GATE 2021 EC Q4

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Nyquist-Shannon Sampling Theorem

• If a function $x(t)$ contains no frequencies higher than B hertz, it is completely determined by giving its ordinates at a series of points spaced $\frac{1}{2B}$ seconds apart.
• The threshold 2B is called the Nyquist rate.

Question

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Consider a real-valued base-band signal $x(t)$, band limited to 10 kHz. The Nyquist rate for the signal $y(t) = x(t)\, x\left(1 + \frac{t}{2}\right)$ is

1. 15 kHz
2. 30 kHz
3. 60 kHz
4. 20 kHz

Solution

$$\text{bandwidth of } x(t) = 10kHz \tag{1}$$

$$\text{bandwidth of bandwidth of } y(t) = \frac{10+5}{2}kHz \tag{2}$$

$$= 15kHz \tag{3}$$

$$\text{Nyquist rate} = 2 \times \text{maximum frequency} \tag{4}$$

$$= 30kHz \tag{5}$$

Dirac-delta Impulse

Dirac-delta Impulse

$$\delta(t) = \begin{cases} \infty, & t = 0 \\ 0, & otherwise \end{cases} \tag{6}$$

Shifting property of $\delta(t)$

If $g(t)$ is a continuous and finite function at $t = a$ then

$$\int_{-\infty}^{\infty} \delta(t-a)\, g(t)\, dt = g(a) \tag{7}$$

Theorem-1

Fourier transform of shifted impulse is the complex exponential.

$$G(f) = \mathcal{F}\left\{\delta(t-a)\right\} = e^{-i2\pi fa} \tag{8}$$

Proof

$$\begin{aligned} G(f) &= \int_{-\infty}^{\infty} \delta(t-a)\, e^{-i2\pi ft} dt \quad (9) \\ &= e^{-i2\pi fa} \quad (10) \end{aligned}$$

Corollary-1

Inverse Fourier Transform of the complex exponential must be the shifted impulse. So

$$\mathcal{F}^{-1}\left\{e^{-2\pi fa}\right\} = \int_{-\infty}^{\infty} e^{-2\pi fa} e^{i2\pi ft} df \tag{11}$$

$$= \int_{-\infty}^{\infty} e^{i2\pi f(t-a)} df \tag{12}$$

$$= \int_{-\infty}^{\infty} e^{-i2\pi f(t-a)} df \tag{13}$$

$$\implies \int_{-\infty}^{\infty} e^{-i2\pi f(t-a)} df = \delta\left(t-a\right) \tag{14}$$

Theorem-2

The Fourier transform of $g\left(t\right) = e^{i2\pi at}$ is given by

$$G\left(f\right) = \mathcal{F}\left\{e^{i2\pi at}\right\} = \delta\left(f - a\right) \tag{15}$$

Proof

$$G\left(f\right) = \int_{-\infty}^{\infty} e^{i2\pi at} e^{-i2\pi ft} dt \tag{16}$$

$$= \int_{-\infty}^{\infty} e^{i2\pi t(a-f)} dt \tag{17}$$

$$= \delta\left(f - a\right) \tag{18}$$

Linearity of Fourier Transform

$$\mathcal{F}\left\{c_1 g\left(t\right) + c_2 h\left(t\right)\right\} = c_1 \mathcal{F}\left\{g\left(t\right)\right\} + c_2 \mathcal{F}\left\{h\left(t\right)\right\} \tag{19}$$

Lemma-1

Let $x\left(t\right)$ be a signal, its Fourier Transform be of the form

$$G_x\left(f\right) = c_1 \delta\left(f - a_1 A\right) + c_2 \delta\left(f - a_2 A\right) + \ldots \tag{20}$$

where $c_i \in \mathbb{C}$ and $a_i \in \mathbb{R}$. Then the frequencies present in the signal are $a_j A$ where $a_j \in \mathbb{R}^+$

Fourier Transform of Cosine function

Let $x\left(t\right) = \cos\left(2\pi At\right)$, where $A = 10kHz$.

$$\cos\left(2\pi At\right) = \frac{e^{i2\pi At} + e^{-i2\pi At}}{2} \tag{21}$$

The Fourier transform of $x\left(t\right)$

$$G_x\left(f\right) = \int_{-\infty}^{\infty} \frac{e^{i2\pi At} + e^{-i2\pi At}}{2} e^{-i2\pi ft} dt \tag{22}$$

$$= \frac{1}{2}\left[\int_{-\infty}^{\infty} e^{-i2\pi t(f-A)} dt + \int_{-\infty}^{\infty} e^{-i2\pi t(A+F)}\right] \tag{23}$$

$$= \frac{1}{2}\left[\delta\left(f - A\right) + \delta\left(f + A\right)\right] \tag{24}$$

$\therefore$ All the energy of the sinusoidal wave is entirely localized at the frequencies given by $\left|f\right| = A$.

Solution

$$y(t) = \cos(2\pi At)\cos\left(2\pi A\left(1+\frac{t}{2}\right)\right) \tag{25}$$
$$= \cos(2\pi A + 3\pi At) + \cos(2\pi A - \pi At) \tag{26}$$
$$= \cos(3\pi At) + \cos(\pi At) \tag{27}$$

Using the linearity of Fourier Transform. Fourier Transform of $y(t)$ is given by

$$G_y(f) = \frac{1}{4}\left[\delta\left(f-\frac{3A}{2}\right)+\delta\left(f+\frac{3A}{2}\right)\right] + \frac{1}{4}\left[\delta\left(f-\frac{A}{2}\right)+\delta\left(f+\frac{A}{2}\right)\right] \tag{28}$$

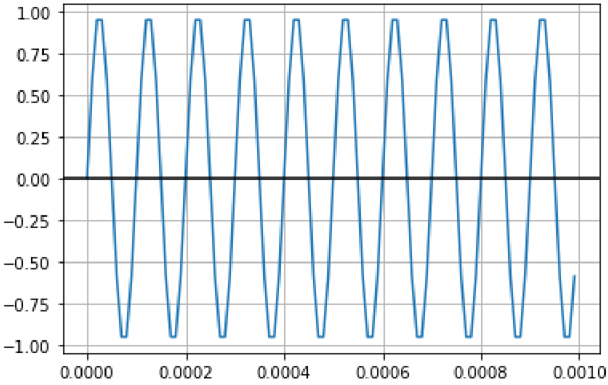

Figure: $x(t)$:Sinusoidal signal with freq=10kHz

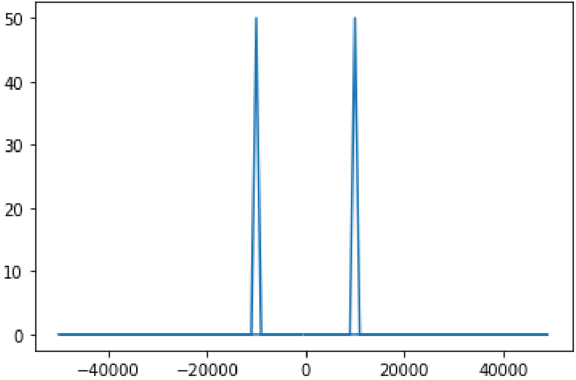

Figure: Fourier transform of $x(t)$. The only frequency present in the signal is 10kHz

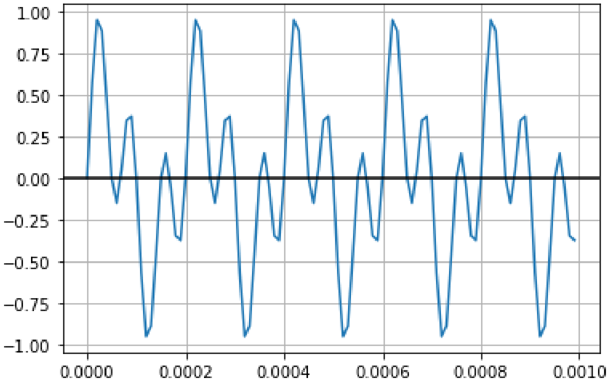

Figure: Plot of $y(t)$

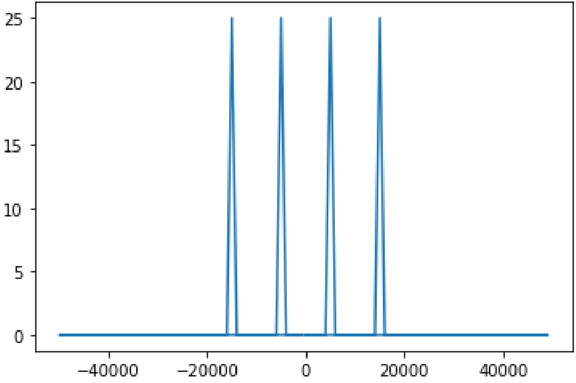

Figure: Fourier transform of $y(t)$

Convolution

Multiplication property of Fourier Transform

$$\text{If } x(t) \overset{\mathcal{F}}{\rightleftharpoons} X(f) \tag{29}$$

$$y(t) \overset{\mathcal{F}}{\rightleftharpoons} Y(f) \tag{30}$$

Then

$$x(t)\, y(t) \overset{\mathcal{F}}{\rightleftharpoons} X(f) * Y(f) \tag{31}$$

where $*$ represents convolution

Lemma

$$\delta(t - t_0) * g(t) = g(t - t_0) \tag{32}$$

$$x(t) = \cos(2\pi At) \tag{33}$$

$$x(t) \overset{\mathcal{F}}{\rightleftharpoons} X_1(f) = \frac{1}{2}\left[\delta(f-A) + \delta(f+A)\right] \tag{34}$$

$$x\left(1+\frac{t}{2}\right) = \cos(\pi At) \tag{35}$$

$$x\left(1+\frac{t}{2}\right) \overset{\mathcal{F}}{\rightleftharpoons} X_2(f) = \frac{1}{2}\left[\delta\left(f-\frac{A}{2}\right) + \delta\left(f+\frac{A}{2}\right)\right] \tag{36}$$

Using convolution

$$G_y(f) = X_1(f) * X_2(f) \tag{37}$$

$$= \left(\frac{1}{2}\left[\delta(f-A) + \delta(f+A)\right]\right) * X_2(f) \tag{38}$$

$$= \frac{1}{2}\left\{\delta(f-A) * X_2(f) + \delta(f+A) * X_2(f)\right\} \tag{39}$$

Using (32)

$$G_y(f) = \frac{1}{2}\left(X_2(f - A) + X_2(f + A)\right) \tag{40}$$

$$G_y(f) = \frac{1}{4}\left[\delta\left(f - \frac{3A}{2}\right) + \delta\left(f + \frac{3A}{2}\right)\right] + \frac{1}{4}\left[\delta\left(f - \frac{A}{2}\right) + \delta\left(f + \frac{A}{2}\right)\right] \tag{41}$$

Shannon Interpolation

Let $x[nT]$ represents samples, sampling rate $= \frac{1}{T}$, of a continuous signal then

$$x(t) = \sum_{n=-\infty}^{\infty} x[nT] sinc\left(\frac{t-nT}{T}\right) \tag{42}$$

is the perfect reconstruction of the continuous signal.

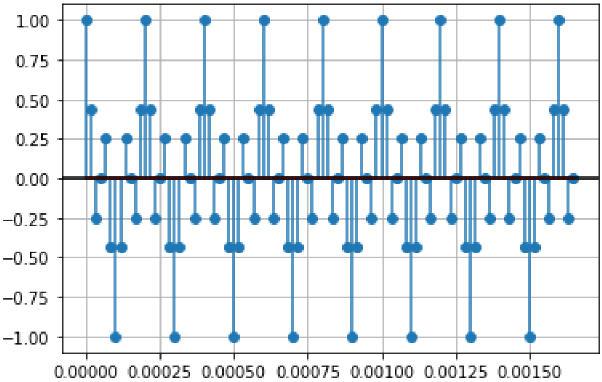

Figure: Stem plot of $y(t)$ sampled at 60kHz

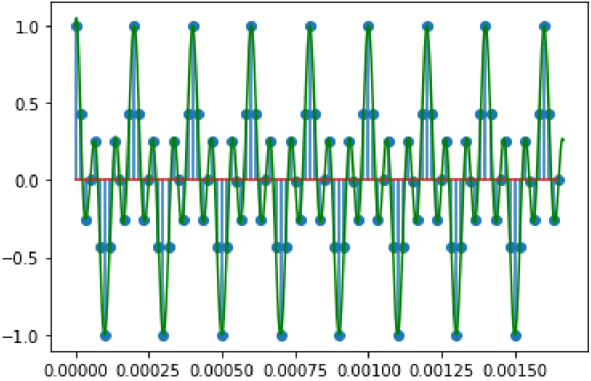

Figure: Interpolation of $y(t)$ sampled at 60kHz

- The interpolated graph passes through all the sampled points.

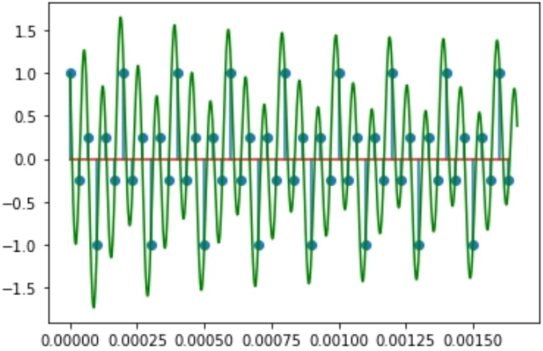

Figure: Interpolation of $y(t)$ sampled at 30kHz

• If we sample $y(t)$ at a frequency less than 30kHz, then we can't obtain the original $y(t)$ from the sampled one.

• The above graph is not perfectly reconstructed as the sample rate is less than Nyquist rate.

Concepts Learnt

1. Properties of Dirac impulse
2. Fourier Transform of a sinusoid
3. Convolution
4. Sampling Theorem
5. Interpolation