Assignment 12

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The link to the solution is

https://github.com/Adarsh1310/EE5609

***Abstract*—This documents solves a problem based on Row Echelon form.**

1 Problem

Suppose $\mathbf{R}$ and $\mathbf{R}'$ are 2×3 row-reduced echelon matrices and that the system $\mathbf{RX}$=0 and $\mathbf{R}'\mathbf{X}$=0 have exactly the same solutions. Prove that $\mathbf{R}$=$\mathbf{R}'$.

2 Solution

Since $\mathbf{R}$ and $\mathbf{R}'$ are 2×3 row-reduced echelon matrices they can be of following three types:-

1) Suppose matrix $\mathbf{R}$ has one non-zero row then $\mathbf{RX}$=0 will have two free variables.Since $\mathrm{R}'\mathbf{X}$=0 will have the exact same solution as $\mathbf{RX}$=0, $\mathrm{R}'\mathbf{X}$=0 will also have two free variables. Thus $\mathbf{R}'$ have one non zero row.Now let's consider a matrix $\mathbf{A}$ with the first row as the non-zero row $\mathbf{R}$ and second row as the second row of $\mathbf{R}'$.

$$\mathbf{R} = \begin{pmatrix} 1 & a & b \\ 0 & 0 & 0 \end{pmatrix} \tag{2.0.1}$$

$$\mathbf{R}' = \begin{pmatrix} 1 & c & d \\ 0 & 0 & 0 \end{pmatrix} \tag{2.0.2}$$

$$\mathbf{A} = \mathbf{R} + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\mathbf{R}' \tag{2.0.3}$$

$$\mathbf{A} = \begin{pmatrix} 1 & a & b \\ 1 & c & d \end{pmatrix} \tag{2.0.4}$$

Let $\mathbf{X}$ satisfy

$$\mathbf{RX} = 0 \tag{2.0.5}$$

$$\mathbf{R}'\mathbf{X} = 0 \tag{2.0.6}$$

Now multiplying $\mathbf{A}$ and $\mathbf{X}$ and substituting (2.0.3).

$$= \mathbf{AX} \tag{2.0.7}$$

$$= \left(\mathbf{R} + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\mathbf{R}'\right)\mathbf{X} \tag{2.0.8}$$

$$= \mathbf{RX} + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\mathbf{R}'\mathbf{X} \tag{2.0.9}$$

From (2.0.5) and (2.0.6)

$$\mathbf{AX} = 0 \tag{2.0.10}$$

Now, considering the augmented matrix,

$$\left(\begin{array}{ccc|c} 1 & a & b & 0 \\ 1 & c & d & 0 \end{array}\right) \tag{2.0.11}$$

$$\xleftrightarrow{R_2=R_2-R_1} \left(\begin{array}{ccc|c} 1 & a & b & 0 \\ 0 & c-a & d-b & 0 \end{array}\right) \tag{2.0.12}$$

We know that (2.0.12) will have the same solution as (2.0.5) and (2.0.6).It will also be consistent hence one of the rows should be zero.

$$c - a = 0 \tag{2.0.13}$$

$$c = a \tag{2.0.14}$$

$$d - b = 0 \tag{2.0.15}$$

$$d = b \tag{2.0.16}$$

Hence, $\mathbf{R}$=$\mathbf{R'}$

2) Let $\mathbf{R}$ and $\mathbf{R}'$ have all rows as non zero.

$$\mathbf{R} = \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \end{pmatrix} \tag{2.0.17}$$

$$\mathbf{R}' = \begin{pmatrix} 1 & d & e \\ 0 & 1 & f \end{pmatrix} \tag{2.0.18}$$

Now let's consider another matrix $\mathbf{A}$ whose first two rows are from $\mathbf{R}$ and last two rows are from $\mathbf{R}^{'}$

$$\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{R} + \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{R}^{'} \tag{2.0.19}$$

$$\mathbf{A} = \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 1 & d & e \\ 0 & 1 & f \end{pmatrix} \tag{2.0.20}$$

Let $\mathbf{X}$ satisfy

$$\mathbf{RX} = 0 \tag{2.0.21}$$

$$\mathbf{R}^{'}\mathbf{X} = 0 \tag{2.0.22}$$

Now multiplying $\mathbf{A}$ and $\mathbf{X}$ and substituting (2.0.19).

$$\mathbf{AX} \tag{2.0.23}$$

$$\left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{R} + \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{R}^{'} \right) \mathbf{X} \tag{2.0.24}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{RX} + \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{R}^{'}\mathbf{X} \tag{2.0.25}$$

From (2.0.21) and (2.0.22)

$$\mathbf{AX} = 0 \tag{2.0.26}$$

Now, considering the augmented matrix,

$$\left(\begin{array}{ccc|c} 1 & a & b & 0 \\ 0 & 1 & c & 0 \\ 1 & d & e & 0 \\ 0 & 1 & f & 0 \end{array}\right) \tag{2.0.27}$$

$$\xleftrightarrow{R_3=R_3-R_1} \left(\begin{array}{ccc|c} 1 & a & b & 0 \\ 0 & 1 & c & 0 \\ 1 & d-a & e-b & 0 \\ 0 & 1 & f-c & 0 \end{array}\right) \tag{2.0.28}$$

$$\xleftrightarrow{R_4=R_4-R_2} \left(\begin{array}{ccc|c} 1 & a & b & 0 \\ 0 & 1 & c & 0 \\ 0 & d-a & e-b & 0 \\ 0 & 0 & f-c & 0 \end{array}\right) \tag{2.0.29}$$

Thus, for (2.0.29) will have the same solution as (2.0.21) and (2.0.22).It will also be consistent hence two of the rows should be zero.

$$d - a = 0 \tag{2.0.30}$$

$$d = a \tag{2.0.31}$$

$$e - b = 0 \tag{2.0.32}$$

$$e = b \tag{2.0.33}$$

$$f - c = 0 \tag{2.0.34}$$

$$f = c \tag{2.0.35}$$

Hence, $\mathbf{R}$=$\mathbf{R}'$

3) Suppose matrix $\mathbf{R}$ have all the rows as zero then $\mathbf{RX}$=0 will be satisfied for all values of $\mathbf{X}$. We know that $\mathbf{R}^{'}\mathbf{X}$=0 will have the exact same solution as $\mathbf{RX}$=0 then we can say that for all values of $\mathbf{X}$=0 equation $\mathbf{R}^{'}\mathbf{X}$=0 will be satisfied Hence, $\mathbf{R}^{'}$=$\mathbf{R}$=0.