Assignment 8

Adarsh Sai - AI20BTECH11001

Download all python codes from

https://github.com/Adarsh541/AI1103−prob−and−ranvar/blob/main/Assignment8.1/codes/Assignment8.1.py

and latex-tikz codes from

https://github.com/Adarsh541/AI1103−prob−and−ranvar/blob/main/Assignment8.1/Assignment8.1.tex

1 Problem

Let X_1, X_2, X_3, X_4, X_5 be a random sample of size 5 from a population having standard normal distribution.If $\overline{X} = \frac{1}{5}\sum_{i=1}^{5} X_i$ and $T = \sum_{i=1}^{5}\left(X_i - \overline{X}\right)^2$ then $E[T^2\overline{X}^2]$ is equal to

1) 3
2) 3.6
3) 4.8
4) 5.2

2 Solution

Let $\mathbf{x} \sim N(\mathbf{0}, \mathbf{I})$ be a standard normal random vector

$$\mathbf{x} = \begin{pmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \\ X_5 \end{pmatrix} \tag{2.0.1}$$

Then $\overline{X}$ can be written as

$$\overline{X} = \frac{1}{5}\mathbf{u}^\top \mathbf{x} \tag{2.0.2}$$

where

$$\mathbf{u} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \tag{2.0.3}$$

$$T = \mathbf{x}^\top \mathbf{M} \mathbf{x} \tag{2.0.4}$$

where

$$\mathbf{M} = \begin{pmatrix} \frac{4}{5} & -1/5 & -1/5 & -1/5 & -1/5 \\ -1/5 & \frac{4}{5} & -1/5 & -1/5 & -1/5 \\ -1/5 & -1/5 & \frac{4}{5} & -1/5 & -1/5 \\ -1/5 & -1/5 & -1/5 & \frac{4}{5} & -1/5 \\ -1/5 & -1/5 & -1/5 & -1/5 & \frac{4}{5} \end{pmatrix} \tag{2.0.5}$$

$$\tag{2.0.6}$$

we also have

$$\mathbf{M}^2 = \mathbf{M} \tag{2.0.7}$$

Lemma 2.1.

$$\left(\mathbf{u}^\top \mathbf{x}\right)\left(\mathbf{x}^\top \mathbf{v}\right) = \mathbf{u}^\top \left(\mathbf{x}\mathbf{x}^\top\right)\mathbf{v} \tag{2.0.8}$$

This lemma is verified using python simulation.

Lemma 2.2. *Let* $\mathbf{A}$, $\mathbf{B}$ *are two continuous independent* $l \times 1$ *random vectors then* $\mathbf{A}$, $\mathbf{B}^\top\mathbf{B}$ *are also independent.*

Proof. Since $\mathbf{A}$,$\mathbf{B}$ are independent

$$f_{\mathbf{A},\mathbf{B}}\left(a_1, ..a_l, b_1, ..b_l\right) = f_{\mathbf{A}}\left(a_1, ..a_l\right) f_{\mathbf{B}}\left(b_1, ..b_l\right) \tag{2.0.9}$$

without loss of generality we can assume the solution of the set

$$\{\mathbf{B} | \mathbf{B}^\top\mathbf{B} = c\} = \{\mathbf{b_1}, \mathbf{b_2}, .., \mathbf{b_k}\} \tag{2.0.10}$$

$$f_{\mathbf{A},\mathbf{B}^\top\mathbf{B}}\left(a_1, ..a_l, c\right) = f_{\mathbf{A},\mathbf{B}}\left(a_1, ..a_l, b_{11}, ..b_{1l}\right) + .. + f_{\mathbf{A},\mathbf{B}}\left(a_1, ..a_l, b_{k1}, ..b_{kl}\right) \tag{2.0.11}$$

$$= f_{\mathbf{A}}\left(a_1, ..a_l\right)\left(f_{\mathbf{B}}\left(b_{11}, ..b_{1l}\right) + .. + f_{\mathbf{B}}\left(b_{k1}, ..b_{kl}\right)\right) \tag{2.0.12}$$

$$= f_{\mathbf{A}}\left(a_1, ..a_l\right) f_{\mathbf{B}^\top\mathbf{B}}(c) \tag{2.0.13}$$

□

Definition 2.1 (cross-covariance)**.**

$$Cov\left[\mathbf{x}, \mathbf{y}\right] = E[(\mathbf{x} - E[\mathbf{x}])(\mathbf{y} - E[\mathbf{y}])^\top] \tag{2.0.14}$$

Lemma 2.3. *Two jointly normal vectors are independent if and only if their cross-covariance is zero.*

Theorem 2.4. *Let* $\mathbf{x}$ *be a* 5×1 *standard multivariate normal random vector.Let* $\mathbf{B}$ *be an* $l \times 5$ *real matrix.Then the* $l \times 1$ *random vector* $\mathbf{y}$ *defined by* $\mathbf{y} = \mathbf{Bx}$ *has multivariate normal distribution with mean* $E[\mathbf{y}] = \mathbf{0}$ *and covariance matrix* $Var[\mathbf{y}] = \mathbf{BB}^\top$

Proof. The joint moment generating function of $\mathbf{x}$ is

$$M_{\mathbf{x}}(\mathbf{t}) = exp\left(\mathbf{t}^\top\mu + \frac{1}{2}\mathbf{t}^\top\mathbf{V}\mathbf{t}\right) \tag{2.0.15}$$

since for standard normal distribution $\mu = \mathbf{0}$ and $\mathbf{V} = \mathbf{I}$.So

$$M_{\mathbf{x}}(\mathbf{t}) = exp\left(\frac{1}{2}\mathbf{t}^\top\mathbf{I}\mathbf{t}\right) \tag{2.0.16}$$

Therefore the joint moment generating function of $\mathbf{y}$ is

$$M_{\mathbf{y}}(\mathbf{t}) = M_{\mathbf{x}}\left(\mathbf{B}^\top\mathbf{t}\right) \tag{2.0.17}$$

$$= exp\left(\frac{1}{2}\mathbf{t}^\top\mathbf{BB}^\top\mathbf{t}\right) \tag{2.0.18}$$

on comparing with (2.0.16) we can say $\mathbf{y}$ has multivariate normal distribution. □

Theorem 2.5. *Let* $\mathbf{x}$ *be a* 5×1 *standard multivariate normal random vector.Let* $\mathbf{A}$ *,*$\mathbf{B}$ *be two matrices.Define*

$$\mathbf{T_1} = \mathbf{Ax} \tag{2.0.19}$$

$$\mathbf{T_2} = \mathbf{Bx} \tag{2.0.20}$$

Then $\mathbf{T_1}$ *and* $\mathbf{T_2}$ *are two independent random vectors if and only if* $\mathbf{AB}^\top = 0$

Proof. From theorem 2.4,$\mathbf{T_1}$ and $\mathbf{T_2}$ are jointly normal.Their cross-covariance is

$$Cov[\mathbf{T_1}, \mathbf{T_2}] = E[(\mathbf{T_1} - E[\mathbf{T_1}])(\mathbf{T_2} - E[\mathbf{T_2}])^\top] \tag{2.0.21}$$

$$= E[(\mathbf{Ax} - E[\mathbf{Ax}])(\mathbf{Bx} - E[\mathbf{Bx}])^\top] \tag{2.0.22}$$

$$= \mathbf{A}E[(\mathbf{x} - E[\mathbf{x}])(\mathbf{x} - E[\mathbf{x}])^\top]\mathbf{B}^\top \tag{2.0.23}$$

$$= \mathbf{A}Var[x]\mathbf{B}^\top \tag{2.0.24}$$

$$= \mathbf{AB}^\top \tag{2.0.25}$$

So $\mathbf{T_1}$ and $\mathbf{T_2}$ are independent if and only if $\mathbf{AB}^\top = 0$ □

Theorem 2.6. *Let* $\overline{X}$ *be the sample mean of size 5 from a standard normal distribution.Then*

1) $\overline{X} \sim N(0, \frac{1}{5})$
2) $\overline{X}$ *and* T *are independent.*
3) $T \sim \chi_4^2$

where χ_4^2 *is chi-square distribution with* 4 *degrees of freedom and* T *is defined as*

$$T = \sum_{i=1}^{5}(X_i - \overline{X})^2 \tag{2.0.26}$$

Proof.

1)

$$\overline{X} = \frac{1}{5}\mathbf{u}^\top\mathbf{x} \tag{2.0.27}$$

From theorem 2.4 we can say $\overline{X}$ has normal distribution with mean $E[\overline{X}] = \mathbf{0}$ and covariance matrix

$$Var\left[\overline{X}\right] = \frac{1}{25}\mathbf{u}^\top\mathbf{u} \tag{2.0.28}$$

$$= \frac{1}{5} \tag{2.0.29}$$

2) since $\mathbf{M}$ is symmetric and idempotent we have

$$T = \mathbf{x}^\top\mathbf{M}\mathbf{x} \tag{2.0.30}$$

$$= \mathbf{x}^\top\mathbf{M}\mathbf{M}\mathbf{x} \tag{2.0.31}$$

$$= \mathbf{x}^\top\mathbf{M}^\top\mathbf{M}\mathbf{x} \tag{2.0.32}$$

$$= (\mathbf{Mx})^\top(\mathbf{Mx}) \tag{2.0.33}$$

$$Cov\left[\frac{1}{5}\mathbf{u}^\top\mathbf{x}, \mathbf{Mx}\right] = E[\left(\frac{1}{5}\mathbf{u}^\top\mathbf{x} - E[\frac{1}{5}\mathbf{u}^\top\mathbf{x}]\right)(\mathbf{Mx} - E[\mathbf{Mx}])^\top] \tag{2.0.34}$$

Using lemma 2.1 we get

$$Cov\left[\frac{1}{5}\mathbf{u}^\top\mathbf{x}, \mathbf{Mx}\right] = \frac{1}{5}\mathbf{u}^\top E[(\mathbf{x} - E[\mathbf{x}])(\mathbf{x} - E[\mathbf{x}])^\top]\mathbf{M}^\top \tag{2.0.35}$$

$$= \frac{1}{5}\mathbf{u}^\top Var[\mathbf{x}]\mathbf{M} \tag{2.0.36}$$

$$= \frac{1}{5}\mathbf{u}^\top\mathbf{M} \tag{2.0.37}$$

$$= 0 \tag{2.0.38}$$

So $\mathbf{Mx}$ and $\frac{1}{5}\mathbf{u}^\top\mathbf{x}$ are independent.Using lemma 2.2 we get $\overline{X}$ and T are independent.

3) Since $\mathbf{M}$ is symmetric it can be expressed as

$$\mathbf{M} = \mathbf{PDP}^\top \tag{2.0.39}$$

where $\mathbf{P}$ is orthogonal and $\mathbf{D}$ is diagonal.Then

$$\begin{aligned} T &= \mathbf{x}^\top \mathbf{M}\mathbf{x} && (2.0.40)\\ &= \mathbf{x}^\top \mathbf{P}\mathbf{D}\mathbf{P}^\top \mathbf{x} && (2.0.41)\\ &= \left(\mathbf{P}^\top \mathbf{x}\right)^\top \mathbf{D}\mathbf{P}^\top \mathbf{x} && (2.0.42)\\ &= \mathbf{y}^\top \mathbf{D}\mathbf{y} && (2.0.43)\end{aligned}$$

where $\mathbf{y} = \mathbf{P}^\top \mathbf{x}$.From theorem 2.4 and orthogonality of $\mathbf{P}$ we can say $\mathbf{y} \sim N\left(\mathbf{0}, \mathbf{I}\right)$

a) The eigen values of $\mathbf{M}$ are $1, 1, 1, 1, 0$.So $\mathbf{D}$ can be written as

$$\mathbf{D} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{2.0.44}$$

Let $\mathbf{v_1}, .., \mathbf{v_5}$ be corresponding eigen vectors.Then $\mathbf{P} = (\mathbf{v_1}, .., \mathbf{v_5})$.Since $\mathbf{P}$ is orthogonal the dot product of any two eigen vectors is zero.i.e

$$\mathbf{v_i}^\top \mathbf{v_j} = 0 \tag{2.0.45}$$

for any $i \neq j$

$$\mathbf{y} = \begin{pmatrix} y_1 = \mathbf{v_1}^\top \mathbf{x}\\ y_2 = \mathbf{v_2}^\top \mathbf{x}\\ y_3 = \mathbf{v_3}^\top \mathbf{x}\\ y_4 = \mathbf{v_4}^\top \mathbf{x}\\ y_5 = \mathbf{v_5}^\top \mathbf{x} \end{pmatrix} \tag{2.0.46}$$

(2.0.47)

From (2.0.45) and theorem 2.5,it follows y_1, y_2, y_3, y_4, y_5 are mutually independent.

$$T = y_1^2 + y_2^2 + y_3^2 + y_4^2 \tag{2.0.48}$$

So T is sum of squares of four independent standard normal variables which is chi-square distribution with 4 degrees of freedom.

□

$$E[T^2 \overline{X}^2] = E[T^2]E[\overline{X}^2] \tag{2.0.49}$$

$$\begin{aligned} E[\overline{X}^2] &= Var\left[\overline{X}\right] + \left(E[\overline{X}]\right)^2 && (2.0.50)\\ &= \frac{1}{5} && (2.0.51)\end{aligned}$$

since T is chi-squared distributed with 4 degrees of freedom

$$\begin{aligned} E[T] &= 4 && (2.0.52)\\ Var\left[T\right] &= 8 && (2.0.53)\\ \implies E[T^2] &= Var\left[T\right] + (E[T])^2 && (2.0.54)\\ &= 24 && (2.0.55)\end{aligned}$$

From (2.0.49)

$$E[T^2 \overline{X}^2] = 4.8 \tag{2.0.56}$$