Assignment 9

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Abstract—**This document shows the concept of markov chain state and tranistion matrices**

https://github.com/AddagallaSatyanarayana/AI5106
/tree/master/Assignment9/Assignment9.tex

1 Problem

Consider a Markov chain with state space 0,1,2,3,4 ad transition matrix

$$\mathbf{P} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{1.0.1}$$

Then find $\lim_{n\to\infty} p_{23}^{(n)}$

2 Explanation

The probability of the transition from one state to another state in n steps can be calculated by computing the transition matrix raised to power n, ie $\mathbf{P}^n$.
A matrix $\mathbf{P}$ can be written in the diagonalized form if there exists an invertible matrix $\mathbf{S}$ such that

$$\mathbf{P} = \mathbf{S}\mathbf{D}\mathbf{S}^{-1} \tag{2.0.1}$$

The matrix $\mathbf{S}$ is equal to the eigen vectors of matrix $\mathbf{P}$,where eigen values can be calculated using

$$\left|\mathbf{P} - \lambda\mathbf{I}\right| = 0 \tag{2.0.2}$$

The nth power of the matrix $\mathbf{P}$ can be calculated using

$$\mathbf{P}^n = \mathbf{S}\mathbf{D}^n\mathbf{S}^{-1} \tag{2.0.3}$$

3 Solution

The eigen values for the matrix $\mathbf{P}$ can be found using (2.0.2)

$$\begin{vmatrix} 1-\lambda & 0 & 0 & 0 & 0 \\ \frac{1}{3} & \frac{1}{3}-\lambda & \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{3} & \frac{1}{3}-\lambda & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{3} & \frac{1}{3}-\lambda & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 1-\lambda \end{vmatrix} = 0 \tag{3.0.1}$$

Solving this we get the eigen values of $\mathbf{P}$ as,

$$\lambda_1 = 1 \tag{3.0.2}$$

$$\lambda_2 = \frac{1}{3} \tag{3.0.3}$$

$$\lambda_3 = \frac{-\sqrt{2}+1}{3} \tag{3.0.4}$$

$$\lambda_4 = \frac{\sqrt{2}+1}{3} \tag{3.0.5}$$

The corresponding eigen vectors are:

$$\mathbf{v}_1 = \begin{pmatrix} 4 & 3 & 2 & 1 & 0 \end{pmatrix}^T \tag{3.0.6}$$

$$\mathbf{v}_2 = \begin{pmatrix} -3 & -2 & -1 & 0 & 1 \end{pmatrix}^T \tag{3.0.7}$$

$$\mathbf{v}_3 = \begin{pmatrix} 0 & -1 & 0 & 1 & 0 \end{pmatrix}^T \tag{3.0.8}$$

$$\mathbf{v}_4 = \begin{pmatrix} 0 & 1 & -\sqrt{2} & 1 & 0 \end{pmatrix}^T \tag{3.0.9}$$

$$\mathbf{v}_5 = \begin{pmatrix} 0 & 1 & \sqrt{2} & 1 & 0 \end{pmatrix}^T \tag{3.0.10}$$

The matrix $\mathbf{S}$ is equal to

$$\mathbf{S} = \begin{pmatrix} 4 & -3 & 0 & 0 & 0 \\ 3 & -2 & -1 & 1 & 1 \\ 2 & -1 & 0 & -\sqrt{2} & \sqrt{2} \\ 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix} \tag{3.0.11}$$

The diagonal matrix $\mathbf{D}$ can be expressed using the eigen values in the form:

$$\mathbf{D} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 0 & 0 \\ 0 & 0 & 0 & \frac{-\sqrt{2}+1}{3} & 0 \\ 0 & 0 & 0 & 0 & \frac{\sqrt{2}+1}{3} \end{pmatrix} \tag{3.0.12}$$

The matrix $\mathbf{S}^{-1}$ can be calculated using (3.0.11)

$$\mathbf{S}^{-1} = \begin{pmatrix} \frac{1}{4} & 0 & 0 & 0 & \frac{3}{4} \\ 0 & 0 & 0 & 0 & 1 \\ \frac{1}{4} & \frac{-1}{2} & 0 & \frac{1}{2} & \frac{-1}{4} \\ \frac{\sqrt{2}-2}{8} & \frac{1}{4} & \frac{-\sqrt{2}}{4} & \frac{1}{4} & \frac{\sqrt{2}-2}{8} \\ \frac{-\sqrt{2}-2}{8} & \frac{1}{4} & \frac{\sqrt{2}}{4} & \frac{1}{4} & \frac{-\sqrt{2}-2}{8} \end{pmatrix} \tag{3.0.13}$$

Using equation (2.0.3)

$$\mathbf{P}^n = \mathbf{S}\mathbf{D}^n\mathbf{S}^{-1} \tag{3.0.14}$$

$$\mathbf{P}^n = \begin{pmatrix} 4 & -3 & 0 & 0 & 0 \\ 3 & -2 & -1 & 1 & 1 \\ 2 & -1 & 0 & -\sqrt{2} & \sqrt{2} \\ 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & (\frac{1}{3})^n & 0 & 0 \\ 0 & 0 & 0 & (\frac{-\sqrt{2}+1}{3})^n & 0 \\ 0 & 0 & 0 & 0 & (\frac{\sqrt{2}+1}{3})^n \end{pmatrix} \begin{pmatrix} \frac{1}{4} & 0 & 0 & 0 & \frac{3}{4} \\ 0 & 0 & 0 & 0 & 1 \\ \frac{1}{4} & \frac{-1}{2} & 0 & \frac{1}{2} & \frac{-1}{4} \\ \frac{\sqrt{2}-2}{8} & \frac{1}{4} & \frac{-\sqrt{2}}{4} & \frac{1}{4} & \frac{\sqrt{2}-2}{8} \\ \frac{-\sqrt{2}-2}{8} & \frac{1}{4} & \frac{\sqrt{2}}{4} & \frac{1}{4} & \frac{-\sqrt{2}-2}{8} \end{pmatrix} \tag{3.0.15}$$

Using (3.0.15),it can be shown that

$$\lim_{n\to\infty} p_{23}^{(n)} = 0 \tag{3.0.16}$$