Analysis: Homework 1

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Update: September 8, 2021

Picard's Theorem

For the Initial Value Problem:

$$\begin{cases} y'(x) = F(x,y) \\ y(x_0) = y_0. \end{cases}$$

Prove the existence and uniqueness of the solution on

$$R = \{(x,y) : |x - x_0| \leqslant a, |y - y_0| \leqslant b\}.$$

Assume:

- $F : D \subset \mathbb{R}^2 \to \mathbb{R}$ is Lipschitz continuous for y with constant δ, which means for y_1, y_2:

$$|F(x,y_1) - F(x,y_2)| \leqslant \delta \, |y_1 - y_2| \, .$$

- F is bounded:

$$\forall \, (x,y) \in D, \, \exists \, M \in \mathbb{R}, \, s.t. \; |F(x,y)| \leqslant M.$$

- $\delta a < 1$.

Proof.

Existence, we construct the iteration as following:

$$\begin{cases} y_0(x) = y_0, \\ y_{n+1}(x) = y_0 + \displaystyle\int_{x_0}^{x} F(s, y_n(s)) \, \mathbf{d}s. \end{cases}$$

We can know that $\{y_n\}$ is differentiable. Then, we need to prove the limit exists.

$$\begin{aligned} |y_1(x) - y_0(x)| \leqslant & \left| \int_{x_0}^{x} F(s, y_0(s)) \, \mathbf{d}s \right| \\ \leqslant & M |x - x_0| \\ \leqslant & Ma \end{aligned}$$

$$
\begin{aligned}
|y_2(x)-y_1(x)| \leqslant & \left|\int_{x_0}^{x} F(s,y_1(s))-F(s,y_0(s))\,\mathbf{d}s\right| \\
\leqslant & \left|\delta\int_{x_0}^{x} |y_1(s)-y_0(s)|\,\mathbf{d}s\right| \\
\leqslant & \left|\delta\int_{x_0}^{x}\int_{x_0}^{s_0} |F(s_1,y_0(s_1)|\,\mathbf{d}s_1\right| \\
\leqslant & \delta M\frac{|x-x_0|^2}{2!} \\
\leqslant & \frac{a^2}{2!}\delta M.
\end{aligned}
$$

So, for $|y_n(x)-y_{n-1}(x)|$, we have:

$$
\begin{aligned}
|y_n(x)-y_{n-1}(x)| \leqslant & \left|\int_{x_0}^{x} F(s,y_{n-1}(s))-F(s,y_{n-2}(s))\,\mathbf{d}s\right| \\
\leqslant & \left|\delta^{n-1}\int_{x_0}^{x}\int_{x_0}^{s_0}\cdots\int_{x_0}^{s_{n-2}} |F(s_{n-1},y_0(s_{n-1}))|\,\mathbf{d}s_{n-1}\right| \\
\leqslant & \delta^{n-1}M\frac{|x-x_0|^n}{n!} \\
\leqslant & \frac{a^n}{n!}\delta^{n-1}M.
\end{aligned}
$$

Since $\delta a<1$, then, $\forall\, m>n>0$, we can get:

$$
\begin{aligned}
|y_m(x)-y_n(x)| \leqslant & |y_m(x)-y_{m-1}(x)|+\cdots+|y_{n+1}(x)-y_n(x)| \\
\leqslant & \delta^{n-1}M\sum_{i=n}^{m}\frac{a^n}{n!}.
\end{aligned}
$$

When $n\to\infty$, $|y_m(x)-y_n(x)|\to 0$. So, $\{y_n\}$ is a cauchy sequence, and it is uniformly convergent.

Let $\lim_{n\to\infty} y_n = y$, then:

$$
y(x) = y_0 + \int_{x_0}^{x} F(s,y(s))\,\mathbf{d}s,
$$

And

$$
\begin{cases}
y(x) = y_0, \\
y'(x) = F(x,y(x)).
\end{cases}
$$

Thus, y is the solution of that Initial Value Problem.

□

Proof.

UNIQUENESS, if y_1, y_2 are both the solution of the problem, then, we have:

$$
y_1(x) = y_0 + \int_{x_0}^{x} F(s,y_1(s))\,\mathbf{d}s,
$$

$$y_2(x) = y_0 + \int_{x_0}^{x} F(s, y_2(s))\,\mathbf{d}s,$$

Then,

$$\begin{aligned}
|y_1(x) - y_2(x)| &= \left| \int_{x_0}^{x} F(s, y_1(s)) - F(s, y_2(s))\,\mathbf{d}s \right| \\
&\leqslant \left| \delta \int_{x_0}^{x} |y_1(s) - y_2(s)|\,\mathbf{d}s \right| \\
&\leqslant \delta |x - x_0| \max_{s \in [x_0, x_0+h]} |y_1(s) - y_2(s)| \\
&\leqslant \delta a \max_{s \in [x_0, x_0+a]} |y_1(s) - y_2(s)|.
\end{aligned}$$

We can write above inequation as:

$$\max_{x \in [x_0, x_0+a]} |y_1(x) - y_2(x)| \leqslant \delta a \max_{s \in [x_0, x_0+a]} |y_1(s) - y_2(s)|,$$

So, $\max_{x \in [x_0, x_0+a]} |y_1(x) - y_2(x)|$ can only be 0, which means $y_1 = y_2$.

□

Exercise. 1

$$y' = 1 + y^2.$$

Solution.

$$\begin{gathered}
\frac{\mathbf{d}y}{1 + y^2} = \mathbf{d}x \\
\arctan(y) = x + c
\end{gathered}$$

For generality, we can write y as:

$$y(x) = \tan(x + c),$$

If we know the initial value $y(x_0) = y_0$, then

$$c = \arctan(y_0) - x_0.$$

Exercise. 2

$$y' = \sqrt{|y|}.$$

Solution.

If $y > 0$,

$$\begin{gathered}
y^{-1/2}\,\mathbf{d}y = \mathbf{d}x \\
2y^{1/2} = x + c
\end{gathered}$$

$$y = \frac{1}{4}(x+c)^2$$

If we know $y(x_0) = y_0$, then

$$c = 2\sqrt{y_0} - x_0.$$

If $y < 0$,

$$(-y)^{-1/2}\,\mathbf{d}y = \mathbf{d}x$$

$$2(-y)^{1/2} = x + c$$

$$y = -\frac{1}{4}(x+c)^2$$

If we know $y(x_0) = y_0$, then

$$c = 2\sqrt{-y_0} - x_0.$$