Assignment - 4

Thursday, 25 November 2021 5:12 PM

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Q1.a

Working set $W_0 = \{4\}$

$4x + 5y \le -1$

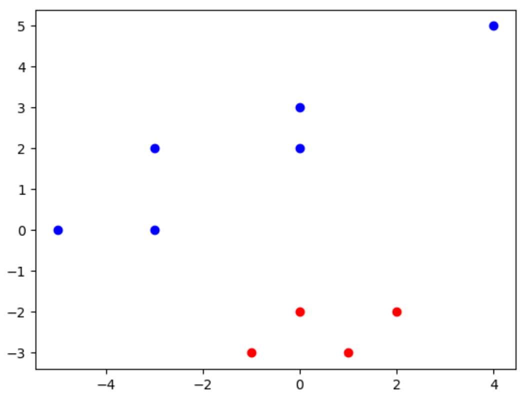

Fig 1: Data Distribution

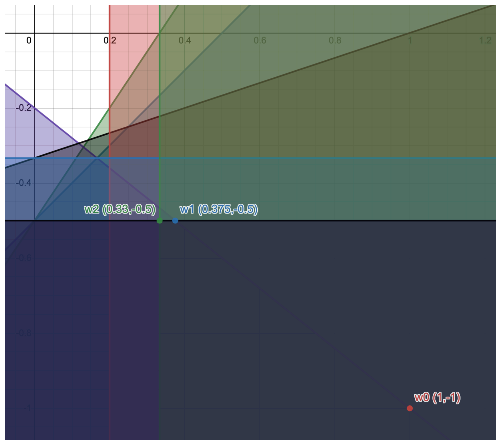

Fig 2: w plot

Q1.b After one iteration

$w_1 = (0.375, -0.5)$

$W_1 = \{4, 10\}$ $\{2y \le -1, 4x+5y \le -1\}$

feasible direction $u = (-0.375, 0)$

$w_1 = w_0 + \alpha p$

$$p = \underset{c^T p = 0}{\operatorname{argmin}} \ \frac{1}{2} p^T p + w_0' p$$

$$x_0 = \begin{bmatrix} 0.375 \\ -0.5 \end{bmatrix} \quad c = \begin{bmatrix} 0 \\ -2 \end{bmatrix}$$

$$L = \frac{1}{2} p^T p + w_0^T p - \mu c^T p$$

$$\nabla L = p + w_0 - \mu c = 0 \Rightarrow p = \mu c - w_0$$

$$c^T [\mu c - w_0] = 0 \quad \therefore p = \frac{1}{4} \begin{bmatrix} 0 \\ -2 \end{bmatrix} - \begin{bmatrix} 0.375 \\ -0.5 \end{bmatrix}$$

$$\mu = \frac{c^T w_0}{\|c\|^2} = \frac{1}{4} \quad p = \begin{bmatrix} -0.375 \\ 0 \end{bmatrix}$$

Q.1 c. After Second Iteration

$$\omega_2 = (0.33, 0.5)$$

p Working Set $W_2 = \{8, 10\}$

$$-3x \leq -1 \qquad 2y \leq -1$$

lot is in Q1.a

Q2.a Linear Program

$$x = \begin{bmatrix} V_1 - V_{min} \\ \vdots \\ V_T - V_{min} \end{bmatrix} \qquad x \geq 0$$

$$A = \begin{bmatrix} \cos\theta_0 & \cdots & \cos\theta_f \\ \sin\theta_0 & \cdots & \sin\theta_f \end{bmatrix}$$

$$b = \begin{bmatrix} x_f - V_{min} \sum_{i=0}^{f} \cos\theta_i \\ y_f - V_{min} \sum_{i=0}^{f} \sin\theta_i \end{bmatrix}$$

$$C = -\mathbb{1}_{T\times 1}$$

$$\min \; c^T x$$
$$Ax = b$$
$$x \geq 0$$

Q2.b

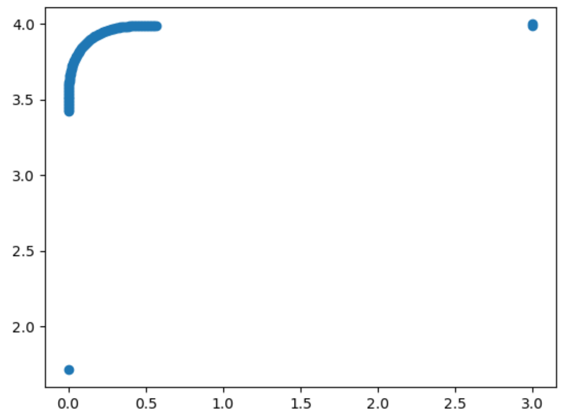

Fig 3: Vmin = 0.01

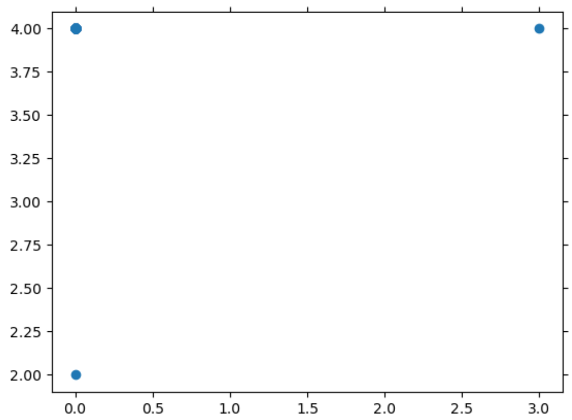

Fig 4: Vmin = 0

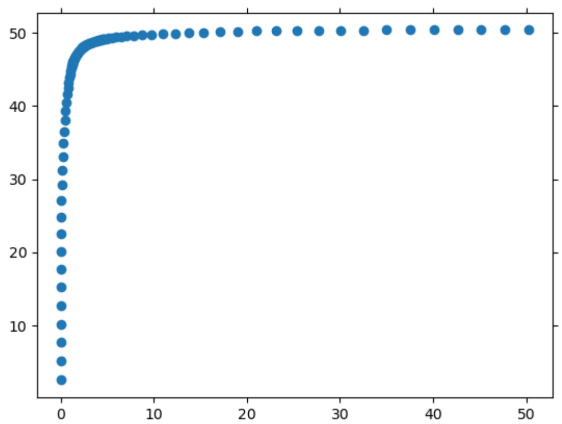

Fig 5: Vmin = 1 (This is infeasible)

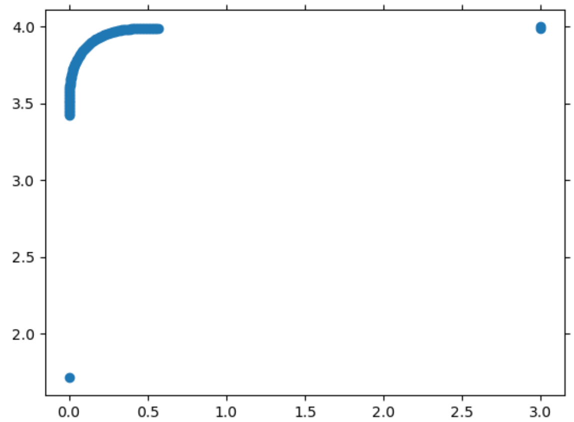

Fig 6: Vmin = 0.01, Theta f = pi/2

Q2 c

$$x_f = \sum_{t=1}^{T} V_t \cos\theta_t$$

$$y_f = \sum_{t=1}^{T} V_t \sin\theta_t$$

$$\therefore \quad x_f + y_f = \sum_{t=1}^{T} V_t (\cos\theta_t + \sin\theta_t) \geq V_{min} \sum_{t=1}^{T} (\cos\theta_t + \sin\theta_t)$$

Claim: $\sum_{1}^{T} (\cos\theta_t + \sin\theta_t) \geq T\sqrt{2}$

$$\therefore \quad V_{min} \leq \frac{x_f + y_f}{T\sqrt{2}}$$

Q2.d Simplex Tableau

x_1	......	x_T	b
$\cos\theta_0$	....	$\cos\theta_f$	$x_f - k_1$
$\sin\theta_0$	....	$\sin\theta_f$	$y_f - k_2$

$k_1 = V_{min}\sum\cos\theta$

$k_2 = V_{min}\sum\sin\theta$

$\Downarrow$

x_1		x_T	b
1	....	0	$x_f - k_1$
0		1	$y_f - k_2$

Let $A = [a_1|..|a_T]$

$\therefore a_1 = \begin{bmatrix}1\\0\end{bmatrix} \quad a_T = \begin{bmatrix}0\\1\end{bmatrix}$

$\therefore B = \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$

$$\therefore \quad x_1 = x_f - k_1$$
$$x_T = y_f - k_2$$

Let $j \in \{2,..T-1\}$

$$z_j = c_j - c_1\cos\theta_j - c_T\sin\theta_j$$
$$= -1 + \underline{\cos\theta_j + \sin\theta_j} > 0$$
$$-1 + \sqrt{1+\sin 2\theta_j} > 0$$

$\therefore$ No feasible descent direction.

$\therefore$ $B = \{1\ T\}$ is optimal BFS.

$$V_1 = x_f + \left(1 - \sum_{i=1}^{T}\cos\theta_i\right)V_{min}$$

$$V_T = y_f + \left(1 - \sum_{i=1}^{T}\sin\theta_i\right)V_{min}$$

$$V_j = V_{min} \qquad j \in \{2,..,T-1\}$$

Q.3) a.

$$\underset{\|w-w^0\|^2 \le r^2}{\operatorname{argmin}} \ \frac{1}{2}\|Aw - b\|^2$$

$$\mathcal{L}(w,\lambda) = \frac{1}{2}w^TA^TAw + \frac{1}{2}b^Tb - b^TAw + \lambda\left[w^Tw + w_0^Tw_0 - 2w_0^Tw - r^2\right]$$

$$= \frac{1}{2} w^T [A^T A - \lambda I] w - [A^T b + 2\lambda w_0]^T w + \frac{1}{2} b^T b + \lambda w_0^T w_0 - \lambda r^2$$

<u>KKT:</u>

① $\nabla L(w, \lambda) = 0 = [A^T A - \lambda I] w - A^T b - 2\lambda w_0 = 0$

② $\lambda \geq 0$

③ $\lambda \cdot [\|w - w_0\|^2 - r^2] = 0$

④ $\|w - w_0\|^2 \leq r^2$

Q.3) b.) If constraint is inactive

$\rightarrow \|w - w_0\|^2 - r^2 \neq 0$

$\rightarrow \lambda = 0$ — ③

$\therefore$ From ① $A^T A w - A^T b = 0$

$\therefore w = [A^T A]^{-1} A^T b$

Q.4.)

(P) $\operatorname{argmin} \; x^T A_0 x + 2 b_0^T x + c_0$

s.t.

$x^T A_i x + 2 b_i^T + c_i \leq 0$

$$L(x, \lambda) = x^T A_0 x + 2 b_0^T x + c_0 + \sum_{i=1}^{m} \lambda_i [x^T A_i x + 2 b_i^T x + c_i]$$

$\lambda \in \mathbb{R}^m$

$$= x^T \left[A_0 + \sum_{i=1}^{m} \lambda_i A_i\right] + 2 \left[b_0 + \sum_{i=1}^{m} \lambda_i b_i\right]^T x + c_0 + \sum_{i=1}^{m} \lambda_i c_i$$

$Q = A_0 + \sum_{i=1}^{m} \lambda_i A_i \succ 0$ <u>P.D.</u> since $\lambda_i \geq 0$
$A_i \succeq 0$ & $A_0 \succ 0$

$h = 2\left[b_0 + \sum_{i=1}^{m} \lambda_i b_i\right]$

$C = c_0 + \sum_{i=1}^{m} \lambda_i c_i$

$\therefore L(x, \lambda) = x^T Q x + h^T x + C$

$$g(\lambda) = \min_{x \in \mathbb{R}^n} L(x, \lambda)$$
$$= -\frac{1}{2} h^T Q^{-1} h + c$$

$\therefore$ Dual :

$$\min_{\lambda \geq 0} \; 2 \left[b_0 + \sum_{i=1}^{m} \lambda_i b_i\right] \left[A_0 + \sum_{j=1}^{m} \lambda_j A_j\right] \left[b_0 + \sum_{k=1}^{m} \lambda_k b_k\right]$$
$$+ \quad c_0 + \sum_{l=1}^{m} \lambda_l c_l$$

Q5) a)
$$P_C(z) = \underset{Ax \leq b}{\text{argmin}} \; \frac{1}{2} \|x - z\|^2$$

$$\mathcal{L}(x, \lambda) = \frac{1}{2} x^T x + \frac{1}{2} z^T z - z^T x + \lambda^T [Ax - b]$$

$$g(\lambda) = \min_{x \in \mathbb{R}^n} \mathcal{L}(x, \lambda)$$

$$\nabla \mathcal{L}(x, \lambda) = 0 : \; x - z + A^T \lambda = 0$$
$$\Rightarrow x = z - A^T \lambda$$

$$\therefore g(\lambda) = \frac{1}{2} [z - A^T \lambda]^T [z - A^T \lambda] + \frac{1}{2} z^T z - z^T [z - A^T \lambda]$$
$$+ \lambda^T [A[z - A^T \lambda] - b]$$

$$\therefore g(\lambda) = -\frac{1}{2} \lambda^T A A^T \lambda + \lambda^T [Az - b]$$

$\therefore$ Dual (D) =
$$\underset{\substack{\lambda_i \geq 0 \\ \lambda \in \mathbb{R}^d}}{\text{argmin}} \; \frac{1}{2} \lambda^T A A^T \lambda - \lambda^T [Az - b]$$

Q5) b)
$$\lambda_{k+1} = P_C\left(\lambda_k - \frac{1}{L} g_k\right)$$

where $g_k = A A^T \lambda_k - Az + b$

$$\therefore \lambda_{k+1} = P_C\left(\lambda_k - \frac{1}{L}\left[AA^T\lambda_k - Az + b\right]\right)$$

$$L = \max_{\|x\|=1} \|A^T x\| = \text{max eigenvalue of } AA^T$$

$$\left[P_{\underset{x\geq 0}{C}}(x)\right]_i = \begin{cases} x_i & ; \ x_i \geq 0 \\ 0 & ; \ x_i < 0 \end{cases}$$

$$\Rightarrow [\lambda_{k+1}]_i = \begin{cases} \left[\lambda_k - \frac{1}{L}\left[AA^T\lambda_k - Az + b\right]\right]_i & \geq 0 \\ 0 & < 0 \end{cases}$$

Q.5) c)

$x_{opt} = [2.667, 0.1667, 0.1667]$

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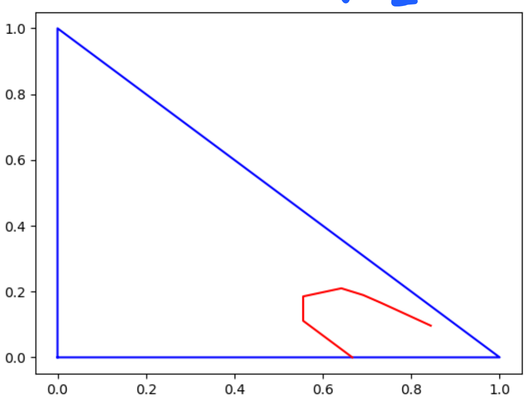
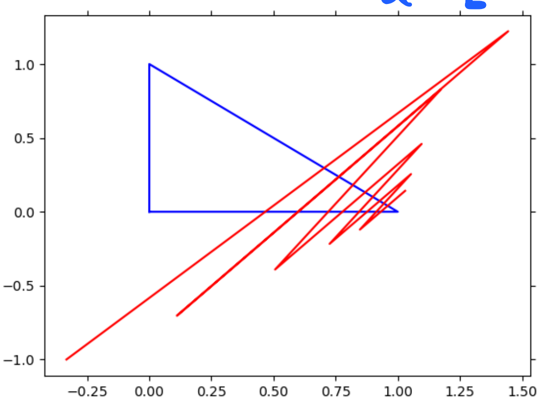
Q.5) d

$\alpha = 1/L$



$\alpha = 2/L$



Taking step $= 1/L$ provides $O(1/T)$ convergence & the deviations are less as compared to $2/L$.

Q.5) e. Analytical!

$$P_C(z) = \underset{\substack{x+3y=1 \\ 2y+z=-1}}{\operatorname{argmin}} \frac{1}{2}\|x-z\|^2$$

$$\mathcal{L}(x,\mu) = \frac{1}{2}x^Tx - z^Tx + \frac{1}{2}z^Tz - \mu_1[x+3y-1] - \mu_2[2y+z=-1]$$

at KKT

$$\nabla \mathcal{L}(x,\mu) = x - z - A^T\mu = 0$$

$$\mu \in \mathbb{R}^2 \;\&\; A = \begin{bmatrix} 1 & 3 & 0 \\ 0 & 2 & 1 \end{bmatrix} \quad b = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$Ax = b \Rightarrow Az + AA^T\mu = b$$

$$\Rightarrow \mu = [AA^T]^{-1}[b - Az]$$

$$\therefore P_c(z) = z + A^T[AA^T]^{-1}[b - Az]$$

$$= [I - A^T[AA^T]^{-1}A]z + A^T[AA^T]^{-1}b$$

$$\therefore P_c(0) = A^T[AA^T]^{-1}b$$

= [0.78571429 0.07142857 -1.14285714]

Gradient Projection:

$$\min \ \frac{1}{2}\|x - z\|^2$$
$$Ax = b$$

Here $f(x) = \frac{1}{2}\|x - z\|^2$

$$\therefore \ g(x) = x - z$$

Here, $\alpha = 2/L \quad L = 1$

$$\therefore \ x_{k+1} = P_C(x_k - \alpha g_k)$$

$$\therefore \ x_{k+1} = [I - A^T[AA^T]^{-1}A]\,[(1-\alpha)x_k - \alpha z\,] + A^T[AA^T]^{-1}b$$
$$= x_k - \alpha(x_k - z) + \alpha\, A^T[AA^T]^{-1} b$$

The result matches ✓