GATE Assignment 1

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Download all python codes from

https://github.com/Adarsh541/EE3900/blob/main/Gate1/codes/Gate1.py

Download latex-tikz codes from

https://github.com/Adarsh541/EE3900/blob/main/Gate1/Gate1.tex

1 Problem(GATE 2021 EC Q4)

Consider a real-valued base-band signal $x(t)$, band limited to 10 kHz. The Nyquist rate for the signal $y(t) = x(t)\,x\left(1 + \frac{t}{2}\right)$ is

1) 15 kHz
2) 30 kHz
3) 60 kHz
4) 20 kHz

2 Solution

Definition 2.1 (Dirac-delta impulse)**.**

$$\delta(t) = \begin{cases} \infty, & t = 0 \\ 0, & otherwise \end{cases} \tag{2.0.1}$$

Lemma 2.1 (Shifting property of $\delta(t)$)**.** *If $g(t)$ is a continuous and finite function at $t = a$ then*

$$\int_{-\infty}^{\infty} \delta(t-a)\, g(t)\, dt = g(a) \tag{2.0.2}$$

We also have

$$\int_{-\infty}^{\infty} \delta(t-a)\, \delta(t-b)\, dt = \delta(a-b) \tag{2.0.3}$$

Theorem 2.2. *Fourier transform of shifted impulse is the complex exponential.*

$$G(f) = \mathcal{F}\{\delta(t-a)\} = e^{-i2\pi fa} \tag{2.0.4}$$

Proof.

$$G(f) = \int_{-\infty}^{\infty} \delta(t-a)\, e^{-i2\pi ft} dt \tag{2.0.5}$$

$$= e^{-i2\pi fa} \tag{2.0.6}$$

□

Corollary 2.2.1. *Inverse Fourier Transform of the complex exponential must be the shifted impulse. So*

$$\mathcal{F}^{-1}\left\{e^{-2\pi fa}\right\} = \int_{-\infty}^{\infty} e^{-2\pi fa} e^{i2\pi ft} df \tag{2.0.7}$$

$$= \int_{-\infty}^{\infty} e^{i2\pi f(t-a)} df \tag{2.0.8}$$

$$= \int_{-\infty}^{\infty} e^{-i2\pi f(t-a)} df \tag{2.0.9}$$

$$= \delta(t-a) \tag{2.0.10}$$

Theorem 2.3. *The Fourier transform of $g(t) = e^{i2\pi at}$ is given by*

$$G(f) = \mathcal{F}\left\{e^{i2\pi at}\right\} = \delta(f-a) \tag{2.0.11}$$

Proof.

$$G(f) = \int_{-\infty}^{\infty} e^{i2\pi at} e^{-i2\pi ft} dt \tag{2.0.12}$$

$$= \int_{-\infty}^{\infty} e^{i2\pi t(a-f)} dt \tag{2.0.13}$$

$$= \delta(f-a) \tag{2.0.14}$$

□

Lemma 2.4 (Linearity of Fourier Transform)**.**

$$\mathcal{F}\{c_1 g(t) + c_2 h(t)\} = c_1 \mathcal{F}\{g(t)\} + c_2 \mathcal{F}\{h(t)\} \tag{2.0.15}$$

Lemma 2.5. *Let $x(t)$ be a signal, its Fourier Transform be of the form*

$$G_x(f) = c_1\delta(f - a_1 A) + c_2\delta(f - a_2 A) + ... \tag{2.0.16}$$

where $c_i \in \mathbb{C}$ and $a_i \in \mathbb{R}$. Then the frequencies present in the signal are $a_j A$ where $a_j \in \mathbb{R}^+$

Let $x(t) = \cos(2\pi At)$, where $A = 10kHz$.

$$\cos(2\pi At) = \frac{e^{i2\pi At} + e^{-i2\pi At}}{2} \tag{2.0.17}$$

The Fourier transform of $x(t)$

$$G_x(f) = \int_{-\infty}^{\infty} \frac{e^{i2\pi At} + e^{-i2\pi At}}{2} e^{-i2\pi ft} dt \tag{2.0.18}$$

$$= \frac{1}{2}\left[\int_{-\infty}^{\infty} e^{-i2\pi t(f-A)} dt + \int_{-\infty}^{\infty} e^{-i2\pi t(A+F)}\right] \tag{2.0.19}$$

$$= \frac{1}{2}\left[\delta(f-A) + \delta(f+A)\right] \tag{2.0.20}$$

$\therefore$ All the energy of the sinusoidal wave is entirely localized at the frequencies given by $|f| = A$.

$$y(t) = \cos(2\pi At)\cos\left(2\pi A\left(1+\frac{t}{2}\right)\right) \tag{2.0.21}$$

$$= \frac{1}{2}\left(\cos(2\pi A + 3\pi At) + \cos(2\pi A - \pi At)\right) \tag{2.0.22}$$

$$= \frac{1}{2}\left(\cos(3\pi At) + \cos(\pi At)\right) \tag{2.0.23}$$

Fourier Transform of $y(t)$ is given by

$$G_y(f) = \frac{1}{4}\left[\delta\left(f-\frac{3A}{2}\right) + \delta\left(f+\frac{3A}{2}\right)\right] + \frac{1}{4}\left[\delta\left(f-\frac{A}{2}\right) + \delta\left(f+\frac{A}{2}\right)\right] \tag{2.0.24}$$

From lemma 2.5 we can conclude that the frequencies present in signal $y(t)$ are $\frac{A}{2}, \frac{3A}{2}$

Lemma 2.6. *Multiplication property of Fourier Transform*

$$\text{If } x(t) \overset{\mathcal{F}}{\rightleftharpoons} X(f) \tag{2.0.25}$$

$$y(t) \overset{\mathcal{F}}{\rightleftharpoons} Y(f) \tag{2.0.26}$$

Then

$$x(t)\,y(t) \overset{\mathcal{F}}{\rightleftharpoons} X(f) * Y(f) \tag{2.0.27}$$

where $*$ represents convolution

Lemma 2.7.

$$\delta(t-t_0) * g(t) = g(t-t_0) \tag{2.0.28}$$

Lemma 2.8. *Computing $G_y(f)$ using convolution*

$$x(t) = \cos(2\pi At) \tag{2.0.29}$$

$$x(t) \overset{\mathcal{F}}{\rightleftharpoons} X_1(f) = \frac{1}{2}\left[\delta(f-A) + \delta(f+A)\right] \tag{2.0.30}$$

$$x\left(1+\frac{t}{2}\right) = \cos(\pi At) \tag{2.0.31}$$

$$x\left(1+\frac{t}{2}\right) \overset{\mathcal{F}}{\rightleftharpoons} X_2(f) = \frac{1}{2}\left[\delta\left(f-\frac{A}{2}\right) + \delta\left(f+\frac{A}{2}\right)\right] \tag{2.0.32}$$

Using lemma 2.6

$$G_y(f) = X_1(f) * X_2(f) \tag{2.0.33}$$

$$= \left(\frac{1}{2}\left[\delta(f-A) + \delta(f+A)\right]\right) * X_2(f) \tag{2.0.34}$$

$$= \frac{1}{2}\left\{\delta(f-A) * X_2(f) + \delta(f+A) * X_2(f)\right\} \tag{2.0.35}$$

Using (2.0.28)

$$G_y(f) = \frac{1}{2}\left(X_2(f-A) + X_2(f+A)\right) \tag{2.0.36}$$

$$G_y(f) = \frac{1}{4}\left[\delta\left(f-\frac{3A}{2}\right) + \delta\left(f+\frac{3A}{2}\right)\right] + \frac{1}{4}\left[\delta\left(f-\frac{A}{2}\right) + \delta\left(f+\frac{A}{2}\right)\right] \tag{2.0.37}$$

$$x(t) = \cos(20k\pi t) \tag{2.0.38}$$

$$\text{bandwidth of } x(t) = 10kHz \tag{2.0.39}$$

$$x\left(1+\frac{t}{2}\right) = \cos(20k\pi + 10k\pi t) \tag{2.0.40}$$

$$\text{bandwidth of } x\left(1+\frac{t}{2}\right) = 5kHz \tag{2.0.41}$$

$$\text{from (2.0.23) } y(t) = \cos(30k\pi t) + \cos(10k\pi t) \tag{2.0.42}$$

$$\text{bandwidth of } y(t) = \frac{30}{2}kHz \tag{2.0.43}$$

$$= 15kHz \tag{2.0.44}$$

$$\text{Nyquist rate} = 2 \times \text{maximum frequency} \tag{2.0.45}$$

$$= 30kHz \tag{2.0.46}$$

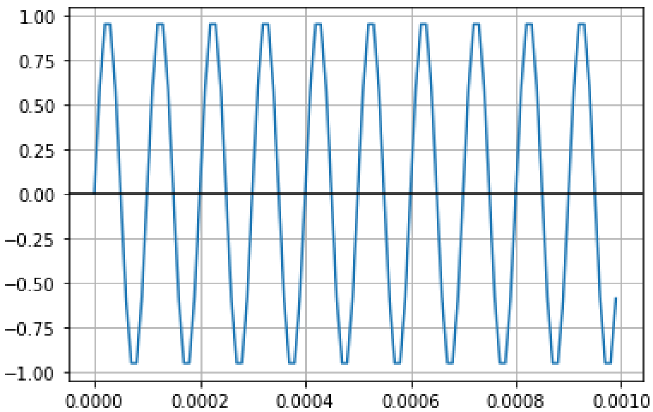

Fig. 4: $x(t)$:Sinusoidal signal with freq=10kHz

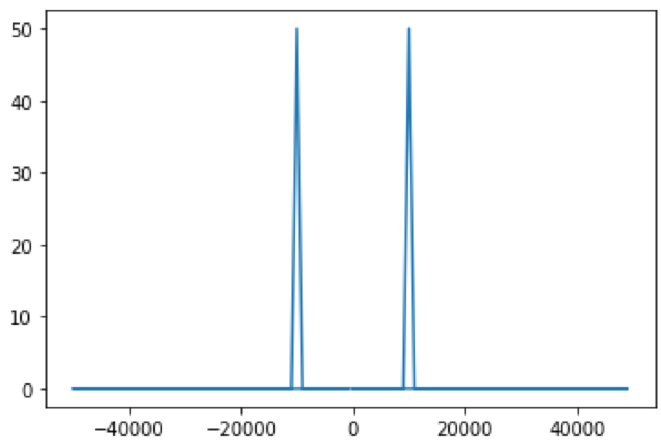

Fig. 4: DFT of $x(t)$. $Bandwidth = 10000$

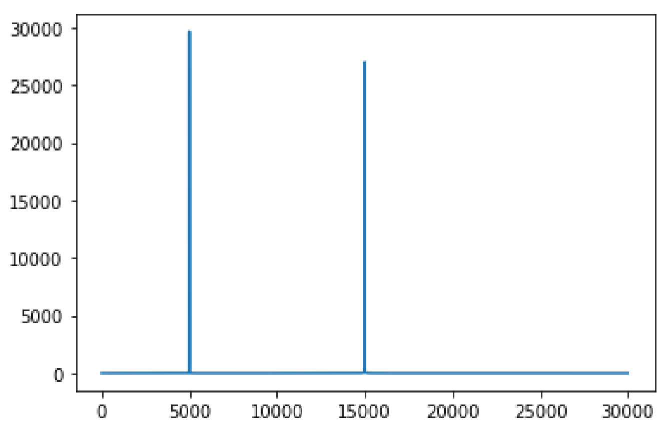

Fig. 4: DFT of $y(t)$.$Bandwidth = 15000$

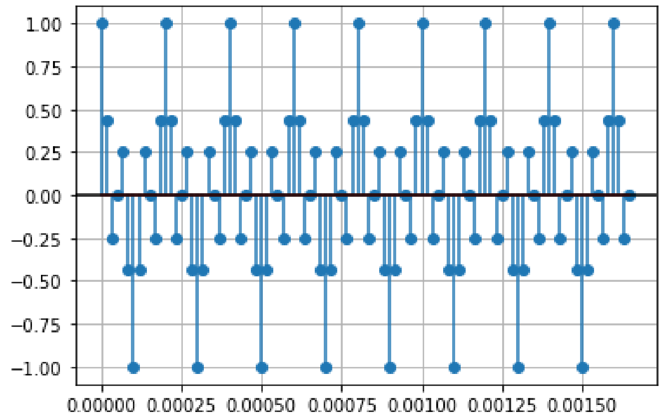

Fig. 4: stem plot of $y(t)$ sampled at 60kHz

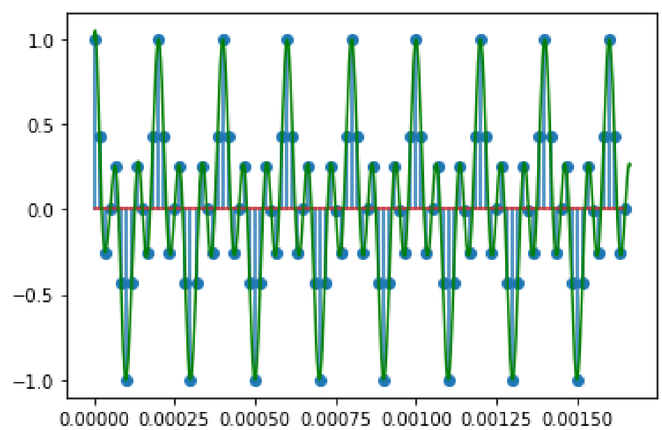

Fig. 4: Shannon interpolation of $y(t)$