Analysis: exercise class 1

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1.1.

Prove that (1.5) implies $y(x) \equiv 0$.

Proof.

For $y'(x) = y(x)$, we can know that $y = Ce^x$, and because $e^x > 0$, $y(x_0) = 0$, we know that $C = 0$. So, $y \equiv 0$.

1.4.

Prove that a differentiable weak solution of (1.4) is a strong solution.

Proof.

We only need to prove that for a weak solution $y(x)$, it satisfies 1) $y'(x) = y(x)$, 2) $y(x_0) = y_0$.

1), $\frac{\partial\, y(x)}{\partial\, x} = \frac{\partial\, \int_{x_0}^{x} y(s)\, \mathbf{d}s}{\partial\, x} = y(x)$.

2), $y(x_0) = \int_{x_0}^{x_0} y(s)\, \mathbf{d}s + y_0 = y_0$.

So, a differentiable weak solution is a strong solution.

1.5.

The iteration defined in (1.10) is called an explicit iteration scheme. Repeat the process described here for the implicit iteration scheme, defined by

$$\widetilde{y}_{n-1}(x) = \int_{x_0}^{x} \widetilde{y}_n(s) ds + y_0,$$

starting at $\widetilde{y}_0(x) \equiv y_0$.

1.6.

Prove that the function $y_n(x)$ is given by

$$y_n(x) = y_0 \sum_{j=0}^{n} \frac{(x-x_0)^j}{j!}. \tag{1}$$

Proof.

Use mathematical induction:

For $n = 1$, $y_1(x) = y_0 + (x - x_0)y_0$. So, (1) holds.

For $n = 2$, $y_2(x) = y_0 + \left((x - x_0) + \frac{1}{2}(x - x_0)^2\right) y_0$. So, (1) holds, too.

Assume that for all $n = k > 2$, (1) holds, then, for $n = k + 1$, we have:

$$\begin{aligned}
y_{k+1} &= \int_{x_0}^{x} y_k(s)\,\mathbf{d}s + y_0 \\
&= y_0 \int_{x_0}^{x} \sum_{j=0}^{k} \frac{(s-x_0)^j}{j!}\,\mathbf{d}s + y_0 \\
&= y_0 \sum_{j=1}^{k+1} \frac{(x-x_0)^j}{j!} + y_0 \\
&= y_0 \sum_{j=0}^{k+1} \frac{(x-x_0)^j}{j!}.
\end{aligned}$$

So, (1) holds, as well. $y_n(x) = y_0 \sum_{j=0}^{n} \frac{(x-x_0)^j}{j!}$.

1.7.

(1) Prove that $\mathcal{L}$ is a linear operator; that is

$$\mathcal{L}\left(\alpha_1 y_1 + \alpha_2 y_2\right) = \alpha_1 \mathcal{L}\left(y_1\right) + \alpha_2 \mathcal{L}\left(y_2\right)$$

(2) Prove that $\mathrm{Ker}(\mathcal{L})$ is a subspace of the vector space where $\mathcal{L}$ is defined. What is its dimension? Can you write the range of $\mathcal{L}$ in some explicit form?

1.18.

Exercise 1.8. Extend the discussion of the equation $y' = y$ to a general first order linear equation; that is, an equation of the form

$$a_0(x)Dy(x) + a_1(x)y(x) = b(x)$$

by imposing conditions on the functions a_0, a_1, b needed for your analysis. Here $D = \frac{\partial}{\partial x}$ is the derivative with respect to the variable x. Discuss existence and uniqueness of solutions.