1 Equations of Motion & Forward Dynamics For a Stewart Platform with Viscoelastic Compliant Legs

The equations of motion for the modified Stewart Platform are derived in the following section. First the conventions are established followed by kinematics, and lastly the equations of motion. The equations of motion are derived by the Lagrangian method, separated between the motion of the upper platform / end effector and the individual leg members as subsystems. This separation is done as the derivation for the legs is non intuitive in using the generalized coordinates of the platform - and vice versa. Each subsystem is derived in its own frame and generalized coordinate system after which, with the use of governing constraints, the entire system is reassembled into a system of equations which can be solved for the accelerations necessary.

1.1 Components of the Modified Stewart Platform

The model developed for the modified Stewart platform consists of 13 rigid components and 6 viscoelastic elements that, together, are coupled with 18 joints. All bodies are represented as point masses.

Each leg member has 2 rigid bodies - one for each end of its prismatic actuator - and an elastic element that has an additional point mass within it. This additional point mass represents the mass of the spring damper element. Thus, in combination with the end effector, there are 19 point masses in total.

Each leg member houses 3 joints. A ball joint (3 degrees of freedom) is located at each end of the leg member along with a single degree of freedom provided by the prismatic actuator within the leg. The bottom and top ball joints connect to the Stewart platform base and effector respectively.

In total the system has 19 point masses and 18 joints while the end effector ultimately has 6 degrees of freedom. Hence the system exhibits a high degree of coupling.

1.2 Forces

The two subsystems, the platform and the legs, interact at the top joints where forces arise that affect both groups. These forces, denoted by $\mathbf{F_r}$, are resultant from the loading of the platform upon the leg members and, vice-versa, loads from the legs being exerted upon the platform. The forces are taken relative to the inertial frame.

$$\mathbf{F_r} = \begin{bmatrix} F_{r_{x_1}} & \dots & F_{r_{x_n}} \\ F_{r_{y_1}} & \dots & F_{r_{y_n}} \\ F_{r_{z_1}} & \dots & F_{r_{z_n}} \end{bmatrix}_{3\times 6} \qquad \text{for } n = 6 \tag{1}$$

The force, corresponding to the joint at leg i, is designated by the vector $\boldsymbol{F_{r_i}}$ with constituent elements defined in the inertial frame. Constituent forces are aligned with the inertial axes and taken as positive on the leg member side of the ball joint.

1.3 Kinematics

Conversely to serial actuators, Stewart platform inverse kinematics are simpler than the forward kinematics. Specifically, in forward kinematics the coupling of all six legs and calculation of platform rotations are fairly complex. In other words it is easier to compute the actuator lengths for a given end effector state than vice versa. Hence inverse kinematics are used.

The end effector, the top platform, of a Stewart platform has six degrees of freedom: three translational and three rotational:

- translation along X,
- translation along Y,
- translation along Z,
- a rotation about the X axis; α,
- a rotation about the Y axis; β,
- a rotation about the upper platform's z axis; γ,

in which the rotation order is defined as first α, followed by β and lastly γ.

For every position and orientation of the Stewart platform end effector, there is a unique corresponding length, L_i of each leg member. Correspondingly, in joint space, the Stewart platform can be controlled by actuating the leg members to achieve the proper combinations of L_i. Here two primary coordinate systems will be used: an inertial system XYZ, located at the origin of the actuator base, and a moving coordinate system xyz, at the center of mass in the top platform. The base of the Stewart Platform, defined by the centers of rotation of the bottom joints, is parallel to the $X-Y$ plane at a height of $Z=0$. The inertial frame is most applicable for defining an effector's task and trajectory, thus it will be the frame in which the final equations of motion are expressed.

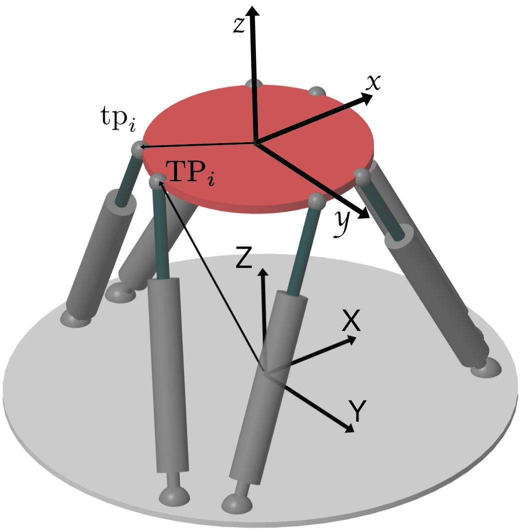

Figure 1: The notation used to model the dynamic actuator member. Compression or extension length of the spring is denoted by δ_{1i}. Displacement of the actuator follower segment (m_3) from the innermost position is denoted by δ_{2i}. The polar angle is θ_i with the reference pole being the X axis (not the Z) and the azimuthul angle is ϕ_i (out of page).

The equations of motion will be applied in a numerical differential equation solver for which the state variables are chosen to be:

$$\boldsymbol{q}_{\text{state}} = \begin{bmatrix} \ddot{X} & \ddot{Y} & \ddot{Z} & \ddot{\alpha} & \ddot{\beta} & \ddot{\gamma} & \boldsymbol{\delta_1} \end{bmatrix} \tag{2}$$

Where $boldsymbol\delta_1$ is the array of displacements in each leg member's spring. These are the variables for which equations of motions must be ultimately obtained at the end of this section.

$$\mathcal{L} = \mathcal{T} - \mathcal{V} \tag{3}$$

1.4 Equations of Motion: Platform

The equations of motion for the end effector of a Stewart platform have been previously studied and modeled for conventional Stewart platforms. The upper platform Lagrange equations used for this work, and reproduced below, are those developed by Liu et al. [1]. The Lagrange equation, Equation 3, is the difference between kinetic (T) and potential (V) energies within a system. This equation is the fundamental component in deriving the equations of motion in the following subsections. Since the addition of viscoelastic elements to the leg members does not affect these equations, no modifications had to be made.

The translational and rotational kinetic energies of the platform are defined by Equations 4 and 5 respectively. Equation 4 is readily expressed in the inertial frame whereas Equation 5 is more easily expressed in the moving frame (designated with mf).

$$\mathcal{T}_{\text{plat (translational)}} = \frac{1}{2} m_{\text{plat}} \left(\dot{X}^2 + \dot{Y}^2 + \dot{Z}^2 \right) \tag{4}$$

$$\mathcal{T}_{\text{plat (rotational)}} = \frac{1}{2} \boldsymbol{\Omega}^T_{\text{plat(mf)}} \boldsymbol{I}_{\text{plat(mf)}} \boldsymbol{\Omega}_{\text{plat(mf)}} \tag{5}$$

Equation 5 is written in matrix notation and $\boldsymbol{\Omega}_{\text{plat(mf)}}$ corresponds to the angular velocity tensor and $\boldsymbol{I}_{\text{plat(mf)}}$ corresponds to the inertia tensor. In the moving frame $\boldsymbol{\Omega}_{\text{plat(mf)}}$ is a diagonal matrix with three rotations and similarly $\boldsymbol{I}_{\text{plat(mf)}}$ is composed of moments of inertia around each local axis:

$$\boldsymbol{I}_{\text{plat(mf)}} = \begin{bmatrix} I_x & 0 & 0 \\ 0 & I_y & 0 \\ 0 & 0 & I_z \end{bmatrix} \tag{6}$$

As the equations of motion are to be functions of Cartesian coordinates in the inertial frame, Equations 5 must be reformulated with rotations α, β, γ. This can be done by substituting $\boldsymbol{\Omega}_{\text{plat(mf)}}$ with Equation 7:

$$\boldsymbol{\Omega}_{\text{plat(mf)}} = R_Z(\gamma)^T R_X(\alpha)^T R_Y(\beta)^T \boldsymbol{\Omega}_{\text{plat(if)}} \tag{7}$$

where $\boldsymbol{\Omega}_{\text{plat(if)}}$ is the angular velocity of the platform in the inertial frame (if), given by Equation 8 in which sin and cosine operators have been replaced with s and c respectively.

$$\boldsymbol{\Omega}_{\text{plat(if)}} = \begin{bmatrix} c\gamma & c\alpha s\gamma & -c\alpha c\gamma s\beta - c\alpha s\alpha s\gamma + c\alpha c\beta s\alpha s\gamma \\ -s\gamma & c\alpha c\gamma & -c\alpha c\gamma s\alpha + c\alpha s\beta s\gamma + c\alpha c\beta s\alpha c\gamma \\ 0 & -s\alpha & s^2\alpha + c^2\alpha c\beta \end{bmatrix} \begin{bmatrix} \dot{\alpha} \\ \dot{\beta} \\ \dot{\gamma} \end{bmatrix} \tag{8}$$

By solving Equation 7 and substituting the result into Equation 5, which can then be combined with Equation 4, the total kinetic energy of the platform is obtained. The total kinetic energy of the platform, expressed in the inertial frame, is given by Equation 9. An advantage of using this formulation is that it retains the platform's moments of inertia, I_x, I_y and I_z, from within the moving frame.

$$T_{\text{plat}} = \frac{1}{2} \begin{bmatrix} \dot{X} \\ \dot{Y} \\ \dot{Z} \\ \dot{\alpha} \\ \dot{\beta} \\ \dot{\gamma} \end{bmatrix}^T \begin{bmatrix} m_{\text{plat}} & 0 & 0 & 0 & 0 & 0 \\ 0 & m_{\text{plat}} & 0 & 0 & 0 & 0 \\ 0 & 0 & m_{\text{plat}} & 0 & 0 & 0 \\ 0 & 0 & 0 & I_x c^2\gamma + I_y s^2\gamma & (I_x - I_y) c\alpha c\gamma s\gamma & 0 \\ 0 & 0 & 0 & (I_x - I_y) c\alpha c\gamma s\gamma & c^2\alpha(I_x s^2\gamma + I_y c^2\gamma) + I_z S^2\alpha & -I_z s^2\alpha \\ 0 & 0 & 0 & 0 & -I_z s^2\alpha & I_z \end{bmatrix} \begin{bmatrix} \dot{X} \\ \dot{Y} \\ \dot{Z} \\ \dot{\alpha} \\ \dot{\beta} \\ \dot{\gamma} \end{bmatrix} \tag{9}$$

The potential energy of the platform is simply:

$$\mathcal{V}_{\text{plat}} = gZm_{\text{plat}} \tag{10}$$

Liu et al. stopped short of deriving the actual Lagrange equations but this derivation has been completed here. The Lagrangian (Equation 11) of the upper platform is given by Equation 11,

$$\begin{aligned} L_{platform} = \frac{1}{2} \left[m_{\text{plat}} \left(\dot{X}^2 + \dot{Y}^2 + \dot{Z}^2 - 2gZ \right) + \dot{\alpha}^2 c\gamma + \dot{\alpha}\dot{\beta}\left(-s\gamma + c\alpha s\gamma\right) + \dot{\beta}^2 c\alpha c\gamma \right] \\ + \frac{1}{2} \left[\dot{\gamma}\dot{\beta}\left[-s\alpha + c\alpha\left(-c\gamma s\alpha + s\beta s\gamma + c\beta s\alpha c\gamma\right)\right] + \dot{\gamma}\dot{\alpha} c\alpha \left(-c\gamma s\beta + s\alpha s\gamma + c\beta s\alpha s\gamma\right) \right] \\ + \frac{\dot{\gamma}^2}{2}\left(s^2\alpha + c^2\alpha c\beta\right) \end{aligned} \tag{11}$$

The Lagrange relation yielding the equations of motion for an undamped system is given by Equation 12 where q_i is a generalized coordinate. For the upper platform, q_i is an element of $q_{\text{plat}} = [X, Y, Z, \alpha, \beta, \gamma]$.

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right) - \left(\frac{\partial \mathcal{L}}{\partial q_i}\right) = \tau_{q_i} \tag{12}$$

Evaluating Equation 12 for q_{plat} yields the components in Equations 13 to 23.

$$\frac{\partial \mathcal{L}}{\partial X} = \frac{\partial \mathcal{L}}{\partial Y} = 0 \tag{13}$$

$$\frac{\partial \mathcal{L}}{\partial Z} = -m_{\text{plat}}\ g \tag{14}$$

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{X}} = m_{\text{plat}}\ \ddot{X} \tag{15}$$

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{Y}} = m_{\text{plat}}\ \ddot{Y} \tag{16}$$

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{Z}} = m_{\text{plat}}\ \ddot{Z} \tag{17}$$

$$\begin{aligned}\frac{\partial \mathcal{L}}{\partial \alpha} = -\dot{\beta}\ \sin(\alpha)\ \Bigg[&\frac{I_x\,\dot{\alpha}\ \sin(2\,\gamma)}{2} - \frac{I_y\,\dot{\alpha}\ \sin(2\,\gamma)}{2} - I_z\,\dot{\beta}\ \cos(\alpha) + 2\,I_z\,\dot{\gamma}\ \cos(\alpha)\\ &- I_x\,\dot{\beta}\ \cos(\alpha)\ \left(\cos(\gamma)^2 - 1\right) + I_y\,\dot{\beta}\ \cos(\alpha)\ \cos(\gamma)^2\Bigg]\end{aligned} \tag{18}$$

$$\begin{aligned}\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\alpha}} = {}&\ddot{\alpha}\ \left(I_x - I_x\sin(\gamma)^2 + I_y\sin(\gamma)^2\right)\\ &- \dot{\gamma}\ (I_x - I_y)\ \left(-2\,\dot{\beta}\ \cos(\alpha)\ \cos(\gamma)^2 + \dot{\beta}\ \cos(\alpha) + \dot{\alpha}\ \sin(2\,\gamma)\right)\\ &+ \ddot{\beta}\ \cos(\alpha)\ \cos(\gamma)\ \sin(\gamma)\ (I_x - I_y)\\ &- \dot{\alpha}\,\dot{\beta}\ \cos(\gamma)\ \sin(\alpha)\ \sin(\gamma)\ (I_x - I_y)\end{aligned} \tag{19}$$

$$\frac{\partial \mathcal{L}}{\partial \beta} = 0 \tag{20}$$

$$\begin{aligned}\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\beta}} = {}&\ddot{\beta}\ \left(I_y\cos(\alpha)^2\cos(\gamma)^2 + I_x\cos(\alpha)^2\sin(\gamma)^2 + I_z\sin(\alpha)^2\right)\\ &- \dot{\alpha}\ \sin(\alpha)\ \Bigg(2\,I_y\,\dot{\beta}\ \cos(\alpha)\ \cos(\gamma)^2 + 2\,I_x\,\dot{\beta}\ \cos(\alpha)\ \sin(\gamma)^2 + \frac{I_x\,\dot{\alpha}\ \sin(2\,\gamma)}{2}\\ &\qquad - \frac{I_y\,\dot{\alpha}\ \sin(2\,\gamma)}{2} - 2\,I_z\,\dot{\beta}\ \cos(\alpha) + 2\,I_z\,\dot{\gamma}\ \cos(\alpha)\Bigg)\\ &- I_z\,\ddot{\gamma}\sin(\alpha)^2 + \dot{\gamma}\ \cos(\alpha)\ (I_x - I_y)\ \left(2\,\dot{\alpha}\cos(\gamma)^2 + 2\,\dot{\beta}\ \cos(\alpha)\ \sin(\gamma)\ \cos(\gamma) - \dot{\alpha}\right)\\ &+ \ddot{\alpha}\ \cos(\alpha)\ \cos(\gamma)\ \sin(\gamma)\ (I_x - I_y)\end{aligned} \tag{21}$$

$$\frac{\partial \mathcal{L}}{\partial \gamma} = \frac{1}{2}(I_x - I_y)\ \left(-1\ \sin(2\,\gamma)\ \dot{\alpha}^2 + 2\ \cos(2\,\gamma)\ \dot{\alpha}\,\dot{\beta}\ \cos(\alpha) + \sin(2\,\gamma)\ \dot{\beta}^2\cos(\alpha)^2\right) \tag{22}$$

$$\begin{aligned}\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\gamma}} &= \frac{d}{dt}\left[I_z\ \left(\dot{\gamma} - \dot{\beta}\sin(\alpha)^2\right)\right]\\ &= -I_z\ \left(\ddot{\beta}\sin(\alpha)^2 - \ddot{\gamma} + \dot{\alpha}\,\dot{\beta}\ \sin(2\,\alpha)\right)\end{aligned} \tag{23}$$

The right hand side term of Equation 12 corresponds to external loadings upon the platform subsystem. In this case there are two components that can be considered as external influences upon the current platform formulation. The first component consists of the internal forces at the ball joints of the platform. The resultant right hand side components, $\boldsymbol{\tau}_{\text{act}}$ are given by Equation 24.

$$\begin{bmatrix}\tau_{\text{act}_X}\\ \tau_{\text{act}_Y}\\ \tau_{\text{act}_Z}\\ \tau_{\text{act}_\alpha}\\ \tau_{\text{act}_\beta}\\ \tau_{\text{act}_\gamma}\end{bmatrix} = \mathbf{J}_{\text{TP}_i}^T\mathbf{F_r} \tag{24}$$

The second component, $\boldsymbol{\tau}_{\text{ext}}$, consists of the external forces $\mathbf{F}_{\text{load}}$ and moments $\mathbf{M}_{\text{load}}$ applied upon the platform. Here these forces and moments are defined in the inertial frame. These respective contributions are calculated via Equations 25 and 27.

$$\begin{bmatrix} \tau_{\text{ext}_X} \\ \tau_{\text{ext}_Y} \\ \tau_{\text{ext}_Z} \end{bmatrix} = \begin{bmatrix} F_{\text{load}_X} \\ F_{\text{load}_Y} \\ F_{\text{load}_Z} \end{bmatrix} \tag{25}$$

Equation 26 is an intermediate step that is necessary to find M_{load_z}, the contribution of moments applied about the inertial axis to the moment about the z axis in the moving frame, which corresponds to γ rotations.

$$\begin{bmatrix} M_{\text{load}_x} \\ M_{\text{load}_y} \\ M_{\text{load}_z} \end{bmatrix} = \begin{bmatrix} R_Z(\gamma)^T \, R_X(\alpha)^T \, R_Y(\beta)^T \end{bmatrix} \begin{bmatrix} M_{\text{load}_X} \\ M_{\text{load}_Y} \\ M_{\text{load}_Z} \end{bmatrix} \tag{26}$$

$$\begin{bmatrix} \tau_{\text{ext}_\alpha} \\ \tau_{\text{ext}_\beta} \\ \tau_{\text{ext}_\gamma} \end{bmatrix} = \begin{bmatrix} M_{\text{load}_X} \\ M_{\text{load}_Y} \\ M_{\text{load}_z} \end{bmatrix} \tag{27}$$

Summing these components yields the net right hand side term for each term in q_{plat}.

$$\begin{bmatrix} \tau_X \\ \tau_Y \\ \tau_Z \\ \tau_\alpha \\ \tau_\beta \\ \tau_\gamma \end{bmatrix} = \begin{bmatrix} \tau_{\text{act}_X} \\ \tau_{\text{act}_Y} \\ \tau_{\text{act}_Z} \\ \tau_{\text{act}_\alpha} \\ \tau_{\text{act}_\beta} \\ \tau_{\text{act}_\gamma} \end{bmatrix} + \begin{bmatrix} \tau_{\text{ext}_X} \\ \tau_{\text{ext}_Y} \\ \tau_{\text{ext}_Z} \\ \tau_{\text{ext}_\alpha} \\ \tau_{\text{ext}_\beta} \\ \tau_{\text{ext}_\gamma} \end{bmatrix} \tag{28}$$

1.5 Equations of Motion: Leg Members

In a conventional Stewart platform the kinetic energy is apparent in the extension or contraction of the member and its rotational movement. The movement and configuration of a leg member is conveniently expressed in spherical coordinates with the origin at the bottom joint center as shown in Figure 2. Extension and contraction of the member is a radial translation, denoted here as δ. The change in length of the spring is further denoted as δ_1 and the expansion or contraction of the actuator from its nominal length is δ_2. The rotational energy is analogous to that of a compound pendulum. Previous works have omitted rotation around the longitudinal axis of the member [1]. In this work the energy contribution of longitudinal rotation is also assumed insignificant and two axis of rotation remain: θ_i and ϕ_i.

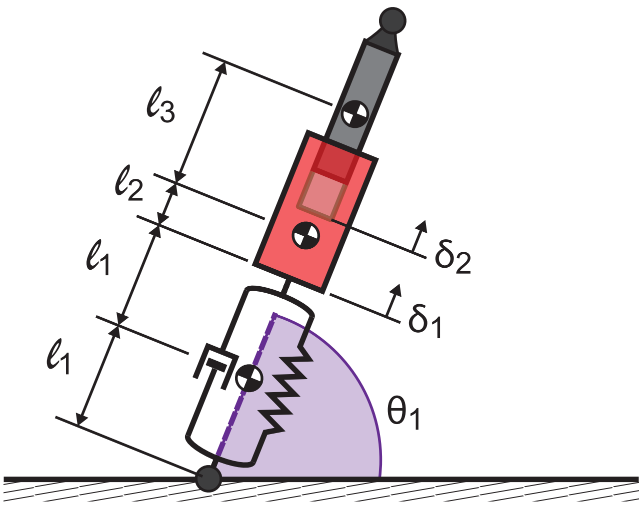

Figure 2: The notation used to model the dynamic actuator member. Compression or extension length of the spring is denoted by δ_{1i}. Displacement of the actuator follower segment (m_3) from the innermost position is denoted by δ_{2i}. The polar angle is θ_i with the reference pole being the X axis (not the Z) and the azimuthul angle is ϕ_i (out of page).

The Lagrange formulation for the dynamics of a conventional Stewart platform has previously been presented by Liu et al. [1]. Their approach calculates an equivalent center of mass between the two segments of the actuator as it elongates and contracts. This method has subsequently appeared in many Lagrange based dynamic analyses of Stewart platforms. Here, an alternative method is proposed that takes into account independent masses between the spring and two segments of the actuator. The reasons for doing so are rooted in the addition of a spring to the modified leg member. In the case of a heavy spring (with additional mass possible due to a housing assembly) and dashpot, there is an additional mass that, upon compression and extension of the spring, can translate independently of the actuator expanding or contracting. Since the spring is compliant, both sides of the actuator can now move relative to the bottom ball joint (in the case that the actuator is connected at the top platform and spring).

The Lagrangian formulation in Equation 29 requires solutions for generalized coordinates:
$q_{\text{leg}} = [\delta_{1_1}, \delta_{1_2}, \delta_{1_3}, \delta_{1_4}, \delta_{1_5}, \delta_{1_6}, \delta_{2_1}, \delta_{2_2}, \delta_{2_3}, \delta_{2_4}, \delta_{2_5}, \delta_{2_6}, \theta_1, \theta_2, \theta_3, \theta_4, \theta_5, \theta_6, \phi_1, \phi_2, \phi_3, \phi_4, \phi_5, \phi_6]$.

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_{\text{leg}}}\right) - \left(\frac{\partial \mathcal{L}}{\partial q_{\text{leg}}}\right) + \frac{\partial}{\partial \dot{q}_{\text{leg}}}\left(\frac{B}{2}\dot{q}_{\text{leg}}^2\right) = \tau \tag{29}$$

The Lagrangian equation of motion formulation for a leg member is given by Equation 29. With the previously noted variables, the Lagrangian of the ith leg member is given by Equation 30. The third term on the right of Equation 29 is the Rayleigh dissipation function, corresponding to a dissipative energy.

$$\begin{aligned}
\mathcal{L}_{leg_i} = {} & \frac{m_1}{2}\left[\left(\frac{\dot{\delta}_{1_i}}{2}\right)^2 + \dot{\theta_i}^2\left(l_{1i} + \frac{\delta_{1_i}}{2}\right)^2 + \dot{\phi_i}^2\cos^2\theta_i\left(l_{1i} + \frac{\delta_{1_i}}{2}\right)^2\right] \\
& + \frac{m_2}{2}\left[\dot{\delta_{1_i}}^2 + \dot{\theta}_i^2\left(2l_{1i} + \delta_{1_i} + l_2\right)^2 + \dot{\phi}_i^2\cos^2\theta_i\left(2l_{1i} + \delta_{1_i} + l_2\right)^2\right] \\
& + \frac{m_3}{2}\left[\left(\dot{\delta_{1_i}} + \dot{\delta_{2_i}}\right)^2 + \dot{\theta}_i^2\left(2l_{1i} + \delta_{1_i} + l_2 + l_3 + \delta_{2_i}\right)^2 + \dot{\phi}_i^2\cos^2\theta_i\left(2l_{1i} + \delta_{1_i} + l_2 + l_3 + \delta_{2_i}\right)^2\right] \\
& - g\sin\theta_i\left[m_1\left(l_{1i} + \frac{\delta_{1_i}}{2}\right) + m_2\left(2l_{1i} + \delta_{1_i} + l_2\right) + m_3\left(2l_{1i} + \delta_{1_i} + l_2 + l_3 + \delta_{2_i}\right)\right] - \frac{1}{2}k_i\delta_{1_i}^2
\end{aligned} \tag{30}$$

Equation 29 requires solutions for $q = [\delta_{1i}, \delta_{2i}, \theta_i, \phi_i]$. These components are derived in Equations 31 to 44.The partial derivatives and time derivatives of Equation 29 are given by Equations 31 through 44. The following sections detail the Lagrangian components of Equation 29 in terms of q_{leg}.

1.6 Lagrangian components for $q_i = \delta_{1i}$

The first term on the left hand side of equation 29 evaluated for $q_i = \delta_{1i}$ yields Equation 31.

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial \delta_{1i}} = {} & \frac{m_1}{2}\left(l_{1i} + \frac{\delta_{1i}}{2}\right)\left(\dot{\theta_i}^2 + \dot{\phi_i}^2\cos^2\theta_i\right) + m_2\left(2l_{1i} + \delta_{1i} + l_2\right)\left(\dot{\theta_i}^2 + \dot{\phi_i}^2\cos^2\theta_i\right) \\
& + m_3\left(2l_{1i} + \delta_{1i} + l_2 + l_3 + \delta_{2i}\right)\left(\dot{\theta_i}^2 + \dot{\phi_i}^2\cos^2\theta_i\right) - g\sin\theta_i\left(\frac{m_1}{2} + m_2 + m_3\right) - k_i\delta_{1i}
\end{aligned} \tag{31}$$

As expected Equation 31 is a radial equation in a rotating reference frame with centrifugal force terms. Continuing to the second left hand side term of the Lagrangian equation yields Equation 32.

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\delta_{1i}}}\right) = \frac{m_1}{4}\ddot{\delta_{1i}} + m_2\ddot{\delta_{1i}} + m_3\ddot{\delta_{1i}} + m_3\ddot{\delta_{2i}} \tag{32}$$

In the case of the modified Stewart platform, dissipation occurs in the damper and is proportional to $\dot{\delta_{1i}}$. Thus the third term on the left hand side of Equation 29 becomes:

$$\frac{\partial \mathcal{L}}{\partial \dot{\delta}_{1_i}}\left(\frac{B_i}{2}\dot{\delta}_{1_i}^2\right) = B_i\dot{\delta_{1_i}} \tag{33}$$

The right hand side of the Lagrangian equation pertaining to $q_i = \delta_{1i}$ is:

$$\tau_{\delta_{1i}} = -F_{\text{act}_i} \tag{34}$$

where F_{act_i} is the force exerted by the actuator, in this case upon the lower actuator piece.

1.7 Lagrangian components for $q_i = \delta_{2i}$

The equations for $q_i = \delta_{2i}$ are similar to those of δ_{1i}:

$$\frac{\partial \mathcal{L}}{\partial \delta_{2i}} = m_3 \left(2l_{1i} + \delta_{1i} + l_2 + l_3 + \delta_{2i}\right) \left(\dot{\theta_i}^2 + \dot{\phi_i}^2 \cos^2 \theta_i\right) - m_3 g \sin \theta_i \tag{35}$$

$$\frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{\delta_{2i}}}\right) = m_3 \ddot{\delta_{1i}} + m_3 \ddot{\delta_{2i}} \tag{36}$$

The right hand side component for δ_{2i} differs from that of δ_{1i} because it receives additional loading from the platform-actuator interface via the connecting joint. The right hand side of the Lagrangian equation pertaining to $q_i = \delta_{2i}$ is therefore:

$$\tau_{\delta_{2i}} = F_{\text{act}_i} + \mathbf{F}_{\mathbf{r}_i} \cdot \frac{\mathbf{L}_i}{|\mathbf{L}_i|} \tag{37}$$

where the second term on the right is the projection of the top joint's reaction forces upon the leg member's length. Knowing a leg member's top position and bottom position, the leg vector $\mathbf{L}_i$ is given via Equation 38.

$$\mathbf{L}_i = \begin{bmatrix} \text{TP}_{i_X} - \text{BP}_{i_X} \\ \text{TP}_{i_Y} - \text{BP}_{i_Y} \\ \text{TP}_{i_Z} - \text{BP}_{i_Z} \end{bmatrix} \tag{38}$$

1.8 Lagrangian components for $q_i = \theta_i$

The Lagrangian components for $q_i = \theta_i$ are given by Equations 39 to 42.

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial \theta_i} = &-\dot{\phi_i}^2 \sin \theta_i \cos \theta_i \left[m_1 \left(l_{1i} + \frac{\delta_{1i}}{2}\right)^2 + m_2 \left(2l_{1i} + \delta_{1i} + l_2\right)^2 + m_3 \left(2l_{1i} + \delta_{1i} + l_2 + l_3 + \delta_{2i}\right)^2\right] \\ &- g \cos \theta_i \left[m_1 \left(l_{1i} + \frac{\delta_{1i}}{2}\right) + m_2 \left(2l_{1i} + \delta_{1i} + l_2\right) + m_3 \left(2l_{1i} + \delta_{1i} + l_2 + l_3 + \delta_{2i}\right)\right] \end{aligned} \tag{39}$$

$$\begin{aligned} \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{\theta}_i} = \ddot{\theta}_i &\left[m_1 \left(l_{1i} + \frac{\delta_{1i}}{2}\right)^2 + m_2 \left(2l_{1i} + \delta_{1i} + l_2\right)^2 + m_3 \left(2l_{1i} + \delta_{1i} + l_2 + l_3 + \delta_{2i}\right)^2\right] \\ &+ 2\dot{\theta}_i \dot{\delta_{1i}} \left[\frac{m_1}{2} \left(l_{1i} + \frac{\delta_{1i}}{2}\right) + m_2 \left(2l_{1i} + \delta_{1i} + l_2\right) + m_3 \left(2l_{1i} + \delta_{1i} + l_2 + l_3 + \delta_{2i}\right)\right] \\ &+ 2\dot{\theta}_i \dot{\delta_{2i}} \left[m_3 \left(2l_{1i} + \delta_{1i} + l_2 + l_3 + \delta_{2i}\right)\right] \end{aligned} \tag{40}$$

As expected, in Equation 40 Coriolis forces become apparent (second and third terms on the right hand side). For the external loading of $q_i = \theta_i$ the top joint reaction forces induce a moment M_{θ_i}.

$$\tau_{\theta_i} = M_{\theta_i} \tag{41}$$

M_{θ_i} is equal to the moment generated by the top joint reaction forces, $\mathbf{F}_{\mathbf{r}_i}$, about the Y' axis which is resultant from a rotation of ϕ degrees about the Z axis of the inertial frame. The moments generated by the reaction forces in the rotated frame are given by $M_{X'Y'Z'_i}$ and can be calculated via Equation 42.

$$\mathbf{M}_{X'Y'Z'_i} = \left(R_Z^T(\phi)\, \mathbf{F}_{\mathbf{r}_i}\right) \times \left(R_Z^T(\phi)\, \mathbf{L}_i\right) \tag{42}$$

1.9 Lagrangian components for $q_i = \phi_i$

The Lagrangian components for $q_i = \phi_i$ are given by Equations 43 to 45.

$$\frac{\partial \mathcal{L}}{\partial \phi_i} = 0 \tag{43}$$

Equation 43 dictates that the angular moment of the ϕ_i rotation is preserved, as expected.

$$\begin{aligned}\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\phi_i}} = \ddot{\phi_i}\cos^2\theta_i &\left[m_1\left(l_{1i}+\frac{\delta_{1i}}{2}\right)^2 + m_2\left(2l_{1i}+\delta_{1i}+l_2\right)^2 + m_3\left(2l_{1i}+\delta_{1i}+l_2+l_3+\delta_{2i}\right)^2\right] \\ &- 2\dot{\phi_i}\dot{\theta_i}\cos\theta_i\sin\theta_i\left[m_1\left(l_{1i}+\frac{\delta_{1i}}{2}\right)^2 + m_2\left(2l_{1i}+\delta_{1i}+l_2\right)^2 + m_3\left(2l_{1i}+\delta_{1i}+l_2+l_3+\delta_{2i}\right)^2\right] \\ &+ 2\dot{\phi_i}\dot{\delta_{1i}}\cos^2\theta_i\left[\frac{m_1}{2}\left(l_{1i}+\frac{\delta_{1i}}{2}\right) + m_2\left(2l_{1i}+\delta_{1i}+l_2\right) + m_3\left(2l_{1i}+\delta_{1i}+l_2+l_3+\delta_{2i}\right)\right] \\ &+ 2\dot{\phi_i}\dot{\delta_{2i}}\cos^2\theta_i\left(m_3\left(2l_{1i}+\delta_{1i}+l_2+l_3+\delta_{2i}\right)\right]\end{aligned} \tag{44}$$

For the external loading of $q_i = \phi_i$ the top joint reaction forces induce a moment M_{ϕ_i}. This moment is equal to the moment generated by the reaction forces about the inertial frame aligned Z axis at the leg's bottom joint. The right hand side element for ϕ_i can be found via Equations 45 and 46.

$$\tau_{\phi_i} = M_{\phi_i} \tag{45}$$

$$M_{\phi_i} = L_{X_i}F_{r_Y} - L_{Y_i}F_{r_Y} \tag{46}$$

1.10 Combining the Platform and Legs

In previous sections the dynamic equations of the legs and the platform were derived separately from one another. This is done as the leg equations are not intuitive in terms of the platform's spatial coordinates. However, the legs and platform are coupled and thus the respective equations must be modified in a manner that reflects this linkage. For this, the generalized coordinates of the legs (δ_{1i}, δ_{2i}, ϕ_i, θ_i) are rewritten in terms of the platform's generalized coordinates: X, Y, Z, α, β, γ. This can be done by relating the length and orientation of each actuator, dictated by the position and orientation of the top platform, with the constituent components/coordinates.

- The $[X_{TP_i}\ Y_{TP_i}\ Z_{TP_i}]$ coordinates of each actuator connection / top point (TP) on the platform can be found by a transformation of the platform's c.o.g. generalized coordinates (X, Y, Z, α, β, γ). Similarly, the velocities and accelerations of the top points can be found.
- Knowing the positions and velocities of the top points, the angles ϕ_i and θ_i and their velocity derivatives can be rewritten in terms of the platform's generalized coordinates (X, Y, Z, α, β, γ) and their derivatives.
- A similar process can be applied for the leg elongation components δ_{1i} and δ_{2i} and their derivatives; rewritten in terms of platform's generalized coordinates (X, Y, Z, α, β, γ) and their derivatives.

Thus, all terms in the Lagrangian equations for the actuators/legs are rewritten in terms of X, Y, Z, α, β, γ.

The interaction between the legs and the platform can be expressed by the addition of forces, those present at the spherical joints between the legs and platform, to the existing equations. At each spherical joint a force vector is present along with it's reacting reciprocal. Decomposing each of these vectors yields an additional 18 unknowns for the system equations, expressed on the right hand side of the Lagrangian equations for the legs and platform. Ultimately, after considerable amounts of formulation, a linear system of equations can be solved for 24 unknowns (48 if leg general coordinates are solved for).

The constraints between the top platform and leg members occur at the ball joint connections. For each joint the movement of the platform at that given point and the movement of the top of the corresponding actuator must be equivalent in order for the joint is to stay together. The position of each top joints is designated by a vector $\mathbf{TP}_i = [\mathrm{TP}_{i_X}\ \mathrm{TP}_{i_Y}\ \mathrm{TP}_{i_Z}]^T$ whose constituents are defined in the inertial frame. In order to formulate the top joint constraints, their position and velocity must be defined in the inertial frame by the variables X, Y, Z, α, β, γ. This position can be found as a transformation of the origin of the top platform (i.e. the moving frame) which itself is defined by those variables. Let the joint positions in the top platform be defined, in the moving frame, by in Equation 47 with matrix notation $\mathbf{tp}_{(\mathrm{mf})}$:

$$\mathbf{tp}_{(\mathrm{mf})} = \begin{bmatrix} \mathrm{tp}1_x & \mathrm{tp}2_x & \mathrm{tp}3_x & \mathrm{tp}4_x & \mathrm{tp}5_x & \mathrm{tp}6_x \\ \mathrm{tp}1_y & \mathrm{tp}2_y & \mathrm{tp}3_y & \mathrm{tp}4_y & \mathrm{tp}5_y & \mathrm{tp}6_y \\ \mathrm{tp}1_z & \mathrm{tp}2_z & \mathrm{tp}3_z & \mathrm{tp}4_z & \mathrm{tp}5_z & \mathrm{tp}6_z \\ 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} \tag{47}$$

where i refers to the leg member / joint and x, y, z is the position in the moving frame. It follows that the position of the joints in the inertial frame is then given by Equation 48:

$$\mathbf{TP} = T_{\mathrm{TP}} \mathbf{tp}_{(\mathrm{mf})} \tag{48}$$

where the first three columns of T_{TP} are equal to the product of the rotations $R_\beta R_\alpha R_\gamma$. $R_\beta R_\alpha$ are extrinsic rotations and R_γ is an intrinsic rotation. The last column corresponds to the position of the top platform's origin in the inertial frame.

$$T_{\mathrm{TP}} = \begin{bmatrix} \cos\beta\,\cos\gamma + \sin\alpha\,\sin\beta\,\sin\gamma & \cos\gamma\,\sin\alpha\,\sin\beta - \cos\beta\,\sin\gamma & \cos\alpha\,\sin\beta & X \\ \cos\alpha\,\sin\gamma & \cos\alpha\,\cos\gamma & -\sin\alpha & Y \\ \cos\beta\,\sin\alpha\,\sin\gamma - \cos\gamma\,\sin\beta & \sin\beta\,\sin\gamma + \cos\beta\,\cos\gamma\,\sin\alpha & \cos\alpha\,\cos\beta & Z \end{bmatrix} \tag{49}$$

Next the velocity of each joint position must be determined in the inertial frame. This is given by the Jacobian of TP.

$$\dot{\mathbf{TP}} = \mathbf{J}_{\mathrm{TP}} \tag{50}$$

Considering a subsection of $\mathbf{J}_{\mathrm{TP}}$, let $\mathbf{J}_{\mathrm{TP}_i}$ be the Jacobian of a single column of TP. Then the Jacobian of $\mathbf{J}_{\mathrm{TP}_i}$ is given by Equation 51.

$$\mathbf{J}_{\mathrm{TP}_i}(X, Y, Z, \alpha, \beta, \gamma) = \begin{bmatrix} 1 & 0 & 0 & a_1 & a_2 & a_3 \\ 0 & 1 & 0 & b_1 & 0 & b_3 \\ 0 & 0 & 1 & c_1 & c_2 & c_3 \end{bmatrix} \tag{51}$$

where:

$$\begin{aligned}
a_1 &= \mathrm{TP}_{i_Y} \cos\alpha \cos\gamma \sin\beta - \mathrm{TP}_{i_Z} \sin\alpha \sin\beta + \mathrm{TP}_{i_X} \cos\alpha \sin\beta \sin\gamma \\
a_2 &= \mathrm{TP}_{i_Y} (\sin\beta \sin\gamma + \cos\beta \cos\gamma \sin\alpha) - \mathrm{TP}_{i_X} (\cos\gamma \sin\beta - \cos\beta \sin\alpha \sin\gamma) + \mathrm{TP}_{i_Z} \cos\alpha \cos\beta) \\
a_3 &= -\mathrm{TP}_{i_X} (\cos\beta \sin\gamma - \cos\gamma \sin\alpha \sin\beta) - \mathrm{TP}_{i_Y} (\cos\beta \cos\gamma + \sin\alpha \sin\beta \sin\gamma) \\
b_1 &= -\mathrm{TP}_{i_Z} \cos\alpha - \mathrm{TP}_{i_Y} \cos\gamma \sin\alpha - \mathrm{TP}_{i_X} \sin\alpha \sin\gamma \\
b_3 &= \mathrm{TP}_{i_X} \cos\alpha \cos\gamma - \mathrm{TP}_{i_Y} \cos\alpha \sin\gamma \\
c_1 &= \mathrm{TP}_{i_Y} \cos\alpha \cos\beta \cos\gamma - \mathrm{TP}_{i_Z} \cos\beta \sin\alpha + \mathrm{TP}_{i_X} \cos\alpha \cos\beta \sin\gamma \\
c_2 &= \mathrm{TP}_{i_Y} (\cos\beta \sin\gamma - \cos\gamma \sin\alpha \sin\beta) - \mathrm{TP}_{i_X} (\cos\beta \cos\gamma + \sin\alpha \sin\beta \sin\gamma) - \mathrm{TP}_{i_Z} \cos\alpha \sin\beta \\
c_3 &= \mathrm{TP}_{i_X} (\sin\beta \sin\gamma + \cos\beta \cos\gamma \sin\alpha) + \mathrm{TP}_{i_Y} (\cos\gamma \sin\beta - \cos\beta \sin\alpha \sin\gamma)
\end{aligned}$$

Subsequently the velocity of the ith top joint is given by:

$$\dot{\mathbf{TP}}_i = \begin{bmatrix} \dot{\mathrm{TP}}_{i_X} \\ \dot{\mathrm{TP}}_{i_Y} \\ \dot{\mathrm{TP}}_{i_Z} \end{bmatrix} = \mathbf{J}_{\mathrm{TP}_i} \begin{bmatrix} \dot{X} \\ \dot{Y} \\ \dot{Z} \\ \dot{\alpha} \\ \dot{\beta} \\ \dot{\gamma} \end{bmatrix} \tag{52}$$

Repeating the procedure, the acceleration of the each top joint position can be found by finding the Jacobian of the $\dot{\mathbf{TP}}_i$.

$$\mathbf{J}_{\dot{\mathrm{TP}}_i}\left(X, Y, Z, \alpha, \beta, \gamma, \dot{X}, \dot{Y}, \dot{Z}, \dot{\alpha}, \dot{\beta}, \dot{\gamma}\right) = \begin{bmatrix} 0 & 0 & 0 & a_1 & a_2 & a_3 & 1 & 0 & 0 & a_7 & a_8 & a_9 \\ 0 & 0 & 0 & b_1 & 0 & b_3 & 0 & 1 & 0 & b_7 & 0 & b_9 \\ 0 & 0 & 0 & c_1 & c_2 & c_3 & 0 & 0 & 1 & c_7 & c_8 & c_9 \end{bmatrix} \tag{53}$$

where:

$$
\begin{aligned}
a_1 &= \dot{\gamma}\ (\mathrm{TP}_{i_X}\ \cos\alpha\,\cos\gamma\,\sin\beta - \mathrm{TP}_{i_Y}\ \cos\alpha\,\sin\beta\,\sin\gamma) + \\
&\quad \dot{\beta}\ (\mathrm{TP}_{i_Y}\ \cos\alpha\,\cos\beta\,\cos\gamma - \mathrm{TP}_{i_Z}\ \cos\beta\,\sin\alpha + \mathrm{TP}_{i_X}\ \cos\alpha\,\cos\beta\,\sin\gamma) - \\
&\quad \dot{\alpha}\ (\mathrm{TP}_{i_Z}\ \cos\alpha\,\sin\beta + \mathrm{TP}_{i_Y}\ \cos\gamma\,\sin\alpha\,\sin\beta + \mathrm{TP}_{i_X}\ \sin\alpha\,\sin\beta\,\sin\gamma) \\
b_1 &= -\dot{\alpha}\ (\mathrm{TP}_{i_Y}\ \cos\alpha\,\cos\gamma - \mathrm{TP}_{i_Z}\ \sin\alpha + \mathrm{TP}_{i_X}\ \cos\alpha\,\sin\gamma) - \dot{\gamma}\ (\mathrm{TP}_{i_X}\ \cos\gamma\,\sin\alpha - \mathrm{TP}_{i_Y}\ \sin\alpha\,\sin\gamma) \\
c_1 &= \dot{\gamma}\ (\mathrm{TP}_{i_X}\ \cos\alpha\,\cos\beta\,\cos\gamma - \mathrm{TP}_{i_Y}\ \cos\alpha\,\cos\beta\,\sin\gamma) - \\
&\quad \dot{\alpha}\ (\mathrm{TP}_{i_Z}\ \cos\alpha\,\cos\beta + \mathrm{TP}_{i_Y}\ \cos\beta\,\cos\gamma\,\sin\alpha + \mathrm{TP}_{i_X}\ \cos\beta\,\sin\alpha\,\sin\gamma) - \\
&\quad \dot{\beta}\ (\mathrm{TP}_{i_Y}\ \cos\alpha\,\cos\gamma\,\sin\beta - \mathrm{TP}_{i_Z}\ \sin\alpha\,\sin\beta + \mathrm{TP}_{i_X}\ \cos\alpha\,\sin\beta\,\sin\gamma) \\
a_2 &= \dot{\alpha}\ (\mathrm{TP}_{i_Y}\ \cos\alpha\,\cos\beta\,\cos\gamma - \mathrm{TP}_{i_Z}\ \cos\beta\,\sin\alpha + \mathrm{TP}_{i_X}\ \cos\alpha\,\cos\beta\,\sin\gamma) - \\
&\quad \dot{\beta}\ (\mathrm{TP}_{i_X}\ (\cos\beta\,\cos\gamma + \sin\alpha\,\sin\beta\,\sin\gamma) - \mathrm{TP}_{i_Y}\ (\cos\beta\,\sin\gamma - \cos\gamma\,\sin\alpha\,\sin\beta) + \mathrm{TP}_{i_Z}\ \cos\alpha\,\sin\beta) + \\
&\quad \dot{\gamma}\ (\mathrm{TP}_{i_X}\ (\sin\beta\,\sin\gamma + \cos\beta\,\cos\gamma\,\sin\alpha) + \mathrm{TP}_{i_Y}\ (\cos\gamma\,\sin\beta - \cos\beta\,\sin\alpha\,\sin\gamma)) \\
c_2 &= \dot{\gamma}\ (\mathrm{TP}_{i_X}\ (\cos\beta\,\sin\gamma - \cos\gamma\,\sin\alpha\,\sin\beta) + \mathrm{TP}_{i_Y}\ (\cos\beta\,\cos\gamma + \sin\alpha\,\sin\beta\,\sin\gamma)) - \\
&\quad \dot{\alpha}\ (\mathrm{TP}_{i_Y}\ \cos\alpha\,\cos\gamma\,\sin\beta - \mathrm{TP}_{i_Z}\ \sin\alpha\,\sin\beta + \mathrm{TP}_{i_X}\ \cos\alpha\,\sin\beta\,\sin\gamma) - \\
&\quad \dot{\beta}\ (\mathrm{TP}_{i_Y}\ (\sin\beta\,\sin\gamma + \cos\beta\,\cos\gamma\,\sin\alpha) - \mathrm{TP}_{i_X}\ (\cos\gamma\,\sin\beta - \cos\beta\,\sin\alpha\,\sin\gamma) + \mathrm{TP}_{i_Z}\ \cos\alpha\,\cos\beta) \\
a_3 &= \dot{\alpha}\ (\mathrm{TP}_{i_X}\ \cos\alpha\,\cos\gamma\,\sin\beta - \mathrm{TP}_{i_Y}\ \cos\alpha\,\sin\beta\,\sin\gamma) + \\
&\quad \dot{\beta}\ (\mathrm{TP}_{i_X}\ (\sin\beta\,\sin\gamma + \cos\beta\,\cos\gamma\,\sin\alpha) + \mathrm{TP}_{i_Y}\ (\cos\gamma\,\sin\beta - \cos\beta\,\sin\alpha\,\sin\gamma)) - \\
&\quad \dot{\gamma}\ (\mathrm{TP}_{i_X}\ (\cos\beta\,\cos\gamma + \sin\alpha\,\sin\beta\,\sin\gamma) - \mathrm{TP}_{i_Y}\ (\cos\beta\,\sin\gamma - \cos\gamma\,\sin\alpha\,\sin\beta)) \\
b_3 &= -\dot{\gamma}\ (\mathrm{TP}_{i_Y}\ \cos\alpha\,\cos\gamma + \mathrm{TP}_{i_X}\ \cos\alpha\,\sin\gamma) - \dot{\alpha}\ (\mathrm{TP}_{i_X}\ \cos\gamma\,\sin\alpha - \mathrm{TP}_{i_Y}\ \sin\alpha\,\sin\gamma) \\
c_3 &= \dot{\alpha}\ (\mathrm{TP}_{i_X}\ \cos\alpha\,\cos\beta\,\cos\gamma - \mathrm{TP}_{i_Y}\ \cos\alpha\,\cos\beta\,\sin\gamma) + \\
&\quad \dot{\beta}\ (\mathrm{TP}_{i_X}\ (\cos\beta\,\sin\gamma - \cos\gamma\,\sin\alpha\,\sin\beta) + \mathrm{TP}_{i_Y}\ (\cos\beta\,\cos\gamma + \sin\alpha\,\sin\beta\,\sin\gamma)) + \\
&\quad \dot{\gamma}\ (\mathrm{TP}_{i_X}\ (\cos\gamma\,\sin\beta - \cos\beta\,\sin\alpha\,\sin\gamma) - \mathrm{TP}_{i_Y}\ (\sin\beta\,\sin\gamma + \cos\beta\,\cos\gamma\,\sin\alpha)) \\
a_7 &= \mathrm{TP}_{i_Y}\ \cos\alpha\,\cos\gamma\,\sin\beta - \mathrm{TP}_{i_Z}\ \sin\alpha\,\sin\beta + \mathrm{TP}_{i_X}\ \cos\alpha\,\sin\beta\,\sin\gamma \\
b_7 &= -\mathrm{TP}_{i_Z}\ \cos\alpha - \mathrm{TP}_{i_Y}\ \cos\gamma\,\sin\alpha - \mathrm{TP}_{i_X}\ \sin\alpha\,\sin\gamma \\
c_7 &= \mathrm{TP}_{i_Y}\ \cos\alpha\,\cos\beta\,\cos\gamma - \mathrm{TP}_{i_Z}\ \cos\beta\,\sin\alpha + \mathrm{TP}_{i_X}\ \cos\alpha\,\cos\beta\,\sin\gamma \\
a_8 &= \mathrm{TP}_{i_Y}\ (\sin\beta\,\sin\gamma + \cos\beta\,\cos\gamma\,\sin\alpha) - \mathrm{TP}_{i_X}\ (\cos\gamma\,\sin\beta - \cos\beta\,\sin\alpha\,\sin\gamma) + \mathrm{TP}_{i_Z}\ \cos\alpha\,\cos\beta \\
c_8 &= \mathrm{TP}_{i_Y}\ (\cos\beta\,\sin\gamma - \cos\gamma\,\sin\alpha\,\sin\beta) - \mathrm{TP}_{i_X}\ (\cos\beta\,\cos\gamma + \sin\alpha\,\sin\beta\,\sin\gamma) - \mathrm{TP}_{i_Z}\ \cos\alpha\,\sin\beta \\
a_9 &= -\mathrm{TP}_{i_X}\ (\cos\beta\,\sin\gamma - \cos\gamma\,\sin\alpha\,\sin\beta) - \mathrm{TP}_{i_Y}\ (\cos\beta\,\cos\gamma + \sin\alpha\,\sin\beta\,\sin\gamma) \\
b_9 &= \mathrm{TP}_{i_X}\ \cos\alpha\,\cos\gamma - \mathrm{TP}_{i_Y}\ \cos\alpha\,\sin\gamma \\
c_9 &= \mathrm{TP}_{i_X}\ (\sin\beta\,\sin\gamma + \cos\beta\,\cos\gamma\,\sin\alpha) + \mathrm{TP}_{i_Y}\ (\cos\gamma\,\sin\beta - \cos\beta\,\sin\alpha\,\sin\gamma)
\end{aligned}
$$

$$
\ddot{\mathbf{TP}}_i = \begin{bmatrix} \ddot{\mathrm{TP}}_{i_X} \\ \ddot{\mathrm{TP}}_{i_Y} \\ \ddot{\mathrm{TP}}_{i_Z} \end{bmatrix} = \mathbf{J}_{\dot{\mathrm{TP}}_i} \begin{bmatrix} \dot{X} & \dot{Y} & \dot{Z} & \dot{\alpha} & \dot{\beta} & \dot{\gamma} & \ddot{X} & \ddot{Y} & \ddot{Z} & \ddot{\alpha} & \ddot{\beta} & \ddot{\gamma} \end{bmatrix}^{\mathrm{T}} \tag{54}
$$

1.11 Leg Member Equations of Motion in Terms of State Variables

1.11.1 Radial Accelerations

The equation for the radial acceleration $\ddot{\delta}_{1_i}$, in terms of the generalized coordinates q_{leg}, is found by substituting Equations 31 and 32 into Equation 29. The result is given by Equation 55.

$$
\begin{aligned}
\ddot{\delta_{1_i}} = \frac{-1}{m_1 + m_2} &\left[F_{\text{act}_i} + K_i \delta_{1_i} + B_i \dot{\delta}_{1_i} - m_2(r_{2_i}\dot{\phi}_i^2 + r_{2_i}\dot{\theta}_i^2 \cos^2\phi_i)\right] \\
&+ \frac{1}{m_1 + m_2}\left[m_1(r_{1_i}\dot{\phi}_i^2 + r_{1_i}\dot{\theta}_i^2\cos^2\phi_i) - g\sin\phi_i(m_1 + m_{2_i})\right]
\end{aligned} \tag{55}
$$

where:

$$
\begin{aligned}
r_1 &= \left(l_{1i} + \tfrac{\delta_{1i}}{2}\right) \\
r_2 &= (2l_{1i} + \delta_{1i} + l_2) \\
r_3 &= (2l_{1i} + \delta_{1i} + l_2 + l_3 + \delta_{2i})
\end{aligned}
$$

Similarly, $\ddot{\delta}_{2_i}$ is found by substituting Equations 35 and 36 into Equation 29. The result is given by Equation 56:

$$
\ddot{\delta_{2_i}} = \frac{1}{m_3}\left[F_{\text{act}_i} + \mathbf{F}_{\mathbf{r}_i} \cdot \frac{\mathbf{L}_i}{|\mathbf{L}_i|} - m_3\ddot{\delta_{1_i}} + m_3 r_{3_i}\left(\dot{\phi_i}^2 + \dot{\theta_i}^2 \cos^2\phi_i\right) - m_3 g \sin\phi_i\right] \tag{56}
$$

where the vectorized dot product term: $\mathbf{F}_{\mathbf{r}_i} \cdot \mathbf{L}_i / |\mathbf{L}_i|$, is the projection of the upper joint reaction force upon the leg member.

1.11.2 Angular Accelerations

The equation for the angular acceleration $\ddot{\phi}_i$, in terms of the generalized coordinates q_{leg}, is found by substituting Equations 44 and 43 into Equation 29. The result is given by Equation 57.

$$\begin{aligned}\ddot{\phi}_i = \frac{1}{a_1}&\left[4M_{\phi_i} + {l_2}^2 m_1 \dot{\theta}_i \dot{\phi}_i \sin(2\theta_i) + m_1 {r_{2_i}}^2 \dot{\theta}_i \dot{\phi}_i \sin(2\theta_i) + 4m_2 {r_{2_i}}^2 \dot{\theta}_i \dot{\phi}_i \sin(2\theta_i)\right] \\ &+ \frac{1}{a_2}\left[-4m_1 \dot{\delta}_{1_i} r_{2_i} \dot{\phi}_i - 4l_2 m_1 \dot{\delta}_{1_i} \dot{\phi}_i + 8m_2 \dot{\delta}_{1_i} r_{2_i} \dot{\phi}_i + 8m_3 \dot{\delta}_{1_i} r_{3_i} \dot{\phi}_i + 8m_3 \dot{\delta}_{2_i} r_{3_i} \dot{\phi}_i\right] \\ &+ \frac{1}{a_1}\left[4m_3 {r_{3_i}}^2 \dot{\theta}_i \dot{\phi}_i \sin(2\theta_i) - 2l_2 m_1 r_{2_i} \dot{\theta}_i \dot{\phi}_i \sin(2\theta_i)\right] \qquad (57)\end{aligned}$$

where:

$$\begin{aligned} a_1 &= \cos^2\theta_i \left({l_2}^2 m_1 + m_1 {r_{2_i}}^2 + 4m_2 {r_{2_i}}^2 + 4m_3 {r_{3_i}}^2 - 2l_2 m_1 r_{2_i}\right) \\ a_2 &= {l_2}^2 m_1 + m_1 {r_{2_i}}^2 + 4m_2 {r_{2_i}}^2 + 4m_3 {r_{3_i}}^2 - 2l_2 m_1 r_{2_i} \\ M_{\phi_i} &= L_{X_i} F_{r_Y} - L_{Y_i} F_{r_Y} \end{aligned}$$

Analogously, the angular acceleration $\ddot{\theta}_i$, in terms of the generalized coordinates q_{leg}, is found by substituting Equations 40 and 39 into Equation 29. The result is given by Equation 58.

$$\begin{aligned}\left[m_2 {r_{2_i}}^2 + m_3 {r_{3_i}}^2 + \frac{m_1}{2}(l_2 - r_{2_i})^2\right]\ddot{\theta}_i = \\ &- \cos(\theta_i)\sin(\theta_i)\left(m_2 {r_{2_i}}^2 + m_3 {r_{3_i}}^2 + m_1\left(\frac{l_2}{2} - \frac{r_{2_i}}{2}\right)^2\right){\dot{\phi}_i}^2 \\ &+ M_{\theta_i} - g\cos(\theta_i)\left(m_2 r_{2_i} + m_3 r_{3_i} - \frac{m_1(l_2 - r_{2_i})}{2}\right) \\ &- 2\dot{\delta}_{1_i}\dot{\theta}_i\left(m_2 r_{2_i} + m_3 r_{3_i} - \frac{m_1(l_2 - r_{2_i})}{4}\right) - 2m_3 \dot{\delta}_{2_i} r_{3_i} \dot{\theta}_i\end{aligned} \qquad (58)$$

where:

$$M_{\theta_i} = F_{r_{z_i}}\left[(\text{TP}_{i_X} - \text{BP}_{i_X})\cos\theta_i + (\text{TP}_{i_Y} - \text{BP}_{i_Y})\sin\theta_i\right] - (\text{TP}_{i_Z} - \text{BP}_{i_Z})\left(F_{r_{x_i}}\cos\theta_i + F_{r_{y_i}}\sin\theta_i\right)$$

In Equations 57 and 58 the components M_{ϕ_i} and M_{θ_i} correspond to the moments generated by the reaction forces, at the top joint i, $\mathbf{F}_{r_i}$, around the relevant axis of ϕ and θ.

1.11.3 Conversion to State Variables

In order to solve Equations 57 and 58, the variables θ_i, $\dot{\theta}_i$, $\dot{\phi}_i$, δ_{1_i}, $\dot{\delta_{1_i}}$, δ_{2_i}, $\dot{\delta_{2_i}}$ must be known. The chosen state variables for the model are $\boldsymbol{q}_{\text{state}}$ and hence δ_{1_i} and $\dot{\delta_{1_i}}$ remain while the other variables are ultimately rewritten in terms of X, Y, Z, α, β, γ and their derivatives and secondary derivatives. The following section provides the expressions for substitution given in terms of TP_i, BP_i, δ_{1_i} and $\dot{\delta_{1_i}}$. Note, the top points TP_i are intermediate variables for expressions derived in terms of $\boldsymbol{q}_{\text{state}}$ in Section 1.10.

Knowing position, velocity and acceleration of each top joint point, it is relatively straight forward to derive each leg member's angles and angular velocity components: ϕ_i, θ_i, $\dot{\phi}_i$, $\dot{\theta}_i$. The variable ϕ_i can be found via the trigonometric relationship between the top joint point, TP_i, and bottom joint point, BP_i:

$$\phi_i = \text{atan2}\left(\text{TP}_{iY} - \text{BP}_{iY}, \text{TP}_{iX} - \text{BP}_{iX}\right) \qquad (59)$$

Similarly, ϕ_i can be found via:

$$\theta_i = \text{atan2}\left(\text{TP}_{iZ} - \text{BP}_{iZ}, \sqrt{(\text{BP}_{iX} - \text{TP}_{iX})^2 + (\text{BP}_{iY} - \text{BP}_{iY})^2}\right) \qquad (60)$$

Knowing Equations 59 and 60 the velocities of these components can be found via partial differentiation:

$$\dot{\phi}_i = \mathbf{J}_{\phi_i} \begin{bmatrix} \dot{\mathrm{TP}}_{i_X} \\ \dot{\mathrm{TP}}_{i_Y} \\ \dot{\mathrm{TP}}_{i_Z} \end{bmatrix} \tag{61}$$

$$\dot{\phi}_i = \frac{\dot{\mathrm{TP}}_{i_X}(\mathrm{BP}_{i_Y} - \mathrm{TP}_{i_Y})}{(\mathrm{BP}_{i_X} - \mathrm{TP}_{i_X})^2 + (\mathrm{BP}_{i_Y} - \mathrm{TP}_{i_Y})^2} - \frac{\dot{\mathrm{TP}}_{i_Y}(\mathrm{BP}_{i_X} - \mathrm{TP}_{i_X})}{(\mathrm{BP}_{i_X} - \mathrm{TP}_{i_X})^2 + (\mathrm{BP}_{i_Y} - \mathrm{TP}_{i_Y})^2} \tag{62}$$

And for $\dot{\theta}$:

$$\dot{\theta}_i = \mathbf{J}_{\theta_i} \begin{bmatrix} \dot{\mathrm{TP}}_{i_X} \\ \dot{\mathrm{TP}}_{i_Y} \\ \dot{\mathrm{TP}}_{i_Z} \end{bmatrix} \tag{63}$$

$$\begin{aligned} \dot{\theta}_i = & \frac{\dot{\mathrm{TP}}_{i_Z}\sqrt{(\mathrm{BP}_{i_X} - \mathrm{TP}_{i_X})^2 + (\mathrm{BP}_{i_Y} - \mathrm{TP}_{i_Y})^2}}{a_1} \\ & - \frac{\dot{\mathrm{TP}}_{i_X}(2\mathrm{BP}_{i_X} - 2\mathrm{TP}_{i_X})(\mathrm{BP}_{i_Z} - \mathrm{TP}_{i_Z})}{a_2} \\ & - \frac{\dot{\mathrm{TP}}_{i_Y}(2\mathrm{BP}_{i_Y} - 2\mathrm{TP}_{i_Y})(\mathrm{BP}_{i_Z} - \mathrm{TP}_{i_Z})}{a_2} \end{aligned} \tag{64}$$

where:

$$a1 = (\mathrm{BP}_{i_X} - \mathrm{TP}_{i_X})^2 + (\mathrm{BP}_{i_Y} - \mathrm{TP}_{i_Y})^2 + (\mathrm{BP}_{i_Z} - \mathrm{TP}_{i_Z})^2$$
$$a2 = 2\sqrt{(\mathrm{BP}_{i_X} - \mathrm{TP}_{i_X})^2 + (\mathrm{BP}_{i_Y} - \mathrm{TP}_{i_Y})^2}((\mathrm{BP}_{i_X} - \mathrm{TP}_{i_X})^2 + (\mathrm{BP}_{i_Y} - \mathrm{TP}_{i_Y})^2 + (\mathrm{BP}_{i_Z} - \mathrm{TP}_{i_Z})^2)$$

Replacing terms of δ_{2_i} and $\dot{\delta}_{2_i}$ differs slightly from the angle terms in that it additionally requires using δ_{1_i} and $\dot{\delta}_{1_i}$. Knowing the length of leg i at a given point in time, Equation 65 gives δ_{2_i} in terms of state variables.

$$\delta_{2_i} = |\boldsymbol{L}_i| - 2l_{1_i} - l_{2_i} - l_{3_i} - l_{4_i} - \delta_{1_i} \tag{65}$$

where:

$$|\boldsymbol{L}_i| = \sqrt{(\mathrm{BP}_{i_X} - \mathrm{TP}_{i_X})^2 + (\mathrm{BP}_{i_Y} - \mathrm{TP}_{i_Y})^2 + (\mathrm{BP}_{i_Z} - \mathrm{TP}_{i_Z})^2} \tag{66}$$

Knowing the change in length of leg i at a given point in time, Equation 67 gives $\dot{\delta}_{2_i}$ in terms of state variables.

$$\dot{\delta}_{2_i} = |\dot{\boldsymbol{L}}_i| - \dot{\delta}_{1_i} \tag{67}$$

where:

$$\begin{aligned} |\dot{\boldsymbol{L}}_i| = & -\frac{\dot{\mathrm{TP}}_{i_X}\,(2\mathrm{BP}_{iX} - 2\mathrm{TP}_{iX})}{2\sqrt{(\mathrm{BP}_{iX} - \mathrm{TP}_{iX})^2 + (\mathrm{BP}_{iY} - \mathrm{TP}_{iY})^2 + (\mathrm{BP}_{iZ} - \mathrm{TP}_{iZ})^2}} \\ & - \frac{\dot{\mathrm{TP}}_{i_Y}\,(2\mathrm{BP}_{iY} - 2\mathrm{TP}_{iY})}{2\sqrt{(\mathrm{BP}_{iX} - \mathrm{TP}_{iX})^2 + (\mathrm{BP}_{iY} - \mathrm{TP}_{iY})^2 + (\mathrm{BP}_{iZ} - \mathrm{TP}_{iZ})^2}} \\ & - \frac{\dot{\mathrm{TP}}_{i_Z}\,(2\mathrm{BP}_{iZ} - 2\mathrm{TP}_{iZ})}{2\sqrt{(\mathrm{BP}_{iX} - \mathrm{TP}_{iX})^2 + (\mathrm{BP}_{iY} - \mathrm{TP}_{iY})^2 + (\mathrm{BP}_{iZ} - \mathrm{TP}_{iZ})^2}} \end{aligned} \tag{68}$$

1.12 Contstraints

Now that the individual equations of motion for the two subsystems, the leg members and the platform, are known, they require a constraint that that links the two groups. This can be achieved by enforcing an identical acceleration upon a leg member's top joint and the corresponding point on the top platform. Let the acceleration of a leg member's top joint be denoted by $\ddot{\delta}_{L_i}$ then this constraint takes the form of Equation 70.

$$\ddot{\boldsymbol{\delta}}_{\boldsymbol{L}} = \begin{bmatrix} \ddot{\delta}_{L_{1_X}} & \dots & \ddot{\delta}_{L_{n_X}} \\ \ddot{\delta}_{L_{1_Y}} & \dots & \ddot{\delta}_{L_{n_Y}} \\ \ddot{\delta}_{L_{1_Z}} & \dots & \ddot{\delta}_{L_{n_Z}} \end{bmatrix}_{3\times 6} \tag{69}$$

$$\ddot{\boldsymbol{\delta}}_{\boldsymbol{L}} = \mathrm{T}\ddot{\mathrm{P}} \tag{70}$$

The position of the ith leg member's top joint δ_{L_i} is a function of θ_i, ϕ_i , δ_{1_i} and δ_{2_i} as given by Equation 71.

$$\delta_{L_i} = \begin{bmatrix} & \mathrm{BP}_{i_X} \\ R_{\delta_L} & \mathrm{BP}_{i_Y} \\ & \mathrm{BP}_{i_Z} \end{bmatrix} \begin{bmatrix} l_r \\ 0 \\ 0 \\ 1 \end{bmatrix} \tag{71}$$

where:

$$R_{\delta_{L_i}} = \begin{bmatrix} \cos\phi_i & -\sin\phi_i & 0 \\ \sin\phi_i & \cos\phi_i & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta_i & 0 & -\sin\theta_i \\ 0 & 1 & 0 \\ -\sin\theta_i & 0 & \cos\theta_i \end{bmatrix} \tag{72}$$

$$l_{r_i} = 2l_{1_i} + l_{2_i} + l_{3_i} + l_{4_i} + \delta_{1_i} + \delta_{1_i} \tag{73}$$

By differentiating Equation 71 the velocity $\dot{\delta}_{L_i}$ is obtained, presented in Equation 74.

$$\dot{\delta}_{L_i} = \begin{bmatrix} \dot{\delta}_{1_i}\cos\theta_i\cos\phi_i + \dot{\delta}_{2_i}\cos\theta_i\cos\phi_i - \dot{\theta}_i\cos\phi_i\sin(\theta_i)\,l_r - \dot{\phi}_i\cos\theta_i\sin\phi_i l_r \\ \dot{\delta}_{1_i}\cos\theta_i\sin\phi_i + \dot{\delta}_{2_i}\cos\theta_i\sin\phi_i + \dot{\phi}_i\cos\theta_i\cos\phi_i l_r - \dot{\theta}_i\sin(\theta_i)\sin\phi_i l_r \\ \dot{\delta}_{1_i}\sin(\theta_i) + \dot{\delta}_{2_i}\sin(\theta_i) + \dot{\theta}_i\cos\theta_i l_r \end{bmatrix} \tag{74}$$

Differentiating Equation 74 once more yields the desired acceleration $\ddot{\delta}_{L_i}$ as given by Equation 75.

$$\ddot{\delta}_{L_i} = \begin{bmatrix} \ddot{\delta}_{1_i}a_1 - \dot{\phi}_i a_5 - \dot{\delta}_{1_i}\left(\dot{\theta}_i a_4 + \dot{\phi}_i a_2\right) - \dot{\delta}_{2_i}\left(\dot{\theta}_i a_4 + \dot{\phi}_i a_2\right) - \dot{\theta}_i a_6 + \ddot{\delta}_{2_i}a_1 - \ddot{\theta}_i a_4 l_{r_i} - \ddot{\phi}_i a_2 l_{r_i} \\ \dot{\delta}_{1_i}\left(\dot{\phi}_i a_1 - \dot{\theta}_i a_3\right) - \dot{\theta}_i a_7 + \dot{\delta}_{2_i}\left(\dot{\phi}_i a_1 - \dot{\theta}_i a_3\right) + \dot{\phi}_i a_8 + \ddot{\delta}_{1_i}a_2 + \ddot{\delta}_{2_i}a_2 + \ddot{\phi}_i a_1 l_{r_i} - \ddot{\theta}_i a_3 l_{r_i} \\ \ddot{\delta}_{1_i}\sin\theta_i + \ddot{\delta}_{2_i}\sin\theta_i + \dot{\theta}_i a_9 + \dot{\delta}_{1_i}\dot{\theta}_i\cos\theta_i + \dot{\delta}_{2_i}\dot{\theta}_i\cos\theta_i + \ddot{\theta}_i\cos\theta_i l_{r_i} \end{bmatrix} \tag{75}$$

where:

$$\begin{aligned} a1 &= \cos\theta_i\cos\phi_i \\ a2 &= \cos\theta_i\sin\phi_i \\ a3 &= \sin\theta_i\sin\phi_i \\ a4 &= \cos\phi_i\sin\theta_i \\ a5 &= \dot{\delta}_{1_i}a_2 + \dot{\delta}_{2_i}a_2 + \dot{\phi}_i a_1 l_{r_i} - \dot{\theta}_i a_3 l_{r_i} \\ a6 &= \dot{\delta}_{1_i}a_4 + \dot{\delta}_{2_i}a_4 + \dot{\theta}_i a_1 l_{r_i} - \dot{\phi}_i a_3 l_{r_i} \\ a7 &= \dot{\delta}_{1_i}a_3 + \dot{\delta}_{2_i}a_3 + \dot{\theta}_i a_2 l_{r_i} + \dot{\phi}_i a_4 l_{r_i} \\ a8 &= \dot{\delta}_{1_i}a_1 + \dot{\delta}_{2_i}a_1 - \dot{\theta}_i a_4 l_{r_i} - \dot{\phi}_i a_2 l_{r_i} \\ a_9 &= \dot{\delta}_{1_i}\cos\theta_i + \dot{\delta}_{2_i}\cos\theta_i - \dot{\theta}_i\sin\theta_i l_{r_i} \end{aligned}$$

For application in the final system of equations, $\ddot{\delta}_{L_i}$ must be rewritten in terms of the chosen state variables $\boldsymbol{q}_{\text{state}}$. Thus all instances of the variables θ_i, $\dot{\theta}_i$, $\ddot{\theta}_i$, ϕ_i, $\dot{\phi}_i$, $\ddot{\phi}_i$, δ_{2_i}, $\dot{\delta}_{2_i}$ and $\ddot{\delta}_{2_i}$ must be replaced in terms of X, Y, Z, α, β, γ and their derivatives and secondary derivatives. These substitutions can be done via the relations detailed in Section 1.11.3.

1.13 Assembling the Complete System of Equations

Having derived the components necessary for the equations of motion, the final form can be assembled for eventual simulations. As a brief summary of the methodology, the following steps are taken:

1. via Lagrangian mechanics, derive the equations of motion for the upper platform, including external loading $\boldsymbol{F}_{\text{load}}$ and reaction forces at the joint points $\boldsymbol{F}_{\text{r}}$, in terms of $\boldsymbol{q}_{\text{plat}}$;
2. via Lagrangian mechanics, derive the equations of motion for each leg member, including reaction forces at the joint points $\boldsymbol{F}_{\text{r}}$, in terms of $\boldsymbol{q}_{\text{leg}}$;
3. for the leg member equations of motion, substitute terms θ_i, ϕ_i, $\dot{\theta}_i$ and $\dot{\phi}_i$ with functions of $\boldsymbol{q}_{\text{plat}}$ and $\dot{\boldsymbol{q}}_{\text{plat}}$;
4. derive the binding constraints between the platform and leg member subsystems by equating their accelerations at the upper joint points
$$\ddot{\boldsymbol{\delta}}_{\boldsymbol{L}}\left(\boldsymbol{\delta_1}, \dot{\boldsymbol{\delta}}_{\boldsymbol{1}}, \boldsymbol{q}_{\text{leg}}, \dot{\boldsymbol{q}}_{\text{leg}}, \boldsymbol{F}_{\text{r}}\right) = \mathbf{T}\ddot{\mathbf{P}}\left(\boldsymbol{q}_{\text{leg}}, \dot{\boldsymbol{q}}_{\text{leg}}, \boldsymbol{F}_{\text{r}}\right);$$
5. assemble a system of equations using the equations of motion for the upper platform and constraints derived in the previous step yielding a final system in the form of:
$$\boldsymbol{A}_{24\times 24}\,\boldsymbol{X} = \boldsymbol{b}_{24\times 1} \tag{76}$$

The final system of equations takes the form of Equation 77 where:

$$\boldsymbol{X}_{24\times 1} = \left[\begin{array}{ccccccccccc} \ddot{X} & \ddot{Y} & \ddot{Z} & \ddot{\alpha} & \ddot{\beta} & \ddot{\gamma} & F_{r_{X_1}} & F_{r_{Y_1}} & F_{r_{Z_1}} & \ldots & F_{r_{Z_6}} \end{array}\right]^T \tag{77}$$

Once this system is solved, each element of $\ddot{\boldsymbol{\delta}}_{\boldsymbol{1}}$ can then be solved for via Equation 55.

1.14 Variants for Specific Cases

With slight modifications the system of equations can be modified to represent special cases for the Stewart platform. These two cases are locking the actuators and finding the resting length of leg member springs to ensure the platform is at a certain height in its initial undisturbed state.

Actuator Lock

For the first case, locking of the actuators, this enables a compliant Stewart platform in which the only deformation occurs within the springs. This actuator locking requires a force on each actuator that ensures δ_{2_i} is always zero. This can be done by adjusting Equation 77 to Equation 78:

$$\boldsymbol{A}_{30\times 30}\,\boldsymbol{X}_{\text{lock}} = \boldsymbol{b}_{30\times 1} \tag{78}$$

where A now incorporates all six instances of Equation 56 in which the right hand side, δ_{2_i} , is replaced with 0. Following suit $\boldsymbol{X}_{\text{lock}}$ gains 6 more variables to solve for: $\boldsymbol{F}_{\text{act}}$. Once solved for, these are the actuator forces required at that instant to ensure that with the internal actuator force, gravity force, centripetal force and applied force from the top platform, the actuator does not expand or contract.

Spring Solution to Ensure a Still Platform at an Undisturbed State.

Consider the design scenario where the resting height of the platform should be a specified height. If an arbitrary spring length and stiffness is prescribed then within the presence of gravity the platform will sink, ending up below the required height. This can be solved by finding the initial configuration in which there are no accelerations in any of the degrees of freedom. The solution used in this work is, for a given spring stiffness of each spring, to find the resting length of the spring such that, subject to gravity, the erect platform sinks to the desired height without any deformation in the actuators from their desired initial length.

Equation wise this can be done similarly to the Actuator Lock case by expanding Equation 78 to 79:

$$\boldsymbol{A}_{36\times 36}\,\boldsymbol{X}_{\text{lock}} = \boldsymbol{b}_{36\times 1} \tag{79}$$

In which all six instances of Equation 80 are added to the system. Upon solving this expanded version of the system, the resting length of each spring is found to achieve the desired initial condition.

$$2l_{1i} = |\mathbf{L}_i| - \delta_{1i} - l_2 - l_3 - \delta_{2i} - l_4 \tag{80}$$

References

[1] G. Lebret, K. Liu, and F. Lewis, "Dynamic analysis and control of a Stewart platform manipulator," *Journal of Robotic Systems*, vol. 10, no. 3, pp. 629–655, 1993. [Online]. Available: http://onlinelibrary.wiley.com/doi/10.1002/rob.4620100506/abstract

A Simmechanics Wheelchair Model

Figures 3 and 4 provide the dimensions of the CAD humanoid shape used in some of simulation work. This model was developed from scratch using anthropomorphic data made available in to the author at Tokyo Institute of Technology.

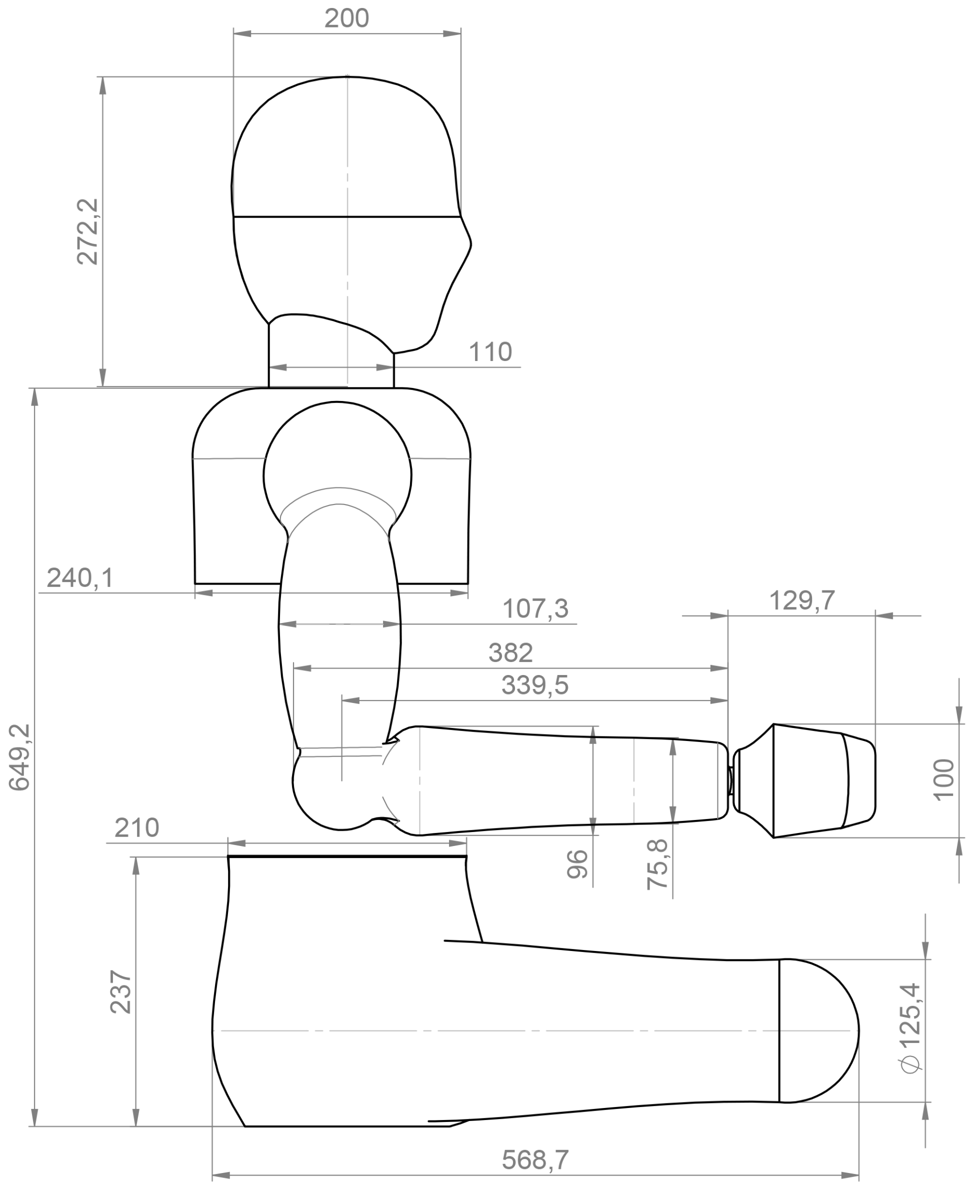

Figure 3: Measurements of various dimensions (in mm) in the CAD model of a humanoid robot without a trunk

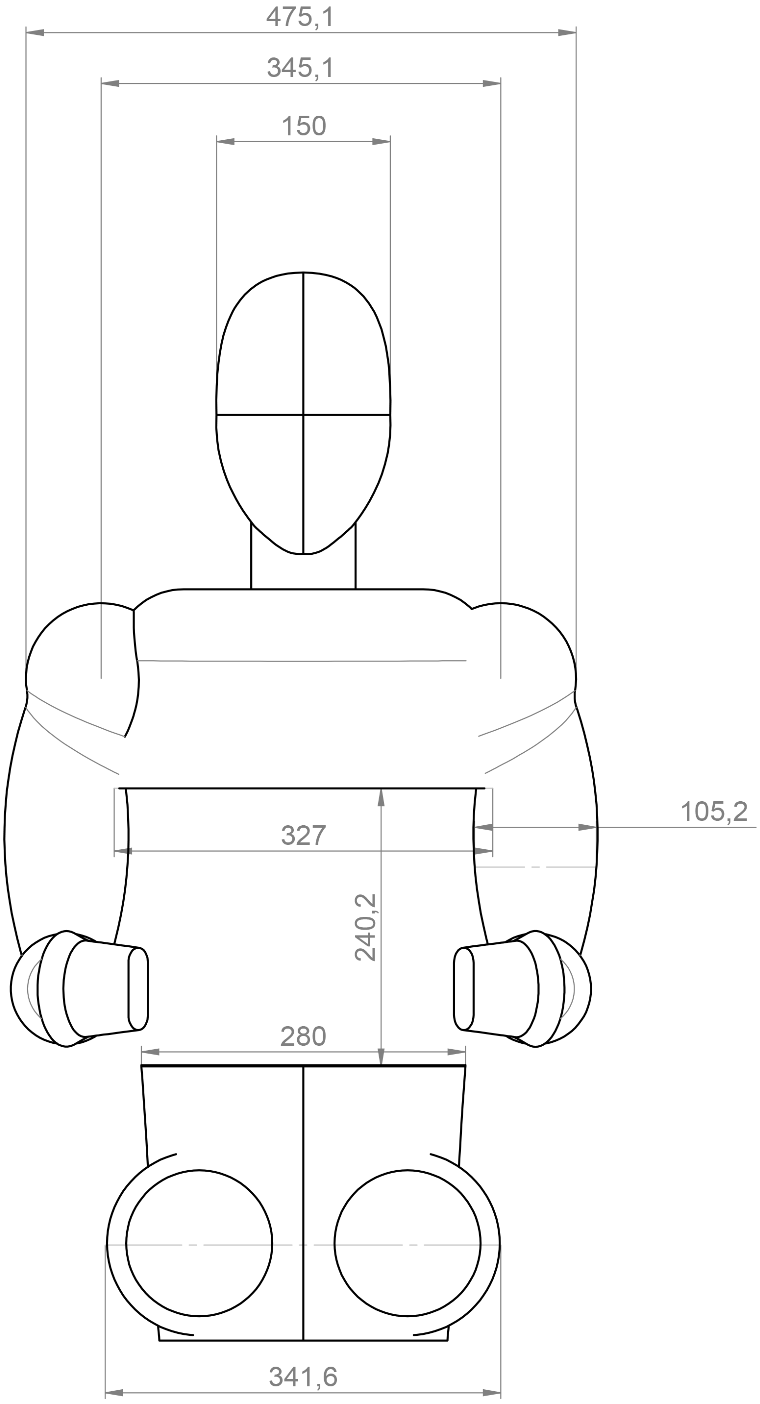

Figure 4: Measurements of various dimensions (in mm) in the CAD model of a humanoid robot without a trunk

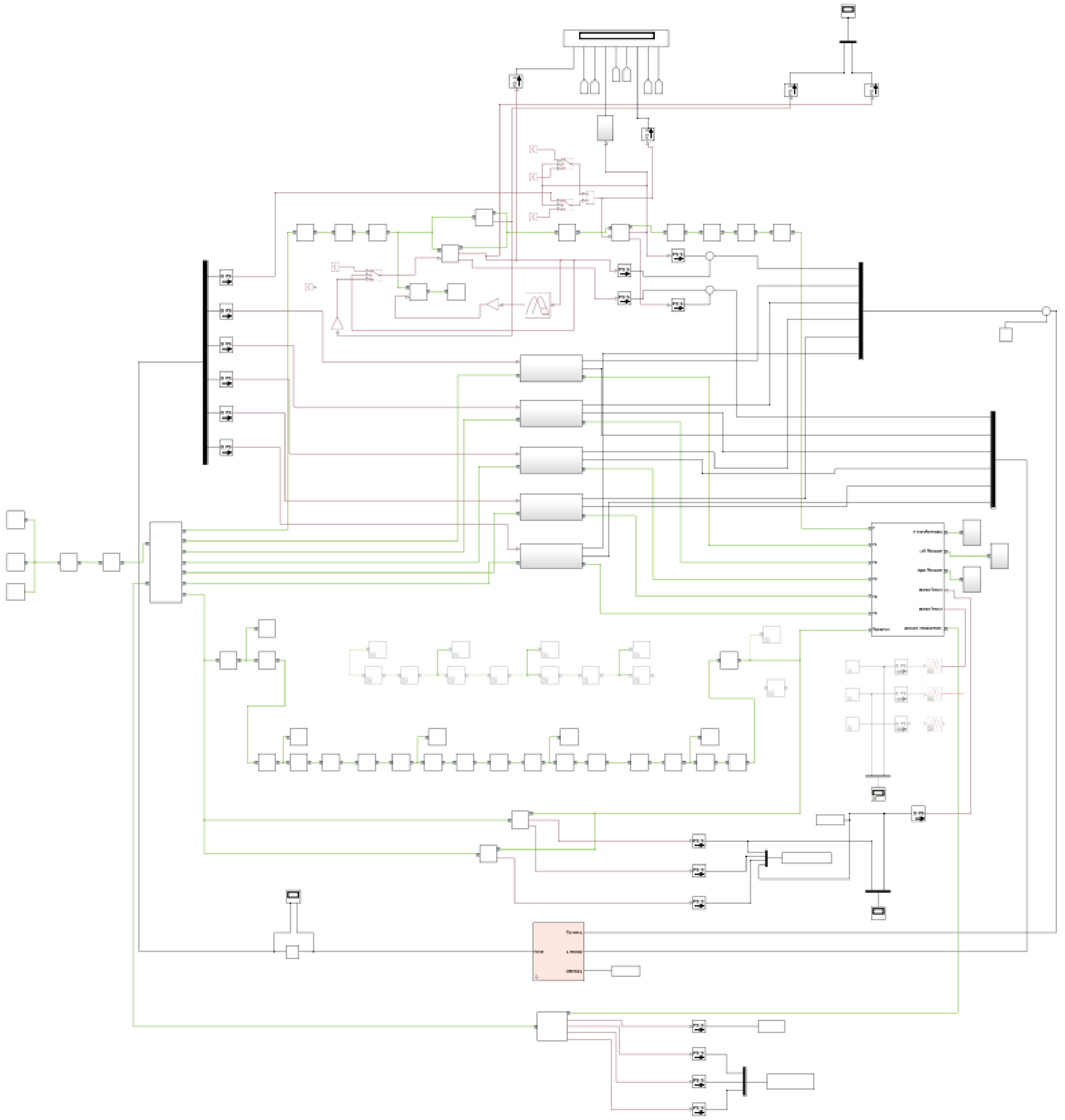

Figure 5: The full Simmechanics model top layer.

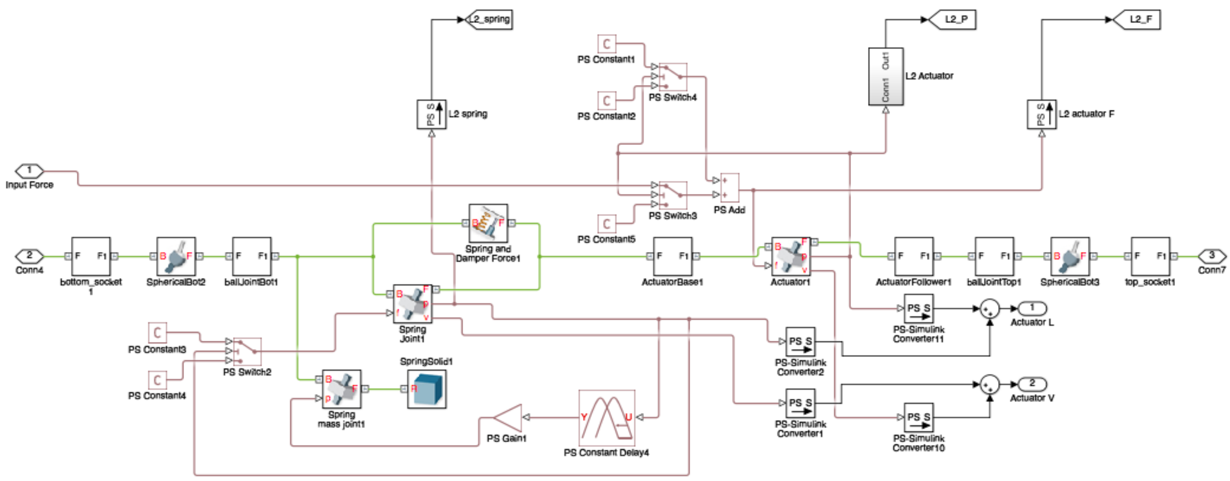

Figure 6: The Simemchanics representation of one actuator leg.