

Assignment 17

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The link to the solution is

<https://github.com/Adarsh1310/EE5609>

Abstract—This documents solves a problem based on coordinates.

1 PROBLEM

Let \mathbf{V} be the real vector space of all polynomial functions from \mathbb{R} to \mathbb{R} of degree 2 or less, i.e, the space of all functions f of the form,

$$f(x) = c_0 + c_1x + c_2x^2$$

Let t be a fixed real number and define

$$g_1(x) = 1, g_2(x) = x + t, g_3(x) = (x + t)^2$$

Prove that $\beta = \{g_1, g_2, g_3\}$ is a basis for \mathbf{V} . If

$$f(x) = c_0 + c_1x + c_2x^2$$

what are the coordinates of f in the ordered basis β

2 SOLUTION

Let's start by proving that $\{g_1, g_2, g_3\}$ are linearly independent,

$$\begin{pmatrix} a & b & c \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ t & 1 & 0 \\ t^2 & 2t & 1 \end{pmatrix} \begin{pmatrix} 1 \\ x \\ x^2 \end{pmatrix} = 0 \quad (2.0.1)$$

$$\begin{pmatrix} 1 & x & x^2 \end{pmatrix} \begin{pmatrix} 1 & t & t^2 \\ 0 & 1 & 2t \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0 \quad (2.0.2)$$

$$\mathbf{ABc} = 0 \quad (2.0.3)$$

Now, taking \mathbf{B} and applying row reduce operations,

$$\begin{pmatrix} 1 & t & t^2 \\ 0 & 1 & 2t \\ 0 & 0 & 1 \end{pmatrix} \quad (2.0.4)$$

$$\xleftrightarrow{R_1 = R_1 - tR_2} \begin{pmatrix} 1 & 0 & t^2 \\ 0 & 1 & 2t \\ 0 & 0 & 1 \end{pmatrix} \quad (2.0.5)$$

$$\xleftrightarrow{R_1 = R_1 - t^2R_3} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 2t \\ 0 & 0 & 1 \end{pmatrix} \quad (2.0.6)$$

$$\xleftrightarrow{R_2 = R_2 - 2tR_3} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (2.0.7)$$

Since \mathbf{B} is linearly independent so \mathbf{c} will be zero vector. We know that \mathbf{c} is in the nullspace of \mathbf{B} and hence it will also be in the nullspace of \mathbf{AB} . So, $\{g_1, g_2, g_3\}$ are linearly independent. Now, to find the coordinates,

$$\begin{pmatrix} c_0 & c_1 & c_2 \end{pmatrix} \begin{pmatrix} 1 \\ x \\ x^2 \end{pmatrix} = \begin{pmatrix} a & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ x \\ x^2 \end{pmatrix} + \begin{pmatrix} bt & b & 0 \end{pmatrix} \begin{pmatrix} 1 \\ x \\ x^2 \end{pmatrix} + \begin{pmatrix} ct^2 & 2ct & c \end{pmatrix} \begin{pmatrix} 1 \\ x \\ x^2 \end{pmatrix} \quad (2.0.8)$$

$$\begin{pmatrix} c_0 & c_1 & c_2 \end{pmatrix} \begin{pmatrix} 1 \\ x \\ x^2 \end{pmatrix} = \begin{pmatrix} a + bt + ct^2 & b + 2ct & c \end{pmatrix} \begin{pmatrix} 1 \\ x \\ x^2 \end{pmatrix} \quad (2.0.9)$$

From above we can see that,

$$c_2 = c \quad (2.0.10)$$

$$c_1 = b + 2ct \quad (2.0.11)$$

$$c_0 = a + bt + ct^2 \quad (2.0.12)$$

Solving the above equation we get,

$$c = c_2 \quad (2.0.13)$$

$$b = c_1 - 2c_2t \quad (2.0.14)$$

$$a = c_0 - c_1t + c_2t^2 \quad (2.0.15)$$

So, finally the coordinates of f in ordered basis of β ,

$$\begin{pmatrix} c_0 - c_1t + c_2t^2 \\ c_1 - 2c_2t \\ c_2 \end{pmatrix} \quad (2.0.16)$$