Joint Estimator of Frequency and Phase of Short Signals Based on Maximum Correlation Coefficient

Zhiyu Shen

Nanjing University

Prof. Hongxing Liu

1 Introduction

In the field of signal processing, an important research direction is the frequency and initial phase estimation of sinusoidal signals. In application scenarios such as power systems and biomedical electronics, parameter estimation is often required for extremely low frequency signals. Such estimation requires the most accurate results possible in a short time, meaning that the sampling sequence cannot be too long or even less than the period of the signal under test.

For example, a sinusoidal signal with a frequency of 0.01Hz is to be estimated. Many spectrum analyzers require at least two cycles of sampling when estimating signal frequency, implying a minimum of 200 seconds of sampling when estimating, which is unacceptable in practical applications. Therefore, achieving high precision signal parameter estimation with as few cycles is a topic of significant research value.

Currently, most frequency and phase estimators make the two estimations separately, i.e., one of the parameters is always estimated first, and then the other parameter is estimated on the basis, which inevitably introduces cumulative errors.

Furthermore, frequency estimation in traditional methods is usually done in the frequency domain, and the most classical frequency domain estimation method is the DTFT (Discrete-Time Fourier Transform)-based spectrum peak search. Researchers have made a lot of attempts to improve the frequency estimation accuracy by performing DFT (Discrete Fourier Transform) with big zero-padding, zoom FFT, or DFT combined with other local maximum searching algorithms to overcome the fence effect. Their essential assumption is that the frequency f_{peak} at the DTFT amplitude spectrum peak is equal to the real harmonic frequency f_0 to be solved, which is not a perfect assumption and limits their effects.

In addition, DTFT-based estimation method generally requires at least two cycles of sampling. For low-frequency signals, such as signals with frequency lower than 1 Hz, it takes a long time for the conventional methods to complete an estimation.

In this research, we propose a method for joint estimation of frequency and phase of short-time sinusoidal signal based on maximum correlation coefficient.

2 Theoretical Foundation

2.1 Pearson Correlation Coefficient

Pearson Correlation Coefficient is used in the estimation:

$$\rho(A,B)=\frac{1}{N-1}\sum_{i=1}^{N}\left(\frac{A_i-\mu_A}{\sigma_A}\right)\left(\frac{B_i-\mu_B}{\sigma_B}\right) \tag{0.1}$$

Where N is the number of samples, A_i and B_i are the ith samples of sequence A and B, μ_A and μ_B are signal's means, σ_A and σ_B are signal's standard variances.

An important property of *Pearson Correlation Coefficient* is that it's range is $[-1,1]$.

Prove:

According to the definition of Pearson Correlation Coefficient:

$$\begin{aligned}\rho(A,B)&=\frac{1}{N-1}\frac{1}{\sigma_A\sigma_B}\sum_{i=1}^{N}(A_i-\mu_A)(B_i-\mu_B)\\&=\frac{1}{N-1}\sqrt{\frac{N}{\sum_{i=1}^{N}(A_i-\mu_A)^2}}\sqrt{\frac{N}{\sum_{i=1}^{N}(B_i-\mu_B)^2}}\sum_{i=1}^{N}(A_i-\mu_A)(B_i-\mu_B)\end{aligned}$$

Then,

$$[\rho(A,B)]^2=\left(\frac{N}{N-1}\right)^2\frac{\left[\sum_{i=1}^{N}(A_i-\mu_A)(B_i-\mu_B)\right]^2}{\left[\sum_{i=1}^{N}(A_i-\mu_A)^2\right]\left[\sum_{i=1}^{N}(B_i-\mu_B)^2\right]}$$

According to the *Cauchy-Schwarz Inequality*:

$$\left[\sum_{i=1}^{N}(A_i-\mu_A)(B_i-\mu_B)\right]^2\le\left[\sum_{i=1}^{N}(A_i-\mu_A)^2\right]\left[\sum_{i=1}^{N}(B_i-\mu_B)^2\right]$$

So,

$$[\rho(A,B)]^2\le\left(\frac{N}{N-1}\right)^2\approx 1$$

Conditions for the equals sign to hold is:

$$\frac{A_1}{B_1} = \frac{A_2}{B_2} = \cdots = \frac{A_i}{B_i} = \cdots = \frac{A_N}{B_N} = k \neq 0$$

If $k > 0$, $\rho(A,B) = 1$; if $k < 0$, $\rho(A,B) = -1$.

Thus, only when the constructed sinusoidal signal has the same frequency and phase with the signal to be estimated, the Pearson correlation coefficient obtains the maximum value.

2.2 Maximum Correlation Coefficient

Signal to be estimated can be expressed as:

$$y_{test}(t) = A\cos(\omega_0 t + \phi_0) \tag{0.2}$$

Sample the signal for T seconds with sampling rate F_s, the sampled sequence would be:

$$s_{test}(n) = A\cos\left(\frac{\omega_0}{F_s} n + \phi_0\right), \quad n = 0,1,\ldots,TF_s \tag{0.3}$$

To estimate the frequency and phase parameters of the input signal, construct a series of sinusoldal signals with different frequencies and phases.

$$s_{cons}(n;\mathbf{x}) = \cos\left(\frac{\omega}{F_s} n + \phi\right), \quad n = 1,2,\ldots,TF_s \tag{0.4}$$

Where $\mathbf{x} = (\omega,\phi)^\top$. Then calculate the correlation coefficient of the signal to be estimated and constructed signals:

$$R(\mathbf{x}) = \rho\left[s_{test}(n), s_{cons}(n;\mathbf{x})\right] \tag{0.5}$$

According to the conclusion in last section, when $s_{test}(n) = s_{cons}(n;\mathbf{x})$, $R(\mathbf{x})$ takes the maximum value.

2.3 Searching for the Maximum Correlation Coefficient

The conclusion in last section proves that the maximum correlation coefficient method is reasonable in estimating the frequency and phase of sinusoidal signal. The next step is to find a way to fetch the maximum correlation coefficient as soon as possible.

To view the problem more explicitly, we plot the correlation coefficients in a figure with changing frequency and phase of the constructed signal. Figure 1 shows the variation of the correlation coefficient with four different input signal.

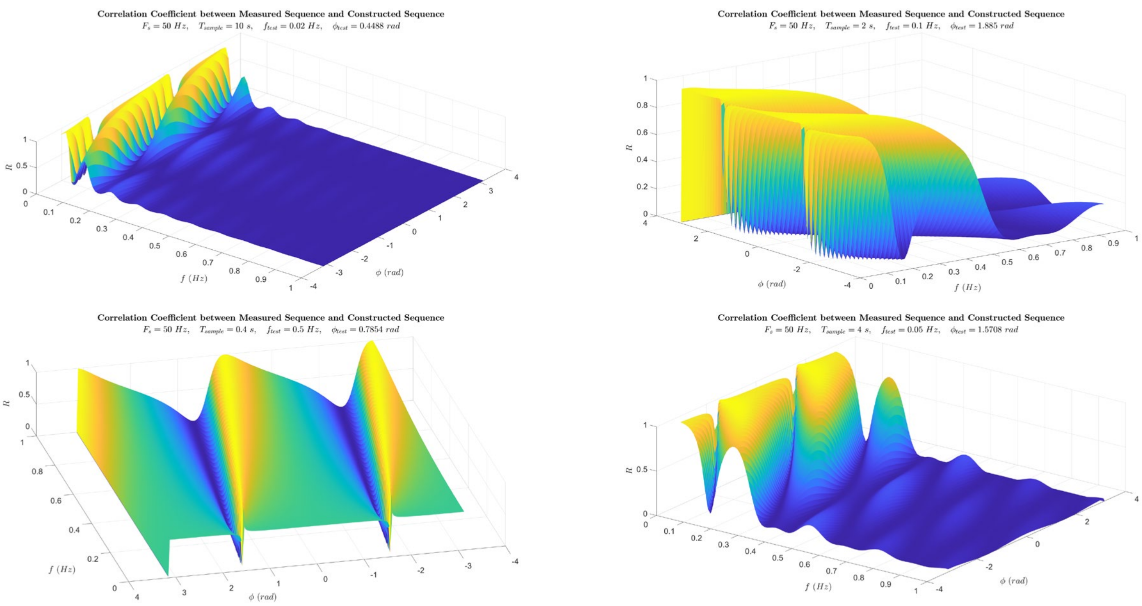


Figure 1　Correlation coefficient variance with changing frequency and phase of constructed signal

Multi-dimension optimization algorithms can be adopted to find the maximum value of the correlation coefficient and its corresponding variable value. However, according to Figure 1, the searching problem is not a convex optimization problem. Thus, we need to do a global search in the first place to find the rough position of the maximum correlation coefficient.

3　Multi-Dimension Search Optimization

3.1　Objective Function

Before discussing the optimization algorithm, we need to determine the objective function. Define the objective function:

$$f(\mathbf{x}) = 8 - e^{R(\mathbf{x})+1} \tag{0.6}$$

According to Chapter 1, the range of correlation coefficient $R(\mathbf{x})$ is $[-1,1]$. So, the range of the objective function $f(\mathbf{x})$ is $[8-e^2,8]$. The range is above 0. Also, it is obvious that $f(\mathbf{x})$ increases monotonically with $R(\mathbf{x})$. The maximal value search problem turns into a minimal value search problem, which is more common. Moreover, since $\left|\frac{\mathrm{d}f}{\mathrm{d}R}\right| = e^{R+1} \geq 1$, the new function value resolution is larger compared to the original one, which means the algorithm is easier to distinguish between two different searched points.

3.2 Hybrid Search Optimization Algorithm

A hybrid optimization algorithm is introduced to solve the muti-dimension search problem which was mentioned in Chapter 1.

To reduce the computational effort, part of the calculation of *Pearson Correlation Coefficient* is done before optimization.

According to Equation (1.1) and (1.5):

$$R(m) = \frac{1}{N-1}\sum_{n=1}^{N}\left[\frac{s_{test}(n)-\mu_{test}}{\sigma_{test}}\right]\left[\frac{s_{cons}(n;m)-\mu_{cons}}{\sigma_{cons}}\right] \tag{0.7}$$

Where N is the number of samples in the sequence. Define

$$\mathbf{C_{test}} = \left(\frac{s_{test}(1)-\mu_{test}}{\sigma_{test}}, \frac{s_{test}(2)-\mu_{test}}{\sigma_{test}}, \ldots, \frac{s_{test}(N)-\mu_{test}}{\sigma_{test}}\right) \tag{0.8}$$

$$\mathbf{C_{cons}}(m) = \left(\frac{s_{cons}(1;m)-\mu_{cons}}{\sigma_{cons}}, \frac{s_{cons}(2;m)-\mu_{cons}}{\sigma_{cons}}, \ldots, \frac{s_{cons}(N;m)-\mu_{cons}}{\sigma_{cons}}\right) \tag{0.9}$$

Then Equation (1.7) can be written as

$$R(m) = \frac{1}{N-1}\mathbf{C_{test}}\left[\mathbf{C_{cons}}(m)\right]^{\top} \tag{0.10}$$

We notice that the variables in Equation (1.8) to calculate $\mathbf{C_{test}}$ are only related to $s_{test}(n)$ and the variables in Equation (1.9) to calculate $\mathbf{C_{cons}}(m)$ are only related to $s_{cons}(n;m)$. So, $\mathbf{C_{test}}$ can be calculated before search optimization and passed into the objective function as a known constant.

The pseudo code of the hybrid search optimization algorithm is shown below.

Algorithm 1 Hybrid Search Optimization

Input: Maximum iteration: $maxIter$

Calculate $\mathbf{C_{test}}$ according to Eq. (1.7)
$\mathbf{x_{iter}} = \mathbf{0}$, $y_{iter} = 1000$
for $i = 1, 2, \ldots, maxIter$ **do**
    Start Particle Swarm Optimization
    Find the global optimum $\mathbf{x_{glob}}$ and its function value y_{glob}
    Start Conjugate Gradient method
    Find the local optimum $\mathbf{x_{loc}}$ and its function value y_{loc}
    if $y_{loc} < y_{iter}$ **then**
        $\mathbf{x_{iter}} = \mathbf{x_{loc}}$, $y_{iter} = y_{loc}$
    end if
end for

Return: Optimal point $\mathbf{x_{iter}}$

3.3 Particle Swarm Optimization

The PSO algorithm is based on the algorithm described in Kennedy and Eberhart, using modifications suggested in Mezura-Montes and Coello Coello and in Pedersen.

The formula for updating particle velocities is shown below

$$\mathbf{v_{new}} = w\mathbf{v} + c_1 u_1 (\mathbf{p} - \mathbf{x}) + c_2 u_2 (\mathbf{g} - \mathbf{x}) \tag{0.11}$$

Where $\mathbf{v}$ is previous velocities, c_1 is self adjustment weight, c_2 is social adjustment weight, $\mathbf{p} - \mathbf{x}$ is the difference between the current position and the best position the particle has seen, $\mathbf{g} - \mathbf{x}$ is the difference between the current position and the best position in the current neighborhood, $u_1, u_2 \sim U(0,1)$.

The pseudo code of the PSO algorithm is shown below.

Algorithm 2 Particle Swarm Optimization

Initialize particle states: positions $\mathbf{x}$, velocities $\mathbf{v}$
Calculate fitness value $\mathbf{f}$
Best position in particles $\mathbf{p} = \mathbf{x}$, best fitness value in particles $\mathbf{f_{best}} = \mathbf{f}$
Calculate best fitness value among all particles $b = \min(\mathbf{f_{best}})$, the particle position $d = x$
Initialize fixed parameters: c_1, c_2
Initialize adaptive parameters: neighborhoodSize N, inertia w
Initialize stall counter $c = 0$
while Exit flag is empty **do**
    Generate best neighborhood position $\mathbf{g}$
    Update particle velocities $\mathbf{v}$ with Eq. (1.11)
    Update particle positions $\mathbf{x} = \mathbf{x} + \mathbf{v}$
    For particles out of bounds, enforce $v = 0$
    Update fitness values $\mathbf{f}$
    Update best fitness values and particle positions in particles:
    if $f < fun(p)$ **then**
        $p = x$
    end if
    Update best fitness value and best position among all particles:
    if $\min(\mathbf{f}) < b$ **then**
        $b = \min(\mathbf{f})$, $d = x$
        $c = \max(0, c - 1)$, $N = minNeighborSize$
    else
        $c = c + 1$, $N = \min(N + minNeighborSize, SwarmSize)$
    end if
    Update inertia factor:
    if $c < 2$ **then**
        $w = 2w$
    else
        $w = w/2$
    end if
    Check exit condition, update exit flag
end while
Return: Best particle position d, best fitness value b

4 Experiments

4.1 Test on Joint Estimation Method

4.1.1 Experiment Condition

The conditions of the experiment are shown in Table 1.

Method	Joint Estimation
Frequency Range	0 ~ 0.5 Hz
Frequency Increment	0.01 Hz
Phase Range	0 ~ 2π rad
Phase Increment	$\pi/50$

Table 1 Conditions of the experiment on Joint Estimation Method with Varying Parameters

4.1.2 Experiment Results

Figure 2 shows the MSE of frequency and phase estimation with Joint Estimation Method under different signal frequency and phase.

	MSE of Frequency Estimation	**MSE of Phase Estimation**

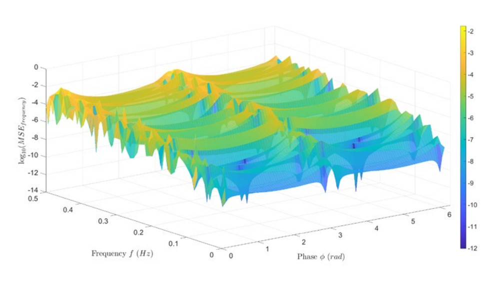
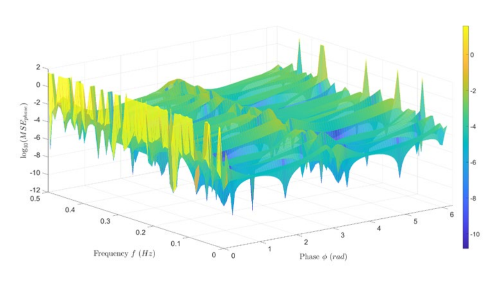
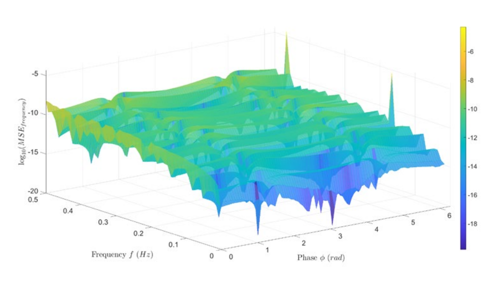
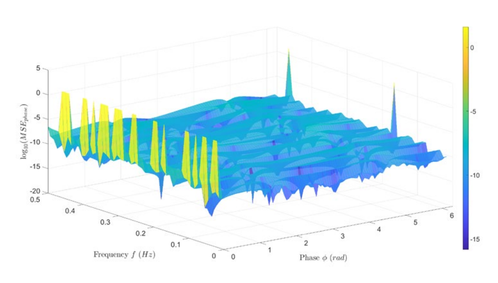
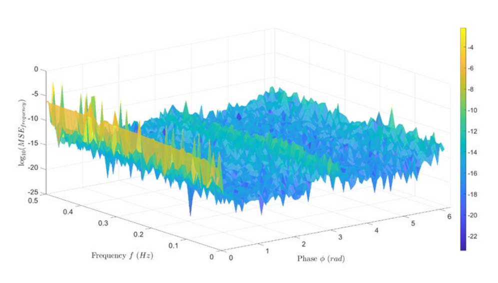
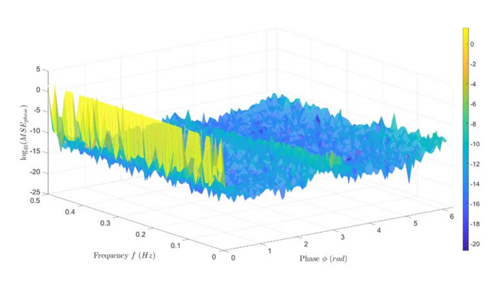
SNR = 40 dB		
SNR = 80 dB		
Without noise		

Figure 2 The estimation MSE with Joint Estimation Method under different signal frequency and phase

From the figures above, we can see that the estimation MSE stays at a low level under conditions with different SNR. The MSE only rises at several points where the phase of the tested signal is near its search boarder.

4.2 Compariosn between Estimation Methods (Varying SNR)

4.2.1 Experiment Condition

The conditions of the experiment are shown in Table 2.

Sampling Frequency (F_s)	10 Hz
Sampling Time (t_s)	0.5T

Table 2 Conditions of the experiment Comparing Estimation Methods with Varying Parameters

4.2.2 Experiment Results

Figure 3 shows the MSE of estimation and mean estimation time with different estimation method under different SNR.

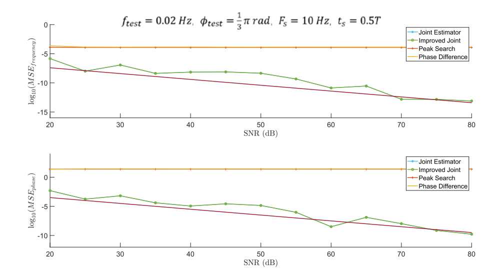
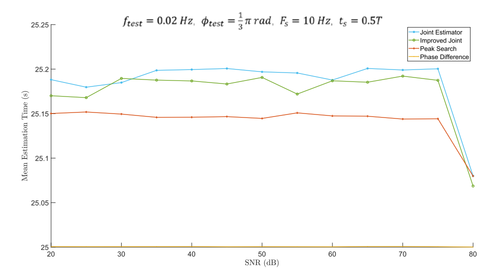
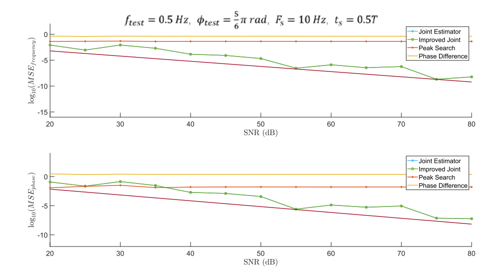
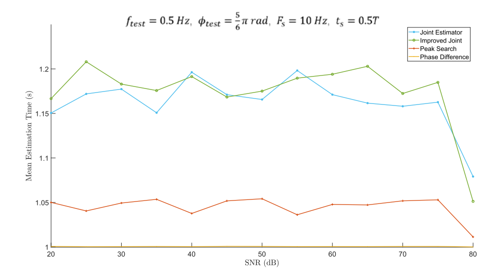
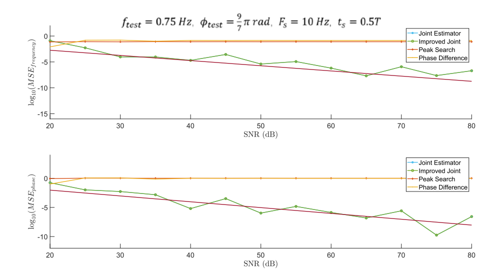
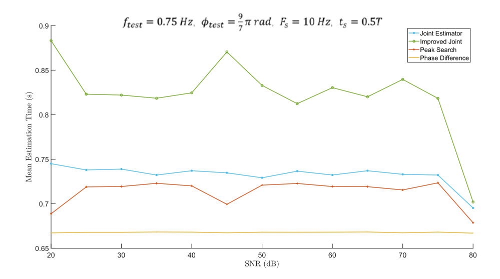
	MSE - SNR Curve	Mean Estimation Time - SNR Curve
0.01 Hz $\frac{1}{3}\pi\ rad$		
0.5 Hz $\frac{5}{6}\pi\ rad$		
0.75 Hz $\frac{9}{7}\pi\ rad$		

Figure 3　MSE - SNR and time - SNR curve with different estimation method

From the curves above we can see that the MSE of the Joint Estimation Method is much smaller than that of conventinal methods when the sampling time is extremely short, less than a full cycle.