Assignment 7

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***Abstract*—This document uses the properties of a parabola**

Download latex-tikz codes from

https://github.com/AddagallaSatyanarayana/AI5106/tree/master/Assignment7/Assignment7.tex

1 Problem

Trace the parabola

$$144x^2 - 120xy + 25y^2 + 619x - 272y + 663 = 0 \tag{1.0.1}$$

2 Explanation

The general equation of second degree is given by

$$ax^2 + 2bxy + cy^2 + 2dx + 2ey + f = 0 \tag{2.0.1}$$

and can be expressed as

$$\mathbf{x}^T\mathbf{V}\mathbf{x} + 2\mathbf{u}^T\mathbf{x} + f = 0 \tag{2.0.2}$$

where

$$\mathbf{V} = \mathbf{V}^T = \begin{pmatrix} a & b \\ b & c \end{pmatrix} \tag{2.0.3}$$

$$\mathbf{u}^T = \begin{pmatrix} d & e \end{pmatrix} \tag{2.0.4}$$

From equation (1.0.1) , we get

$$\mathbf{V} = \begin{pmatrix} 144 & -60 \\ -60 & 25 \end{pmatrix} \tag{2.0.5}$$

$$\mathbf{u} = \begin{pmatrix} \frac{619}{2} \\ -\frac{272}{2} \end{pmatrix} \tag{2.0.6}$$

$$f = 663 \tag{2.0.7}$$

Expanding the determinant of $\mathbf{V}$ we observe,

$$\begin{vmatrix} 144 & -60 \\ -60 & 25 \end{vmatrix} = 0 \tag{2.0.8}$$

The characteristic equation of $\mathbf{V}$ is given as follows,

$$\left|\lambda\mathbf{I} - \mathbf{V}\right| = \begin{vmatrix} \lambda - 144 & 60 \\ 60 & \lambda - 25 \end{vmatrix} = 0 \tag{2.0.9}$$

$$\implies \lambda^2 - 169\lambda = 0 \tag{2.0.10}$$

the eigenvalues are given by

$$\lambda_1 = 0, \lambda_2 = 169 \tag{2.0.11}$$

For $\lambda_1 = 0$, the eigen vector $\mathbf{p}$ is given by

$$\mathbf{V}\mathbf{p} = 0 \tag{2.0.12}$$

Row reducing $\mathbf{V}$

$$\implies \begin{pmatrix} -144 & 60 \\ 60 & -25 \end{pmatrix} \xleftrightarrow[R_2=R_2+5R_1]{R_1=\frac{R_1}{12}} \begin{pmatrix} -12 & 5 \\ 0 & 0 \end{pmatrix} \tag{2.0.13}$$

$$\implies \mathbf{p}_1 = \frac{1}{13}\begin{pmatrix} 5 \\ 12 \end{pmatrix} \tag{2.0.14}$$

Similarly,

$$\mathbf{p}_2 = \frac{1}{13}\begin{pmatrix} 12 \\ -5 \end{pmatrix} \tag{2.0.15}$$

Thus,

$$\mathbf{P} = \begin{pmatrix} \mathbf{p}_1 & \mathbf{p}_2 \end{pmatrix} = \frac{1}{13}\begin{pmatrix} 5 & 12 \\ 12 & -5 \end{pmatrix} \tag{2.0.16}$$

and its equation is

$$\mathbf{y}^{\mathbf{T}}\mathbf{D}\mathbf{y} = -2\eta\begin{pmatrix} 1 & 0 \end{pmatrix}\mathbf{y} \tag{2.0.17}$$

where

$$\eta = \mathbf{u}^T\mathbf{p_1} = -6.5 \tag{2.0.18}$$

2.1 *Finding QR decompostion of V*

The QR decomposition of $\mathbf{V}$ can be written as,

$$\mathbf{V} = \begin{pmatrix} \mathbf{a} & \mathbf{b} \end{pmatrix} \tag{2.1.1}$$

where $\mathbf{a}$ and $\mathbf{b}$ and are column vectors,

$$\mathbf{a} = \begin{pmatrix} 144 \\ -60 \end{pmatrix} \tag{2.1.2}$$

$$\mathbf{b} = \begin{pmatrix} -60 \\ 25 \end{pmatrix} \tag{2.1.3}$$

$$\mathbf{V} = \mathbf{QR} \tag{2.1.4}$$

where $\mathbf{R}$ is a upper triangular matrix and $\mathbf{Q}$ such that,

$$\mathbf{Q}^{\mathbf{T}}\mathbf{Q} = \mathbf{I} \tag{2.1.5}$$

and

$$\mathbf{Q} = \begin{pmatrix} \mathbf{q_1} & \mathbf{q_2} \end{pmatrix} \textit{ and } \mathbf{R} = \begin{pmatrix} r_1 & r_2 \\ 0 & r_3 \end{pmatrix} \tag{2.1.6}$$

$\mathbf{q_1}$, $\mathbf{q_2}$ and the values in $\mathbf{R}$ are given by,

$$r_1 = \|\mathbf{a}\| \tag{2.1.7}$$

$$\mathbf{q_1} = \frac{\mathbf{a}}{r_1} \tag{2.1.8}$$

$$r_2 = \frac{\mathbf{q_1^T b}}{\|\mathbf{q_1}\|^2} \tag{2.1.9}$$

$$\mathbf{q_2} = \frac{\mathbf{b} - r_2\mathbf{q_1}}{\|\mathbf{b} - r_2\mathbf{q_1}\|} \tag{2.1.10}$$

$$r_3 = \mathbf{q_2^T b} \tag{2.1.11}$$

Hence,

$$r_1 = \sqrt{24336} = 156 \tag{2.1.12}$$

$$\mathbf{q_1} = \frac{1}{156}\begin{pmatrix} 144 \\ -60 \end{pmatrix} = \begin{pmatrix} \frac{12}{13} \\ \frac{-5}{13} \end{pmatrix} \tag{2.1.13}$$

$$r_2 = -65 \tag{2.1.14}$$

$$\mathbf{q_2} = \sqrt{5}\begin{pmatrix} \frac{1}{5} \\ \frac{2}{5} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{2.1.15}$$

$$r_3 = 0 \tag{2.1.16}$$

Therefore,

$$\mathbf{QR} = \begin{pmatrix} \frac{12}{13} & 0 \\ \frac{-5}{13} & 0 \end{pmatrix}\begin{pmatrix} 156 & -65 \\ 0 & 0 \end{pmatrix} \tag{2.1.17}$$

$$\implies \mathbf{QR} = \begin{pmatrix} 144 & -60 \\ -60 & 25 \end{pmatrix} \tag{2.1.18}$$

As (2.0.5) and (2.1.18) are equal, the QR decomposition holds.

2.2 *Finding vertex using SVD*

The equation of perpendicular line passing through focus and intersecting parabola at vertex $\mathbf{c}$ is given as

$$\begin{pmatrix} \mathbf{u^T} + \frac{\eta}{2}\mathbf{p_1^T} \\ \mathbf{V} \end{pmatrix}\mathbf{c} = \begin{pmatrix} -f \\ \frac{\eta}{2}\mathbf{p_1} - \mathbf{u} \end{pmatrix} \tag{2.2.1}$$

$$\begin{pmatrix} \frac{1233}{4} & -139 \\ 144 & -60 \\ -60 & 25 \end{pmatrix}\mathbf{c} = \begin{pmatrix} -663 \\ \frac{1243}{4} \\ 133 \end{pmatrix} \tag{2.2.2}$$

$$\implies \mathbf{Mc} = \mathbf{b} \tag{2.2.3}$$

To solve (2.2.3), we perform Singular Value Decomposition on $\mathbf{M}$ given as

$$\mathbf{M} = \mathbf{USV^T} \tag{2.2.4}$$

Putting this value of $\mathbf{M}$ in (2.2.3), we get

$$\mathbf{USV^T c} = \mathbf{b} \tag{2.2.5}$$

$$\implies \mathbf{c} = \mathbf{VS_+U^T b} \tag{2.2.6}$$

where, $\mathbf{S_+}$ is Moore-Penrose pseudo-inverse of $\mathbf{S}$. Columns of $\mathbf{U}$ are eigen-vectors of $\mathbf{MM^T}$, columns of $\mathbf{V}$ are eigen-vectors of $\mathbf{M^TM}$ and $\mathbf{S}$ is diagonal matrix of singular value of eigenvalues of $\mathbf{M^TM}$.

$$\mathbf{MM^T} = \begin{pmatrix} 308 & -139 \\ 144 & -60 \\ -60 & 25 \end{pmatrix}\begin{pmatrix} 308 & 144 & -60 \\ -139 & -60 & 25 \end{pmatrix} \tag{2.2.7}$$

$$= \begin{pmatrix} 114340 & 52728 & -21970 \\ 52728 & 24336 & -10140 \\ -21970 & -10140 & 4225 \end{pmatrix} \tag{2.2.8}$$

$$\mathbf{M^TM} = \begin{pmatrix} 308 & 144 & -60 \\ -139 & -60 & 25 \end{pmatrix}\begin{pmatrix} 308 & -139 \\ 144 & -60 \\ -60 & 25 \end{pmatrix} \tag{2.2.9}$$

$$= \begin{pmatrix} 119354 & -52986 \\ -52986 & 23546 \end{pmatrix} \tag{2.2.10}$$

Eigen values of $\mathbf{M^TM}$ can be found out as

$$\left|\mathbf{M^TM} - \lambda\mathbf{I}\right| = 0 \tag{2.2.11}$$

$$\implies \begin{vmatrix} 119354 - \lambda & -52986 \\ -52986 & 23546 - \lambda \end{vmatrix} = 0 \tag{2.2.12}$$

Solving this we get the eigen values of $\mathbf{M^TM}$ as,

$$\lambda_1 = 19 \tag{2.2.13}$$

$$\lambda_2 = 142880 \tag{2.2.14}$$

The corresponding normalized eigen vectors are:

$$\mathbf{v}_1 = \begin{pmatrix} -0.9140 \\ 0.4058 \end{pmatrix} \tag{2.2.15}$$

$$\mathbf{v}_2 = \begin{pmatrix} -0.4058 \\ -0.9140 \end{pmatrix} \tag{2.2.16}$$

Hence,

$$\mathbf{V} = \begin{pmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{pmatrix} = \begin{pmatrix} -0.9140 & -0.4058 \\ 0.4058 & -0.9140 \end{pmatrix} \tag{2.2.17}$$

Eigen values of $\mathbf{MM^T}$ can be found by solving:

$$\left|\mathbf{MM^T} - \lambda\mathbf{I}\right| = 0 \tag{2.2.18}$$

$$\implies \begin{vmatrix} 114340 - \lambda & 52728 & -21970 \\ 52728 & 24336 - \lambda & -10140 \\ -21970 & -10140 & 4225 - \lambda \end{vmatrix} = 0 \tag{2.2.19}$$

Solving this, we get the eigen values of $\mathbf{MM^T}$ as:

$$\lambda_3 = 19 \tag{2.2.20}$$

$$\lambda_4 = 142880 \tag{2.2.21}$$

$$\lambda_5 = 0 \tag{2.2.22}$$

The corresponding eigen vectors after normalizing are:

$$\mathbf{u}_1 = \begin{pmatrix} -0.8945 \\ -0.4126 \\ 0.1719 \end{pmatrix} \tag{2.2.23}$$

$$\mathbf{u}_2 = \begin{pmatrix} 0.4470 \\ -0.8257 \\ 0.3441 \end{pmatrix} \tag{2.2.24}$$

$$\mathbf{u}_3 = \begin{pmatrix} 0 \\ 0.3846 \\ 0.9231 \end{pmatrix} \tag{2.2.25}$$

$$\therefore \mathbf{U} = \begin{pmatrix} -0.8945 & 0.4470 & 0 \\ -0.4126 & -0.8257 & 0.3846 \\ 0.1719 & 0.3441 & 0.9231 \end{pmatrix} \tag{2.2.26}$$

After computing the singular values from the eigen values,

$$\mathbf{S} = \begin{pmatrix} \sqrt{\lambda_1} & 0 \\ 0 & \sqrt{\lambda_2} \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 378 & 0 \\ 0 & 4.35 \\ 0 & 0 \end{pmatrix} \tag{2.2.27}$$

Therefore we get the SVD of $\mathbf{M}$ as:

$$\mathbf{M} = \begin{pmatrix} -0.8945 & 0.4470 & 0 \\ -0.4126 & -0.8257 & 0.3846 \\ 0.1719 & 0.3441 & 0.9231 \end{pmatrix} \begin{pmatrix} 378 & 0 \\ 0 & 4.35 \\ 0 & 0 \end{pmatrix} \tag{2.2.28}$$

$$\begin{pmatrix} -0.9140 & -0.4058 \\ 0.4058 & -0.9140 \end{pmatrix}^T \tag{2.2.29}$$

$$= \begin{pmatrix} 308 & -139 \\ 144 & -60 \\ -60 & 25 \end{pmatrix} \tag{2.2.30}$$

Moore- penrose pseudo inverse of $\mathbf{S}$ is:

$$\mathbf{S}_+ = \begin{pmatrix} 0.0026 & 0 & 0 \\ 0 & 0.2294 & 0 \end{pmatrix} \tag{2.2.31}$$

Putting the values in (2.2.6),

$$\mathbf{U^T b} = \begin{pmatrix} 744.132 \\ 5.99 \\ 3.25 \end{pmatrix} \tag{2.2.32}$$

$$\mathbf{S_+ U^T b} = \begin{pmatrix} 1.964 \\ 1.3736 \end{pmatrix} \tag{2.2.33}$$

$$\mathbf{c} = \mathbf{S_+ U^T b} = \begin{pmatrix} -2.35 \\ -0.458 \end{pmatrix} \tag{2.2.34}$$

2.3 Verification using least square method

$$\begin{pmatrix} 308 & -139 \\ 144 & -60 \\ -60 & 25 \end{pmatrix} \mathbf{c} = \begin{pmatrix} -663 \\ \frac{1243}{4} \\ 133 \end{pmatrix} \tag{2.3.1}$$

This is in the form of

$$A\mathbf{c} = \mathbf{b} \tag{2.3.2}$$

$$A^T A\mathbf{c} = A^T \mathbf{b} \implies \mathbf{c} = (A^T A)^{-1} A^T \mathbf{b} \tag{2.3.3}$$

$$A^T A = \begin{pmatrix} 308 & 144 & -60 \\ -139 & -60 & 25 \end{pmatrix} \begin{pmatrix} 308 & -139 \\ 144 & -60 \\ -60 & 25 \end{pmatrix} \tag{2.3.4}$$

$$= \begin{pmatrix} 119354 & -52986.8 \\ -52986.8 & 23546 \end{pmatrix} \tag{2.3.5}$$

The inverse can be written as

$$(A^T A)^{-1} = \begin{pmatrix} 0.00867 & 0.01951 \\ 0.01951 & 0.04395 \end{pmatrix} \tag{2.3.6}$$

$$A^T \mathbf{b} = \begin{pmatrix} -257098 \\ 114127 \end{pmatrix} \tag{2.3.7}$$

using 2.3.6 and 2.3.7 in 2.3.3,the center $\mathbf{c}$

$$\mathbf{c} = \begin{pmatrix} 0.00867 & 0.01951 \\ 0.01951 & 0.04395 \end{pmatrix} \begin{pmatrix} -257098 \\ 114127 \end{pmatrix} \tag{2.3.8}$$

$$\implies \mathbf{c} = \begin{pmatrix} -2.363 \\ -0.453 \end{pmatrix} \tag{2.3.9}$$

Comparing (2.2.34) and (2.3.9), it can be said that the solution of $\mathbf{c}$ is verified.