

Assignment 12

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The link to the solution is

<https://github.com/Adarsh1310/EE5609>

Abstract—This documents solves a problem based on Row Echelon form.

1 PROBLEM

Suppose \mathbf{R} and \mathbf{R}' are 2×3 row-reduced echelon matrices and that the system $\mathbf{R}\mathbf{X}=0$ and $\mathbf{R}'\mathbf{X}=0$ have exactly the same solutions. Prove that $\mathbf{R}=\mathbf{R}'$.

2 SOLUTION

Since \mathbf{R} and \mathbf{R}' are 2×3 row-reduced echelon matrices they can be of following three types:-

- Suppose matrix \mathbf{R} has one non-zero row then $\mathbf{R}\mathbf{X}=0$ will have two free variables. Since $\mathbf{R}'\mathbf{X}=0$ will have the exact same solution as $\mathbf{R}\mathbf{X}=0$, $\mathbf{R}'\mathbf{X}=0$ will also have two free variables. Thus \mathbf{R}' have one non zero row. Now let's consider a matrix \mathbf{A} with the first row as the non-zero row \mathbf{R} and second row as the second row of \mathbf{R}' .

$$\mathbf{R} = \begin{pmatrix} 1 & a & b \\ 0 & 0 & 0 \end{pmatrix} \quad (2.0.1)$$

$$\mathbf{R}' = \begin{pmatrix} 1 & c & d \\ 0 & 0 & 0 \end{pmatrix} \quad (2.0.2)$$

$$\mathbf{A} = \mathbf{R} + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \mathbf{R}' \quad (2.0.3)$$

$$\mathbf{A} = \begin{pmatrix} 1 & a & b \\ 1 & c & d \end{pmatrix} \quad (2.0.4)$$

Let \mathbf{X} satisfy

$$\mathbf{R}\mathbf{X} = 0 \quad (2.0.5)$$

$$\mathbf{R}'\mathbf{X} = 0 \quad (2.0.6)$$

Now multiplying \mathbf{A} and \mathbf{X} and substituting (2.0.3).

$$= \mathbf{A}\mathbf{X} \quad (2.0.7)$$

$$= \left(\mathbf{R} + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \mathbf{R}' \right) \mathbf{X} \quad (2.0.8)$$

$$= \mathbf{R}\mathbf{X} + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \mathbf{R}'\mathbf{X} \quad (2.0.9)$$

From (2.0.5) and (2.0.6)

$$\mathbf{A}\mathbf{X} = 0 \quad (2.0.10)$$

Now, considering the augmented matrix,

$$\left(\begin{array}{ccc|c} 1 & a & b & 0 \\ 1 & c & d & 0 \end{array} \right) \quad (2.0.11)$$

$$\xleftrightarrow{R_2=R_2-R_1} \left(\begin{array}{ccc|c} 1 & a & b & 0 \\ 0 & c-a & d-b & 0 \end{array} \right) \quad (2.0.12)$$

Now, Assuming $c-a \neq 0$ and reducing (2.0.12) to row echelon form.

$$\left(\begin{array}{ccc|c} 1 & a & b & 0 \\ 0 & c-a & d-b & 0 \end{array} \right) \quad (2.0.13)$$

$$\xleftrightarrow{R_2=\frac{R_2}{R_1}} \left(\begin{array}{ccc|c} 1 & a & b & 0 \\ 0 & 1 & \frac{d-b}{c-a} & 0 \end{array} \right) \quad (2.0.14)$$

We can see that if $c-a \neq 0$ then second row wouldn't come out to be zero which isn't possible because rows will be linear combination of each other as \mathbf{R} and \mathbf{R}' have the same solution. So, $c-a=0$,

$$c-a=0 \quad (2.0.15)$$

$$c=a \quad (2.0.16)$$

$$d-b=0 \quad (2.0.17)$$

$$d=b \quad (2.0.18)$$

Hence, $\mathbf{R}=\mathbf{R}'$

2) Let \mathbf{R} and \mathbf{R}' have all rows as non zero.

$$\mathbf{R} = \begin{pmatrix} 1 & 0 & b \\ 0 & 1 & c \end{pmatrix} \quad (2.0.19)$$

$$\mathbf{R}' = \begin{pmatrix} 1 & 0 & e \\ 0 & 1 & f \end{pmatrix} \quad (2.0.20)$$

Now let's consider another matrix \mathbf{A} whose first two rows are from \mathbf{R} and last two rows are from \mathbf{R}'

$$\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{R} + \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{R}' \quad (2.0.21)$$

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & b \\ 0 & 1 & c \\ 1 & 0 & e \\ 0 & 1 & f \end{pmatrix} \quad (2.0.22)$$

Let \mathbf{X} satisfy

$$\mathbf{R}\mathbf{X} = 0 \quad (2.0.23)$$

$$\mathbf{X}^T \mathbf{R} = 0 \quad (2.0.24)$$

Here,

$$\mathbf{R} = (\mathbf{I} \quad \mathbf{a}) \quad (2.0.25)$$

$$\mathbf{a} = \begin{pmatrix} b \\ c \end{pmatrix} \quad (2.0.26)$$

Let,

$$\mathbf{X} = (\mathbf{y}^T \quad z) \quad (2.0.27)$$

where z is a scalar constant. Now, substituting (2.0.27) and (2.0.25) in (2.0.24)

$$\begin{pmatrix} \mathbf{y}^T \\ z \end{pmatrix} (\mathbf{I} \quad \mathbf{a}) = 0 \quad (2.0.28)$$

$$\begin{pmatrix} \mathbf{y}^T \mathbf{I} & \mathbf{y}^T \mathbf{a} \\ z \mathbf{I} & z \mathbf{a} \end{pmatrix} = 0 \quad (2.0.29)$$

Now for,

$$\mathbf{R}' \mathbf{X} = 0 \quad (2.0.30)$$

$$\mathbf{X}^T \mathbf{R}' = 0 \quad (2.0.31)$$

Here,

$$\mathbf{R}' = (\mathbf{I} \quad \mathbf{a}') \quad (2.0.32)$$

$$\mathbf{a}' = \begin{pmatrix} e \\ f \end{pmatrix} \quad (2.0.33)$$

Let,

$$\mathbf{X} = (\mathbf{y}^T \quad z) \quad (2.0.34)$$

where z is a scalar constant. Now, substituting (2.0.34) and (2.0.32) in (2.0.31)

$$\begin{pmatrix} \mathbf{y}^T \\ z \end{pmatrix} (\mathbf{I} \quad \mathbf{a}') = 0 \quad (2.0.35)$$

$$\begin{pmatrix} \mathbf{y}^T \mathbf{I} & \mathbf{y}^T \mathbf{a}' \\ z \mathbf{I} & z \mathbf{a}' \end{pmatrix} = 0 \quad (2.0.36)$$

Now multiplying \mathbf{A} and \mathbf{X} and substituting (2.0.21).

$$= \mathbf{A}\mathbf{X} \quad (2.0.37)$$

$$= \left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{R} + \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{R}' \right) \mathbf{X} \quad (2.0.38)$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{R}\mathbf{X} + \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{R}' \mathbf{X} \quad (2.0.39)$$

From (2.0.23) and (2.0.25)

$$\mathbf{A}\mathbf{X} = 0 \quad (2.0.40)$$

Now, considering the augmented matrix,

$$\left(\begin{array}{ccc|c} 1 & 0 & b & 0 \\ 0 & 1 & c & 0 \\ 1 & 0 & e & 0 \\ 0 & 1 & f & 0 \end{array} \right) \quad (2.0.41)$$

$$\xleftrightarrow{R_3=R_3-R_1} \left(\begin{array}{ccc|c} 1 & 0 & b & 0 \\ 0 & 1 & c & 0 \\ 1 & 0 & e-b & 0 \\ 0 & 1 & f-c & 0 \end{array} \right) \quad (2.0.42)$$

$$\xleftrightarrow{R_4=R_4-R_2} \left(\begin{array}{ccc|c} 1 & 0 & b & 0 \\ 0 & 1 & c & 0 \\ 0 & 0 & e-b & 0 \\ 0 & 0 & f-c & 0 \end{array} \right) \quad (2.0.43)$$

$$\xleftrightarrow{R_3=\frac{R_3}{e-b}} \left(\begin{array}{ccc|c} 1 & a & b & 0 \\ 0 & 1 & c & 0 \\ 0 & 0 & e-b & 0 \\ 0 & 0 & f-c & 0 \end{array} \right) \quad (2.0.44)$$

$$\xleftrightarrow{R_3 = \frac{R_3}{f-c}} \left(\begin{array}{ccc|c} 1 & 0 & b & 0 \\ 0 & 1 & c & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{array} \right) \quad (2.0.45)$$

$$\xleftrightarrow{R_4 = \frac{R_4}{f-c}} \left(\begin{array}{ccc|c} 1 & 0 & b & 0 \\ 0 & 1 & c & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{array} \right) \quad (2.0.46)$$

$$\xleftrightarrow{R_4 = R_4 - R_3} \left(\begin{array}{ccc|c} 1 & 0 & b & 0 \\ 0 & 1 & c & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right) \quad (2.0.47)$$

$$\xleftrightarrow{R_1 = R_1 - bR_3} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & c & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right) \quad (2.0.48)$$

$$\xleftrightarrow{R_2 = R_2 - cR_3} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right) \quad (2.0.49)$$

We can see that if $e-b \neq 0$ then third row wouldn't come out to be zero which isn't possible because rows will be linear combination of each other as \mathbf{R} and \mathbf{R}' have the same solution. So,

$$d = 0 \quad (2.0.50)$$

$$a = 0 \quad (2.0.51)$$

$$e - b = 0 \quad (2.0.52)$$

$$e = b \quad (2.0.53)$$

$$f - c = 0 \quad (2.0.54)$$

$$f = c \quad (2.0.55)$$

Hence, $\mathbf{R} = \mathbf{R}'$

- 3) Suppose matrix \mathbf{R} have all the rows as zero then $\mathbf{RX} = 0$ will be satisfied for all values of \mathbf{X} . We know that $\mathbf{R}'\mathbf{X} = 0$ will have the exact same solution as $\mathbf{RX} = 0$ then we can say that for all values of $\mathbf{X} = 0$ equation $\mathbf{R}'\mathbf{X} = 0$ will be satisfied Hence, $\mathbf{R}' = \mathbf{R} = 0$.