Generalizing Strong Parallel Repetition for Monogamy-of-Entanglement Games

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1 Introduction

Monogamy-of-entanglement (MoE) games are a fundamental class of extended nonlocal games in quantum information theory. Recent work by Tomamichel, Fehr, Kaniewski, and Wehner (TFKW) established strong parallel repetition theorems for MoE games with:

- Two-question case (|Σ|=2)
- Uniform question distribution (π)
- Projective referee measurements ($R(a|x)$)

Specifically, the theorem in Theorem 6.4 of [1] shows that for such games, the parallel repetition bound is tight:

$$\omega^*(G^r) = \left(\frac{1}{2} + \frac{1}{2}\sqrt{c(G)}\right)^r \tag{1}$$

where c(G) is the maximal overlap of the referee's measurements.

Key Open Questions

1. **Extension to Non-Uniform Distributions**: Does strong parallel repetition hold when π is non-uniform? If so, what is the modified bound?
2. **Generalization to POVMs**: Can the theorem mentioned be extended to games where $R(a|x)$ are POVMs rather than projective measurements?
3. **Beyond Two Questions**: What happens when |Σ| >2? Does a similar tight bound exist?

2 Technical Approach

The project will combine theoretical derivations with numerical simulations using the nonlocal_games.py and extended_nonlocal_games.py from toqito.

(A) Non-Uniform Question Distributions

1. **Theoretical Investigation**: toqito's nonlocal_games module can be used to model MoE games with a biased distribution, for instance $\pi = (0.6, 0.4)$. Then, we can generalise the lemma $||\pi_0 + \pi_1|| = 1 + ||\pi_0\pi_1||$ where π_0 and π_1 are non-zero projection operators on C^r, using symbolic optimization to handle weighted projection operators.

2. **Numerical Verification**: We could implement a custom script to compute the game values for non-uniform π. It could also involve comparing the empirical results with the conjectured bound.

(B) **Generalization to POVMs**

1. **Theoretical Framework**: First, we represent POVMs and then we adapt the proposition

$$\omega(G) = \frac{1}{2} + \frac{1}{2} \max_{a,b\in\Gamma} \|R(a|0)R(b|1)\| \tag{2}$$

by testing Naimark-dilated POVMs.

2. **Numerical Experiments**: Parallel repetition is simulated by sampling random POVMs for $|\Sigma|$=2, compute c(G) and check if TFKW's bound still holds empirically.

(C) Beyond Two Questions ($|\Sigma| > 2$): We will face the following obstacles while tackling this problem: The $|\Sigma|$=2 case exploits symmetry and for $|\Sigma|$=k, the maximal overlap c(G) becomes more complex.

Potential Directions: First approach is the **Inductive Approach** where k-question games are related to nested 2-question subgames and **Graph-Theoretic Methods** by representing c(G) as edge weights in a compatibility graph. For Graph-Theoretic Analysis, we could represent c(G) as a compatibility graph and then compute bounds via SDP solvers.

3 Expected Outcomes

Following are the goals for this project:

1. **Generalized Parallel Repetition Theorem**: This outcome includes a proof (or counter-proof) for non-uniform π and an extension to POVMs, possibly with modified bounds.

2. **Numerical Insights**: It will have helped us analyse empirical trends (if any) for $|\Sigma| > 2$ and transitions in c(G) for different types of measurements.

4 Schedule of Deliverables

Table 1: Phase 1: Theoretical Foundation & Implementation (Weeks 1–5)

Week	**Deliverable**
1	• Community Bonding Period to understand toqito's modules and literature review of TFKW • Setup of development environment (GitHub) • Implementation of basic MoE game framework
2–3	• Extension to non-uniform distributions ($\pi = (p, 1-p)$) • Derivation of conjectured bound: $\omega^*(G^r) \leq \left(p + (1-p)\sqrt{c(G)}\right)^r$
4–5	• Implementation of POVM for MoE class • Benchmarking of Naimark vs. direct POVM approaches

Table 2: Phase 2: Advanced Exploration & Dissemination (Weeks 6–11)

Week	**Deliverable**
6–8	• Large-scale simulation (10^5 instances) of POVM games • Compatibility graph design for $\lvert\Sigma\rvert > 2$ cases
9–10	• SDP experiments • Analysis of multi-question game behavior
11	• Drafting of research preprint • Final code optimization and documentation

5 Information About The Applicant

1. **Why apply for this project?** Having engaged in research in multiple disciplines within Quantum Technologies, what motivates me to continue to work in this field is how interdisciplinary it can be. Foundational Quantum Information can help us better understand what happens to information inside a black hole or devise better communication methods. Already having gained some theoretical knowledge, I know for a fact that this is one of the best programs for experience in the field of open-source software development. I would like to contribute to the toqito package to help researchers do more with nonlocal and extended nonlocal games.

2. **Relevant Experience**: The following academic and research projects have equipped me with the skills necessary for this role. During my previous internships, I worked on the mathematical foundations of Quantum Information Theory at ISI, Kolkata under Prof. Guruprashad Kar and studied open quantum systems in QIntern 2024 under Dr. Fadwa Benabdallah. I have also worked in QClairvoyance as an Intern, learning about and implementing Quantum Machine Learning methods for Quantum Chemistry. Recently, I finished a project on Benchmarking Quantum Problems with Reinforcement Learning with QOSF under Dr. Akash Kundu, further helping me understand Quantum Algorithms, Machine Learning and the vast world of research possible because of these two. Beyond my research, I am quite familiar with Python, including implementing quantum algorithms and simulating quantum systems using Classiq and libraries such as Qiskit, Qulacs and PennyLane.

References

[1] Vincent Russo(2017) *Extended nonlocal games*, arXiv:1704.07375v1 [quant-ph] 24 Apr 2017

[2] Johnston Nathaniel, Mittal Rajat, Russo Vincent and Watrous John 2016 *Extended non-local games and monogamy-of-entanglement games* Proc. R. Soc. A.47220160003

[3] Buhrman, Harry et al. *On the Parallel Repetition of Multi-Player Games: The No-Signaling Case*, Theory of Quantum Computation, Communication, and Cryptography (2013).

[4] Yuen, Henry S.. *A parallel repetition theorem for all entangled games.* Electron. Colloquium Comput. Complex. TR16 (2016): n. pag.