Probability: Problem Set 1

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1.1.1

Let $\Omega = \mathbf{R}$, $\mathcal{F} =$ all subsets so that A or A^c is countable, $P(A) = 0$ in the first case and $= 1$ in the second. Show that $(\Omega, \mathcal{F}, P)$ is a probability space.

Proof.

a), For the case A is countable, A^c must be uncountable, because $\mathbb{R}$ is uncountable. But $(A^c)^c = A$ is countable, which means $A^c \in \mathcal{F}$. So, if $A \in \mathcal{F}$, then $A^c \in \mathcal{F}$. For the case A^c is countable, which means $A \in \mathcal{F}$. And because A^c is countable, $A^c \in \mathcal{F}$. In summary, if $A \in \mathcal{F}$, then we have $A^c \in \mathcal{F}$.

b), For any countable sequence sets $A_i \in \mathcal{F}$, A_i must be either the subset of $\mathbb{Q}$ or including all irrational numbers, which cloud make sure $A^c \in \mathcal{F}$, because only rational numbers in real numbers are countable. So, $\cup_i A_i$ is either the subset of the $\mathbb{Q}$ or including all irrational numbers. In this case, Either $\cup_i A_i$ is countable or $(\cup_i A_i)^c$ is countable. i.e. $\cup_i A_i \in \mathcal{F}$.

So, $\mathcal{F}$ is a σ-algebra, and $(\Omega, \mathcal{F}, P)$ is a probability space.

1.1.3

A σ-field $\mathcal{F}$ is said to be countably generated if there is a countable collection $\mathcal{C} \subset \mathcal{F}$ so that $\sigma(\mathcal{C}) = \mathcal{F}$. Show that $\mathcal{R}^d$ is countably generated.

1.1.4

(i) Show that if $\mathcal{F}_1 \subset \mathcal{F}_2 \subset \cdots$ are σ-algebras, then $\cup_i \mathcal{F}_i$ is an algebra.
(ii) Give an example to show that $\cup_i \mathcal{F}_i$ need not be a σ-algebra.

1.1.5

A set $A \subset \{1, 2, \cdots\}$ is said to have asymptotic density θ if

$$\lim_{n\to\infty} |A \cap \{1, 2, \cdots, n\}|/n = \theta$$

Let $\mathcal{A}$ be the collection of sets for which the asymptotic density exists. Is $\mathcal{A}$ a σ-algebra? an algebra?

Solution.

For any finite subset $A \subset \{1, 2, \cdots\}$, the asymptotic density is 0. And for any infinite subset $A \subset \{1, 2, \cdots\}$, the asymptotic density is 1. So, $\mathcal{A}$ is the power set of $\{1, 2, \cdots\}$.

$$\forall A \in \mathcal{A}, A^c \in \mathcal{A},$$

For a countable sequence sets $A_n \in \mathcal{F}$, $\cup_n A_n \in \mathcal{F}$.

So, $\mathcal{A}$ is a σ-algebra. Further, $\mathcal{A}$ is an algebra.

1.2.1

Suppose X and Y are random variables on $(\Omega, \mathcal{F}, P)$ and let $A \in \mathcal{F}$. Show that if we let $Z(\omega) = X(\omega)$ for $\omega \in A$ and $Z(\omega) = Y(\omega)$ for $\omega \in A^c$, then Z is a random variable.

1.2.4

Show that if $F(x) = P(X \leq x)$ is continuous, then $Y = F(X)$ has a uniform distribution on $(0, 1)$, that is, if $y \in [0, 1]$, $P(Y \leq y) = y$.

1.2.7

(i) Suppose X has density function f. Compute the distribution function of X^2 and then differentiate to find its density function.
(ii) Work out the answer when X has a standard normal distribution to find the density of the chi-square distribution.

1.3.1

Show that if $\mathcal{A}$ generates $\mathcal{S}$, then $X^{-1}(\mathcal{A}) \equiv \{\{X \in A\} : A \in \mathcal{A}\}$ generates $\sigma(X) = \{\{X \in B\} : B \in \mathcal{S}\}$.

1.3.5

A function f is said to be lower semicontinuous or l.s.c. if

$$\liminf_{y \to x} f(y) \geq f(x)$$

and upper semicontinuous (u.s.c.) if $-f$ is l.s.c. Show that f is l.s.c. if and only if $\{x : f(x) \leq a\}$ is closed for each $a \in \mathbf{R}$ and conclude that semicontinuous functions are measurable.