Opto Stim DMFT

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1 DMF Equations for Network without Optogenetic Stimulation

$$\left(\frac{1}{\tau_A}+\frac{\mathrm{d}}{\mathrm{d}t}\right)\mu_A^i(t)=\sum_B W_{AB}\sum_j c_{AB}^{ij}\phi_B^j(t)+\frac{1}{\tau_A}\xi_A^i(t) \tag{1}$$

$$\begin{aligned}\left(\frac{1}{\tau_A}+\frac{\mathrm{d}}{\mathrm{d}t}\right)\mu_A(t)&=\left\langle\sum_B W_{AB}\sum_j c_{AB}^{ij}\phi_B^j(t)+\frac{1}{\tau_A}\xi_A^i(t)\right\rangle\\&=\sum_B W_{AB}\sum_j\left\langle c_{AB}^{ij}\right\rangle\left\langle\phi_B^j(t)\right\rangle\\&=\sum_B W_{AB}K_B r_B(t)\end{aligned} \tag{2}$$

$$\begin{aligned}&\left(\frac{1}{\tau_A}+\frac{\mathrm{d}}{\mathrm{d}s}\right)\left(\frac{1}{\tau_A}+\frac{\mathrm{d}}{\mathrm{d}t}\right)\sigma_{\mu_A}^2(s,t)\\&\quad=\left\langle\left(\sum_B W_{AB}\sum_j c_{AB}^{ij}\phi_B^j(s)+\frac{1}{\tau_A}\xi_A^i(s)\right)\left(\sum_{B'} W_{AB}\sum_{j'} c_{AB}^{ij}\phi_B^j(t)+\frac{1}{\tau_A}\xi_A^i(t)\right)\right\rangle\\&\qquad\qquad-\left(\sum_B W_{AB}K_B r_B(s)\right)\left(\sum_{B'} W_{AB'}K_{B'}r_{B'}(t)\right)\\&\quad=\sum_B\sum_j W_{AB}^2\left\langle c_{AB}^{ij\;2}\right\rangle\left\langle\phi_B^j(s)\phi_B^j(t)\right\rangle\\&\qquad+\sum_B\sum_{B'}\sum_j\sum_{j'}W_{AB}W_{AB'}\left\langle c_{AB}^{ij}\right\rangle\left\langle c_{AB'}^{ij'}\right\rangle\left\langle\phi_B^j(s)\right\rangle\left\langle\phi_{B'}^{j'}(t)\right\rangle(1-\delta_{BB'}\delta jj')\\&\qquad\qquad+\frac{1}{\tau_A^2}\left\langle\xi_A^i(s)\xi_A^i(t)\right\rangle-\left(\sum_B W_{AB}K_B r_B(s)\right)\left(\sum_{B'} W_{AB'}K_{B'}r_{B'}(t)\right)\\&\quad=\sum_B W_{AB}^2K_B C_{r_B}(s,t)+\sum_B\sum_{B'}W_{AB}W_{AB'}K_BK_{B'}r_B(s)r_{B'}(t)-\sum_B W_{AB}^2K_Bp_Br_B(s)r_B(t)\\&\qquad\qquad+\frac{1}{\tau_A^2}D_A(s,t)-\left(\sum_B W_{AB}K_B r_B(s)\right)\left(\sum_{B'} W_{AB'}K_{B'}r_{B'}(t)\right)\\&\quad=\sum_B W_{AB}^2K_B[C_{r_B}(s,t)-p_Br_B(s)r_B(t)]+\frac{1}{\tau_A^2}D_A(s,t)\end{aligned} \tag{3}$$

2 DMF Equations for Optogenetically Stimulated Network

$$\left(\frac{1}{\tau_A}+\frac{\mathrm{d}}{\mathrm{d}t}\right)\tilde{\mu}_A^i(t)=\sum_B W_{AB}\sum_j c_{AB}^{ij}\tilde{\phi}_B^j(t)+\frac{1}{\tau_A}\lambda_A^i L+\frac{1}{\tau_A}\xi_A^i(t) \tag{4}$$

$$\begin{aligned}\left(\frac{1}{\tau_A}+\frac{\mathrm{d}}{\mathrm{d}t}\right)\tilde{\mu}_A(t)&=\left\langle\sum_B W_{AB}\sum_j c_{AB}^{ij}\tilde{\phi}_B^j(t)+\frac{1}{\tau_A}\lambda_A^i L+\frac{1}{\tau_A}\xi_A^i(t)\right\rangle\\&=\sum_B W_{AB}\sum_j\left\langle c_{AB}^{ij}\right\rangle\left\langle\tilde{\phi}_B^j(t)\right\rangle+\frac{1}{\tau_A}\left\langle\lambda_A^i\right\rangle L\\&=\sum_B W_{AB}K_B\tilde{r}_B(t)+\frac{1}{\tau_A}\left\langle\lambda_A^i\right\rangle L\end{aligned} \tag{5}$$

$$\Delta\mu_A^i(t)=\tilde{\mu}_A^i(t)-\mu_A^i(t)-\lambda_A^i L \tag{6}$$

$$\begin{aligned}\left(\frac{1}{\tau_A}+\frac{\mathrm{d}}{\mathrm{d}t}\right)\Delta\mu_A^i(t)&=\sum_B W_{AB}\sum_j c_{AB}^{ij}\tilde{\phi}_B^j(t)+\frac{1}{\tau_A}\lambda_A^i L+\frac{1}{\tau_A}\xi_A^i(t)\\&\qquad-\left(\sum_B W_{AB}\sum_j c_{AB}^{ij}\phi_B^j(t)+\frac{1}{\tau_A}\xi_A^i(t)\right)-\frac{1}{\tau_A}\lambda_A^i L\\&=\sum_B W_{AB}\sum_j c_{AB}^{ij}[\tilde{\phi}_B^j(t)-\phi_B^j(t)]\\&=\sum_B W_{AB}\sum_j c_{AB}^{ij}\Delta\phi_B^j(t)\end{aligned} \tag{7}$$

$$\begin{aligned}\left(\frac{1}{\tau_A}+\frac{\mathrm{d}}{\mathrm{d}t}\right)\Delta\mu_A(t)&=\left\langle\sum_B W_{AB}\sum_j c_{AB}^{ij}\Delta\phi_B^j(t)\right\rangle\\&=\sum_B W_{AB}\sum_j\left\langle c_{AB}^{ij}\right\rangle\left\langle\Delta\phi_B^j(t)\right\rangle\\&=\sum_B W_{AB}K_B\Delta r_B(t)\end{aligned} \tag{8}$$

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\begin{aligned}
&\left(\frac{1}{\tau_A}+\frac{\mathrm{d}}{\mathrm{d}s}\right)\left(\frac{1}{\tau_A}+\frac{\mathrm{d}}{\mathrm{d}t}\right)\sigma^2_{\Delta\mu_A}(s,t)\\
&\quad=\left\langle\left(\sum_B W_{AB}\sum_j c^{ij}_{AB}\Delta\phi^j_B(s)\right)\left(\sum_{B'} W_{AB'}\sum_{j'} c^{ij'}_{AB'}\Delta\phi^{j'}_{B'}(t)\right)\right\rangle\\
&\qquad\qquad-\left(\sum_B W_{AB}K_B\Delta r_B(s)\right)\left(\sum_{B'} W_{AB'}K_{B'}\Delta r_{B'}(t)\right)\\
&\quad=\sum_B\sum_j W^2_{AB}\left\langle c^{ij\;2}_{AB}\right\rangle\left\langle\Delta\phi^j_B(s)\Delta\phi^j_B(t)\right\rangle\\
&\qquad+\sum_B\sum_{B'}\sum_j\sum_{j'} W_{AB}W_{AB'}\left\langle c^{ij}_{AB}\right\rangle\left\langle c^{ij'}_{AB'}\right\rangle\left\langle\Delta\phi^j_B(s)\right\rangle\left\langle\Delta\phi^{j'}_{B'}(t)\right\rangle(1-\delta_{BB'}\delta jj')\\
&\qquad\qquad-\left(\sum_B W_{AB}K_B\Delta r_B(s)\right)\left(\sum_{B'} W_{AB'}K_{B'}\Delta r_{B'}(t)\right)\\
&\quad=\sum_B W^2_{AB}K_B C_{\Delta r_B}(s,t)+\sum_B\sum_{B'} W_{AB}W_{AB'}K_BK_{B'}\Delta r_B(s)\Delta r_{B'}(t)\\
&\qquad-\sum_B W^2_{AB}K_B p_B\Delta r_B(s)\Delta r_B(t)-\left(\sum_B W_{AB}K_B\Delta r_B(s)\right)\left(\sum_{B'} W_{AB'}K_{B'}\Delta r_{B'}(t)\right)\\
&\quad=\sum_B W^2_{AB}K_B[C_{\Delta r_B}(s,t)-p_B\Delta r_B(s)\Delta r_B(t)]
\end{aligned}
\tag{9}
$$

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\begin{aligned}
&\left(\frac{1}{\tau_A}+\frac{\mathrm{d}}{\mathrm{d}s}\right)\left(\frac{1}{\tau_A}+\frac{\mathrm{d}}{\mathrm{d}t}\right)\rho_{\mu_A,\Delta\mu_A}(s,t)\\
&\quad=\left\langle\left(\sum_B W_{AB}\sum_j c^{ij}_{AB}\phi^j_B(s)+\frac{1}{\tau_A}\xi^i_A(s)\right)\left(\sum_{B'} W_{AB'}\sum_{j'} c^{ij'}_{AB'}\Delta\phi^{j'}_{B'}(t)\right)\right\rangle\\
&\qquad\qquad-\left(\sum_B W_{AB}K_B r_B(s)\right)\left(\sum_{B'} W_{AB'}K_{B'}\Delta r_{B'}(t)\right)\\
&\quad=\sum_B\sum_j W^2_{AB}\left\langle c^{ij\;2}_{AB}\right\rangle\left\langle\phi^j_B(s)\Delta\phi^j_B(t)\right\rangle\\
&\qquad+\sum_B\sum_{B'}\sum_j\sum_{j'} W_{AB}W_{AB'}\left\langle c^{ij}_{AB}\right\rangle\left\langle c^{ij'}_{AB'}\right\rangle\left\langle\phi^j_B(s)\right\rangle\left\langle\Delta\phi^{j'}_{B'}(t)\right\rangle(1-\delta_{BB'}\delta jj')\\
&\qquad\qquad-\left(\sum_B W_{AB}K_B r_B(s)\right)\left(\sum_{B'} W_{AB'}K_{B'}\Delta r_{B'}(t)\right)\\
&\quad=\sum_B W^2_{AB}K_B R_{r_B,\Delta r_B}(s,t)+\sum_B\sum_{B'} W_{AB}W_{AB'}K_BK_{B'}r_B(s)\Delta r_{B'}(t)\\
&\qquad-\sum_B W^2_{AB}K_B p_B r_B(s)\Delta r_B(t)-\left(\sum_B W_{AB}K_B r_B(s)\right)\left(\sum_{B'} W_{AB'}K_{B'}\Delta r_{B'}(t)\right)\\
&\quad=\sum_B W^2_{AB}K_B[R_{r_B,\Delta r_B}(s,t)-p_B r_B(s)\Delta r_B(t)]
\end{aligned}
\tag{10}
$$

3 Calculating Rate Moments

$$r_A(t) = \int \mathrm{d}x\, P_{\mathcal{N}}(x)\, \phi_A[\mu_A(t) + \sigma_{\mu_A}(t,t)\, x] \tag{11}$$

$$\begin{aligned} C_{r_A}(s,t) = \int \mathrm{d}x\, \mathrm{d}y_s\, \mathrm{d}y_t\, P_{\mathcal{N}}(x) P_{\mathcal{N}}(y_s) P_{\mathcal{N}}(y_t)\, \phi_A\Big[\mu_A(s) + \sigma_{\mu_A}(s,s)\sqrt{c_{r_A}}\, x + \sigma_{\mu_A}(s,s)\sqrt{1-c_{r_A}}\, y_s\Big] \\ \times \phi_A\Big[\mu_A(t) + \sigma_{\mu_A}(t,t)\sqrt{c_{r_A}}\, x + \sigma_{\mu_A}(t,t)\sqrt{1-c_{r_A}}\, y_t\Big] \end{aligned} \tag{12}$$

$$c_{r_A} = \frac{\sigma^2_{\mu_A}(s,t)}{\sigma_{\mu_A}(s,s)\sigma_{\mu_A}(t,t)}$$

$$\tilde{r}_A(t) = \int \mathrm{d}x\, P_{\mathcal{N}}(x) P(\lambda)\, \phi_A\Big[\mu^L_A(t) + \sigma_{\mu^L_A}(t,t)\, x + \lambda L\Big] \tag{13}$$

$$\mu^L_A(t) = \mu_A(t) + \Delta\mu_A(t)$$

$$\sigma^2_{\mu^L_A}(t,t) = \sigma^2_{\mu_A}(t,t) + \sigma^2_{\Delta\mu_A}(t,t) + \rho_{\mu_A,\Delta\mu_A}(t,t) + \rho_{\Delta\mu_A,\mu_A}(t,t)$$

$$\begin{aligned} C_{\tilde{r}_A}(s,t) = \int \mathrm{d}x\, \mathrm{d}y_s\, \mathrm{d}y_t\, P_{\mathcal{N}}(x) P_{\mathcal{N}}(y_s) P_{\mathcal{N}}(y_t) P(\lambda) \\ \times \phi_A\Big[\mu^L_A(s) + \sigma_{\mu^L_A}(s,s)\sqrt{c_{\tilde{r}_A}}\, x + \sigma_{\mu^L_A}(s,s)\sqrt{1-c_{\tilde{r}_A}}\, y_s + \lambda L\Big] \\ \times \phi_A\Big[\mu^L_A(t) + \sigma_{\mu^L_A}(t,t)\sqrt{c_{\tilde{r}_A}}\, x + \sigma_{\mu^L_A}(t,t)\sqrt{1-c_{\tilde{r}_A}}\, y_t + \lambda L\Big] \end{aligned} \tag{14}$$

$$c_{\tilde{r}_A} = \frac{\sigma^2_{\mu^L_A}(s,t)}{\sigma_{\mu^L_A}(s,s)\sigma_{\mu^L_A}(t,t)}$$

$$\begin{aligned} R_{r_A,\tilde{r}_A}(s,t) = \int \mathrm{d}x\, \mathrm{d}y_s\, \mathrm{d}y_t\, P_{\mathcal{N}}(x) P_{\mathcal{N}}(y_s) P_{\mathcal{N}}(y_t) P(\lambda) \\ \times \phi_A\big[\mu_A(s) + \sigma_{\mu_A}(s,s)\sqrt{c_{r_A,\tilde{r}_A}}\, x + \sigma_{\mu_A}(s,s)\sqrt{1-c_{r_A,\tilde{r}_A}}\, y_s\big] \\ \times \phi_A\Big[\mu^L_A(t) + \sigma_{\mu^L_A}(t,t)\sqrt{c_{r_A,\tilde{r}_A}}\, x + \sigma_{\mu^L_A}(t,t)\sqrt{1-c_{r_A,\tilde{r}_A}}\, y_t + \lambda L\Big] \end{aligned} \tag{15}$$

$$c_{r_A,\tilde{r}_A} = \frac{\sigma^2_{\mu_A}(s,t) + \rho_{\mu_A,\Delta\mu_A}(s,t)}{\sigma_{\mu_A}(s,s)\sigma_{\mu^L_A}(t,t)}$$

$$\Delta r_A(t) = \tilde{r}_A(t) - r_A(t) \tag{16}$$

$$C_{\Delta r_A}(s,t) = C_{\tilde{r}_A}(s,t) - R_{r_A,\tilde{r}_A}(s,t) - R_{\tilde{r}_A,r_A}(s,t) + C_{r_A}(s,t) \tag{17}$$

$$R_{r_A,\Delta r_A}(s,t) = R_{r_A,\tilde{r}_A}(s,t) - C_{r_A}(s,t) \tag{18}$$

For time-independent external input ($r_X(t) = r_X$), $\rho_{\mu_A,\Delta\mu_A}(s,t) = \rho_{\mu_A,\Delta\mu_A}(t,s)$ and $R_{r_B,\Delta r_B}(s,t) = R_{r_B,\Delta r_B}(t,s)$. Thus, we can simplify our later calculations with

$$\begin{aligned} \rho_{\mu_A,\Delta\mu_A}(s,t) &= \rho_{\mu_A,\Delta\mu_A}(t,s) \\ &= \rho_{\Delta\mu_A,\mu_A}(s,t) \end{aligned} \tag{19}$$

$$\begin{aligned} R_{r_B,\Delta r_B}(s,t) &= R_{r_B,\Delta r_B}(t,s) \\ &= R_{\Delta r_B,r_B}(s,t) \end{aligned} \tag{20}$$