Probability: Problem Set 2

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1.4.1.

Show that if $f \geq 0$ and $\int f \, \mathbf{d}\mu = 0$, then $f = 0$ a.e.

1.4.2.

Let $f \geq 0$ and $E_{n,m} = \{x : m/2^n \leq f(x) < (m+1)/2^n\}$. As $n \uparrow \infty$,

$$\sum \sum_{m=1}^{\infty} \frac{m}{2^n} \mu\left(E_{n,m}\right) \uparrow \int f \, \mathbf{d}\mu$$

1.4.3.

Let g be an integrable function on $\mathbf{R}$ and $\epsilon > 0$.

(i) Use the definition of the integral to conclude there is a simple function $\varphi = \sum_k b_k 1_{A_k}$ with $\int |g - \varphi| dx < \epsilon$.

(ii) Use Exercise A.2.1 to approximate the A_k by finite unions of intervals to get a step function

$$q = \sum_{j=1}^{k} c_j 1_{(a_{j-1}, a_j)}$$

with $a_0 < a_1 < \ldots < a_k$, so that $\int |\varphi - q| < \epsilon$.

(iii) Round the corners of q to get a continuous function r so that $\int |q - r| \, \mathbf{d}x < \epsilon$.

(iv) To make a continuous function replace each $c_j 1_{(a_{j-1}, a_j)}$ by a function that is 0 $\left(a_{j-1}, a_j\right)^c$, c_j on $[a_{j-1} + \delta - j, a_j - \delta_j]$, and linear otherwise. If the δ_j are small enough and we let $r(x) = \sum_{j=1}^{k} r_j(x)$, then

$$\int |q(x) - r(x)| \, \mathbf{d}\mu = \sum_{j=1}^{k} \delta_j c_j < \epsilon.$$

1.4.4.

Prove the Riemann-Lebesgue lemma. If g is integrable, then

$$\lim_{n \to \infty} \int g(x) \cos nx \, \mathbf{d}x = 0,$$

Hint: If g is a step function, this is easy. Now use the previous exercise.

1.5.2.

Show that if μ is a probability measure, then

$$\|f\|_\infty = \lim_{p\to\infty} \|f\|_p.$$

1.5.7.

Let $f \geq 0$.

(i) Show that $\int f \wedge n \, \mathbf{d}\mu \uparrow \int f \, \mathbf{d}\mu$ as $n \to \infty$.

(ii) Use (i) to conclude that if g is integrable and $\epsilon > 0$, then we can pick $\delta > 0$ so that $\mu(A) < \delta$ implies $\int_A |g| \, \mathbf{d}\mu < \epsilon$.

1.6.1.

Suppose φ is strictly convex, i.e., $>$ holds for $\lambda \in (0,1)$. Show that, under the assumptions of Theorem 1.6.2, $\varphi(EX) = E\varphi(X)$ implies $X = EX$ a.s.

1.6.2.

Suppose $\varphi : \mathbf{R}^n \to \mathbf{R}$ is convex. Imitate the proof of Theorem 1.5.1 to show

$$E\varphi\left(X_1, \ldots, X_n\right) \geq \varphi\left(EX_1, \ldots, EX_n\right)$$

provided $E\left|\varphi\left(X_1, \ldots, X_n\right)\right| < \infty$ and $E\left|X_i\right| < \infty$ for all i

1.6.3.

Chebyshev's inequality is and is not sharp.

(i) Show that Theorem 1.6.4 is sharp by showing that if $0 < b \leq a$ are fixed there is an X with $EX^2 = b^2$ for which $P(|X| \geq a) = b^2/a^2$.

(ii) Show that Theorem 1.6.4 is not sharp by showing that if X has $0 < EX^2 < \infty$, then

$$\lim_{a\to\infty} a^2 P(|X| \geq a)/EX^2 = 0.$$

1.6.9.

Inclusion-exclusion formula. Let $A_1, A_2, \ldots A_n$ be events and $A = \cup_{i=1}^n A_i$. Prove that $1_A = 1 - \prod_{i=1}^n \left(1 - 1_{A_i}\right)$. Expand out the right-hand side, then take expected value to conclude

$$\begin{aligned} P\left(\cup_{i=1}^n A_i\right) = & \sum_{i=1}^n P\left(A_i\right) - \sum_{i<j} P\left(A_i \cap A_j\right) \\ & + \sum_{i<j<k} P\left(A_i \cap A_j \cap A_k\right) - \cdots + (-1)^{n-1} P\left(\cap_{i=1}^n A_i\right). \end{aligned}$$

1.6.10.

Bonferroni inequalities. Let $A_1, A_2, \ldots A_n$ be events and $A = \cup_{i=1}^{n} A_i$. Show that $1_A \leq \sum_{i=1}^{n} 1_{A_i}$, etc. and then take expected values to conclude

$$P\left(\cup_{i=1}^{n} A_i\right) \leq \sum_{i=1}^{n} P\left(A_i\right)$$

$$P\left(\cup_{i=1}^{n} A_i\right) \geq \sum_{i=1}^{n} P\left(A_i\right) - \sum_{i<j} P\left(A_i \cap A_j\right)$$

$$P\left(\cup_{i=1}^{n} A_i\right) \leq \sum_{i=1}^{n} P\left(A_i\right) - \sum_{i<j} P\left(A_i \cap A_j\right) + \sum_{i<j<k} P\left(A_i \cap A_j \cap A_k\right)$$

In general, if we stop the inclusion-exclusion formula after an even (odd) number of sums, we get a(n) lower (upper) bound.

1.6.11.

If $E|X|^k < \infty$, then for $0 < j < k$, $E|X|^j < \infty$, and furthermore

$$E|X|^j \leq \left(E|X|^k\right)^{j/k}.$$