Assignment 17

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The link to the solution is

https://github.com/Adarsh1310/EE5609

***Abstract*—This documents solves a problem based on coordinates.**

1 Problem

Let $\mathbf{V}$ be the real vector space of all polynomial functions from $\mathbb{R}$ to $\mathbb{R}$ of degree 2 or less, i.e, the space of all functions f of the form,

$$f(x) = c_0 + c_1x + c_2x^2$$

Let t be a fixed real number and define

$$g_1(x) = 1, g_2(x) = x + t, g_3(x) = (x + t)^2$$

Prove that β={g1,g2,g3} is a basis for V. If

$$f(x) = c_0 + c_1x + c_2x^2$$

what are the coordinates of f in the ordered basis β

2 Solution

We start by taking,

$$\mathbf{f} = \begin{pmatrix} 1 \\ x \\ x^2 \end{pmatrix} \tag{2.0.1}$$

Let's start by proving that $\mathbf{g}$ is linearly independent.

$$\mathbf{g} = \mathbf{Bf} \tag{2.0.2}$$

where,

$$\mathbf{B} = \begin{pmatrix} 1 & 0 & 0 \\ t & 1 & 0 \\ t^2 & 2t & 1 \end{pmatrix} \tag{2.0.3}$$

Now,

$$\mathbf{v}^T\mathbf{g} = 0 \tag{2.0.4}$$

$$\implies \mathbf{v}^T\mathbf{Bf} = 0 \tag{2.0.5}$$

Since $\mathbf{f}$ is linearly independent,

$$\mathbf{v}^T\mathbf{B} = 0 \tag{2.0.6}$$

$$\mathbf{B}^T\mathbf{v} = 0 \tag{2.0.7}$$

Let's prove that $\mathbf{B}^T$ is invertible,

$$\begin{pmatrix} 1 & t & t^2 \\ 0 & 1 & 2t \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 2t \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & t^2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & t & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{2.0.8}$$

$$= \mathbf{E_1E_2E_3} \tag{2.0.9}$$

Since $\mathbf{B}^T$ can be written as a combination of elementary matrices, it is invertible matrix and hence $\mathbf{v}$ will be zero vector. Now, to find the coordinates,

$$f(x) = \mathbf{w}^T\mathbf{g} \tag{2.0.10}$$

So,

$$\mathbf{c}^T\mathbf{f} = \mathbf{w}^T\mathbf{g} \tag{2.0.11}$$

$$\mathbf{c}^T\mathbf{f} = \mathbf{w}^T\mathbf{Bf} \tag{2.0.12}$$

$$(\mathbf{c}^T - \mathbf{w}^T\mathbf{B})\mathbf{f} = 0 \tag{2.0.13}$$

Since, $\mathbf{f}$ is linearly independent,

$$\mathbf{c}^T - \mathbf{w}^T\mathbf{B} = 0 \tag{2.0.14}$$

$$\mathbf{c}^T = \mathbf{w}^T\mathbf{B} \tag{2.0.15}$$

$$\mathbf{c}^T\mathbf{B}^{-1} = \mathbf{w}^T \tag{2.0.16}$$

$$(\mathbf{B}^{-1})^T\mathbf{c} = \mathbf{w} \tag{2.0.17}$$

$$(\mathbf{E_3}^{-1}\mathbf{E_2}^{-1}\mathbf{E_1}^{-1})^T\mathbf{c} = \mathbf{w} \tag{2.0.18}$$

$$\begin{pmatrix} 1 & -t & t^2 \\ 0 & 1 & -2t \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} c_0 \\ c_1 \\ c_2 \end{pmatrix} = \mathbf{w} \tag{2.0.19}$$

So, finally the coordinates will be,

$$\begin{pmatrix} c_0 - c_1t + c_2t^2 \\ c_1 - 2c_2t \\ c_2 \end{pmatrix} \tag{2.0.20}$$