Assignment 12

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The link to the solution is

https://github.com/Adarsh1310/EE5609

***Abstract*—This documents solves a problem based on Row Echelon form.**

1 Problem

Suppose $\mathbf{R}$ and $\mathbf{R}'$ are 2×3 row-reduced echelon matrices and that the system $\mathbf{RX}$=0 and $\mathbf{R}'\mathbf{X}$=0 have exactly the same solutions. Prove that $\mathbf{R}$=$\mathbf{R}'$.

2 Solution

Since $\mathbf{R}$ and $\mathbf{R}'$ are 2×3 row-reduced echelon matrices they can be of following three types:-

1) Suppose matrix $\mathbf{R}$ has one non-zero row then $\mathbf{RX}$=0 will have two free variables.Since R$'\mathbf{X}$=0 will have the exact same solution as $\mathbf{RX}$=0, R$'\mathbf{X}$=0 will also have two free variables. Thus $\mathbf{R}'$ have one non zero row.Now let's consider a matrix $\mathbf{A}$ with the first row as the non-zero row $\mathbf{R}$ and second row as the second row of $\mathbf{R}'$.

$$\mathbf{R} = \begin{pmatrix} 1 & a & b \\ 0 & 0 & 0 \end{pmatrix} \tag{2.0.1}$$

$$\mathbf{R}' = \begin{pmatrix} 1 & c & d \\ 0 & 0 & 0 \end{pmatrix} \tag{2.0.2}$$

$$\mathbf{A} = \mathbf{R} + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \mathbf{R}' \tag{2.0.3}$$

$$\mathbf{A} = \begin{pmatrix} 1 & a & b \\ 1 & c & d \end{pmatrix} \tag{2.0.4}$$

Let,

$$\mathbf{X} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \tag{2.0.5}$$

If $\mathbf{X}$ satisfies

$$\mathbf{R} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0 \tag{2.0.6}$$

$$x_1 + x_2 a + x_3 b = 0 \tag{2.0.7}$$

$$\mathbf{R}' \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0 \tag{2.0.8}$$

$$x_1 + x_2 c + x_3 d = 0 \tag{2.0.9}$$

then, Now multiplying $\mathbf{A}$ with $\mathbf{X}$.

$$\begin{pmatrix} 1 & a & b \\ 1 & c & d \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \tag{2.0.10}$$

$$\begin{pmatrix} x_1 + x_2 a + x_3 b \\ x_1 + x_2 c + x_3 d \end{pmatrix} \tag{2.0.11}$$

So from equation (2.0.7) and (2.0.9)

$$\mathbf{A} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0 \tag{2.0.12}$$

Thus, $\mathbf{A}$ in it's reduced form must have one non-zero row which is possible only when the rows of $\mathbf{A}$ are equal because leading entries in both the vectors equals one.Hence,$\mathbf{R} = \mathbf{R}'$.

2) Let $\mathbf{R}$ and $\mathbf{R}'$ have all rows as non zero. Let

$$\mathbf{R} = \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \end{pmatrix} \tag{2.0.13}$$

$$\mathbf{R}' = \begin{pmatrix} 1 & d & e \\ 0 & 1 & f \end{pmatrix} \tag{2.0.14}$$

Now let's consider another matrix $\mathbf{A}$ whose first two rows are from $\mathbf{R}$ and last two rows are from $\mathbf{R}'$.

$$\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{R} + \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{R}' \tag{2.0.15}$$

$$\mathbf{A} = \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 1 & d & e \\ 0 & 1 & f \end{pmatrix} \tag{2.0.16}$$

If $\mathbf{X}$ satisfies

$$\mathbf{R} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0 \tag{2.0.17}$$

$$x_1 + x_2 a + x_3 b = 0 \tag{2.0.18}$$

$$x_2 + x_3 c = 0 \tag{2.0.19}$$

$$\mathbf{R}' \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0 \tag{2.0.20}$$

$$x_1 + x_2 d + x_3 e = 0 \tag{2.0.21}$$

$$x_2 + x_3 f = 0 \tag{2.0.22}$$

$$\tag{2.0.23}$$

Now multiplying $\mathbf{A}$ with $\mathbf{X}$,

$$\begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 1 & d & e \\ 0 & 1 & f \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \tag{2.0.24}$$

$$\begin{pmatrix} x_1 + x_2 a + x_3 b \\ x_2 + x_3 c \\ x_1 + x_2 d + x_3 e \\ x_2 + x_3 f \end{pmatrix} \tag{2.0.25}$$

So from equations (2.0.18),(2.0.19),(2.0.21) and (2.0.22).

$$\mathbf{A} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0 \tag{2.0.26}$$

Therefore row reduced echelon form of $\mathbf{A}$ must have two non-zero rows, which implies the rows of $\mathbf{R}$ and $\mathbf{R}'$ must be a linear combination of each other. It is possible only when the leading coefficients of the first row of $\mathbf{R}$ and $\mathbf{R}'$ occur in the same column.By similar argument, the leading coefficients of the second rows must also occur in the same column. Thus, the only way the rows of $\mathbf{R}$ and $\mathbf{R}'$ are linear combination of one another is that the respective rows coincide and hence $\mathbf{R} = \mathbf{R}'$.

3) Suppose matrix $\mathbf{R}$ have all the rows as zero then $\mathbf{RX}$=0 will be satisfied for all values of $\mathbf{X}$. We know that $\mathbf{R}'\mathbf{X}$=0 will have the exact same solution as $\mathbf{RX}$=0 then we can say that for all values of $\mathbf{X}$=0 equation $\mathbf{R}'\mathbf{X}$=0 will be satisfied hence, $\mathbf{R}'$=$\mathbf{R}$=0.