thus  $\mathbf{R}'$  have one non zero row. Now let's consider a matrix  $\mathbf{A}$  with the first row as the non-zero row  $\mathbf{R}$  and second row as the second row of  $\mathbf{R}'$ .

$$\mathbf{R} = \begin{pmatrix} 1 & a & b \\ 0 & 0 & 0 \end{pmatrix} \quad (2.0.1)$$

$$\mathbf{R}' = \begin{pmatrix} 1 & c & d \\ 0 & 0 & 0 \end{pmatrix} \quad (2.0.2)$$

$$\mathbf{A} = \mathbf{R} + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \mathbf{R}' \quad (2.0.3)$$

$$\mathbf{A} = \begin{pmatrix} 1 & a & b \\ 1 & c & d \end{pmatrix} \quad (2.0.4)$$

Let  $\mathbf{X}$  satisfy

$$\mathbf{R}\mathbf{X} = 0 \quad (2.0.5)$$

$$\mathbf{R}'\mathbf{X} = 0 \quad (2.0.6)$$

Now multiplying  $\mathbf{A}$  and  $\mathbf{X}$  and substituting (2.0.3).

$$= \mathbf{A}\mathbf{X} \quad (2.0.7)$$

$$= \left( \mathbf{R} + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \mathbf{R}' \right) \mathbf{X} \quad (2.0.8)$$

$$= \mathbf{R}\mathbf{X} + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \mathbf{R}'\mathbf{X} \quad (2.0.9)$$

From (2.0.5) and (2.0.6)

$$\mathbf{A}\mathbf{X} = 0 \quad (2.0.10)$$

Now, considering the augmented matrix,

$$\left( \begin{array}{ccc|c} 1 & a & b & 0 \\ 1 & c & d & 0 \end{array} \right) \quad (2.0.11)$$

$$\xleftrightarrow{R_2=R_2-R_1} \left( \begin{array}{ccc|c} 1 & a & b & 0 \\ 0 & c-a & d-b & 0 \end{array} \right) \quad (2.0.12)$$

Now, Assuming  $c-a \neq 0$  and reducing (2.0.12) to row echelon form.

$$\left( \begin{array}{ccc|c} 1 & a & b & 0 \\ 0 & c-a & d-b & 0 \end{array} \right) \quad (2.0.13)$$

$$\xleftrightarrow{R_2=\frac{R_2}{R_1}} \left( \begin{array}{ccc|c} 1 & a & b & 0 \\ 0 & 1 & \frac{d-b}{c-a} & 0 \end{array} \right) \quad (2.0.14)$$

We can see that if  $c-a \neq 0$  then second row wouldn't come out to be zero which isn't possible because rows will be linear combination of each other as  $\mathbf{R}$  and  $\mathbf{R}'$  have the same solution. So,  $c-a=0$ ,

$$c-a=0 \quad (2.0.15)$$

$$c=a \quad (2.0.16)$$

$$d-b=0 \quad (2.0.17)$$

$$d=b \quad (2.0.18)$$

Hence,  $\mathbf{R}=\mathbf{R}'$

2) Let  $\mathbf{R}$  and  $\mathbf{R}'$  have all rows as non zero.

$$\mathbf{R} = \begin{pmatrix} 1 & 0 & b \\ 0 & 1 & c \end{pmatrix} \quad (2.0.19)$$

$$\mathbf{R}' = \begin{pmatrix} 1 & 0 & e \\ 0 & 1 & f \end{pmatrix} \quad (2.0.20)$$

Now let's consider another matrix  $\mathbf{A}$  whose first two rows are from  $\mathbf{R}$  and last two rows are from  $\mathbf{R}'$

$$\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{R} + \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{R}' \quad (2.0.21)$$

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & b \\ 0 & 1 & c \\ 1 & 0 & e \\ 0 & 1 & f \end{pmatrix} \quad (2.0.22)$$

Let  $\mathbf{X}$  satisfy

$$\mathbf{R}\mathbf{X} = 0 \quad (2.0.23)$$

$$\mathbf{X}^T \mathbf{R}^T = 0 \quad (2.0.24)$$

Here,

$$\mathbf{R} = (\mathbf{I} \quad \mathbf{a}) \quad (2.0.25)$$

$$\mathbf{a} = \begin{pmatrix} b \\ c \end{pmatrix} \quad (2.0.26)$$

$$\mathbf{R}^T = \begin{pmatrix} \mathbf{I} \\ \mathbf{a}^T \end{pmatrix} \quad (2.0.27)$$

Let,

$$\mathbf{X} = \begin{pmatrix} \mathbf{y}^T \\ z \end{pmatrix} \quad (2.0.28)$$

where  $z$  is a scalar constant. Now, substituting (2.0.28) and (2.0.25) in (2.0.24)

$$(\mathbf{y}^T \quad z) \begin{pmatrix} \mathbf{I} \\ \mathbf{a}^T \end{pmatrix} = 0 \quad (2.0.29)$$

$$\mathbf{y}^T + z\mathbf{a}^T = 0 \quad (2.0.30)$$

Now for,

$$\mathbf{R}'\mathbf{X} = 0 \quad (2.0.31)$$

$$\mathbf{X}^T \mathbf{R}'^T = 0 \quad (2.0.32)$$

Here,

$$\mathbf{R}' = (\mathbf{I} \quad \mathbf{b}) \quad (2.0.33)$$

$$\mathbf{b} = \begin{pmatrix} e \\ f \end{pmatrix} \quad (2.0.34)$$

Let,

$$\mathbf{X} = \begin{pmatrix} \mathbf{y}^T \\ z \end{pmatrix} \quad (2.0.35)$$

where  $z$  is a scalar constant. Now, substituting (2.0.35) and (2.0.33) in (2.0.32)

$$(\mathbf{y}^T \quad z) \begin{pmatrix} \mathbf{I} \\ \mathbf{b}^T \end{pmatrix} = 0 \quad (2.0.36)$$

$$\mathbf{y}^T + z\mathbf{b}^T = 0 \quad (2.0.37)$$

Now multiplying  $\mathbf{A}$  and  $\mathbf{X}$  and substituting (2.0.21).

$$= \mathbf{A}\mathbf{X} \quad (2.0.38)$$

$$= \left( \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{R} + \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{R}' \right) \mathbf{X} \quad (2.0.39)$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{R}\mathbf{X} + \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{R}'\mathbf{X} \quad (2.0.40)$$

$$= \begin{pmatrix} \mathbf{I} & \mathbf{a} \\ \mathbf{I} & \mathbf{b} \end{pmatrix} \mathbf{X} \quad (2.0.41)$$

Now using,  $\mathbf{X}^T \mathbf{A}^T$

$$= (\mathbf{y}^T \quad z) \begin{pmatrix} \mathbf{I} & \mathbf{I} \\ \mathbf{a}^T & \mathbf{b}^T \end{pmatrix} \quad (2.0.42)$$

$$= (\mathbf{y}^T + z\mathbf{a}^T \quad \mathbf{y} + z\mathbf{b}^T) \quad (2.0.43)$$

Since (2.0.43) is coming out to be a zero vector. We can say that  $\mathbf{R}$  and  $\mathbf{R}'$  are linear combination of each other. Also from (2.0.30) and (2.0.37)

$$\mathbf{y}^T + z\mathbf{a}^T = \mathbf{y}^T + z\mathbf{b}^T \quad (2.0.44)$$

$$\mathbf{a}^T = \mathbf{b}^T \quad (2.0.45)$$

$$b = e \quad (2.0.46)$$

$$c = f \quad (2.0.47)$$

Hence,  $\mathbf{R} = \mathbf{R}'$

3) Suppose matrix  $\mathbf{R}$  have all the rows as zero then  $\mathbf{R}\mathbf{X} = 0$  will be satisfied for all values of

**X.** We know that  $\mathbf{R}'\mathbf{X}=0$  will have the exact same solution as  $\mathbf{R}\mathbf{X}=0$  then we can say that for all values of  $\mathbf{X}=0$  equation  $\mathbf{R}'\mathbf{X}=0$  will be satisfied Hence,  $\mathbf{R}'=\mathbf{R}=0$ .