Probability: Problem Set 1

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1.1.1

Let $\Omega = \mathbf{R}$, $\mathcal{F} =$ all subsets so that A or A^c is countable, $P(A) = 0$ in the first case and $= 1$ in the second. Show that $(\Omega, \mathcal{F}, P)$ is a probability space.

Proof.

a), For the case A is countable, A^c must be uncountable, because $\mathbb{R}$ is uncountable. But $(A^c)^c = A$ is countable, which means $A^c \in \mathcal{F}$. So, if $A \in \mathcal{F}$, then $A^c \in \mathcal{F}$. For the case A^c is countable, which means $A \in \mathcal{F}$. In summary, if $A \in \mathcal{F}$, then we have $A^c \in \mathcal{F}$.

b), For any countable sequence sets $A_i \in \mathcal{F}$, A_i must be either the subset of $\mathbb{Q}$ or at least including all irrational numbers, which cloud make sure A or A^c is countable, because only rational numbers in real numbers are countable. So, $\cup_i A_i$ is either the subset of the $\mathbb{Q}$ or at least including all irrational numbers. In this case, Either $\cup_i A_i$ or $(\cup_i A_i)^c$ is countable. i.e. $\cup_i A_i \in \mathcal{F}$.

So, $\mathcal{F}$ is σ-algebra. For P, because Ω is uncountable, $P(\Omega) = 1$. Thus, $(\Omega, \mathcal{F}, P)$ is a probability space.

□

1.1.3

A σ-field $\mathcal{F}$ is said to be countably generated if there is a countable collection $\mathcal{C} \subset \mathcal{F}$ so that $\sigma(\mathcal{C}) = \mathcal{F}$. Show that $\mathcal{R}^d$ is countably generated.

Proof.

With the hint from exercise 1.1.2, we can prove that $\sigma(\mathcal{S}_d) = \mathcal{R}^d$. $\mathcal{S}_d = \{(a_1, b_1] \times (a_2, b_2] \times \cdots \times (a_n, b_n] : -\infty \leq a_1 < b_1 \leq +\infty\}$. Let $\mathcal{O}$ denotes all open subsets in $\mathcal{R}^d$. So, $\mathcal{R}^d$ can be generated by $\mathcal{O}$. Then, we need to prove $\sigma(\mathcal{O}) \subset \sigma(\mathcal{S}_d)$ and $\sigma(\mathcal{S}_d) \subset \sigma(\mathcal{O})$.

a), Notice that

$$(a_1, b_1) \times \cdots \times (a_d, b_d) = \cup_{n\to\infty} (a_1, b_1 - 1/n] \times \cdots \times (a_d, b_d - 1/n]$$

which means the open rectangles $(a_1, b_1) \times \cdots \times (a_d, b_d) \subset \sigma(\mathcal{S}_d)$. For any open set in $\sigma(\mathcal{O})$, it can be represented as countable union of open rectangles of rational numbers, because rational numbers are dense and countable. So, we have $\sigma(\mathcal{O}) \subset \sigma(\mathcal{S}_d)$.

b), Observe that

$$(a_1, b_1] \times \cdots \times (a_d, b_d] = \cap_{n\to\infty} (a_1, b_1 + 1/n) \times \cdots \times (a_d, b_d + 1/n)$$

Because the union of open sets is open set, as well, we can get that $\mathcal{S}_d \subset \sigma(\mathcal{O})$. Futher, $\sigma(\mathcal{S}_d) \subset \sigma(\mathcal{O})$.

c), Combine a) and b), we know that $\sigma(\mathcal{S}_d) = \mathcal{R}^d$. Let $\mathcal{Q} = \{(q_1, \infty) \times \cdots \times (q_n, \infty) : q_i \in \mathbb{Q}\}$. For any $\mathcal{S} = (a_1, b_1] \times \cdots \times (a_d, b_d] \in \mathcal{S}_d$, since $\mathbb{Q}$ is countable, $\mathcal{S}$ can be represented as countable intersection and union operations of the subsets in $\mathcal{Q}$. So, $\sigma(\mathcal{Q}) = \sigma(\mathcal{S}_d)$. Then, we can get that $\mathcal{R}^d$ is countably generated.

□

1.1.4

(i) Show that if $\mathcal{F}_1 \subset \mathcal{F}_2 \subset \cdots$ are σ-algebras, then $\cup_i \mathcal{F}_i$ is an algebra.

(ii) Give an example to show that $\cup_i \mathcal{F}_i$ need not be a σ-algebra.

Proof.

i),

- For any $A \in \cup_i \mathcal{F}_i$, there must be a k, such that $A \in \mathcal{F}_k$. So, $A^c \in \mathcal{F}_k$. Thus, $A^c \in \cup_i \mathcal{F}_i$.
- For $A, B \in \cup_i \mathcal{F}_i$, there must be a, b, such that $A \in \mathcal{F}_a$, $B \in \mathcal{F}_b$. So, $A \cup B \subset \mathcal{F}_a \cup \mathcal{F}_b \subset \cup_i \mathcal{F}_i$.

So, $\cup_i \mathcal{F}_i$ is an algebra.

□

Solution.

ii), Let $\mathcal{F}_i = \{-i, \cdots, i\}$, and $\mathcal{F}_1 \subset \mathcal{F}_2 \subset \cdots$. For $\mathcal{A}_i = \{i\} \in \cup_i \mathcal{F}_i$, $\cup_i^{\infty} \mathcal{A}_i = \mathbb{N} \backslash \{0\}$. If $\cup_i \mathcal{F}_i$ is a σ-algebra, there should be a k, such that $\mathbb{N} \backslash \{0\} \subset \mathcal{F}_k$, however, it is impossible. So, $\cup_i \mathcal{F}_i$ need not be a σ-algebra. [a]

[a] Maybe not correct.

1.1.5

A set $A \subset \{1, 2, \cdots\}$ is said to have asymptotic density θ if

$$\lim_{n\to\infty} |A \cap \{1, 2, \cdots, n\}|/n = \theta$$

Let $\mathcal{A}$ be the collection of sets for which the asymptotic density exists. Is $\mathcal{A}$ a σ-algebra? an algebra?

Solution.

Let $A_1 = \{1, \cdots, n_1, \cancel{n_1+1, \cdots, 2n_1}\}$,

$|A_1 \cap \{1, 2, \cdots, 2n_1\}|/(2n_1) = \frac{1}{2}$;

Let $A_2 = \{1, \cdots, n_1, \cancel{n_1+1, \cdots, 4n_1}\}$,

$|A_2 \cap \{1, 2, \cdots, 4n_1\}|/(4n_1) = \frac{1}{4}$;

Let $A_3 = \{1, \cdots, n_1, \cancel{n_1+1, \cdots, 4n_1}, 4n_1+1, \cdots, 6n_1\}$,

$|A_3 \cap \{1, 2, \cdots, 6n_1\}|/(6n_1) = \frac{3}{6} = \frac{1}{2}$;

Let $A_4 = \{1, \cdots, n_1, \cancel{n_1+1, \cdots, 4n_1}, 4n_1+1, \cdots, 6n_1, \cancel{6n_1+1, \cdots, 12n_1}\}$,

$|A_4 \cap \{1, 2, \cdots, 12n_1\}|/(12n_1) = \frac{3}{12} = \frac{1}{4}$;

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Like this, we can get an infinite countable set sequence $\{A_i\}$, and its limit oscillates between $\frac{1}{2}$ and $\frac{1}{4}$. So, the limit of $\{A_i\}$ doesn't exist, i.e. $\cup_i^\infty A_i \notin \mathcal{A}$, however, $A_i \in \mathcal{A}$. Hence, $\mathcal{A}$ isn't a σ-algebra.

I guess it may not be an algebra. But I can not get an example to prove it.

1.2.1

Suppose X and Y are random variables on $(\Omega, \mathcal{F}, P)$ and let $A \in \mathcal{F}$. Show that if we let $Z(\omega) = X(\omega)$ for $\omega \in A$ and $Z(\omega) = Y(\omega)$ for $\omega \in A^c$, then Z is a random variable.

Proof.

First, $Z : \Omega \to \mathbb{R}$,

$$z(\omega) = \begin{cases} X(\omega) \in \mathbb{R}, \omega \in A \\ Y(\omega) \in \mathbb{R}, \omega \in A^c \end{cases},$$

So, $Z(\omega) \in \mathbb{R}$.

Second, for borel set $B \in \mathbb{R}$, we need to prove that $Z^{-1} \in \mathcal{F}$.

$$\begin{aligned} Z^{-1}(B) &= \{\omega \in \Omega : Z(\omega) \in B\} \\ &= \{\omega \in A : X(\omega) \in B\} \cup \{\omega \in A^c : Y(\omega) \in B\} \\ &= \{A \cap X^{-1}(B)\} \cup \{A^c \cap Y^{-1}(B)\}. \end{aligned}$$

Because A and $A^c \in \mathcal{F}$, $Z^{-1} \in \mathcal{F}$.

□

1.2.4

Show that if $F(x) = P(X \leq x)$ is continuous, then $Y = F(X)$ has a uniform distribution on $(0, 1)$, that is, if $y \in [0, 1]$, $P(Y \leq y) = y$.

1.2.7

(i) Suppose X has density function f. Compute the distribution function of X^2 and then differentiate to find its density function.
(ii) Work out the answer when X has a standard normal distribution to find the density of the chi-square distribution.

1.3.1

Show that if $\mathcal{A}$ generates $\mathcal{S}$, then $X^{-1}(\mathcal{A}) \equiv \{\{X \in A\} : A \in \mathcal{A}\}$ generates $\sigma(X) = \{\{X \in B\} : B \in \mathcal{S}\}$.

1.3.5

A function f is said to be lower semicontinuous or l.s.c. if

$$\liminf_{y \to x} f(y) \geq f(x)$$

and upper semicontinuous (u.s.c.) if $-f$ is l.s.c. Show that f is l.s.c. if and only if $\{x : f(x) \leq a\}$ is closed for each $a \in \mathbf{R}$ and conclude that semicontinuous functions are measurable.