A Comparative Study of SMT and MILP for the Nurse Rostering Problem

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Appendix: Details on the SMT and MILP Model Formulations

In this section, we provide additional descriptions of the generic constraints (GCs) along with detailed SMT and MILP formulations.

A. Descriptions of the Generic Constraints

The constraints are modelled in a generic fashion, where each constraint defines a framework within which input parameters apply specificity. Each *generic constraint* (GC) can be instanced any number of times with their own parameters. By taking this approach, the GCs are able to model a wide range of constraints and can therefore be used in organisations with specific requirements without needing specific mathematical formulations. The input parameters take the forms $\widetilde{\mathcal{P}} \subseteq \mathcal{P}$, $\widetilde{\mathcal{S}} \subseteq \mathcal{S}$, $x, y \in \mathbb{N}$, and $u, v \in \Re$, determining the specific constraints that they model.

GC1: *The number of shifts in $\widetilde{\mathcal{S}}$ that are not assigned at least one person from $\widetilde{\mathcal{P}}$ must be greater than or equal to x and less than or equal to y.* GC1 models the degree to which a certain set of staff members is assigned to a certain set of shifts. There are a wide range of requirements that can be modelled using this GC, such as that all shifts should be assigned at least one person ($x = y = 0$), that none of a certain group of shifts should be assigned anyone from a group of staff members, that only two shifts during a specific week may be assigned to a certain person, etc.

GC2: *The number of shifts in $\widetilde{\mathcal{S}}$ that have been assigned an unqualified person from $\widetilde{\mathcal{P}}$ must be greater than or equal to x and less than or equal to y.* GC2 defines the degree to which staff members are assigned to shifts that they are not qualified for. Perhaps the most common example of a constraint that can be modelled with this GC, is that all staff members should be qualified for shifts that they are assigned to ($x = y = 0$). Another example is that a certain number of shifts may be assigned unqualified staff members, as may be the case if e.g. it is sufficient that one qualified nurse is present at any ward at a given moment.

GC3: *The number of times any person $j \in \widetilde{\mathcal{P}}$ is assigned to all shifts in a set of overlapping shifts $k \in \mathcal{O}$ for which they are not allowed to be assigned to ($j \notin \mathcal{P}_k$) must be greater than or equal to x and less than or equal to y.* GC3 concerns how staff members are assigned to shifts that occur at the same time. The overlapping combinations $k \in \mathcal{O}$ contain shifts that overlap and staff members that are allowed to be assigned to them or any subset of them. GC3 constrains the number of times a staff member is assigned to overlapping shifts that they are not allowed to be assigned to. By distinguishing between staff members being *allowed* to work overlapping shifts and being *assigned* to overlapping shifts, it is possible to model more types of requirements. For example, it may be acceptable for staff members to work overlapping shifts that they are not allowed to work, provided it happens a limited number of times. However, the typical use of this constraint is to ensure that no staff member is assigned to overlapping shifts that they are not allowed to work.

GC4: *For each person in $\widetilde{\mathcal{P}}$, the fraction of their assigned workload from shifts in $\widetilde{\mathcal{S}}$, if they are assigned to any at all, must be greater than or equal to u and less than or equal to v.* GC4 gives control over individual staff member's ratios of shifts that they are assigned to that belong to a certain set. For example, a certain percentage of the shifts a person is assigned to could be required to be of a specific type, such as if they have specific skills that require regular use to maintain.

GC5: *If any person in $\widetilde{\mathcal{P}}_1$ is assigned to a shift in $\widetilde{\mathcal{S}}_1$, then the number of assignments of people in $\widetilde{\mathcal{P}}_2$ to shifts in $\widetilde{\mathcal{S}}_2$ must be greater than or equal to x and less than or equal to y.* GC5 addresses how staff members are assigned to shifts in relation to each other. If any staff member of one group is assigned to any shift in a set, then the degree to which staff members of another group are assigned to another set of shifts can be constrained with this GC. Note that the groups of staff members may overlap, as may the sets of shifts. For example, GC5 can ensure that a certain staff member does or does not work on the same day as someone else by including all shifts on that day in both shift sets, each person in each staff set, and setting x and y to either 1 or 0. It can also ensure that if a staff member is assigned to one shift, they should not be assigned to another.

GC6: *The number of consecutive days of shifts in $\widetilde{\mathcal{S}}$ assigned to a person in $\widetilde{\mathcal{P}}$ must be greater than or equal to x and less than or equal to y.* GC6 constrains the number of consecutive days of assignments to certain shifts for certain people. Besides simply limiting the minimum and maximum number of consecutive days a staff member can work, there are other useful ways to use GC6. For example, by setting the minimum and maximum to 2 for weekend shifts, it can be used to define that a staff member assigned to one weekend day must also be assigned to the other.

GC7: *For a person in $\widetilde{\mathcal{P}}$ that is assigned to shifts in $\widetilde{\mathcal{S}}$ for x to y consecutive days, the n days immediately before must be without assignments to shifts in $\widetilde{\mathcal{S}}_1$ and m days immediately after must be without assignments to shifts in $\widetilde{\mathcal{S}}_2$.* GC7 is related

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to GC6, but addresses the days preceding and following consecutive days of being assigned to certain shifts. Essentially, if a staff member has been assigned shifts in a set for a number of consecutive days between a minimum and maximum, then they cannot be assigned to shifts in another set for a certain number of preceding days. Similarly, they cannot be assigned to shifts in a third set for a certain number of following days. Less straightforwardly, GC7 can be used to ensure that when a staff member is assigned to a weekend shift, they will not be assigned to other weekend shift some weeks before and after.

GC8: *If a staff member in $\widetilde{\mathcal{P}}$ is assigned to a shift in $\widetilde{\mathcal{S}}$, then, on the day before the assigned shift's start day, the staff member must either be assigned to a shift in $\widetilde{\mathcal{S}}$ with the same shift type or not be assigned to any shift in $\widetilde{\mathcal{S}}$.* GC8 ensures continuity of shift types over consecutive days of assignments. In many situations, it could be beneficial to avoid switching between shift types during consecutive days of work to achieve a more efficient workflow. One approach could be to ensure that all shifts during consecutive days of work are of the same type. However, this does not allow for multiple shift assignments on the same day, such as being assigned to overlapping shifts. Therefore, the constraint is formulated such that for each assignment to a shift, the staff member must, on the day before, either be assigned to a shift of the same type or not assigned to any shifts at all. This is the same as ensuring that the set of assigned shifts on one day is a subset of the set of assigned shifts on the day before, unless there are no assigned shifts during that day. It is then possible to be assigned to multiple shifts on each day over consecutive days. This formulation, however, does not allow for switching from consecutive days of one shift type to another without a day with no assignments in between, since then the following days assignments will not be a subset of the previous days assignments.

GC9: *For a staff member in $\widetilde{\mathcal{P}}$, the workload that they are assigned to from shifts in $\widetilde{\mathcal{S}}$ divided by their desired workload must be within v percent of the expected workload ratio. The expected workload ratio is the total workload of all shifts in $\widetilde{\mathcal{S}}$ divided by the total desired workload of all staff members in $\widetilde{\mathcal{P}}$.* GC9 addresses fairness of shift assignments between staff members. Each shift has a workload and each staff member has a desired workload, thus a measurement of a staff member's satisfaction is here defined as the total workload from their assigned shifts in $\widetilde{\mathcal{S}9}$ divided by their desired workload. Therefore, a measurement of fairness between the personnel in $\widetilde{\mathcal{P}9}$ with respect to the shifts in $\widetilde{\mathcal{S}9}$ is defined here as the difference between a staff member's workload ratio and the expected workload ratio. Limiting this difference therefore increases the fairness between staff members.

B. Notation

In this section, we define helpful notation. *Start Day* returns the set of all shifts in a given set that start on a given day,

$$\mathcal{S}^{sd}(\widetilde{\mathcal{S}}, k) = \left\{ i \in \widetilde{\mathcal{S}} \mid sd_i = k \right\}.$$

Relative Start Day is similar to Start Day but relative to a given shift,

$$\mathcal{S}_i^{sd}(\widetilde{\mathcal{S}}, k) = \left\{ i' \in \widetilde{\mathcal{S}} \mid sd_{i'} = sd_i + k \right\}.$$

Type Set returns the subset of shifts with a given type,

$$\mathcal{S}^{t}(\widetilde{\mathcal{S}}, t) = \left\{ i' \in \widetilde{\mathcal{S}} \mid t_{i'} = t \right\}.$$

Finally, *Relative Type Set* returns the subset of shifts of the same shift type as a given shift,

$$\mathcal{S}_i^{t}(\widetilde{\mathcal{S}}) = \left\{ i' \in \widetilde{\mathcal{S}} \mid t_{i'} = t_i \right\}.$$

The aim is to find an assignment of variables such that all constraints (every instance of every GC) is satisfied, as well as the initial problem constraint of each shift only being assigned one staff member.

C. SMT Formulation

The SMT formulation of the GCs are described in this section. For each person $j \in \mathcal{P}$ and shift $i \in \mathcal{S}$, the Boolean variable $a_{j,i}$ denotes if j is assigned to i. Every instance of a GC formulation (with specific parameter values) adds constraints to the problem. Thus, in the following formulations, all parameters should be interpreted as specific for a GC instance. The *Boolean to Binary* operator maps a Boolean evaluation to binary,

$$\oplus\omega = \begin{cases} 1 & \text{if } \omega \text{ is true} \\ 0 & \text{otherwise.} \end{cases}$$

Boolean Summation extends this to count the number of *True* evaluations for a function over a set Ω,

$$\bigoplus_{\omega \in \Omega} f(\omega) = \sum_{\omega \in \Omega} \oplus f(\omega)$$

We now formulate the constraints. Only one staff member can be assigned to a shift, formulated in (1).

$$\left(\bigoplus_{j\in\mathcal{P}} a_{j,i}\right) \leq 1, \quad \forall i \in \mathcal{S} \tag{1}$$

GC1: *The number of shifts in $\widetilde{\mathcal{S}}$ that are not assigned at least one person from $\widetilde{\mathcal{P}}$ must be greater than or equal to x and less than or equal to y.*

$$x \leq \left(\bigoplus_{i\in\widetilde{\mathcal{S}}} \bigwedge_{j\in\widetilde{\mathcal{P}}} \neg a_{j,i}\right) \leq y \tag{2}$$

GC2: *The number of shifts in $\widetilde{\mathcal{S}}$ that have been assigned an unqualified person from $\widetilde{\mathcal{P}}$ must be greater than or equal to x and less than or equal to y.*

$$x \leq \left(\bigoplus_{i\in\widetilde{\mathcal{S}}} \bigvee_{j\in\widetilde{\mathcal{P}}\setminus QP_i} a_{j,i}\right) \leq y \tag{3}$$

GC3: *The number of times any person $j \in \widetilde{\mathcal{P}}$ is assigned to all shifts in a set of overlapping shifts $k \in \mathcal{O}$ for which they are not allowed to be assigned to ($j \notin \mathcal{P}_k$) must be greater than or equal to x and less than or equal to y.* We first define the set $\mathcal{O}_j^A \subseteq \mathcal{O}$ containing every 2-size $k \in \mathcal{O}$ that staff member j is *allowed* to be assigned to ($j \in \mathcal{P}_k$),

$$\mathcal{O}_j^A = \left\{\mathcal{S}_k \,\middle|\, |\mathcal{S}_k| = 2, j \in \mathcal{P}_k, k \in \mathcal{O}\right\}, \ \forall j \in \widetilde{\mathcal{P}}. \tag{4}$$

The set $\mathcal{O}_j^F$ contains the 2-size overlapping combinations in $k \in \mathcal{O}$ that j is *forbidden* from being assigned to,

$$\mathcal{O}_j^F = \left\{k \in \mathcal{O} \,\middle|\, |\mathcal{S}_k| = 2, \mathcal{S}_k \notin \mathcal{O}_j^A\right\}, \quad \forall j \in \widetilde{\mathcal{P}}. \tag{5}$$

Finally, this formulation is concluded by constraining the sum over all staff members in $j \in \widetilde{\mathcal{P}}$ and overlapping combinations in $k \in \mathcal{O}_j^F$ where j is assigned to all shifts in $\mathcal{S}_k$:

$$x \leq \left(\sum_{j\in\widetilde{\mathcal{P}}} \bigoplus_{k\in\mathcal{O}_j^F} \bigwedge_{i\in\mathcal{S}_k} a_{j,i}\right) \leq y. \tag{6}$$

GC4: *For each person in $\widetilde{\mathcal{P}}$, the fraction of their assigned workload from shifts in $\widetilde{\mathcal{S}}$, if they are assigned to any at all, must be greater than or equal to u and less than or equal to v.* This formulation ensures integer values are used, since it was found during implementation to lead to significantly better performance of the SMT solver. The parameter $\eta = 1000$ is used to scale values, which are then rounded.

$$\omega_{u,j} = \sum_{i\in\mathcal{S}} \lfloor \eta w_i u \rfloor \oplus a_{j,i}, \quad \forall j \in \widetilde{\mathcal{P}} \tag{7}$$

$$\omega_{v,j} = \sum_{i\in\mathcal{S}} \lfloor \eta w_i v \rfloor \oplus a_{j,i}, \quad \forall j \in \widetilde{\mathcal{P}} \tag{8}$$

$$\omega_{u,j} \leq \left(\sum_{i\in\widetilde{\mathcal{S}}} \lfloor \eta w_i \rfloor \oplus i a_{j,i}\right) \leq \omega_{v,j}, \quad \forall j \in \widetilde{\mathcal{P}} \tag{9}$$

GC5: *If any person in $\widetilde{\mathcal{P}}_1$ is assigned to a shift in $\widetilde{\mathcal{S}}_1$, then the number of assignments of people in $\widetilde{\mathcal{P}}_2$ to shifts in $\widetilde{\mathcal{S}}_2$ must be greater than or equal to x and less than or equal to y.*

$$\omega = \bigoplus_{i\in\widetilde{\mathcal{S}}_2} \bigvee_{j\in\widetilde{\mathcal{P}}_2} a_{j,i} \tag{10}$$

$$\left(\bigvee_{i\in\widetilde{\mathcal{S}}_1} \bigvee_{j\in\widetilde{\mathcal{P}}_1} a_{j,i}\right) \Rightarrow (x \leq \omega \leq y) \tag{11}$$

GC6: *The number of consecutive days of shifts in $\widetilde{\mathcal{S}}$ assigned to a person in $\widetilde{\mathcal{P}}$ must be greater than or equal to x and less than or equal to y.*

$$\left(a_{j,i} \wedge \bigwedge_{i' \in \mathcal{S}_i^{sd}(\widetilde{\mathcal{S}},-1)} \neg a_{j,i'}\right) \Rightarrow \left(\bigwedge_{c=1}^{x-1} \bigvee_{i' \in \mathcal{S}_i^{sd}(\widetilde{\mathcal{S}},c)} a_{j,i'}\right), \qquad \forall j \in \widetilde{\mathcal{P}} \quad \forall i \in \widetilde{\mathcal{S}} \tag{12}$$

$$\left(a_{j,i} \wedge \bigwedge_{i' \in \mathcal{S}_i^{sd}(\widetilde{\mathcal{S}},-1)} \neg a_{j,i'}\right) \Rightarrow \left(\bigvee_{c=1}^{y} \bigwedge_{i' \in \mathcal{S}_i^{sd}(\widetilde{\mathcal{S}},c)} \neg a_{j,i'}\right), \qquad \forall j \in \widetilde{\mathcal{P}} \quad \forall i \in \widetilde{\mathcal{S}} \tag{13}$$

GC7: *For a person in $\widetilde{\mathcal{P}}$ that is assigned to shifts in $\widetilde{\mathcal{S}}$ for x to y consecutive days, the n days immediately before must be without assignments to shifts in $\widetilde{\mathcal{S}}_1$ and m days immediately after must be without assignments to shifts in $\widetilde{\mathcal{S}}_2$.*

$$\begin{gathered}\left(\left(\bigwedge_{\widetilde{c}=d}^{d+c-1} \bigvee_{i \in \mathcal{S}^{sd}(\widetilde{\mathcal{S}},\widetilde{c})} a_{j,i}\right) \wedge \left(\bigwedge_{i \in \mathcal{S}^{sd}(\widetilde{\mathcal{S}},d-1) \cup \mathcal{S}^{sd}(\widetilde{\mathcal{S}},d+c)} \neg a_{j,i}\right)\right) \Rightarrow \\ \left(\left(\bigwedge_{\widetilde{n}=d-n}^{d-1} \bigwedge_{i \in \mathcal{S}^{sd}(\widetilde{\mathcal{S}}_1,\widetilde{n})} \neg a_{j,i}\right) \wedge \left(\bigwedge_{\widetilde{m}=d+c}^{d+c+m-1} \bigwedge_{i \in \mathcal{S}^{sd}(\widetilde{\mathcal{S}}_2,\widetilde{m})} \neg a_{j,i}\right)\right), \\ \forall c \in [x\,..\,y] \quad \forall d \in \mathcal{D} \quad \forall j \in \widetilde{\mathcal{P}}\end{gathered} \tag{14}$$

GC8: *If a staff member in $\widetilde{\mathcal{P}}$ is assigned to a shift in $\widetilde{\mathcal{S}}$, then, on the day before the assigned shift's start day, the staff member must either be assigned to a shift in $\widetilde{\mathcal{S}}$ with the same shift type or not be assigned to any shift in $\widetilde{\mathcal{S}}$.*

$$\begin{gathered} a_{j,i} \Rightarrow \left(\left(\bigvee_{i' \in \mathcal{S}_i^{sd}(\widetilde{\mathcal{S}},-1) \cap \mathcal{S}_i^{t}(\widetilde{\mathcal{S}})} a_{j,i'}\right) \vee \left(\bigwedge_{i' \in \mathcal{S}_i^{sd}(\widetilde{\mathcal{S}},-1)} \neg a_{j,i'}\right)\right), \\ \forall i \in \widetilde{\mathcal{S}} \quad \forall j \in \widetilde{\mathcal{P}} \end{gathered} \tag{15}$$

GC9: *For a staff member in $\widetilde{\mathcal{P}}$, the workload that they are assigned to from shifts in $\widetilde{\mathcal{S}}$ divided by their desired workload must be within v percent of the expected workload ratio. The expected workload ratio is the total workload of all shifts in $\widetilde{\mathcal{S}}$ divided by the total desired workload of all staff members in $\widetilde{\mathcal{P}}$.* The parameter $\eta = 1000$ is also used here.

$$r_e = \left(\sum_{i \in \widetilde{\mathcal{S}}} w_i\right) \Bigg/ \left(\sum_{j \in \widetilde{\mathcal{P}}} dw_j\right) \tag{16}$$

$$r_j = \sum_{i \in \widetilde{\mathcal{S}}} \left\lfloor \eta \frac{w_i}{dw_j} \right\rfloor \oplus a_{j,i}, \quad \forall j \in \widetilde{\mathcal{P}} \tag{17}$$

$$\lfloor \eta r_e (1 - v) \rfloor \le r_j \le \lfloor \eta r_e (1 + v) \rfloor, \quad \forall j \in \widetilde{\mathcal{P}} \tag{18}$$

D. MILP Formulation

The MILP formulation uses 0-1-variables: $a_{j,i} = 1$ if person j is assigned to shift i, 0 otherwise. Besides these, additional auxiliary 0-1-variables δ are used in all GC formulations (expect GC9), which should be interpreted as specific to each GC instance. Whenever the big-M method is used, an appropriate value is given. First, the constraint that only one person can be assigned to a shift is formulated as

$$\sum_{j \in \mathcal{P}} a_{j,i} \le 1, \quad \forall i \in \mathcal{S} \tag{19}$$

GC1: *The number of shifts in $\widetilde{\mathcal{S}}$ that are not assigned at least one person from $\widetilde{\mathcal{P}}$ must be greater than or equal to x and less than or equal to y.* This formulation uses auxiliary variables $\delta_i, \forall i \in \widetilde{\mathcal{S}}$, which are modeled in (20) using the big-M method ($M \ge |\widetilde{\mathcal{P}}|$) such that $\delta_i = 1$ if no one in $\widetilde{\mathcal{P}}$ is assigned to shift i, 0 otherwise.

$$1 - \delta_i \le \sum_{j \in \widetilde{\mathcal{P}}} a_{j,i} \le M\left(1 - \delta_i\right), \quad \forall i \in \widetilde{\mathcal{S}} \tag{20}$$

$$x \leq \sum_{i \in \widetilde{\mathcal{S}}} \delta_i \leq y \tag{21}$$

GC2: *The number of shifts in $\widetilde{\mathcal{S}}$ that have been assigned an unqualified person from $\widetilde{\mathcal{P}}$ must be greater than or equal to x and less than or equal to y.* This formulation uses auxiliary variables $\delta_i, \forall i \in \widetilde{\mathcal{S}}$, which are modeled in (22) using the big-M method ($M \geq |\widetilde{\mathcal{P}}|$) such that $\delta_i = 1$ if at least one unqualified person in $\widetilde{\mathcal{P}}$ is assigned to shift i.

$$\delta_i \leq \sum_{j \in \widetilde{\mathcal{P}} \setminus QP_i} a_{j,i} \leq M\delta_i, \quad \forall i \in \widetilde{\mathcal{S}} \tag{22}$$

$$x \leq \sum_{i \in \widetilde{\mathcal{S}}} \delta_i \leq y \tag{23}$$

GC3: *The number of times any person $j \in \widetilde{\mathcal{P}}$ is assigned to all shifts in a set of overlapping shifts $k \in \mathcal{O}$ for which they are not allowed to be assigned to ($j \notin \mathcal{P}_k$) must be greater than or equal to x and less than or equal to y.* This formulation uses the sets $\mathcal{O}_j^A$ and $\mathcal{O}_j^F$ from (4) and (5), respectively. Auxiliary variables $\delta_{k,j}$, $\forall k \in \mathcal{O}_j^F$, $\forall j \in \widetilde{\mathcal{P}}$, are modeled in (24) such that $\delta_{k,j} = 1$ if j is assigned to all shifts in $\mathcal{S}_k$, 0 otherwise.

$$2\delta_{k,j} \leq \sum_{i \in \mathcal{S}_k} a_{j,i} \leq \delta_{k,j} + 1, \quad \forall k \in \mathcal{O}_j^F,\ \forall j \in \widetilde{\mathcal{P}} \tag{24}$$

$$x \leq \sum_{j \in \widetilde{\mathcal{P}}} \sum_{o_k \in \mathcal{O}_j^F} \delta_{k,j} \leq y \tag{25}$$

GC4: *For each person in $\widetilde{\mathcal{P}}$, the fraction of their assigned workload from shifts in $\widetilde{\mathcal{S}}$, if they are assigned to any at all, must be greater than or equal to u and less than or equal to v.* Only one auxiliary variable is used in this formulation: $\delta = 1$ if any person in $\widetilde{\mathcal{P}}$ is assigned to any shift in $\widetilde{\mathcal{S}}$, 0 otherwise. The parameter $\eta = 1000$ is used here, similar to the SMT formulation, simply for the sake of applying the same model to both solvers for a fair comparison. The use of η and rounding lead to a negligible impact on the performance of Gurobi.

$$\omega_{u,j} = \sum_{i \in \mathcal{S}} \lfloor \eta w_i u \rfloor a_{j,i}, \qquad \forall j \in \widetilde{\mathcal{P}} \tag{26}$$

$$\omega_{v,j} = \sum_{i \in \mathcal{S}} \lfloor \eta w_i v \rfloor a_{j,i}, \qquad \forall j \in \widetilde{\mathcal{P}} \tag{27}$$

$$\omega_{u,j} \leq \left(\sum_{i \in \widetilde{\mathcal{S}}} \lfloor \eta w_i \rfloor a_{j,i} \right) \leq \omega_{v,j}, \qquad \forall j \in \widetilde{\mathcal{P}} \tag{28}$$

GC5: *If any person in $\widetilde{\mathcal{P}}_1$ is assigned to a shift in $\widetilde{\mathcal{S}}_1$, then the number of assignments of people in $\widetilde{\mathcal{P}}_2$ to shifts in $\widetilde{\mathcal{S}}_2$ must be greater than or equal to x and less than or equal to y.* The auxiliary variables used here are:

- $\delta = 1$ if any person in $\widetilde{\mathcal{P}}_1$ is assigned to any shift in $\widetilde{\mathcal{S}}_1$, 0 otherwise.
- $\forall j \in \widetilde{\mathcal{P}}_2, \forall i \in \widetilde{\mathcal{S}}_2$: $\delta_{j,i} = 1$ if person j is assigned to shift i and $\delta = 1$, 0 otherwise.

$$x \leq \left(\sum_{i \in \widetilde{\mathcal{S}}_2} \sum_{j \in \widetilde{\mathcal{P}}_2} \delta_{j,i} \right) \leq y \tag{29}$$

$$\left(\sum_{i \in \widetilde{\mathcal{S}}_1} \sum_{j \in \widetilde{\mathcal{P}}_1} a_{j,i} \right) \geq \delta \tag{30}$$

$$\left(\sum_{i \in \widetilde{\mathcal{S}}_1} \sum_{j \in \widetilde{\mathcal{P}}_1} a_{j,i} \right) \leq M\delta \tag{31}$$

$$\delta + a_{j,i} \leq 1 + \delta_{j,i}, \qquad \forall i \in \widetilde{\mathcal{S}}_2 \quad \forall j \in \widetilde{\mathcal{P}}_2 \tag{32}$$

$$\delta \geq \delta_{j,i}, \qquad \forall i \in \widetilde{\mathcal{S}}_2 \quad \forall j \in \widetilde{\mathcal{P}}_2 \tag{33}$$

$$a_{j,i} \geq \delta_{j,i}, \qquad \forall i \in \widetilde{\mathcal{S}}_2 \quad \forall j \in \widetilde{\mathcal{P}}_2 \tag{34}$$

$$M \geq |\widetilde{\mathcal{P}}_1||\widetilde{\mathcal{S}}_1| \tag{35}$$

GC6: *The number of consecutive days of shifts in $\widetilde{\mathcal{S}}$ assigned to a person in $\widetilde{\mathcal{P}}$ must be greater than or equal to x and less than or equal to y.* The auxiliary variables used here are:

- $\forall j \in \widetilde{\mathcal{P}}$, $\forall i \in \widetilde{\mathcal{S}}$: $\delta_{j,i} = 1$ if person j is assigned to shift i and not assigned to any shifts in $\widetilde{\mathcal{S}}$ on the day before, 0 otherwise.
- $\forall j \in \widetilde{\mathcal{P}}$, $\forall i \in \widetilde{\mathcal{S}}$, $\forall c \in [1\,..\,x]$: $\delta_{A,j,i,c} = 1$ if person j is assigned to any shift in $\widetilde{\mathcal{S}}$ starting c days after shift i, 0 otherwise.
- $\forall j \in \widetilde{\mathcal{P}}$, $\forall i \in \widetilde{\mathcal{S}}$: $\delta_{A,j,i} = 1$ if person j is assigned to shift in $\widetilde{\mathcal{S}}$ every 1 to $x-1$ days after shift i, 0 otherwise.
- $\forall j \in \widetilde{\mathcal{P}}$, $\forall i \in \widetilde{\mathcal{S}}$, $\forall c \in [1\,..\,y]$: $\delta_{B,j,i,c} = 1$ if person j is not assigned to any shift in $\widetilde{\mathcal{S}}$ c days after shift i, 0 otherwise.
- $\forall j \in \widetilde{\mathcal{P}}$, $\forall i \in \widetilde{\mathcal{S}}$: $\delta_{B,j,i} = 1$ if on any of the 1 to $x-1$ days after shift i, person j is not assigned to any shift in $\widetilde{\mathcal{S}}$, 0 otherwise.

$$(1 - a_{j,i}) + \left(\sum_{i' \in \mathcal{S}_i^{sd}(\widetilde{\mathcal{S}}, -1)} a_{j,i'} \right) \geq (1 - \delta_{j,i}), \qquad \forall i \in \widetilde{\mathcal{S}} \quad \forall j \in \widetilde{\mathcal{P}} \tag{36}$$

$$(1 - a_{j,i}) + \left(\sum_{i' \in \mathcal{S}_i^{sd}(\widetilde{\mathcal{S}}, -1)} a_{j,i'} \right) \leq M_1 (1 - \delta_{j,i}), \qquad \forall s_i \in \widetilde{\mathcal{S}} \quad \forall j \in \widetilde{\mathcal{P}} \tag{37}$$

$$\left(\sum_{i' \in \mathcal{S}_i^{sd}(\widetilde{\mathcal{S}}, c)} a_{j,i'} \right) \geq \delta_{A,j,i,c}, \qquad \forall c \in [1\,..\,x-1] \quad \forall i \in \widetilde{\mathcal{S}} \quad \forall j \in \widetilde{\mathcal{P}} \tag{38}$$

$$\left(\sum_{i' \in \mathcal{S}_i^{sd}(\widetilde{\mathcal{S}}, c)} a_{j,i'} \right) \leq M_1 \delta_{A,j,i,c}, \qquad \forall c \in [1\,..\,x-1] \quad \forall i \in \widetilde{\mathcal{S}} \quad \forall j \in \widetilde{\mathcal{P}} \tag{39}$$

$$\left(\sum_{c=1}^{x-1} 1 - \delta_{A,j,i,c} \right) \geq 1 - \delta_{A,j,i}, \qquad \forall i \in \widetilde{\mathcal{S}} \quad \forall j \in \widetilde{\mathcal{P}} \tag{40}$$

$$\left(\sum_{c=1}^{x-1} 1 - \delta_{A,j,i,c} \right) \leq M_2 (1 - \delta_{A,j,i}), \qquad \forall i \in \widetilde{\mathcal{S}} \quad \forall j \in \widetilde{\mathcal{P}} \tag{41}$$

$$\delta_{j,i} - \delta_{A,j,i} \leq 0, \qquad \forall i \in \widetilde{\mathcal{S}} \quad \forall j \in \widetilde{\mathcal{P}} \tag{42}$$

$$\left(\sum_{i' \in \mathcal{S}_i^{sd}(\widetilde{\mathcal{S}}, c)} a_{j,i'} \right) \geq 1 - \delta_{B,j,i,c}, \qquad \forall c \in [1\,..\,y] \quad \forall i \in \widetilde{\mathcal{S}} \quad \forall j \in \widetilde{\mathcal{P}} \tag{43}$$

$$\left(\sum_{i' \in \mathcal{S}_i^{sd}(\widetilde{\mathcal{S}}, c)} a_{j,i'} \right) \leq M_1 (1 - \delta_{B,j,i,c}), \qquad \forall c \in [1\,..\,y] \quad \forall i \in \widetilde{\mathcal{S}} \quad \forall j \in \widetilde{\mathcal{P}} \tag{44}$$

$$\left(\sum_{c=1}^{y} \delta_{B,j,i,c} \right) \geq \delta_{B,j,i}, \qquad \forall i \in \widetilde{\mathcal{S}} \quad \forall j \in \widetilde{\mathcal{P}} \tag{45}$$

$$\left(\sum_{c=1}^{y} \delta_{B,j,i,c} \right) \leq M_3 \delta_{B,j,i}, \qquad \forall i \in \widetilde{\mathcal{S}} \quad \forall j \in \widetilde{\mathcal{P}} \tag{46}$$

$$\delta_{j,i} - \delta_{B,j,i} \leq 0, \qquad \forall i \in \widetilde{\mathcal{S}} \quad \forall j \in \widetilde{\mathcal{P}} \tag{47}$$

$$M_1 \geq |\widetilde{\mathcal{S}}|, \quad M_2 \geq x - 1, \quad M_3 \geq y \tag{48}$$

GC7: *For a person in $\widetilde{\mathcal{P}}$ that is assigned to shifts in $\widetilde{\mathcal{S}}$ for x to y consecutive days, the n days immediately before must be without assignments to shifts in $\widetilde{\mathcal{S}}_1$ and m days immediately after must be without assignments to shifts in $\widetilde{\mathcal{S}}_2$.* The auxiliary variables used here are:

- $\forall j \in \widetilde{\mathcal{P}}$, $\forall d \in \mathcal{D}$: $\delta_{j,d} = 1$ if person j is assigned to at least one shift in $\widetilde{\mathcal{S}}$ on day d, 0 otherwise.
- $\forall j \in \widetilde{\mathcal{P}}$, $\forall d \in \mathcal{D}$, $\forall c \in [x\,..\,y]$: $\delta_{A,j,d,c} = 1$ if person j is assigned to at least one shift in $\widetilde{\mathcal{S}}$ on every day from d to $d + c - 1$, and not assigned to any shifts in $\widetilde{\mathcal{S}}$ on the days $d - 1$ or $d + c$, 0 otherwise.

- $\forall j \in \widetilde{\mathcal{P}}$, $\forall d \in \mathcal{D}$, $\forall c \in [x\,..\,y]$: $\delta_{B,j,d,c} = 1$ if person j is not assigned to any shifts in $\widetilde{\mathcal{S}}_1$ on any of the days from $d-n$ to $d-1$, and not assigned to any shifts in $\widetilde{\mathcal{S}}_2$ on any of the days from $d+c$ to $d+c+m-1$, otherwise 0.

$$\left(\sum_{i \in \mathcal{S}^{sd}(\widetilde{\mathcal{S}},d)} a_{j,i}\right) \geq \delta_{j,d}, \qquad \forall d \in \mathcal{D} \quad \forall j \in \widetilde{\mathcal{P}} \tag{49}$$

$$\left(\sum_{i \in \mathcal{S}^{sd}(\widetilde{\mathcal{S}},d)} a_{j,i}\right) \leq M_1 \delta_{j,d}, \qquad \forall d \in \mathcal{D} \quad \forall j \in \widetilde{\mathcal{P}} \tag{50}$$

$$\left(\sum_{\widetilde{c}=d}^{d+c-1} 1 - \delta_{j,\widetilde{c}}\right) + \left(\sum_{i \in \mathcal{S}^{sd}(\widetilde{\mathcal{S}},d-1) \cup \mathcal{S}^{sd}(\widetilde{\mathcal{S}},d+c)} a_{j,i}\right) \geq 1 - \delta_{A,j,d,c}, \\ \forall c \in [x\,..\,y] \quad \forall d \in \mathcal{D} \quad \forall j \in \widetilde{\mathcal{P}} \tag{51}$$

$$\left(\sum_{\widetilde{c}=d}^{d+c-1} 1 - \delta_{j,\widetilde{c}}\right) + \left(\sum_{i \in \mathcal{S}^{sd}(\widetilde{\mathcal{S}},d-1) \cup \mathcal{S}^{sd}(\widetilde{\mathcal{S}},d+c)} a_{j,i}\right) \leq M_2 \left(1 - \delta_{A,j,d,c}\right), \\ \forall c \in [x\,..\,y] \quad \forall d \in \mathcal{D} \quad \forall j \in \widetilde{\mathcal{P}} \tag{52}$$

$$\left(\sum_{\widetilde{n}=d-n}^{d-1} \sum_{i \in \mathcal{S}^{sd}(\widetilde{\mathcal{S}}_1,\widetilde{n})} a_{j,i}\right) + \left(\sum_{\widetilde{m}=d+c}^{d+c+m-1} \sum_{i \in \mathcal{S}^{sd}(\widetilde{\mathcal{S}}_2,\widetilde{m})} a_{j,i}\right) \geq 1 - \delta_{B,j,d,c}, \\ \forall c \in [x\,..\,y] \quad \forall d \in \mathcal{D} \quad \forall j \in \widetilde{\mathcal{P}} \tag{53}$$

$$\left(\sum_{\widetilde{n}=d-n}^{d-1} \sum_{i \in \mathcal{S}^{sd}(\widetilde{\mathcal{S}}_1,\widetilde{n})} a_{j,i}\right) + \left(\sum_{\widetilde{m}=d+c}^{d+c+m-1} \sum_{i \in \mathcal{S}^{sd}(\widetilde{\mathcal{S}}_2,\widetilde{m})} a_{j,i}\right) \leq M_3 \left(1 - \delta_{B,j,d,c}\right), \\ \forall c \in [x\,..\,y] \quad \forall d \in \mathcal{D} \quad \forall j \in \widetilde{\mathcal{P}} \tag{54}$$

$$\delta_{A,j,d,c} - \delta_{B,j,d,c} \leq 0, \qquad \forall c \in [x\,..\,y] \quad \forall d \in \mathcal{D} \quad \forall j \in \widetilde{\mathcal{P}} \tag{55}$$

$$M_1 \geq |\widetilde{\mathcal{S}}|, \qquad M_2 \geq y + |\widetilde{\mathcal{S}}|, \qquad M_3 \geq |\widetilde{\mathcal{S}}_1| + |\widetilde{\mathcal{S}}_2| \tag{56}$$

GC8: *If a staff member in $\widetilde{\mathcal{P}}$ is assigned to a shift in $\widetilde{\mathcal{S}}$, then, on the day before the assigned shift's start day, the staff member must either be assigned to a shift in $\widetilde{\mathcal{S}}$ with the same shift type or not be assigned to any shift in $\widetilde{\mathcal{S}}$.* The auxiliary variables used here are:

- $\forall j \in \widetilde{\mathcal{P}}$, $\forall i \in \widetilde{\mathcal{S}}$: $\delta_{A,j,i} = 1$ if person j is assigned to a shift in $\widetilde{\mathcal{S}}$ of type t_i on the day before shift i, 0 otherwise.
- $\forall j \in \widetilde{\mathcal{P}}$, $\forall i \in \widetilde{\mathcal{S}}$: $\delta_{B,j,i} = 1$ if person j is not assigned to any shift in $\widetilde{\mathcal{S}}$ on the day before shift i, 0 otherwise.

$$\left(\sum_{i' \in \mathcal{S}_i^{sd}(\widetilde{\mathcal{S}},-1) \cap \mathcal{S}_i^{t}(\widetilde{\mathcal{S}})} a_{j,i'}\right) \geq \delta_{A,j,i}, \qquad \forall i \in \widetilde{\mathcal{S}} \quad \forall j \in \widetilde{\mathcal{P}} \tag{57}$$

$$\left(\sum_{i' \in \mathcal{S}_i^{sd}(\widetilde{\mathcal{S}},-1) \cap \mathcal{S}_i^{t}(\widetilde{\mathcal{S}})} a_{j,i'}\right) \leq M \delta_{A,j,i}, \qquad \forall i \in \widetilde{\mathcal{S}} \quad \forall j \in \widetilde{\mathcal{P}} \tag{58}$$

$$\left(\sum_{i' \in \mathcal{S}_i^{sd}(\widetilde{\mathcal{S}},-1)} a_{j,i'}\right) \geq 1 - \delta_{B,j,i}, \qquad \forall i \in \widetilde{\mathcal{S}} \quad \forall j \in \widetilde{\mathcal{P}} \tag{59}$$

$$\left(\sum_{i' \in \mathcal{S}_i^{sd}(\widetilde{\mathcal{S}},-1)} a_{j,i'}\right) \leq M \left(1 - \delta_{B,j,i}\right), \qquad \forall i \in \widetilde{\mathcal{S}} \quad \forall j \in \widetilde{\mathcal{P}} \tag{60}$$

$$a_{j,i} - \delta_{A,j,i} - \delta_{B,j,i} \leq 0, \qquad \forall i \in \widetilde{\mathcal{S}} \quad \forall j \in \widetilde{\mathcal{P}} \tag{61}$$

$$M \geq |\widetilde{\mathcal{S}}| \tag{62}$$

GC9: *For a staff member in $\widetilde{\mathcal{P}}$, the workload that they are assigned to from shifts in $\widetilde{\mathcal{S}}$ divided by their desired workload must be within v percent of the expected workload ratio. The expected workload ratio is the total workload of all shifts in $\widetilde{\mathcal{S}}$ divided by the total desired workload of all staff members in $\widetilde{\mathcal{P}}$.* The parameter $\eta = 1000$ is also used here. The auxiliary variables used here are

$$r_e = \left(\sum_{i \in \widetilde{\mathcal{S}}} w_i\right) \Big/ \left(\sum_{j \in \widetilde{\mathcal{P}}} dw_j\right) \tag{63}$$

$$\lfloor \eta r_e (1 - v) \rfloor \leq \sum_{i \in \widetilde{\mathcal{S}}} a_{j,i} \left\lfloor \eta \frac{w_i}{dw_j} \right\rfloor, \qquad \forall j \in \widetilde{\mathcal{P}} \tag{64}$$

$$\lfloor \eta r_e (1 + v) \rfloor \geq \sum_{i \in \widetilde{\mathcal{S}}} a_{j,i} \left\lfloor \eta \frac{w_i}{dw_j} \right\rfloor, \qquad \forall j \in \widetilde{\mathcal{P}} \tag{65}$$