$$\underset{25\times 1}{x(t+1)} = \underset{25\times 25}{A}\ \underset{25\times 1}{x}$$

The optimal decoder

Let $D_0(y^{(0)}, y^{(1)} \dots y^{(T-1)}) = \hat{x}$

D_0 is the optimal decoder if $\hat{x}$ is the solution to following optimization problem

$$\underset{\hat{x}\in\mathbb{R}^n,\ \hat{K}\subset\{1,\dots,p\}}{\text{minimize}}\ |\hat{K}|$$

subject to

$$\text{supp}(y^{(t)} - CA^t\hat{x}) \subset K$$

Let $\hat{x} \longrightarrow [Cx \mid CAx \mid \dots \mid CA^{T-1}x]$

be a linear map called $\phi^{(T)}$

$$\|x\|_p = \left(\sum_{i=1}^{n} |x_i|^p \right)^{1/p}$$

$$\therefore D_{1,r}(y^{(0)}, \dots y^{(T-1)}) = \underset{\hat{x} \in \mathbb{R}^n}{\arg\min} \|Y^T - \phi^T \hat{x}\|_{l_1/l_r}$$

Assume random $\hat{x} = \begin{bmatrix} x_{11} & x_{12} & \dots & x_{1n} \\ \vdots & & & \vdots \\ x_{n1} & - & - & x_{nn} \end{bmatrix}$

$$L = Y^T - \phi^T \hat{x}$$

$C \to p \times n$

$A \to n \times n$

$CA \to p \times n$

$x \to n \times 1$

$CAx \to p \times 1$

$$[Cx \mid CAx \mid \dots \mid CA^{T-1}x]$$

$$\left.\begin{aligned} \text{let } C &= k_0 \\ CA &= k_1 \\ CA^2 &= k_2 \\ &\vdots \\ CA^{T-1} &= k_{T-1} \end{aligned}\right\}$$ All known

where $k_i \rightarrow p \times n$

$$\Rightarrow \left[Cx \mid CAx \mid \dots \mid CA^{T-1}x \right]$$

$$= \left[\underbrace{k_0 x}_{p\times 1} \mid k_1 x \mid \dots \mid k_{T-1} x \right]$$

$$k_0 x = \underbrace{\begin{bmatrix} - & k_{00} & - \\ - & k_{01} & - \\ & \vdots & \\ - & k_{0,p-1} & - \end{bmatrix}}_{k_0} \underbrace{\begin{bmatrix} x_0 \\ \vdots \\ x_{n-1} \end{bmatrix}}_{\hat{x} \rightarrow n \times 1}$$ $k_{0i} \rightarrow 1 \times n$

$$= \begin{bmatrix} k_{00} x \\ k_{01} x \\ \vdots \\ k_{0,p-1} x \end{bmatrix}$$

$$L = \left[y^{(0)} \middle| y^{(1)} \middle| \cdots \middle| y^{(T-1)} \right] - \left[K_0 x \middle| K_1 x \middle| \cdots \middle| K_{T-1} x \right]$$

$$\Rightarrow L = \left[y^{(0)} - K_0 x \middle| y^{(1)} - K_1 x \middle| \cdots \middle| y^{(T-1)} - K_{T-1} x \right]$$

$$y^{(t)} = \begin{bmatrix} y_0^{(t)} \\ y_1^{(t)} \\ \vdots \\ y_{p-1}^{(t)} \end{bmatrix} \qquad K_t = \underbrace{\begin{bmatrix} - k_0^{(t)} - \\ - k_1^{(t)} - \\ \vdots \\ - k_{p-1}^{(t)} - \end{bmatrix}}_{n} \ p \times n \qquad x = \begin{bmatrix} x_0 \\ x_1 \\ \vdots \\ x_n \end{bmatrix}_{n \times 1}$$

$1 \times n$

$$\Rightarrow K^{(t)} x = \begin{bmatrix} k_0^{(t)} x \\ k_1^{(t)} x \\ \vdots \\ k_{p-1}^{(t)} x \end{bmatrix}_{p \times 1}$$

$$\Rightarrow L = \begin{bmatrix} y_0^{(0)} - k_0^{(0)} x & \cdots & y_0^{(T-1)} - k_0^{(T-1)} x \\ y_1^{(0)} - k_1^{(0)} x & \cdots & y_1^{(T-1)} - k_1^{(T-1)} x \\ \vdots & & \vdots \\ y_{p-1}^{(0)} - k_{p-1}^{(0)} x & \cdots & y_{p-1}^{(T-1)} - k_{p-1}^{(T-1)} x \end{bmatrix}_{p \times T}$$

To minimize :

$$L = \sqrt{\left(y_0^{(0)} - k_0^{(0)}x\right)^2 + \left(y_0^{(1)} - k_0^{(1)}x\right)^2 + \dots + \left(y_0^{(T-1)} - k_0^{(T-1)}x\right)^2}$$

$$+$$

$$\sqrt{\left(y_1^{(0)} - k_1^{(0)}x\right)^2 + \left(y_1^{(1)} - k_1^{(1)}x\right)^2 + \dots + \left(y_1^{(T-1)} - k_1^{(T-1)}x\right)^2}$$

$$+$$

$$\vdots$$

$$+$$

$$\sqrt{\left(y_{p-1}^{(0)} - k_{p-1}^{(0)}x\right)^2 + \left(y_{p-1}^{(1)} - k_{p-1}^{(1)}x\right)^2 + \dots + \left(y_{p-1}^{(T-1)} - k_{p-1}^{(T-1)}x\right)^2}$$

To find $\frac{dL}{dx}$. $\frac{dL}{dx} =$

$$\frac{1}{2}\sum_{p=0}^{P-1} \frac{1}{\sqrt{2\left(y_p^{(0)} - k_p^{(0)}x\right)\left(-k_p^{(0)}\right) + \dots + 2\left(y_p^{(T-1)} - k_p^{(T-1)}x\right)\left(-k_p^{(T-1)}\right)}}$$

$$L = \sum_{p=0}^{P-1} \sqrt{\sum_{t=0}^{T-1} \underbrace{\left(y_p^{(t)} - k_p^{(t)} x\right)^2}_{m_p^{(t)}}}$$

$$m_p^{(t)} = \big(\underset{\substack{1 \times 1 \\ (\text{const})}}{y_p^{(t)}} - \underset{\substack{(1 \times n) \\ (\text{const})}}{k_p^{(t)}} \underset{(n \times 1)}{x}\big)^2$$

$$= y_p^{2(t)} - 2 \cdot y_p^{(t)} k_p^{(t)} x + \left(k_p^{(t)} x\right)^2$$

$$\frac{\delta m_p^{(t)}}{\delta x_i} = -2 \cdot y_p^{(t)} k_{pi}^{(t)} + 2\left(k_p^{(t)} x\right) k_{pi}^{(t)}$$

$$\begin{bmatrix} k_{p_0}^{(t)} & k_{p_1}^{(t)} & \cdots & k_{p_{n-1}}^{(t)} \end{bmatrix} \begin{bmatrix} \vdots \\ x_i \\ \vdots \end{bmatrix}$$

$$\left(k_p^{(t)} x\right)^2 = \left(\ldots + \ldots + k_{p_i}^{(t)} x_i + \ldots + \ldots\right)^2$$

$$\begin{bmatrix} \frac{\delta m_p^{(t)}}{\delta x_0} \\ \frac{\delta m_p^{(t)}}{\delta x_1} \\ \vdots \\ \frac{\delta m_p^{(t)}}{\delta x_{n-1}} \end{bmatrix} = \underbrace{\left(2\left(k_p^{(t)} x\right) - 2y_p^{(t)}\right)}_{T \times P \times 1} \begin{bmatrix} k_{p0}^{(t)} \\ k_{p_1}^{(t)} \\ \vdots \\ k_{p,n-1}^{(t)} \end{bmatrix} \quad T \times P \times N$$

Now,

$$L = \sum_{p=0}^{P-1} \sqrt{\sum_{t=0}^{T-1} \underbrace{\left(y_p^{(t)} - k_p^{(t)} x\right)^2}_{m_p^{(t)}}}$$

$$\Rightarrow \quad L = \sum_{p=0}^{P-1} \sqrt{\sum_{t=0}^{T-1} m_p^{(t)}}$$

$$\Rightarrow \frac{\delta L}{\delta x_i} = \sum_{p=0}^{P-1} \frac{\delta}{\delta x_i} \sqrt{\sum_{t=0}^{T-1} m_p^{(t)}}$$

$$\Rightarrow \frac{\delta L}{\delta x_i} = \frac{1}{2} \sum_{p=0}^{P-1} \left(\sum_{t=0}^{T-1} m_p^{(t)}\right)^{-1/2} \sum_{t=0}^{T-1} \frac{\delta m_p^{(t)}}{\delta x_i}$$

$$\Rightarrow \frac{\delta L}{\delta x_i} = \frac{1}{\cancel{2}} \sum_{p=0}^{P-1} \left(\sum_{t=0}^{T-1} m_p(t) \right)^{-1/2} \sum_{t=0}^{T-1} \cancel{2} k_{p_i}^{(t)} \overbrace{\left[\overbrace{k_p^{(t)} x - y_p^{(t)}}^{T \times P} \right]}^{T \times P}$$

$$\begin{bmatrix} \frac{\delta L}{\delta x_0} \\ \frac{\delta L}{\delta x_1} \\ \vdots \\ \frac{\delta L}{\delta x_{n-1}} \end{bmatrix} = \begin{bmatrix} \frac{1}{\cancel{2}} \sum_{p=0}^{P-1} \left(\sum_{t=0}^{T-1} m_p(t) \right)^{-1/2} \sum_{t=0}^{T-1} \cancel{2} k_{p_0}^{(t)} \left[k_p^{(t)} x - y_p^{(t)} \right] \\ \frac{1}{\cancel{2}} \sum_{p=0}^{P-1} \left(\sum_{t=0}^{T-1} m_p(t) \right)^{-1/2} \sum_{t=0}^{T-1} \cancel{2} k_{p_1}^{(t)} \left[k_p^{(t)} x - y_p^{(t)} \right] \\ \vdots \\ \frac{1}{\cancel{2}} \sum_{p=0}^{P-1} \left(\sum_{t=0}^{T-1} m_p(t) \right)^{-1/2} \sum_{t=0}^{T-1} \cancel{2} k_{p_{n-1}}^{(t)} \left[k_p^{(t)} x - y_p^{(t)} \right] \end{bmatrix}$$

$T \times P$