Assignment 7

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***Abstract*—This document uses the properties of a parabola**

Download latex-tikz codes from

https://github.com/AddagallaSatyanarayana/AI5106/tree/master/Assignment7/Assignment7.tex

1 Problem

Trace the parabola

$$144x^2 - 120xy + 25y^2 + 619x - 272y + 663 = 0 \tag{1.0.1}$$

2 Explanation

The general equation of second degree is given by

$$ax^2 + 2bxy + cy^2 + 2dx + 2ey + f = 0 \tag{2.0.1}$$

and can be expressed as

$$\mathbf{x}^T\mathbf{V}\mathbf{x} + 2\mathbf{u}^T\mathbf{x} + f = 0 \tag{2.0.2}$$

where

$$\mathbf{V} = \mathbf{V}^T = \begin{pmatrix} a & b \\ b & c \end{pmatrix} \tag{2.0.3}$$

$$\mathbf{u}^T = \begin{pmatrix} d & e \end{pmatrix} \tag{2.0.4}$$

From equation (1.0.1) , we get

$$\mathbf{V} = \begin{pmatrix} 144 & -60 \\ -60 & 25 \end{pmatrix} \tag{2.0.5}$$

$$\mathbf{u} = \begin{pmatrix} \frac{619}{2} \\ -\frac{272}{2} \end{pmatrix} \tag{2.0.6}$$

$$f = 663 \tag{2.0.7}$$

Expanding the determinant of $\mathbf{V}$ we observe,

$$\begin{vmatrix} 144 & -60 \\ -60 & 25 \end{vmatrix} = 0 \tag{2.0.8}$$

The characteristic equation of $\mathbf{V}$ is given as follows,

$$\left|\lambda\mathbf{I} - \mathbf{V}\right| = \begin{vmatrix} \lambda - 144 & 60 \\ 60 & \lambda - 25 \end{vmatrix} = 0 \tag{2.0.9}$$

$$\implies \lambda^2 - 169\lambda = 0 \tag{2.0.10}$$

the eigenvalues are given by

$$\lambda_1 = 0, \lambda_2 = 169 \tag{2.0.11}$$

For $\lambda_1 = 0$, the eigen vector $\mathbf{p}$ is given by

$$\mathbf{V}\mathbf{p} = 0 \tag{2.0.12}$$

Row reducing $\mathbf{V}$

$$\implies \begin{pmatrix} -144 & 60 \\ 60 & -25 \end{pmatrix} \xleftrightarrow[R_2 = R_2 + 5R_1]{R_1 = \frac{R_1}{12}} \begin{pmatrix} -12 & 5 \\ 0 & 0 \end{pmatrix} \tag{2.0.13}$$

$$\implies \mathbf{p}_1 = \frac{1}{13}\begin{pmatrix} 5 \\ 12 \end{pmatrix} \tag{2.0.14}$$

Similarly,

$$\mathbf{p}_2 = \frac{1}{13}\begin{pmatrix} 12 \\ -5 \end{pmatrix} \tag{2.0.15}$$

Thus,

$$\mathbf{P} = \begin{pmatrix} \mathbf{p}_1 & \mathbf{p}_2 \end{pmatrix} = \frac{1}{13}\begin{pmatrix} 5 & 12 \\ 12 & -5 \end{pmatrix} \tag{2.0.16}$$

and its equation is

$$\mathbf{y}^\mathbf{T}\mathbf{D}\mathbf{y} = -2\eta\begin{pmatrix} 1 & 0 \end{pmatrix}\mathbf{y} \tag{2.0.17}$$

where

$$\eta = \mathbf{u}^T\mathbf{p_1} = -6.5 \tag{2.0.18}$$

The focal length of the parabola is given by

$$\frac{\left|2\mathbf{u}^T\mathbf{p_1}\right|}{\lambda_2} = \frac{1}{13} \tag{2.0.19}$$

$$\begin{pmatrix} \mathbf{u^T} + \eta\mathbf{p_1^T} \\ \mathbf{V} \end{pmatrix}\mathbf{c} = \begin{pmatrix} -f \\ \eta\mathbf{p_1} - \mathbf{u} \end{pmatrix} \tag{2.0.20}$$

using equations (2.0.5),(2.0.6) and (2.0.20)

$$\begin{pmatrix} 307 & -142 \\ 144 & -60 \\ -60 & 25 \end{pmatrix}\mathbf{c} = \begin{pmatrix} -663 \\ -312 \\ 130 \end{pmatrix} \tag{2.0.21}$$

Forming the augmented matrix and row reducing it:

$$\begin{pmatrix} 307 & -142 & -663 \\ 144 & -60 & -312 \\ -60 & 25 & 130 \end{pmatrix} \tag{2.0.22}$$

$$\xleftrightarrow{R_3 \leftarrow R_3 + (5/12)R_2} \begin{pmatrix} 307 & -142 & -663 \\ 144 & -60 & -312 \\ 0 & 0 & 0 \end{pmatrix} \tag{2.0.23}$$

$$\xleftrightarrow{R_2 \leftarrow R_2/12} \begin{pmatrix} 307 & -142 & -663 \\ 12 & -5 & -26 \\ 0 & 0 & 0 \end{pmatrix} \tag{2.0.24}$$

$$\xleftrightarrow{R_1 \leftarrow R_1/(307)} \begin{pmatrix} 1 & -142/307 & -663/307 \\ 12 & -5 & -26 \\ 0 & 0 & 0 \end{pmatrix} \tag{2.0.25}$$

$$\xleftrightarrow{R_2 \leftarrow R_2 - 12R_1} \begin{pmatrix} 1 & -142/307 & -663/307 \\ 0 & 169/307 & -26/307 \\ 0 & 0 & 0 \end{pmatrix} \tag{2.0.26}$$

$$\xleftrightarrow{R_2 \leftarrow R_2(307/169)} \begin{pmatrix} 1 & -142/307 & -663/307 \\ 0 & 1 & -2/13 \\ 0 & 0 & 0 \end{pmatrix} \tag{2.0.27}$$

$$\xleftrightarrow{R_1 \leftarrow R_1 + (142/307)R_2} \begin{pmatrix} 1 & 0 & -8903/3991 \\ 0 & 1 & -2/13 \\ 0 & 0 & 0 \end{pmatrix} \tag{2.0.28}$$

Thus the vertex $\mathbf{c}$ is:

$$\mathbf{c} = \begin{pmatrix} -2.23 \\ -0.153 \end{pmatrix} \tag{2.0.29}$$

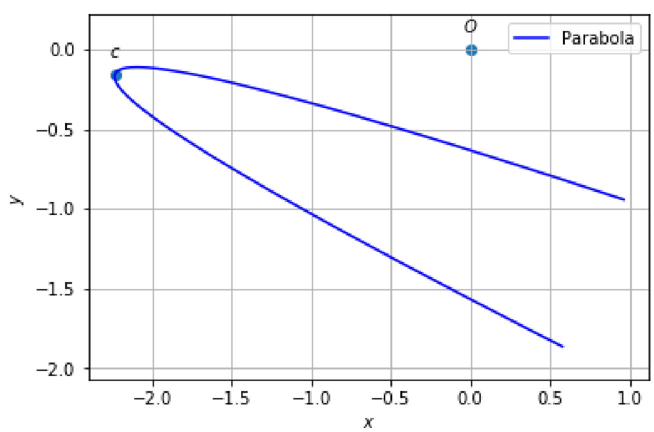

Fig. 0: Parabola