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4.1 EXERCISES

1. Let V be the first quadrant in the xy-plane; that is, let

$$V = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} : x \geq 0, y \geq 0 \right\}$$

a. If $\mathbf{u}$ and $\mathbf{v}$ are in V, is $\mathbf{u} + \mathbf{v}$ in V? Why?

b. Find a specific vector $\mathbf{u}$ in V and a specific scalar c such that $c\mathbf{u}$ is *not* in V. (This is enough to show that V is *not* a vector space.)

2. Let W be the union of the first and third quadrants in the xy-plane. That is, let $W = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} : xy \geq 0 \right\}$.

a. If $\mathbf{u}$ is in W and c is any scalar, is $c\mathbf{u}$ in W? Why?

b. Find specific vectors $\mathbf{u}$ and $\mathbf{v}$ in W such that $\mathbf{u}+\mathbf{v}$ is not in W. This is enough to show that W is *not* a vector space.

3. Let H be the set of points inside and on the unit circle in the xy-plane. That is, let $H = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} : x^2 + y^2 \le 1 \right\}$. Find a specific example—two vectors or a vector and a scalar—to show that H is not a subspace of $\mathbb{R}^2$.

4. Construct a geometric figure that illustrates why a line in $\mathbb{R}^2$ *not* through the origin is not closed under vector addition.

In Exercises 5–8, determine if the given set is a subspace of $\mathbb{P}_n$ for an appropriate value of n. Justify your answers.

5. All polynomials of the form $\mathbf{p}(t) = at^2$, where a is in $\mathbb{R}$.

6. All polynomials of the form $\mathbf{p}(t) = a + t^2$, where a is in $\mathbb{R}$.

7. All polynomials of degree at most 3, with integers as coefficients.

8. All polynomials in $\mathbb{P}_n$ such that $\mathbf{p}(0) = 0$.

9. Let H be the set of all vectors of the form $\begin{bmatrix} s \\ 3s \\ 2s \end{bmatrix}$. Find a vector $\mathbf{v}$ in $\mathbb{R}^3$ such that $H = \text{Span}\,\{\mathbf{v}\}$. Why does this show that H is a subspace of $\mathbb{R}^3$?

10. Let H be the set of all vectors of the form $\begin{bmatrix} 2t \\ 0 \\ -t \end{bmatrix}$. Show that H is a subspace of $\mathbb{R}^3$. (Use the method of Exercise 9.)

11. Let W be the set of all vectors of the form $\begin{bmatrix} 5b+2c \\ b \\ c \end{bmatrix}$, where b and c are arbitrary. Find vectors $\mathbf{u}$ and $\mathbf{v}$ such that $W = \text{Span}\,\{\mathbf{u}, \mathbf{v}\}$. Why does this show that W is a subspace of $\mathbb{R}^3$?

12. Let W be the set of all vectors of the form $\begin{bmatrix} s+3t \\ s-t \\ 2s-t \\ 4t \end{bmatrix}$. Show that W is a subspace of $\mathbb{R}^4$. (Use the method of Exercise 11.)

13. Let $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} 4 \\ 2 \\ 6 \end{bmatrix}$, and $\mathbf{w} = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$.

a. Is $\mathbf{w}$ in $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$? How many vectors are in $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$?

b. How many vectors are in Span $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$?

c. Is $\mathbf{w}$ in the subspace spanned by $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$? Why?

14. Let $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ be as in Exercise 13, and let $\mathbf{w} = \begin{bmatrix} 8 \\ 4 \\ 7 \end{bmatrix}$. Is $\mathbf{w}$ in the subspace spanned by $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$? Why?

In Exercises 15–18, let W be the set of all vectors of the form shown, where a, b, and c represent arbitrary real numbers. In each case, either find a set S of vectors that spans W or give an example to show that W is *not* a vector space.

15. $\begin{bmatrix} 3a+b \\ 4 \\ a-5b \end{bmatrix}$

16. $\begin{bmatrix} -a+1 \\ a-6b \\ 2b+a \end{bmatrix}$

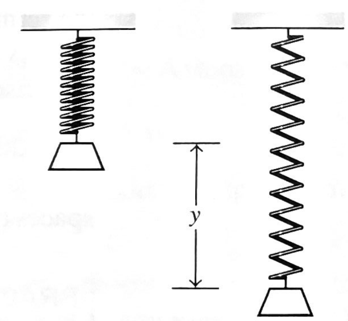
17. $\begin{bmatrix} a-b \\ b-c \\ c-a \\ b \end{bmatrix}$

18. $\begin{bmatrix} 4a+3b \\ 0 \\ a+b+c \\ c-2a \end{bmatrix}$

19. If a mass m is placed at the end of a spring, and if the mass is pulled downward and released, the mass–spring system will begin to oscillate. The displacement y of the mass from its resting position is given by a function of the form

$$y(t) = c_1 \cos \omega t + c_2 \sin \omega t \tag{5}$$

where ω is a constant that depends on the spring and the mass. (See the figure below.) Show that the set of all functions described in (5) (with ω fixed and c_1, c_2 arbitrary) is a vector space.



20. The set of all continuous real-valued functions defined on a closed interval $[a, b]$ in $\mathbb{R}$ is denoted by $C[a, b]$. This set is a subspace of the vector space of all real-valued functions defined on $[a, b]$.

a. What facts about continuous functions should be proved in order to demonstrate that $C[a, b]$ is indeed a subspace as claimed? (These facts are usually discussed in a calculus class.)

b. Show that $\{\mathbf{f} \text{ in } C[a,b] : \mathbf{f}(a) = \mathbf{f}(b)\}$ is a subspace of $C[a, b]$.

For fixed positive integers m and n, the set $M_{m\times n}$ of all $m \times n$ matrices is a vector space, under the usual operations of addition of matrices and multiplication by real scalars.

21. Determine if the set H of all matrices of the form $\begin{bmatrix} a & b \\ 0 & d \end{bmatrix}$ is a subspace of $M_{2\times 2}$.

22. Let F be a fixed 3×2 matrix, and let H be the set of all matrices A in $M_{2\times 4}$ with the property that $FA = 0$ (the zero matrix in $M_{3\times 4}$). Determine if H is a subspace of $M_{2\times 4}$.

…tement True or False. Justify each answer.

23. a. If $\mathbf{f}$ is a function in the vector space V of all real-valued functions on $\mathbb{R}$ and if $\mathbf{f}(t) = 0$ for some t, then $\mathbf{f}$ is the zero vector in V.

b. A vector is an arrow in three-dimensional space.

c. A subset H of a vector space V is a subspace of V if the zero vector is in H.

d. A subspace is also a vector space.

e. Analog signals are used in the major control systems for the space shuttle, mentioned in the introduction to the chapter.

24. a. A vector is any element of a vector space.

b. If $\mathbf{u}$ is a vector in a vector space V, then $(-1)\mathbf{u}$ is the same as the negative of $\mathbf{u}$.

c. A vector space is also a subspace.

d. $\mathbb{R}^2$ is a subspace of $\mathbb{R}^3$.

e. A subset H of a vector space V is a subspace of V if the following conditions are satisfied: (i) the zero vector of V is in H, (ii) $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{u}+\mathbf{v}$ are in H, and (iii) c is a scalar and $c\mathbf{u}$ is in H.

Exercises 25–29 show how the axioms for a vector space V can be used to prove the elementary properties described after the definition of a vector space. Fill in the blanks with the appropriate axiom numbers. Because of Axiom 2, Axioms 4 and 5 imply, respectively, that $\mathbf{0}+\mathbf{u}=\mathbf{u}$ and $-\mathbf{u}+\mathbf{u}=\mathbf{0}$ for all $\mathbf{u}$.

25. Complete the following proof that the zero vector is unique. Suppose that $\mathbf{w}$ in V has the property that $\mathbf{u}+\mathbf{w}=\mathbf{w}+\mathbf{u}=\mathbf{u}$ for all $\mathbf{u}$ in V. In particular, $\mathbf{0}+\mathbf{w}=\mathbf{0}$. But $\mathbf{0}+\mathbf{w}=\mathbf{w}$, by Axiom ____. Hence $\mathbf{w}=\mathbf{0}+\mathbf{w}=\mathbf{0}$.

26. Complete the following proof that $-\mathbf{u}$ is the *unique vector* in V such that $\mathbf{u}+(-\mathbf{u})=\mathbf{0}$. Suppose that $\mathbf{w}$ satisfies $\mathbf{u}+\mathbf{w}=\mathbf{0}$. Adding $-\mathbf{u}$ to both sides, we have

$(-\mathbf{u})+[\mathbf{u}+\mathbf{w}] = (-\mathbf{u})+\mathbf{0}$

$[(-\mathbf{u})+\mathbf{u}]+\mathbf{w} = (-\mathbf{u})+\mathbf{0}$ by Axiom ____ (a)

$\mathbf{0}+\mathbf{w} = (-\mathbf{u})+\mathbf{0}$ by Axiom ____ (b)

$\mathbf{w} = -\mathbf{u}$ by Axiom ____ (c)

27. Fill in the missing axiom numbers in the following proof that $0\mathbf{u}=\mathbf{0}$ for every $\mathbf{u}$ in V.

$0\mathbf{u} = (0+0)\mathbf{u} = 0\mathbf{u}+0\mathbf{u}$ by Axiom ____ (a)

Add the negative of $0\mathbf{u}$ to both sides:

$0\mathbf{u}+(-0\mathbf{u}) = [0\mathbf{u}+0\mathbf{u}]+(-0\mathbf{u})$

$0\mathbf{u}+(-0\mathbf{u}) = 0\mathbf{u}+[0\mathbf{u}+(-0\mathbf{u})]$ by Axiom ____ (b)

$\mathbf{0} = 0\mathbf{u}+\mathbf{0}$ by Axiom ____ (c)

$\mathbf{0} = 0\mathbf{u}$ by Axiom ____ (d)

28. Fill in the missing axiom numbers in the following proof that $c\mathbf{0}=\mathbf{0}$ for every scalar c.

$c\mathbf{0} = c(\mathbf{0}+\mathbf{0})$

$= c\mathbf{0}+c\mathbf{0}$ by Axiom ____ (a)

by Axiom ____ (b)

Add the negative of $c\mathbf{0}$ to both sides:

$c\mathbf{0}+(-c\mathbf{0}) = [c\mathbf{0}+c\mathbf{0}]+(-c\mathbf{0})$

$c\mathbf{0}+(-c\mathbf{0}) = c\mathbf{0}+[c\mathbf{0}+(-c\mathbf{0})]$ by Axiom ____ (c)

$\mathbf{0} = c\mathbf{0}+\mathbf{0}$ by Axiom ____ (d)

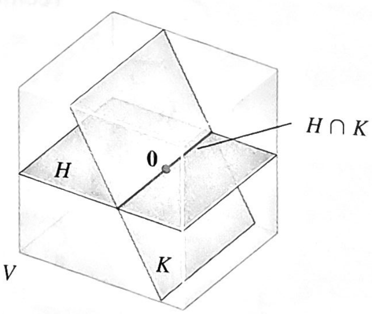
$\mathbf{0} = c\mathbf{0}$ by Axiom ____ (e)

29. Prove that $(-1)\mathbf{u} = -\mathbf{u}$. [*Hint:* Show that $\mathbf{u}+(-1)\mathbf{u}=\mathbf{0}$. Use some axioms and the results of Exercises 26 and 27.]

30. Suppose $c\mathbf{u}=\mathbf{0}$ for some nonzero scalar c. Show that $\mathbf{u}=\mathbf{0}$. Mention the axioms or properties you use.

31. Let $\mathbf{u}$ and $\mathbf{v}$ be vectors in a vector space V, and let H be any subspace of V that contains both $\mathbf{u}$ and $\mathbf{v}$. Explain why H also contains Span $\{\mathbf{u},\mathbf{v}\}$. This shows that Span $\{\mathbf{u},\mathbf{v}\}$ is the smallest subspace of V that contains both $\mathbf{u}$ and $\mathbf{v}$.

32. Let H and K be subspaces of a vector space V. The **intersection** of H and K, written as $H \cap K$, is the set of $\mathbf{v}$ in V that belong to both H and K. Show that $H \cap K$ is a subspace of V. (See the figure.) Give an example in $\mathbb{R}^2$ to show that the union of two subspaces is not, in general, a subspace.



33. Given subspaces H and K of a vector space V, the **sum** of H and K, written as $H+K$, is the set of all vectors in V that can be written as the sum of two vectors, one in H and the other in K; that is,

$$H+K = \{\mathbf{w} : \mathbf{w}=\mathbf{u}+\mathbf{v} \text{ for some } \mathbf{u} \text{ in } H \text{ and some } \mathbf{v} \text{ in } K\}$$

a. Show that $H+K$ is a subspace of V.

b. Show that H is a subspace of $H+K$ and K is a subspace of $H+K$.

34. Suppose $\mathbf{u}_1,\dots,\mathbf{u}_p$ and $\mathbf{v}_1,\dots,\mathbf{v}_q$ are vectors in a vector space V, and let

$$H = \text{Span}\{\mathbf{u}_1,\dots,\mathbf{u}_p\} \text{ and } K = \text{Span}\{\mathbf{v}_1,\dots,\mathbf{v}_q\}$$

Show that $H+K = \text{Span}\{\mathbf{u}_1,\dots,\mathbf{u}_p,\mathbf{v}_1,\dots,\mathbf{v}_q\}$.

In Exercises 3–6, find an explicit description of Nul A by listing vectors that span the null space.

3. $A = \begin{bmatrix} 1 & 3 & 5 & 0 \\ 0 & 1 & 4 & -2 \end{bmatrix}$

4. $A = \begin{bmatrix} 1 & -6 & 4 & 0 \\ 0 & 0 & 2 & 0 \end{bmatrix}$

5. $A = \begin{bmatrix} 1 & -2 & 0 & 4 & 0 \\ 0 & 0 & 1 & -9 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$

6. $A = \begin{bmatrix} 1 & 5 & -4 & -3 & 1 \\ 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

In Exercises 7–14, either use an appropriate theorem to show that the given set, W, is a vector space, or find a specific example to the contrary.

7. $\left\{ \begin{bmatrix} a \\ b \\ c \end{bmatrix} : a + b + c = 2 \right\}$

8. $\left\{ \begin{bmatrix} r \\ s \\ t \end{bmatrix} : 5r - 1 = s + 2t \right\}$

9. $\left\{ \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} : \begin{matrix} a - 2b = 4c \\ 2a = c + 3d \end{matrix} \right\}$

10. $\left\{ \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} : \begin{matrix} a + 3b = c \\ b + c + a = d \end{matrix} \right\}$

11. $\left\{ \begin{bmatrix} b - 2d \\ 5 + d \\ b + 3d \\ d \end{bmatrix} : b, d \text{ real} \right\}$

12. $\left\{ \begin{bmatrix} b - 5d \\ 2b \\ 2d + 1 \\ d \end{bmatrix} : b, d \text{ real} \right\}$

13. $\left\{ \begin{bmatrix} c - 6d \\ d \\ c \end{bmatrix} : c, d \text{ real} \right\}$

14. $\left\{ \begin{bmatrix} -a + 2b \\ a - 2b \\ 3a - 6b \end{bmatrix} : a, b \text{ real} \right\}$

In Exercises 15 and 16, find A such that the given set is Col A.

15. $\left\{ \begin{bmatrix} 2s + 3t \\ r + s - 2t \\ 4r + s \\ 3r - s - t \end{bmatrix} : r, s, t \text{ real} \right\}$

16. $\left\{ \begin{bmatrix} b - c \\ 2b + c + d \\ 5c - 4d \\ d \end{bmatrix} : b, c, d \text{ real} \right\}$

For the matrices in Exercises 17–20, (a) find k such that Nul A is a subspace of $\mathbb{R}^k$, and (b) find k such that Col A is a subspace of $\mathbb{R}^k$.

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