A Comparative Study of SMT and MILP for the Nurse Rostering Problem

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APPENDIX: DETAILS ON THE SMT AND MILP MODEL FORMULATIONS

In this section, we provide detailed SMT and MILP formulations of the GCs. We begin by defining helpful notation. *Start Day* returns the set of all shifts in a given set that start on a given day,

$$\mathcal{S}^{sd}(\widetilde{\mathcal{S}}, k) = \left\{ i \in \widetilde{\mathcal{S}} \mid s_i^{sd} = k \right\}.$$

Relative Start Day is similar to Start Day but relative to a given shift,

$$\mathcal{S}_i^{sd}(\widetilde{\mathcal{S}}, k) = \left\{ i' \in \widetilde{\mathcal{S}} \mid s_{i'}^{sd} = s_i^{sd} + k \right\}.$$

Type Set returns the subset of shifts with a given type,

$$\mathcal{S}^{t}(\widetilde{\mathcal{S}}, t) = \left\{ i' \in \widetilde{\mathcal{S}} \mid s_{i'}^{t} = t \right\}.$$

Finally, *Relative Type Set* returns the subset of shifts of the same shift type as a given shift,

$$\mathcal{S}_i^{t}(\widetilde{\mathcal{S}}) = \left\{ i' \in \widetilde{\mathcal{S}} \mid s_{i'}^{t} = s_i^{t} \right\}.$$

The aim is to find an assignment of variables such that all constraints (every instance of every GC) is satisfied, as well as the initial problem constraint of each shift only being assigned one staff member.

A. SMT Formulation

The SMT formulation of the GCs are described in this section. For each person $j \in \mathcal{P}$ and shift $i \in \mathcal{S}$, the Boolean variable $a_{j,i}$ denotes if j is assigned to i. Every instance of a GC formulation (with specific parameter values) adds constraints to the problem. Thus, in the following formulations, all parameters should be interpreted as specific for a GC instance. The *Boolean to Binary* operator maps a Boolean evaluation to binary,

$$\oplus\omega = \begin{cases} 1 & \text{if } \omega \text{ is true} \\ 0 & \text{otherwise.} \end{cases}$$

Boolean Summation extends this to count the number of *True* evaluations for a function over a set Ω,

$$\bigoplus_{\omega \in \Omega} f(\omega) = \sum_{\omega \in \Omega} \oplus f(\omega)$$

We now formulate the constraints. Only one staff member can be assigned to a shift, formulated in (1).

$$\left(\bigoplus_{j \in \mathcal{P}} a_{j,i} \right) \leq 1, \quad \forall i \in \mathcal{S} \tag{1}$$

GC1: *The number of shifts in $\widetilde{\mathcal{S}}$ that are not assigned at least one person from $\widetilde{\mathcal{P}}$ must be greater than or equal to x and less than or equal to y.*

$$x \leq \left(\bigoplus_{i \in \widetilde{\mathcal{S}}} \bigwedge_{j \in \widetilde{\mathcal{P}}} \neg a_{j,i} \right) \leq y \tag{2}$$

GC2: *The number of shifts in $\widetilde{\mathcal{S}}$ that have been assigned an unqualified person from $\widetilde{\mathcal{P}}$ must be greater than or equal to x and less than or equal to y.*

$$x \leq \left(\bigoplus_{i \in \widetilde{\mathcal{S}}} \bigvee_{j \in \widetilde{\mathcal{P}} \setminus s_i^{QP}} a_{j,i} \right) \leq y \tag{3}$$

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GC3: *The number of times any person $j \in \widetilde{\mathcal{P}}$ is assigned to all shifts in a set of overlapping shifts $k \in \mathcal{O}$ for which they are not allowed to be assigned to ($j \notin o_k^{\mathcal{P}}$) must be greater than or equal to x and less than or equal to y.* We first define the set $\mathcal{O}_j^A \subseteq \mathcal{O}$ containing every 2-size $k \in \mathcal{O}$ that staff member j is *allowed* to be assigned to ($j \in o_k^{\mathcal{P}}$),

$$\mathcal{O}_j^A = \left\{ o_k^{\mathcal{S}} \,\middle|\, |o_k^{\mathcal{S}}| = 2, j \in o_k^{\mathcal{P}}, k \in \mathcal{O} \right\}, \ \forall j \in \widetilde{\mathcal{P}}. \tag{4}$$

The set $\mathcal{O}_j^F$ contains the 2-size overlapping combinations in $k \in \mathcal{O}$ that j is *forbidden* from being assigned to,

$$\mathcal{O}_j^F = \left\{ k \in \mathcal{O} \,\middle|\, |o_k^{\mathcal{S}}| = 2, o_k^{\mathcal{S}} \notin \mathcal{O}_j^A \right\}, \quad \forall j \in \widetilde{\mathcal{P}}. \tag{5}$$

Finally, this formulation is concluded by constraining the sum over all staff members in $j \in \widetilde{\mathcal{P}}$ and overlapping combinations in $k \in \mathcal{O}_j^F$ where j is assigned to all shifts in $o_k^{\mathcal{S}}$:

$$x \leq \left(\sum_{j \in \widetilde{\mathcal{P}}} \bigoplus_{k \in \mathcal{O}_j^F} \bigwedge_{i \in o_k^{\mathcal{S}}} a_{j,i} \right) \leq y. \tag{6}$$

GC4: *For each person in $\widetilde{\mathcal{P}}$, the fraction of their assigned workload from shifts in $\widetilde{\mathcal{S}}$, if they are assigned to any at all, must be greater than or equal to u and less than or equal to v.* This formulation ensures integer values are used, since it was found during implementation to lead to significantly better performance of the SMT solver. The parameter $\eta = 1000$ is used to scale values, which are then rounded.

$$\omega_{u,j} = \sum_{i \in \mathcal{S}} \lfloor \eta s_i^w u \rfloor \oplus a_{j,i}, \quad \forall j \in \widetilde{\mathcal{P}} \tag{7}$$

$$\omega_{v,j} = \sum_{i \in \mathcal{S}} \lfloor \eta s_i^w v \rfloor \oplus a_{j,i}, \quad \forall j \in \widetilde{\mathcal{P}} \tag{8}$$

$$\omega_{u,j} \leq \left(\sum_{i \in \widetilde{\mathcal{S}}} \lfloor \eta s_i^w \rfloor \oplus i a_{j,i} \right) \leq \omega_{v,j}, \quad \forall j \in \widetilde{\mathcal{P}} \tag{9}$$

GC5: *If any person in $\widetilde{\mathcal{P}}_1$ is assigned to a shift in $\widetilde{\mathcal{S}}_1$, then the number of assignments of people in $\widetilde{\mathcal{P}}_2$ to shifts in $\widetilde{\mathcal{S}}_2$ must be greater than or equal to x and less than or equal to y.*

$$\omega = \bigoplus_{i \in \widetilde{\mathcal{S}}_2} \bigvee_{j \in \widetilde{\mathcal{P}}_2} a_{j,i} \tag{10}$$

$$\left(\bigvee_{i \in \widetilde{\mathcal{S}}_1} \bigvee_{j \in \widetilde{\mathcal{P}}_1} a_{j,i} \right) \Rightarrow (x \leq \omega \leq y) \tag{11}$$

GC6: *The number of consecutive days of shifts in $\widetilde{\mathcal{S}}$ assigned to a person in $\widetilde{\mathcal{P}}$ must be greater than or equal to x and less than or equal to y.*

$$\left(a_{j,i} \wedge \bigwedge_{i' \in \mathcal{S}_i^{sd}(\widetilde{\mathcal{S}}, -1)} \neg a_{j,i'} \right) \Rightarrow \left(\bigwedge_{c=1}^{x-1} \bigvee_{i' \in \mathcal{S}_i^{sd}(\widetilde{\mathcal{S}}, c)} a_{j,i'} \right), \qquad \forall j \in \widetilde{\mathcal{P}} \quad \forall i \in \widetilde{\mathcal{S}} \tag{12}$$

$$\left(a_{j,i} \wedge \bigwedge_{i' \in \mathcal{S}_i^{sd}(\widetilde{\mathcal{S}}, -1)} \neg a_{j,i'} \right) \Rightarrow \left(\bigvee_{c=1}^{y} \bigwedge_{i' \in \mathcal{S}_i^{sd}(\widetilde{\mathcal{S}}, c)} \neg a_{j,i'} \right), \qquad \forall j \in \widetilde{\mathcal{P}} \quad \forall i \in \widetilde{\mathcal{S}} \tag{13}$$

GC7: *For a person in $\widetilde{\mathcal{P}}$, x to y consecutive days of assignments to shifts in $\widetilde{\mathcal{S}}$ must be preceded by n days without assignments to shifts in $\widetilde{\mathcal{S}}_1$ and followed by m days without assignments to shifts in $\widetilde{\mathcal{S}}_2$.*

$$\left(\left(\bigwedge_{\widetilde{c}=d}^{d+c-1} \bigvee_{i\in\mathcal{S}^{sd}(\widetilde{\mathcal{S}},\widetilde{c})} a_{j,i}\right) \wedge \left(\bigwedge_{i\in\mathcal{S}^{sd}(\widetilde{\mathcal{S}},d-1)\cup\mathcal{S}^{sd}(\widetilde{\mathcal{S}},d+c)} \neg a_{j,i}\right)\right) \Rightarrow$$
$$\left(\left(\bigwedge_{\widetilde{n}=d-n}^{d-1} \bigwedge_{i\in\mathcal{S}^{sd}(\widetilde{\mathcal{S}}_1,\widetilde{n})} \neg a_{j,i}\right) \wedge \left(\bigwedge_{\widetilde{m}=d+c}^{d+c+m-1} \bigwedge_{i\in\mathcal{S}^{sd}(\widetilde{\mathcal{S}}_2,\widetilde{m})} \neg a_{j,i}\right)\right), \tag{14}$$
$$\forall c\in[x\,..\,y] \quad \forall d\in\mathcal{D} \quad \forall j\in\widetilde{\mathcal{P}}$$

GC8: *If a staff member in $\widetilde{\mathcal{P}}$ is assigned to a shift in $\widetilde{\mathcal{S}}$, then, on the day before the assigned shift's start day, the staff member must either be assigned to a shift in $\widetilde{\mathcal{S}}$ with the same shift type or not be assigned to any shift in $\widetilde{\mathcal{S}}$.*

$$a_{j,i} \Rightarrow \left(\left(\bigvee_{i'\in\mathcal{S}_i^{sd}(\widetilde{\mathcal{S}},-1)\cap\mathcal{S}_i^{t}(\widetilde{\mathcal{S}})} a_{j,i'}\right) \vee \left(\bigwedge_{i'\in\mathcal{S}_i^{sd}(\widetilde{\mathcal{S}},-1)} \neg a_{j,i'}\right)\right), \tag{15}$$
$$\forall i\in\widetilde{\mathcal{S}} \quad \forall j\in\widetilde{\mathcal{P}}$$

GC9: *For a staff member in $\widetilde{\mathcal{P}}$, the workload that they are assigned to from shifts in $\widetilde{\mathcal{S}}$ divided by their desired workload must be within v percent of the expected workload ratio. The expected workload ratio is the total workload of all shifts in $\widetilde{\mathcal{S}}$ divided by the total desired workload of all staff members in $\widetilde{\mathcal{P}}$.* The parameter $\eta = 1000$ is also used here.

$$r_e = \left(\sum_{i\in\widetilde{\mathcal{S}}} s_i^w\right) \Big/ \left(\sum_{j\in\widetilde{\mathcal{P}}} p_j^{dw}\right) \tag{16}$$

$$r_j = \sum_{i\in\widetilde{\mathcal{S}}} \left\lfloor \eta \frac{s_i^w}{p_j^{dw}} \right\rfloor \oplus a_{j,i}, \quad \forall j\in\widetilde{\mathcal{P}} \tag{17}$$

$$\lfloor \eta r_e(1-v)\rfloor \le r_j \le \lfloor \eta r_e(1+v)\rfloor, \quad \forall j\in\widetilde{\mathcal{P}} \tag{18}$$

B. MILP Formulation

The MILP formulation uses 0-1-variables: $a_{j,i}=1$ if person j is assigned to shift i, 0 otherwise. Besides these, additional auxiliary 0-1-variables δ are used in all GC formulations (expect GC9), which should be interpreted as specific to each GC instance. Whenever the big-M method is used, an appropriate value is given. First, the constraint that only one person can be assigned to a shift is formulated as

$$\sum_{j\in\mathcal{P}} a_{j,i} \le 1, \quad \forall i\in\mathcal{S} \tag{19}$$

GC1: *The number of shifts in $\widetilde{\mathcal{S}}$ that are not assigned at least one person from $\widetilde{\mathcal{P}}$ must be greater than or equal to x and less than or equal to y.* This formulation uses auxiliary variables $\delta_i, \forall i\in\widetilde{\mathcal{S}}$, which are modeled in (20) using the big-M method ($M \ge |\widetilde{\mathcal{P}}|$) such that $\delta_i = 1$ if no one in $\widetilde{\mathcal{P}}$ is assigned to shift i, 0 otherwise.

$$1-\delta_i \le \sum_{j\in\widetilde{\mathcal{P}}} a_{j,i} \le M(1-\delta_i), \quad \forall i\in\widetilde{\mathcal{S}} \tag{20}$$

$$x \le \sum_{i\in\widetilde{\mathcal{S}}} \delta_i \le y \tag{21}$$

GC2: *The number of shifts in $\widetilde{\mathcal{S}}$ that have been assigned an unqualified person from $\widetilde{\mathcal{P}}$ must be greater than or equal to x and less than or equal to y.* This formulation uses auxiliary variables $\delta_i, \forall i\in\widetilde{\mathcal{S}}$, which are modeled in (22) using the big-M method ($M \ge |\widetilde{\mathcal{P}}|$) such that $\delta_i = 1$ if at least one unqualified person in $\widetilde{\mathcal{P}}$ is assigned to shift i.

$$\delta_i \le \sum_{j\in\widetilde{\mathcal{P}}\setminus s_i^{QP}} a_{j,i} \le M\delta_i, \quad \forall i\in\widetilde{\mathcal{S}} \tag{22}$$

$$x \le \sum_{i\in\widetilde{\mathcal{S}}} \delta_i \le y \tag{23}$$

GC3: *The number of times any person $j \in \widetilde{\mathcal{P}}$ is assigned to all shifts in a set of overlapping shifts $k \in \mathcal{O}$ for which they are not allowed to be assigned to ($j \notin o_k^{\mathcal{P}}$) must be greater than or equal to x and less than or equal to y.* This formulation uses the sets $\mathcal{O}_j^A$ and $\mathcal{O}_j^F$ from (4) and (5), respectively. Auxiliary variables $\delta_{k,j}$, $\forall k \in \mathcal{O}_j^F$, $\forall j \in \widetilde{\mathcal{P}}$, are modeled in (24) such that $\delta_{k,j} = 1$ if j is assigned to all shifts in $o_k^{\mathcal{S}}$, 0 otherwise.

$$2\delta_{k,j} \leq \sum_{i \in o_k^{\mathcal{S}}} a_{j,i} \leq \delta_{k,j} + 1, \quad \forall k \in \mathcal{O}_j^F, \ \forall j \in \widetilde{\mathcal{P}} \tag{24}$$

$$x \leq \sum_{j \in \widetilde{\mathcal{P}}} \sum_{o_k \in \mathcal{O}_j^F} \delta_{k,j} \leq y \tag{25}$$

GC4: *For each person in $\widetilde{\mathcal{P}}$, the fraction of their assigned workload from shifts in $\widetilde{\mathcal{S}}$, if they are assigned to any at all, must be greater than or equal to u and less than or equal to v.* Only one auxiliary variable is used in this formulation: $\delta = 1$ if any person in $\widetilde{\mathcal{P}}$ is assigned to any shift in $\widetilde{\mathcal{S}}$, 0 otherwise. The parameter $\eta = 1000$ is used here, similar to the SMT formulation, simply for the sake of applying the same model to both solvers for a fair comparison. The use of η and rounding lead to a negligible impact on the performance of Gurobi.

$$\omega_{u,j} = \sum_{i \in \mathcal{S}} \lfloor \eta s_i^w u \rfloor a_{j,i}, \qquad \forall j \in \widetilde{\mathcal{P}} \tag{26}$$

$$\omega_{v,j} = \sum_{i \in \mathcal{S}} \lfloor \eta s_i^w v \rfloor a_{j,i}, \qquad \forall j \in \widetilde{\mathcal{P}} \tag{27}$$

$$\omega_{u,j} \leq \left(\sum_{i \in \widetilde{\mathcal{S}}} \lfloor \eta s_i^w \rfloor a_{j,i} \right) \leq \omega_{v,j}, \qquad \forall j \in \widetilde{\mathcal{P}} \tag{28}$$

GC5: *If any person in $\widetilde{\mathcal{P}}_1$ is assigned to a shift in $\widetilde{\mathcal{S}}_1$, then the number of assignments of people in $\widetilde{\mathcal{P}}_2$ to shifts in $\widetilde{\mathcal{S}}_2$ must be greater than or equal to x and less than or equal to y.* The auxiliary variables used here are:

- $\delta = 1$ if any person in $\widetilde{\mathcal{P}}_1$ is assigned to any shift in $\widetilde{\mathcal{S}}_1$, 0 otherwise.
- $\forall j \in \widetilde{\mathcal{P}}_2, \forall i \in \widetilde{\mathcal{S}}_2$: $\delta_{j,i} = 1$ if person j is assigned to shift i and $\delta = 1$, 0 otherwise.

$$x \leq \left(\sum_{i \in \widetilde{\mathcal{S}}_2} \sum_{j \in \widetilde{\mathcal{P}}_2} \delta_{j,i} \right) \leq y \tag{29}$$

$$\left(\sum_{i \in \widetilde{\mathcal{S}}_1} \sum_{j \in \widetilde{\mathcal{P}}_1} a_{j,i} \right) \geq \delta \tag{30}$$

$$\left(\sum_{i \in \widetilde{\mathcal{S}}_1} \sum_{j \in \widetilde{\mathcal{P}}_1} a_{j,i} \right) \leq M\delta \tag{31}$$

$$\delta + a_{j,i} \leq 1 + \delta_{j,i}, \qquad \forall i \in \widetilde{\mathcal{S}}_2 \quad \forall j \in \widetilde{\mathcal{P}}_2 \tag{32}$$

$$\delta \geq \delta_{j,i}, \qquad \forall i \in \widetilde{\mathcal{S}}_2 \quad \forall j \in \widetilde{\mathcal{P}}_2 \tag{33}$$

$$a_{j,i} \geq \delta_{j,i}, \qquad \forall i \in \widetilde{\mathcal{S}}_2 \quad \forall j \in \widetilde{\mathcal{P}}_2 \tag{34}$$

$$M \geq |\widetilde{\mathcal{P}}_1||\widetilde{\mathcal{S}}_1| \tag{35}$$

GC6: *The number of consecutive days of shifts in $\widetilde{\mathcal{S}}$ assigned to a person in $\widetilde{\mathcal{P}}$ must be greater than or equal to x and less than or equal to y.* The auxiliary variables used here are:

- $\forall j \in \widetilde{\mathcal{P}}$, $\forall i \in \widetilde{\mathcal{S}}$: $\delta_{j,i} = 1$ if person j is assigned to shift i and not assigned to any shifts in $\widetilde{\mathcal{S}}$ on the day before, 0 otherwise.
- $\forall j \in \widetilde{\mathcal{P}}$, $\forall i \in \widetilde{\mathcal{S}}$, $\forall c \in [1\,..\,x]$: $\delta_{A,j,i,c} = 1$ if person j is assigned to any shift in $\widetilde{\mathcal{S}}$ starting c days after shift i, 0 otherwise.
- $\forall j \in \widetilde{\mathcal{P}}$, $\forall i \in \widetilde{\mathcal{S}}$: $\delta_{A,j,i} = 1$ if person j is assigned to shift in $\widetilde{\mathcal{S}}$ every 1 to $x - 1$ days after shift i, 0 otherwise.
- $\forall j \in \widetilde{\mathcal{P}}$, $\forall i \in \widetilde{\mathcal{S}}$, $\forall c \in [1\,..\,y]$: $\delta_{B,j,i,c} = 1$ if person j is not assigned to any shift in $\widetilde{\mathcal{S}}$ c days after shift i, 0 otherwise.
- $\forall j \in \widetilde{\mathcal{P}}$, $\forall i \in \widetilde{\mathcal{S}}$: $\delta_{B,j,i} = 1$ if on any of the 1 to $x - 1$ days after shift i, person j is not assigned to any shift in $\widetilde{\mathcal{S}}$, 0 otherwise.

$$(1 - a_{j,i}) + \left(\sum_{i' \in \mathcal{S}_i^{sd}(\widetilde{\mathcal{S}}, -1)} a_{j,i'} \right) \geq (1 - \delta_{j,i}), \qquad \forall i \in \widetilde{\mathcal{S}} \quad \forall j \in \widetilde{\mathcal{P}} \tag{36}$$

$$(1 - a_{j,i}) + \left(\sum_{i' \in \mathcal{S}_i^{sd}(\widetilde{\mathcal{S}}, -1)} a_{j,i'} \right) \leq M (1 - \delta_{j,i}), \qquad \forall s_i \in \widetilde{\mathcal{S}} \quad \forall j \in \widetilde{\mathcal{P}} \tag{37}$$

$$\left(\sum_{i' \in \mathcal{S}_i^{sd}(\widetilde{\mathcal{S}}, c)} a_{j,i'} \right) \geq \delta_{A,j,i,c}, \qquad \forall c \in [1\,..\,x-1] \quad \forall i \in \widetilde{\mathcal{S}} \quad \forall j \in \widetilde{\mathcal{P}} \tag{38}$$

$$\left(\sum_{i' \in \mathcal{S}_i^{sd}(\widetilde{\mathcal{S}}, c)} a_{j,i'} \right) \leq M \delta_{A,j,i,c}, \qquad \forall c \in [1\,..\,x-1] \quad \forall i \in \widetilde{\mathcal{S}} \quad \forall j \in \widetilde{\mathcal{P}} \tag{39}$$

$$\left(\sum_{c=1}^{x-1} 1 - \delta_{A,j,i,c} \right) \geq 1 - \delta_{A,j,i}, \qquad \forall i \in \widetilde{\mathcal{S}} \quad \forall j \in \widetilde{\mathcal{P}} \tag{40}$$

$$\left(\sum_{c=1}^{x-1} 1 - \delta_{A,j,i,c} \right) \leq M (1 - \delta_{A,j,i}), \qquad \forall i \in \widetilde{\mathcal{S}} \quad \forall j \in \widetilde{\mathcal{P}} \tag{41}$$

$$\delta_{j,i} - \delta_{A,j,i} \leq 0, \qquad \forall i \in \widetilde{\mathcal{S}} \quad \forall j \in \widetilde{\mathcal{P}} \tag{42}$$

$$\left(\sum_{i' \in \mathcal{S}_i^{sd}(\widetilde{\mathcal{S}}, c)} a_{j,i'} \right) \geq 1 - \delta_{B,j,i,c}, \qquad \forall c \in [1\,..\,y] \quad \forall i \in \widetilde{\mathcal{S}} \quad \forall j \in \widetilde{\mathcal{P}} \tag{43}$$

$$\left(\sum_{i' \in \mathcal{S}_i^{sd}(\widetilde{\mathcal{S}}, c)} a_{j,i'} \right) \leq M (1 - \delta_{B,j,i,c}), \qquad \forall c \in [1\,..\,y] \quad \forall i \in \widetilde{\mathcal{S}} \quad \forall j \in \widetilde{\mathcal{P}} \tag{44}$$

$$\left(\sum_{c=1}^{y} \delta_{B,j,i,c} \right) \geq \delta_{B,j,i}, \qquad \forall i \in \widetilde{\mathcal{S}} \quad \forall j \in \widetilde{\mathcal{P}} \tag{45}$$

$$\left(\sum_{c=1}^{y} \delta_{B,j,i,c} \right) \leq M \delta_{B,j,i}, \qquad \forall i \in \widetilde{\mathcal{S}} \quad \forall j \in \widetilde{\mathcal{P}} \tag{46}$$

$$\delta_{j,i} - \delta_{B,j,i} \leq 0, \qquad \forall i \in \widetilde{\mathcal{S}} \quad \forall j \in \widetilde{\mathcal{P}} \tag{47}$$

$$M_1 \geq |\widetilde{\mathcal{S}}|, \quad M \geq x - 1, \quad M \geq y \tag{48}$$

GC7: *For a person in $\widetilde{\mathcal{P}}$, x to y consecutive days of assignments to shifts in $\widetilde{\mathcal{S}}$ must be preceded by n days without assignments to shifts in $\widetilde{\mathcal{S}}_1$ and followed by m days without assignments to shifts in $\widetilde{\mathcal{S}}_2$.* The auxiliary variables used here are:

- $\forall j \in \widetilde{\mathcal{P}}$, $\forall d \in \mathcal{D}$: $\delta_{j,d} = 1$ if person j is assigned to at least one shift in $\widetilde{\mathcal{S}}$ on day d, 0 otherwise.
- $\forall j \in \widetilde{\mathcal{P}}$, $\forall d \in \mathcal{D}$, $\forall c \in [x\,..\,y]$: $\delta_{A,j,d,c} = 1$ if person j is assigned to at least one shift in $\widetilde{\mathcal{S}}$ on every day from d to $d + c - 1$, and not assigned to any shifts in $\widetilde{\mathcal{S}}$ on the days $d - 1$ or $d + c$, 0 otherwise.
- $\forall j \in \widetilde{\mathcal{P}}$, $\forall d \in \mathcal{D}$, $\forall c \in [x\,..\,y]$: $\delta_{B,j,d,c} = 1$ if person j is not assigned to any shifts in $\widetilde{\mathcal{S}}_n$ on any of the days from $d - n$ to $d - 1$, and not assigned to any shifts in $\widetilde{\mathcal{S}}_m$ on any of the days from $d + c$ to $d + c + m - 1$, otherwise 0.

$$\left(\sum_{i \in \mathcal{S}^{sd}(\widetilde{\mathcal{S}}, d)} a_{j,i} \right) \geq \delta_{j,d}, \qquad \forall d \in \mathcal{D} \quad \forall j \in \widetilde{\mathcal{P}} \tag{49}$$

$$\left(\sum_{i \in \mathcal{S}^{sd}(\widetilde{\mathcal{S}}, d)} a_{j,i} \right) \leq M_1 \delta_{j,d}, \qquad \forall d \in \mathcal{D} \quad \forall j \in \widetilde{\mathcal{P}} \tag{50}$$

$$\left(\sum_{\widetilde{c}=d}^{d+c-1} 1-\delta_{j,\widetilde{c}}\right)+\left(\sum_{i\in\mathcal{S}^{sd}(\widetilde{\mathcal{S}},d-1)\cup\mathcal{S}^{sd}(\widetilde{\mathcal{S}},d+c)} a_{j,i}\right)\geq 1-\delta_{A,j,d,c},$$
$$\forall c\in[x\,..\,y]\quad\forall d\in\mathcal{D}\quad\forall j\in\widetilde{\mathcal{P}} \tag{51}$$

$$\left(\sum_{\widetilde{c}=d}^{d+c-1} 1-\delta_{j,\widetilde{c}}\right)+\left(\sum_{i\in\mathcal{S}^{sd}(\widetilde{\mathcal{S}},d-1)\cup\mathcal{S}^{sd}(\widetilde{\mathcal{S}},d+c)} a_{j,i}\right)\leq M_2\left(1-\delta_{A,j,d,c}\right),$$
$$\forall c\in[x\,..\,y]\quad\forall d\in\mathcal{D}\quad\forall j\in\widetilde{\mathcal{P}} \tag{52}$$

$$\left(\sum_{\widetilde{n}=d-n}^{d-1}\ \sum_{i\in\mathcal{S}^{sd}(\widetilde{\mathcal{S}}_n,\widetilde{n})} a_{j,i}\right)+\left(\sum_{\widetilde{m}=d+c}^{d+c+m-1}\ \sum_{i\in\mathcal{S}^{sd}(\widetilde{\mathcal{S}}_m,\widetilde{m})} a_{j,i}\right)\geq 1-\delta_{B,j,d,c},$$
$$\forall c\in[x\,..\,y]\quad\forall d\in\mathcal{D}\quad\forall j\in\widetilde{\mathcal{P}} \tag{53}$$

$$\left(\sum_{\widetilde{n}=d-n}^{d-1}\ \sum_{i\in\mathcal{S}^{sd}(\widetilde{\mathcal{S}}_n,\widetilde{n})} a_{j,i}\right)+\left(\sum_{\widetilde{m}=d+c}^{d+c+m-1}\ \sum_{i\in\mathcal{S}^{sd}(\widetilde{\mathcal{S}}_m,\widetilde{m})} a_{j,i}\right)\leq M_3\left(1-\delta_{B,j,d,c}\right),$$
$$\forall c\in[x\,..\,y]\quad\forall d\in\mathcal{D}\quad\forall j\in\widetilde{\mathcal{P}} \tag{54}$$

$$\delta_{A,j,d,c}-\delta_{B,j,d,c}\leq 0,\qquad \forall c\in[x\,..\,y]\quad\forall d\in\mathcal{D}\quad\forall j\in\widetilde{\mathcal{P}} \tag{55}$$

$$M_1\geq|\widetilde{\mathcal{S}}|,\qquad M_2\geq y+|\widetilde{\mathcal{S}}|,\qquad M_3\geq|\widetilde{\mathcal{S}}_n|+|\widetilde{\mathcal{S}}_m| \tag{56}$$

GC8: *If a staff member in $\widetilde{\mathcal{P}}$ is assigned to a shift in $\widetilde{\mathcal{S}}$, then, on the day before the assigned shift's start day, the staff member must either be assigned to a shift in $\widetilde{\mathcal{S}}$ with the same shift type or not be assigned to any shift in $\widetilde{\mathcal{S}}$.* The auxiliary variables used here are:

- $\forall j\in\widetilde{\mathcal{P}}$, $\forall i\in\widetilde{\mathcal{S}}$: $\delta_{A,j,i}=1$ if person j is assigned to a shift in $\widetilde{\mathcal{S}}$ of type s_i^t on the day before shift i, 0 otherwise.
- $\forall j\in\widetilde{\mathcal{P}}$, $\forall i\in\widetilde{\mathcal{S}}$: $\delta_{B,j,i}=1$ if person j is not assigned to any shift in $\widetilde{\mathcal{S}}$ on the day before shift i, 0 otherwise.

$$\left(\sum_{i'\in\mathcal{S}_i^{sd}(\widetilde{\mathcal{S}},-1)\cap\mathcal{S}_i^{t}(\widetilde{\mathcal{S}})} a_{j,i'}\right)\geq\delta_{A,j,i},\qquad\forall i\in\widetilde{\mathcal{S}}\quad\forall j\in\widetilde{\mathcal{P}} \tag{57}$$

$$\left(\sum_{i'\in\mathcal{S}_i^{sd}(\widetilde{\mathcal{S}},-1)\cap\mathcal{S}_i^{t}(\widetilde{\mathcal{S}})} a_{j,i'}\right)\leq M\delta_{A,j,i},\qquad\forall i\in\widetilde{\mathcal{S}}\quad\forall j\in\widetilde{\mathcal{P}} \tag{58}$$

$$\left(\sum_{i'\in\mathcal{S}_i^{sd}(\widetilde{\mathcal{S}},-1)} a_{j,i'}\right)\geq 1-\delta_{B,j,i},\qquad\forall i\in\widetilde{\mathcal{S}}\quad\forall j\in\widetilde{\mathcal{P}} \tag{59}$$

$$\left(\sum_{i'\in\mathcal{S}_i^{sd}(\widetilde{\mathcal{S}},-1)} a_{j,i'}\right)\leq M\left(1-\delta_{B,j,i}\right),\qquad\forall i\in\widetilde{\mathcal{S}}\quad\forall j\in\widetilde{\mathcal{P}} \tag{60}$$

$$a_{j,i}-\delta_{A,j,i}-\delta_{B,j,i}\leq 0,\qquad\forall i\in\widetilde{\mathcal{S}}\quad\forall j\in\widetilde{\mathcal{P}} \tag{61}$$

$$M\geq|\widetilde{\mathcal{S}}| \tag{62}$$

GC9: *For a staff member in $\widetilde{\mathcal{P}}$, the workload that they are assigned to from shifts in $\widetilde{\mathcal{S}}$ divided by their desired workload must be within v percent of the expected workload ratio. The expected workload ratio is the total workload of all shifts in $\widetilde{\mathcal{S}}$ divided by the total desired workload of all staff members in $\widetilde{\mathcal{P}}$.* The parameter $\eta=1000$ is also used here. The auxiliary variables used here are

$$r_e=\left(\sum_{i\in\widetilde{\mathcal{S}}} s_i^w\right)\bigg/\left(\sum_{j\in\widetilde{\mathcal{P}}} p_j^{dw}\right) \tag{63}$$

$$\lfloor\eta r_e(1-v)\rfloor\leq\sum_{i\in\widetilde{\mathcal{S}}} a_{j,i}\left\lfloor\eta\frac{s_i^w}{p_j^{dw}}\right\rfloor,\qquad\forall j\in\widetilde{\mathcal{P}} \tag{64}$$

$$\lfloor \eta r_e(1+v) \rfloor \geq \sum_{i \in \widetilde{\mathcal{S}}} a_{j,i} \left\lfloor \eta \frac{s_i^w}{p_j^{dw}} \right\rfloor, \qquad \forall j \in \widetilde{\mathcal{P}} \tag{65}$$