Nonhomogeneous linear equation solution set

The solution of the nonhomogeneous linear equation is consist of two parts; one is a any special solution, and the other is the general solution of the homogeneous linear equation. That means:

$$Soltion\ Set = \{Special\ Solution + General\ Solution\} \tag{1}$$

Range

The range of a matrix (A) is the column space of the matrix.

Diagonalizable matrix

If a n-dimension matrix can be diagonalized, it must have n eigenvectors, which are independent with each other.

Precedure:

1. Find its eigenvalues λ_1, λ_2, $\cdots$, λ_m;
2. Find the correspoding eigenvectors $\vec{a_1}$, $\vec{a_2}$, $\cdots$, $\vec{a_n}$;
3. $V = \{\vec{a_1}, \cdots, \vec{a_n}\}$;
4. $A = V\ diag(\lambda_1, \cdots, \lambda_m)\ V^{-1}$.

Inverse matrix

1. $\{A|I\} \Rightarrow \{I|A^{-1}\}$;
2. $A^{-1} = \frac{A^*}{|A|}$, A^* is adjugate matrix.

Orthogonal complement

- Row $(A)^\perp$ = Null (A);
- Range $(A)^\perp$ = Null (A^T).

Geometric multiplicity and algebraic multiplicity

- Geometric multiplicity of λ is the dimension of the Null$(A - \lambda I)$;
- Algebraic multiplicity of λ is the number of times λ appears in the equation $|\lambda I - A| = 0$.
- Geometric multiplicity is always not exceed Algebraic multiplicity; And all algebraic multiplicities of different eigenvalues sum up should be n (for a n×n matrix);
- if every Geometric multiplicity equals to Algebraic multiplicity, we say that the matrix is diagonalizable.

Vandermonde matrix

$$V = \begin{pmatrix} 1 & x_0 & x_0^2 & \cdots & x_0^n \\ 1 & x_1 & x_1^2 & \cdots & x_1^n \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^n \end{pmatrix}$$

$$V \Rightarrow \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 1 & x_1 - x_0 & x_1(x_1 - x_0) & \cdots & x_1^{n-1}(x_1 - x_0) \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & x_n - x_0 & x_n(x_n - x_0) & \cdots & x_n^{n-1}(x_n - x_0) \end{pmatrix}$$

$$\det(V) = \begin{vmatrix} x_1 - x_0 & x_1(x_1 - x_0) & \cdots & x_1^{n-1}(x_1 - x_0) \\ \vdots & \vdots & & \vdots \\ x_n - x_0 & x_n(x_n - x_0) & \cdots & x_n^{n-1}(x_n - x_0) \end{vmatrix} = (x_n - x_0)\cdots(x_1 - x_0) \begin{vmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^n \\ 1 & x_2 & x_2^2 & \cdots & x_2^n \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^n \end{vmatrix}$$

$$\det(V) = \prod_{0 \le i < j \le n} (x_j - x_i)$$

Characteristic polynomial

The characteristic polynomial of a matrix A is $|A - \lambda I|$.

Representing linear transformations by matrices

For a transformation T and a basis a, it can be represented by a matrix T as follows:

$$T(x) = Tx = x^T T^T,\ T = \{T(a_1), \cdots, T(a_n)\}^T$$

Vector projection

For a vector u, the projection of u on v is as follows:

$$a_1 = \frac{< u, v >}{< v, v >} v$$

So, the projection of u on a plane with normal vector v is as follows:

$$a_2 = u - a_1$$

So, the reflection of u on a plane with normal vector v is as follows:

$$a_2 = u - 2a_1$$

Dimension formula

$$\dim(V_1 + V_2) = \dim(V_1) + \dim(V_2) - \dim(V_1 \cap V_2)$$

Rotation and reflection

For a transformation, if its matrix is orthogonal and the determinant is 1, then, it is a Rotation. Otherwise, if the determinant is -1, then, it is a reflection.

$$\begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}$$

Gram–Schmidt process

For a basis $(\alpha_1, \cdots, \alpha_n)$, the orthogonal basis can be gotten with the following steps:

1. $\beta_1 = \alpha_1$;
2. $\beta_2 = \alpha_2 - k\beta_1,\ k = \frac{<\alpha_2, \beta_1>}{<\beta_1, \beta_1>}$;
3. $\cdots\cdots$
4. $\beta_n = \alpha_n - k_1\beta_1 - k_2\beta_2 - \cdots - k_{n-1}\beta_{n-1},\ k_i = \frac{<\alpha_n, \beta_i>}{<\beta_i, \beta_i>}$.