Geometric structure of thermal cones

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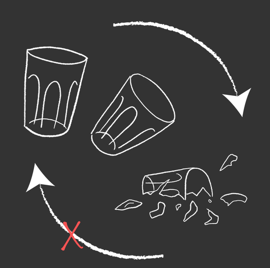
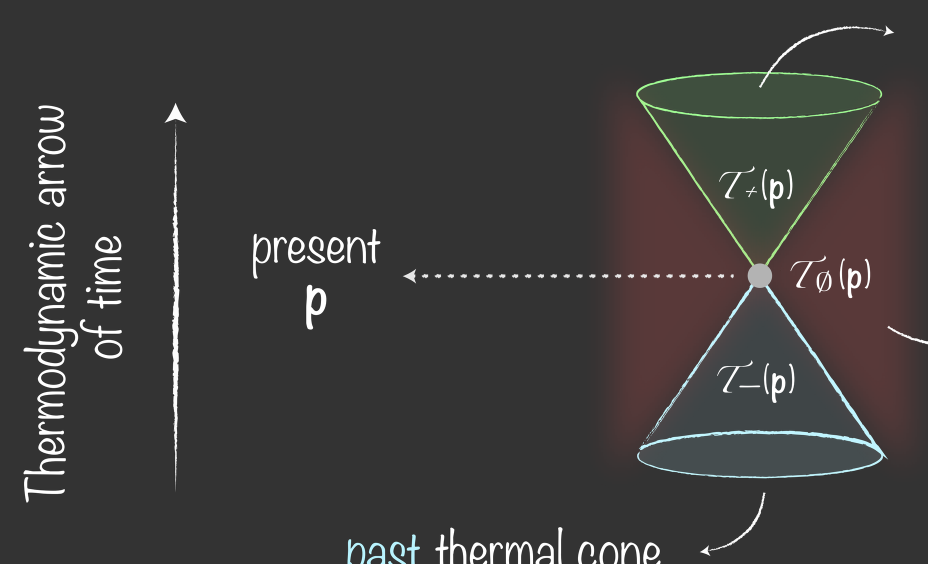
Thermodynamic arrow of time

The 2^{nd} law ⟶ fundamental asymetry:



$\Delta S > 0$ ⟶ ordering in the space of states ⟶ structure of the thermodynamic arrow of time ⟶ Q. Knowing the initial state, what can we say about its past and future?

Statement of the problem



Resource theory of thermodynamics

Indentifying the set of thermodynamically-free states

(ρ, H) + (γ_E, H_E) ⟶ $\gamma = \dfrac{e^{-\beta H}}{Z}$, $Z = \mathrm{tr}(e^{-\beta H})$,

Thermodynamic transformations are modelled by thermal operations

$$\mathcal{E}(\rho) = \mathrm{tr}_E[U(\rho \otimes \gamma_E)U^\dagger] \quad \text{with} \quad [U, H \otimes \mathbb{1}_E + \mathbb{1} \otimes H_E] = 0$$

Energy-conserving interaction

Setting the scene

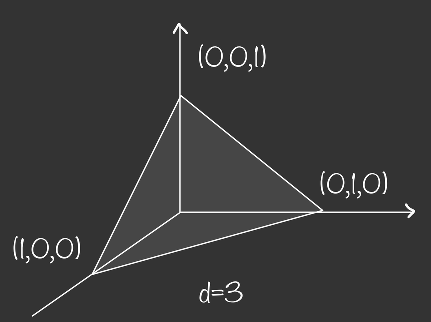
Finite dimensional system
d

$$(\rho, H) = \left(\sum_i p_i |e_i\rangle\langle e_i|, \sum_i E_i |e_i\rangle\langle e_i|\right) \longrightarrow \mathbf{p} = (p_1, \ldots, p_d)$$

energy-incoherent state

vector of population

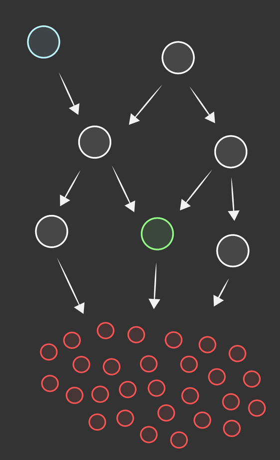
The space of states is the probability simplex



Fundamental question ⟶

State interconversion: given $\mathbf{p}$, $\mathbf{q}$, and a thermal bath at β: $\mathcal{E}(\mathbf{p}) = \mathbf{q}$

Out of equilibrium



Equilibrium

$$\boxed{\mathcal{E}(\rho) = \sigma : \mathbf{p} \succ_\beta \mathbf{q}}$$

M. Horodecki & Jonathan Oppenheim, Nat.Commun (2013)

Thermal cones

Infinite temperature limit $T \longrightarrow \infty / \beta = 0$: $\boldsymbol{\gamma} = \boldsymbol{\eta} := (1/d, \ldots, 1/d)$

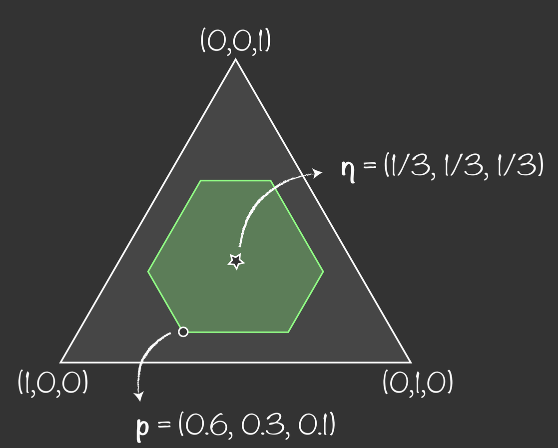
B: The set of n x n bistochastic matrices is a convex set whose extreme points are permutation matrices

+

HLP: There exists a bistochastic matrix mapping $\mathbf{p}$ into $\mathbf{q}$ if and only if $\mathbf{p}$ majorises $\mathbf{q}$: $\Lambda\,\mathbf{p} = \mathbf{q}$, $\Lambda\,\boldsymbol{\eta} = \boldsymbol{\eta}$

⟶ Corollary: Future cone

$$\boxed{\mathcal{T}_+(\mathbf{p}) = \mathrm{conv}[\{\Pi\mathbf{p}, \Pi \in S_d\}]}$$



Q. Given a present state $\mathbf{p}$, how to characterise its incomparable and past cone?

Lemma: Incomparable region

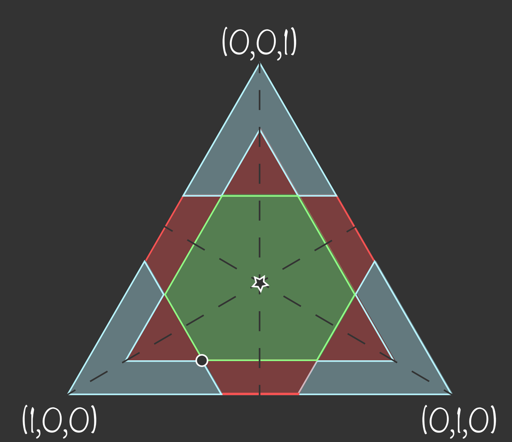
$$\mathcal{T}_\emptyset(\mathbf{p}) = [\mathrm{int}(T) \setminus \mathcal{T}_+(\mathbf{p})] \cap \Delta_d \quad \text{where} \quad T = \bigcup_{j=1}^{d-1} \mathrm{conv}[\mathcal{T}_+(\mathbf{t}^{(j)}) \cup \mathcal{T}_+(\mathbf{t}^{(j+1)})],$$

$$\text{and} \quad \mathbf{t}^{(n)} = \left(\sum_i^{n-1} p_i - (n-2)p_n, p_n, \ldots, p_n, 1 - \sum_i^{n-1} p_i + (n-2)p_n - (d-2)p_n\right).$$

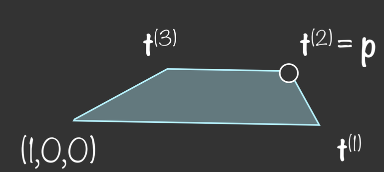
Theorem: Past cone

$$\boxed{\mathcal{T}_-(\mathbf{p}) = \Delta_d \setminus \mathrm{int}(T).}$$

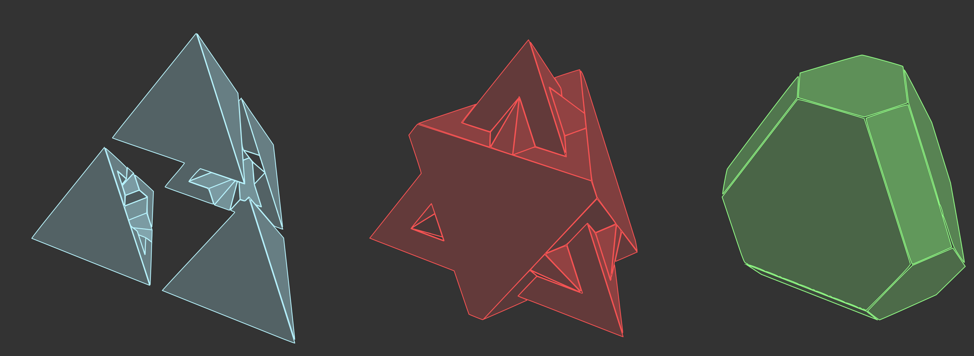
Example. $\mathbf{p}$ = (0.6, 0.3, 0.1)



- Only the future is convex.
- The past is the union of d! convex pieces.

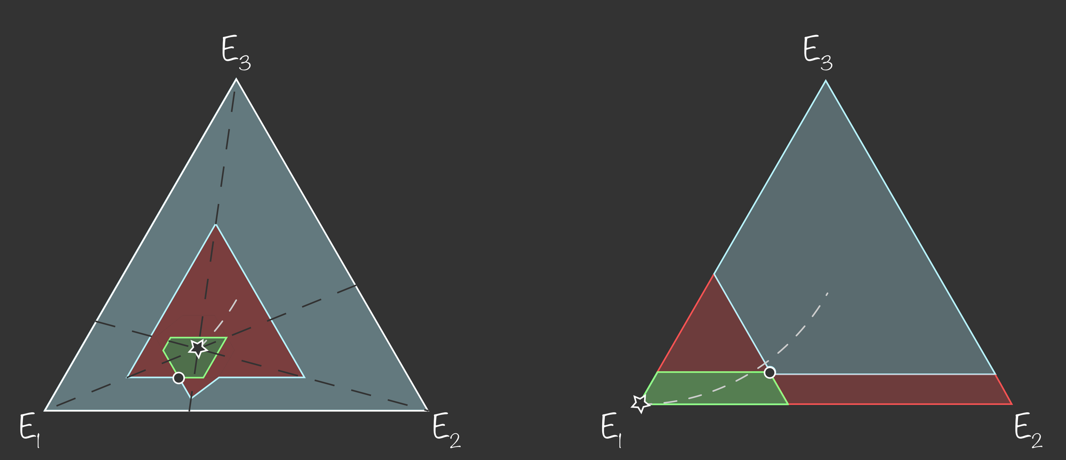


Example. $\mathbf{p}$ = (0.37 , 0.31, 0.23, 0.09)

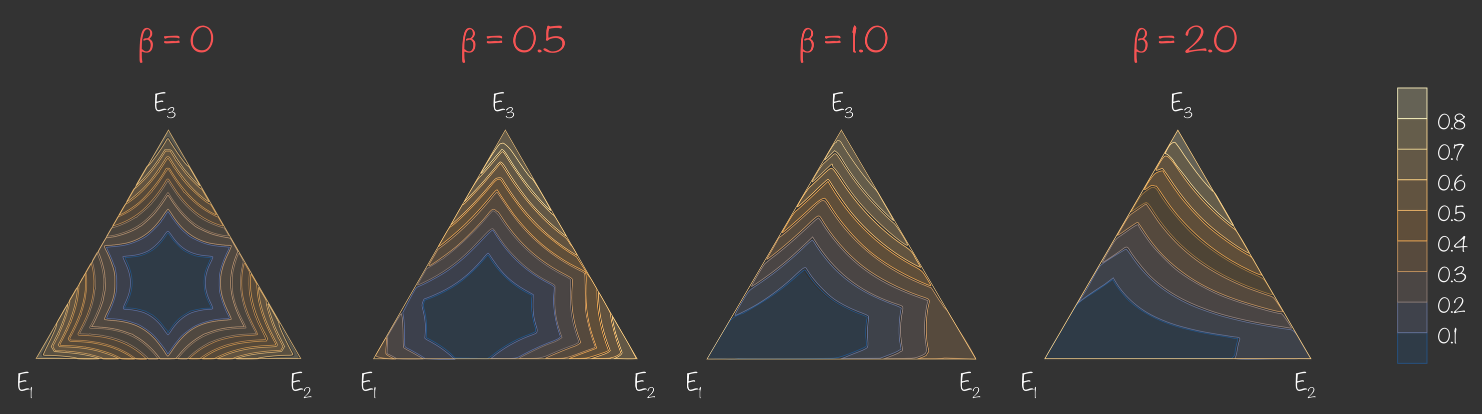


Finite temperatures $\beta > 0$: $\gamma = \dfrac{e^{-\beta H}}{Z}$ ⟶ The above results do not hold anymore! But, they can be generalised.

Example. $\mathbf{p}$ = (0.6, 0.3, 0.1) and E_s = (0,1,2)



Q. How does V_+ behave within the space of state?



Q. What is the role played by the volumes of the thermal cones?

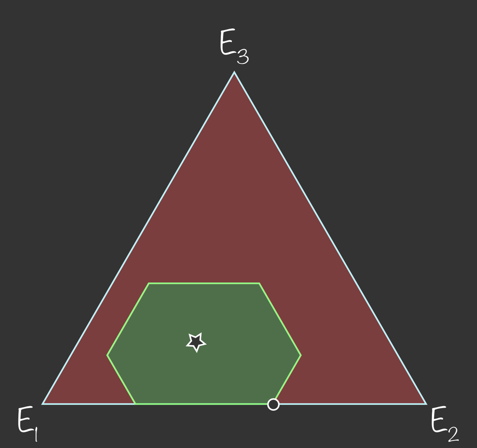
A. V_+ is a thermodynamic monotone:

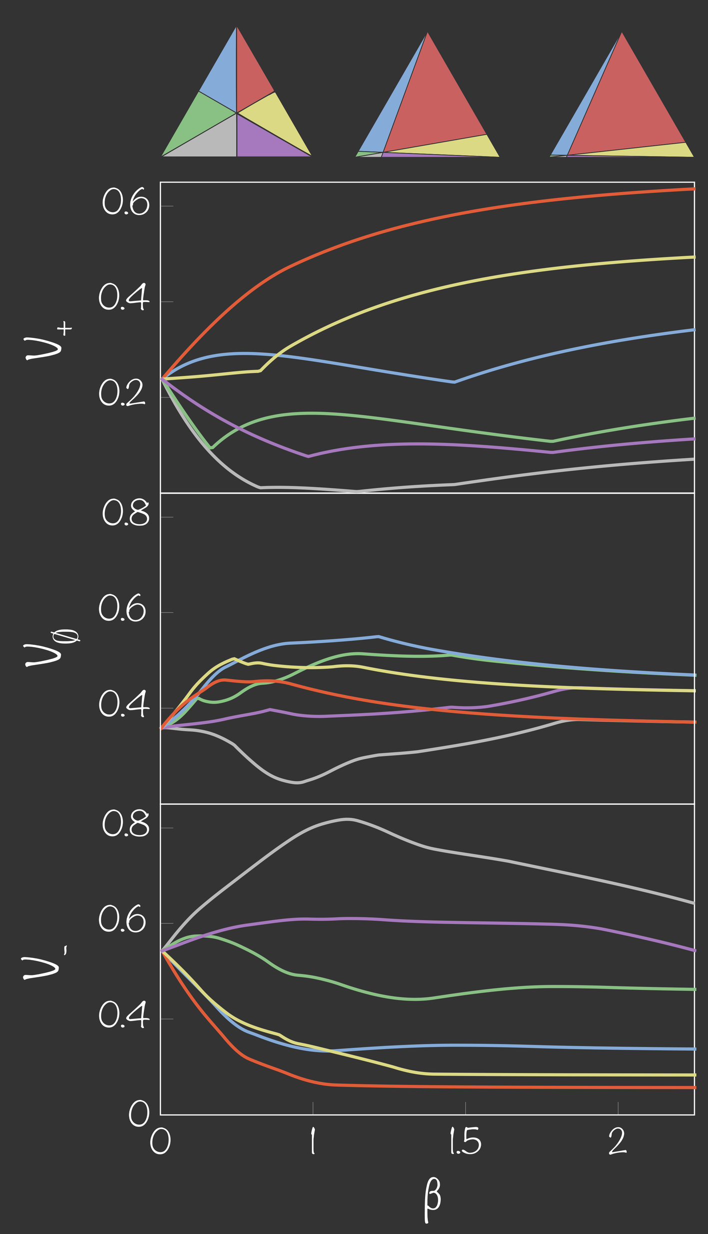
i. $V_+(\mathcal{E}(\rho)) \leq V_+(\rho)$

ii. $V_+(\gamma) = 0$

- Non-full rank state has $V_- = 0$

Example. $\mathbf{p}$ = (0.4, 0.6, 0), E_s = (0,1,2) and β = 0.5:





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