Assignment 7

Adarsh Sai - AI20BTECH11001

Download all python codes from

```
https://github.com/Adarsh541/AI1103-prob-and-ranvar/blob/%main/Assignment6/codes/Assignment6.py
```

Download latex-tikz codes from

```
https://github.com/Adarsh541/AI1103-prob-and-ranvar/blob/main/Assignment7/Assignment7.tex
```

1 Problem(CSIR UGC NET EXAM(June 2012)Q.112)

Let $X_1, X_2, X_3, ..., X_n$ be i.i.d observations from a distribution with continuous probability density function f which is symmetric around θ i.e

$$f(x-\theta) = f(\theta - x) \tag{1.0.1}$$

for all real x.Consider the test $H_0 : \theta = 0$ vs $H_1 : \theta > 0$ and the sign test statistic

$$S_n = \sum_{i=1}^{n} sign(X_i) \tag{1.0.2}$$

where

$$sign(x) = \begin{cases} 1 & x > 0 \\ 0 & x = 0 \\ -1 & x < 0 \end{cases} \tag{1.0.3}$$

Let z_α be the upper $100(1-\alpha)^{th}$ percentile of the standard normal distribution where $0 < \alpha < 1$. Which of the following is/are correct?

1) If $\theta = 0$ then $\lim_{n\to\infty} P\left\{S_n > \sqrt{n} z_\alpha\right\} = 1$
2) If $\theta = 0$ then $\lim_{n\to\infty} P\left\{S_n > \sqrt{n} z_\alpha\right\} = \alpha$
3) If $\theta > 0$ then $\lim_{n\to\infty} P\left\{S_n > \sqrt{n} z_\alpha\right\} = 1$
4) If $\theta > 0$ then $\lim_{n\to\infty} P\left\{S_n > \sqrt{n} z_\alpha\right\} = \alpha$

2 Solution(CSIR UGC NET EXAM(June 2012)Q.112)

2.1 $H_0 : \theta = 0$

Assume hypothesis $H_0 : \theta = 0$ is true.

1) Given X is symmetric around zero.

$$f_X(x) = f_X(-x) \tag{2.1.1}$$

$$\int_0^\infty f_X(x)dx = \int_0^\infty f_X(-x)dx \tag{2.1.2}$$

a) Solving LHS of (2.1.2)

$$\int_0^\infty f_X(x)dx = \Pr(X \geq 0) \tag{2.1.3}$$

b) Solving RHS of (2.1.2)

$$\int_0^\infty f_X(-x)dx \tag{2.1.4}$$

Changing $-x \to x$ we get

$$\int_0^\infty f_X(-x)dx = \int_{-\infty}^0 f_X(x)dx \tag{2.1.5}$$

$$= \Pr(X \leq 0) \tag{2.1.6}$$

but

$$\int_{-\infty}^0 f_X(x)dx + \int_0^\infty f_X(x)dx = 1 \tag{2.1.7}$$

from (2.1.2) , (2.1.5) and (2.1.7)

$$\int_{-\infty}^0 f_X(x)dx = \int_0^\infty f_X(x)dx = \frac{1}{2} \tag{2.1.8}$$

$$\implies \Pr(X \leq 0) = \Pr(X \geq 0) = \frac{1}{2} \tag{2.1.9}$$

2) Let Y be a random variable such that

$$Y = sign(X) \tag{2.1.10}$$

$$Y = \begin{cases} 1 & X > 0 \\ -1 & X < 0 \end{cases} \tag{2.1.11}$$

From (2.1.9) and (2.2.4) we have

$$\Pr(Y=-1)=\Pr(Y=1)=\frac{1}{2} \tag{2.1.12}$$

So $Y = sign(X)$ is also symmetric around zero.

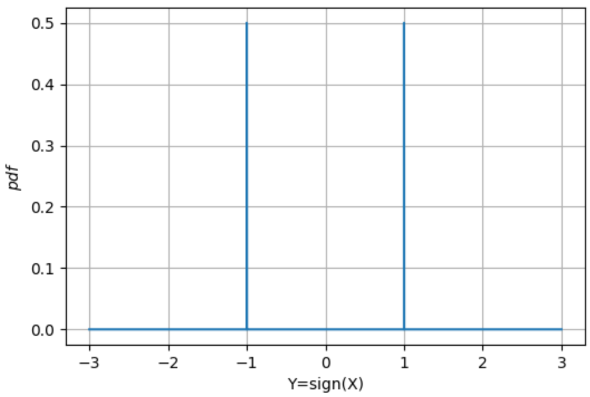

Fig. 2: pdf of $Y = sign(X)$

$$\implies \mu_y = 0 \tag{2.1.13}$$

and variance is

$$\sigma_y^2 = (-1)^2\left(\frac{1}{2}\right)+(1)^2\left(\frac{1}{2}\right) \tag{2.1.14}$$

$$= 1 \tag{2.1.15}$$

3) Given

$$S_n = \sum_{i=1}^{n} sign(X_i) \tag{2.1.16}$$

$$S_n(\theta=0) = \sum_{i=1}^{n} Y_i \tag{2.1.17}$$

From central limit theorem

$$Z = \lim_{n\to\infty} \sqrt{n}\left(\frac{\frac{S_n}{n}-\mu_y}{\sigma_y}\right) \tag{2.1.18}$$

$$= \lim_{n\to\infty} \sqrt{n}\left(\frac{S_n}{n}\right) \tag{2.1.19}$$

$$= \lim_{n\to\infty} \left(\frac{S_n}{\sqrt{n}}\right) \tag{2.1.20}$$

where Z is a standard normal variable N(0,1).

a) Given

$$\alpha = P\{Z > z_\alpha\} \tag{2.1.21}$$

So from (2.1.20) and (2.1.21)

$$\lim_{n\to\infty} P\left\{\frac{S_n}{\sqrt{n}} > z_\alpha\right\} = \alpha \tag{2.1.22}$$

$$\implies \lim_{n\to\infty} P\left\{S_n > \sqrt{n}z_\alpha\right\} = \alpha \tag{2.1.23}$$

2.2 *$H_1 : \theta > 0$ is true*

1) Given X is symmetric around $\theta > 0$.Let us assume $\theta = \theta_0 > 0$.

$$f_X(\theta_0 - x) = f_X(\theta_0 + x) \tag{2.2.1}$$

$$\int_{\theta_0}^{\infty} f_X(\theta_0 - x)dx = \int_{\theta_0}^{\infty} f_X(\theta_0 + x)dx \tag{2.2.2}$$

a) Solving LHS of (2.2.2).Changinng $(\theta_0 - x) \to t$

$$\int_{\theta_0}^{\infty} f_X(\theta_0 - x)dx = \int_{-\infty}^{0} f_X(t)dt \tag{2.2.3}$$

$$= \Pr(X \le 0) \tag{2.2.4}$$

b) Solving RHS of (2.2.2).Changing $(\theta_0 + x) \to t$

$$\int_{\theta_0}^{\infty} f_X(\theta_0 + x)dx = \int_{2\theta_0}^{\infty} f_X(t)dt \tag{2.2.5}$$

$$= \int_{0}^{\infty} f_X(t)dt - \int_{0}^{2\theta_0} f_X(t)dt \tag{2.2.6}$$

$$= \Pr(X \ge 0) - k \tag{2.2.7}$$

where

$$k = \int_{0}^{2\theta_0} f_X(t)dt > 0 \tag{2.2.8}$$

From (2.2.2),(2.2.4) and (2.2.7)

$$\Pr(X \ge 0) > \Pr(X \le 0) \tag{2.2.9}$$

2) So

$$\Pr(Y=1) > \Pr(Y=-1) \tag{2.2.10}$$

Therefore,if we perform the experiment and find the value of $\left(\frac{S_n}{\sqrt{n}}\right)$,it is most likely to occur on the right side of the distribution of $\left(\frac{S_n}{\sqrt{n}}\right)$.In (2.1.20) it is shown that distrubution of the random variable $\left(\frac{S_n}{\sqrt{n}}\right)$ is $N(0,1)$ when n is very large.So

$$\lim_{n\to\infty} P\left\{\frac{S_n}{\sqrt{n}} > Z_\alpha\right\} = 1 \tag{2.2.11}$$