Assignment 9

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The link to the solution is

https://github.com/Adarsh1310/EE5609

***Abstract*—This documents solves a Singular Value decomposition problem.**

1 Problem

Find the shortest distance between the lines

$$\mathbf{x} = \begin{pmatrix}1\\1\\0\end{pmatrix} + \lambda_1 \begin{pmatrix}2\\-1\\1\end{pmatrix} \tag{1.0.1}$$

$$\mathbf{x} = \begin{pmatrix}2\\1\\-1\end{pmatrix} + \lambda_2 \begin{pmatrix}3\\-5\\2\end{pmatrix} \tag{1.0.2}$$

2 Solution

The lines will intersect if

$$\begin{pmatrix}1\\1\\0\end{pmatrix} + \lambda_1 \begin{pmatrix}2\\-1\\1\end{pmatrix} = \begin{pmatrix}2\\1\\-1\end{pmatrix} + \lambda_2 \begin{pmatrix}3\\-5\\2\end{pmatrix} \tag{2.0.1}$$

$$\begin{pmatrix}2 & 3\\-1 & -5\\1 & 2\end{pmatrix}\begin{pmatrix}\lambda_1\\\lambda_2\end{pmatrix} = \begin{pmatrix}1\\0\\-1\end{pmatrix} \tag{2.0.2}$$

$$\mathbf{A}\lambda = \mathbf{b} \tag{2.0.3}$$

Since the rank of augmented matrix will be 3. We can say that lines do not intersect.

$$\mathbf{A} = \mathbf{USV}^T \tag{2.0.4}$$

Where the columns of $\mathbf{V}$ are the eigenvectors of $\mathbf{A}^T\mathbf{A}$,the columns of $\mathbf{U}$ are the eigenvectors of $\mathbf{AA}^T$ and $\mathbf{S}$ is diagonal matrix of singular value of eigenvalues of $\mathbf{A}^T\mathbf{A}$.

$$\mathbf{A}^T\mathbf{A} = \begin{pmatrix}6 & 13\\13 & 38\end{pmatrix} \tag{2.0.5}$$

$$\mathbf{AA}^T = \begin{pmatrix}13 & -17 & 8\\1-17 & 26 & -11\\8 & -11 & 5\end{pmatrix} \tag{2.0.6}$$

Eigen vectors of $\mathbf{A}^T$A.

$$\begin{vmatrix}6-\lambda & 13\\13 & 38-\lambda\end{vmatrix} \lambda^2 - 44\lambda + 59 = 0 \tag{2.0.7}$$

$$\lambda_1 = 42.615, \lambda_2 = 1.3844 \tag{2.0.8}$$

Hence,The eigenvectors will be

$$\mathbf{v}_1 = \begin{pmatrix}0.35504\\1\end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix}-2.8165\\1\end{pmatrix} \tag{2.0.9}$$

Normalising the eigenvectors

$$l_1 = \sqrt{0.3550^2 + 1^2} = 1.0611 \tag{2.0.10}$$

$$\mathbf{v}_1 = \frac{1}{1.0611}\begin{pmatrix}0.35504\\1\end{pmatrix} \tag{2.0.11}$$

$$\mathbf{v}_1 = \begin{pmatrix}0.3345\\0.9423\end{pmatrix} \tag{2.0.12}$$

$$l_2 = \sqrt{-2.8165^2 + 1^2} = 2.9888 \tag{2.0.13}$$

$$\mathbf{v}_2 = \frac{1}{2.9888}\begin{pmatrix}-2.8165\\1\end{pmatrix} \tag{2.0.14}$$

$$\mathbf{v}_2 = \begin{pmatrix}-0.9423\\0.3345\end{pmatrix} \tag{2.0.15}$$

From here we can say that

$$\mathbf{V} = \begin{pmatrix}0.3345 & -0.9423\\0.9423 & 0.3345\end{pmatrix} \tag{2.0.16}$$

Eigen vectors of $\mathbf{AA}^T$.

$$\begin{vmatrix}13-\lambda & -17 & 8\\17 & 26-\lambda & -11\\8 & -11 & 5-\lambda\end{vmatrix} - \lambda^3 + 44\lambda^2 - 59\lambda = 0 \tag{2.0.17}$$

$$\lambda_1 = 0, \lambda_2 = 0.13844, \lambda_3 = 42.61552 \tag{2.0.18}$$

Hence,The eigenvectors will be

$$\mathbf{v}_1 = \begin{pmatrix}1.5753\\-2.273\\1\end{pmatrix} \mathbf{v}_2 = \begin{pmatrix}3.2273\\2.6738\\1\end{pmatrix} \mathbf{v}_3 = \begin{pmatrix}-0.4285\\0.1428\\1\end{pmatrix} \tag{2.0.19}$$

Normalising the eigenvectors

$$l_1 = \sqrt{1.5753^2 + -2.2738^2 + 1^2} = 2.9414 \quad (2.0.20)$$

$$\mathbf{v}_1 = \frac{1}{2.9414}\begin{pmatrix} 1.5753 \\ -2.273 \\ 1 \end{pmatrix} \quad (2.0.21)$$

$$\mathbf{v}_1 = \begin{pmatrix} 0.5355 \\ -0.7730 \\ 0.3399 \end{pmatrix} \quad (2.0.22)$$

$$l_2 = \sqrt{3.2246^2 + -2.6738^2 + 1^2} = 4.3067 \quad (2.0.23)$$

$$\mathbf{v}_2 = \frac{1}{4.3067}\begin{pmatrix} 3.2273 \\ 2.6738 \\ 1 \end{pmatrix} \quad (2.0.24)$$

$$\mathbf{v}_2 = \begin{pmatrix} 0.7487 \\ 0.6208 \\ 0.2321 \end{pmatrix} \quad (2.0.25)$$

$$l_3 = \sqrt{-0.4285^2 + 0.1428^2 + 1^2} = 1.0973 \quad (2.0.26)$$

$$\mathbf{v}_3 = \frac{1}{1.0973}\begin{pmatrix} -0.4285 \\ 0.1428 \\ 1 \end{pmatrix} \quad (2.0.27)$$

$$\mathbf{v}_3 = \begin{pmatrix} -0.3905 \\ 0.1301 \\ 0.9113 \end{pmatrix} \quad (2.0.28)$$

$$\mathbf{U} = \begin{pmatrix} 0.5355 & -0.7487 & -0.3905 \\ -0.7730 & -0.6208 & 0.1301 \\ 0.3399 & -0.2321 & 0.9113 \end{pmatrix} \quad (2.0.29)$$

Now,

$$\mathbf{S} = \begin{pmatrix} \sqrt{42.615} & 0 \\ 0 & \sqrt{1.3844} \\ 0 & 0 \end{pmatrix} \quad (2.0.30)$$

So,from equation (2.0.4)

$$\begin{pmatrix} 2 & 3 \\ -1 & -5 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 0.5355 & -0.7487 & -0.3905 \\ -0.7730 & -0.6208 & 0.1301 \\ 0.3399 & -0.2321 & 0.9113 \end{pmatrix} \begin{pmatrix} \sqrt{42.615} & 0 \\ 0 & \sqrt{1.3844} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0.3345 & -0.9423 \\ 0.9423 & 0.3345 \end{pmatrix}^T \quad (2.0.31)$$