Assignment 8

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Download all python codes from

https://github.com/Adarsh541/AI1103−prob−and−ranvar/blob/main/Assignment8.1/codes/Assignment8.1.py

and latex-tikz codes from

https://github.com/Adarsh541/AI1103−prob−and−ranvar/blob/main/Assignment8.1/Assignment8.1.tex

1 Problem

Let X_1, X_2, X_3, X_4, X_5 be a random sample of size 5 from a population having standard normal distribution.If $\overline{X} = \frac{1}{5}\sum_{i=1}^{5} X_i$ and $T = \sum_{i=1}^{5}\left(X_i - \overline{X}\right)^2$ then $E[T^2\overline{X}^2]$ is equal to

1) 3
2) 3.6
3) 4.8
4) 5.2

2 Solution

Let $\mathbf{x} \sim N(\mathbf{0}, \mathbf{I})$ be a standard normal random vector

$$\mathbf{x} = \begin{pmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \\ X_5 \end{pmatrix} \tag{2.0.1}$$

Then $\overline{X}$ can be written as

$$\overline{X} = \frac{1}{5}\mathbf{u}^\top \mathbf{x} \tag{2.0.2}$$

where

$$\mathbf{u} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \tag{2.0.3}$$

$$T = \mathbf{x}^\top \mathbf{M} \mathbf{x} \tag{2.0.4}$$

where

$$\mathbf{M} = \begin{pmatrix} \frac{4}{5} & -1/5 & -1/5 & -1/5 & -1/5 \\ -1/5 & \frac{4}{5} & -1/5 & -1/5 & -1/5 \\ -1/5 & -1/5 & \frac{4}{5} & -1/5 & -1/5 \\ -1/5 & -1/5 & -1/5 & \frac{4}{5} & -1/5 \\ -1/5 & -1/5 & -1/5 & -1/5 & \frac{4}{5} \end{pmatrix} \tag{2.0.5}$$

(2.0.6)

we also have

$$\mathbf{M}^2 = \mathbf{M} \tag{2.0.7}$$

Definition 2.1. *chi-square distribution*
Let $X_1, X_2, ..X_k$ be i.i.d standard normal random variables.Define a random variable Y as

$$Y = X_1^2 + X_2^2 + ... + X_k^2 \tag{2.0.8}$$

We say Y is chi-square distributed with k degrees of freedom.The mean and variance is given by

$$E[Y] = k \tag{2.0.9}$$

$$Var[Y] = 2k \tag{2.0.10}$$

conjecture 2.1.

$$\left(\mathbf{u}^\top \mathbf{x}\right)\left(\mathbf{x}^\top \mathbf{v}\right) = \mathbf{u}^\top\left(\mathbf{x}\mathbf{x}^\top\right)\mathbf{v} \tag{2.0.11}$$

This conjecture is verified using python simulation.

conjecture 2.2. *Let $\mathbf{y}$ and $\mathbf{z}$ be two independent normal random vectors then $\mathbf{y}$ and $\|\mathbf{z}\|$ are also independent.*

Theorem 2.3. *Functions of independent random variables are themselves independent.*

Proof. A random variable X is a real-valued function defined on the "sample space" Ω (the set of outcomes being studied via probability).

1) A random variable X is studied by means of the probabilities that its value lies within various intervals of real numbers (or, more generally, sets constructed in simple ways out of intervals: these are the Borel measurable sets of real numbers).

2) Corresponding to any Borel measurable set I is the event $X^*(I)$ consisting of all outcomes ω for which $X(\omega)$ lies in I.
3) The sigma-algebra generated by X is determined by the collection of all such events.
4) The naive definition says two random variables X and Y are independent "when their probabilities multiply." That is, when I is one Borel measurable set and J is another, then

$$\Pr(X(\omega) \in I, Y(\omega) \in J) = \Pr(X(\omega) \in I) \Pr(Y(\omega) \in J). \quad (2.0.12)$$

5) But in the language of events (and sigma algebras) that's the same as

$$\Pr(\omega \in X^*(I), \omega \in Y^*(J)) = \Pr(\omega \in X^*(I)) \Pr(\omega \in Y^* J)). \quad (2.0.13)$$

Consider now two functions $f, g : \mathbb{R} \to \mathbb{R}$ and suppose that $f \circ X$ and $g \circ Y$ are random variables. (The circle is functional composition: $(f \circ X)(\omega) = f(X(\omega))$. This is what it means for f to be a "function of a random variable".) Notice–this is just elementary set theory–that

$$(f \circ X)^*(I) = X^*(f^*(I)) \quad (2.0.14)$$

In other words, every event generated by $f \circ X$ (which is on the left) is automatically an event generated by X (as exhibited by the form of the right hand side). Therefore (5) automatically holds for $f \circ X$ and $g \circ Y$: there's nothing to check! **Note**: You may replace "real-valued" everywhere by "with values in $\mathbb{R}^d$" without needing to change anything else in any material way. This covers the case of vector-valued random variables. □

Definition 2.2 (cross-covariance)**.**

$$Cov[\mathbf{x}, \mathbf{y}] = E[(\mathbf{x} - E[\mathbf{x}])(\mathbf{y} - E[\mathbf{y}])^\top] \quad (2.0.15)$$

Lemma 2.4. *Two jointly normal vectors are independent if and only if their cross-covariance is zero.*

Theorem 2.5. *Let* $\mathbf{x}$ *be a* $k \times 1$ *standard multivariate normal random vector.Let* $\mathbf{B}$ *be an* $l \times k$ *real matrix.Then the* $l \times 1$ *random vector* $\mathbf{y}$ *defined by* $\mathbf{y} = \mathbf{Bx}$ *has multivariate normal distribution with mean* $E[\mathbf{y}] = \mathbf{0}$ *and covariance matrix* $Var[\mathbf{y}] = \mathbf{BB}^\top$

Proof. The joint moment generating function of $\mathbf{x}$ is

$$M_{\mathbf{x}}(\mathbf{t}) = exp\left(\mathbf{t}^\top \mu + \frac{1}{2}\mathbf{t}^\top \mathbf{V}\mathbf{t}\right) \quad (2.0.16)$$

since for standard normal distribution $\mu = \mathbf{0}$ and $\mathbf{V} = \mathbf{I}$.So

$$M_{\mathbf{x}}(\mathbf{t}) = exp\left(\frac{1}{2}\mathbf{t}^\top \mathbf{I}\mathbf{t}\right) \quad (2.0.17)$$

Therefore the joint moment generating function of $\mathbf{y}$ is

$$M_{\mathbf{y}}(\mathbf{t}) = M_{\mathbf{x}}(\mathbf{B}^\top \mathbf{t}) \quad (2.0.18)$$

$$= exp\left(\frac{1}{2}\mathbf{t}^\top \mathbf{BB}^\top \mathbf{t}\right) \quad (2.0.19)$$

on comparing with (2.0.17) we can say $\mathbf{y}$ has multivariate normal distribution. □

Theorem 2.6. *Let* $\mathbf{x}$ *be a* $k \times 1$ *standard multivariate normal random vector.Let* $\mathbf{A}$ *,*$\mathbf{B}$ *be two matrices.Define*

$$\mathbf{T_1} = \mathbf{Ax} \quad (2.0.20)$$

$$\mathbf{T_2} = \mathbf{Bx} \quad (2.0.21)$$

Then $\mathbf{T_1}$ *and* $\mathbf{T_2}$ *are two independent random vectors if and only if* $\mathbf{AB}^\top = 0$

Proof. From theorem 2.5,$\mathbf{T_1}$ and $\mathbf{T_2}$ are jointly normal.Their cross-covariance is

$$Cov[\mathbf{T_1}, \mathbf{T_2}] = E[(\mathbf{T_1} - E[\mathbf{T_1}])(\mathbf{T_2} - E[\mathbf{T_2}])^\top] \quad (2.0.22)$$

$$= E[(\mathbf{Ax} - E[\mathbf{Ax}])(\mathbf{Bx} - E[\mathbf{Bx}])^\top] \quad (2.0.23)$$

$$= \mathbf{A}E[(\mathbf{x} - E[\mathbf{x}])(\mathbf{x} - E[\mathbf{x}])^\top]\mathbf{B}^\top \quad (2.0.24)$$

$$= \mathbf{A}Var[x]\mathbf{B}^\top \quad (2.0.25)$$

$$= \mathbf{AB}^\top \quad (2.0.26)$$

So $\mathbf{T_1}$ and $\mathbf{T_2}$ are independent if and only if $\mathbf{AB}^\top = 0$ □

Theorem 2.7. *Let* $\overline{X}$ *be the sample mean of size 5 from a standard normal distribution.Then*

1) $\overline{X} \sim N(0, \frac{1}{5})$

Proof.

$$\overline{X} = \frac{1}{5}\mathbf{u}^\top \mathbf{x} \quad (2.0.27)$$

From theorem 2.5 we can say $\overline{X}$ has normal distribution with mean $E[\overline{X}] = \mathbf{0}$ and covariance matrix

$$Var\left[\overline{X}\right] = \frac{1}{25}\mathbf{u}^\top\mathbf{u} \tag{2.0.28}$$
$$= \frac{1}{5} \tag{2.0.29}$$

□

Theorem 2.8. *Let X_1, X_2, X_3, X_4, X_5 be a random sample of size 5 from a standard normal population.Define*

$$\overline{X} = \frac{1}{5}\sum_{i=1}^{5} X_i \tag{2.0.30}$$
$$T = \sum_{i=1}^{5}\left(X_i - \overline{X}\right)^2 \tag{2.0.31}$$

then $\overline{X}$ and T are independent.

Proof.

$$T = \mathbf{x}^\top\mathbf{M}\mathbf{x} \tag{2.0.32}$$

since $\mathbf{M}$ is symmetric and idempotent we have

$$T = \mathbf{x}^\top\mathbf{M}\mathbf{M}\mathbf{x} \tag{2.0.33}$$
$$= \mathbf{x}^\top\mathbf{M}^\top\mathbf{M}\mathbf{x} \tag{2.0.34}$$
$$= (\mathbf{M}\mathbf{x})^\top(\mathbf{M}\mathbf{x}) \tag{2.0.35}$$
$$= ||\mathbf{M}\mathbf{x}|| \tag{2.0.36}$$

$$Cov\left[\frac{1}{5}\mathbf{u}^\top\mathbf{x}, \mathbf{M}\mathbf{x}\right] = E[\left(\frac{1}{5}\mathbf{u}^\top\mathbf{x} - E[\frac{1}{5}\mathbf{u}^\top\mathbf{x}]\right) (\mathbf{M}\mathbf{x} - E[\mathbf{M}\mathbf{x}])^\top] \tag{2.0.37}$$

Using conjecture 2.1 we get

$$Cov\left[\frac{1}{5}\mathbf{u}^\top\mathbf{x}, \mathbf{M}\mathbf{x}\right] = \frac{1}{5}\mathbf{u}^\top E[(\mathbf{x} - E[\mathbf{x}])(\mathbf{x} - E[\mathbf{x}])^\top]\mathbf{M}^\top \tag{2.0.38}$$

$$= \frac{1}{5}\mathbf{u}^\top Var\left[\mathbf{x}\right]\mathbf{M} \tag{2.0.39}$$
$$= \frac{1}{5}\mathbf{u}^\top\mathbf{M} \tag{2.0.40}$$
$$= 0 \tag{2.0.41}$$

So $\mathbf{M}\mathbf{x}$ and $\frac{1}{5}\mathbf{u}^\top\mathbf{x}$ are independent.From conjecture 2.2 we can say $\overline{X}$ and T are independent. □

Theorem 2.9. *Let X_1, X_2, X_3, X_4, X_5 be a random sample of size 5 from a standard normal population.Then T is chi-square distributed with 4 degrees of freedom,i.e $T \sim \chi_4^2$*

Proof. Since $\mathbf{M}$ is symmetric it can be expressed as

$$\mathbf{M} = \mathbf{P}\mathbf{D}\mathbf{P}^\top \tag{2.0.42}$$

where $\mathbf{P}$ is orthogonal and $\mathbf{D}$ is diagonal.Then

$$T = \mathbf{x}^\top\mathbf{M}\mathbf{x} \tag{2.0.43}$$
$$= \mathbf{x}^\top\mathbf{P}\mathbf{D}\mathbf{P}^\top\mathbf{x} \tag{2.0.44}$$
$$= \left(\mathbf{P}^\top\mathbf{x}\right)^\top\mathbf{D}\mathbf{P}^\top\mathbf{x} \tag{2.0.45}$$
$$= \mathbf{y}^\top\mathbf{D}\mathbf{y} \tag{2.0.46}$$

where $\mathbf{y} = \mathbf{P}^\top\mathbf{x}$.Since $\mathbf{x}$ is standard normal, from theorem 2.5 we can say $\mathbf{y}$ is also jointly normal with

$$E[\mathbf{y}] = 0 \tag{2.0.47}$$
$$Var\left[\mathbf{y}\right] = \mathbf{P}^\top\left(\mathbf{P}^\top\right)^\top \tag{2.0.48}$$
$$= \mathbf{P}^\top\mathbf{P} \tag{2.0.49}$$
$$= \mathbf{I}\,(\text{Since P is orthogonal}) \tag{2.0.50}$$

So $\mathbf{y} \sim N\left(0, \mathbf{I}\right)$ is standard normal.

1) The eigen values of $\mathbf{M}$ are $1, 1, 1, 1, 0$.So $\mathbf{D}$ can be written as

$$\mathbf{D} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{2.0.51}$$

Let $\mathbf{v_1}, .., \mathbf{v_5}$ be corresponding eigen vectors.Then $\mathbf{P} = (\mathbf{v_1}, .., \mathbf{v_5})$.Since $\mathbf{P}$ is orthogonal the dot product of any two eigen vectors is zero.i.e

$$\mathbf{v_i}^\top\mathbf{v_j} = 0 \tag{2.0.52}$$

for any $i \neq j$

$$\mathbf{y} = \mathbf{P}^\top\mathbf{x} \tag{2.0.53}$$

$$\implies \mathbf{y} = \begin{pmatrix} y_1 = \mathbf{v_1}^\top\mathbf{x} \\ y_2 = \mathbf{v_2}^\top\mathbf{x} \\ y_3 = \mathbf{v_3}^\top\mathbf{x} \\ y_4 = \mathbf{v_4}^\top\mathbf{x} \\ y_5 = \mathbf{v_5}^\top\mathbf{x} \end{pmatrix} \tag{2.0.54}$$

(2.0.55)

From (2.0.52) and theorem 2.6,it follows

y_1, y_2, y_3, y_4, y_5 are mutually independent.

$$T = \mathbf{y}^\top \mathbf{D}\mathbf{y} \tag{2.0.56}$$

$$\implies T = y_1^2 + y_2^2 + y_3^2 + y_4^2 \tag{2.0.57}$$

So T is sum of squares of four independent standard normal variables which is chi-square distribution with 4 degrees of freedom.

□

$$E[T^2\overline{X}^2] = E[T^2]E[\overline{X}^2] \tag{2.0.58}$$

$$E[\overline{X}^2] = Var\left[\overline{X}\right] + \left(E[\overline{X}]\right)^2 \tag{2.0.59}$$

$$= \frac{1}{5} \tag{2.0.60}$$

since T is chi-squared distributed with 4 degrees of freedom

$$E[T] = 4 \tag{2.0.61}$$

$$Var[T] = 8 \tag{2.0.62}$$

$$\implies E[T^2] = Var[T] + (E[T])^2 \tag{2.0.63}$$

$$= 24 \tag{2.0.64}$$

From (2.0.58)

$$E[T^2\overline{X}^2] = 4.8 \tag{2.0.65}$$