

1.1.1

Let $\Omega = \mathbb{R}$, \mathcal{F} = all subsets so that A or A^c is countable, $P(A) = 0$ in the first case and $= 1$ in the second. Show that (Ω, \mathcal{F}, P) is a probability space.

Proof.

a). For the case A is countable, A^c must be uncountable, because \mathbb{R} is uncountable. But $(A^c)^c = A$ is countable, which means $A^c \in \mathcal{F}$. So, if $A \in \mathcal{F}$, then $A^c \in \mathcal{F}$. For the case A^c is countable, which means $A \in \mathcal{F}$. And because A^c is countable, $A^c \in \mathcal{F}$. In summary, if $A \in \mathcal{F}$, then we have $A^c \in \mathcal{F}$.

b). For any countable sequence $A_i \in \mathcal{F}$, A_i must be the subset of \mathbb{Q} , because only rational numbers in real numbers are countable. So, $\cup_i A_i$ is also countable because $\cup_i A_i$ is the subset of \mathbb{Q} , as well. i.e. $\cup_i A_i \in \mathcal{F}$.

So, (Ω, \mathcal{F}, P) is a probability space.

1.1.3

A σ -field \mathcal{F} is said to be countably generated if there is a countable collection $\mathcal{C} \subset \mathcal{F}$ so that $\sigma(\mathcal{C}) = \mathcal{F}$. Show that \mathcal{R}^d is countably generated.

1.1.4

- (i) Show that if $\mathcal{F}_1 \subset \mathcal{F}_2 \subset \dots$ are σ -algebras, then $\cup_i \mathcal{F}_i$ is an algebra.
- (ii) Give an example to show that $\cup_i \mathcal{F}_i$ need not be a σ -algebra.

1.1.5

A set $A \subset \{1, 2, \dots\}$ is said to have asymptotic density θ if

$$\lim_{n \rightarrow \infty} |A \cap \{1, 2, \dots, n\}|/n = \theta$$

Let \mathcal{A} be the collection of sets for which the asymptotic density exists. Is \mathcal{A} a σ -algebra? an algebra?

1.2.1

Suppose X and Y are random variables on (Ω, \mathcal{F}, P) and let $A \in \mathcal{F}$. Show that if we let $Z(\omega) = X(\omega)$ for $\omega \in A$ and $Z(\omega) = Y(\omega)$ for $\omega \in A^c$, then Z is a random variable.

1.2.4

Show that if $F(x) = P(X \leq x)$ is continuous, then $Y = F(X)$ has a uniform distribution on $(0, 1)$, that is, if $y \in [0, 1]$, $P(Y \leq y) = y$.

1.2.7

- (i) Suppose X has density function f . Compute the distribution function of X^2 and then differentiate to find its density function.
- (ii) Work out the answer when X has a standard normal distribution to find the density of the chi-square distribution.

1.3.1

Show that if \mathcal{A} generates \mathcal{S} , then $X^{-1}(\mathcal{A}) \equiv \{\{X \in A\} : A \in \mathcal{A}\}$ generates $\sigma(X) = \{\{X \in B\} : B \in \mathcal{S}\}$.

1.3.5

A function f is said to be lower semicontinuous or l.s.c. if

$$\liminf_{y \rightarrow x} f(y) \geq f(x)$$

and upper semicontinuous (u.s.c.) if $-f$ is l.s.c. Show that f is l.s.c. if and only if $\{x : f(x) \leq a\}$ is closed for each $a \in \mathbf{R}$ and conclude that semicontinuous functions are measurable.