BABY RUDIN CHAPTER 1

Problem 1
If $r \in \mathbb{I} = \mathbb{R} \setminus \mathbb{Q}$ and $x \in \mathbb{Q}$ prove that $r + x, rx \in \mathbb{I}$.

By hypothesis there are no integers a, b such that $r = a/b$, but there are integers such that $x = a/b$. In particular let's say $x = a/b$ where $b \neq 0$ and $gcd(a, b) = 1$. For contradiction suppose $r + x = p/q$ for integers p, q. This would imply

$$r = p/q - x = \frac{pb - aq}{bq} \in \mathbb{Q}$$

Also for contradiction assume $rx \in \mathbb{Q}$ and $rx = p/q$ to lazily re-use our old symbols. Then so long as $a \neq 0$ we have

$$r = \frac{p}{q}\frac{1}{x} = \frac{pb}{qa} \in \mathbb{Q}$$

and if $a = 0$ then of course $rx = ra/b = 0 \in \mathbb{Q}$.

Problem 2

Prove there is no rational square-root of 12.

Suppose for contradiction $a/b = \sqrt{12}$ with $gcd(a, b) = 1$ and $b \neq 0$. Then $a^2 = 12b^2$ which implies a has a factor of 3. Therefore a^2 has two factors of 3. Therefore b must be furnishing at least one factor of 3, since 12 is only furnishing one. But then $gcd(a, b) > 1$.

Problem 4

Show that lowerbounds are less than upperbounds.

Let $\emptyset \neq E \subseteq \mathbb{R}$ with lower bound α and upper bound β. Let $x \in E$ so that $\alpha \leq x \leq \beta$.

Problem 5.

Show that the inf is the negative sup of the negative set.

Let $\emptyset \neq A \subseteq \mathbb{R}$. We want to argue that

$$\inf A = -\sup(-A)$$

and to show that anything is the infimum of a set, we can attempt to show that it is the greatest lower bound. That means showing 1) it's a lower bound and 2) showing that it's greater than any other lower bound.

First let's see that $-\sup(-A)$ is a lower bound of A (i.e. part (1)). So let $x \in A$ and for brevity let's call $s = -\sup(-A)$. Then $-s = \sup(-A)$ and by definition of $\sup(-A)$ we know that $-s$ is an upper bound of $-A$. Since $-x \in -A$ it follows that $-x \leq -s$ and so $s \leq x$ as desired.

Now let's see that s is the greatest lower bound of A, so consider some s' which is a lower bound of A. As before we know $-s = \sup(-A)$ and so $-s$ is the least upper bound of $-A$. We can also claim that $-s'$ is an upper bound of $-A$ because it's a lower bound of A. In the paragraph below I prove this claim.

We want to show that $-s'$ is an upper bound of $-A$ as stated above. So let $-x \in -A$, then we want $-x \leq -s'$. This is equivalent to $s' \leq x \in A$. But since s' was assumed to be a lower bound of A then this inequality must be true.

To summarize, we now know that $-s'$ is an upper bound of $-A$ and $-s$ is the least upper bound. These together imply $-s \leq -s'$ so that $s' \leq s$. But this then demonstrates that s is greater than all other lower bounds of A. That makes s the greatest lower bound, as desired.