Solutions

Reminder: The general form of the determinant is:

$$\det(A) = \sum_{(i \text{ or } j)=1}^{n} (-1)^{i+j} a_{ij} A(i|j)$$

Where you either go down a column or across a row.

1. Given the system:

$$\begin{aligned} x_1 + 2x_2 - 3x_3 &= 5 \\ 2x_1 + x_2 - 3x_3 &= 13 \\ -x_1 + x_2 \qquad &= -8 \end{aligned}$$

(a) (5 points) Write the system in vector form.

$$x_1 \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ -3 \\ 0 \end{bmatrix} = \begin{bmatrix} 5 \\ 13 \\ -8 \end{bmatrix}$$

(b) (5 points) Write the system in matrix form.

$$\begin{bmatrix} 1 & 2 & -3 \\ 2 & 1 & -3 \\ -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5 \\ 13 \\ -8 \end{bmatrix}$$

(c) (10 points) Solve the system and write the solution in parametric form.

$$\left[\begin{array}{ccc|c} 1 & 2 & -3 & 5 \\ 2 & 1 & -3 & 13 \\ -1 & 1 & 0 & -8 \end{array}\right] \sim \left[\begin{array}{cccc} 1 & 0 & -1 & 7 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$V = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad v = \begin{bmatrix} 7 \\ -1 \\ 0 \end{bmatrix}, \quad t = x_3$$

$$\boxed{\{ t\vec{v} + \vec{v} \mid t \in \mathbb{C} \}}$$

2. Let $A = \begin{bmatrix} 1 & -3 & 4 & -1 \\ -2 & 6 & -6 & -1 \\ -3 & 9 & -6 & -6 \\ 3 & -9 & 4 & 9 \end{bmatrix}$

(a) (10 points) Find $\mathcal{N}(A)$.

$$A \sim \begin{bmatrix} 1 & -3 & 0 & 5 \\ 0 & 0 & 1 & -1.5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$x_1 = 3x_2 - 5x_4$$
$$x_2 = x_2$$
$$x_3 = 1.5x_4$$
$$x_4 = x_4$$

$$\Rightarrow N(A) = \left\langle \left\{ \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -5 \\ 0 \\ 1.5 \\ 1 \end{bmatrix} \right\} \right\rangle$$

(b) (10 points) Find $\mathcal{C}(A)$.

$$C(A) = \left\langle \left\{ \begin{bmatrix} 1 \\ -2 \\ -3 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ -6 \\ -6 \\ 4 \end{bmatrix} \right\} \right\rangle$$

4. (10 points) Find the matrix representation for the linear transformation $T : \mathbb{P}_2 \to \mathbb{P}_2$, where

$$T(\mathbf{p}) = 1 + x\mathbf{p}' = 1 + x(0 + \alpha_1 + 2\alpha_2 x)$$

$$= 1 + \alpha_1 x + 2\alpha_2 x^2$$

$\mathbf{p}'$ is the first derivative of the polynomial.

$$M^{T}_{B,B} = \left[\rho(T(1)) \,\middle|\, \rho(T(x)) \,\middle|\, \rho(T(x^2)) \right]$$

$$= \left[\rho_B(1) \,\middle|\, \rho_B(x) \,\middle|\, \rho_B(2x^2) \right]$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

5. Let $A = \begin{bmatrix} 1 & 0 & 1 \\ -2 & 3 & 0 \\ 1 & 0 & 0 \end{bmatrix}$.

(a) (5 points) Find the determinant of A by hand. Show all work.

$$0 + 3 \begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} - 0$$

$$= 3(1 \cdot 0 - 1 \cdot 1) = -3$$

(b) (5 points) Is A is invertible? Why or why not?

$\det(A) \neq 0 \Rightarrow$ invertible

6. Let $A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$.

(a) (10 points) Find an orthonormal basis for $\mathcal{C}(A)$

$$U_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$U_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} - \frac{0+1+1}{1+1+1} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} - \frac{2}{3} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -2/3 \\ 1/3 \\ 1/3 \end{bmatrix}$$

$$U_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} - \frac{0+0+1}{3} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - \frac{0+0+\frac{1}{3}}{\frac{4}{9}+\frac{1}{9}+\frac{1}{9}} \begin{bmatrix} -2/3 \\ 1/3 \\ 1/3 \end{bmatrix}$$

$$= \begin{bmatrix} -1/3 \\ -1/3 \\ 2/3 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} -2/3 \\ 1/3 \\ 1/3 \end{bmatrix} = \begin{bmatrix} 0 \\ -1/2 \\ 1/2 \end{bmatrix}$$

$$\frac{U_1}{\|U_1\|} = \frac{1}{\sqrt{1+1+1}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix}$$

$\frac{6}{9}$

$$\frac{U_2}{\|U_2\|} = \frac{1}{\sqrt{\frac{6}{9}}} \begin{bmatrix} -2/3 \\ 1/3 \\ 1/3 \end{bmatrix} = \frac{3}{\sqrt{6}} \begin{bmatrix} -2/3 \\ 1/3 \\ 1/3 \end{bmatrix} = \begin{bmatrix} -2/\sqrt{6} \\ 1/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix}$$

$$\frac{U_3}{\|U_3\|} = \frac{1}{\sqrt{1/2}} \begin{bmatrix} 0 \\ -1/2 \\ 1/2 \end{bmatrix} = \begin{bmatrix} 0 \\ -\sqrt{2}/2 \\ \sqrt{2}/2 \end{bmatrix}$$

$$B = \left\{ \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \frac{1}{\sqrt{6}} \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}, \frac{1}{2} \begin{bmatrix} 0 \\ -\sqrt{2} \\ \sqrt{2} \end{bmatrix} \right\}$$

(b) (10 points) Find a QR factorization of A.

$$R = Q^T A = \begin{bmatrix} \frac{1}{\sqrt{3}} & -\frac{2}{\sqrt{6}} & 0 \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & -\frac{\sqrt{2}}{2} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & \frac{\sqrt{2}}{2} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} \sqrt{3} & \frac{2}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ 0 & \sqrt{\frac{2}{3}} & \frac{1}{\sqrt{6}} \\ 0 & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}$$

7. Let $A = \begin{bmatrix} 1 & 5 \\ 5 & 1 \end{bmatrix}$.

(a) (5 points) Find the eigenvalues of A.

$$\begin{bmatrix} 1-\lambda & 5 \\ 5 & 1-\lambda \end{bmatrix} = (1-\lambda)(1-\lambda) - 25 = 1 - 2\lambda + \lambda^2 - 25$$

$$= -24 - 2\lambda + \lambda^2$$

$$= (\lambda - 6)(\lambda + 4)$$

$$\lambda = 6, -4$$

(b) (10 points) Find a basis for the eigenspace corresponding to each eigenvalue.

$\lambda = 6$

$$\begin{bmatrix} -5 & 5 \\ 5 & -5 \end{bmatrix} \Rightarrow \left\langle \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\} \right\rangle = \mathcal{E}_A(6)$$

$$\begin{bmatrix} 5 & 5 \\ 5 & 5 \end{bmatrix} \Rightarrow \left\langle \left\{ \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\} \right\rangle = \mathcal{E}_A(-4)$$

(c) (5 points) Orthogonally diagonalize A.

$$\left\langle \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\rangle = -1 + 1 = 0$$

$$\left\| \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\| = 2 \qquad \left\| \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\| = 2$$

$$A = \left(\frac{1}{2} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \right) \begin{bmatrix} 6 & 0 \\ 0 & -4 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

8. Let $A = \begin{bmatrix} 0 & 5 & 2 \\ 4 & 3 & 3 \\ 1 & -2 & 0 \\ 3 & 1 & -1 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 8 \\ 9 \\ 1 \\ 3 \end{bmatrix}$, find the normal equations and the least squares solution.

Normal equation

$$A^T A \hat{x} = A^T b$$

$$\Rightarrow \begin{bmatrix} 26 & 13 & 9 \\ 13 & 39 & 18 \\ 15 & 20 & 12 \end{bmatrix} \hat{x} = \begin{bmatrix} 46 \\ 68 \\ 40 \end{bmatrix}$$

Finding $\tilde{x}$,

$$\left[A^T A \mid A^T b \right] \sim \left[I \;\middle|\; \begin{matrix} \frac{128}{91} \\ 216/91 \\ -50/21 \end{matrix} \right]$$

$$\Rightarrow \hat{x} = \begin{bmatrix} 128/91 \\ 216/91 \\ -50/21 \end{bmatrix}$$