Assignment 5

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The link to the solution is

https://github.com/Adarsh1310/EE5609

***Abstract*—This documents solves a problem based on circles.**

1 Problem

Find the area of the region bounded by the circle $\mathbf{x}^{\mathbf{T}}\ \mathbf{x}= 4$ and $\left\|\mathbf{x}-\begin{pmatrix}2\\0\end{pmatrix}\right\| = 2.$

2 Solution

$\|\mathbf{x}\|^2 + 2\mathbf{u}^T\mathbf{x} + f = 0$
$\mathbf{x}^T\mathbf{x} + 2\mathbf{u}^T\mathbf{x} + f = 0$
So from above equation we can say that,

2.1 Circle 1

Taking equation of the first circle to be,

$$\|\mathbf{x}\|^2 + 2\mathbf{u}_1^T\mathbf{x} + f_1 = 0 \tag{2.1.1}$$

$$\mathbf{x}^T\mathbf{x} - 4 = 0(\text{given}) \tag{2.1.2}$$

$$\mathbf{u_1} = \begin{pmatrix}0\\0\end{pmatrix} \tag{2.1.3}$$

$$f_1 = -4 \tag{2.1.4}$$

$$\mathbf{O_1} = \begin{pmatrix}0\\0\end{pmatrix} \tag{2.1.5}$$

2.2 Circle 2

Taking equation of the second circle to be,

$$\left\|\mathbf{x}-\begin{pmatrix}2\\0\end{pmatrix}\right\|^2 = 2^2(\text{given}) \tag{2.2.1}$$

$$\mathbf{x}^T\mathbf{x} + 2\mathbf{u_2}^T\mathbf{x} = 0 \tag{2.2.2}$$

$$\mathbf{u_2} = \begin{pmatrix}-2\\0\end{pmatrix} \tag{2.2.3}$$

$$f_2 = 0 \tag{2.2.4}$$

$$\mathbf{O_2} = \begin{pmatrix}2\\0\end{pmatrix} \tag{2.2.5}$$

Now,Subtracting equation (2.2.2) from (2.1.2) We get,

$$\mathbf{x}^T\mathbf{x} - 2\mathbf{u_2}^T\mathbf{x} + f_1 - \mathbf{x}^T\mathbf{x} = 0 \tag{2.2.6}$$

$$2\mathbf{u}^T\mathbf{x} = -4 \tag{2.2.7}$$

$$\begin{pmatrix}-4 & 0\end{pmatrix}\mathbf{x} = -4 \tag{2.2.8}$$

Which can be written as:-

$$\begin{pmatrix}1 & 0\end{pmatrix}\mathbf{x} = 1 \tag{2.2.9}$$

$$\mathbf{x} = \begin{pmatrix}1\\0\end{pmatrix} + \lambda\begin{pmatrix}0\\1\end{pmatrix} \tag{2.2.10}$$

$$\mathbf{x} = \mathbf{q} + \lambda\mathbf{m} \tag{2.2.11}$$

$$\mathbf{q} = \begin{pmatrix}1\\0\end{pmatrix} \tag{2.2.12}$$

$$\mathbf{m} = \begin{pmatrix}0\\1\end{pmatrix} \tag{2.2.13}$$

Substituting (2.2.11) in (2.1.1)

$$\|\mathbf{x}\|^2 + 2\mathbf{u}_1^T\mathbf{x} + f_1 = 0 \tag{2.2.14}$$

$$\|\mathbf{q} + \lambda\mathbf{m}\|^2 + f_1 = 0 \tag{2.2.15}$$

$$(\mathbf{q} + \lambda\mathbf{m})^T(\mathbf{q} + \lambda\mathbf{m}) + f_1 = 0 \tag{2.2.16}$$

$$\mathbf{q}^T(\mathbf{q} + \lambda\mathbf{m}) + \lambda\mathbf{m}^T(\mathbf{q} + \lambda\mathbf{m}) + f_1 = 0 \tag{2.2.17}$$

$$\|\mathbf{q}\|^2 + \lambda\mathbf{q}^T\mathbf{m} + \lambda\mathbf{m}^T\mathbf{q} + \lambda^2\|\mathbf{m}\|^2 + f_1 = 0 \tag{2.2.18}$$

$$\|\mathbf{q}\|^2 + 2\lambda\mathbf{q}^T\mathbf{m} + \lambda^2\|\mathbf{m}\|^2 + f_1 = 0 \tag{2.2.19}$$

$$\lambda(\lambda\|\mathbf{m}\|^2 + 2\mathbf{q}^T\mathbf{m}) = -f_1 - \|\mathbf{q}\|^2 \tag{2.2.20}$$

$$\lambda^2\|\mathbf{m}\|^2 = -f_1 - \|\mathbf{q}\|^2 \tag{2.2.21}$$

$$\lambda^2 = \frac{-f_1 - \|\mathbf{q}\|^2}{\|\mathbf{m}\|^2} \tag{2.2.22}$$

$$\lambda^2 = 3 \tag{2.2.23}$$

$$\lambda = +\sqrt{3}, -\sqrt{3} \tag{2.2.24}$$

Substituting the value of λ in(2.2.11)

$$\mathbf{x} = \mathbf{q} + \lambda \mathbf{m} \tag{2.2.25}$$

$$\mathbf{A} = \begin{pmatrix} 1 \\ \sqrt{3} \end{pmatrix} \tag{2.2.26}$$

$$\mathbf{B} = \begin{pmatrix} 1 \\ -\sqrt{3} \end{pmatrix} \tag{2.2.27}$$

Now finding the direction vector $\mathbf{O_1A}$,$\mathbf{O_1B}$,$\mathbf{O_2A}$ and $\mathbf{O_2B}$.

$$\mathbf{O_1A} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 1 \\ \sqrt{3} \end{pmatrix} = \begin{pmatrix} -1 \\ -\sqrt{3} \end{pmatrix} \tag{2.2.28}$$

$$\mathbf{O_1B} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 1 \\ -\sqrt{3} \end{pmatrix} = \begin{pmatrix} -1 \\ \sqrt{3} \end{pmatrix} \tag{2.2.29}$$

$$\mathbf{O_2A} = \begin{pmatrix} 2 \\ 0 \end{pmatrix} - \begin{pmatrix} 1 \\ \sqrt{3} \end{pmatrix} = \begin{pmatrix} 1 \\ -\sqrt{3} \end{pmatrix} \tag{2.2.30}$$

$$\mathbf{O_2B} = \begin{pmatrix} 2 \\ 0 \end{pmatrix} - \begin{pmatrix} 1 \\ -\sqrt{3} \end{pmatrix} = \begin{pmatrix} 1 \\ \sqrt{3} \end{pmatrix} \tag{2.2.31}$$

Now finding the angle $\angle O_1AB$.

$$\mathbf{O_1AO_1B} = \|O_1A\| \, \|O_1B\| \cos\theta_1 \tag{2.2.32}$$

$$\frac{\mathbf{O_1AO_1B}}{\|O_1A\| \, \|O_1B\|} = \cos\theta_1 \tag{2.2.33}$$

$$\frac{-1}{2} = \cos\theta_1 \tag{2.2.34}$$

$$\theta_1 = 120^\circ \tag{2.2.35}$$

Now finding the angle $\angle O_2AB$.

$$\mathbf{O_2AO_2B} = \|O_2A\| \, \|O_2B\| \cos\theta_2 \tag{2.2.36}$$

$$\frac{\mathbf{O_2AO_2B}}{\|O_2A\| \, \|O_2B\|} = \cos\theta_2 \tag{2.2.37}$$

$$\frac{-1}{2} = \cos\theta_2 \tag{2.2.38}$$

$$\theta_2 = 120^\circ \tag{2.2.39}$$

Finding area of $\mathbf{O_1AB}$ and $\mathbf{O_2AB}$.

$$Area_1 = \frac{\theta_1}{360} r^2 - \frac{1}{2} 2\sqrt{3} \tag{2.2.40}$$

$$Area_1 = \frac{120}{360} 4\pi - \frac{1}{2} 2\sqrt{3} \tag{2.2.41}$$

$$Area_2 = \frac{\pi\theta_2}{360} r^2 - \frac{1}{2} 2\sqrt{3} \tag{2.2.42}$$

$$Area_2 = \frac{120}{360} 4\pi - \frac{1}{2} 2\sqrt{3} \tag{2.2.43}$$

$$TotalArea = \frac{8\pi}{3} - 2\sqrt{3} (\because Area_1 + Area_2) \tag{2.2.44}$$

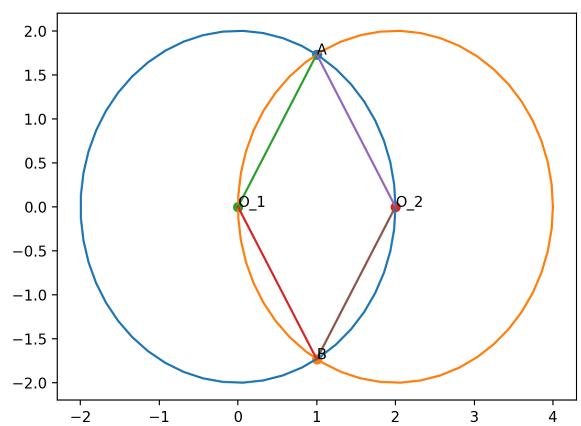

Fig. 0: Figure depicting intersection points of circle