

1-13 odd

4.7 EXERCISES

1. Let $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$ and $\mathcal{C} = \{\mathbf{c}_1, \mathbf{c}_2\}$ be bases for a vector space V , and suppose $\mathbf{b}_1 = 6\mathbf{c}_1 - 2\mathbf{c}_2$ and $\mathbf{b}_2 = 9\mathbf{c}_1 - 4\mathbf{c}_2$.
 - a. Find the change-of-coordinates matrix from \mathcal{B} to \mathcal{C} .
 - b. Find $[\mathbf{x}]_{\mathcal{C}}$ for $\mathbf{x} = -3\mathbf{b}_1 + 2\mathbf{b}_2$. Use part (a).
2. Let $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$ and $\mathcal{C} = \{\mathbf{c}_1, \mathbf{c}_2\}$ be bases for a vector space V , and suppose $\mathbf{b}_1 = -\mathbf{c}_1 + 4\mathbf{c}_2$ and $\mathbf{b}_2 = 5\mathbf{c}_1 - 3\mathbf{c}_2$.
 - a. Find the change-of-coordinates matrix from \mathcal{B} to \mathcal{C} .
 - b. Find $[\mathbf{x}]_{\mathcal{C}}$ for $\mathbf{x} = 5\mathbf{b}_1 + 3\mathbf{b}_2$.

3. Let $\mathcal{U} = \{\mathbf{u}_1, \mathbf{u}_2\}$ and $\mathcal{W} = \{\mathbf{w}_1, \mathbf{w}_2\}$ be bases for V , and let P be a matrix whose columns are $[\mathbf{u}_1]_{\mathcal{W}}$ and $[\mathbf{u}_2]_{\mathcal{W}}$. Which of the following equations is satisfied by P for all \mathbf{x} in V ?

(i) $[\mathbf{x}]_{\mathcal{U}} = P[\mathbf{x}]_{\mathcal{W}}$ (ii) $[\mathbf{x}]_{\mathcal{W}} = P[\mathbf{x}]_{\mathcal{U}}$

4. Let $\mathcal{A} = \{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$ and $\mathcal{D} = \{\mathbf{d}_1, \mathbf{d}_2, \mathbf{d}_3\}$ be bases for V , and let $P = \begin{bmatrix} [\mathbf{d}_1]_{\mathcal{A}} & [\mathbf{d}_2]_{\mathcal{A}} & [\mathbf{d}_3]_{\mathcal{A}} \end{bmatrix}$. Which of the following equations is satisfied by P for all \mathbf{x} in V ?

(i) $[\mathbf{x}]_{\mathcal{A}} = P[\mathbf{x}]_{\mathcal{D}}$ (ii) $[\mathbf{x}]_{\mathcal{D}} = P[\mathbf{x}]_{\mathcal{A}}$

5. Let $\mathcal{A} = \{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$ and $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$ be bases for a vector space V , and suppose $\mathbf{a}_1 = 4\mathbf{b}_1 - \mathbf{b}_2$, $\mathbf{a}_2 = -\mathbf{b}_1 + \mathbf{b}_2 + \mathbf{b}_3$, and $\mathbf{a}_3 = \mathbf{b}_2 - 2\mathbf{b}_3$.

a. Find the change-of-coordinates matrix from \mathcal{A} to \mathcal{B} .

b. Find $[\mathbf{x}]_{\mathcal{B}}$ for $\mathbf{x} = 3\mathbf{a}_1 + 4\mathbf{a}_2 + \mathbf{a}_3$.

6. Let $\mathcal{D} = \{\mathbf{d}_1, \mathbf{d}_2, \mathbf{d}_3\}$ and $\mathcal{F} = \{\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3\}$ be bases for a vector space V , and suppose $\mathbf{f}_1 = 2\mathbf{d}_1 - \mathbf{d}_2 + \mathbf{d}_3$, $\mathbf{f}_2 = 3\mathbf{d}_2 + \mathbf{d}_3$, and $\mathbf{f}_3 = -3\mathbf{d}_1 + 2\mathbf{d}_3$.

a. Find the change-of-coordinates matrix from \mathcal{F} to \mathcal{D} .

b. Find $[\mathbf{x}]_{\mathcal{D}}$ for $\mathbf{x} = \mathbf{f}_1 - 2\mathbf{f}_2 + 2\mathbf{f}_3$.

In Exercises 7–10, let $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$ and $\mathcal{C} = \{\mathbf{c}_1, \mathbf{c}_2\}$ be bases for \mathbb{R}^2 . In each exercise, find the change-of-coordinates matrix from \mathcal{B} to \mathcal{C} and the change-of-coordinates matrix from \mathcal{C} to \mathcal{B} .

7. $\mathbf{b}_1 = \begin{bmatrix} 7 \\ 5 \end{bmatrix}, \mathbf{b}_2 = \begin{bmatrix} -3 \\ -1 \end{bmatrix}, \mathbf{c}_1 = \begin{bmatrix} 1 \\ -5 \end{bmatrix}, \mathbf{c}_2 = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$

8. $\mathbf{b}_1 = \begin{bmatrix} -1 \\ 8 \end{bmatrix}, \mathbf{b}_2 = \begin{bmatrix} 1 \\ -5 \end{bmatrix}, \mathbf{c}_1 = \begin{bmatrix} 1 \\ 4 \end{bmatrix}, \mathbf{c}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

9. $\mathbf{b}_1 = \begin{bmatrix} -6 \\ -1 \end{bmatrix}, \mathbf{b}_2 = \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \mathbf{c}_1 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}, \mathbf{c}_2 = \begin{bmatrix} 6 \\ -2 \end{bmatrix}$

10. $\mathbf{b}_1 = \begin{bmatrix} 7 \\ -2 \end{bmatrix}, \mathbf{b}_2 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}, \mathbf{c}_1 = \begin{bmatrix} 4 \\ 1 \end{bmatrix}, \mathbf{c}_2 = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$

In Exercises 11 and 12, \mathcal{B} and \mathcal{C} are bases for a vector space V . Mark each statement True or False. Justify each answer.

11. a. The columns of the change-of-coordinates matrix ${}_{\mathcal{C}}P_{\mathcal{B}}$ are \mathcal{B} -coordinate vectors of the vectors in \mathcal{C} .

b. If $V = \mathbb{R}^n$ and \mathcal{C} is the standard basis for V , then ${}_{\mathcal{C}}P_{\mathcal{B}}$ is the same as the change-of-coordinates matrix $P_{\mathcal{B}}$ introduced in Section 4.4.

12. a. The columns of ${}_{\mathcal{C}}P_{\mathcal{B}}$ are linearly independent.

b. If $V = \mathbb{R}^2$, $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$, and $\mathcal{C} = \{\mathbf{c}_1, \mathbf{c}_2\}$, then row reduction of $[\mathbf{c}_1 \ \mathbf{c}_2 \ \mathbf{b}_1 \ \mathbf{b}_2]$ to $[I \ P]$ produces a matrix P that satisfies $[\mathbf{x}]_{\mathcal{B}} = P[\mathbf{x}]_{\mathcal{C}}$ for all \mathbf{x} in V .

13. In \mathbb{P}_2 , find the change-of-coordinates matrix from the basis $\mathcal{B} = \{1 - 2t + t^2, 3 - 5t + 4t^2, 2t + 3t^2\}$ to the standard basis $\mathcal{C} = \{1, t, t^2\}$. Then find the \mathcal{B} -coordinate vector for $-1 + 2t$.

14. In \mathbb{P}_2 , find the change-of-coordinates matrix from the basis $\mathcal{B} = \{1 - 3t^2, 2 + t - 5t^2, 1 + 2t\}$ to the standard basis. Then write t^2 as a linear combination of the polynomials in \mathcal{B} .

Exercises 15 and 16 provide a proof of Theorem 15. Fill in a justification for each step.

15. Given \mathbf{v} in V , there exist scalars x_1, \dots, x_n , such that

$$\mathbf{v} = x_1\mathbf{b}_1 + x_2\mathbf{b}_2 + \cdots + x_n\mathbf{b}_n$$

because (a) _____. Apply the coordinate mapping determined by the basis \mathcal{C} , and obtain

$$[\mathbf{v}]_{\mathcal{C}} = x_1[\mathbf{b}_1]_{\mathcal{C}} + x_2[\mathbf{b}_2]_{\mathcal{C}} + \cdots + x_n[\mathbf{b}_n]_{\mathcal{C}}$$

because (b) _____. This equation may be written in the form

$$[\mathbf{v}]_{\mathcal{C}} = \begin{bmatrix} [\mathbf{b}_1]_{\mathcal{C}} & [\mathbf{b}_2]_{\mathcal{C}} & \cdots & [\mathbf{b}_n]_{\mathcal{C}} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \quad (8)$$

by the definition of (c) _____. This shows that the matrix ${}_{\mathcal{C}}P_{\mathcal{B}}$ shown in (5) satisfies $[\mathbf{v}]_{\mathcal{C}} = {}_{\mathcal{C}}P_{\mathcal{B}}[\mathbf{v}]_{\mathcal{B}}$ for each \mathbf{v} in V , because the vector on the right side of (8) is (d) _____.

16. Suppose Q is any matrix such that

$$[\mathbf{v}]_{\mathcal{C}} = Q[\mathbf{v}]_{\mathcal{B}} \quad \text{for each } \mathbf{v} \text{ in } V \quad (9)$$

Set $\mathbf{v} = \mathbf{b}_1$ in (9). Then (9) shows that $[\mathbf{b}_1]_{\mathcal{C}}$ is the first column of Q because (a) _____. Similarly, for $k = 2, \dots, n$, the k th column of Q is (b) _____ because (c) _____. This shows that the matrix ${}_{\mathcal{C}}P_{\mathcal{B}}$ defined by (5) in Theorem 15 is the only matrix that satisfies condition (4).

17. [M] Let $\mathcal{B} = \{\mathbf{x}_0, \dots, \mathbf{x}_6\}$ and $\mathcal{C} = \{\mathbf{y}_0, \dots, \mathbf{y}_6\}$, where \mathbf{x}_k is the function $\cos^k t$ and \mathbf{y}_k is the function $\cos kt$. Exercise 34 in Section 4.5 showed that both \mathcal{B} and \mathcal{C} are bases for the vector space $H = \text{Span}\{\mathbf{x}_0, \dots, \mathbf{x}_6\}$.

- a. Set $P = \begin{bmatrix} [\mathbf{y}_0]_{\mathcal{B}} & \cdots & [\mathbf{y}_6]_{\mathcal{B}} \end{bmatrix}$, and calculate P^{-1} .
b. Explain why the columns of P^{-1} are the \mathcal{C} -coordinate vectors of $\mathbf{x}_0, \dots, \mathbf{x}_6$. Then use these coordinate vectors to write trigonometric identities that express powers of $\cos t$ in terms of the functions in \mathcal{C} .

See the Study Guide.

18. [M] (Calculus required)³ Recall from calculus that integrals such as

$$\int (5 \cos^3 t - 6 \cos^4 t + 5 \cos^5 t - 12 \cos^6 t) dt \quad (10)$$

are tedious to compute. (The usual method is to apply integration by parts repeatedly and use the half-angle formula.) Use the matrix P or P^{-1} from Exercise 17 to transform (10); then compute the integral.

³ The idea for Exercises 17 and 18 and five related exercises in earlier sections came from a paper by Jack W. Rogers, Jr., of Auburn University, presented at a meeting of the International Linear Algebra Society, August 1995. See "Applications of Linear Algebra in Calculus," *American Mathematical Monthly* 104 (1), 1997.