

# Assignment 12

Adarsh Srivastava

The link to the solution is

<https://github.com/Adarsh1310/EE5609>

**Abstract**—This documents solves a problem based on Row Echelon form.

## 1 PROBLEM

Suppose  $\mathbf{R}$  and  $\mathbf{R}'$  are  $2 \times 3$  row-reduced echelon matrices and that the system  $\mathbf{RX}=0$  and  $\mathbf{R}'\mathbf{X}=0$  have exactly the same solutions. Prove that  $\mathbf{R}=\mathbf{R}'$ .

## 2 SOLUTION

Since  $\mathbf{R}$  and  $\mathbf{R}'$  are  $2 \times 3$  row-reduced echelon matrices they can be of following three types:-

- 1) Suppose matrix  $\mathbf{R}$  has one non-zero row then  $\mathbf{RX}=0$  will have two free variables. Since  $\mathbf{R}'\mathbf{X}=0$  will have the exact same solution as  $\mathbf{RX}=0$ ,  $\mathbf{R}'\mathbf{X}=0$  will also have two free variables. Thus  $\mathbf{R}'$  have one non zero row. Now let's consider a matrix  $\mathbf{A}$  with the first row as the non-zero row  $\mathbf{R}$  and second row as the second row of  $\mathbf{R}'$ .

$$\mathbf{R} = \begin{pmatrix} 1 & a & b \\ 0 & 0 & 0 \end{pmatrix} \quad (2.0.1)$$

$$\mathbf{R}' = \begin{pmatrix} 1 & c & d \\ 0 & 0 & 0 \end{pmatrix} \quad (2.0.2)$$

$$\mathbf{A} = \mathbf{R} + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \mathbf{R}' \quad (2.0.3)$$

$$\mathbf{A} = \begin{pmatrix} 1 & a & b \\ 1 & c & d \end{pmatrix} \quad (2.0.4)$$

Let,

$$\mathbf{X} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \quad (2.0.5)$$

If  $\mathbf{X}$  satisfies

$$\mathbf{R} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0 \quad (2.0.6)$$

$$\mathbf{R}' \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0 \quad (2.0.7)$$

then,

$$\mathbf{A} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0 \quad (2.0.8)$$

Thus,  $\mathbf{A}$  in it's reduced form must have one non-zero row which is possible only when the rows of  $\mathbf{A}$  are equal because leading entries in both the vectors equals one. Hence,  $\mathbf{R} = \mathbf{R}'$ .

- 2) Let  $\mathbf{R}$  and  $\mathbf{R}'$  have all rows as non zero. Let

$$\mathbf{R} = \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \end{pmatrix} \quad (2.0.9)$$

$$\mathbf{R}' = \begin{pmatrix} 1 & d & e \\ 0 & 1 & f \end{pmatrix} \quad (2.0.10)$$

Now let's consider another matrix  $\mathbf{A}$  whose first two rows are from  $\mathbf{R}$  and last two rows are from  $\mathbf{R}'$ .

$$\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{R} + \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{R}' \quad (2.0.11)$$

$$\mathbf{A} = \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 1 & d & e \\ 0 & 1 & f \end{pmatrix} \quad (2.0.12)$$

If  $\mathbf{X}$  satisfies

$$\mathbf{R} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0 \quad (2.0.13)$$

$$\mathbf{R}' \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0 \quad (2.0.14)$$

then,

$$\mathbf{A} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0 \quad (2.0.15)$$

Therefore row reduced echelon form of  $\mathbf{A}$  must have two non-zero rows, which implies the rows of  $\mathbf{R}$  and  $\mathbf{R}'$  must be a linear combination of each other. It is possible only when the leading coefficients of the first row of  $\mathbf{R}$  and  $\mathbf{R}'$  occur in the same column. By similar

argument, the leading coefficients of the second rows must also occur in the same column. Thus, the only way the rows of  $\mathbf{R}$  and  $\mathbf{R}'$  are linear combination of one another is that the respective rows coincide and hence  $\mathbf{R} = \mathbf{R}'$ .

- 3) Suppose matrix  $\mathbf{R}$  have all the rows as zero then  $\mathbf{RX}=0$  will be satisfied for all values of  $\mathbf{X}$ . We know that  $\mathbf{R}'\mathbf{X}=0$  will have the exact same solution as  $\mathbf{RX}=0$  then we can say that for all values of  $\mathbf{X}=0$  equation  $\mathbf{R}'\mathbf{X}=0$  will be satisfied hence,  $\mathbf{R}'=\mathbf{R}=0$ .