Assignment 12

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The link to the solution is

https://github.com/Adarsh1310/EE5609

***Abstract*—This documents solves a problem based on Row Echelon form.**

1 Problem

Suppose **R** and $\mathbf{R}'$ are 2×3 row-reduced echelon matrices and that the system **RX**=0 and $\mathbf{R}'\mathbf{X}$=0 have exactly the same solutions. Prove that **R**=$\mathbf{R}'$.

2 Solution

Since **R** and $\mathbf{R}'$ are 2×3 row-reduced echelon matrices they can be of following three types:-

1) Suppose matrix **R** has one non-zero row then **RX**=0 will have two free variables.Since $R'\mathbf{X}$=0 will have the exact same solution as **RX**=0, $R'\mathbf{X}$=0 will also have two free variables. Thus $\mathbf{R}'$ have one non zero row.Now let's consider a matrix **A** with the first row as the non-zero row **R** and second row as the second row of $\mathbf{R}'$.

$$\mathbf{R} = \begin{pmatrix} 1 & a & b \\ 0 & 0 & 0 \end{pmatrix} \tag{2.0.1}$$

$$\mathbf{R}' = \begin{pmatrix} 1 & c & d \\ 0 & 0 & 0 \end{pmatrix} \tag{2.0.2}$$

$$\mathbf{A} = \mathbf{R} + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \mathbf{R}' \tag{2.0.3}$$

$$\mathbf{A} = \begin{pmatrix} 1 & a & b \\ 1 & c & d \end{pmatrix} \tag{2.0.4}$$

Let **X** satisfy

$$\mathbf{RX} = 0 \tag{2.0.5}$$

$$\mathbf{R}'\mathbf{X} = 0 \tag{2.0.6}$$

Now multiplying **A** and **X** and substituting (2.0.3).

$$= \mathbf{AX} \tag{2.0.7}$$

$$= \left(\mathbf{R} + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \mathbf{R}'\right) \mathbf{X} \tag{2.0.8}$$

$$= \mathbf{RX} + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \mathbf{R}'\mathbf{X} \tag{2.0.9}$$

From (2.0.5) and (2.0.6)

$$\mathbf{AX} = 0 \tag{2.0.10}$$

Now,reducing **A**,

$$\begin{pmatrix} 1 & a & b \\ 1 & c & d \end{pmatrix} \tag{2.0.11}$$

$$\begin{pmatrix} 1 & a & b \\ 0 & c-a & d-b \end{pmatrix} \tag{2.0.12}$$

$$\begin{pmatrix} 1 & a & b \\ 0 & 1 & \frac{d-b}{c-a} \end{pmatrix} \tag{2.0.13}$$

Thus, **A** in it's reduced form must have one non-zero row which is possible only when the rows of **A** are equal because leading entries in both the vectors equals one.Hence,$\mathbf{R} = \mathbf{R}'$.

2) Let **R** and $\mathbf{R}'$ have all rows as non zero.

$$\mathbf{R} = \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \end{pmatrix} \tag{2.0.14}$$

$$\mathbf{R}' = \begin{pmatrix} 1 & d & e \\ 0 & 1 & f \end{pmatrix} \tag{2.0.15}$$

Now let's consider another matrix **A** whose first two rows are from **R** and last two rows are from $\mathbf{R}'$

$$\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{R} + \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{R}' \tag{2.0.16}$$

$$\mathbf{A} = \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 1 & d & e \\ 0 & 1 & f \end{pmatrix} \tag{2.0.17}$$

Let $\mathbf{X}$ satisfy

$$\mathbf{R}\mathbf{X} = 0 \tag{2.0.18}$$

$$\mathbf{R}'\mathbf{X} = 0 \tag{2.0.19}$$

Now multiplying $\mathbf{A}$ and $\mathbf{X}$ and substituting (2.0.16).

$$\mathbf{A}\mathbf{X} \tag{2.0.20}$$

$$\left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}\mathbf{R} + \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}\mathbf{R}'\right)\mathbf{X} \tag{2.0.21}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}\mathbf{R}\mathbf{X} + \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}\mathbf{R}'\mathbf{X} \tag{2.0.22}$$

From (2.0.18) and (2.0.19)

$$\mathbf{A}\mathbf{X} = 0 \tag{2.0.23}$$

Now,bringing,reducing $\mathbf{A}$

$$\begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 1 & d & e \\ 0 & 1 & f \end{pmatrix} \tag{2.0.24}$$

$$\begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & d-a & e-b \\ 0 & 0 & f-c \end{pmatrix} \tag{2.0.25}$$

Therefore row reduced echelon form of $\mathbf{A}$ must have two non-zero rows, which implies the rows of $\mathbf{R}$ and $\mathbf{R}'$ must be a linear combination of each other. It is possible only when the leading coefficients of the first row of $\mathbf{R}$ and $\mathbf{R}'$ occur in the same column.By similar argument, the leading coefficients of the second rows must also occur in the same column. Thus, the only way the rows of $\mathbf{R}$ and $\mathbf{R}'$ are linear combination of one another is that the respective rows coincide and hence $\mathbf{R} = \mathbf{R}'$.

3) Suppose matrix $\mathbf{R}$ have all the rows as zero then $\mathbf{R}\mathbf{X}$=0 will be satisfied for all values of $\mathbf{X}$. We know that $\mathbf{R}'\mathbf{X}$=0 will have the exact same solution as $\mathbf{R}\mathbf{X}$=0 then we can say that for all values of $\mathbf{X}$=0 equation $\mathbf{R}'\mathbf{X}$=0 will be satisfied hence, $\mathbf{R}'$=$\mathbf{R}$=0.