Assignment 8

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Download latex-tikz codes from

https://github.com/Adarsh541/AI1103−prob−and−ranvar/blob/main/Assignment8/Assignment8.tex

1 Problem(gov/stats/2015/statistics-I(1),Q.3(a))

Let $X_1, X_2, ..., X_n$ be independent Poisson variates with $E[X_i] = \mu_i$.Find the conditional distribution of $X_1, ..., X_n \Big| \sum_{i=1}^{n} X_i = y$

2 Solution(gov/stats/2015/statistics-I(1),Q.3(a))

$$F\left\{x_1, ..., x_n \middle| \sum_{i=1}^{n} X_i = y\right\} = \Pr\left(X_1 \leq x_1, ..., X_n \leq x_n \middle| \sum_{i=1}^{n} X_i = y\right) \tag{2.0.1}$$

$$= \frac{\Pr\left(X_1 \leq x_1, ..., X_n \leq x_n, \sum_{i=1}^{n} X_i = y\right)}{\Pr\left(\sum_{i=1}^{n} X_i = y\right)} \tag{2.0.2}$$

1) To find numerator of (2.0.2),we solve for two random variables and extend it to n random variables.
   a) Solving for two variables. Consider the sets

$$\{X_1 \leq x_1\} \tag{2.0.3}$$
$$\{X_2 \leq x_2\} \tag{2.0.4}$$
$$\{X_1 + X_2 = y\} \tag{2.0.5}$$

   i) If $x_1 + x_2 = y$.Adding (2.0.3) and (2.0.4)

$$X_1 + X_2 \leq x_1 + x_2 = y \tag{2.0.6}$$

   and the equality holds when

$$X_1 = x_1 \tag{2.0.7}$$
$$X_2 = x_2 \tag{2.0.8}$$

   From (2.0.5) and (2.0.6)

$$X_1 + X_2 = y \tag{2.0.9}$$

   So the intersection of the sets is

$$\begin{gathered} \{X_1 \leq x_1, X_2 \leq x_2, X_1 + X_2 = y\} \\ = \{X_1 = x_1, X_2 = x_2\} \\ \text{Since } X_1 \text{ and } X_2 \text{ are independent} \\ \implies \Pr(X_1 \leq x_1, X_2 \leq x_2, X_1 + X_2 = y) \\ = \Pr(X_1 = x_1)\Pr(X_2 = x_2) \end{gathered} \tag{2.0.10}$$

$$= \prod_{i=1}^{2}\left(\frac{\mu_i^{x_i} e^{-\mu_i}}{x_i!}\right) \tag{2.0.11}$$

$$= \left(e^{-\sum \mu_i}\right)\prod_{i=1}^{2}\left(\frac{\mu_i^{x_i}}{x_i!}\right) \tag{2.0.12}$$

   ii) if $x_1 + x_2 < y$.Adding (2.0.3) and (2.0.4)

$$X_1 + X_2 \leq x_1 + x_2 < y \tag{2.0.13}$$

   but from (2.0.5)

$$X_1 + X_2 = y \tag{2.0.14}$$

   So there exist no values for X_1 and X_2 Therefore

$$\Pr(X_1 \leq x_1, X_2 \leq x_2, X_1 + X_2 = y) = \begin{cases} 0 & x_1 + x_2 < y \\ \left(e^{-\sum \mu_i}\right)\prod_{i=1}^{2}\left(\frac{\mu_i^{x_i}}{x_i!}\right) & x_1 + x_2 = y \end{cases} \tag{2.0.15}$$

   Extending to n variables we get

$$\Pr\left(X_1 \leq x_1, ..., X_n \leq x_n, \sum_{i=1}^{n} X_i = y\right) = \begin{cases} \left(e^{-\sum \mu_i}\right)\prod_{i=1}^{n}\left(\frac{\mu_i^{x_i}}{x_i!}\right) & \sum_{i=1}^{n} x_i = y \\ 0 & \sum_{i=1}^{n} x_i < y \end{cases} \tag{2.0.16}$$

2) **Theorem**:If the random variables X_1, X_2 are independent and poisson distributed with parameters λ_1, λ_2,then their sum $Z = X_1 + X_2$ is also poisson distributed with parameter $\lambda_1 + \lambda_2$.**proof**:The characteristic function of a pois-

son random variable is given by

$$\Phi_X(\omega) = e^{-\lambda(1-e^{j\omega})} \tag{2.0.17}$$

Using convolution

$$\Phi_Z(\omega) = \Phi_{X_1}(\omega)\Phi_{X_2}(\omega) \tag{2.0.18}$$
$$= e^{-\lambda_1(1-e^{j\omega})}e^{-\lambda_2(1-e^{j\omega})} \tag{2.0.19}$$
$$= e^{-(\lambda_1+\lambda_2)(1-e^{j\omega})} \tag{2.0.20}$$

This theorem can be extended to n variables.So

$$\Pr\left(\sum_{i=1}^{n} X_i = y\right) = \frac{(\sum \mu_i)^y \, e^{-\sum \mu_i}}{y!} \tag{2.0.21}$$

From (2.0.2),(2.0.16) and (2.0.21)

$$F\left\{x_1, ..., x_n \middle| \sum_{i=1}^{n} X_i = y\right\}$$
$$= \begin{cases} \frac{y!}{(\sum \mu_i)^y} \prod_{i=1}^{n} \left(\frac{\mu_i^{x_i}}{x_i!}\right) & \sum_{i=1}^{n} x_i = y \\ 0 & \sum_{i=1}^{n} x_i < y \end{cases} \tag{2.0.22}$$