Assignment 12

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The link to the solution is

https://github.com/Adarsh1310/EE5609

***Abstract*—This documents solves a problem based on Row Echelon form.**

1 Problem

Suppose **R** and **R**$'$ are 2×3 row-reduced echelon matrices and that the system **RX**=0 and **R**$'$**X**=0 have exactly the same solutions. Prove that **R**=**R**$'$.

2 Solution

Since **R** and **R**$'$ are 2×3 row-reduced echelon matrices they can be of following three types:-

1) Suppose matrix **R** has one non-zero row then **RX**=0 will have two free variables.Since R$'$**X**=0 will have the exact same solution as **RX**=0, R$'$**X**=0 will also have two free variables. Thus **R**$'$ have one non zero row.Now let's consider a matrix **A** with the first row as the non-zero row **R** and second row as the second row of **R**$'$.

$$\mathbf{R} = \begin{pmatrix} 1 & a & b \\ 0 & 0 & 0 \end{pmatrix} \tag{2.0.1}$$

$$\mathbf{R}' = \begin{pmatrix} 1 & c & d \\ 0 & 0 & 0 \end{pmatrix} \tag{2.0.2}$$

$$\tag{2.0.3}$$

Let **X** satisfy

$$\mathbf{RX} = 0 \tag{2.0.4}$$

$$\begin{pmatrix} 1 & \mathbf{a}^T \end{pmatrix} \begin{pmatrix} x \\ \mathbf{y} \end{pmatrix} = 0 \tag{2.0.5}$$

$$x + \mathbf{a}^T \mathbf{y} = 0 \tag{2.0.6}$$

where

$$\mathbf{a} = \begin{pmatrix} a \\ b \end{pmatrix} \tag{2.0.7}$$

$$\mathbf{R}'\mathbf{X} = 0 \tag{2.0.8}$$

$$\begin{pmatrix} 1 & \mathbf{b}^T \end{pmatrix} \begin{pmatrix} x \\ \mathbf{y} \end{pmatrix} = 0 \tag{2.0.9}$$

$$x + \mathbf{b}^T \mathbf{y} = 0 \tag{2.0.10}$$

where

$$\mathbf{b} = \begin{pmatrix} c \\ d \end{pmatrix} \tag{2.0.11}$$

Subtracting (2.0.10) from (2.0.6),

$$x + \mathbf{a}^T \mathbf{y} - x - \mathbf{b}^T \mathbf{y} = 0 \tag{2.0.12}$$

$$(\mathbf{a}^T - \mathbf{b}^T)\mathbf{y} = 0 \tag{2.0.13}$$

Since **y** is a 2×1 vector,

$$\implies y_1 \mathbf{a} - y_2 \mathbf{b} = 0 \tag{2.0.14}$$

Which can be written as,

$$\mathbf{a} = k\mathbf{b} \tag{2.0.15}$$

where,k=$\frac{\mathbf{y_2}}{\mathbf{y_1}}$ assuming $y_1 \neq 0$.

Hence for (2.0.13) to be always valid

$$\mathbf{a}^T - \mathbf{b}^T = 0 \tag{2.0.16}$$

$$\mathbf{a}^T = \mathbf{b}^T \tag{2.0.17}$$

Hence, **R**=**R′**

2) Let **R** and **R**$'$ have all rows as non zero.

$$\mathbf{R} = \begin{pmatrix} 1 & 0 & b \\ 0 & 1 & c \end{pmatrix} \tag{2.0.18}$$

$$\mathbf{R}' = \begin{pmatrix} 1 & 0 & e \\ 0 & 1 & f \end{pmatrix} \tag{2.0.19}$$

Let **X** satisfy

$$\mathbf{R}\mathbf{X} = 0 \tag{2.0.20}$$
$$\mathbf{X}^T\mathbf{R}^T = 0 \tag{2.0.21}$$

Here,

$$\mathbf{R} = \begin{pmatrix} \mathbf{I} & \mathbf{a} \end{pmatrix} \tag{2.0.22}$$
$$\mathbf{a} = \begin{pmatrix} b \\ c \end{pmatrix} \tag{2.0.23}$$
$$\mathbf{R}^T = \begin{pmatrix} \mathbf{I} \\ \mathbf{a}^T \end{pmatrix} \tag{2.0.24}$$

Let,

$$\mathbf{X}^T = \begin{pmatrix} \mathbf{y}^T & z \end{pmatrix} \tag{2.0.25}$$

where z is a scalar constant. Now,substituting (2.0.25) and (2.0.22) in (2.0.21)

$$\begin{pmatrix} \mathbf{y}^T & z \end{pmatrix}\begin{pmatrix} \mathbf{I} \\ \mathbf{a}^T \end{pmatrix} = 0 \tag{2.0.26}$$
$$\mathbf{y}^T + z\mathbf{a}^T = 0 \tag{2.0.27}$$

Now for,

$$\mathbf{R}'\mathbf{X} = 0 \tag{2.0.28}$$
$$\mathbf{X}^T\mathbf{R}'^T = 0 \tag{2.0.29}$$

Here,

$$\mathbf{R}' = \begin{pmatrix} \mathbf{I} & \mathbf{b} \end{pmatrix} \tag{2.0.30}$$
$$\mathbf{b} = \begin{pmatrix} e \\ f \end{pmatrix} \tag{2.0.31}$$

Let,

$$\mathbf{X}^T = \begin{pmatrix} \mathbf{y}^T & z \end{pmatrix} \tag{2.0.32}$$

where z is a scalar constant.Now,substituting (2.0.32) and (2.0.30) in (2.0.29)

$$\begin{pmatrix} \mathbf{y}^T & z \end{pmatrix}\begin{pmatrix} \mathbf{I} \\ \mathbf{b}^T \end{pmatrix} = 0 \tag{2.0.33}$$
$$\mathbf{y}^T + z\mathbf{b}^T = 0 \tag{2.0.34}$$

Subtracting (2.0.34) from (2.0.27)

$$\mathbf{y}^T + z\mathbf{a}^T - \mathbf{y}^T - z\mathbf{b}^T = 0 \tag{2.0.35}$$
$$(\mathbf{a}^T - \mathbf{b}^T)y = 0 \tag{2.0.36}$$

Hence (2.0.36) to be always valid

$$\mathbf{a}^T - \mathbf{b}^T = 0 \tag{2.0.37}$$
$$\mathbf{a}^T = \mathbf{b}^T \tag{2.0.38}$$

Hence, **R**=**R′**

3) Suppose matrix **R** have all the rows as zero then **RX**=0 will be satisfied for all values of **X**. We know that $\mathbf{R}'\mathbf{X}$=0 will have the exact same solution as **RX**=0 then we can say that for all values of **X**=0 equation $\mathbf{R}'\mathbf{X}$=0 will be satisfied Hence, $\mathbf{R}'$=**R**=0.

Since

$$\mathbf{y}$$

is a

$$2 \times 1$$

vector,

$$(a^T - b^T)\mathbf{y} = 0 \implies$$

$y_1\mathbf{a} - y_2\mathbf{b} = 0, or, \mathbf{a} = k\mathbf{b}, k = \frac{y_2}{y_1} assuming$ that

$$y_1 \neq 0$$

. Then proceed with your argument. Also, consider the case when

$$y_1 = 0$$

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