My Presentation

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chi-square distribution

Let $X_1, X_2, ..X_k$ be i.i.d standard normal random variables.Define a random variable Y as

$$Y = X_1^2 + X_2^2 + ... + X_k^2 \tag{1}$$

We say Y is chi-square distributed with k degrees of freedom.The mean and variance is given by

$$E[Y] = k \tag{2}$$
$$Var[Y] = 2k \tag{3}$$

cross-covariance

$$Cov[\mathrm{x}, \mathrm{y}] = E[(\mathrm{x} - E[\mathrm{x}])(\mathrm{y} - E[\mathrm{y}])^\top] \tag{4}$$

Theorem-1

Let x be a $k \times 1$ standard multivariate normal random vector.Let B be an $l \times k$ real matrix.Then the $l \times 1$ random vector y defined by $y = Bx$ has multivariate normal distribution with mean $E[y] = 0$ and covariance matrix $Var[y] = BB^\top$

PROOF:

The joint moment generating function of x is

$$M_x(t) = exp\left(t^\top \mu + \frac{1}{2}t^\top Vt\right) \tag{5}$$

since for standard normal distribution $\mu = 0$ and $V = I$.So

$$M_x(t) = exp\left(\frac{1}{2}t^\top It\right) \tag{6}$$

Therefore the joint moment generating function of y is

$$M_{\mathrm{y}}(\mathrm{t}) = M_{\mathrm{x}}\left(\mathrm{B}^{\top}\mathrm{t}\right) \tag{7}$$

$$= exp\left(\frac{1}{2}\mathrm{t}^{\top}\mathrm{B}\mathrm{B}^{\top}\mathrm{t}\right) \tag{8}$$

on comparing with (6) we can say y has multivariate normal distribution.

Theorem-2

Let x be a $k \times 1$ standard multivariate normal random vector.Let A ,B be two matrices.Define

$$\mathrm{T}_1 = \mathrm{Ax} \tag{9}$$

$$\mathrm{T}_2 = \mathrm{Bx} \tag{10}$$

Then T_1 and T_2 are two independent random vectors if and only if $\mathrm{AB}^{\top} = 0$

PROOF

From theorem-1, T_1 and T_2 are jointly normal vectors.Their cross-covariance is

$$\begin{aligned} Cov\left[T_1, T_2\right] &= E[(T_1 - E[T_1])\,(T_2 - E[T_2])^\top] && (11)\\ &= E[(Ax - E[Ax])\,(Bx - E[Bx])^\top] && (12)\\ &= AE[(x - E[x])\,(x - E[x])^\top]B^\top && (13)\\ &= A\,Var\left[x\right]B^\top && (14)\\ &= AB^\top && (15) \end{aligned}$$

Two jointly normal vectors are independent if and only if their cross-covariance is zero. So T_1 and T_2 are independent if and only if $AB^\top = 0$

Theorem-3

Let $\overline{X}$ be the sample mean of size 5 from a standard normal distribution.Then

1. $\overline{X} \sim N(0, \frac{1}{5})$
2. $\overline{X}$ and T are independent.
3. $T \sim \chi_4^2$

where χ_4^2 is chi-square distribution with 4 degrees of freedom and T(Sample variance) is given by

$$T = \sum_{i=1}^{5} (X_i - \overline{X})^2 \tag{16}$$

PROOF

$$\overline{X} = \frac{1}{5}\mathrm{u}^\top \mathrm{x} \tag{17}$$

From theorem-1 we can say $\overline{X}$ has normal distribution with mean $E[\overline{X}] = 0$ and covariance matrix

$$Var\left[\overline{X}\right] = \frac{1}{25}\mathrm{u}^\top \mathrm{u} \tag{18}$$

$$= \frac{1}{5} \tag{19}$$

$$T = \mathrm{x}^\top \mathrm{M}\mathrm{x} \tag{20}$$

where

$$\mathrm{M} = \begin{pmatrix} \frac{4}{5} & -1/5 & -1/5 & -1/5 & -1/5 \\ -1/5 & \frac{4}{5} & -1/5 & -1/5 & -1/5 \\ -1/5 & -1/5 & \frac{4}{5} & -1/5 & -1/5 \\ -1/5 & -1/5 & -1/5 & \frac{4}{5} & -1/5 \\ -1/5 & -1/5 & -1/5 & -1/5 & \frac{4}{5} \end{pmatrix} \tag{21}$$

$$\tag{22}$$

we also have

$$\mathrm{M}^2 = \mathrm{M} \tag{23}$$

$$
\begin{aligned}
T &= \mathrm{x}^\top \mathrm{M}\mathrm{x} && (24)\\
&= \mathrm{x}^\top \mathrm{M}\mathrm{M}\mathrm{x} && (25)\\
&= \mathrm{x}^\top \mathrm{M}^\top \mathrm{M}\mathrm{x} && (26)\\
&= (\mathrm{M}\mathrm{x})^\top (\mathrm{M}\mathrm{x}) && (27)
\end{aligned}
$$

$$
\begin{aligned}
Cov\left[\frac{1}{5}\mathrm{u}^\top \mathrm{x}, \mathrm{M}\mathrm{x}\right] &= E[\left(\frac{1}{5}\mathrm{u}^\top \mathrm{x} - E[\frac{1}{5}\mathrm{u}^\top \mathrm{x}]\right)(\mathrm{M}\mathrm{x} - E[\mathrm{M}\mathrm{x}])^\top] && (28)\\
&= \frac{1}{5}\mathrm{u}^\top E[(\mathrm{x} - E[\mathrm{x}])(\mathrm{x} - E[\mathrm{x}])^\top]\mathrm{M}^\top && (29)\\
&= \frac{1}{5}\mathrm{u}^\top Var[\mathrm{x}]\,\mathrm{M} && (30)\\
&= \frac{1}{5}\mathrm{u}^\top \mathrm{M} && (31)\\
&= 0 && (32)
\end{aligned}
$$

Lemma

Functions of independent random variables are themselves independent.

proof that $y = x^\top x$ is a function

Assume that it is not a function. So there exist $x_1 \in \mathbb{R}^n$, $a, b \in \mathbb{R}$, $a \neq b$ such that

$$x_1^\top x_1 = a \tag{33}$$

$$x_1^\top x_1 = b \tag{34}$$

But from (33) and (34) it can clearly be seen that

$$a = b \tag{35}$$

which is a contradiction. So our assumption is wrong. Therefore $y : \mathbb{R}^n \to \mathbb{R}$ is a function.

So Mx and $\frac{1}{5}\mathbf{u}^\top\mathbf{x}$ are independent.Since functions of two independent variables are also independent, we can say $\overline{X}$ and T(A function of Mx)are independent.
Since M is symmetric it can be expressed as

$$\mathbf{M} = \mathbf{P}\mathbf{D}\mathbf{P}^\top \tag{36}$$

where P is orthogonal and D is diagonal.Then

$$\begin{aligned} T &= \mathbf{x}^\top\mathbf{M}\mathbf{x} && (37)\\ &= \mathbf{x}^\top\mathbf{P}\mathbf{D}\mathbf{P}^\top\mathbf{x} && (38)\\ &= \left(\mathbf{P}^\top\mathbf{x}\right)^\top \mathbf{D}\mathbf{P}^\top\mathbf{x} && (39)\\ &= \mathbf{y}^\top\mathbf{D}\mathbf{y} && (40) \end{aligned}$$

where $\mathbf{y} = \mathbf{P}^\top\mathbf{x}$.Since x is standard normal, from theorem-1 we can say y is also jointly normal with

$$E[\mathrm{y}] = 0 \tag{41}$$

$$Var[\mathrm{y}] = \mathrm{P}^\top \left(\mathrm{P}^\top\right)^\top \tag{42}$$

$$= \mathrm{P}^\top \mathrm{P} \tag{43}$$

$$= \mathrm{I} \qquad \text{(Since P is orthogonal)} \tag{44}$$

So $\mathrm{y} \sim N(0, \mathrm{I})$ is standard normal
The eigen values of M are $1, 1, 1, 1, 0$.So D can be written as

$$\mathrm{D} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{45}$$

Let $\mathrm{v}_1, .., \mathrm{v}_5$ be corresponding eigen vectors.Then $\mathrm{P} = (\mathrm{v}_1, .., \mathrm{v}_5)$.Since P is orthogonal the dot product of any two eigen vectors is zero.i.e

$$\mathrm{v_i}^\top \mathrm{v_j} = 0 \qquad \text{for any } i \neq j \tag{46}$$

$$y = P^{\top}x \tag{47}$$

$$\implies y = \begin{pmatrix} y_1 = v_1{}^{\top}x \\ y_2 = v_2{}^{\top}x \\ y_3 = v_3{}^{\top}x \\ y_4 = v_4{}^{\top}x \\ y_5 = v_5{}^{\top}x \end{pmatrix} \tag{48}$$

$$\tag{49}$$

From (46) and theorem-2, it follows y_1, y_2, y_3, y_4, y_5 are mutually independent.

$$T = y^{\top}Dy \tag{50}$$

$$\implies T = y_1^2 + y_2^2 + y_3^2 + y_4^2 \tag{51}$$

So T is sum of squares of four independent standard normal variables which is chi-square distribution with 4 degrees of freedom.

Question

gov/stats/2019/STATS-P1-IESISS,(Q.25)

Let X_1, X_2, X_3, X_4, X_5 be a random sample of size 5 from a population having standard normal distribution.If $\overline{X} = \frac{1}{5}\sum_{i=1}^{5} X_i$ and $T = \sum_{i=1}^{5}\left(X_i - \overline{X}\right)^2$ then $E[T^2\overline{X}^2]$ is equal to

1. 3
2. 3.6
3. 4.8
4. 5.2

Solution

$$E[T^2\overline{X}^2] = E[T^2]E[\overline{X}^2] \tag{52}$$

$$E[\overline{X}^2] = Var\left[\overline{X}\right] + \left(E[\overline{X}]\right)^2 \tag{53}$$

$$= \frac{1}{5} \tag{54}$$

since T is chi-squared distributed with 4 degrees of freedom

$$E[T] = 4 \tag{55}$$

$$Var[T] = 8 \tag{56}$$

$$\implies E[T^2] = Var[T] + (E[T])^2 \tag{57}$$

$$= 24 \tag{58}$$

From (52)

$$E[T^2\overline{X}^2] = 4.8 \tag{59}$$