

Assignment 9

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The link to the solution is

<https://github.com/Adarsh1310/EE5609>

Abstract—This documents solves a Singular Value decomposition problem.

1 PROBLEM

Find the shortest distance between the lines

$$\mathbf{x} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \lambda_1 \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} \quad (1.0.1)$$

$$\mathbf{x} = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} + \lambda_2 \begin{pmatrix} 3 \\ -5 \\ 2 \end{pmatrix} \quad (1.0.2)$$

2 SOLUTION

The lines will intersect if

$$\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \lambda_1 \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} + \lambda_2 \begin{pmatrix} 3 \\ -5 \\ 2 \end{pmatrix} \quad (2.0.1)$$

$$\begin{pmatrix} 2 & 3 \\ -1 & -5 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \quad (2.0.2)$$

$$\mathbf{M}\mathbf{x} = \mathbf{b} \quad (2.0.3)$$

Since the rank of augmented matrix will be 3. We can say that lines do not intersect.

$$\mathbf{M} = \mathbf{U}\mathbf{S}\mathbf{V}^T \quad (2.0.4)$$

Where the columns of \mathbf{V} are the eigenvectors of $\mathbf{A}^T\mathbf{A}$, the columns of \mathbf{U} are the eigenvectors of $\mathbf{A}\mathbf{A}^T$ and \mathbf{S} is diagonal matrix of singular value of eigenvalues of $\mathbf{A}^T\mathbf{A}$.

$$\mathbf{M}^T\mathbf{M} = \begin{pmatrix} 6 & 13 \\ 13 & 38 \end{pmatrix} \quad (2.0.5)$$

$$\mathbf{M}\mathbf{M}^T = \begin{pmatrix} 13 & -17 & 8 \\ -17 & 26 & -11 \\ 8 & -11 & 5 \end{pmatrix} \quad (2.0.6)$$

Calculating eigen value of $\mathbf{M}^T\mathbf{M}$.

$$\begin{vmatrix} 6-\lambda & 13 \\ 13 & 38-\lambda \end{vmatrix} \lambda^2 - 44\lambda + 59 = 0 \quad (2.0.7)$$

$$\lambda_1 = -5\sqrt{17} + 22, \lambda_2 = 5\sqrt{17} + 22 \quad (2.0.8)$$

Eigen vectors of $\mathbf{M}\mathbf{M}^T$.

$$\begin{vmatrix} 13-\lambda & -17 & 8 \\ 17 & 26-\lambda & -11 \\ 8 & -11 & 5-\lambda \end{vmatrix} - \lambda^3 + 44\lambda^2 - 59\lambda = 0 \quad (2.0.9)$$

$$\lambda_4 = -5\sqrt{17} + 22, \lambda_3 = 5\sqrt{17} + 22, \lambda_5 = 0, \quad (2.0.10)$$

Hence, The eigenvectors will be

$$\mathbf{u}_2 = \begin{pmatrix} \frac{\sqrt{17}+12}{5} \\ \frac{3\sqrt{17}+1}{5} \\ 1 \end{pmatrix} \mathbf{u}_1 = \begin{pmatrix} \frac{-\sqrt{17}+12}{5} \\ \frac{-3\sqrt{17}+1}{5} \\ 1 \end{pmatrix} \mathbf{u}_3 = \begin{pmatrix} \frac{-3}{7} \\ \frac{1}{7} \\ 1 \end{pmatrix} \quad (2.0.11)$$

Normalising the eigenvectors

$$l_1 = \sqrt{\left(\frac{12-\sqrt{17}}{5}\right)^2 + \left(\frac{1-3\sqrt{17}}{5}\right)^2 + 1^2} \quad (2.0.12)$$

$$\mathbf{u}_1 = \begin{pmatrix} \frac{-\sqrt{17}+12}{\sqrt{340-20\sqrt{17}}} \\ \frac{\sqrt{340-20\sqrt{17}}}{\sqrt{340-20\sqrt{17}}} \\ \frac{-3\sqrt{17}+1}{\sqrt{340-20\sqrt{17}}} \\ \frac{5}{\sqrt{340-20\sqrt{17}}} \end{pmatrix} \quad (2.0.13)$$

$$(2.0.14)$$

$$l_2 = \sqrt{\left(\frac{\sqrt{17}+12}{5}\right)^2 + \left(\frac{3\sqrt{17}+1}{5}\right)^2 + 1^2} \quad (2.0.15)$$

$$\mathbf{u}_2 = \frac{5}{\sqrt{340+20\sqrt{17}}} \begin{pmatrix} \frac{\sqrt{17}+12}{5} \\ \frac{3\sqrt{17}+1}{5} \\ 1 \end{pmatrix} \quad (2.0.16)$$

$$\mathbf{u}_2 = \begin{pmatrix} \frac{\sqrt{17}+12}{\sqrt{340+20\sqrt{17}}} \\ \frac{\sqrt{340+20\sqrt{17}}}{\sqrt{340+20\sqrt{17}}} \\ \frac{3\sqrt{17}+1}{\sqrt{340+20\sqrt{17}}} \\ \frac{5}{\sqrt{340+20\sqrt{17}}} \end{pmatrix} \quad (2.0.17)$$

$$l_3 = \sqrt{\left(\frac{-3}{7}\right)^2 + \left(\frac{1}{7}\right)^2 + 1^2} \quad (2.0.18)$$

$$\mathbf{u}_3 = \frac{7}{\sqrt{59}} \begin{pmatrix} \frac{-3}{7} \\ \frac{1}{7} \\ 1 \end{pmatrix} \quad (2.0.19)$$

$$\mathbf{u}_3 = \begin{pmatrix} \frac{-3}{\sqrt{59}} \\ \frac{1}{\sqrt{59}} \\ \frac{7}{\sqrt{59}} \end{pmatrix} \quad (2.0.20)$$

$$\mathbf{U} = \begin{pmatrix} \frac{-\sqrt{17}+12}{\sqrt{340-20\sqrt{17}}} & \frac{\sqrt{17}+12}{\sqrt{340+20\sqrt{17}}} & \frac{-3}{\sqrt{59}} \\ \frac{-3\sqrt{17}+1}{\sqrt{340-20\sqrt{17}}} & \frac{3\sqrt{17}+1}{\sqrt{340+20\sqrt{17}}} & \frac{1}{\sqrt{59}} \\ \frac{7}{\sqrt{59}} & \frac{5}{\sqrt{340+20\sqrt{17}}} & \frac{7}{\sqrt{59}} \end{pmatrix} \quad (2.0.21)$$

Now,

$$\mathbf{S} = \begin{pmatrix} \sqrt{5\sqrt{17}+22} & 0 \\ 0 & \sqrt{-5\sqrt{17}+22} \\ 0 & 0 \end{pmatrix} \quad (2.0.22)$$

Now, $\mathbf{V} = \mathbf{M}^T \frac{\mathbf{u}_i}{\sqrt{\lambda_i}}$

$$\mathbf{V} = \begin{pmatrix} \frac{\sqrt{17}+28}{\sqrt{340-20\sqrt{17}}\sqrt{5\sqrt{17}+22}} & \frac{-\sqrt{17}+26}{\sqrt{340+20\sqrt{17}}\sqrt{-5\sqrt{17}+22}} \\ \frac{-\sqrt{17}+26}{\sqrt{340+20\sqrt{17}}\sqrt{-5\sqrt{17}+22}} & \frac{-\sqrt{17}-28}{\sqrt{340-20\sqrt{17}}\sqrt{5\sqrt{17}+22}} \end{pmatrix} \quad (2.0.23)$$

So, from equation (2.0.4)

$$\begin{pmatrix} 2 & 3 \\ -1 & -5 \\ 1 & 2 \end{pmatrix} =$$

$$\begin{pmatrix} \frac{-\sqrt{17}+12}{\sqrt{340-20\sqrt{17}}} & \frac{\sqrt{17}+12}{\sqrt{340+20\sqrt{17}}} & \frac{-3}{\sqrt{59}} \\ \frac{-3\sqrt{17}+1}{\sqrt{340-20\sqrt{17}}} & \frac{3\sqrt{17}+1}{\sqrt{340+20\sqrt{17}}} & \frac{1}{\sqrt{59}} \\ \frac{7}{\sqrt{59}} & \frac{5}{\sqrt{340+20\sqrt{17}}} & \frac{7}{\sqrt{59}} \end{pmatrix} \begin{pmatrix} \sqrt{5\sqrt{17}+22} & 0 \\ 0 & \sqrt{-5\sqrt{17}+22} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{\sqrt{17}+28}{\sqrt{340-20\sqrt{17}}\sqrt{5\sqrt{17}+22}} & \frac{-\sqrt{17}+26}{\sqrt{340+20\sqrt{17}}\sqrt{-5\sqrt{17}+22}} \\ \frac{-\sqrt{17}+26}{\sqrt{340+20\sqrt{17}}\sqrt{-5\sqrt{17}+22}} & \frac{-\sqrt{17}-28}{\sqrt{340-20\sqrt{17}}\sqrt{5\sqrt{17}+22}} \end{pmatrix}^T \quad (2.0.24)$$

Now, Finding Moore-Penrose Pseudo inverse of \mathbf{S}

$$\mathbf{S}_+ = \begin{pmatrix} \frac{1}{\sqrt{5\sqrt{17}+22}} & 0 & 0 \\ 0 & \frac{1}{\sqrt{-5\sqrt{17}+22}} & 0 \end{pmatrix} \quad (2.0.25)$$

We know that, $\mathbf{x} = \mathbf{V}(\mathbf{S}_+(\mathbf{U}^T \mathbf{b}))$

$$\mathbf{U}^T \mathbf{b} = \begin{pmatrix} \frac{-\sqrt{17}+7}{\sqrt{340-20\sqrt{17}}} \\ \frac{\sqrt{17}+7}{\sqrt{340+20\sqrt{17}}} \\ \frac{-10}{\sqrt{59}} \end{pmatrix} \quad (2.0.26)$$

$$\mathbf{S}_+(\mathbf{U}^T \mathbf{b}) = \begin{pmatrix} \frac{-\sqrt{17}+7}{\sqrt{340-20\sqrt{17}}\sqrt{5\sqrt{17}+22}} \\ \frac{\sqrt{17}+7}{\sqrt{340+20\sqrt{17}}\sqrt{-5\sqrt{17}+22}} \end{pmatrix} \quad (2.0.27)$$

$$\mathbf{x} = \begin{pmatrix} \frac{\sqrt{17}+28}{\sqrt{340-20\sqrt{17}}\sqrt{5\sqrt{17}+22}} & \frac{-\sqrt{17}+26}{\sqrt{340+20\sqrt{17}}\sqrt{-5\sqrt{17}+22}} \\ \frac{-\sqrt{17}+26}{\sqrt{340+20\sqrt{17}}\sqrt{-5\sqrt{17}+22}} & \frac{-\sqrt{17}-28}{\sqrt{340-20\sqrt{17}}\sqrt{5\sqrt{17}+22}} \end{pmatrix} \begin{pmatrix} \frac{-\sqrt{17}+7}{\sqrt{340-20\sqrt{17}}\sqrt{5\sqrt{17}+22}} \\ \frac{\sqrt{17}+7}{\sqrt{340+20\sqrt{17}}\sqrt{-5\sqrt{17}+22}} \end{pmatrix} \quad (2.0.28)$$

$$\mathbf{x} = \begin{pmatrix} \frac{971\sqrt{17}+127323}{320960} \\ \frac{-68\sqrt{17}-14}{\sqrt{108800}\sqrt{59}} \end{pmatrix} \quad (2.0.29)$$

Simplifying the values of x_1 and x_2

$$x_2 = \frac{-68\sqrt{17}-14}{\sqrt{108800}\sqrt{59}} \quad (2.0.30)$$

$$= \frac{-34\sqrt{17}-7}{\sqrt{27200}\sqrt{59}} \quad (2.0.31)$$

$$= \frac{-34\sqrt{17}}{\sqrt{27200}\sqrt{59}} - \frac{7}{\sqrt{27200}\sqrt{59}} \quad (2.0.32)$$

$$= \frac{1}{40\sqrt{59}} \left(-34 - \frac{7}{\sqrt{17}} \right) \quad (2.0.33)$$

$$\approx \frac{1}{40\sqrt{59}} \left(-34 - \frac{70}{41} \right) \quad (2.0.34)$$

$$\approx \frac{1}{40\sqrt{59}} \left(-\frac{1464}{41} \right) \quad (2.0.35)$$

$$\approx -\frac{7}{59} \quad (2.0.36)$$

$$x_1 = \frac{971\sqrt{17} + 127323}{320960} \quad (2.0.37)$$

$$= \frac{16507}{320960\sqrt{17}} + \frac{127323}{320960} \quad (2.0.38)$$

$$= \frac{971}{18880\sqrt{17}} + \frac{127323}{320960} \quad (2.0.39)$$

$$\approx \frac{9710}{778442} + \frac{127323}{320960} \quad (2.0.40)$$

$$\approx \frac{25}{59} \quad (2.0.41)$$

Now, Verifying the values using

$$\mathbf{M}^T \mathbf{M} \mathbf{x} = \mathbf{M}^T \mathbf{b} \quad (2.0.42)$$

Solving R.H.S

$$\mathbf{M}^T \mathbf{M} \mathbf{x} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad (2.0.43)$$

Now using equation (2.0.5) in (2.0.41)

$$\begin{pmatrix} 6 & 13 \\ 13 & 38 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad (2.0.44)$$

Solving the augmented matrix.

$$\begin{pmatrix} 6 & 13 & 1 \\ 13 & 38 & 1 \end{pmatrix} \xleftarrow{R_2 - \frac{13}{6}R_1} \begin{pmatrix} 6 & 13 & 1 \\ 0 & \frac{59}{6} & -\frac{7}{6} \end{pmatrix} \quad (2.0.45)$$

$$\frac{59}{6}x_2 = -\frac{7}{6} \quad (2.0.46)$$

$$6x_1 + 13x_2 = 1 \quad (2.0.47)$$

$$x_1 = \frac{25}{59}, x_2 = -\frac{7}{59} \quad (2.0.48)$$

$$\mathbf{x} = \begin{pmatrix} \frac{25}{59} \\ -\frac{7}{59} \end{pmatrix} \quad (2.0.49)$$