AARHUS UNIVERSITY

Methods 4 - 2

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BSc Programme in Cognitive Science

Spring 2022

Recap: Bayes' rule

- The product rule of probability states that

$$p(A|B)p(B) = p(B|A)p(A)$$

- If we divide by $p(B)$, we get **Bayes' rule:**

$$p(A|B) = \frac{p(B|A)p(A)}{p(B)} = \frac{p(B|A)p(A)}{\sum_a p(B|a)p(a)}$$

What???

- The last equality comes from unpacking $p(B)$ according to the product and sum rules:

$$p(B) = \sum_a p(B, a) = \sum_a p(B|a)p(a)$$

Bayes' rule: what problem does it solve?

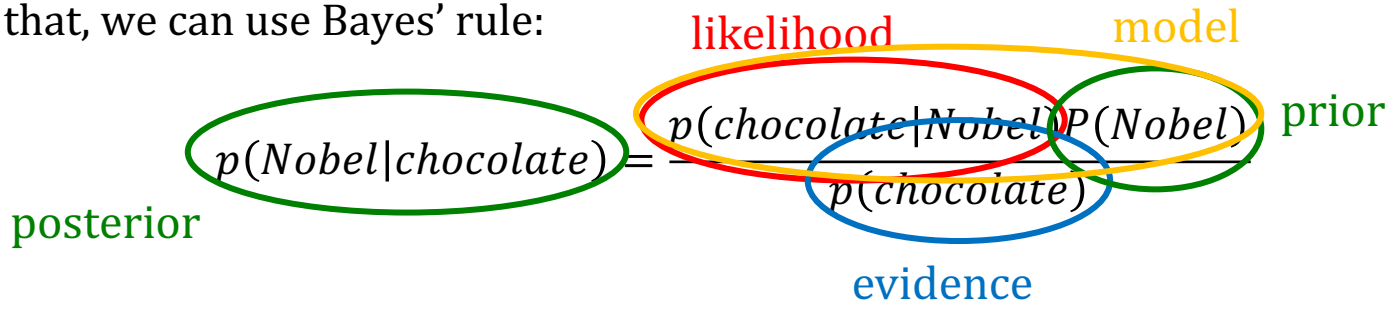
- Why is Bayes' rule important?
- It allows us to invert conditional probabilities, ie to pass from $p(B|A)$ to $p(A|B)$:

$$p(A|B) = \frac{p(B|A)p(A)}{p(B)}$$

- In other words, it allows us to update our belief about A in light of observation B

Bayes' rule: the chocolate example

In our example, it is immediately clear that $P(Nobel|chocolate)$ is very different from $P(chocolate|Nobel)$. While the first is hopeless to determine directly, the second is much easier to find out: ask Nobel laureates how much chocolate they eat. Once we know that, we can use Bayes' rule:



However: *note that no amount of statistical analysis will tell you anything about the causal mechanism behind this if you don't have* ***a hypothesis about that mechanism and a causal scientific model of it****!*

A simple example of Bayesian inference

(adapted from Jaynes (1976))

Two manufacturers, *A* and *B*, deliver the same kind of components that turn out to have the following lifetimes (in hours):

A:

59.5814
37.3953
47.5956
40.5607
48.6468
36.2789
31.5110
31.3606
45.6517

B:

48.8506
48.7296
59.1971
51.8895

Assuming prices are comparable, from which manufacturer would you buy?

A simple example of Bayesian inference

How do we compare such samples?

- By comparing their arithmetic means

Why do we take means?

- If we take the mean as our estimate, the error in our estimate is the mean of the errors in the individual measurements
- Taking the mean as maximum-likelihood estimate implies a **Gaussian error distribution**
- A Gaussian error distribution appropriately reflects our **prior** knowledge about the errors whenever we know nothing about them except perhaps their variance

A simple example of Bayesian inference

What next?

- Let's do a *t*-test (but first, let's compare variances with an *F*-test):

```
>> [fh,fp,fci,fstats] = vartest2(xa,xb)

fh =          fp =          fci =          fstats =

     0             0.3297        0.2415          fstat: 3.5114
                                19.0173            df1: 8
                                                   df2: 3
```

Variances not significantly different!

```
>> [h, p, ci, stats]= ttest2(xa,xb)

h =           p =           ci =           stats =

     0             0.0665      -21.0191          tstat: -2.0367
                                 0.8151             df: 11
                                                    sd: 8.2541
```

Means not significantly different!

Is this satisfactory? No, so what can we learn by turning to probability theory (i.e., Bayesian inference)?

A simple example of Bayesian inference

The procedure in brief:

- Determine your question of interest («What is the probability that...?»)
- Specify your statistical model (likelihood and prior)
- Calculate the posterior using Bayes' theorem
- Ask your question of interest of the posterior

All you need is the rules of probability theory.

(Sometimes you'll encounter a nasty integral. But that's only a technical difficulty, not a conceptual one, and software packages like SPM will solve it for you – normally).

A simple example of Bayesian inference

The question:

- What is the probability that the components from manufacturer *B* have a longer lifetime than those from manufacturer *A*?
- More specifically: given how much more expensive they are, how much longer do I require the components from *B* to live.
- Example of a *decision rule:* **if the components from *B* live 3 hours longer than those from *A* with a probability of at least 80%, I will choose those from *B*.**

A simple example of Bayesian inference

The model:

Likelihood (Gaussian):

$$p(\{y_i\}|\mu,\lambda) = \prod_{i=1}^{n} \left(\frac{\lambda}{2\pi}\right)^{\frac{1}{2}} \exp\left(-\frac{\lambda}{2}(y_i-\mu)^2\right)$$

This is the probability of making observations $\{y_i\}_{i=1,\dots,n}$ if the **mean** of the sampling distribution is μ and its **precision** is λ.

Prior (Gaussian-gamma):

$$p(\mu,\lambda|\mu_0,\kappa_0 a_0,b_0) = \mathcal{N}(\mu|\mu_0,(\kappa_0\lambda)^{-1})\mathrm{Gam}(\lambda|a_0,b_0)$$

This is our assumption about the realistic range in which we expect to find μ and λ, determined by the **hyperparameters** μ_0, κ_0, a_0, and b_0.

A simple example of Bayesian inference

- Applying Bayes' rule gives us the **posterior hyperparameters** μ_n, κ_n, a_n and b_n

- If we choose prior hyperparameters $\kappa_0 = 0$, $a_0 = 0$, $b_0 = 0$, the posterior hyperparameters are:

$$\mu_n = \bar{y} \qquad \kappa_n = n \qquad a_n = \frac{n}{2} \qquad b_n = \frac{n}{2} s^2$$

- This means that all we need is n, the number of data points; $\bar{y}$, their mean; and s^2, their variance.

- If we choose different prior hyperparameters, the equations for the posterior hyperparameters look a bit more complicated, but in any case they can easily be calculated for our example model.

- In many applications of Bayesian inference, the posterior cannot be calculated analytically and written in terms of a function determined by hyperparameters. In these cases, **approximate Bayesian inference** has to be used, using for example *Monte Carlo sampling* or *variational calculus*.

A simple example of Bayesian inference

The joint posterior distributions of lifetimes μ_A of products from manufacturer A and μ_B are $p(\mu_A|\{y_i\}_A)$ and $p(\mu_B|\{y_k\}_B)$, respectively.

We can now use them to answer our question: what is the probability that parts from B live at least 3 hours longer than parts from A?

$$p(\mu_B - \mu_A > 3) = \int_{-\infty}^{\infty} p(\mu_A|\{y_i\}_A) \int_{\mu_A+3}^{\infty} p(\mu_B|\{y_k\}_B)\mathrm{d}\mu_B \, \mathrm{d}\mu_A = 0.9501$$

Note that **the classical procedure with the *t*-test told us that there was «no significant difference»** even though according to our Bayesian calculation **there is a >95% probability that the parts from *B* will last at least 3 hours longer than those from *A***.

Bayesian inference

The procedure in brief:

- Determine your question of interest («What is the probability that...?»)
- Specify your statistical model (likelihood and prior)
- Ask your question of interest of the posterior

All you need is the rules of probability theory

[– and a causal model if you want to do science and not just statistics.]

Flipped classroom ☺

The procedure in brief:

- You've done the readings (hopefully…)
- You may also have watched the videos

Now we go through the content again and explain it to each other.

Bayesian data analysis

For each possible explanation of the data,

Count all the ways data can happen.

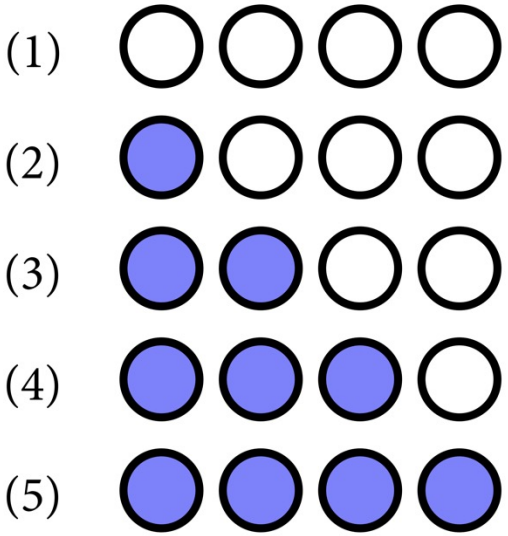
Explanations with more ways to produce the data are more plausible.

Source: Richard McElraeth, https://github.com/rmcelreath/stat_rethinking_2022

Garden of Forking Data

Contains 4 marbles

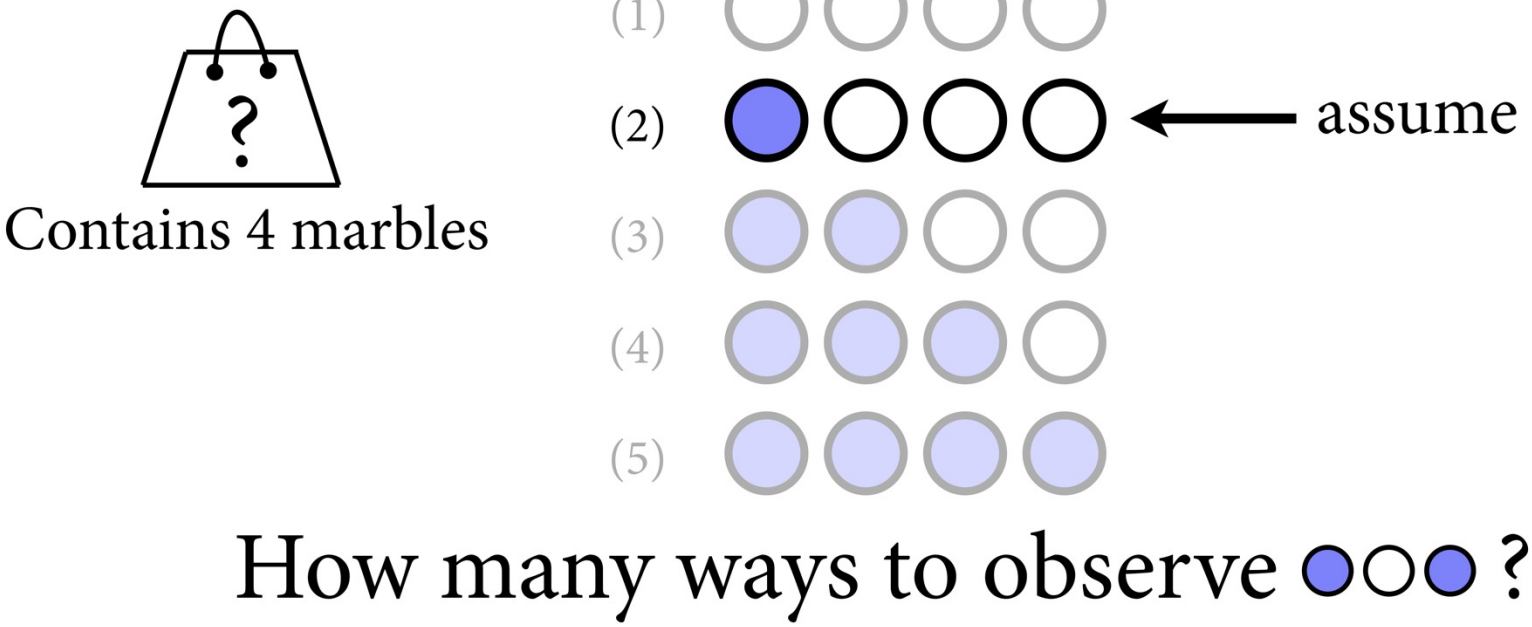
Possible contents:



Observe:

Source: Richard McElraeth, https://github.com/rmcelreath/stat_rethinking_2022

Garden of Forking Data



Source: Richard McElreath, https://github.com/rmcelreath/stat_rethinking_2022

3 Ways to see

if the bag contains

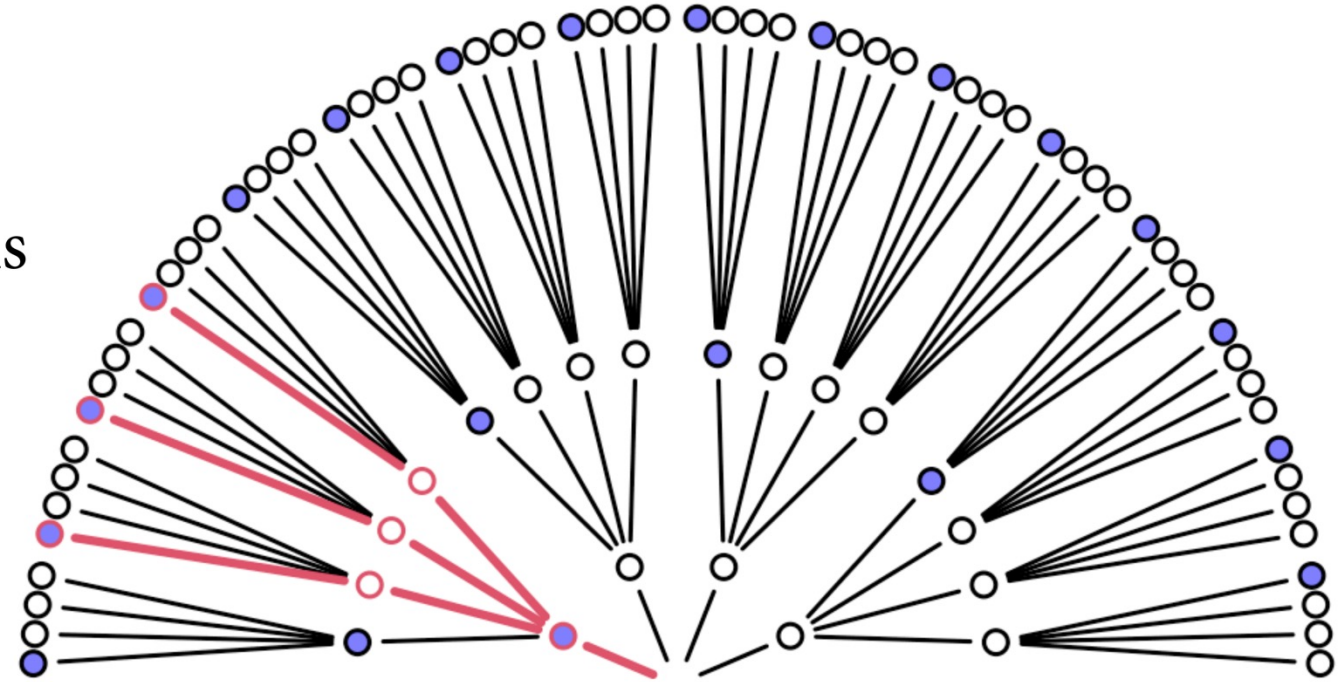
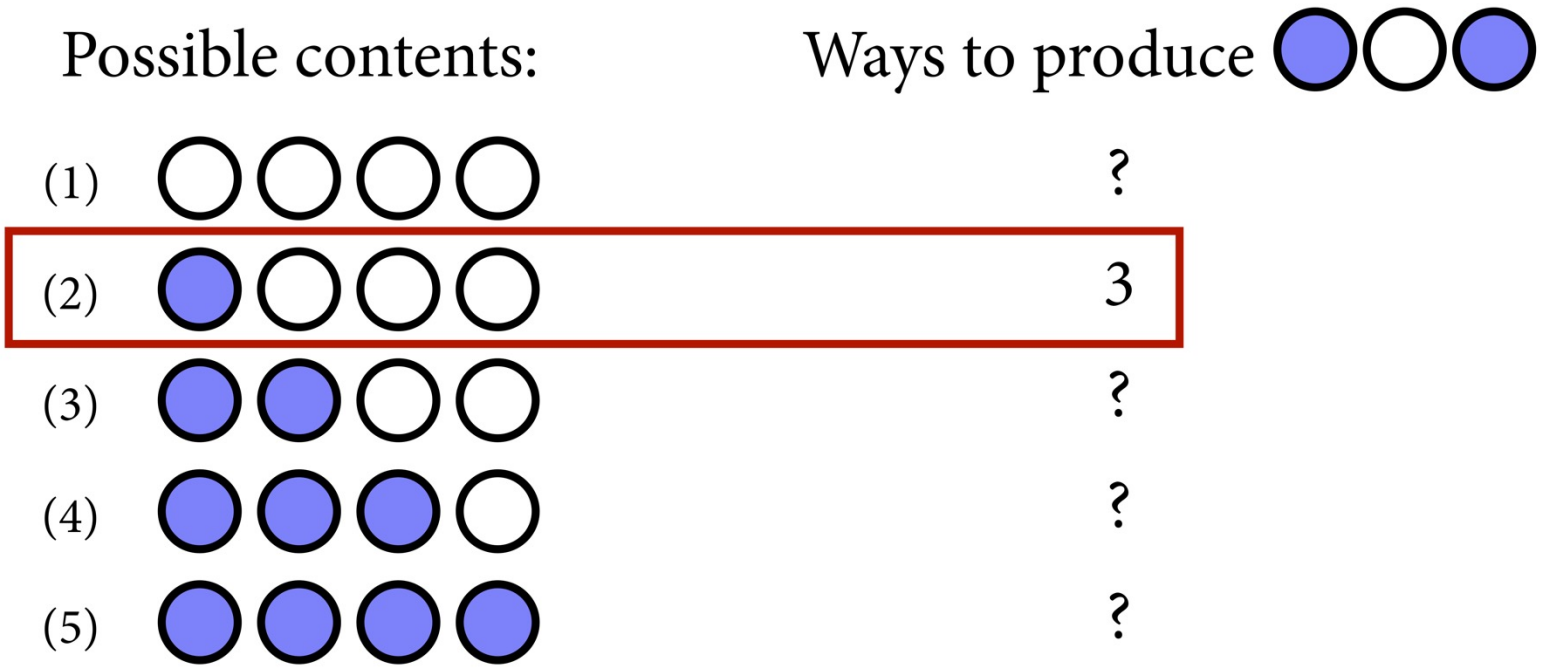


Figure 2.2

Source: Richard McElraeth, https://github.com/rmcelreath/stat_rethinking_2022

Garden of Forking Data



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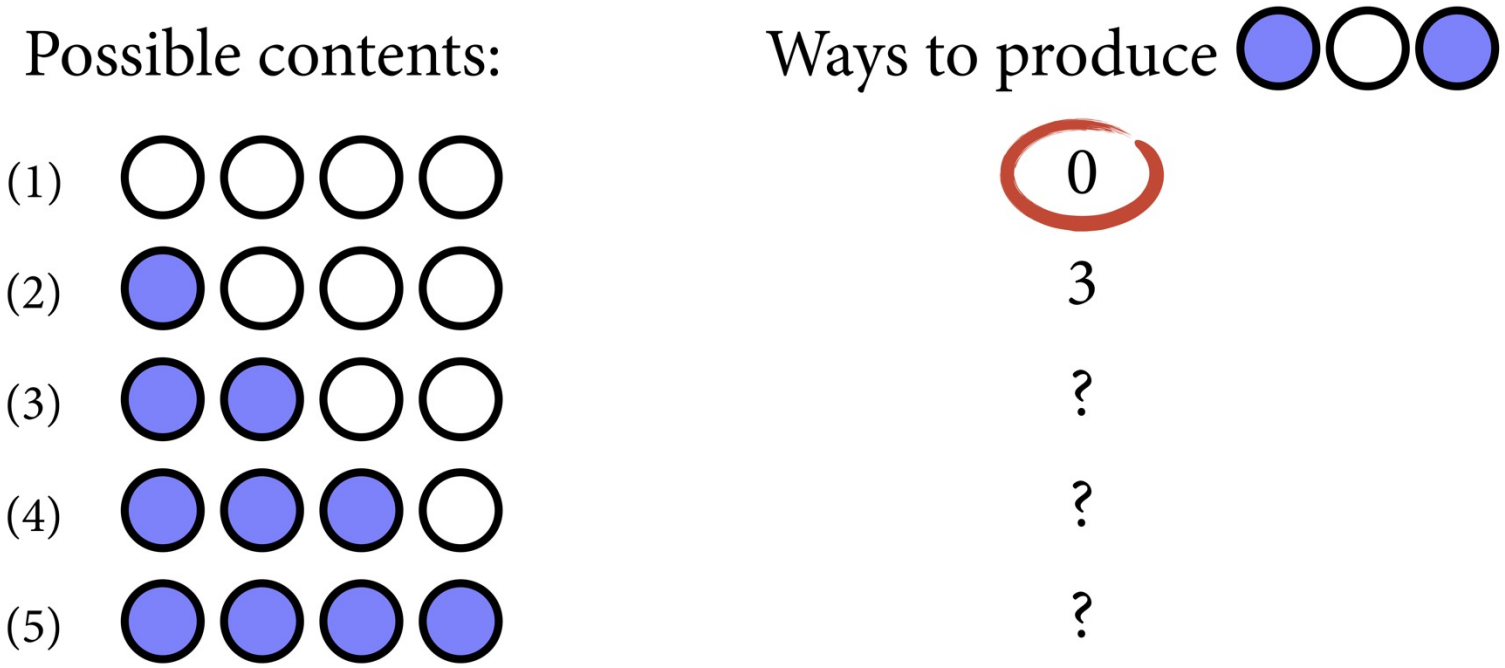

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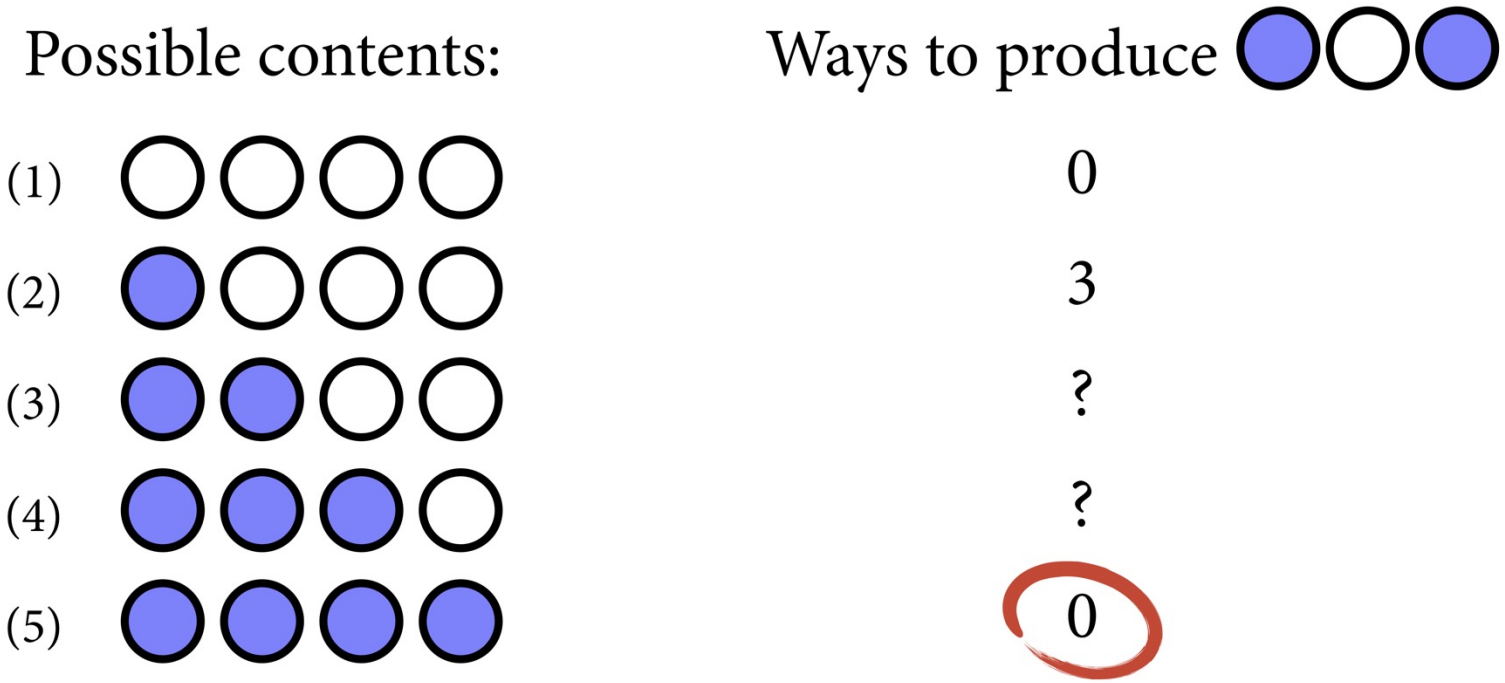
Garden of Forking Data



Source: Richard McElraeth, https://github.com/rmcelreath/stat_rethinking_2022

Garden of Forking Data

Possible contents:	Ways to produce ●○●
(1) ○○○○	0
(2) ●○○○	3
(3) ●●○○	8
(4) ●●●○	9
(5) ●●●●	0

Source: Richard McElraeth, https://github.com/rmcelreath/stat_rethinking_2022

Counts to plausibility

Unglamorous basis of applied probability:
Things that can happen more ways are more plausible.

Possible composition
[○○○○]
[●○○○]
[●●○○]
[●●●○]
[●●●●]

Source: Richard McElraeth, https://github.com/rmcelreath/stat_rethinking_2022

Counts to plausibility

Unglamorous basis of applied probability:
Things that can happen more ways are more plausible.

Possible composition	p	ways to produce data	plausibility
[○○○○]	0	0	0
[●○○○]	0.25	3	0.15
[●●○○]	0.5	8	0.40
[●●●○]	0.75	9	0.45
[●●●●]	1	0	0

Source: Richard McElraeth, https://github.com/rmcelreath/stat_rethinking_2022

Counts to plausibility

Possible composition	p	ways to produce data	plausibility
[○○○○]	0	0	0
[●○○○]	0.25	3	0.15
[●●○○]	0.5	8	0.40
[●●●○]	0.75	9	0.45
[●●●●]	1	0	0

R code 2.1

```
ways <- c( 3 , 8 , 9 )
ways/sum(ways)
```

```
[1] 0.15 0.40 0.45
```

Source: Richard McElraeth, https://github.com/rmcelreath/stat_rethinking_2022

Updating

Another draw from the bag: ●

Conjecture
[○○○○]
[●○○○]
[●●○○]
[●●●○]
[●●●●]

Source: Richard McElraeth, https://github.com/rmcelreath/stat_rethinking_2022

Updating

Another draw from the bag: ●

Conjecture	Ways to produce ●
[○○○○]	0
[●○○○]	1
[●●○○]	2
[●●●○]	3
[●●●●]	4

Source: Richard McElraeth, https://github.com/rmcelreath/stat_rethinking_2022

Updating

Another draw from the bag: ●

Conjecture	Ways to produce ●	Previous counts
[○○○○]	0	0
[●○○○]	1	3
[●●○○]	2	8
[●●●○]	3	9
[●●●●]	4	0

Source: Richard McElraeth, https://github.com/rmcelreath/stat_rethinking_2022

Updating

Another draw from the bag: ●

Conjecture	Ways to produce ●	Previous counts	New count
[○○○○]	0	0	$0 \times 0 = 0$
[●○○○]	1	3	$3 \times 1 = 3$
[●●○○]	2	8	$8 \times 2 = 16$
[●●●○]	3	9	$9 \times 3 = 27$
[●●●●]	4	0	$0 \times 4 = 0$

Source: Richard McElraeth, https://github.com/rmcelreath/stat_rethinking_2022

Bayesian updating

The rules:

1. State a causal model for how the observations arise, given each possible explanation

2. Count ways data could arise for each explanation

3. Relative plausibility is relative value from (2)

Source: Richard McElraeth, https://github.com/rmcelreath/stat_rethinking_2022

Globe of Forking Water

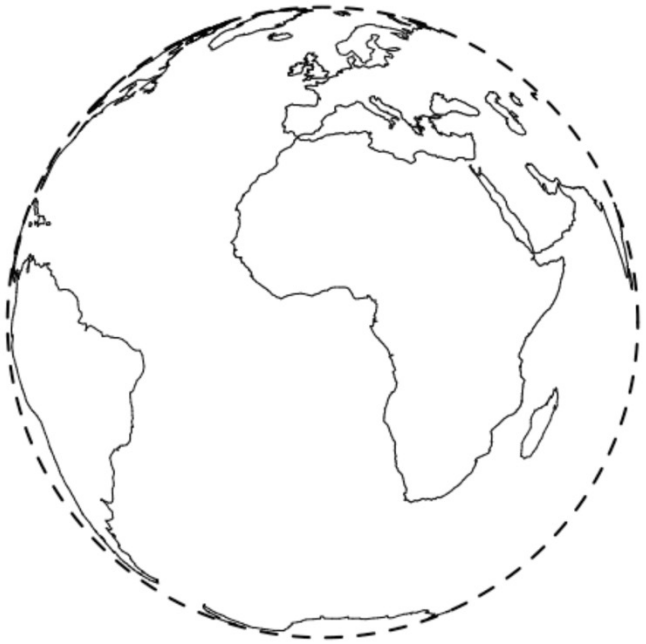
For each possible proportion of water,

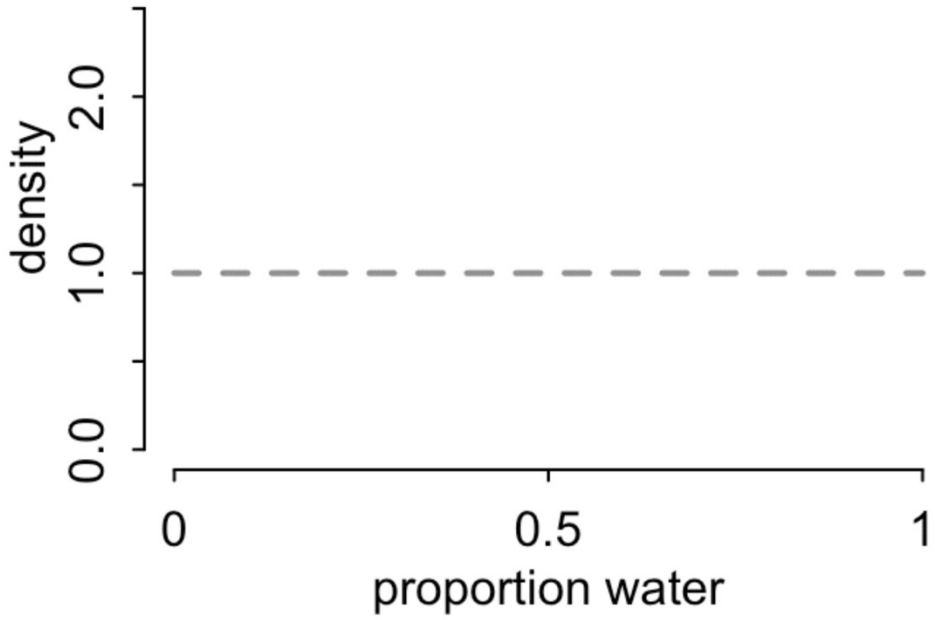
Count number of ways data could happen.

Must state how observations are generated

Source: Richard McElraeth, https://github.com/rmcelreath/stat_rethinking_2022

Toss The First





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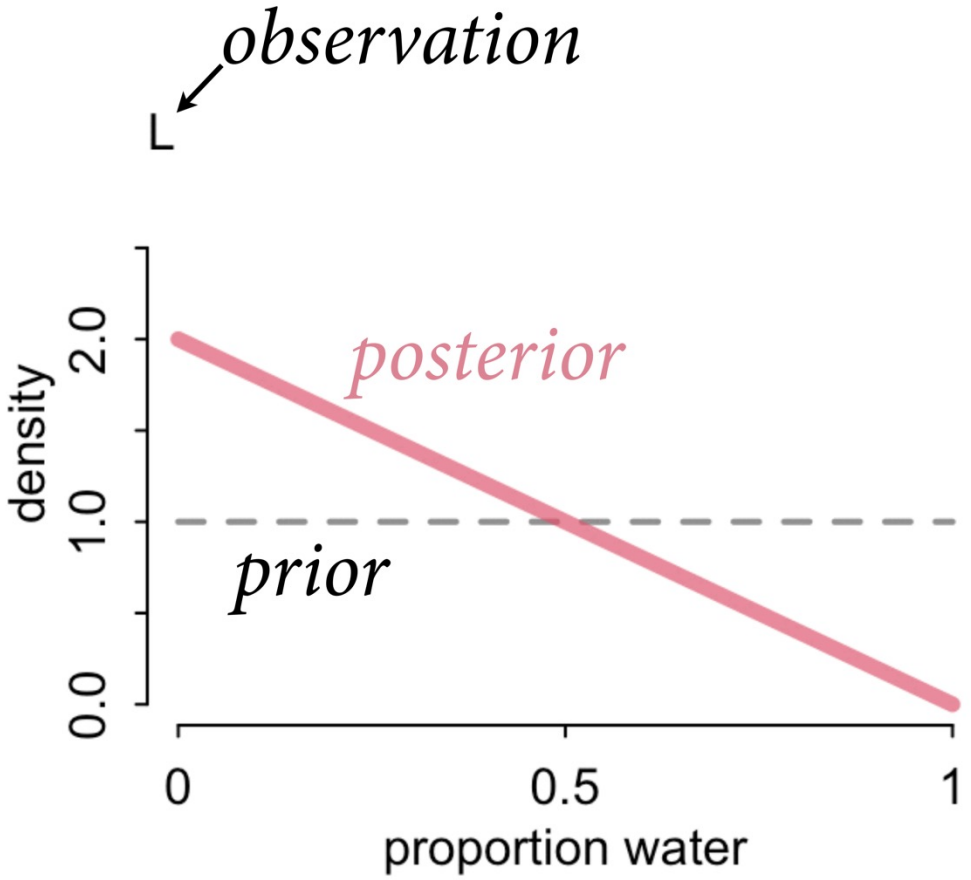

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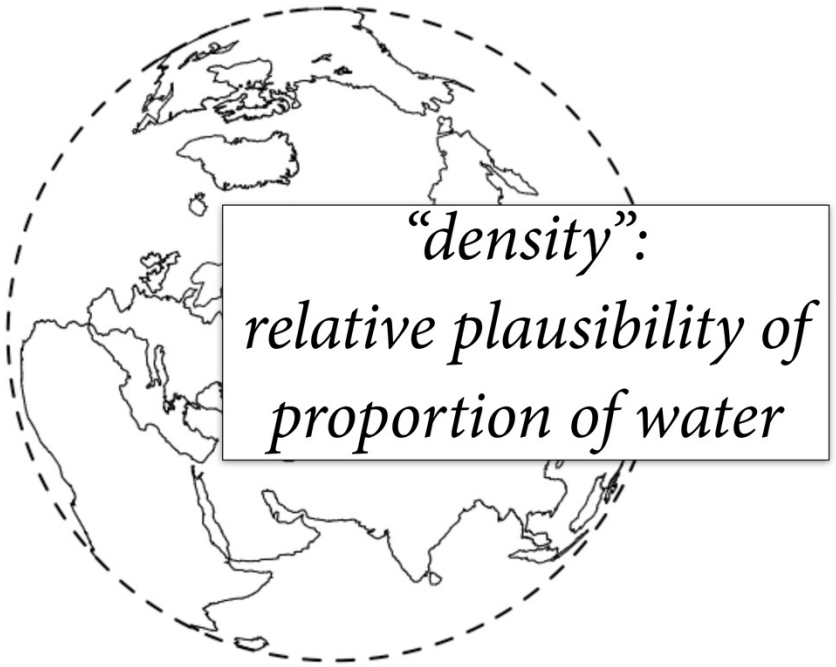

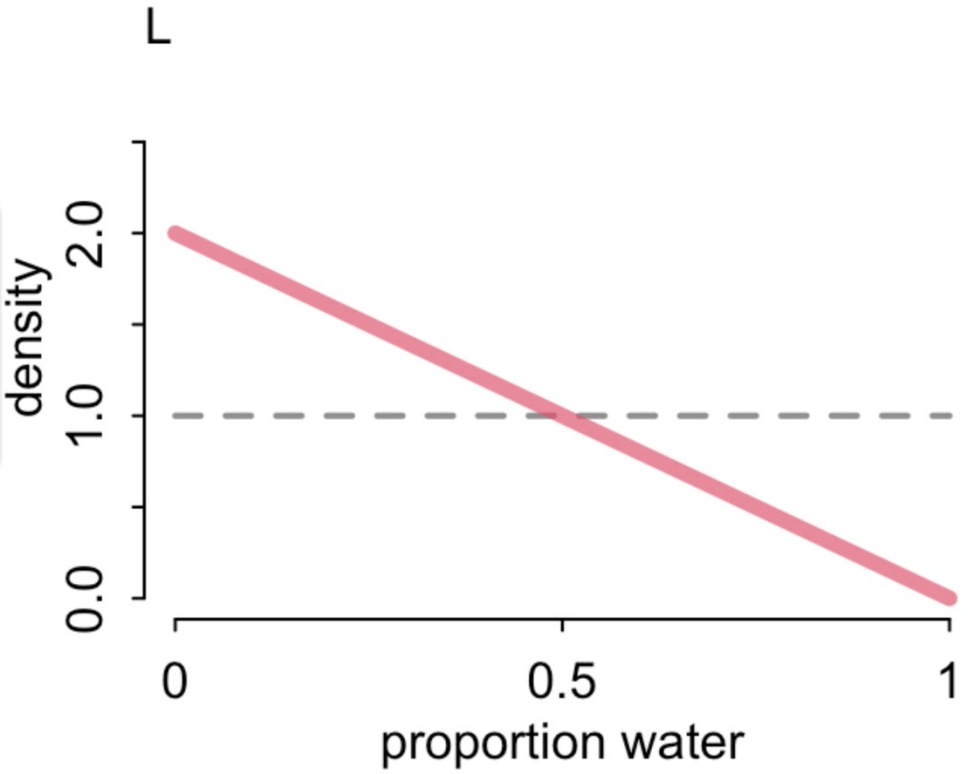

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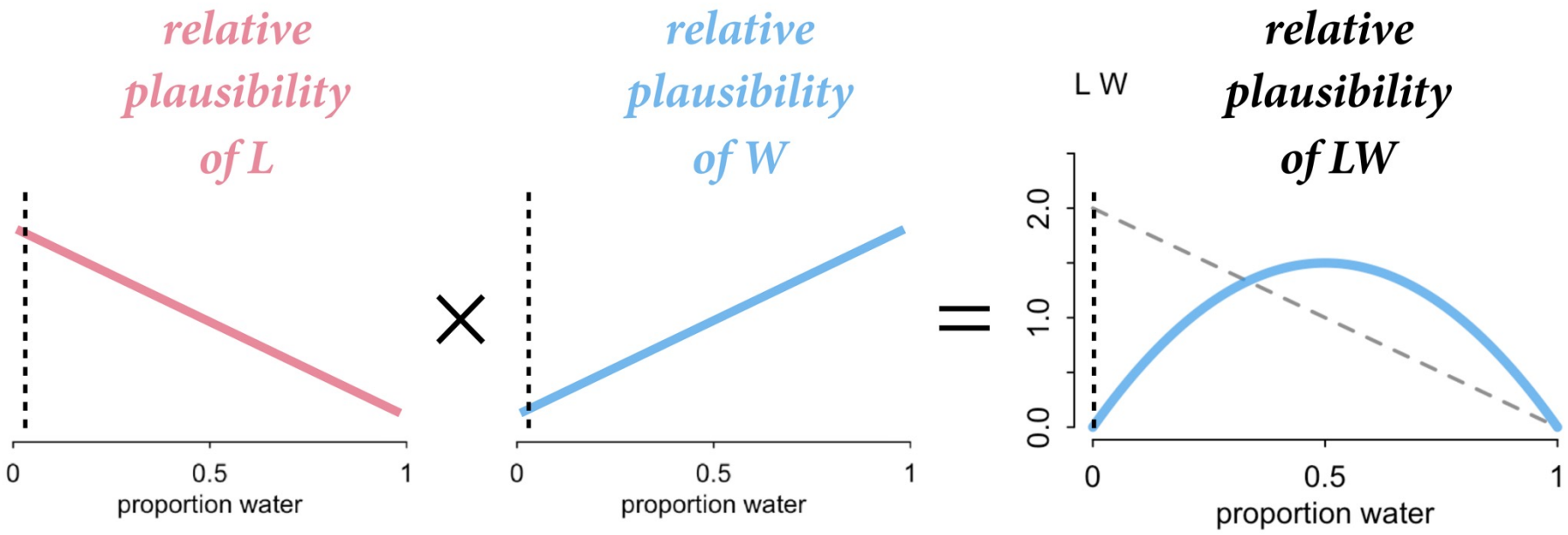
Toss The Second



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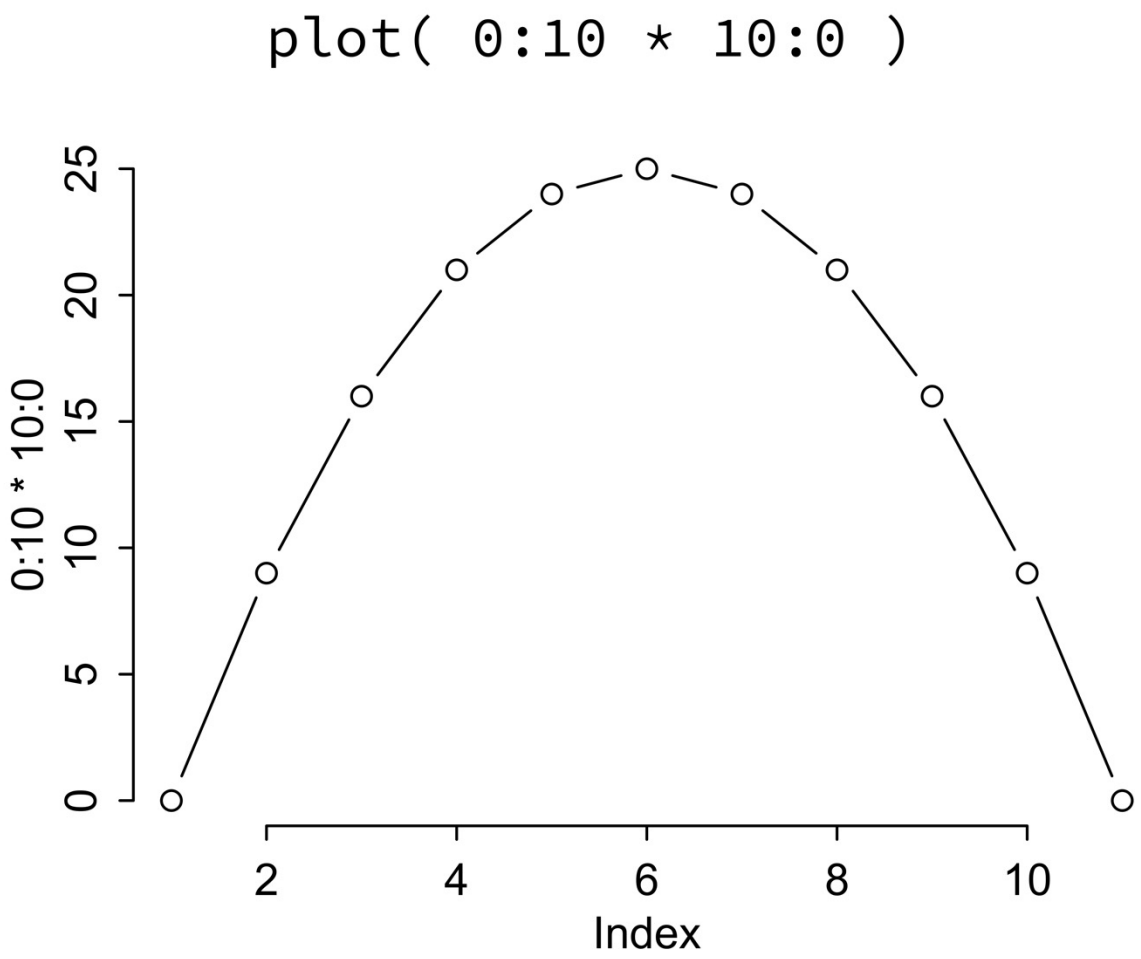
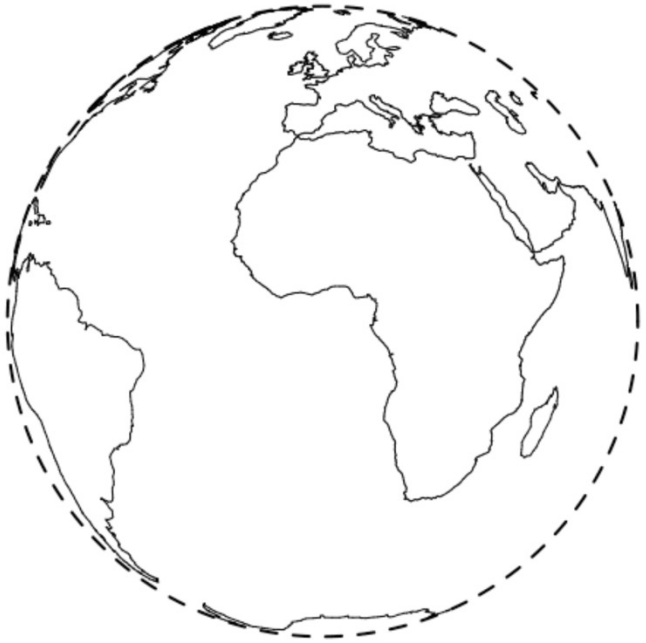
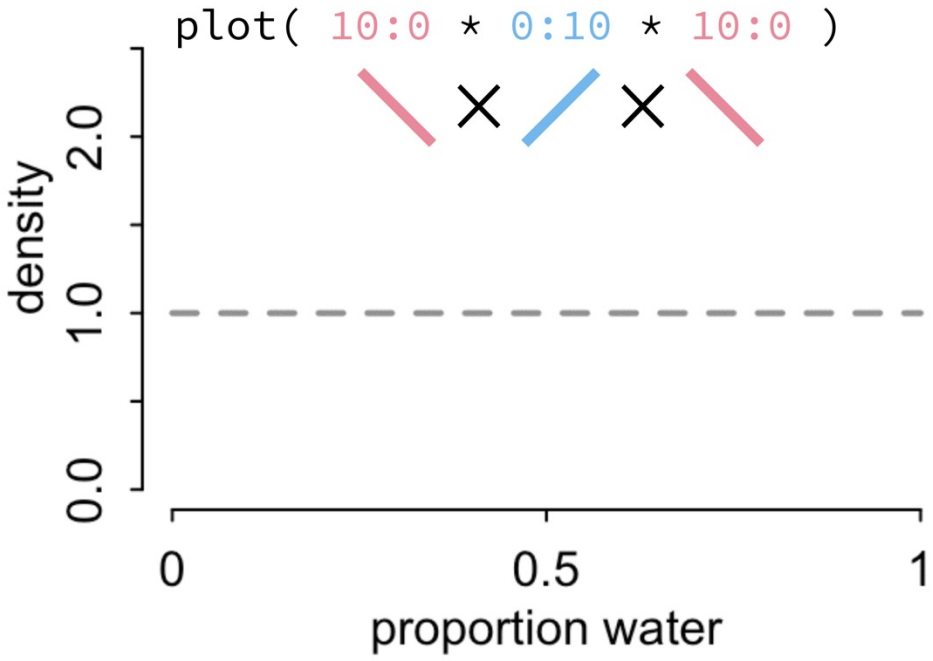


Source: Richard McElraeth, https://github.com/rmcelreath/stat_rethinking_2022

Toss The Third





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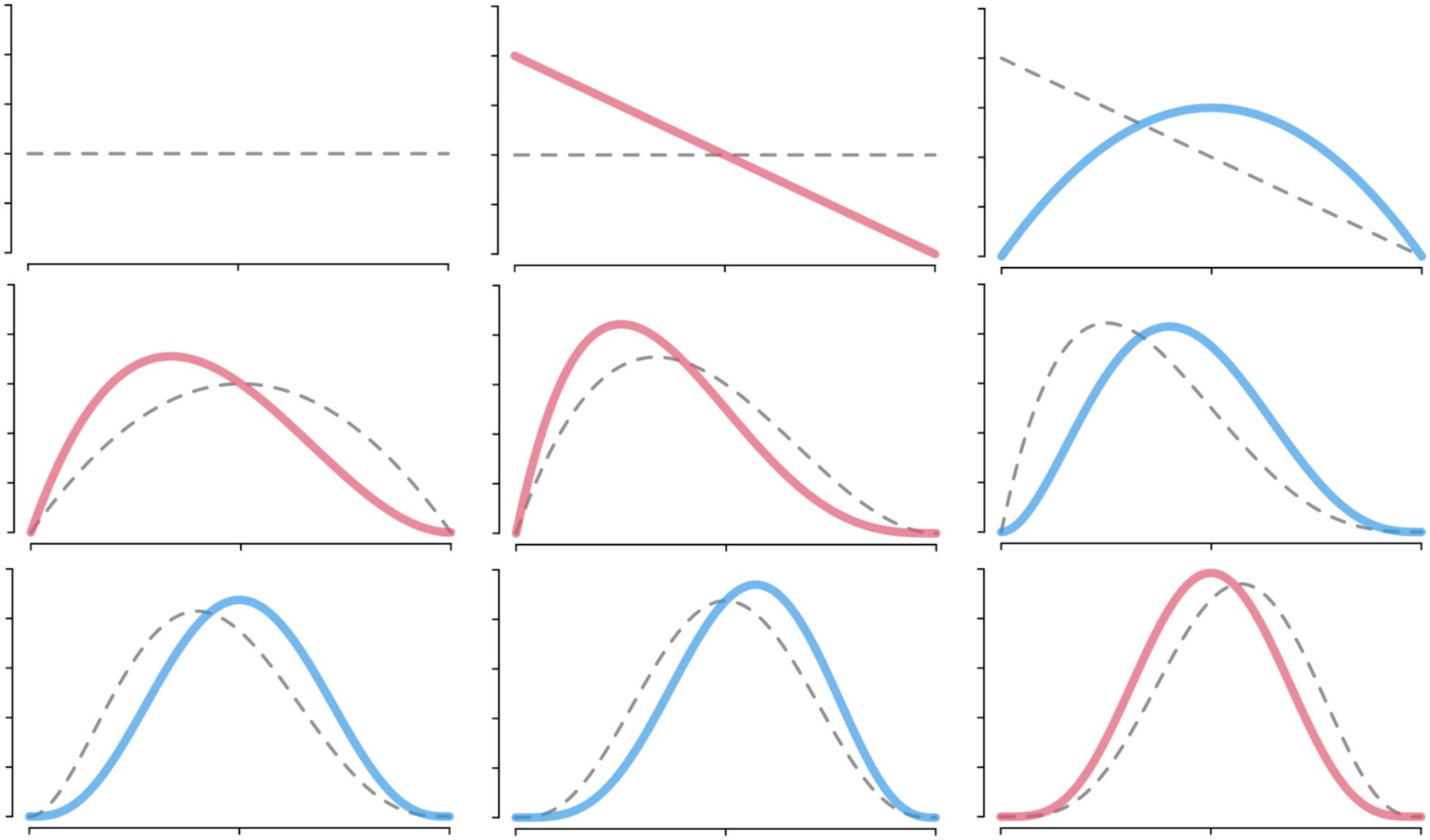
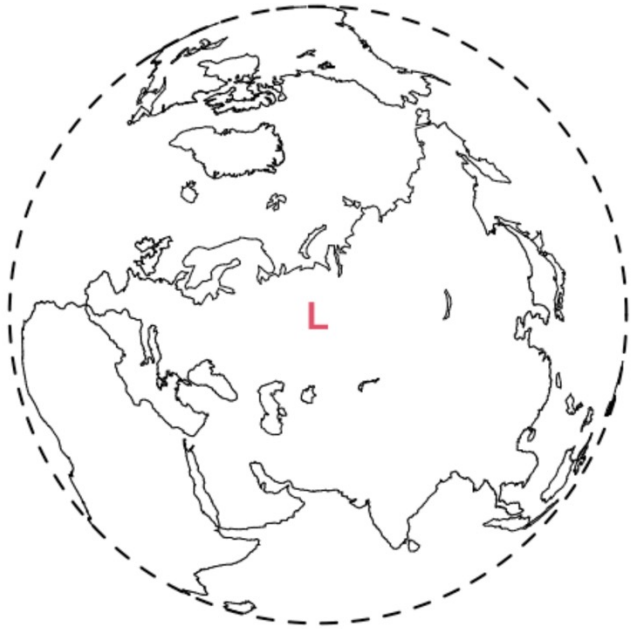
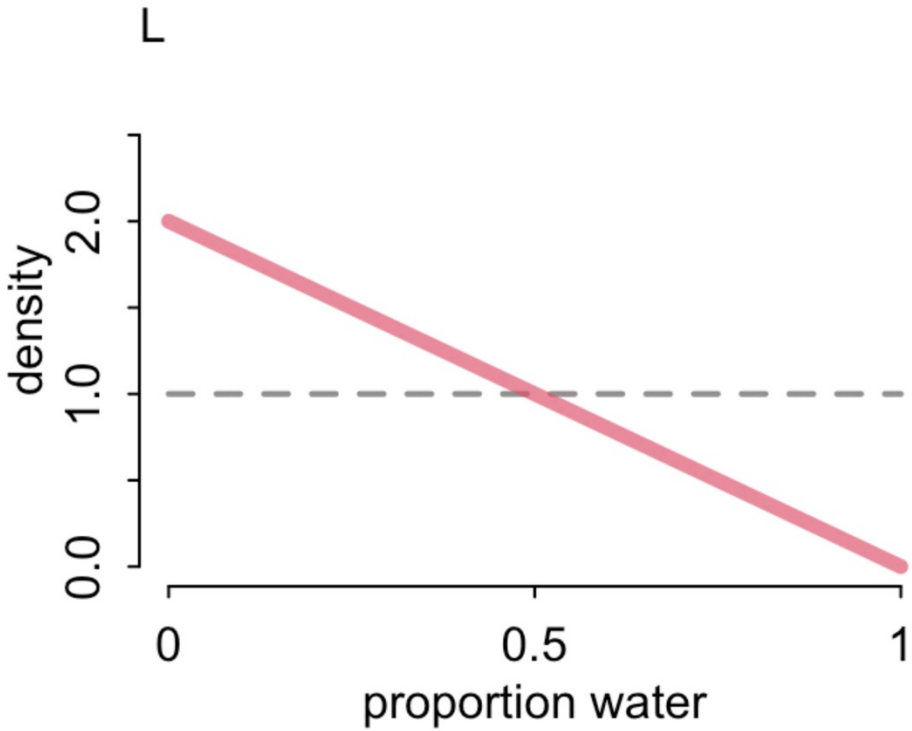


Source: Richard McElraeth, https://github.com/rmcelreath/stat_rethinking_2022

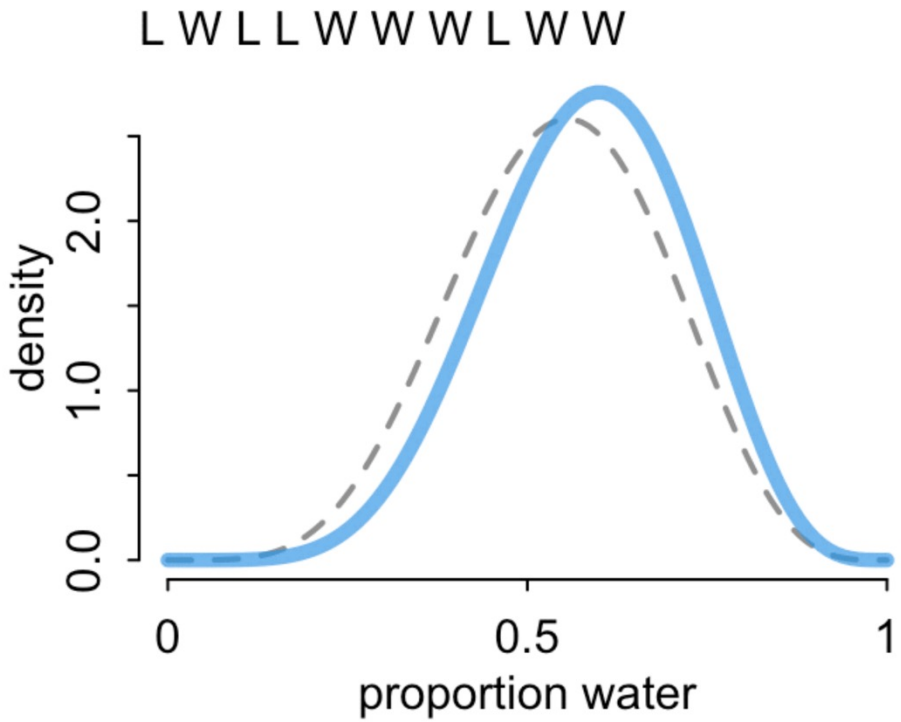
(1) No minimum sample size





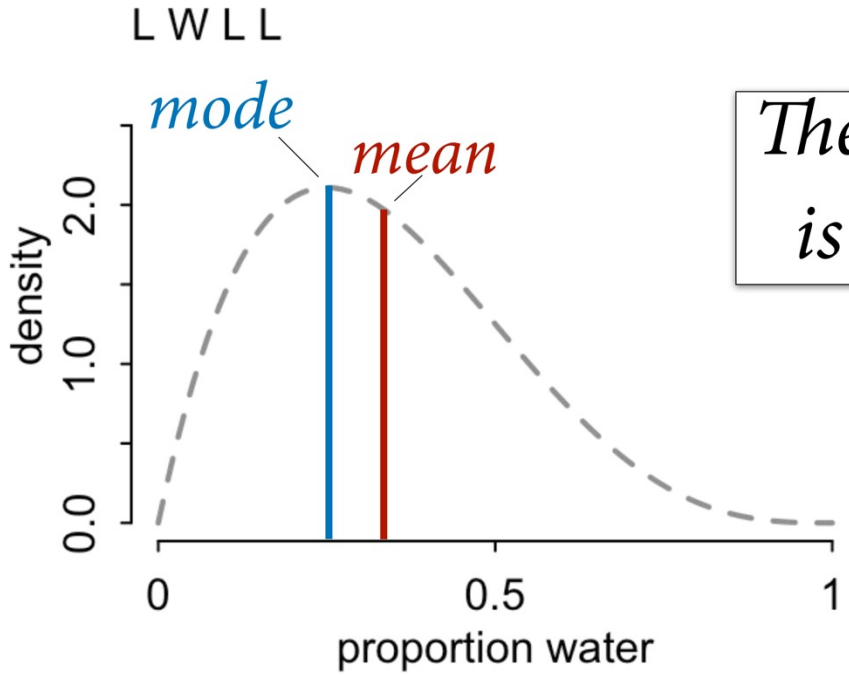
Source: Richard McElraeth, https://github.com/rmcelreath/stat_rethinking_2022

(2) Shape embodies sample size



Source: Richard McElraeth, https://github.com/rmcelreath/stat_rethinking_2022

(3) No point estimate

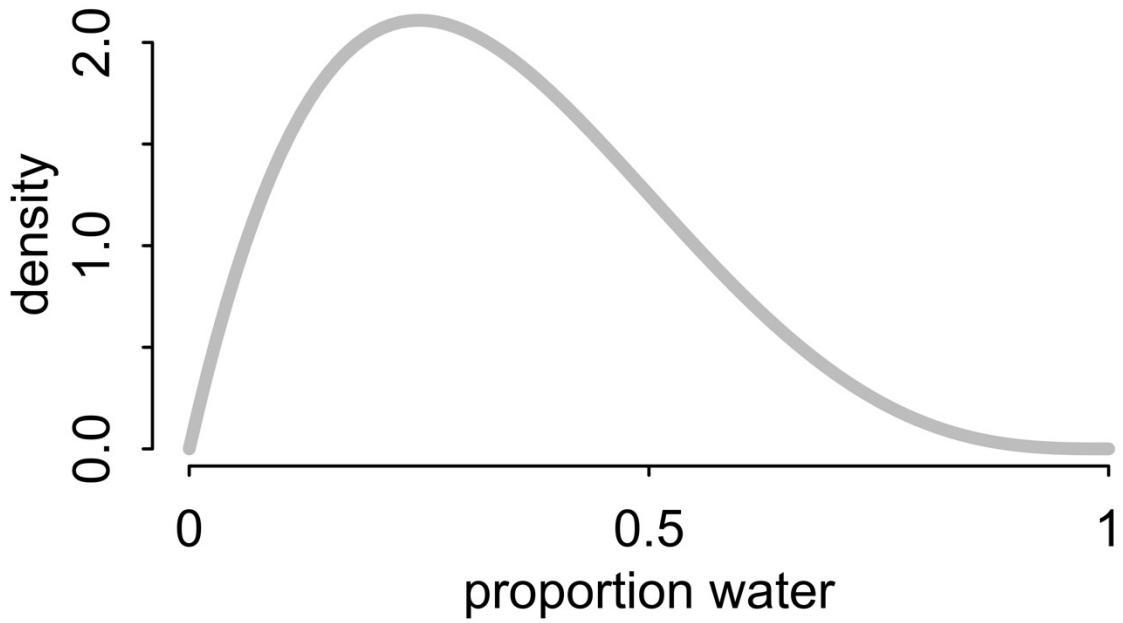


*The **distribution** is the estimate*

Always use the entire distribution

Source: Richard McElraeth, https://github.com/rmcelreath/stat_rethinking_2022

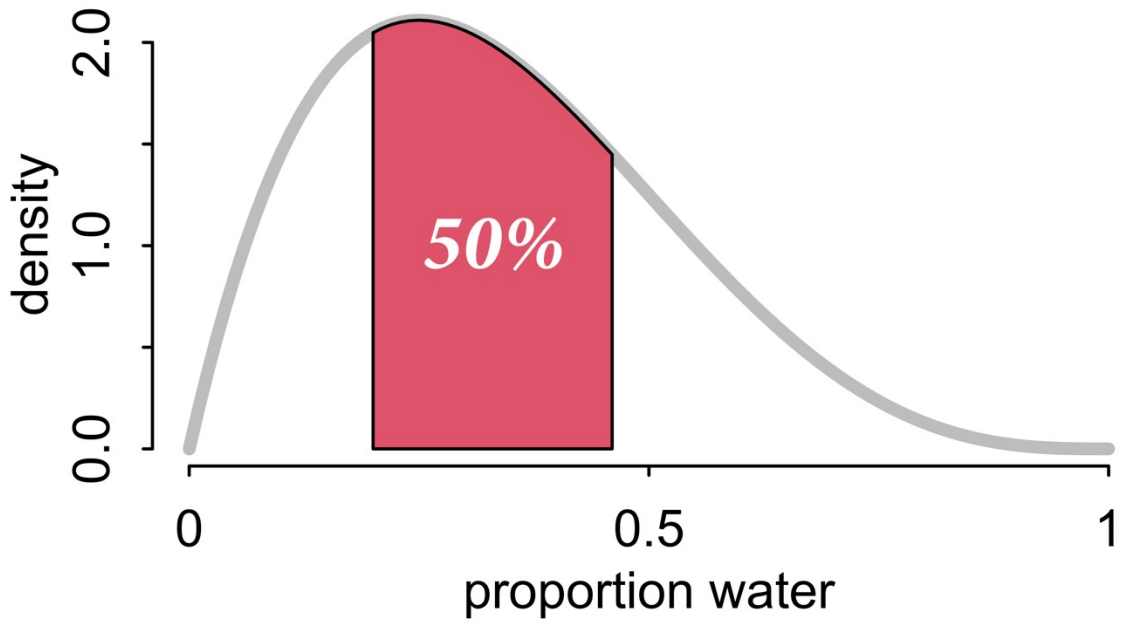
(4) No one true interval



Intervals communicate shape of posterior

Source: Richard McElraeth, https://github.com/rmcelreath/stat_rethinking_2022

(4) No one true interval



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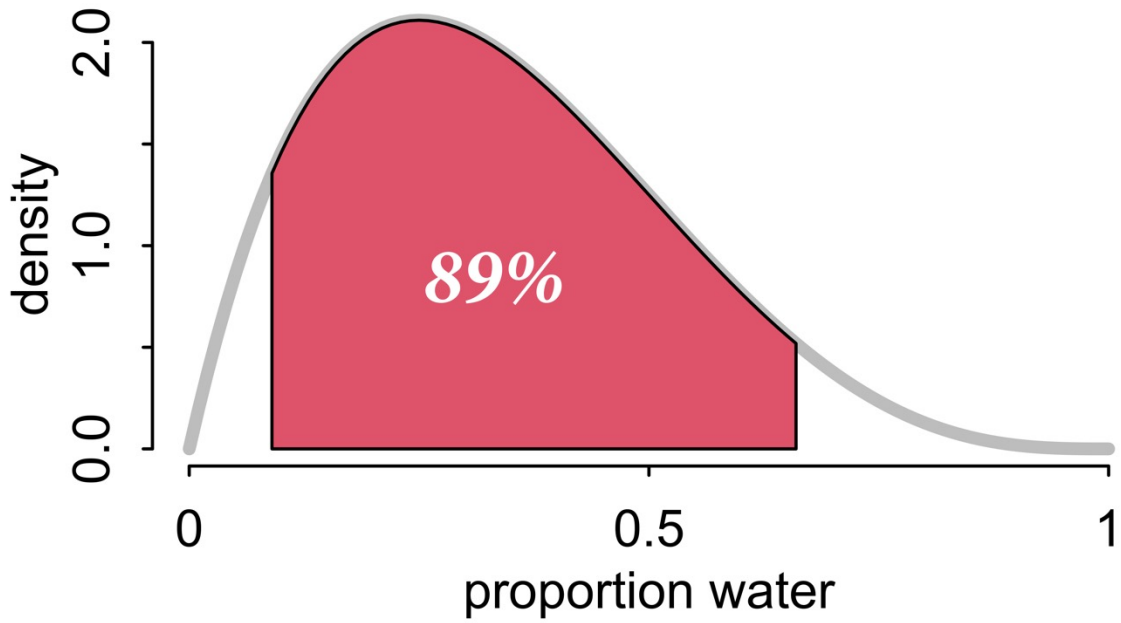

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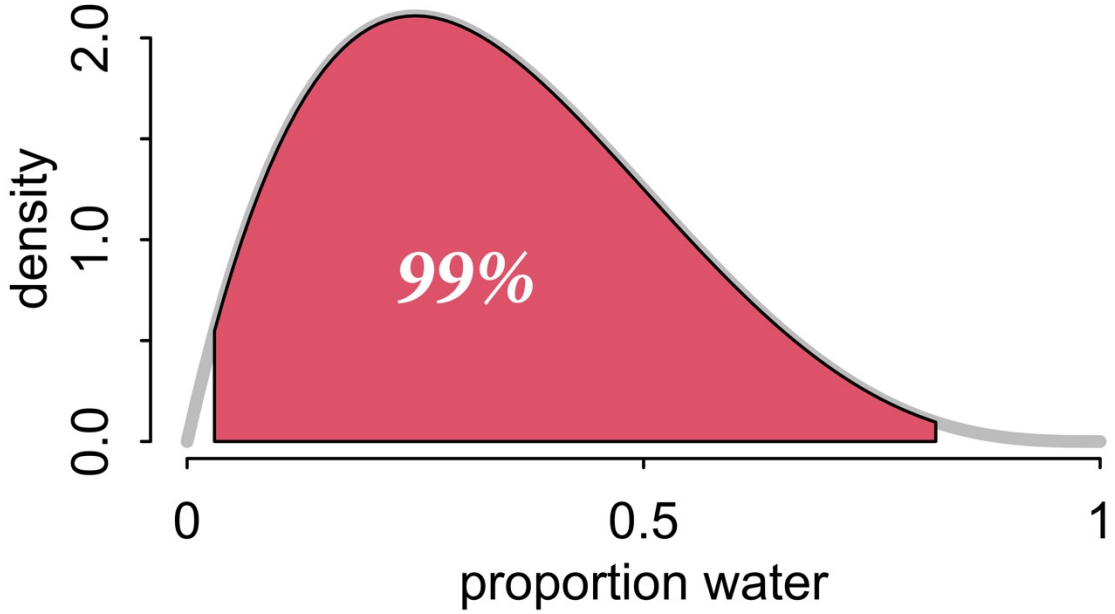


Intervals communicate shape of posterior

95% is obvious superstition. Nothing magical happens at the boundary.

Source: Richard McElraeth, https://github.com/rmcelreath/stat_rethinking_2022

Letters From My Reviewers

"The author uses these cute 89% intervals, but we need to see the 95% intervals so we can tell whether any of the effects are robust."

That an arbitrary interval contains an arbitrary value is not meaningful. Use the whole distribution.

Source: Richard McElraeth, https://github.com/rmcelreath/stat_rethinking_2022

The Formalities

Data: W and L, the number of water and land observations

$$\Pr(W, L|p) = \frac{(W+L)!}{W!L!} p^W (1-p)^L$$

The number of ways to realize W,L given p

Binomial probability function

```
dbinom( W , W+L , p )
```

```
> dbinom( 6 , 9 , 0.7 )
[1] 0.2668279
>
```

Source: Richard McElraeth, https://github.com/rmcelreath/stat_rethinking_2022

The Formalities

Data: W and L, the number of water and land observations

$$\Pr(W, L|p) = \frac{(W+L)!}{W!L!} p^W (1-p)^L$$

The number of ways to realize W,L given p

Parameters: p, the proportion of water on the globe

$$\Pr(p) = \frac{1}{1-0} = 1.$$

Relative plausibility of each possible p

Source: Richard McElreath, https://github.com/rmcelreath/stat_rethinking_2022

The Formalities

$$\Pr(W, L|p) = \frac{(W+L)!}{W!L!} p^W (1-p)^L \qquad \Pr(p) = \frac{1}{1-0} = 1.$$

Posterior is (normalized) product:

$$\Pr(p|W, L) = \frac{\Pr(W, L|p)\Pr(p)}{\Pr(W, L)}$$

Relative plausibility of each possible p, after learning W,L

We multiply because that's how the garden counts!

Source: Richard McElraeth, https://github.com/rmcelreath/stat_rethinking_2022

With Numbers

Ignore the mathematics for the moment and just draw the owl with numbers

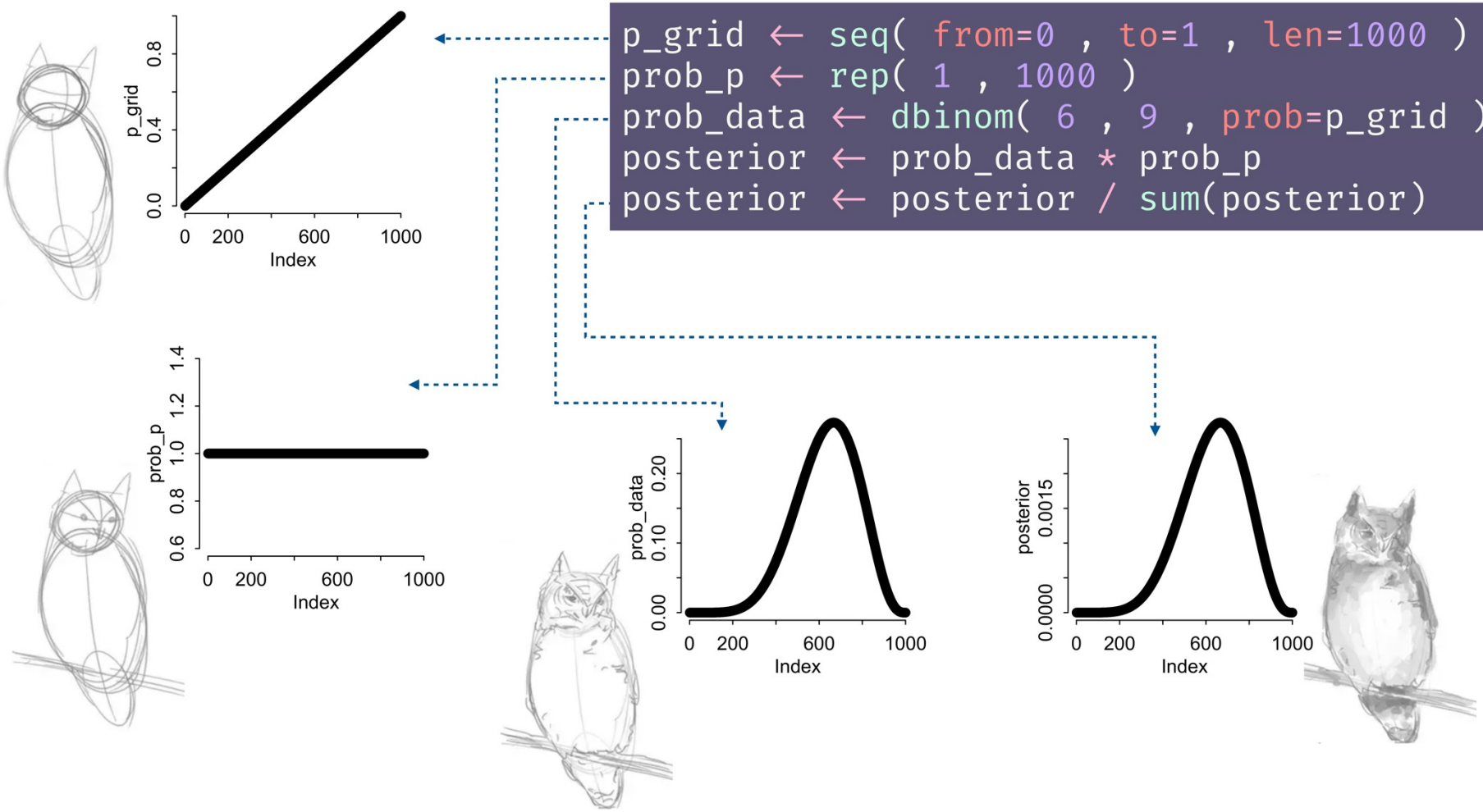
1. For each possible value of p

2. Compute product $\Pr(W,L|p)\Pr(p)$

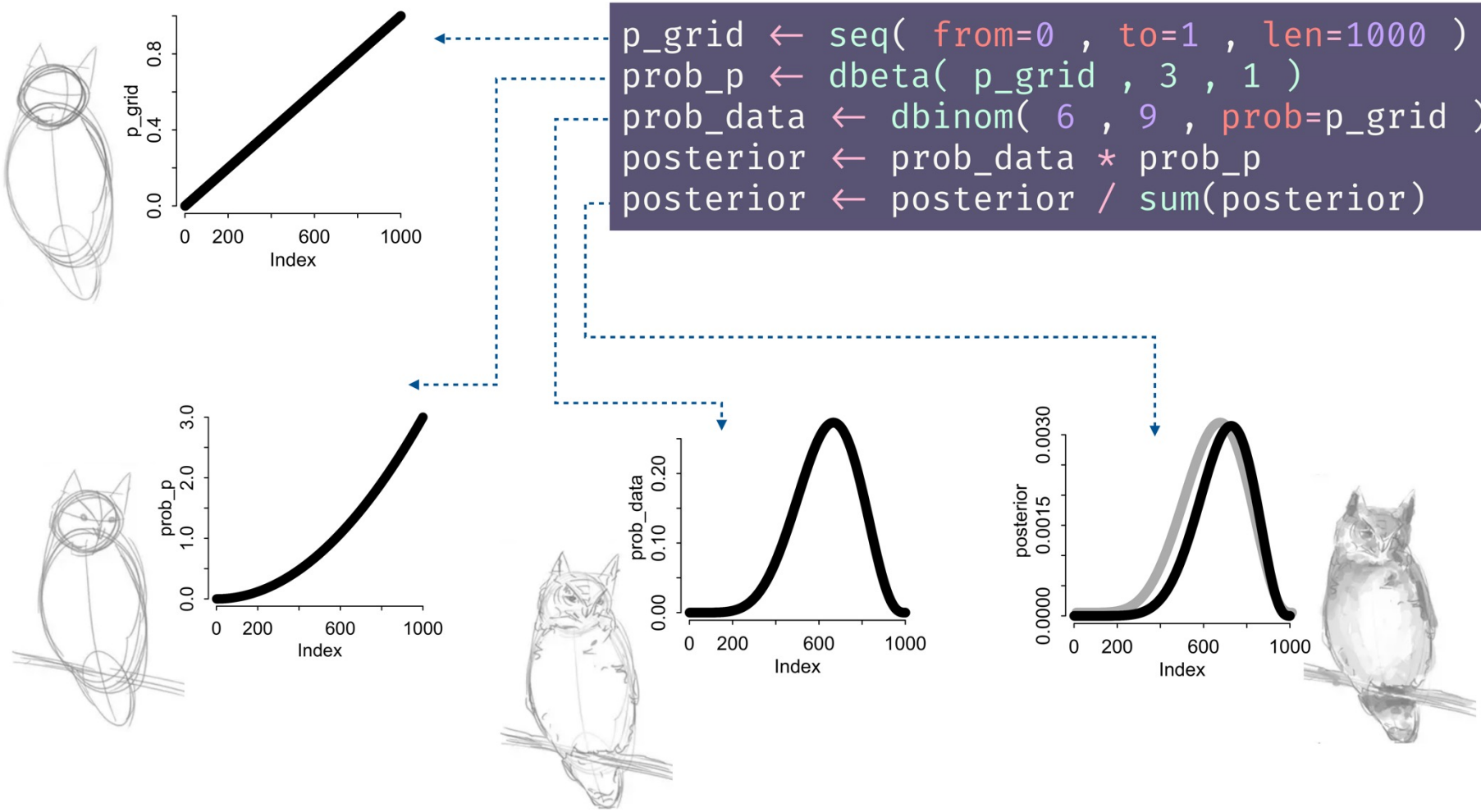
3. Relative sizes of products in (2) are posterior probabilities

Bayesian owl

Source: Richard McElraeth, https://github.com/rmcelreath/stat_rethinking_2022



Source: Richard McElreath, https://github.com/rmcelreath/stat_rethinking_2022



Source: Richard McElreath, https://github.com/rmcelreath/stat_rethinking_2022

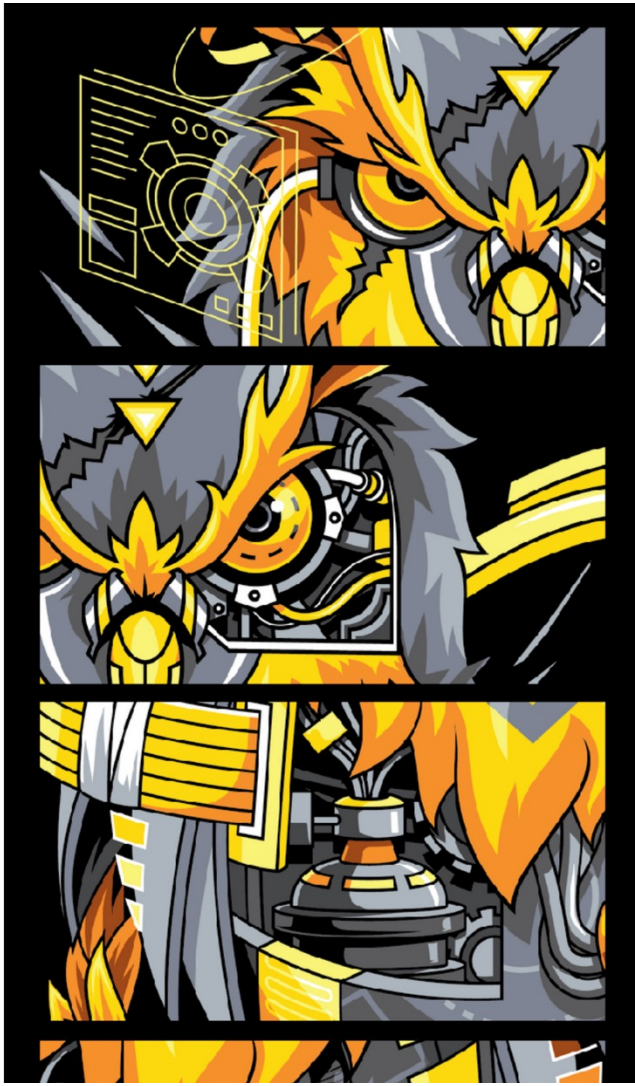
Many Ways to Count

Grid Approximation inefficient

Other methods:

Quadratic approximation

Markov chain Monte Carlo (MCMC)



Source: Richard McElraeth, https://github.com/rmcelreath/stat_rethinking_2022

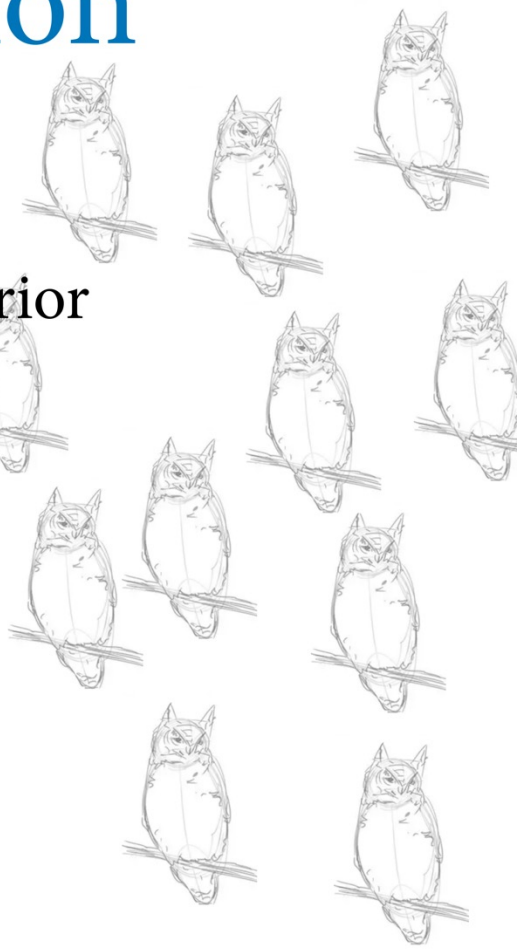
From Posterior to Prediction

Implications of model depend upon **entire** posterior

Must average any inference over entire posterior

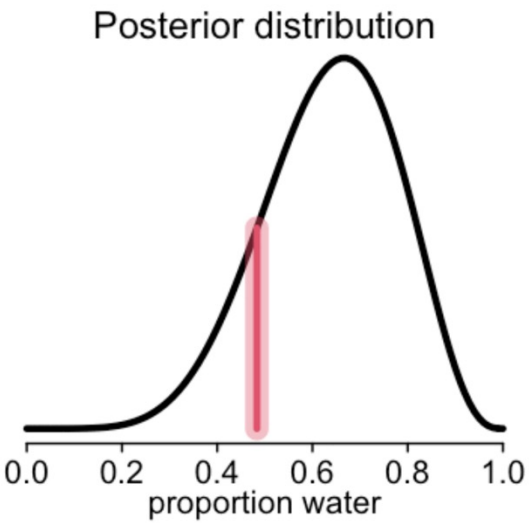
This usually requires integral calculus

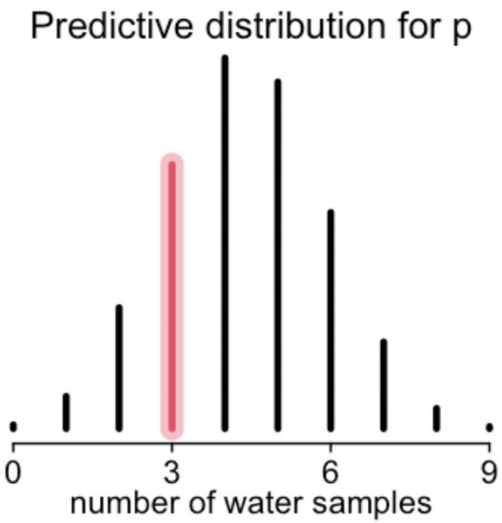
OR we can just take samples from the posterior

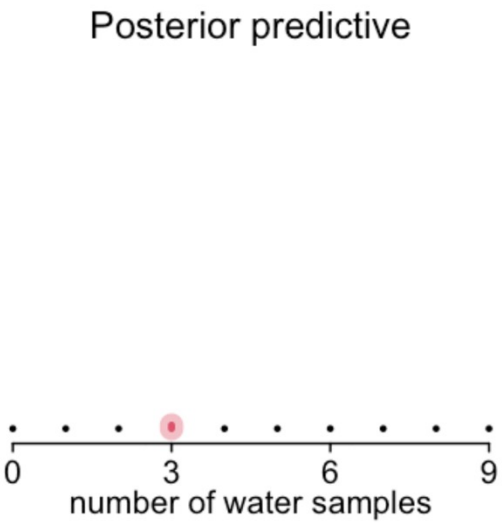


Source: Richard McElraeth, https://github.com/rmcelreath/stat_rethinking_2022

Uncertainty ⇨ Causal model ⇨ Implications







Source: Richard McElraeth, https://github.com/rmcelreath/stat_rethinking_2022

Sample from posterior

R code 3.2

```
p_grid <- seq( from=0 , to=1 , length.out=1000 )
prob_p <- rep( 1 , 1000 )
prob_data <- dbinom( 6 , size=9 , prob=p_grid )
posterior <- prob_data * prob_p
posterior <- posterior / sum(posterior)
```

R code 3.3

```
samples <- sample( p_grid , prob=posterior , size=1e4 , replace=TRUE )
```

Source: Richard McElraeth, https://github.com/rmcelreath/stat_rethinking_2022

Sample from posterior

R code 3.3

```
samples <- sample( p_grid , prob=posterior , size=1e4 , replace=TRUE )
```

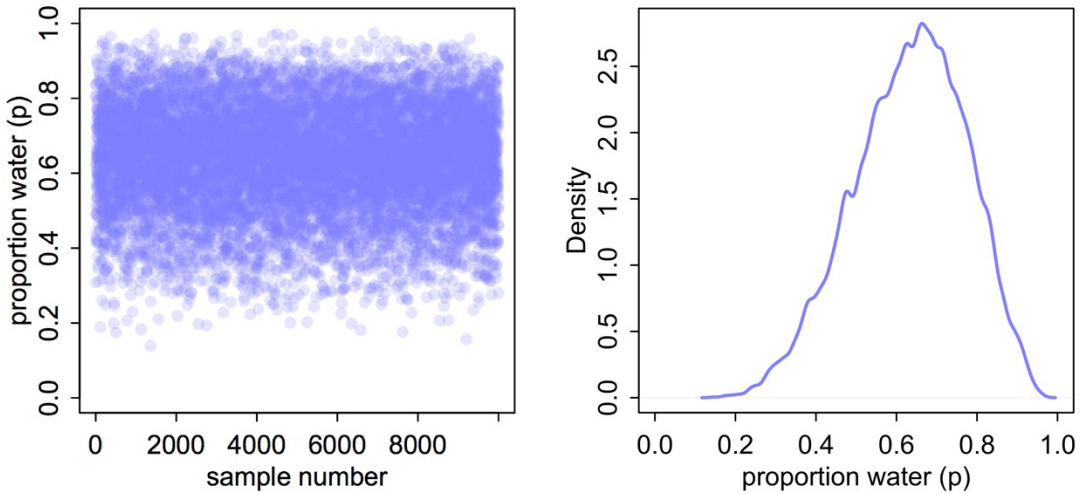


Figure 3.1

Source: Richard McElraeth, https://github.com/rmcelreath/stat_rethinking_2022

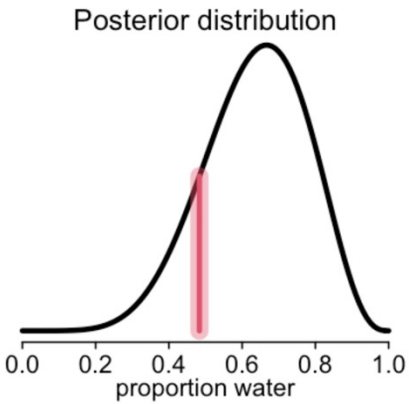
Sample predictions

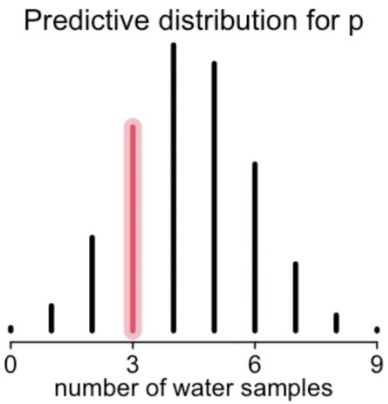
R code 3.3

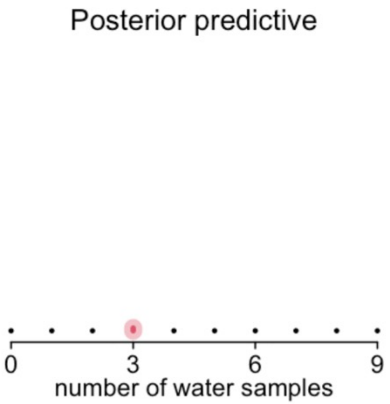
```
samples <- sample( p_grid , prob=posterior , size=1e4 , replace=TRUE )
```

R code 3.26

```
w <- rbinom( 1e4 , size=9 , prob=samples )
```







Source: Richard McElreath, https://github.com/rmcelreath/stat_rethinking_2022

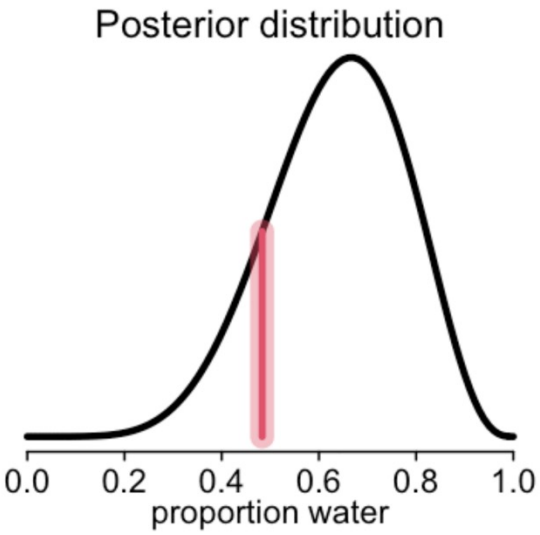
Sampling is Fun & Easy

Sample from posterior, compute desired quantity for each sample, profit

Much easier than doing integrals

Turn a **calculus problem** into a **data summary problem**

MCMC produces only samples anyway



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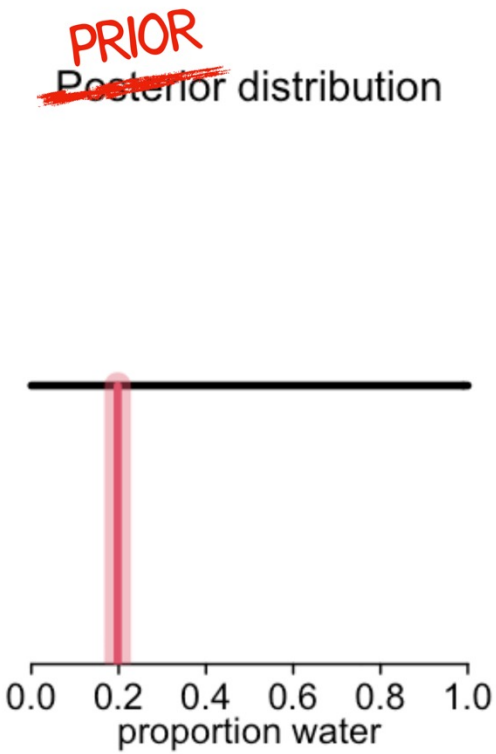
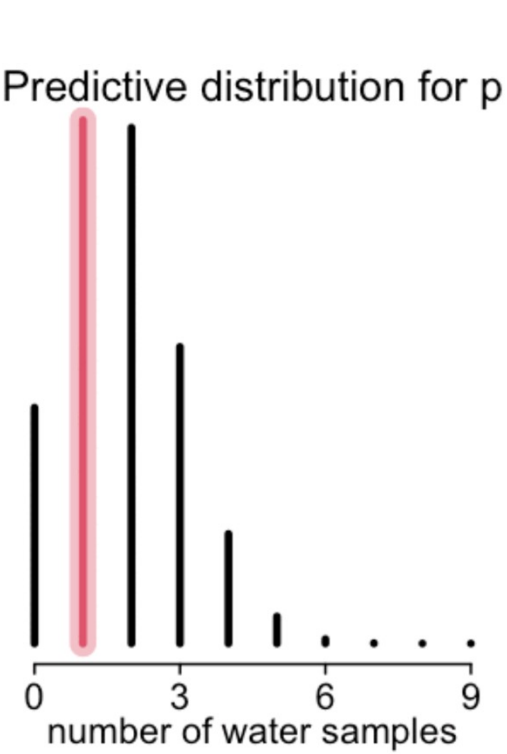




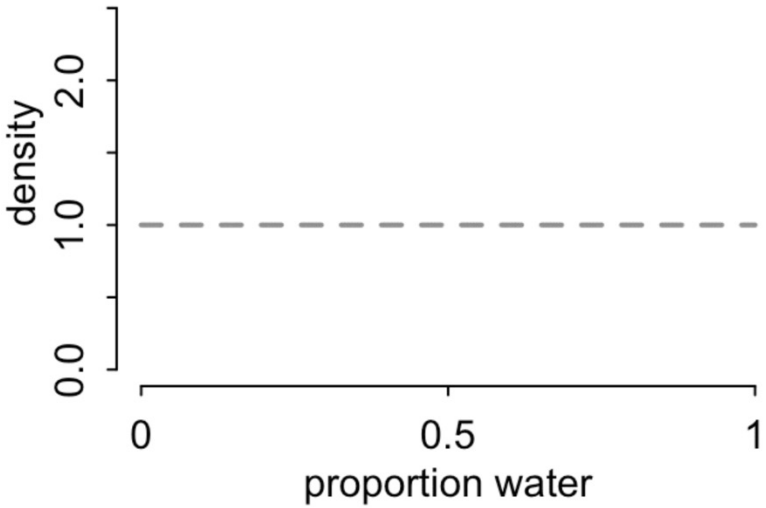
Source: Richard McElraeth, https://github.com/rmcelreath/stat_rethinking_2022

Bayesian modesty

No guarantees except ***logical***

Probability theory is a method of logically deducing ***implications of data*** *under assumptions that you must choose*

Any framework selling you more is hiding assumptions



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