Assignment 8

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Download all python codes from

```
https://github.com/Adarsh541/AI1103−prob−and−ranvar/blob/main/Assignment8.1/codes/Assignment8.1.py
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and latex-tikz codes from

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https://github.com/Adarsh541/AI1103−prob−and−ranvar/blob/main/Assignment8.1/Assignment8.1.tex
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1 Problem

Let X_1, X_2, X_3, X_4, X_5 be a random sample of size 5 from a population having standard normal distribution.If $\overline{X} = \frac{1}{5}\sum_{i=1}^{5} X_i$ and $T = \sum_{i=1}^{5}\left(X_i - \overline{X}\right)^2$ then $E[T^2\overline{X}^2]$ is equal to

1) 3
2) 3.6
3) 4.8
4) 5.2

2 Solution

2.1 Terminology

Let $\mathbf{x} \sim N(\mathbf{0}, \mathbf{I})$ be a standard normal random vector

$$\mathbf{x} = \begin{pmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \\ X_5 \end{pmatrix} \tag{2.1.1}$$

Then $\overline{X}$ can be written as

$$\overline{X} = \frac{1}{5}\mathbf{u}^\top \mathbf{x} \tag{2.1.2}$$

where

$$\mathbf{u} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \tag{2.1.3}$$

$$T = \mathbf{x}^\top \mathbf{M}\mathbf{x} \tag{2.1.4}$$

where

$$\mathbf{M} = \begin{pmatrix} \frac{4}{5} & -1/5 & -1/5 & -1/5 & -1/5 \\ -1/5 & \frac{4}{5} & -1/5 & -1/5 & -1/5 \\ -1/5 & -1/5 & \frac{4}{5} & -1/5 & -1/5 \\ -1/5 & -1/5 & -1/5 & \frac{4}{5} & -1/5 \\ -1/5 & -1/5 & -1/5 & -1/5 & \frac{4}{5} \end{pmatrix} \tag{2.1.5}$$

$$\tag{2.1.6}$$

we also have

$$\mathbf{M}^2 = \mathbf{M} \tag{2.1.7}$$

2.2 Theorem

Let $\overline{X_n}$ be the sample mean of size n from a normal distribution with mean μ and variance σ^2.Then

1) $\overline{X_n} \sim N(\mu, \frac{\sigma^2}{n})$
2) $\overline{X}$ and S^2 are independent.
3) $\frac{(n-1)S^2}{\sigma^2} \sim \chi^2_{n-1}$

where χ^2_{n-1} is chi-square distribution with $(n-1)$ degrees of freedom and S^2 is defined as

$$S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \overline{X})^2 \tag{2.2.1}$$

2.3 Useful concepts

If X,Y are independent random variables.then

$$E[XY] = E[X]E[Y] \tag{2.3.1}$$

$$E[X^2] = Var[X] + (E[X])^2 \tag{2.3.2}$$

If X is chi-square distributed with parameter k,then

$$E[X] = k \tag{2.3.3}$$

$$Var[X] = 2k \tag{2.3.4}$$

2.4 solution

For standard normal distribution $\mu = 0, \sigma^2 = 1$.So from the above theorem for $n = 5$ we have

1) T/4 and $\overline{X}$ are independent.
2) $\overline{X} \sim N(0, 1/5)$

3) $T \sim \chi^2_4$

So from (2.3.3) and (2.3.4)

$$E[T] = 4 \tag{2.4.1}$$
$$Var[T] = 8 \tag{2.4.2}$$

Since $\frac{T}{4}$ and $\overline{X}$ are independent,T and $\overline{X}$ are also independent

$$E[T^2\overline{X}^2] = E[T^2]E[\overline{X}^2] \tag{2.4.3}$$

from (2.3.2)

$$E[\overline{X}^2] = \frac{1}{5} \tag{2.4.4}$$
$$E[T^2] = 24 \tag{2.4.5}$$

So from (2.4.3)

$$E[T^2\overline{X}^2] = 4.8 \tag{2.4.6}$$

3 Proof for Independency

$$\overline{X} = \frac{1}{5}\mathbf{u}^\top\mathbf{x} \tag{3.0.1}$$

since $\mathbf{M}$ is symmetric and idempotent we have

$$T = \mathbf{x}^\top\mathbf{M}\mathbf{x} \tag{3.0.2}$$
$$= \mathbf{x}^\top\mathbf{M}\mathbf{M}\mathbf{x} \tag{3.0.3}$$
$$= \mathbf{x}^\top\mathbf{M}^\top\mathbf{M}\mathbf{x} \tag{3.0.4}$$
$$= (\mathbf{M}\mathbf{x})^\top(\mathbf{M}\mathbf{x}) \tag{3.0.5}$$

From (3.0.1) it can be seen that $\overline{X}$ depends only on $\mathbf{u}^\top\mathbf{x}$ and from (3.0.5) it can be seen that T depends only on $\mathbf{M}\mathbf{x}$. So if $\mathbf{M}\mathbf{x}$ and $\mathbf{u}^\top\mathbf{x}$ are independent then T and $\overline{X}$ are also independent

3.1 Identity

$$(\mathbf{u}^\top\mathbf{x})(\mathbf{x}^\top\mathbf{v}) = \mathbf{u}^\top(\mathbf{x}\mathbf{x}^\top)\mathbf{v} \tag{3.1.1}$$

This identity is verified using python.

3.2 Cross-Covariance

$$Cov[\mathbf{x},\mathbf{y}] = E[(\mathbf{x} - E[\mathbf{x}])(\mathbf{y} - E[\mathbf{y}])^\top] \tag{3.2.1}$$

Two jointly normal vectors are independent if and only if their cross-covariance is zero.

$$Cov[\mathbf{u}^\top\mathbf{x},\mathbf{M}\mathbf{x}] = E[(\mathbf{u}^\top\mathbf{x} - E[\mathbf{u}^\top\mathbf{x}])(\mathbf{M}\mathbf{x} - E[\mathbf{M}\mathbf{x}])^\top] \tag{3.2.2}$$

using the identity stated in section 3.1 we get

$$Cov[\mathbf{u}^\top\mathbf{x},\mathbf{M}\mathbf{x}] = \mathbf{u}^\top E[(\mathbf{x} - E[\mathbf{x}])(\mathbf{x} - E[\mathbf{x}])^\top]\mathbf{M}^\top \tag{3.2.3}$$
$$= \mathbf{u}^\top Var[\mathbf{x}]\mathbf{M} \tag{3.2.4}$$
$$= \mathbf{u}^\top\mathbf{M} \tag{3.2.5}$$
$$= 0 \tag{3.2.6}$$

So $\mathbf{M}\mathbf{x}$ and $\mathbf{u}^\top\mathbf{x}$ are independent.This implies $\overline{X}$ and T are independent.