# Assignment 7

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***Abstract*—This document uses the properties of a parabola**

Download latex-tikz codes from

https://github.com/AddagallaSatyanarayana/AI5106/tree/master/Assignment7/Assignment7.tex

## 1 Problem

Trace the parabola

$$144x^2 - 120xy + 25y^2 + 619x - 272y + 663 = 0 \tag{1.0.1}$$

## 2 Explanation

The general equation of second degree is given by

$$ax^2 + 2bxy + cy^2 + 2dx + 2ey + f = 0 \tag{2.0.1}$$

and can be expressed as

$$\mathbf{x}^T\mathbf{V}\mathbf{x} + 2\mathbf{u}^T\mathbf{x} + f = 0 \tag{2.0.2}$$

where

$$\mathbf{V} = \mathbf{V}^T = \begin{pmatrix} a & b \\ b & c \end{pmatrix} \tag{2.0.3}$$

$$\mathbf{u}^T = \begin{pmatrix} d & e \end{pmatrix} \tag{2.0.4}$$

From equation (1.0.1) , we get

$$\mathbf{V} = \begin{pmatrix} 144 & -60 \\ -60 & 25 \end{pmatrix} \tag{2.0.5}$$

$$\mathbf{u} = \begin{pmatrix} \frac{619}{2} \\ -\frac{272}{2} \end{pmatrix} \tag{2.0.6}$$

$$f = 663 \tag{2.0.7}$$

Expanding the determinant of $\mathbf{V}$ we observe,

$$\begin{vmatrix} 144 & -60 \\ -60 & 25 \end{vmatrix} = 0 \tag{2.0.8}$$

The characteristic equation of $\mathbf{V}$ is given as follows,

$$\left|\lambda\mathbf{I} - \mathbf{V}\right| = \begin{vmatrix} \lambda - 144 & 60 \\ 60 & \lambda - 25 \end{vmatrix} = 0 \tag{2.0.9}$$

$$\implies \lambda^2 - 169\lambda = 0 \tag{2.0.10}$$

the eigenvalues are given by

$$\lambda_1 = 0, \lambda_2 = 169 \tag{2.0.11}$$

For $\lambda_1 = 0$, the eigen vector $\mathbf{p}$ is given by

$$\mathbf{V}\mathbf{p} = 0 \tag{2.0.12}$$

Row reducing $\mathbf{V}$

$$\implies \begin{pmatrix} -144 & 60 \\ 60 & -25 \end{pmatrix} \xleftrightarrow[R_2 = R_2 + 5R_1]{R_1 = \frac{R_1}{12}} \begin{pmatrix} -12 & 5 \\ 0 & 0 \end{pmatrix} \tag{2.0.13}$$

$$\implies \mathbf{p}_1 = \frac{1}{13}\begin{pmatrix} 5 \\ 12 \end{pmatrix} \tag{2.0.14}$$

Similarly,

$$\mathbf{p}_2 = \frac{1}{13}\begin{pmatrix} 12 \\ -5 \end{pmatrix} \tag{2.0.15}$$

Thus,

$$\mathbf{P} = \begin{pmatrix} \mathbf{p}_1 & \mathbf{p}_2 \end{pmatrix} = \frac{1}{13}\begin{pmatrix} 5 & 12 \\ 12 & -5 \end{pmatrix} \tag{2.0.16}$$

and its equation is

$$\mathbf{y^T D y} = -2\eta\begin{pmatrix} 1 & 0 \end{pmatrix}\mathbf{y} \tag{2.0.17}$$

where

$$\eta = \mathbf{u}^T\mathbf{p_1} = -6.5 \tag{2.0.18}$$

The focal length of the parabola is given by

$$\frac{\left|2\mathbf{u}^T\mathbf{p_1}\right|}{\lambda_2} = \frac{1}{13} \tag{2.0.19}$$

$$\begin{pmatrix} \mathbf{u^T} + \eta\mathbf{p_1^T} \\ \mathbf{V} \end{pmatrix}\mathbf{c} = \begin{pmatrix} -f \\ \eta\mathbf{p_1} - \mathbf{u} \end{pmatrix} \tag{2.0.20}$$

using equations (2.0.5),(2.0.6) and (2.0.14)

$$\begin{pmatrix} 307 & -142 \\ 144 & -60 \\ -60 & 25 \end{pmatrix}\mathbf{c} = \begin{pmatrix} -663 \\ -312 \\ 130 \end{pmatrix} \tag{2.0.21}$$

Forming the augmented matrix and row reducing it:

$$\begin{pmatrix} 307 & -142 & -663 \\ 144 & -60 & -312 \\ -60 & 25 & 130 \end{pmatrix} \tag{2.0.22}$$

$$\xleftrightarrow{R_3 \leftarrow R_3 + (5/12)R_2} \begin{pmatrix} 307 & -142 & -663 \\ 144 & -60 & -312 \\ 0 & 0 & 0 \end{pmatrix} \tag{2.0.23}$$

$$\xleftrightarrow{R_2 \leftarrow R_2/12} \begin{pmatrix} 307 & -142 & -663 \\ 12 & -5 & -26 \\ 0 & 0 & 0 \end{pmatrix} \tag{2.0.24}$$

$$\xleftrightarrow{R_1 \leftarrow R_1/(307)} \begin{pmatrix} 1 & -142/307 & -663/307 \\ 12 & -5 & -26 \\ 0 & 0 & 0 \end{pmatrix} \tag{2.0.25}$$

$$\xleftrightarrow{R_2 \leftarrow R_2 - 12R_1} \begin{pmatrix} 1 & -142/307 & -663/307 \\ 0 & 169/307 & -26/307 \\ 0 & 0 & 0 \end{pmatrix} \tag{2.0.26}$$

$$\xleftrightarrow{R_2 \leftarrow R_2(307/169)} \begin{pmatrix} 1 & -142/307 & -663/307 \\ 0 & 1 & -2/13 \\ 0 & 0 & 0 \end{pmatrix} \tag{2.0.27}$$

$$\xleftrightarrow{R_1 \leftarrow R_1 + (142/307)R_2} \begin{pmatrix} 1 & 0 & -8903/3991 \\ 0 & 1 & -2/13 \\ 0 & 0 & 0 \end{pmatrix} \tag{2.0.28}$$

Thus the vertex $\mathbf{c}$ is:

$$\mathbf{c} = \begin{pmatrix} -2.23 \\ -0.153 \end{pmatrix} \tag{2.0.29}$$

The QR decomposition of $\mathbf{V}$ can be written as,

$$\mathbf{V} = \begin{pmatrix} \mathbf{a} & \mathbf{b} \end{pmatrix} \tag{2.0.30}$$

where $\mathbf{a}$ and $\mathbf{b}$ and are column vectors,

$$\mathbf{a} = \begin{pmatrix} 144 \\ -60 \end{pmatrix} \tag{2.0.31}$$

$$\mathbf{b} = \begin{pmatrix} -60 \\ 25 \end{pmatrix} \tag{2.0.32}$$

$$\mathbf{V} = \mathbf{QR} \tag{2.0.33}$$

where $\mathbf{R}$ is a upper triangular matrix and $\mathbf{Q}$ such that,

$$\mathbf{Q^TQ} = \mathbf{I} \tag{2.0.34}$$

and

$$\mathbf{Q} = \begin{pmatrix} \mathbf{q_1} & \mathbf{q_2} \end{pmatrix} \textit{ and } \mathbf{R} = \begin{pmatrix} r_1 & r_2 \\ 0 & r_3 \end{pmatrix} \tag{2.0.35}$$

$\mathbf{q_1}$, $\mathbf{q_2}$ and the values in $\mathbf{R}$ are given by,

$$r_1 = \|\mathbf{a}\| \tag{2.0.36}$$

$$\mathbf{q_1} = \frac{\mathbf{a}}{r_1} \tag{2.0.37}$$

$$r_2 = \frac{\mathbf{q_1^T b}}{\|\mathbf{q_1}\|^2} \tag{2.0.38}$$

$$\mathbf{q_2} = \frac{\mathbf{b} - r_2\mathbf{q_1}}{\|\mathbf{b} - r_2\mathbf{q_1}\|} \tag{2.0.39}$$

$$r_3 = \mathbf{q_2^T b} \tag{2.0.40}$$

Hence,

$$r_1 = \sqrt{24336} = 156 \tag{2.0.41}$$

$$\mathbf{q_1} = \frac{1}{156}\begin{pmatrix} 144 \\ -60 \end{pmatrix} = \begin{pmatrix} \frac{12}{13} \\ \\ \frac{-5}{13} \end{pmatrix} \tag{2.0.42}$$

$$r_2 = -65 \tag{2.0.43}$$

$$\mathbf{q_2} = \sqrt{5}\begin{pmatrix} \frac{1}{5} \\ \\ \frac{2}{5} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{2.0.44}$$

$$r_3 = 0 \tag{2.0.45}$$

Therefore,

$$\mathbf{QR} = \begin{pmatrix} \frac{12}{13} & 0 \\ \frac{-5}{13} & 0 \end{pmatrix}\begin{pmatrix} 156 & -65 \\ 0 & 0 \end{pmatrix} \tag{2.0.46}$$

$$\implies \mathbf{QR} = \begin{pmatrix} 144 & -60 \\ -60 & 25 \end{pmatrix} \tag{2.0.47}$$

As (2.0.5) and (2.0.47) are equal, the QR decomposition holds.

Verifying the solution using least square method

$$\begin{pmatrix} 307 & -142 \\ 144 & -60 \\ -60 & 25 \end{pmatrix}\mathbf{c} = \begin{pmatrix} -663 \\ -312 \\ 130 \end{pmatrix} \tag{2.0.48}$$

This is in the form of

$$A\mathbf{c} = \mathbf{b} \tag{2.0.49}$$

using least squares solution of linear system

$$A^TA\mathbf{c} = A^T\mathbf{b} \implies \mathbf{c} = (A^TA)^{-1}A^T\mathbf{b} \tag{2.0.50}$$

$$A^TA = \begin{pmatrix} 307 & 144 & --60 \\ -142 & -60 & 25 \end{pmatrix}\begin{pmatrix} 307 & -142 \\ 144 & -60 \\ -60 & 25 \end{pmatrix}$$

$$\implies A^TA = \begin{pmatrix} 118585 & -53734 \\ -53734 & 24389 \end{pmatrix} \tag{2.0.51}$$

The inverse can be written as

$$(A^TA)^{-1} = \frac{1}{4826809}\begin{pmatrix} 24389 & 53734 \\ 53734 & 118585 \end{pmatrix} \tag{2.0.52}$$

$$A^T\mathbf{b} = \begin{pmatrix} 307 & 144 & --60 \\ -142 & -60 & 25 \end{pmatrix}\begin{pmatrix} -663 \\ -312 \\ 130 \end{pmatrix}$$

$$A^T\mathbf{b} = \begin{pmatrix} -256269 \\ 116116 \end{pmatrix} \tag{2.0.53}$$

using 2.0.52 and 2.0.53 in 2.0.50,the center $\mathbf{c}$

$$\mathbf{c} = \frac{1}{4826809}\begin{pmatrix} 24389 & 53734 \\ 53734 & 118585 \end{pmatrix}\begin{pmatrix} -256269 \\ 116116 \end{pmatrix} \tag{2.0.54}$$

$$\implies \mathbf{c} = \frac{1}{4826809}\begin{pmatrix} -10767497 \\ -742586 \end{pmatrix} \tag{2.0.55}$$

$$\implies \mathbf{c} = \begin{pmatrix} -2.23 \\ -0.153 \end{pmatrix} \tag{2.0.56}$$

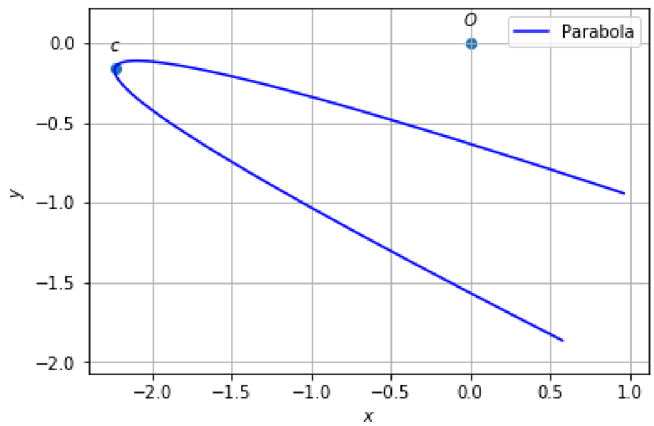


Fig. 0: Parabola