

GATE 2021 EC Q4

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Dirac-delta Impulse

Dirac-delta Impulse

$$\delta(t) = \begin{cases} \infty, & t = 0 \\ 0, & \text{otherwise} \end{cases} \quad (1)$$

Shifting property of $\delta(t)$

If $g(t)$ is a continuous and finite function at $t = a$ then

$$\int_{-\infty}^{\infty} \delta(t - a) g(t) dt = g(a) \quad (2)$$

Theorem-1

Fourier transform of shifted impulse is the complex exponential.

$$G(f) = \mathcal{F}\{\delta(t - a)\} = e^{-i2\pi fa} \quad (3)$$

Proof

$$G(f) = \int_{-\infty}^{\infty} \delta(t - a) e^{-i2\pi ft} dt \quad (4)$$

$$= e^{-i2\pi fa} \quad (5)$$

Corollary-1

Inverse Fourier Transform of the complex exponential must be the shifted impulse. So

$$\mathcal{F}^{-1} \left\{ e^{-2\pi fa} \right\} = \int_{-\infty}^{\infty} e^{-2\pi fa} e^{i2\pi ft} df \quad (6)$$

$$= \int_{-\infty}^{\infty} e^{i2\pi f(t-a)} df \quad (7)$$

$$= \int_{-\infty}^{\infty} e^{-i2\pi f(t-a)} df \quad (8)$$

$$\Rightarrow \int_{-\infty}^{\infty} e^{-i2\pi f(t-a)} df = \delta(t-a) \quad (9)$$

Theorem-2

The Fourier transform of $g(t) = e^{i2\pi at}$ is given by

$$G(f) = \mathcal{F}\{e^{i2\pi at}\} = \delta(f - a) \quad (10)$$

Proof

$$G(f) = \int_{-\infty}^{\infty} e^{i2\pi at} e^{-i2\pi ft} dt \quad (11)$$

$$= \int_{-\infty}^{\infty} e^{i2\pi t(a-f)} dt \quad (12)$$

$$= \delta(f - a) \quad (13)$$

Linearity of Fourier Transform

$$\mathcal{F}\{c_1g(t) + c_2h(t)\} = c_1\mathcal{F}\{g(t)\} + c_2\mathcal{F}\{h(t)\} \quad (14)$$

Lemma-1

Let $x(t)$ be a signal, its Fourier Transform be of the form

$$G_x(f) = c_1\delta(f - a_1A) + c_2\delta(f - a_2A) + \dots \quad (15)$$

where $c_i \in \mathbb{C}$ and $a_i \in \mathbb{R}$. Then the frequencies present in the signal are a_jA where $a_j \in \mathbb{R}^+$

Fourier Transform of Cosine function

Let $x(t) = \cos(2\pi At)$, where $A = 10\text{kHz}$.

$$\cos(2\pi At) = \frac{e^{i2\pi At} + e^{-i2\pi At}}{2} \quad (16)$$

The Fourier transform of $x(t)$

$$G_x(f) = \int_{-\infty}^{\infty} \frac{e^{i2\pi At} + e^{-i2\pi At}}{2} e^{-i2\pi ft} dt \quad (17)$$

$$= \frac{1}{2} \left[\int_{-\infty}^{\infty} e^{-i2\pi t(f-A)} dt + \int_{-\infty}^{\infty} e^{-i2\pi t(A+F)} dt \right] \quad (18)$$

$$= \frac{1}{2} [\delta(f-A) + \delta(f+A)] \quad (19)$$

\therefore All the energy of the sinusoidal wave is entirely localized at the frequencies given by $|f| = A$.

Question

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Consider a real-valued base-band signal $x(t)$, band limited to 10 kHz. The Nyquist rate for the signal $y(t) = x(t) \times \left(1 + \frac{t}{2}\right)$ is

- ① 15 kHz
- ② 30 kHz
- ③ 60 kHz
- ④ 20 kHz

Solution

$$y(t) = \cos(2\pi At) \cos\left(2\pi A\left(1 + \frac{t}{2}\right)\right) \quad (20)$$

$$= \cos(2\pi A + 3\pi At) + \cos(2\pi A - \pi At) \quad (21)$$

$$= \cos(3\pi At) + \cos(\pi At) \quad (22)$$

Using the linearity of Fourier Transform. Fourier Transform of $y(t)$ is given by

$$G_y(f) = \frac{1}{4} \left[\delta\left(f - \frac{3A}{2}\right) + \delta\left(f + \frac{3A}{2}\right) \right] + \frac{1}{4} \left[\delta\left(f - \frac{A}{2}\right) + \delta\left(f + \frac{A}{2}\right) \right] \quad (23)$$

Convolution

Multiplication property of Fourier Transform

$$\text{If } x(t) \xrightarrow{\mathcal{F}} X(f) \quad (24)$$

$$y(t) \xrightarrow{\mathcal{F}} Y(f) \quad (25)$$

Then

$$x(t)y(t) \xrightarrow{\mathcal{F}} X(f) * Y(f) \quad (26)$$

where $*$ represents convolution

Lemma

$$\delta(t - t_0) * g(t) = g(t - t_0) \quad (27)$$

$$x(t) = \cos(2\pi At) \quad (28)$$

$$x(t) \xrightarrow{\mathcal{F}} X_1(f) = \frac{1}{2} [\delta(f - A) + \delta(f + A)] \quad (29)$$

$$x\left(1 + \frac{t}{2}\right) = \cos(\pi At) \quad (30)$$

$$x\left(1 + \frac{t}{2}\right) \xrightarrow{\mathcal{F}} X_2(f) = \frac{1}{2} \left[\delta\left(f - \frac{A}{2}\right) + \delta\left(f + \frac{A}{2}\right) \right] \quad (31)$$

Using convolution

$$G_y(f) = X_1(f) * X_2(f) \quad (32)$$

$$= \left(\frac{1}{2} [\delta(f - A) + \delta(f + A)] \right) * X_2(f) \quad (33)$$

$$= \frac{1}{2} \{ \delta(f - A) * X_2(f) + \delta(f + A) * X_2(f) \} \quad (34)$$

Using (27)

$$G_y(f) = \frac{1}{2} (X_2(f - A) + X_2(f + A)) \quad (35)$$

$$G_y(f) = \frac{1}{4} \left[\delta \left(f - \frac{3A}{2} \right) + \delta \left(f + \frac{3A}{2} \right) \right] \\ + \frac{1}{4} \left[\delta \left(f - \frac{A}{2} \right) + \delta \left(f + \frac{A}{2} \right) \right] \quad (36)$$

$$x(t) = \cos(20k\pi t) \quad (37)$$

$$\text{bandwidth of } x(t) = 10\text{kHz} \quad (38)$$

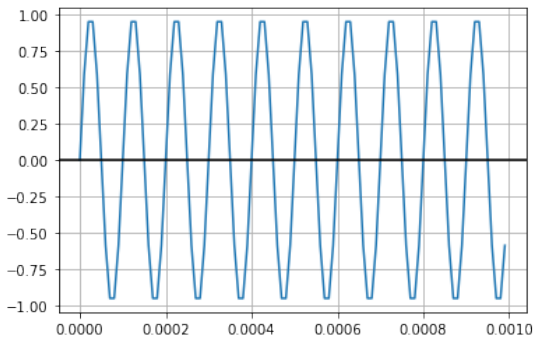


Figure: $x(t)$: Sinusoidal signal with freq=10kHz

$$x\left(1 + \frac{t}{2}\right) = \cos(20k\pi + 10k\pi t) \quad (39)$$

$$\text{bandwidth of } x\left(1 + \frac{t}{2}\right) = 5kHz \quad (40)$$

$$\text{from (22) } y(t) = \cos(30k\pi t) + \cos(10k\pi t) \quad (41)$$

$$\text{bandwidth of } y(t) = \frac{30}{2} kHz \quad (42)$$

$$= 15kHz \quad (43)$$

$$\text{Nyquist rate} = 2 \times \text{maximum frequency} \quad (44)$$

$$= 30kHz \quad (45)$$

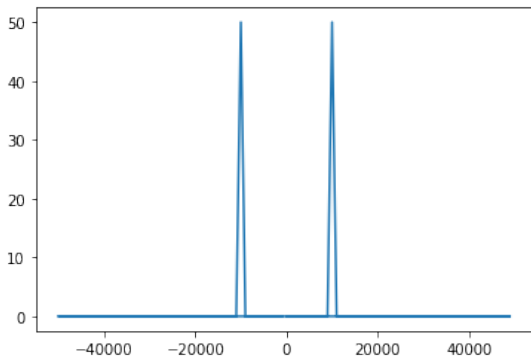


Figure: DFT of $x(t)$. *Bandwidth = 10000*

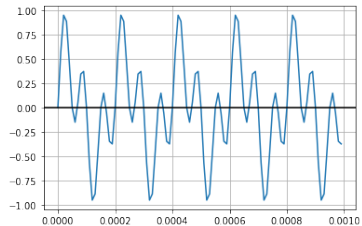


Figure: $y(t)$

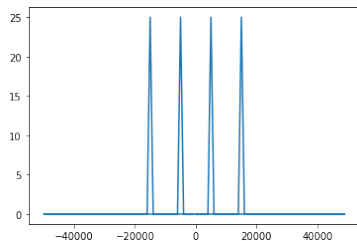


Figure: DFT of $y(t)$. $Bandwidth = 15000$