GATE 2021 EC Q4

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September 7, 2021

Dirac-delta Impulse

Dirac-delta Impulse

$$\delta(t) = \begin{cases} \infty, & t = 0 \\ 0, & otherwise \end{cases} \tag{1}$$

Shifting property of $\delta(t)$

If $g(t)$ is a continuous and finite function at $t = a$ then

$$\int_{-\infty}^{\infty} \delta(t-a)\, g(t)\, dt = g(a) \tag{2}$$

Theorem-1

Fourier transform of shifted impulse is the complex exponential.

$$G(f) = \mathcal{F}\left\{\delta(t-a)\right\} = e^{-i2\pi fa} \tag{3}$$

Proof

$$G(f) = \int_{-\infty}^{\infty} \delta(t-a)\, e^{-i2\pi ft}\, dt \tag{4}$$

$$= e^{-i2\pi fa} \tag{5}$$

Corollary-1

Inverse Fourier Transform of the complex exponential must be the shifted impulse. So

$$\mathcal{F}^{-1}\left\{e^{-2\pi fa}\right\} = \int_{-\infty}^{\infty} e^{-2\pi fa} e^{i2\pi ft} df \tag{6}$$

$$= \int_{-\infty}^{\infty} e^{i2\pi f(t-a)} df \tag{7}$$

$$= \int_{-\infty}^{\infty} e^{-i2\pi f(t-a)} df \tag{8}$$

$$\implies \int_{-\infty}^{\infty} e^{-i2\pi f(t-a)} df = \delta(t-a) \tag{9}$$

Theorem-2

The Fourier transform of $g(t) = e^{i2\pi at}$ is given by

$$G(f) = \mathcal{F}\left\{e^{i2\pi at}\right\} = \delta(f-a) \tag{10}$$

Proof

$$G(f) = \int_{-\infty}^{\infty} e^{i2\pi at} e^{-i2\pi ft} dt \tag{11}$$

$$= \int_{-\infty}^{\infty} e^{i2\pi t(a-f)} dt \tag{12}$$

$$= \delta(f-a) \tag{13}$$

Linearity of Fourier Transform

$$\mathcal{F}\left\{c_1 g\left(t\right) + c_2 h\left(t\right)\right\} = c_1 \mathcal{F}\left\{g\left(t\right)\right\} + c_2 \mathcal{F}\left\{h\left(t\right)\right\} \tag{14}$$

Lemma-1

Let $x\left(t\right)$ be a signal, its Fourier Transform be of the form

$$G_x\left(f\right) = c_1 \delta\left(f - a_1 A\right) + c_2 \delta\left(f - a_2 A\right) + \ldots \tag{15}$$

where $c_i \in \mathbb{C}$ and $a_i \in \mathbb{R}$. Then the frequencies present in the signal are $a_j A$ where $a_j \in \mathbb{R}^+$

Fourier Transform of Cosine function

Let $x\left(t\right) = \cos\left(2\pi At\right)$, where $A = 10kHz$.

$$\cos\left(2\pi At\right) = \frac{e^{i2\pi At} + e^{-i2\pi At}}{2} \tag{16}$$

The Fourier transform of $x\left(t\right)$

$$G_x\left(f\right) = \int_{-\infty}^{\infty} \frac{e^{i2\pi At} + e^{-i2\pi At}}{2} e^{-i2\pi ft} dt \tag{17}$$

$$= \frac{1}{2}\left[\int_{-\infty}^{\infty} e^{-i2\pi t(f-A)} dt + \int_{-\infty}^{\infty} e^{-i2\pi t(A+F)}\right] \tag{18}$$

$$= \frac{1}{2}\left[\delta\left(f - A\right) + \delta\left(f + A\right)\right] \tag{19}$$

$\therefore$ All the energy of the sinusoidal wave is entirely localized at the frequencies given by $\left|f\right| = A$.

Question

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Consider a real-valued base-band signal $x(t)$, band limited to 10 kHz. The Nyquist rate for the signal $y(t) = x(t)\,x\left(1 + \frac{t}{2}\right)$ is

1. 15 kHz
2. 30 kHz
3. 60 kHz
4. 20 kHz

Solution

$$y(t) = \cos(2\pi At)\cos\left(2\pi A\left(1+\frac{t}{2}\right)\right) \quad (20)$$

$$= \cos(2\pi A + 3\pi At) + \cos(2\pi A - \pi At) \quad (21)$$

$$= \cos(3\pi At) + \cos(\pi At) \quad (22)$$

Using the linearity of Fourier Transform. Fourier Transform of $y(t)$ is given by

$$G_y(f) = \frac{1}{2}\left[\delta\left(f-\frac{3A}{2}\right)+\delta\left(f+\frac{3A}{2}\right)\right] + \frac{1}{2}\left[\delta\left(f-\frac{A}{2}\right)+\delta\left(f+\frac{A}{2}\right)\right] \quad (23)$$

$$x(t) = \cos(20k\pi t) \tag{24}$$

$$\text{bandwidth of } x(t) = 10kHz \tag{25}$$

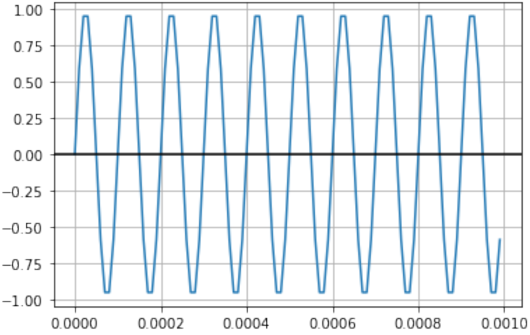

Figure: $x(t)$:Sinusoidal signal with freq=10kHz

$$x\left(1+\frac{t}{2}\right) = \cos\left(20k\pi + 10k\pi t\right) \tag{26}$$

$$\text{bandwidth of } x\left(1+\frac{t}{2}\right) = 5kHz \tag{27}$$

$$\text{from (22) } y\left(t\right) = \cos\left(30k\pi t\right) + \cos\left(10k\pi t\right) \tag{28}$$

$$\text{bandwidth of } y\left(t\right) = \frac{30}{2}kHz \tag{29}$$

$$= 15kHz \tag{30}$$

$$\text{Nyquist rate} = 2 \times \text{maximum frequency} \tag{31}$$

$$= 30kHz \tag{32}$$

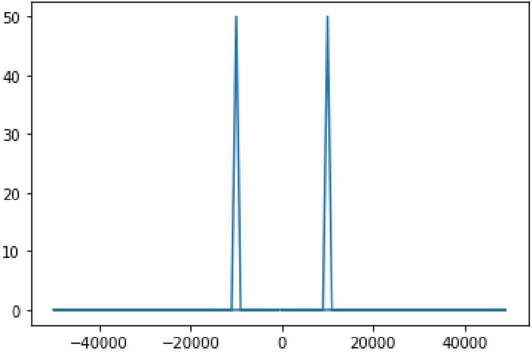

Figure: DFT of $x(t)$. $Bandwidth = 10000$

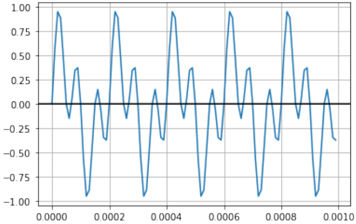

Figure: $y(t)$

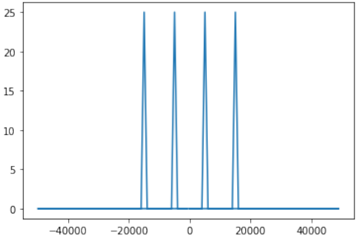

Figure: DFT of $y(t).Bandwidth = 15000$