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Essential properties for integer-valued polynomial rings

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Some papers

- M. Griffin, *Some results on v-multiplication rings*, Canad. J. Math. **19** (1967).
- J. Brewer and W. Heinzer, *Associated primes of principal ideals* Duke Math. J. (1974).
- J. Mott and M. Zafrullah, *On Prüfer v-multiplication domains*, Manuscripta Math. **35** (1981).
- D.D. Anderson, J. Mott and M. Zafrullah, *Finite character representations for integral domains*, Boll. Un. Mat. Ital. **6** (1992).
- P.J. Cahen, K. A. Loper and F.T., *Integer-valued polynomials and Prüfer v-multiplication domains*, J. Algebra (2000).
- A. Tamoussit and F. T., *Essential-type properties for integer-valued polynomial rings*, New York J. Math., to appear.
- A. Tamoussit and F. T., *On Weakly-Krull Domains of Integer-Valued Polynomials*, Ricerche di Matematica, to appear.

Let D be an integral domain with quotient field K.
The t-closure of a nonzero fractional ideal I of D is defined as follows: $I_t = \bigcup\{J_v; J \text{ is finitely generated, } J \subseteq I\}$
where $J_v = (J^{-1})^{-1}$ and $J^{-1} = (D\colon J) = \{x \in K; xJ \subseteq D\}$.
The ideal I is a t-ideal if $I = I_t$.
An ideal I is t-prime if it is prime and $I = I_t$ and it is t-maximal if it is maximal among the proper t-ideals of D.
Each t-maximal ideal is t-prime and each t-ideal is contained in a t-maximal ideal.

PvMD

An integral domain D is Prüfer v-multiplication (PvMD) if D_P is a valuation domain for each t-prime (or t-maximal) ideal of D.

Essential property

Given a subset $\mathcal{P}$ of $\mathrm{Spec}(D)$, we say that D is an essential domain with defining family $\mathcal{P}$ if $D = \cap_{\mathfrak{p} \in \mathcal{P}} D_{\mathfrak{p}}$ and each $D_{\mathfrak{p}}$ is a valuation domain.

PvMDs are essential domains

Following Lazard (Autour de la platitude, 1969) a prime ideal $\mathfrak{p}$ of D is an associated prime of an ideal I if there exists $b \in D \setminus I$ such that P is minimal over $(I : bD)$.

By Brewer-Heinzer (1974) a prime ideal $\mathfrak{p}$ of D is an associated prime (of a principal ideal aD) if $\mathfrak{p}$ is minimal over $(aD : bD)$ for some $b \in D \setminus aD$.

ce qui montre que $P' Y_i Y_{i+1} \ldots Y_n$ est dans l'idéal (primaire) engendré par les Y_h^2. Comme a est l'image de P' dans A, on obtient l'égalité cherchée $a y_i y_{i+1} \ldots y_n = 0$.

Chapitre II. — **Assassins.**

Dans ce chapitre et les suivants, tous les anneaux sont commutatifs.

1. **Généralités.**

Définition 1.1. — *Soient A un anneau, M un A-module et $\mathfrak{p}$ un idéal premier. On dit que $\mathfrak{p}$ est associé à M, s'il existe $x \in M$ tel que $\mathfrak{p}$ soit minimal parmi les idéaux premiers contenant l'annulateur de x. On appelle « assassin » de M, et on note $\mathrm{Ass}_A(M)$ ou $\mathrm{Ass}(M)$, l'ensemble des idéaux premiers associés à M.*

Bourbaki (dans [11], chap. IV, § 1, exerc. 17) ajoute le qualificatif « faible » à ces notions. Cela nous semble inutile, car, dans le cas noethérien, elles redonnent les notions classiques, et, dans le cas général, les notions classiques ont très peu d'intérêt.

Voici les principales propriétés de « Ass » telles qu'elles sont données par Bourbaki ([11], chap. IV, § 1, exerc. 17). Pour une démonstration détaillée, voir l'article de Merker [54].

Propriété 1.2. — La relation $M = 0$ équivaut à $\mathrm{Ass}(M) = \emptyset$.

Propriété 1.3. — Pour qu'un élément de A n'annule aucun élément de M, il faut et il suffit qu'il n'appartienne à aucun élément de $\mathrm{Ass}(M)$.

Propriété 1.4. — Soit $a \in A$; pour que tout élément de M soit annulé

Theorem [Lemma 1, BH74]

Let I be an ideal of a domain D, P be a prime ideal and S a multiplicative system of D with $P \cap S = \emptyset$. If P is an associated prime of I, then PD_S is an associated prime of ID_S. Conversely, if PD_P is an associated prime of ID_P, then P is an associated prime of I.

Theorem [Corollary 8 , BH74]

Let P be an associated prime in $D[X]$ and suppose that $\mathfrak{p} = P \cap D \neq (0)$. Then $\mathfrak{p}$ is an associated prime of D and $P = \mathfrak{p}[X]$.

P-domain (J. Mott and M. Zafrullah - 1981)

A domain D is a P-domain if D_P is a valuation domain for every associated prime.

In particular, a P-domain D is essential with defining family $\mathrm{Ass}(D)$ and Mott-Zafrullah showed that P-domains are exactly the integral domains such that their rings of fractions are essential domains.

Thus, these domains are also called Locally Essential domains.

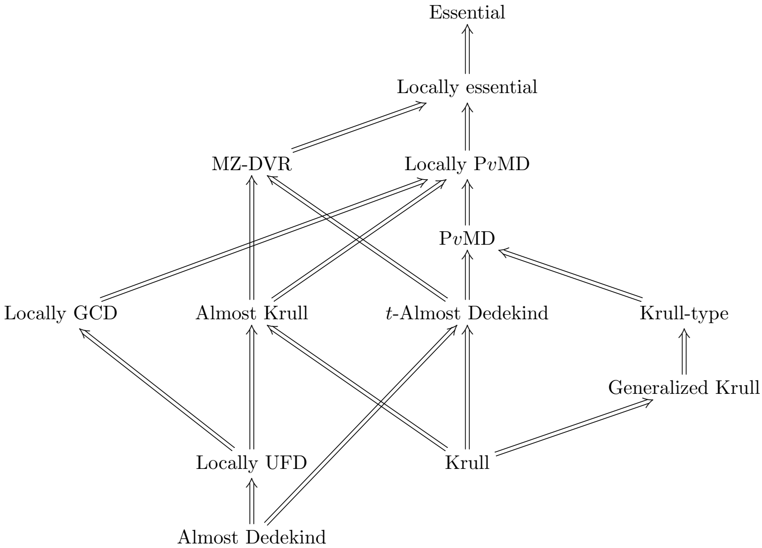
Locally finite intersection : given $D = \bigcap_{P \in \mathcal{P}} D_P$, $\mathcal{P} \subseteq \mathrm{Spec}(D)$, the intersection is locally finite if every nonzero element of D is contained in finitely many $P \in \mathcal{P}$.

A domain D is Krull if $D = \cap_{\mathfrak{p} \in X^1(D)} D_{\mathfrak{p}}$, where $X^1(D)$ is the set of the height-one prime ideals of D, $D_{\mathfrak{p}}$ are DVR and the intersection is locally finite.

A domain D is almost Krull if it is locally Krull.

A domain D is generalized Krull if $D = \cap_{\mathfrak{p} \in X^1(D)} D_{\mathfrak{p}}$, $D_{\mathfrak{p}}$ are one-dimensional valuation domains and the intersection is locally finite.

A domain D is Krull-type if $D = \bigcap_{P \in \mathcal{P}} D_P$, $\mathcal{P} \subseteq \mathrm{Spec}(D)$, D_P is a valuation domain for each $P \in \mathcal{P}$ and the intersection is locally finite.


Essential
Locally essential
MZ-DVR
Locally PvMD
PvMD
Locally GCD
Almost Krull
t-Almost Dedekind
Krull-type
Generalized Krull
Locally UFD
Krull
Almost Dedekind

A domain D is *t-almost Dedekind* (*almost Dedekind*) if it is t-locally DVR (locally DVR).

- Krull domains and almost Dedekind domains are t-almost Dedekind and t-almost Dedekind domains are PvMDs.

- PvMD $\not\Rightarrow$ t-almost Dedekind

  $\mathrm{Int}(\mathbb{Z})$ or $\mathbb{Z} + X\mathbb{Q}[X]$ because they are two-dimensional Prüfer domains

- t-almost Dedekind $\not\Rightarrow$ almost Dedekind

  $\mathbb{Z}[X]$ is a Krull domain, hence it is t-almost Dedekind but, since it is two-dimensional, it is not almost Dedekind.

- PvMD $\not\Rightarrow$ Krull-type

  $\mathrm{Int}(\mathbb{Z})$ is Prüfer, hence PvMD, but it is not Krull-type (because it has not the t-finite character)

The Integer-valued polynomial ring over D is the ring

$$\mathrm{Int}(D) := \{f \in K[X]; f(D) \subseteq D\}.$$

Given D we make a distinction between two types of prime ideals of D:

- int primes $\rightarrow$ primes $\mathfrak{p}$ such that $\mathrm{Int}(D) \not\subseteq D_{\mathfrak{p}}[X]$ (necessarily maximal with finite residue field);
- polynomial primes $\rightarrow$ primes $\mathfrak{p}$ such that $\mathrm{Int}(D) \subseteq D_{\mathfrak{p}}[X]$. In this case $\mathrm{Int}(D)_{\mathfrak{p}} = \mathrm{Int}(D)_{D \setminus \mathfrak{p}} = D_{\mathfrak{p}}[X]$.

int prime $\Rightarrow$ associated prime $\Rightarrow$ t-maximal

We partition the spectrum $\mathrm{Spec}(D)$ into two subsets:

- Δ_0 is the set of int prime ideals (these are all maximal);
- Δ_1 is the set of polynomial prime ideals.

We then set

$$D_0 := \bigcap_{\mathfrak{m} \in \Delta_0} D_{\mathfrak{m}}, \quad D_1 := \bigcap_{\mathfrak{p} \in \Delta_1} D_{\mathfrak{p}}.$$

- $D = D_0 \cap D_1$
- $\mathrm{Int}(D) = \mathrm{Int}(D_0) \cap \mathrm{Int}(D_1) = \mathrm{Int}(D_0) \cap D_1[X]$.
- $\mathrm{Int}(D_0)$ is the non polynomial part of $\mathrm{Int}(D)$ and $D_1[X]$ is the polynomial part of $\mathrm{Int}(D)$.

Theorem

Let D be an integral domain. Then D is

- essential, locally essential
- PvMD, locally PvMD
- Krull, Krull-type, almost Krull, generalized Krull

if and only if $D[X]$ so is.

The good behaviour of $D[X]$ with respect to these properties helps a lot when we focus on prime ideals of $\mathrm{Int}(D)$ contracting to polynomial primes of D.

An integral domain D has the finite character on a set of prime ideals $\mathcal{P}$ if every nonzero element of D belongs, at most, to finitely many primes of $\mathcal{P}$.
This is equivalent to say that the intersection $\cap_{\mathfrak{p}\in\mathcal{P}} D_{\mathfrak{p}}$ is locally finite (every nonzero element of D belongs to finitely many ideals of $\mathcal{P}$.

If $\mathcal{P}$ is the set of maximal ideals, then D is said to have the finite character.

If $\mathcal{P}$ is the set of t- maximal ideals, then D is said to be of t-finite character.

Noetherian domains has always the t-finite character but they may fail to have the finite character on maximal ideals (for instance $\mathbb{Z}[X]$ has not the finite character).

It is well-known that $D[X]$ has the t-finite character if and only if D has it.

This is not true, in general, for $\mathrm{Int}(D)$. In fact, if D has the t-finite character $\mathrm{Int}(D)$ may not have it. For instance $\mathrm{Int}(\mathbb{Z})$ does not have the t-finite character (and $\mathbb{Z}$ has it). Indeed $\mathrm{Int}(\mathbb{Z})$ is Prüfer, so each ideal is a t-ideal and the t-finite character is equivalent to the finite character on maximal ideals.

Proposition

If $\mathrm{Int}(D)$ is a locally finite intersection of a family of its localizations then D so is.

Corollary

If $\mathrm{Int}(D)$ has the t-finite (resp., finite) character then D has it too.

It is known that the polynomial ring $D[X]$ never has the finite character on maximal ideals, unless D is a field.

However, $\mathrm{Int}(D)$ may have the finite character on maximal ideals. Indeed, consider a one-dimensional, local, non unibranched Noetherian domain D. If $\mathfrak{m}$ is the maximal ideal of D, then the prime spectrum of $\mathrm{Int}(D)$ is made of the primes above $\mathfrak{m}$ and the primes above (0) and the primes ideals above $\mathfrak{m}$ are finitely many. The set of nonzero primes above (0) has the finite character (since they correspond to the nonzero primes of $K[X]$ which is Dedekind). So, we have that $\mathrm{Int}(D)$ has the finite character on maximal ideals

PvMD may not have the t-finite character: $\mathrm{Int}(\mathbb{Z})$ is Prüfer, hence PvMD, and it does not have the t-finite character.

Theorem (Griffin, 1967)

The following conditions are equivalent:

(i) D is a Krull type domain.

(ii) D is a PvMD with t-finite character.

Theorem (F.T.)

If D is a Krull-type domain, then $\mathrm{Int}(D)$ is a PvMD if and only if the following conditions hold:

(a) D is Krull-type;

(b) each int prime ideal of D is height-one.

Theorem(Tamoussit - T.)

$\mathrm{Int}(D)$ is Krull-type if and only if D is Krull-type and $\mathrm{Int}(D) = D[X]$.

Theorem (F.T.)

Let D be a Noetherian domain. Then $\mathrm{Int}(D)$ is PvMD if and only if D is Dedekind.

Theorem (Cahen-Loper-T.)

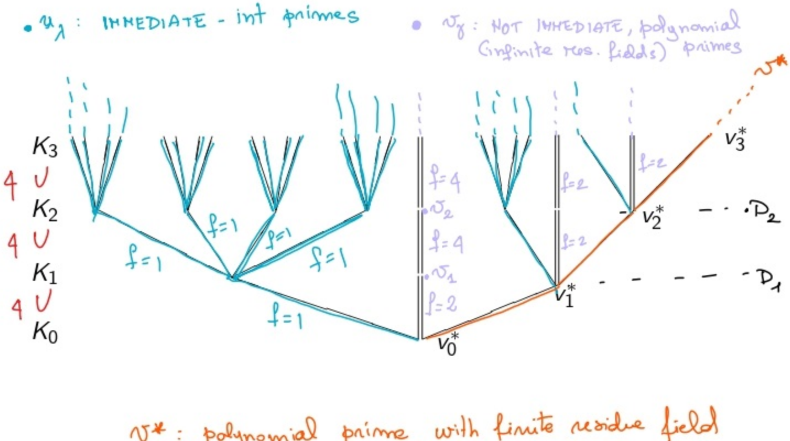
Let D be a domain. Then $\mathrm{Int}(D)$ is a PvMD if and only if the following conditions hold:

(a) D is a PvMD

(b) each int prime ideal of D is an height-one prime ideal

(c) each nonzero polynomial t-prime ideal of D contains a finitely generated ideal which is not contained in any int prime ideal

Using the above characterization for PvMDs it is possible to construct a domain $\mathrm{Int}(D)$ that is locally essential (and locally PvMD) but not PvMD:



All the results that follow are a part of a collaboration with A. Tamoussit.

Theorem 1

Let D be an integrally closed domain and suppose that for each int prime ideal $\mathfrak{p}$ of D, $D_{\mathfrak{p}}$ is a DVR. Then for each prime ideal $\mathfrak{P}$ of $\mathrm{Int}(D)$ above an int prime of D we have that $\mathrm{Int}(D)_{\mathfrak{P}}$ is a valuation domain.

This result was proved by Cahen-Loper-T. when D is a domain such that $\mathrm{Int}(D)$ is a PvMD. The properties of D really needed by the arguments of the proof are that D is an integrally closed domain and that $D_{\mathfrak{p}}$ is a DVR for each int prime ideal $\mathfrak{p}$ of D .

Proposition 2

Let D be a locally essential domain. Then, for each int prime ideal $\mathfrak{m}$ of D, the integral domain $D_{\mathfrak{m}}$ is a valuation domain with maximal principal ideal (in particular, $\mathfrak{m}$ is a t-ideal).

The argument used to prove Proposition 2 strongly needs that, for any int prime ideal $\mathfrak{m}$ of D, $D_{\mathfrak{m}}$ is essential and so $D_{\mathfrak{m}} = \cap D_{\mathfrak{p}}$ where this intersection is taken over a subset of the defining family $\mathcal{P}$ of D. This condition may not be in general satisfied for an essential domain that is not locally essential.
(The point is whether an int prime ideal $\mathfrak{m}$ belongs to the defining family of prime ideals of D.)

The example that we know of an essential domain that is not locally essential is given by W. Heinzer (An essential integral domain with a nonessential localization, Canadian J. Math. (1981)).

In this example it is not known whether $\mathfrak{m}$ is an int prime. Thus, it is an open question whether Proposition 2 may work for essential domains (with a different proof). This is crucial, because the consistence of Proposition 2 for essential domains would allow us to give a complete characterization of domains D such that $\mathrm{Int}(D)$ is essential.

Proposition

Let D be an integral domain. If $\mathrm{Int}(D)$ is a locally essential domain then the following statements hold:

(a) D is locally essential;

(b) for each int prime ideal $\mathfrak{m}$ of D, $D_{\mathfrak{m}}$ is a DVR with finite residue field.

(b) From Proposition 2, $D_{\mathfrak{m}}$ is a valuation domain with maximal principal ideal. The ideal $\mathfrak{P}_{\mathfrak{m},0} := \{f \in \mathrm{Int}(D);\ f(0) \in \mathfrak{m}\}$ is an associated prime (because it is int prime containing $\mathrm{Int}(D, \mathfrak{p})$). Then $\mathrm{Int}(D)_{\mathfrak{P}_{\mathfrak{m},0}}$ is a valuation domain. Assume, by way of contradiction, that $\mathfrak{m}$ is of height at least 2. Then there is some nonzero prime ideal $\mathfrak{p}$ of D contained in $\mathfrak{m}$. $\mathrm{Int}(D)_{\mathfrak{P}_{\mathfrak{p},0}}$ is also a valuation domain since $\mathfrak{P}_{\mathfrak{p},0} := \{f \in \mathrm{Int}(D);\ f(0) \in \mathfrak{p}\} \subset \mathfrak{P}_{\mathfrak{m},0}$. Since $\mathrm{Int}(D) \subseteq D_{\mathfrak{p}}[X]$, $\mathrm{Int}(D)_{\mathfrak{P}_{\mathfrak{p},0}} = D[X]_{(\mathfrak{p},X)}$ and then the contradiction follows from the fact that $D[X]_{(\mathfrak{p},X)}$ is never a valuation domain. Thus $\mathfrak{m}$ is height-one

Theorem - Characterization locally essential

Let D be an integral domain. Then $\mathrm{Int}(D)$ is locally essential if and only if the following conditions hold:

(a) D is locally essential;

(b) each int prime ideal of D is height-one.

$(\Leftarrow)$ Let $P \in Ass(D)$, $P \cap D = \mathfrak{p}$.
If $\mathfrak{p}$ is an int prime ideal, the conclusion follows from Theorem 1.
If $\mathfrak{p}$ is a polynomial prime then $\mathrm{Int}(D)_{\mathfrak{p}} = D_{\mathfrak{p}}[X]$, $PD_{\mathfrak{p}}[X]$ is an associated prime of $D_{\mathfrak{p}}[X]$ and so it is a t-prime. Since $D_{\mathfrak{p}}$ is integrally closed, the t-primes of $D_{\mathfrak{p}}[X]$ are the uppers to zero and the extended ideals of t-primes of $D_{\mathfrak{p}}$. Thus $PD_{\mathfrak{p}}[X] = \mathfrak{p}D_{\mathfrak{p}}[X]$ and $P = \mathfrak{p}D_{\mathfrak{p}}[X] \cap \mathrm{Int}(D)$. Then $\mathrm{Int}(D)_P = D_{\mathfrak{p}}[X]_{\mathfrak{p}D_{\mathfrak{p}}[X]} = D_{\mathfrak{p}}(X)$ which is the Nagata ring of $D_{\mathfrak{p}}$ and it is a valuation domain since we can show that $D_{\mathfrak{p}}$ is a valuation domain.

Corollary

Let D be an integral domain with $\mathrm{Ass}(D) = X^1(D)$. Then $\mathrm{Int}(D)$ is locally essential if and only if so is D.

It is known that int primes are t-ideals. So we have the following corollary.

Corollary

For any integral domain D that is either t-almost Dedekind or almost Krull, $\mathrm{Int}(D)$ is locally essential.

This corollary allows to construct other examples of locally essential domains that are not PvMD. If D is an almost Krull domain that is not PvMD (such example is given by Arnold-Matsuda) then $\mathrm{Int}(D)$ is locally essential but not PvMD.

Proposition

Let D be a Krull-type domain. Then the following statements are equivalent.

(1) $\mathrm{Int}(D)$ is a PvMD;

(2) $\mathrm{Int}(D)$ is a locally essential domain;

(3) $D_{\mathfrak{p}}$ is a DVR, for each int prime ideal $\mathfrak{p}$ of D;

(4) $\mathrm{Int}(D_0)$ is a Prüfer domain, where D_0.

We recall that an integral domain D is *strong Mori* if it satisfies the ascending chain condition (a.c.c.) on integral w-ideals. Thus, the class of strong Mori domains includes that of Noetherian domains.

Proposition

Let D be a strong Mori domain. Then the following statements are equivalent.

(i) $\mathrm{Int}(D)$ is a PvMD;

(ii) $\mathrm{Int}(D)$ is a locally essential domain;

(iii) D is an integrally closed domain (i.e. a Krull domain).

Proposition

Let V be a valuation domain. Then the following statements are equivalent.

(i) $\mathrm{Int}(V)$ is a PvMD;

(ii) $\mathrm{Int}(V)$ is a locally essential domain;

(iii) $\mathrm{Int}(V) = V[X]$ or V is a DVR with finite residue field. In this last case, $\mathrm{Int}(V)$ is Prüfer.

Remark. If V be a valuation domain such that $\mathrm{Int}(V) \neq V[X]$:

- If $\dim(V) = 1$, V is a DVR with finite residue field and hence $\mathrm{Int}(V)$ is a Prüfer domain, so it is locally essential.
- If $\dim(V) \geqslant 2$, then $\mathrm{Int}(V)$ is not locally essential.

Proposition

Let D be a locally essential domain. If t-dim$(\mathrm{Int}(D)) = 1$ then $\mathrm{Int}(D)$ is a PvMD.

(t-dim$(\mathrm{Int}(D)) = 1 \Rightarrow D$ is either a field or of t-dimension 1)

The converse of this Proposition is not, in general, true. Indeed, $\mathrm{Int}(\mathbb{Z})$ is a two-dimensional Prüfer domain and hence it is of t-dimension two (since all ideals of a Prüfer domain are t-ideals).

Theorem

Let D be an integral domain such that $D_{\mathfrak{p}}$ is a DVR for each int prime ideal $\mathfrak{p}$ of D. Then $\mathrm{Int}(D)$ is essential if and only if D is essential.

In the proof of this Theorem the hypothesis that $D_{\mathfrak{p}}$ is a DVR for each int prime ideal cannot be dropped off for the sufficient condition (D essential $\Rightarrow \mathrm{Int}(D)$ essential). The argument used is based on Theorem 1.

Indeed, for the necessary condition, it would be enough to assume that $D_{\mathfrak{p}}$ is a valuation domain when $\mathfrak{p}$ is int prime.

Exampe CLT contructs a domain $\mathrm{Int}(D)$ that is locally PvMD but not PvMD.

Theorem

Let D be an integral domain. Then D is a locally PvMD if and only if $D[X]$ is a locally PvMD.

Theorem

Let D be an integral domain. Then $\mathrm{Int}(D)$ is locally PvMD if and only if D is locally PvMD and each int prime ideal of D is height-one.

Corollary

For any integral domain D that is either almost Krull or t-almost Dedekind or PvMD of t-dimension one $\mathrm{Int}(D)$ is a locally PvMD.

Theorem

Let D be an integral domain. Then $\mathrm{Int}(D)$ is locally GCD if and only if D is locally GCD and each int prime ideal of D is height-one.

Proposition

If D is locally UFD then $\mathrm{Int}(D)$ is locally GCD.

From the previous corollary we deduce that we can construct non-trivial rings $\mathrm{Int}(D)$ that are locally GCD by taking D almost Dedekind such that $\mathrm{Int}(D) \neq D[X]$. For instance, Example CLT verifies this condition and it is an example of a locally GCD domain that is not PvMD.

A domain is called MZ-DVR if $D_{\mathfrak{p}}$ is a DVR for each $\mathfrak{p} \in Ass(D)$. Their definition is inspired by the P-domains of Mott and Zafrullah. Notice that almost Krull domains and t-almost Dedekind domains are MZ-DVRs.
J. Arnold and R. Matsuda (1986) constructs an example of an almost Krull domain that is not PvMD, whence it is a MZ-DVR but not t-almost Dedekind.

Proposition

Let D be MZ-DVR, then $\mathrm{Int}(D)$ is a locally essential domain.

Theorem

Let $(\mathcal{P})$ denote one of the following properties for integral domains:

- locally essential
- locally PvMD
- locally GCD

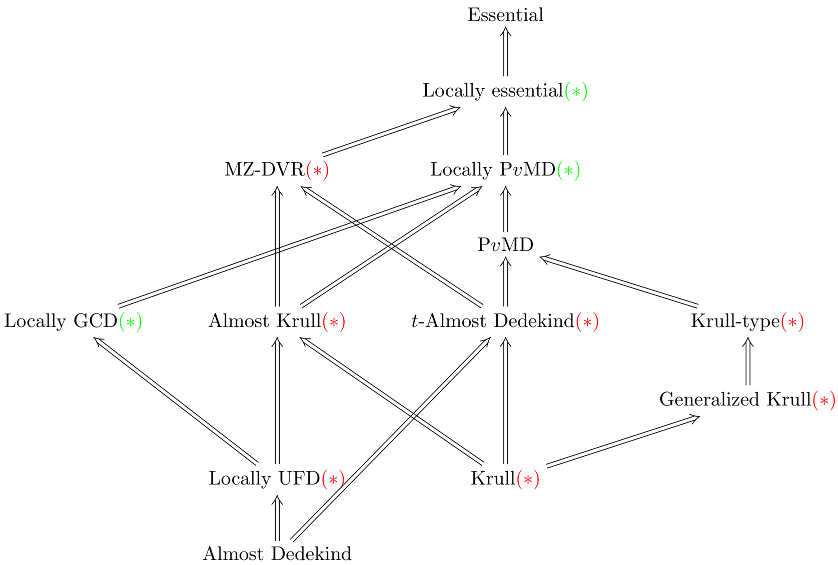
Then $\mathrm{Int}(D)$ has the property $(\mathcal{P})$ if and only if D has the same property and each int prime is height-one.

Theorem

Let $(\mathcal{P})$ denote one of the following properties for integral domains:

- Krull, Krull-type, almost Krull, generalized Krull
- t-almost Dedekind,
- locally UFD,
- MZ-DVR.

Then $\mathrm{Int}(D)$ has the property $(\mathcal{P})$ if and only if D has the same property and $\mathrm{Int}(D) = D[X]$.


Essential
Locally essential(∗)
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