My Presentation

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April 3, 2021

Fourier Transforms

There are several common conventions for defining the Fourier transform of an integrable function $f : \mathbb{R} \to \mathbb{C}$.One of them is

$$F(t) = \int_{-\infty}^{\infty} f(x) e^{-i2\pi xt} dx \tag{1}$$

and the inverse Fourier transform of F is

$$f(x) = \int_{-\infty}^{\infty} F(t) e^{i2\pi xt} dt \tag{2}$$

Theorem

Let $g : \mathbb{R} \to \mathbb{C}$ and $G : \mathbb{R} \to \mathbb{C}$ be two functions such that

$$g(x) \overset{\mathcal{F}}{\rightleftharpoons} G(t) \tag{3}$$

$$\implies G(t) \overset{\mathcal{F}}{\rightleftharpoons} g(-x) \tag{4}$$

where $\overset{\mathcal{F}}{\rightleftharpoons}$ represents Fourier transform and

$$G(t) = \int_{-\infty}^{\infty} g(x) e^{-i2\pi xt} dx \tag{5}$$

Proof

$$g(x) = \int_{-\infty}^{\infty} G(t)e^{i2\pi xt}dt \tag{6}$$

changing $x \to -x$

$$g(-x) = \int_{-\infty}^{\infty} G(t)e^{-i2\pi xt}dt \tag{7}$$

Comparing with (5)

$$\implies G(t) \overset{\mathcal{F}}{\rightleftharpoons} g(-x) \tag{8}$$

Some Useful Functions

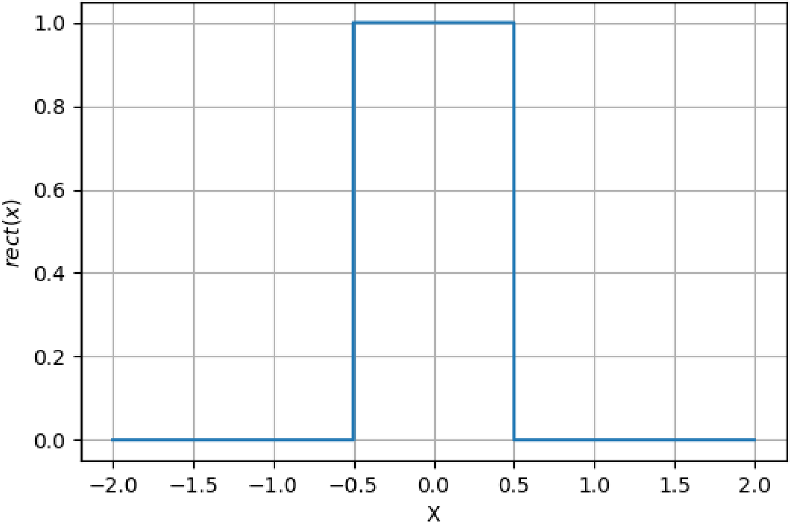
Rectangular Function

$$rect\left(\frac{x}{\tau}\right) = \begin{cases} 1 & , \frac{-\tau}{2} < x < \frac{\tau}{2} \\ \frac{1}{2} & , |x| = \frac{\tau}{2} \\ 0 & , otherwise \end{cases} \tag{9}$$

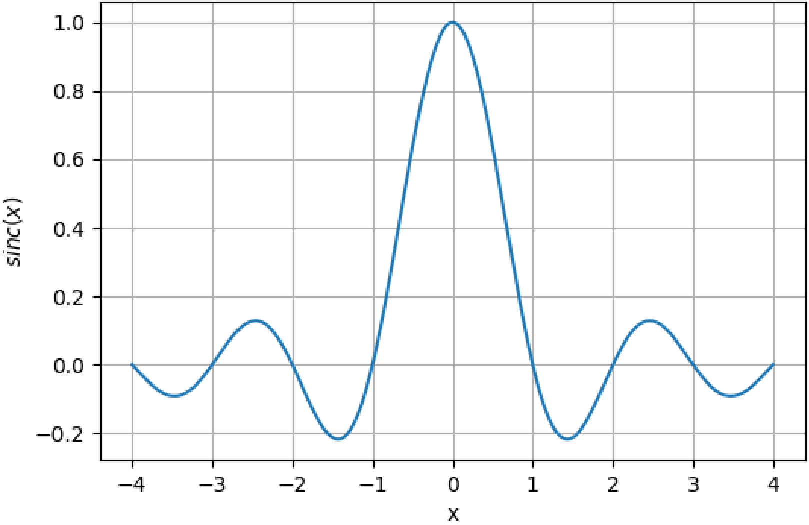
Sinc function

$$Sinc(x) = \begin{cases} 1 & x = 0 \\ \frac{\sin \pi x}{\pi x} & otherwise \end{cases} \tag{10}$$

Figures



Figures



Fourier transform of rectangular function

$$\int_{-\infty}^{\infty} rect\left(\frac{x}{\tau}\right) e^{-i2\pi xt} dx = \int_{\frac{-\tau}{2}}^{\frac{\tau}{2}} e^{-i2\pi xt} dx \tag{11}$$

$$= \left[\frac{e^{-i2\pi xt}}{-i2\pi t}\right]_{\frac{-\tau}{2}}^{\frac{\tau}{2}} \tag{12}$$

$$= \frac{e^{-i\pi t\tau} - e^{i\pi t\tau}}{-i2\pi t} \tag{13}$$

$$= \tau \frac{\sin \pi t\tau}{\pi t\tau} \tag{14}$$

$$= \tau sinc(t\tau) \tag{15}$$

So

$$rect\left(\frac{x}{\tau}\right) \overset{\mathcal{F}}{\rightleftharpoons} \tau sinc(t\tau) \tag{16}$$

Question

(GATE 2020(ST) Q16)

The characteristic function of a random variable X is given by

$$\phi_X(t) = \begin{cases} \frac{\sin t \cos t}{t} & t \neq 0 \\ 1 & t = 0 \end{cases} \tag{17}$$

Then $P\left(|X| \leq \frac{3}{2}\right) =$

Solution

The characteristic function of a random variable is defined as

$$\phi_X(t) = \int_{-\infty}^{\infty} f_X(x) e^{itx} dx \tag{18}$$

where $f_X(x)$ is pdf of X.
eq(18) is one of the conventions of Fourier transform.So $\phi_X(t)$ is Fourier transform of $f_X(x)$.From Inversion Theorem

$$\implies f_X(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \phi_X\left(t\right) e^{-ixt} dt \tag{19}$$

Solution

We know that

$$rect\left(\frac{x}{\tau}\right) \overset{\mathcal{F}}{\rightleftharpoons} \tau sinc(t\tau) \tag{20}$$

from (4) we have

$$\tau sinc(t\tau) \overset{\mathcal{F}}{\rightleftharpoons} rect\left(-\frac{x}{\tau}\right) \tag{21}$$

$$\implies rect\left(-\frac{x}{\tau}\right) = \int_{-\infty}^{\infty} \tau \frac{\sin \pi t\tau}{\pi t\tau} e^{-i2\pi xt} dt \tag{22}$$

substituting $\tau = \frac{2}{\pi}$ and changing $2\pi x \to x$ we get

$$\frac{1}{4} rect\left(\frac{-x}{4}\right) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(\frac{\sin 2t}{2t}\right) e^{-jxt} dt \tag{23}$$

Solution

From (23) the pdf is given by

$$f_X(x) = \frac{1}{4} rect\left(\frac{-x}{4}\right) dx \tag{24}$$

So

$$P\left(|X| \leq \frac{3}{2}\right) = \int_{-\frac{3}{2}}^{\frac{3}{2}} \frac{1}{4} rect\left(\frac{-x}{4}\right) dx \tag{25}$$

$$= \int_{\frac{-3}{2}}^{\frac{3}{2}} \frac{1}{4} dx \tag{26}$$

$$= \frac{3}{4} \tag{27}$$