Assignment 12

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The link to the solution is

Abstract—**This documents solves a problem based on Row Echelon form.**

1 Problem

Suppose $\mathbf{R}$ and $\mathbf{R}'$ are 2×3 row-reduced echelon matrices and that the system $\mathbf{RX}=0$ and $\mathbf{R}'\mathbf{X}=0$ have exactly the same solutions. Prove that $\mathbf{R}=\mathbf{R}'$.

2 Solution

Since $\mathbf{R}$ and $\mathbf{R}'$ are 2×3 row-reduced echelon matrices they can be of following three types:-

1) Suppose matrix $\mathbf{R}$ has one non-zero row then $\mathbf{RX}=0$ will have two free variables. Since $\mathbf{R}'\mathbf{X}=0$ will have the exact same solution as $\mathbf{RX}=0$, $\mathbf{R}'\mathbf{X}=0$ will also have two free variables. Thus $\mathbf{R}'$ have one non zero row. Now let's consider a matrix $\mathbf{A}$ with the first row as the non-zero row $\mathbf{R}$ and second row as the second row of $\mathbf{R}'$.

$$\mathbf{R} = \begin{pmatrix} 1 & a & b \\ 0 & 0 & 0 \end{pmatrix} \tag{2.0.1}$$

$$\mathbf{R}' = \begin{pmatrix} 1 & c & d \\ 0 & 0 & 0 \end{pmatrix} \tag{2.0.2}$$

$$\mathbf{A} = \mathbf{R} + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\mathbf{R}' \tag{2.0.3}$$

$$\mathbf{A} = \begin{pmatrix} 1 & a & b \\ 1 & c & d \end{pmatrix} \tag{2.0.4}$$

Let $\mathbf{X}$ satisfy

$$\mathbf{RX} = 0 \tag{2.0.5}$$

$$\mathbf{R}'\mathbf{X} = 0 \tag{2.0.6}$$

Now multiplying $\mathbf{A}$ and $\mathbf{X}$ and substituting (2.0.3).

$$= \mathbf{AX} \tag{2.0.7}$$

$$= \left(\mathbf{R} + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\mathbf{R}'\right)\mathbf{X} \tag{2.0.8}$$

$$= \mathbf{RX} + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\mathbf{R}'\mathbf{X} \tag{2.0.9}$$

From (2.0.5) and (2.0.6)

$$\mathbf{AX} = 0 \tag{2.0.10}$$

Now, considering the augmented matrix,

$$\left(\begin{array}{ccc|c} 1 & a & b & 0 \\ 1 & c & d & 0 \end{array}\right) \tag{2.0.11}$$

$$\xrightarrow{R_2 = R_2 - R_1} \left(\begin{array}{ccc|c} 1 & a & b & 0 \\ 0 & c-a & d-b & 0 \end{array}\right) \tag{2.0.12}$$

For (2.0.12) to be consistent,

$$c - a = 0 \tag{2.0.13}$$

$$c = a \tag{2.0.14}$$

$$d - b = 0 \tag{2.0.15}$$

$$d = b \tag{2.0.16}$$

Hence, $\mathbf{R}=\mathbf{R}'$

2) Let $\mathbf{R}$ and $\mathbf{R}'$ have all rows as non zero.

$$\mathbf{R} = \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \end{pmatrix} \tag{2.0.17}$$

$$\mathbf{R}' = \begin{pmatrix} 1 & d & e \\ 0 & 1 & f \end{pmatrix} \tag{2.0.18}$$

Now let's consider another matrix $\mathbf{A}$ whose first two rows are from $\mathbf{R}$ and last two rows

are from $\mathbf{R}'$

$$\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{R} + \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{R}' \qquad (2.0.19)$$

$$\mathbf{A} = \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 1 & d & e \\ 0 & 1 & f \end{pmatrix} \qquad (2.0.20)$$

Let $\mathbf{X}$ satisfy

$$\mathbf{R}\mathbf{X} = 0 \qquad (2.0.21)$$

$$\mathbf{R}'\mathbf{X} = 0 \qquad (2.0.22)$$

Now multiplying $\mathbf{A}$ and $\mathbf{X}$ and substituting (2.0.19).

$$\mathbf{A}\mathbf{X} \qquad (2.0.23)$$

$$\left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{R} + \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{R}' \right) \mathbf{X} \qquad (2.0.24)$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{R}\mathbf{X} + \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{R}'\mathbf{X} \qquad (2.0.25)$$

From (2.0.21) and (2.0.22)

$$\mathbf{A}\mathbf{X} = 0 \qquad (2.0.26)$$

Now, considering the augmented matrix,

$$\left(\begin{array}{ccc|c} 1 & a & b & 0 \\ 0 & 1 & c & 0 \\ 1 & d & e & 0 \\ 0 & 1 & f & 0 \end{array} \right) \qquad (2.0.27)$$

$$\xleftarrow{R_3 = R_3 - R_1} \left(\begin{array}{ccc|c} 1 & a & b & 0 \\ 0 & 1 & c & 0 \\ 1 & d-a & e-b & 0 \\ 0 & 1 & f-c & 0 \end{array} \right)$$
$$(2.0.28)$$

$$\xleftarrow{R_4 = R_4 - R_2} \left(\begin{array}{ccc|c} 1 & a & b & 0 \\ 0 & 1 & c & 0 \\ 0 & d-a & e-b & 0 \\ 0 & 0 & f-c & 0 \end{array} \right)$$
$$(2.0.29)$$

For (2.0.29) to be consistent,

$$d - a = 0 \qquad (2.0.30)$$
$$d = a \qquad (2.0.31)$$
$$e - b = 0 \qquad (2.0.32)$$
$$e = b \qquad (2.0.33)$$
$$f - c = 0 \qquad (2.0.34)$$
$$f = c \qquad (2.0.35)$$

Hence, $\mathbf{R} = \mathbf{R}'$

3) Suppose matrix $\mathbf{R}$ have all the rows as zero then $\mathbf{R}\mathbf{X} = 0$ will be satisfied for all values of $\mathbf{X}$. We know that $\mathbf{R}'\mathbf{X} = 0$ will have the exact same solution as $\mathbf{R}\mathbf{X} = 0$ then we can say that for all values of $\mathbf{X} = 0$ equation $\mathbf{R}'\mathbf{X} = 0$ will be satisfied Hence, $\mathbf{R}' = \mathbf{R} = 0$.