

# Assignment 8

Adarsh Sai - AI20BTECH11001

Download all python codes from

<https://github.com/Adarsh541/AI1103-prob-and-ranvar/blob/main/Assignment8.1/codes/Assignment8.1.py>

and latex-tikz codes from

<https://github.com/Adarsh541/AI1103-prob-and-ranvar/blob/main/Assignment8.1/Assignment8.1.tex>

$$T = \mathbf{x}^T \mathbf{M} \mathbf{x} \quad (2.1.4)$$

where

$$\mathbf{M} = \begin{pmatrix} \frac{4}{5} & -1/5 & -1/5 & -1/5 & -1/5 \\ -1/5 & \frac{4}{5} & -1/5 & -1/5 & -1/5 \\ -1/5 & -1/5 & \frac{4}{5} & -1/5 & -1/5 \\ -1/5 & -1/5 & -1/5 & \frac{4}{5} & -1/5 \\ -1/5 & -1/5 & -1/5 & -1/5 & \frac{4}{5} \end{pmatrix} \quad (2.1.5)$$

$$(2.1.6)$$

we also have

$$\mathbf{M}^2 = \mathbf{M} \quad (2.1.7)$$

## 1 PROBLEM

Let  $X_1, X_2, X_3, X_4, X_5$  be a random sample of size 5 from a population having standard normal distribution. If  $\bar{X} = \frac{1}{5} \sum_{i=1}^5 X_i$  and  $T = \sum_{i=1}^5 (X_i - \bar{X})^2$  then  $E[T^2 \bar{X}^2]$  is equal to

- 1) 3
- 2) 3.6
- 3) 4.8
- 4) 5.2

## 2 SOLUTION

### 2.1 Terminology

Let  $\mathbf{x} \sim N(\mathbf{0}, \mathbf{I})$  be a standard normal random vector

$$\mathbf{x} = \begin{pmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \\ X_5 \end{pmatrix} \quad (2.1.1)$$

Then  $\bar{X}$  can be written as

$$\bar{X} = \frac{1}{5} \mathbf{u}^T \mathbf{x} \quad (2.1.2)$$

where

$$\mathbf{u} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \quad (2.1.3)$$

### 2.2 Theorem

Let  $\bar{X}_n$  be the sample mean of size n from a normal distribution with mean  $\mu$  and variance  $\sigma^2$ . Then

- 1)  $\bar{X}_n \sim N(\mu, \frac{\sigma^2}{n})$
- 2)  $\bar{X}$  and  $S^2$  are independent.
- 3)  $\frac{(n-1)S^2}{\sigma^2} \sim \chi_{n-1}^2$

where  $\chi_{n-1}^2$  is chi-square distribution with  $(n-1)$  degrees of freedom and  $S^2$  is defined as

$$S^2 = \frac{1}{n-1} \sum_{i=1}^n (X_i - \bar{X})^2 \quad (2.2.1)$$

### 2.3 Useful concepts

If X, Y are independent random variables, then

$$E[XY] = E[X]E[Y] \quad (2.3.1)$$

$$E[X^2] = \text{Var}[X] + (E[X])^2 \quad (2.3.2)$$

If X is chi-square distributed with parameter k, then

$$E[X] = k \quad (2.3.3)$$

$$\text{Var}[X] = 2k \quad (2.3.4)$$

### 2.4 solution

For standard normal distribution  $\mu = 0, \sigma^2 = 1$ . So from the above theorem for  $n = 5$  we have

- 1)  $T/4$  and  $\bar{X}$  are independent.
- 2)  $\bar{X} \sim N(0, 1/5)$

$$3) T \sim \chi_4^2$$

So from (2.3.3) and (2.3.4)

$$E[T] = 4 \quad (2.4.1)$$

$$Var[T] = 8 \quad (2.4.2)$$

Since  $\frac{T}{4}$  and  $\bar{X}$  are independent,  $T$  and  $\bar{X}$  are also independent

$$E[T^2 \bar{X}^2] = E[T^2]E[\bar{X}^2] \quad (2.4.3)$$

from (2.3.2)

$$E[\bar{X}^2] = \frac{1}{5} \quad (2.4.4)$$

$$E[T^2] = 24 \quad (2.4.5)$$

So from (2.4.3)

$$E[T^2 \bar{X}^2] = 4.8 \quad (2.4.6)$$

### 3 PROOF FOR INDEPENDENCY

$$\bar{X} = \frac{1}{5} \mathbf{u}^\top \mathbf{x} \quad (3.0.1)$$

since  $\mathbf{M}$  is symmetric and idempotent we have

$$T = \mathbf{x}^\top \mathbf{M} \mathbf{x} \quad (3.0.2)$$

$$= \mathbf{x}^\top \mathbf{M} \mathbf{M} \mathbf{x} \quad (3.0.3)$$

$$= \mathbf{x}^\top \mathbf{M}^\top \mathbf{M} \mathbf{x} \quad (3.0.4)$$

$$= (\mathbf{M} \mathbf{x})^\top (\mathbf{M} \mathbf{x}) \quad (3.0.5)$$

So if  $\mathbf{M} \mathbf{x}$  and  $\mathbf{u}^\top \mathbf{x}$  are independent then  $\bar{X}$  and  $T$  are also independent

#### 3.1 Cross-Covariance

$$Cov[\mathbf{x}, \mathbf{y}] = E[(\mathbf{x} - E[\mathbf{x}]) (\mathbf{y} - E[\mathbf{y}])^\top] \quad (3.1.1)$$

Two jointly normal vectors are independent if and only if their cross-covariance is zero.

$$Cov[\mathbf{u}^\top \mathbf{x}, \mathbf{M} \mathbf{x}] = E[(\mathbf{u}^\top \mathbf{x} - E[\mathbf{u}^\top \mathbf{x}]) (\mathbf{M} \mathbf{x} - E[\mathbf{M} \mathbf{x}])^\top] \quad (3.1.2)$$

$$= \mathbf{u}^\top E[(\mathbf{x} - E[\mathbf{x}]) (\mathbf{x} - E[\mathbf{x}])^\top] \mathbf{M}^\top \quad (3.1.3)$$

$$= \mathbf{u}^\top Var[\mathbf{x}] \mathbf{M} \quad (3.1.4)$$

$$= \mathbf{u}^\top \mathbf{M} \quad (3.1.5)$$

$$= 0 \quad (3.1.6)$$

So  $\mathbf{M} \mathbf{x}$  and  $\mathbf{u}^\top \mathbf{x}$  are independent. This implies  $\bar{X}$  and  $T$  are independent.