Assignment 8

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Download all python codes from

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https://github.com/Adarsh541/AI1103-prob-and-ranvar/blob/main/Assignment8.1/codes/Assignment8.1.py
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and latex-tikz codes from

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https://github.com/Adarsh541/AI1103-prob-and-ranvar/blob/main/Assignment8.1/Assignment8.1.tex
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1 Problem

Let X_1, X_2, X_3, X_4, X_5 be a random sample of size 5 from a population having standard normal distribution.If $\overline{X} = \frac{1}{5}\sum_{i=1}^{5} X_i$ and $T = \sum_{i=1}^{5}\left(X_i - \overline{X}\right)^2$ then $E[T^2\overline{X}^2]$ is equal to

1) 3
2) 3.6
3) 4.8
4) 5.2

2 Solution

Let $\mathbf{x} \sim N(\mathbf{0}, \mathbf{I})$ be a standard normal random vector

$$\mathbf{x} = \begin{pmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \\ X_5 \end{pmatrix} \tag{2.0.1}$$

Then $\overline{X}$ can be written as

$$\overline{X} = \frac{1}{5}\mathbf{u}^\top \mathbf{x} \tag{2.0.2}$$

where

$$\mathbf{u} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \tag{2.0.3}$$

$$T = \mathbf{x}^\top \mathbf{M}\mathbf{x} \tag{2.0.4}$$

where

$$\mathbf{M} = \begin{pmatrix} \frac{4}{5} & -1/5 & -1/5 & -1/5 & -1/5 \\ -1/5 & \frac{4}{5} & -1/5 & -1/5 & -1/5 \\ -1/5 & -1/5 & \frac{4}{5} & -1/5 & -1/5 \\ -1/5 & -1/5 & -1/5 & \frac{4}{5} & -1/5 \\ -1/5 & -1/5 & -1/5 & -1/5 & \frac{4}{5} \end{pmatrix} \tag{2.0.5}$$

$$\tag{2.0.6}$$

we also have

$$\mathbf{M}^2 = \mathbf{M} \tag{2.0.7}$$

Definition 2.1. *chi-square distribution*
Let $X_1, X_2, ..X_k$ be i.i.d standard normal random variables.Define a random variable Y as

$$Y = X_1^2 + X_2^2 + ... + X_k^2 \tag{2.0.8}$$

We say Y is chi-square distributed with k degrees of freedom.The mean and variance is given by

$$E[Y] = k \tag{2.0.9}$$

$$Var[Y] = 2k \tag{2.0.10}$$

Lemma 2.1.

$$\left(\mathbf{u}^\top \mathbf{x}\right)\left(\mathbf{x}^\top \mathbf{v}\right) = \mathbf{u}^\top \left(\mathbf{x}\mathbf{x}^\top\right)\mathbf{v} \tag{2.0.11}$$

This lemma is verified using python simulation.

Theorem 2.2. *If the random variables X and Y are independent,then the random variables $Z = g(X)$ and $W = h(Y)$ are also independent.*

Proof. We denote by A_z the set of points such that $g(X) \le z$ and by B_w the set of points such that $h(Y) \le w$.Clearly,

$$\{Z \le z\} = \{X \in A_z\} \tag{2.0.12}$$

$$\{W \le w\} = \{Y \in B_w\} \tag{2.0.13}$$

Therefore the events $\{Z \le z\}$ and $\{W \le w\}$ are independent because the events $\{X \in A_z\}$ and $\{Y \in B_w\}$ are independent. □

Definition 2.2 (cross-covariance)**.**

$$Cov[\mathbf{x}, \mathbf{y}] = E[(\mathbf{x} - E[\mathbf{x}])(\mathbf{y} - E[\mathbf{y}])^\top] \tag{2.0.14}$$

Lemma 2.3. *Two jointly normal vectors are independent if and only if their cross-covariance is zero.*

Theorem 2.4. *Let* $\mathbf{x}$ *be a* 5×1 *standard multivariate normal random vector.Let* $\mathbf{B}$ *be an* $l\times5$ *real matrix.Then the* $l\times1$ *random vector* $\mathbf{y}$ *defined by* $\mathbf{y} = \mathbf{Bx}$ *has multivariate normal distribution with mean* $E[\mathbf{y}] = \mathbf{0}$ *and covariance matrix* $Var[\mathbf{y}] = \mathbf{BB}^\top$

Proof. The joint moment generating function of $\mathbf{x}$ is

$$M_{\mathbf{x}}(\mathbf{t}) = exp\left(\mathbf{t}^\top\mu + \frac{1}{2}\mathbf{t}^\top\mathbf{Vt}\right) \tag{2.0.15}$$

since for standard normal distribution $\mu = \mathbf{0}$ and $\mathbf{V} = \mathbf{I}$.So

$$M_{\mathbf{x}}(\mathbf{t}) = exp\left(\frac{1}{2}\mathbf{t}^\top\mathbf{It}\right) \tag{2.0.16}$$

Therefore the joint moment generating function of $\mathbf{y}$ is

$$M_{\mathbf{y}}(\mathbf{t}) = M_{\mathbf{x}}\left(\mathbf{B}^\top\mathbf{t}\right) \tag{2.0.17}$$

$$= exp\left(\frac{1}{2}\mathbf{t}^\top\mathbf{BB}^\top\mathbf{t}\right) \tag{2.0.18}$$

on comparing with (2.0.16) we can say $\mathbf{y}$ has multivariate normal distribution. □

Theorem 2.5. *Let* $\mathbf{x}$ *be a* 5×1 *standard multivariate normal random vector.Let* $\mathbf{A}$ *,*$\mathbf{B}$ *be two matrices.Define*

$$\mathbf{T_1} = \mathbf{Ax} \tag{2.0.19}$$

$$\mathbf{T_2} = \mathbf{Bx} \tag{2.0.20}$$

Then $\mathbf{T_1}$ *and* $\mathbf{T_2}$ *are two independent random vectors if and only if* $\mathbf{AB}^\top = 0$

Proof. From theorem 2.4,$\mathbf{T_1}$ and $\mathbf{T_2}$ are jointly normal.Their cross-covariance is

$$Cov[\mathbf{T_1},\mathbf{T_2}] = E[(\mathbf{T_1} - E[\mathbf{T_1}])(\mathbf{T_2} - E[\mathbf{T_2}])^\top] \tag{2.0.21}$$

$$= E[(\mathbf{Ax} - E[\mathbf{Ax}])(\mathbf{Bx} - E[\mathbf{Bx}])^\top] \tag{2.0.22}$$

$$= \mathbf{A}E[(\mathbf{x} - E[\mathbf{x}])(\mathbf{x} - E[\mathbf{x}])^\top]\mathbf{B}^\top \tag{2.0.23}$$

$$= \mathbf{A}Var[x]\mathbf{B}^\top \tag{2.0.24}$$

$$= \mathbf{AB}^\top \tag{2.0.25}$$

So $\mathbf{T_1}$ and $\mathbf{T_2}$ are independent if and only if $\mathbf{AB}^\top = 0$ □

Theorem 2.6. *Let* $\overline{X}$ *be the sample mean of size 5 from a standard normal distribution.Then*

1) $\overline{X} \sim N(0, \frac{1}{5})$
2) $\overline{X}$ *and* T *are independent.*
3) $T \sim \chi_4^2$

where χ_4^2 *is chi-square distribution with 4 degrees of freedom and* T *is defined as*

$$T = \sum_{i=1}^{5}(X_i - \overline{X})^2 \tag{2.0.26}$$

Proof. 1)

$$\overline{X} = \frac{1}{5}\mathbf{u}^\top\mathbf{x} \tag{2.0.27}$$

From theorem 2.4 we can say $\overline{X}$ has normal distribution with mean $E[\overline{X}] = \mathbf{0}$ and covariance matrix

$$Var\left[\overline{X}\right] = \frac{1}{25}\mathbf{u}^\top\mathbf{u} \tag{2.0.28}$$

$$= \frac{1}{5} \tag{2.0.29}$$

2) since $\mathbf{M}$ is symmetric and idempotent we have

$$T = \mathbf{x}^\top\mathbf{Mx} \tag{2.0.30}$$

$$= \mathbf{x}^\top\mathbf{MMx} \tag{2.0.31}$$

$$= \mathbf{x}^\top\mathbf{M}^\top\mathbf{Mx} \tag{2.0.32}$$

$$= (\mathbf{Mx})^\top(\mathbf{Mx}) \tag{2.0.33}$$

$$Cov\left[\frac{1}{5}\mathbf{u}^\top\mathbf{x},\mathbf{Mx}\right] = E[\left(\frac{1}{5}\mathbf{u}^\top\mathbf{x} - E[\frac{1}{5}\mathbf{u}^\top\mathbf{x}]\right)(\mathbf{Mx} - E[\mathbf{Mx}])^\top] \tag{2.0.34}$$

Using lemma 2.1 we get

$$Cov\left[\frac{1}{5}\mathbf{u}^\top\mathbf{x},\mathbf{Mx}\right] = \frac{1}{5}\mathbf{u}^\top E[(\mathbf{x} - E[\mathbf{x}])(\mathbf{x} - E[\mathbf{x}])^\top]\mathbf{M}^\top \tag{2.0.35}$$

$$= \frac{1}{5}\mathbf{u}^\top Var[\mathbf{x}]\mathbf{M} \tag{2.0.36}$$

$$= \frac{1}{5}\mathbf{u}^\top\mathbf{M} \tag{2.0.37}$$

$$= 0 \tag{2.0.38}$$

So $\mathbf{Mx}$ and $\frac{1}{5}\mathbf{u}^\top\mathbf{x}$ are independent.From theorem 2.2(functions of two independent variables are also independent), we can say $\overline{X}$ and T(A function of $\mathbf{Mx}$)are independent

3) Since $\mathbf{M}$ is symmetric it can be expressed as

$$\mathbf{M} = \mathbf{PDP}^{\top} \tag{2.0.39}$$

where $\mathbf{P}$ is orthogonal and $\mathbf{D}$ is diagonal.Then

$$T = \mathbf{x}^{\top}\mathbf{Mx} \tag{2.0.40}$$
$$= \mathbf{x}^{\top}\mathbf{PDP}^{\top}\mathbf{x} \tag{2.0.41}$$
$$= \left(\mathbf{P}^{\top}\mathbf{x}\right)^{\top}\mathbf{DP}^{\top}\mathbf{x} \tag{2.0.42}$$
$$= \mathbf{y}^{\top}\mathbf{Dy} \tag{2.0.43}$$

where $\mathbf{y} = \mathbf{P}^{\top}\mathbf{x}$.Since $\mathbf{x}$ is standard normal, from theorem 2.4 we can say $\mathbf{y}$ is also jointly normal with

$$E[\mathbf{y}] = 0 \tag{2.0.44}$$
$$Var[\mathbf{y}] = \mathbf{P}^{\top}\left(\mathbf{P}^{\top}\right)^{\top} \tag{2.0.45}$$
$$= \mathbf{P}^{\top}\mathbf{P} \tag{2.0.46}$$
$$= \mathbf{I} \qquad \text{(Since P is orthogonal)} \tag{2.0.47}$$

So $\mathbf{y} \sim N(0, \mathbf{I})$ is standard normal.

a) The eigen values of $\mathbf{M}$ are $1, 1, 1, 1, 0$.So $\mathbf{D}$ can be written as

$$\mathbf{D} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{2.0.48}$$

Let $\mathbf{v_1}, .., \mathbf{v_5}$ be corresponding eigen vectors.Then $\mathbf{P} = (\mathbf{v_1}, .., \mathbf{v_5})$.Since $\mathbf{P}$ is orthogonal the dot product of any two eigen vectors is zero.i.e

$$\mathbf{v_i}^{\top}\mathbf{v_j} = 0 \tag{2.0.49}$$

for any $i \neq j$

$$\mathbf{y} = \mathbf{P}^{\top}\mathbf{x} \tag{2.0.50}$$

$$\implies \mathbf{y} = \begin{pmatrix} y_1 = \mathbf{v_1}^{\top}\mathbf{x} \\ y_2 = \mathbf{v_2}^{\top}\mathbf{x} \\ y_3 = \mathbf{v_3}^{\top}\mathbf{x} \\ y_4 = \mathbf{v_4}^{\top}\mathbf{x} \\ y_5 = \mathbf{v_5}^{\top}\mathbf{x} \end{pmatrix} \tag{2.0.51}$$

(2.0.52)

From (2.0.49) and theorem 2.5,it follows y_1, y_2, y_3, y_4, y_5 are mutually independent.

$$T = \mathbf{y}^{\top}\mathbf{Dy} \tag{2.0.53}$$
$$\implies T = y_1^2 + y_2^2 + y_3^2 + y_4^2 \tag{2.0.54}$$

So T is sum of squares of four independent standard normal variables which is chi-square distribution with 4 degrees of freedom.

□

$$E[T^2\overline{X}^2] = E[T^2]E[\overline{X}^2] \tag{2.0.55}$$

$$E[\overline{X}^2] = Var\left[\overline{X}\right] + \left(E[\overline{X}]\right)^2 \tag{2.0.56}$$
$$= \frac{1}{5} \tag{2.0.57}$$

since T is chi-squared distributed with 4 degrees of freedom

$$E[T] = 4 \tag{2.0.58}$$
$$Var[T] = 8 \tag{2.0.59}$$
$$\implies E[T^2] = Var[T] + (E[T])^2 \tag{2.0.60}$$
$$= 24 \tag{2.0.61}$$

From (2.0.55)

$$E[T^2\overline{X}^2] = 4.8 \tag{2.0.62}$$