Assignment 8

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Download all python codes from

https://github.com/Adarsh541/AI1103−prob−and−ranvar/blob/main/Assignment8.1/codes/Assignment8.1.py

and latex-tikz codes from

https://github.com/Adarsh541/AI1103−prob−and−ranvar/blob/main/Assignment8.1/Assignment8.1.tex

1 Problem

Let X_1, X_2, X_3, X_4, X_5 be a random sample of size 5 from a population having standard normal distribution.If $\overline{X} = \frac{1}{5}\sum_{i=1}^{5} X_i$ and $T = \sum_{i=1}^{5}\left(X_i - \overline{X}\right)^2$ then $E[T^2\overline{X}^2]$ is equal to

1) 3
2) 3.6
3) 4.8
4) 5.2

2 Solution

Let $\mathbf{x} \sim N(\mathbf{0}, \mathbf{I})$ be a standard normal random vector

$$\mathbf{x} = \begin{pmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \\ X_5 \end{pmatrix} \tag{2.0.1}$$

Then $\overline{X}$ can be written as

$$\overline{X} = \frac{1}{5}\mathbf{u}^\top\mathbf{x} \tag{2.0.2}$$

where

$$\mathbf{u} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \tag{2.0.3}$$

$$T = \mathbf{x}^\top\mathbf{M}\mathbf{x} \tag{2.0.4}$$

where

$$\mathbf{M} = \begin{pmatrix} \frac{4}{5} & -1/5 & -1/5 & -1/5 & -1/5 \\ -1/5 & \frac{4}{5} & -1/5 & -1/5 & -1/5 \\ -1/5 & -1/5 & \frac{4}{5} & -1/5 & -1/5 \\ -1/5 & -1/5 & -1/5 & \frac{4}{5} & -1/5 \\ -1/5 & -1/5 & -1/5 & -1/5 & \frac{4}{5} \end{pmatrix} \tag{2.0.5}$$

(2.0.6)

we also have

$$\mathbf{M}^2 = \mathbf{M} \tag{2.0.7}$$

Lemma 2.1.

$$\left(\mathbf{u}^\top\mathbf{x}\right)\left(\mathbf{x}^\top\mathbf{v}\right) = \mathbf{u}^\top\left(\mathbf{x}\mathbf{x}^\top\right)\mathbf{v} \tag{2.0.8}$$

This lemma is verified using python simulation.

Definition 2.1 (cross-covariance)**.**

$$Cov\left[\mathbf{x}, \mathbf{y}\right] = E[(\mathbf{x} - E[\mathbf{x}])\left(\mathbf{y} - E[\mathbf{y}]\right)^\top] \tag{2.0.9}$$

Theorem 2.2. *Let $\overline{X}$ be the sample mean of size 5 from a standard normal distribution.Then*

1) $\overline{X} \sim N(0, \frac{1}{5})$
2) *$\overline{X}$ and T are independent.*
3) $T \sim \chi_4^2$

where χ_4^2 is chi-square distribution with 4 degrees of freedom and T is defined as

$$T = \sum_{i=1}^{5}(X_i - \overline{X})^2 \tag{2.0.10}$$

Proof.

$$\overline{X} = \frac{1}{5}\mathbf{u}^\top\mathbf{x} \tag{2.0.11}$$

since $\mathbf{M}$ is symmetric and idempotent we have

$$\begin{aligned} T &= \mathbf{x}^\top\mathbf{M}\mathbf{x} && (2.0.12)\\ &= \mathbf{x}^\top\mathbf{M}\mathbf{M}\mathbf{x} && (2.0.13)\\ &= \mathbf{x}^\top\mathbf{M}^\top\mathbf{M}\mathbf{x} && (2.0.14)\\ &= \left(\mathbf{M}\mathbf{x}\right)^\top\left(\mathbf{M}\mathbf{x}\right) && (2.0.15) \end{aligned}$$

From (2.0.11) it can be seen that $\overline{X}$ depends only on $\mathbf{u}^\top\mathbf{x}$ and from (2.0.15) it can be seen that T depends only on $\mathbf{Mx}$. So if $\mathbf{Mx}$ and $\mathbf{u}^\top\mathbf{x}$ are independent then T and $\overline{X}$ are also independent.

$$Cov\left[\mathbf{u}^\top\mathbf{x}, \mathbf{Mx}\right] = E[(\mathbf{u}^\top\mathbf{x} - E[\mathbf{u}^\top\mathbf{x}]) \\ (\mathbf{Mx} - E[\mathbf{Mx}])^\top] \tag{2.0.16}$$

From the lemma stated above we get

$$Cov\left[\mathbf{u}^\top\mathbf{x}, \mathbf{Mx}\right] = \mathbf{u}^\top E[(\mathbf{x} - E[\mathbf{x}])(\mathbf{x} - E[\mathbf{x}])^\top]\mathbf{M}^\top \tag{2.0.17}$$

$$= \mathbf{u}^\top Var\left[\mathbf{x}\right]\mathbf{M} \tag{2.0.18}$$

$$= \mathbf{u}^\top\mathbf{M} \tag{2.0.19}$$

$$= 0 \tag{2.0.20}$$

Two jointly normal vectors are independent if and only if their cross-covariance is zero.So $\mathbf{Mx}$ and $\mathbf{u}^\top\mathbf{x}$ are independent.This implies $\overline{X}$ and T are independent. □

$$E[T^2\overline{X}^2] = E[T^2]E[\overline{X}^2] \tag{2.0.21}$$

$$E[\overline{X}^2] = Var\left[\overline{X}\right] + \left(E[\overline{X}]\right)^2 \tag{2.0.22}$$

$$= \frac{1}{5} \tag{2.0.23}$$

since T is chi-squared distributed with 4 degrees of freedom

$$E[T] = 4 \tag{2.0.24}$$

$$Var\left[T\right] = 8 \tag{2.0.25}$$

$$\implies E[T^2] = Var\left[T\right] + (E[T])^2 \tag{2.0.26}$$

$$= 24 \tag{2.0.27}$$

From (2.0.21)

$$E[T^2\overline{X}^2] = 4.8 \tag{2.0.28}$$