

Assignment 8

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Download all python codes from

<https://github.com/Adarsh541/AI1103-prob-and-ranvar/blob/main/Assignment8.1/codes/Assignment8.1.py>

and latex-tikz codes from

<https://github.com/Adarsh541/AI1103-prob-and-ranvar/blob/main/Assignment8.1/Assignment8.1.tex>

where

$$\mathbf{M} = \begin{pmatrix} \frac{4}{5} & -1/5 & -1/5 & -1/5 & -1/5 \\ -1/5 & \frac{4}{5} & -1/5 & -1/5 & -1/5 \\ -1/5 & -1/5 & \frac{4}{5} & -1/5 & -1/5 \\ -1/5 & -1/5 & -1/5 & \frac{4}{5} & -1/5 \\ -1/5 & -1/5 & -1/5 & -1/5 & \frac{4}{5} \end{pmatrix} \quad (2.0.5)$$

(2.0.6)

we also have

$$\mathbf{M}^2 = \mathbf{M} \quad (2.0.7)$$

1 PROBLEM

Let X_1, X_2, X_3, X_4, X_5 be a random sample of size 5 from a population having standard normal distribution. If $\bar{X} = \frac{1}{5} \sum_{i=1}^5 X_i$ and $T = \sum_{i=1}^5 (X_i - \bar{X})^2$ then $E[T^2 \bar{X}^2]$ is equal to

- 1) 3
- 2) 3.6
- 3) 4.8
- 4) 5.2

2 SOLUTION

Let $\mathbf{x} \sim N(\mathbf{0}, \mathbf{I})$ be a standard normal random vector

$$\mathbf{x} = \begin{pmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \\ X_5 \end{pmatrix} \quad (2.0.1)$$

Then \bar{X} can be written as

$$\bar{X} = \frac{1}{5} \mathbf{u}^T \mathbf{x} \quad (2.0.2)$$

where

$$\mathbf{u} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \quad (2.0.3)$$

$$T = \mathbf{x}^T \mathbf{M} \mathbf{x} \quad (2.0.4)$$

Lemma 2.1.

$$(\mathbf{u}^T \mathbf{x})(\mathbf{x}^T \mathbf{v}) = \mathbf{u}^T (\mathbf{x} \mathbf{x}^T) \mathbf{v} \quad (2.0.8)$$

This lemma is verified using python simulation.

Lemma 2.2. If \mathbf{A}, \mathbf{B} are two independent random vectors then $\mathbf{A}, \mathbf{B}^T \mathbf{B}$ are independent.

Proof. Since \mathbf{A}, \mathbf{B} are independent

$$\Pr(\mathbf{A} = \mathbf{a}, \mathbf{B} = \mathbf{b}) = \Pr(\mathbf{A} = \mathbf{a}) \Pr(\mathbf{B} = \mathbf{b}) \quad (2.0.9)$$

Let the solution of the set

$$\{\mathbf{B}^T \mathbf{B} = \mathbf{c}\} = \{\mathbf{b}_1, \mathbf{b}_2\} \quad (2.0.10)$$

$$\Pr(\mathbf{A} = \mathbf{a}, \mathbf{B}^T \mathbf{B} = \mathbf{c}) = \Pr(\mathbf{A} = \mathbf{a}, \mathbf{B} = \mathbf{b}_1) + \Pr(\mathbf{A} = \mathbf{a}, \mathbf{B} = \mathbf{b}_2) \quad (2.0.11)$$

$$= \Pr(\mathbf{A} = \mathbf{a}) (\Pr(\mathbf{B} = \mathbf{b}_1) + \Pr(\mathbf{B} = \mathbf{b}_2)) \quad (2.0.12)$$

$$= \Pr(\mathbf{A} = \mathbf{a}) \Pr(\mathbf{B}^T \mathbf{B} = \mathbf{c}) \quad (2.0.13)$$

□

Definition 2.1 (cross-covariance).

$$\text{Cov}[\mathbf{x}, \mathbf{y}] = E[(\mathbf{x} - E[\mathbf{x}])(\mathbf{y} - E[\mathbf{y}])^T] \quad (2.0.14)$$

Lemma 2.3. Two jointly normal vectors are independent if and only if their cross-covariance is zero.

Theorem 2.4. Let \mathbf{x} be a 5×1 standard multivariate normal random vector. Let \mathbf{B} be an $l \times 5$ real matrix. Then the $l \times 1$ random vector \mathbf{y} defined by $\mathbf{y} =$

\mathbf{Bx} has multivariate normal distribution with mean $E[\mathbf{y}] = \mathbf{0}$ and covariance matrix $\text{Var}[\mathbf{y}] = \mathbf{BB}^\top$

Proof. The joint moment generating function of \mathbf{x} is

$$M_{\mathbf{x}}(\mathbf{t}) = \exp\left(\mathbf{t}^\top \mu + \frac{1}{2} \mathbf{t}^\top \mathbf{V} \mathbf{t}\right) \quad (2.0.15)$$

since for standard normal distribution $\mu = \mathbf{0}$ and $\mathbf{V} = \mathbf{I}$. So

$$M_{\mathbf{x}}(\mathbf{t}) = \exp\left(\frac{1}{2} \mathbf{t}^\top \mathbf{I} \mathbf{t}\right) \quad (2.0.16)$$

Therefore the joint moment generating function of \mathbf{y} is

$$M_{\mathbf{y}}(\mathbf{t}) = M_{\mathbf{x}}(\mathbf{B}^\top \mathbf{t}) \quad (2.0.17)$$

$$= \exp\left(\frac{1}{2} \mathbf{t}^\top \mathbf{B} \mathbf{B}^\top \mathbf{t}\right) \quad (2.0.18)$$

on comparing with (2.0.16) we can say \mathbf{y} has multivariate normal distribution. \square

Theorem 2.5. Let \mathbf{x} be a 5×1 standard multivariate normal random vector. Let \mathbf{A}, \mathbf{B} be two matrices. Define

$$\mathbf{T}_1 = \mathbf{Ax} \quad (2.0.19)$$

$$\mathbf{T}_2 = \mathbf{Bx} \quad (2.0.20)$$

Then \mathbf{T}_1 and \mathbf{T}_2 are two independent random vectors if and only if $\mathbf{AB}^\top = \mathbf{0}$

Proof. From theorem 2.4, \mathbf{T}_1 and \mathbf{T}_2 are jointly normal. Their cross-covariance is

$$\text{Cov}[\mathbf{T}_1, \mathbf{T}_2] = E[(\mathbf{T}_1 - E[\mathbf{T}_1])(\mathbf{T}_2 - E[\mathbf{T}_2])^\top] \quad (2.0.21)$$

$$= E[(\mathbf{Ax} - E[\mathbf{Ax}])(\mathbf{Bx} - E[\mathbf{Bx}])^\top] \quad (2.0.22)$$

$$= \mathbf{AE}[(\mathbf{x} - E[\mathbf{x}])(\mathbf{x} - E[\mathbf{x}])^\top] \mathbf{B}^\top \quad (2.0.23)$$

$$= \mathbf{A} \text{Var}[\mathbf{x}] \mathbf{B}^\top \quad (2.0.24)$$

$$= \mathbf{AB}^\top \quad (2.0.25)$$

So \mathbf{T}_1 and \mathbf{T}_2 are independent if and only if $\mathbf{AB}^\top = \mathbf{0}$ \square

Theorem 2.6. Let \bar{X} be the sample mean of size 5 from a standard normal distribution. Then

$$1) \bar{X} \sim N(0, \frac{1}{5})$$

$$2) \bar{X} \text{ and } T \text{ are independent.}$$

$$3) T \sim \chi_4^2$$

where χ_4^2 is chi-square distribution with 4 degrees of freedom and T is defined as

$$T = \sum_{i=1}^5 (X_i - \bar{X})^2 \quad (2.0.26)$$

Proof. 1)

$$\bar{X} = \frac{1}{5} \mathbf{u}^\top \mathbf{x} \quad (2.0.27)$$

From theorem 2.4 we can say \bar{X} has normal distribution with mean $E[\bar{X}] = \mathbf{0}$ and covariance matrix

$$\text{Var}[\bar{X}] = \frac{1}{25} \mathbf{u}^\top \mathbf{u} \quad (2.0.28)$$

$$= \frac{1}{5} \quad (2.0.29)$$

2) since \mathbf{M} is symmetric and idempotent we have

$$T = \mathbf{x}^\top \mathbf{M} \mathbf{x} \quad (2.0.30)$$

$$= \mathbf{x}^\top \mathbf{M} \mathbf{M} \mathbf{x} \quad (2.0.31)$$

$$= \mathbf{x}^\top \mathbf{M}^\top \mathbf{M} \mathbf{x} \quad (2.0.32)$$

$$= (\mathbf{Mx})^\top (\mathbf{Mx}) \quad (2.0.33)$$

$$\text{Cov}\left[\frac{1}{5} \mathbf{u}^\top \mathbf{x}, \mathbf{Mx}\right] = E\left[\left(\frac{1}{5} \mathbf{u}^\top \mathbf{x} - E\left[\frac{1}{5} \mathbf{u}^\top \mathbf{x}\right]\right)(\mathbf{Mx} - E[\mathbf{Mx}])^\top\right] \quad (2.0.34)$$

Using lemma 2.1 we get

$$\begin{aligned} \text{Cov}\left[\frac{1}{5} \mathbf{u}^\top \mathbf{x}, \mathbf{Mx}\right] &= \frac{1}{5} \mathbf{u}^\top E[(\mathbf{x} - E[\mathbf{x}])(\mathbf{x} - E[\mathbf{x}])^\top] \mathbf{M}^\top \quad (2.0.35) \end{aligned}$$

$$= \frac{1}{5} \mathbf{u}^\top \text{Var}[\mathbf{x}] \mathbf{M} \quad (2.0.36)$$

$$= \frac{1}{5} \mathbf{u}^\top \mathbf{M} \quad (2.0.37)$$

$$= 0 \quad (2.0.38)$$

So \mathbf{Mx} and $\frac{1}{5} \mathbf{u}^\top \mathbf{x}$ are independent. Using lemma 2.2 we get \bar{X} and T are independent.

3) Since \mathbf{M} is symmetric it can be expressed as

$$\mathbf{M} = \mathbf{P} \mathbf{D} \mathbf{P}^\top \quad (2.0.39)$$

where \mathbf{P} is orthogonal and \mathbf{D} is diagonal. Then

$$T = \mathbf{x}^\top \mathbf{M} \mathbf{x} \quad (2.0.40)$$

$$= \mathbf{x}^\top \mathbf{P} \mathbf{D} \mathbf{P}^\top \mathbf{x} \quad (2.0.41)$$

$$= (\mathbf{P}^\top \mathbf{x})^\top \mathbf{D} \mathbf{P}^\top \mathbf{x} \quad (2.0.42)$$

$$= \mathbf{y}^\top \mathbf{D} \mathbf{y} \quad (2.0.43)$$

where $\mathbf{y} = \mathbf{P}^\top \mathbf{x}$. From theorem 2.4 and orthogonality of \mathbf{P} we can say $\mathbf{y} \sim N(\mathbf{0}, \mathbf{I})$

a) The eigen values of \mathbf{M} are 1, 1, 1, 1, 0. So \mathbf{D} can be written as

$$\mathbf{D} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad (2.0.44)$$

Let $\mathbf{v}_1, \dots, \mathbf{v}_5$ be corresponding eigen vectors. Then $\mathbf{P} = (\mathbf{v}_1, \dots, \mathbf{v}_5)$. Since \mathbf{P} is orthogonal the dot product of any two eigen vectors is zero. i.e

$$\mathbf{v}_i^\top \mathbf{v}_j = 0 \quad (2.0.45)$$

for any $i \neq j$

$$\mathbf{y} = \begin{pmatrix} y_1 = \mathbf{v}_1^\top \mathbf{x} \\ y_2 = \mathbf{v}_2^\top \mathbf{x} \\ y_3 = \mathbf{v}_3^\top \mathbf{x} \\ y_4 = \mathbf{v}_4^\top \mathbf{x} \\ y_5 = \mathbf{v}_5^\top \mathbf{x} \end{pmatrix} \quad (2.0.46)$$

$$(2.0.47)$$

From (2.0.45) and theorem 2.5, it follows y_1, y_2, y_3, y_4, y_5 are mutually independent.

$$T = y_1^2 + y_2^2 + y_3^2 + y_4^2 \quad (2.0.48)$$

So T is sum of squares of four independent standard normal variables which is chi-square distribution with 4 degrees of freedom.

□

$$E[T^2 \bar{X}^2] = E[T^2] E[\bar{X}^2] \quad (2.0.49)$$

$$E[\bar{X}^2] = \text{Var}[\bar{X}] + (E[\bar{X}])^2 \quad (2.0.50)$$

$$= \frac{1}{5} \quad (2.0.51)$$

since T is chi-squared distributed with 4 degrees of freedom

$$E[T] = 4 \quad (2.0.52)$$

$$\text{Var}[T] = 8 \quad (2.0.53)$$

$$\implies E[T^2] = \text{Var}[T] + (E[T])^2 \quad (2.0.54)$$

$$= 24 \quad (2.0.55)$$

From (2.0.49)

$$E[T^2 \bar{X}^2] = 4.8 \quad (2.0.56)$$